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SKEWED TURBULENT BOUNDARY LAYERS

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by

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TABLE OF CONTENTS

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ABSTRACT

The problem of predicting the growth of incompressible, skewed, turbulent boundary layers on smooth surfaces is considered. It is supposed that sufficient initial information and a complete knowledge of the external pressure distribution is given.

A computation scheme is devised which makes use of the two momentum integral relations and an auxiliary equation. The formulation of the latter is based on the empirical observation that if a velocity profile is plotted in a hodograph plane the outer portion is nearly always linear.

A theoretical means of describing velocity distributions is formulated. These theoretical profiles contain two "universal" functions and five parameters that are functions of the surface coordinates only. The "universal" functions are derived analytically and compared with experiment.

Finally, a comparison of theoretical and experimental velocity profiles suggests that the proposed computation scheme is only applicable to situations where gradients of flow quantities in the cross-flow direction are not too large. Therefore, the scheme is probably better suited to the external flow over wings, etc., than to flows in turbomachinery.

PREFACE

The purpose of this preface is merely to say a word about the manner in which references are indicated. When encountered in the text references are shown by the names of the authors followed by the date of the publication in parentheses, e.q. Clauser (1956). The references are listed at the end of the paper alphabetically according to the last name of the author and then by date in case more than one paper by the same man is noted. The purpose of this system is to avoid all footnotes and discontinuities in the text -- so dear to the hearts of humanities scholars and so distressing to read.

No attempt has been made to compile an extensive bibliography and so for any omissions, the author's apologies -- no slight was intended.

NOMENCLATURE

c	constant appearing in "law of the wall"
C	parameter appearing in the derivation of $F_2(\eta)$
C_1	constant appearing in the formulation of $F_1(\eta)$
$\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$	unit base vectors along the coordinate lines
$\underline{e}_o$, $\underline{e}_w$	unit base vectors appearing in the theoretical velocity profile
$F(\eta)$	two-dimensional self-preserving velocity profile
$F_1(\eta)$, $F_2(\eta)$	"universal" functions appearing in theoretical velocity profiles
G_1, G_2	quantities related to the "universality" of $F_1(\eta)$ and $F_2(\eta)$
h_1, h_2, h_3	scale factors in the curvilinear coordinate system
$Hh_n(t)$	function appearing in the derivation of $F_2(\eta)$
I_1, I_3, I_{11}, etc.	integrals appearing in the momentum integral relations
i_1, i_2, i_{11}, etc.	integrals involving the functions $F_1(\eta)$ and $F_2(\eta)$
$\bar{k}$	constant appearing in the expression for eddy viscosity
k_1	constant appearing in the formulation of $F_1(\eta)$
K_1, K_3	curvatures of S in the directions of x_1 and x_3 respectively
K_I, K_{II}	principal curvatures of S
$\lvert K\rvert_{max}$	maximum curvature of S without regard to sign
ℓ	scale of variation of mean quantities in the direction x_2
L	scale of variation of mean quantities in the directions x_1 and x_3
m_1, m_2	arbitrary numbers appearing in the three-dimensional skin-friction relationship
M	quantity appearing in the auxiliary equation
n	parameter appearing in the derivation of $F_2(\eta)$
p	pressure within the boundary layer

P	pressure outside the boundary layer
p_1, p_2, p_3, p_4, p_5	five parameters on which the theoretical profiles depend
$\underline{q}$	boundary layer velocity vector
$\underline{Q}$	free-stream velocity vector
Q	magnitude of the free-stream velocity
R_i	coefficients appearing in the final form of the momentum integral equations
S	surface on which the boundary layer develops
t	variable appearing in the derivation of $F_2(\eta)$
t_o	parameter appearing in the derivation of $F_2(\eta)$
T	parameter appearing in the derivation of $F_2(\eta)$
u_1, u_2, u_3	boundary layer velocity components referred to the coordinate system x_1, x_2, x_3
u'_1, u'_2, u'_3	components of the velocity fluctuation referred to the coordinate system x_1, x_2, x_3
U_1, U_2, U_3	free-stream velocity components referred to the coordinate system x_1, x_2, x_3
u, v, w	boundary layer velocity components in streamline coordinate system or Cartesian coordinate system
U, V, W	main-stream velocity components in streamline coordinate system or Cartesian coordinate system
U_m	order of magnitude of the velocity components u_1 and u_3
u_T^2	order of magnitude of the products $\overline{u'_i u'_j}$
$u^* = \sqrt{\frac{\tau_o}{\rho}}$	boundary layer velocity scale
$w(\eta)$	"wake function" of Coles (1956)
x_1, x_2, x_3	curvilinear coordinates
x, y, z	streamline coordinate system or Cartesian coordinate system
α	angle between $\underline{e}_o$ and $\underline{e}_1$
β	angle between $\underline{e}_w$ and $\underline{e}_1$

γ	$\frac{u^*}{\kappa Q}$
δ	boundary layer length scale
δ^*	two-dimensional displacement thickness
δ_x^* , δ_z^*	displacement thicknesses referred to a streamline co-ordinate system
ϵ	angle between $\underline{e}_1$ and $\underline{e}_3$
η	$\frac{x_2}{\delta}$
η_o	constant appearing in the formulation of $F_1(\eta)$
κ	constant appearing in "law of the wall"
ν	kinematic viscosity of fluid
ν_T	turbulent eddy viscosity
Π	parameter appearing in theoretical velocity profiles
ρ	density of fluid
$\underline{\tau}_o$	wall shear stress-vector
τ_o	magnitude of wall shear-stress
τ_{o1} , τ_{o3}	components of the wall shear-stress vector in the directions x_1 and x_3
φ	angle between $\underline{q}$ and $\underline{e}_1$
Φ	angle between $\underline{Q}$ and $\underline{e}_1$

1. INTRODUCTION

In the fifty five years since Prandtl first introduced the boundary layer concept, boundary layer theory has attained a position of importance and popularity in the field of fluid mechanics. Initially, most of the work on the subject dealt with laminar, two-dimensional, incompressible boundary layers. More recently, compressible flows of that nature have received their share of attention. Much less effort, however, has gone into the investigation of three-dimensional or skewed laminar boundary layers (the two designations will be used interchangeably). A skewed boundary layer is one in which the velocity vectors along a line normal to the surface are not collateral. The velocity vector varies in direction as well as magnitude with distance from the surface. For a good summary of three-dimensional boundary layer work, see Moore (1956).

Although boundary layers encoutered in practice are most often turbulent rather than laminar, turbulent boundary layers are not so well understood. Only recently have some sound theoretical considerations of the two-dimensional case been made; see e.g. Clauser (1956) and Townsend (1956).

Three-dimensional turbulent boundary layers have the greatest practical importance but have received the least attention because of their complexity. Very little experimental information exists and theoretical considerations are even more meagre. The work of Gruschwitz (1935), Prandtl (1946), and Mager (1952) is perhaps the best known. More recently Johnston (1957) and Gardow (1958) have dealt with the problem.

It is the purpose of the present report to propose a scheme for computing the development of skewed turbulent boundary layers. It is supposed that the vector free-stream velocity and its derivatives are known everywhere and, furthermore, that sufficient initial information is given to describe the state of the boundary layer along some line. It is assumed that the fluid is incompressible and that the flow is well described by mean flow quantities which are independent of time. The boundary layer is supposed to develop on a smooth, stationary surface. The assumption of a stationary surface implies that the mechanism producing skewing is curvature of the main-flow streamlines, i.e. the secondary flow is pressure driven. If this last is not the case, as for example near a rotor of an axial compressor, no essential difficulty is involved; the equations merely need be rewritten in the rotating coordinate system. If the surface is not aerodynamically smooth, it may be supposed that only a simple modification is necessary if the roughness is sufficiently great so that the flow near the surface is independent of viscosity. Should the surface be less than "fully rough", the flow near the surface will in general depend on the specific nature of the roughness elements and the situation is no longer simple. See, for example, Clauser (1956) and Townsend (1956).

2. THE EQUATIONS OF MOTION

2.1 The Coordinate System

Suppose that the given surface on which the boundary layer develops is S. A coordinate system may now be constructed in the following way:

Denote the family of surfaces formed by S and all parallel surfaces by x_2 = constant. The given surface S is $x_2 = 0$. Consider any two parametric lines (x_1 = constant, $x_2 = 0$) and (x_3 = constant, $x_2 = 0$) on the surface S. Construct the orthogonal trajectories of the family x_2 = constant. All the orthogonal trajectories which intersect the line (x_1 = constant, $x_2 = 0$) form a surface x_1 = constant. Similarly, a surface x_3 = constant may be generated. The intersections of the three families of surfaces, x_1 = constant, x_2 = constant, x_3 = constant, form the coordinate curves. The curve (x_2 = constant, x_3 = constant) may be called the x_1 curve, the curve (x_3 = constant, x_1 = constant) the x_2 curve, and so on.

Let $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ denote unit base vectors tangent to the coordinate lines at any point. $\underline{e}_1$ and $\underline{e}_3$ lie in the plane tangent to x_2 = constant. Therefore, $\underline{e}_1 \cdot \underline{e}_2 = 0$ and $\underline{e}_2 \cdot \underline{e}_3 = 0$. In general, however, $\underline{e}_1 \cdot \underline{e}_3 \neq 0$, i.e. if the angle between $\underline{e}_1$ and $\underline{e}_3$ is denoted by ϵ, $\epsilon = \frac{\pi}{2}$. The coordinate system is therefore semi-orthogonal. See Weatherburn (1930), p. 74. The main features of the coordinate system are shown in Figure 1.

Let h_1, h_2, h_3 be the relevant scale facors so that the element of arc length ds of any curve is given by

$$ds^2 = \left(h_1 \underline{e}_1 dx_1 + h_2 \underline{e}_2 dx_2 + h_3 \underline{e}_3 dx_3 \right)^2$$

or

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 + 2 h_1 h_3 dx_1 dx_3 \cos\epsilon$$

As has been stated the $\underline{e}_i$ ($i = 1, 2, 3$) are not in general mutually orthogonal. If, however, the parametric surfaces do constitute a triply orthogonal family (i.e. a Lame family of surfaces), Dupin's theorem states that the coordinate lines are lines of curvature on each surface. See Weatherburn (1927), p. 211.

For simplicity, naturally, one would prefer to deal only with orthogonal coordinate systems. However, the restriction that the coordinate lines x_1 and x_3 coincide with the lines of curvature of S may be undesirable. As has been pointed out by Howarth (1955), this is an undue restriction in that it is possible to use a coordinate system which is only locally orthogonal on the surface S without in any way complicating the boundary layer equations of motion. The phrase "boundary layer" is underlined to indicate there is a degree of approximation involved. The equations of motion in the orthogonal and semi-orthogonal coordinate systems have the same form only within the framework of the boundary layer approximations.

In order to see this, consider the surface S. The first order magnitudes see Weatherburn (1927) are

$$E' = h_1^2$$

$$F' = h_1 h_3 \cos\epsilon$$

$$G' = h_3^2$$

The second order magnitudes become

$$L' = -\frac{h_1}{h_2}\frac{\partial h_1}{\partial x_2}$$

$$M' = -\frac{1}{2h_2}\left[\cos\epsilon\,\frac{\partial(h_1 h_3)}{\partial x_2} - h_1 h_3 \sin\epsilon\,\frac{\partial\epsilon}{\partial x_2}\right]$$

$$N' = -\frac{h_3}{h_2}\frac{\partial h_3}{\partial x_2}$$

From here on it is assumed that $\epsilon = \frac{\pi}{2}$ on S. Then the curvature K_1 of S in the direction x_1 is

$$K_1 = -\frac{1}{h_2}\frac{\partial(\ln h_1)}{\partial x_2} \tag{2.1-1a}$$

while the curvature K_3 in the direction x_3 is

$$K_3 = -\frac{1}{h_2}\frac{\partial(\ln h_3)}{\partial x_2} \tag{2.1-1b}$$

Let K_I and K_{II} denote the principal curvatures of S. Then

$$K_I = \frac{1}{2}\left[-\frac{1}{h_2}\frac{\partial(\ln h_1 h_3)}{\partial x_2} + \sqrt{\left[\frac{1}{h_2}\frac{\partial}{\partial x_2}\left(\ln\frac{h_1}{h_3}\right)\right]^2 + \frac{1}{h_2^2}\left(\frac{\partial\epsilon}{\partial x_2}\right)^2}\,\right] \tag{2.1-2a}$$

and

$$K_{II} = \frac{1}{2}\left[-\frac{1}{h_2}\frac{\partial(\ln h_1 h_3)}{\partial x_2} - \sqrt{\left[\frac{1}{h_2}\frac{\partial}{\partial x_2}\left(\ln\frac{h_1}{h_3}\right)\right]^2 + \frac{1}{h_2^2}\left(\frac{\partial\epsilon}{\partial x_2}\right)^2}\,\right] \tag{2.1-2b}$$

Combining (2.1-1) and (2.1-2) one obtains

$$\left(K_{I} - K_{II}\right)^2 = \left(K_1 - K_3\right)^2 + \frac{1}{h_2^2}\left(\frac{\partial \epsilon}{\partial x_2}\right)^2 \qquad (2.1\text{-}3)$$

Since $\quad 0 \leqslant \left(K_1 - K_3\right)^2 \leqslant \left(K_{I} - K_{II}\right)^2$

equation (2.1-3) implies that

$$\frac{1}{h_2^2}\left(\frac{\partial \epsilon}{\partial x_2}\right)^2 \leqslant \left(K_{I} - K_{II}\right)^2$$

Therefore,

$$\left|\frac{1}{h_2}\frac{\partial \epsilon}{\partial x_2}\right| \leqslant \left|K_{I} - K_{II}\right| \leqslant \left|K_{I}\right| + \left|K_{II}\right|$$

If $|K|_{max}$ is the maximum curvature of S without regard to sign, it follows that

$$\left|\frac{1}{h_2}\frac{\partial \epsilon}{\partial x_2}\right| \leqslant 2\left|K\right|_{max} \qquad (2.1\text{-}4)$$

In developing the equations of motion it will be assumed that $h_2 \delta x_2 = O(\ell)$ where ℓ is a measure of the boundary layer thickness. Therefore, $|\Delta \epsilon|$ is of order $2\ell\,|K|_{max}$. If $\ell\,|K|_{max} \ll 1$, i.e. if the boundary layer thickness is small compared to the smallest radius of curvature of S, $\Delta \epsilon$ will be negligible and it may be assumed that $\epsilon = \frac{\pi}{2}$ everywhere even if the lines x_1 and x_3 are not lines of curvature on S .

One other fact may be demonstrated in the same way. Since $|K_1| \leqslant |K|_{max}$, and $|K_3| \leqslant |K|_{max}$, equations (2.1-1 a, b) imply that

$$\left|\frac{1}{h_2}\frac{\partial(\ln h_1)}{\partial x_2}\right| \leqslant |K|_{max} \qquad (2.1\text{-}5a)$$

and
$$\left| \frac{1}{h_2} \frac{\partial(\ln h_3)}{\partial x_2} \right| \leqslant |K|_{max} \qquad (2.1-5b)$$

Therefore, so long as $\ell |K|_{max} \ll 1$, h_1 and h_3 may be considered as functions of x_1 and x_3 only.

2.2 The Boundary Layer Equations of Motion in Differential Form

If we confine our attention to the mean motion only, the governing equations are as follows:

The Continuity Equation

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 u_1) + \frac{\partial}{\partial x_2}(h_3 h_1 u_2) + \frac{\partial}{\partial x_3}(h_1 h_2 u_3)\right] = 0 \qquad (2.2-1)$$

The x_1 - Momentum Equation

$$\frac{1}{h_1}\frac{\partial}{\partial x_1}(u_1^2 + \overline{u_1'^2}) + \frac{1}{h_2}\frac{\partial}{\partial x_2}(u_1 u_2 + \overline{u_1' u_2'}) + \frac{1}{h_3}\frac{\partial}{\partial x_3}(u_1 u_3 + \overline{u_1' u_3'})$$

$$+ \frac{1}{h_1^2 h_2 h_3}\left[(u_1^2 + \overline{u_1'^2})\frac{\partial(h_1 h_2 h_3)}{\partial x_1} + (u_1 u_2 + \overline{u_1' u_2'})\frac{\partial(h_1^2 h_3)}{\partial x_2} + (u_1 u_3 + \overline{u_1' u_3'})\frac{\partial(h_1^2 h_2)}{\partial x_3}\right]$$

$$-\left[\frac{(u_1^2 + \overline{u_1'^2})}{h_1^2}\frac{\partial h_1}{\partial x_1} + \frac{(u_2^2 + \overline{u_2'^2})}{h_1 h_2}\frac{\partial h_2}{\partial x_1} + \frac{(u_3^2 + \overline{u_3'^2})}{h_1 h_3}\frac{\partial h_3}{\partial x_1}\right] = -\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1}$$

$$+ \frac{\nu}{h_2 h_3}\left\{\frac{\partial}{\partial x_3}\left[\frac{h_2}{h_1 h_3}\left(\frac{\partial(h_1 u_1)}{\partial x_3} - \frac{\partial(h_3 u_3)}{\partial x_1}\right)\right] - \frac{\partial}{\partial x_2}\left[\frac{h_3}{h_1 h_2}\left(\frac{\partial(h_2 u_2)}{\partial x_1} - \frac{\partial(h_1 u_1)}{\partial x_2}\right)\right]\right\} \qquad (2.2-2)$$

and two other momentum equations obtained from (2.2-2) by cyclic change of the suffices.

In the above, the u_i are mean velocity components, the u'_i are components of the velocity fluctuation, p is the mean pressure, ρ and ν are the density and kinematic viscosity respectively.

We now suppose that

$$h_1 \delta x_1 = O(L), \qquad h_2 \delta x_2 = O(\ell), \qquad h_3 \delta x_3 = O(L),$$
$$u_1 = O(U_m), \qquad u_3 = O(U_m), \qquad \delta u_1 = O(U_m),$$
$$\delta u_3 = O(U_m), \qquad \overline{u'_i u'_j} = O(u_T^2), \qquad \delta(\overline{u'_i u'_j}) = O(u_T^2) .$$

It has been observed experimentally that ℓ is an order of magnitude less than L and u_T^2 is an order of magnitude less than U_m^2. Equation (2.2-1) implies that u_2 is at most of order $U_m \frac{\ell}{L}$.

Without loss of generality we can take $h_2 = 1$. Then x_2 is simply distance measured normal to the given surface. Furthermore, in accordance with the results of section 2.1 we have assumed that $\ell |K|_{max} \ll 1$. Therefore, h_1 and h_3 may be considered as functions of x_1 and x_3 only.

Retaining only the highest order terms the x_1 - momentum equation becomes

$$\frac{1}{h_1}\frac{\partial(u_1^2)}{\partial x_1} + \frac{\partial}{\partial x_2}\left(u_1 u_2 + \overline{u'_1 u'_2}\right) + \frac{1}{h_3}\frac{\partial(u_1 u_3)}{\partial x_3}$$
$$+ \frac{(u_1^2 - u_3^2)}{h_1 h_3}\frac{\partial h_3}{\partial x_1} + 2\,\frac{u_1 u_3}{h_1 h_3}\frac{\partial h_1}{\partial x_3}$$
$$= -\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1} + \nu \frac{\partial^2 u_1}{\partial x_2^2} \qquad (2.2\text{-}3a)$$

The x_2 - momentum equation is

$$\frac{\partial(\overline{u_2'^2})}{\partial x_2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} - \frac{\nu}{h_1 h_3}\left[\frac{\partial}{\partial x_1}\left(h_3\frac{\partial u_1}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(h_1\frac{\partial u_3}{\partial x_2}\right)\right] \qquad (2.2\text{-}3b)$$

and the x_3 - momentum equation is

$$\frac{1}{h_1}\frac{\partial(u_1 u_3)}{\partial x_1} + \frac{\partial}{\partial x_2}\left(u_2 u_3 + \overline{u_2' u_3'}\right) + \frac{1}{h_3}\frac{\partial(u_3^2)}{\partial x_3}$$
$$-\frac{(u_1^2 - u_3^2)}{h_1 h_3}\frac{\partial h_1}{\partial x_3} + 2\frac{u_1 u_3}{h_1 h_3}\frac{\partial h_3}{\partial x_1}$$
$$= -\frac{1}{\rho h_3}\frac{\partial p}{\partial x_3} + \nu\frac{\partial^2 u_3}{\partial x_2^2} \qquad (2.2\text{-}3c)$$

At high Reynolds numbers the viscous terms will be negligible everywhere except very close to the wall.

Neglecting the viscous term, (2.2-3b) may be integrated to give

$$\overline{u_2'^2} = -\frac{p}{\rho} + \frac{P}{\rho} \qquad (2.2\text{-}4)$$

where P is a function of x_1 and x_3 only and is the pressure outside the boundary layer.

Substituting (2.2-4) into (2.2-3 a, c) and neglecting $\frac{1}{h_1}\frac{\partial \overline{u_2'^2}}{\partial x_1}$ and $\frac{1}{h_3}\frac{\partial \overline{u_2'^2}}{\partial x_3}$ we obtain

$$\frac{1}{h_1}\frac{\partial(u_1^2)}{\partial x_1} + \frac{\partial}{\partial x_2}\left(u_1 u_2 + \overline{u_1' u_2'}\right) + \frac{1}{h_3}\frac{\partial(u_1 u_3)}{\partial x_3} + \frac{(u_1^2 - u_3^2)}{h_1 h_3}\frac{\partial h_3}{\partial x_1}$$

$$+ 2\frac{u_1 u_3}{h_1 h_3}\frac{\partial h_1}{\partial x_3} = -\frac{1}{\rho h_1}\frac{\partial P}{\partial x_1} + \nu\frac{\partial^2 u_1}{\partial x_2^2} \qquad (2.2\text{-}5a)$$

and $$\frac{1}{h_1}\frac{\partial(u_1u_3)}{\partial x_1} + \frac{\partial}{\partial x_2}\left(u_2u_3 + \overline{u_2'u_3'}\right) + \frac{1}{h_3}\frac{\partial(u_3^2)}{\partial x_3} - \frac{(u_1^2 - u_3^2)}{h_1h_3}\frac{\partial h_1}{\partial x_3}$$

$$+ 2\frac{u_1u_3}{h_1h_3}\frac{\partial h_3}{\partial x_1} = -\frac{1}{\rho h_3}\frac{\partial P}{\partial x_3} + \nu\frac{\partial^2 u_3}{\partial x_2^2} \qquad (2.2\text{-}5b)$$

The continuity equation simplifies to

$$\frac{1}{h_1h_3}\frac{\partial(h_3u_1)}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{1}{h_1h_3}\frac{\partial(h_1u_3)}{\partial x_3} = 0 \qquad (2.2\text{-}6)$$

If the free-stream velocity components are U_1, U_2, and U_3, the continuity equation in the free-stream becomes

$$\frac{1}{h_1h_3}\frac{\partial(h_3U_1)}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{1}{h_1h_3}\frac{\partial(h_1U_3)}{\partial x_3} = 0 \qquad (2.2\text{-}7)$$

Further, assuming that as the free-stream is approached $\frac{\partial(\overline{u_1'u_2'})}{\partial x_2}$, $\frac{\partial(\overline{u_2'u_3'})}{\partial x_2}$, $\frac{\partial u_1}{\partial x_2}$, and $\frac{\partial u_3}{\partial x_2}$ all approach zero, the free-stream momentum equations become

$$\frac{1}{h_1}\frac{\partial(U_1^2)}{\partial x_1} + \frac{\partial(U_1U_2)}{\partial x_2} + \frac{1}{h_3}\frac{\partial(U_1U_3)}{\partial x_3} + \frac{(U_1^2 - U_3^2)}{h_1h_3}\frac{\partial h_3}{\partial x_1}$$

$$+ 2\frac{U_1U_3}{h_1h_3}\frac{\partial h_1}{\partial x_3} = -\frac{1}{\rho h_1}\frac{\partial P}{\partial x_1} \qquad (2.2\text{-}8a)$$

and

$$\frac{1}{h_1}\frac{\partial(U_1U_3)}{\partial x_1} + \frac{\partial(U_2U_3)}{\partial x_2} + \frac{1}{h_3}\frac{\partial(U_3^2)}{\partial x_3} - \frac{(U_1^2 - U_3^2)}{h_1h_3}\frac{\partial h_1}{\partial x_3}$$

$$+ 2\frac{U_1U_3}{h_1h_3}\frac{\partial h_3}{\partial x_1} = -\frac{1}{\rho h_3}\frac{\partial P}{\partial x_3} \qquad (2.2\text{-}8b)$$

Finally, the pressure P may be eliminated between (2.2-5) and (2.2-8) to give

$$\frac{1}{h_1}\frac{\partial}{\partial x_1}\left(u_1^2-U_1^2\right) + \frac{\partial}{\partial x_2}\left(u_1u_2-U_1U_2+\overline{u_1'u_2'}\right) + \frac{1}{h_3}\frac{\partial}{\partial x_3}\left(u_1u_3-U_1U_3\right)$$
$$+\left[\left(u_1^2-U_1^2\right)-\left(u_3^2-U_3^2\right)\right]\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1} + 2\left(u_1u_3-U_1U_3\right)\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3}$$
$$= \nu\frac{\partial^2 u_1}{\partial x_2^2} \qquad (2.2\text{-}9\,a)$$

and

$$\frac{1}{h_1}\frac{\partial}{\partial x_1}\left(u_1u_3-U_1U_3\right) + \frac{\partial}{\partial x_2}\left(u_2u_3-U_2U_3+\overline{u_2'u_3'}\right) + \frac{1}{h_3}\frac{\partial}{\partial x_3}\left(u_3^2-U_3^2\right)$$
$$-\left[\left(u_1^2-U_1^2\right)-\left(u_3^2-U_3^2\right)\right]\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3} + 2\left(u_1u_3-U_1U_3\right)\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1}$$
$$= \nu\frac{\partial^2 u_3}{\partial x_2^2} \qquad (2.2\text{-}9\,b)$$

2.3 The Momentum Integral Relations

In order to derive the momentum integral relations we need only integrate equations (2.2-9 a, b) with respect to x_2 from 0 to ∞. Before doing this, however, let us introduce the following notation:

Let Q be the magnitude of the free-stream velocity, i.e.

$$Q^2 = U_1^2 + U_3^2$$

and $\quad U_1 = Q\cos\Phi \quad , \quad U_3 = Q\sin\Phi$

where Φ is the angle between the direction of x_1 and the direction of the free stream velocity vector. It is assumed that Φ is positive when measured towards x_3.

Let η be a new dimensionless variable of integration defined as

$$\eta = \frac{x_2}{\delta}$$

where δ is an appropriate length scale in the boundary layer. δ is a function of x_1 and x_3 only.

The following definitions may now be made:

$$\left.\begin{aligned}
I_1 &= \frac{1}{Q}\int_0^\infty (U_1 - u_1)\, d\eta \\
I_3 &= \frac{1}{Q}\int_0^\infty (U_3 - u_3)\, d\eta \\
I_{11} &= \frac{1}{Q^2}\int_0^\infty (U_1 - u_1)^2\, d\eta \\
I_{13} &= \frac{1}{Q^2}\int_0^\infty (U_1 - u_1)(U_3 - u_3)\, d\eta \\
I_{33} &= \frac{1}{Q^2}\int_0^\infty (U_3 - u_3)^2\, d\eta
\end{aligned}\right\} \qquad (2.3-1)$$

With the aid of (2.3-1) and making some use of the continuity relations (2.2-6) and (2.2-7) the momentum integral equations may be written as

$$\frac{1}{h_1}\frac{\partial I_{11}}{\partial x_1} + \frac{1}{h_3}\frac{\partial I_{13}}{\partial x_3} - \frac{\cos\Phi}{h_1}\frac{\partial I_1}{\partial x_1} - \frac{\sin\Phi}{h_3}\frac{\partial I_1}{\partial x_3}$$

$$+ (I_{11} - I_1\cos\Phi)\frac{1}{\delta h_1}\frac{\partial\delta}{\partial x_1} + (I_{13} - I_1\sin\Phi)\frac{1}{\delta h_3}\frac{\partial\delta}{\partial x_3}$$

$$+ (2I_{11} - 3I_1\cos\Phi)\frac{1}{Qh_1}\frac{\partial Q}{\partial x_1} + (2I_{13} - 2I_1\sin\Phi - I_3\cos\Phi)\frac{1}{Qh_3}\frac{\partial Q}{\partial x_3}$$

$$+ 2I_1\sin\Phi\frac{1}{h_1}\frac{\partial\Phi}{\partial x_1} - (I_1\cos\Phi - I_3\sin\Phi)\frac{1}{h_3}\frac{\partial\Phi}{\partial x_3}$$

$$+ (I_{11} - I_{33} - I_1\cos\Phi + 2I_3\sin\Phi)\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1}$$

$$+ (2I_{13} - 2I_1\sin\Phi - I_3\cos\Phi)\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3} + \frac{\tau_{01}}{\delta\rho Q^2} = 0 \qquad (2.3\text{-}2a)$$

and

$$\frac{1}{h_3}\frac{\partial I_{33}}{\partial x_3} + \frac{1}{h_1}\frac{\partial I_{13}}{\partial x_1} - \frac{\sin\Phi}{h_3}\frac{\partial I_3}{\partial x_3} - \frac{\cos\Phi}{h_1}\frac{\partial I_3}{\partial x_1}$$

$$+ \left(I_{33} - I_3 \sin\Phi\right)\frac{1}{\delta h_3}\frac{\partial \delta}{\partial x_3} + \left(I_{13} - I_3 \cos\Phi\right)\frac{1}{\delta h_1}\frac{\partial \delta}{\partial x_1}$$

$$+ \left(2I_{33} - 3I_3 \sin\Phi\right)\frac{1}{Q h_3}\frac{\partial Q}{\partial x_3} + \left(2I_{13} - 2I_3\cos\Phi - I_1 \sin\Phi\right)\frac{1}{Q h_1}\frac{\partial Q}{\partial x_1}$$

$$- 2I_3 \cos\Phi \frac{1}{h_3}\frac{\partial \Phi}{\partial x_3} + \left(I_3 \sin\Phi - I_1 \cos\Phi\right)\frac{1}{h_1}\frac{\partial \Phi}{\partial x_1}$$

$$+ \left(I_{33} - I_{11} - I_3 \sin\Phi + 2I_1 \cos\Phi\right)\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3}$$

$$+ \left(2I_{13} - 2I_3 \cos\Phi - I_1 \sin\Phi\right)\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1} + \frac{\tau_{03}}{\delta \rho Q^2} = 0 \qquad (2.3-2\,b)$$

τ_{01} and τ_{03} are components of the wall shear-stress vector in the directions of x_1 and x_3 respectively.

3. THE AUXILIARY EQUATION

3.1 Introduction

The essence of the problem as regards turbulent boundary layers, skewed or otherwise, is that, in considering the time averaged equations of motion, one invariably has to deal with more unknowns than there are equations. There is as yet no theoretical connection between the Reynolds stresses and the other mean-flow quantities. Additional information must be introduced. At the present time, of necessity, that information is empirical. The only choice available to the analyst is the manner in which he introduces the empiricism. For example, Von Doenhoff and Tetervin (1943) in their calculation scheme for the two-dimensional turbulent boundary layer, make use of an empirically devised equation for the rate of change of shape factor. Truckenbrodt (1952), on the other hand, uses the energy equation with a dissipation function determined by Rotta (1950) from a number of measurements.

For the present problem, however, an entirely different approach seems more satisfactory. It has long been observed that if one plots a velocity profile from a skewed turbulent boundary layer in a hodograph plane, almost always a straight line is obtained in the outer portion of the boundary layer. Such a straight line relationship may be stated as

$$(u_3 - U_3) = M(u_1 - U_1) \tag{3.1-1}$$

where M is found to be independent of x_2 near the outer edge of the boundary layer. For a fairly extensive collection of evidence testifying to the validity of this relation see Johnston (1957).

Johnston used (3.1-1) to develop an additional equation connecting the various boundary layer parameters and found that his result was well substantiated by his measurements. Gardow (1958) also tested Johnston's equation but with less success. However, it would seem that in Gardow's case better agreement could be obtained by abandoning the untested assumption that initially his boundary layer was collateral.

In any event because of its relative simplicity and its freedom from empirically determined constants, Johnston's equation will be used and his derivation is repeated below in somewhat more general form.

3.2 Derivation of the Auxiliary Equation

The first step is to eliminate u_2 between (2.2-9 a, b). Using (3.1-1) we have

$$\frac{\partial}{\partial x_2}(u_2 u_3 - U_2 U_3) = M\frac{\partial}{\partial x_2}(u_1 u_2 - U_1 U_2) + (U_3 - M U_1)\frac{\partial}{\partial x_2}(u_2 - U_2) \tag{3.2-1}$$

Making use of (3.2-1) we multiply (2.2-9a) by M and (2.2-9b) by -1 and add. The result ~~is~~

$$\begin{aligned}
&\frac{M}{h_1}\frac{\partial}{\partial x_1}(u_1^2 - U_1^2) - \frac{1}{h_1}\frac{\partial}{\partial x_1}(u_1 u_3 - U_1 U_3) - (U_3 - M U_1)\frac{\partial}{\partial x_2}(u_2 - U_2)\\
&+ \frac{M}{h_3}\frac{\partial}{\partial x_3}(u_1 u_3 - U_1 U_3) - \frac{1}{h_3}\frac{\partial}{\partial x_3}(u_3^2 - U_3^2) + M\frac{\partial(\overline{u_1' u_2'})}{\partial x_2} - \frac{\partial(\overline{u_2' u_3'})}{\partial x_2}\\
&+ \left[M(u_1^2 - U_1^2) - M(u_3^2 - U_3^2) - 2(u_1 u_3 - U_1 U_3)\right]\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1}\\
&+ \left[2M(u_1 u_3 - U_1 U_3) + (u_1^2 - U_1^2) - (u_3^2 - U_3^2)\right]\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3}\\
&= \nu\left(M\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2 u_3}{\partial x_2^2}\right)
\end{aligned} \tag{3.2-2}$$

As indicated be Johnston (1957), the outer portion of a cross-flow profile in a skewed turbulent boundary layer can be predicted with a simple shearless analysis if the profile in the direction of the main flow is known. Since, in essence, that is what is being done here, the terms involving Reynolds stresses in (3.2-2) will be ignored. Then, eliminating $\frac{\partial}{\partial x_2}\left(u_2 - U_2\right)$ with the aid of (2.2-6) and (2.2-7) and substituting for u_3 from (3.1-1) equation(3.2-2) becomes

$$
\begin{aligned}
&-\frac{M(U_1-u_1)}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{(U_1-u_1)^2}{h_1}\frac{\partial M}{\partial x_1} + \frac{(U_1-u_1)}{h_1}\frac{\partial U_3}{\partial x_1} + \frac{U_1(U_1-u_1)}{h_1}\frac{\partial M}{\partial x_1} \\
&+\frac{U_3(U_1-u_1)}{h_3}\frac{\partial M}{\partial x_3} - \frac{M(U_1-u_1)^2}{h_3}\frac{\partial M}{\partial x_3} + \frac{M(U_1-u_1)}{h_3}\frac{\partial U_3}{\partial x_3} - \frac{M^2(U_1-u_1)}{h_3}\frac{\partial U_1}{\partial x_3} \\
&+\left[-M^3(U_1-u_1)^2 - M(U_1-u_1)^2 + M U_1(U_1-u_1) + 2M^2 U_3(U_1-u_1) + U_3(U_1-u_1)\right]\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1} \\
&+\left[M^2(U_1-u_1)^2 + (U_1-u_1)^2 - M^2 U_1(U_1-u_1) - 2U_1(U_1-u_1) - M U_3(U_1-u_1)\right]\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3} = 0 \qquad (3.2\text{-}3)
\end{aligned}
$$

Finally, in the outer portion of the boundary layer $(U_1 - u_1)$ is small. Therefore, dividing by $(U_1 - u_1)$ and retaining only the largest terms equation (3.2-3) becomes after substituting for U_1 and U_3 in terms of Q and Φ

$$
\begin{aligned}
&\frac{\cos\Phi}{h_1}\frac{\partial M}{\partial x_1} - \left(M\cos\Phi - \sin\Phi\right)\frac{1}{Q h_1}\frac{\partial Q}{\partial x_1} + \left(M\sin\Phi + \cos\Phi\right)\frac{1}{h_1}\frac{\partial \Phi}{\partial x_1} \\
&+\frac{\sin\Phi}{h_3}\frac{\partial M}{\partial x_3} - \left(M^2\cos\Phi - M\sin\Phi\right)\frac{1}{Q h_3}\frac{\partial Q}{\partial x_3} + \left(M^2\sin\Phi + M\cos\Phi\right)\frac{1}{h_3}\frac{\partial \Phi}{\partial x_3} \\
&+\left(M\cos\Phi + 2M^2\sin\Phi + \sin\Phi\right)\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1} \\
&-\left(M^2\cos\Phi + 2\cos\Phi + M\sin\Phi\right)\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3} = 0 \qquad (3.2\text{-}4)
\end{aligned}
$$

4. VELOCITY PROFILES

4.1 Introduction

Consider at first a two-dimensional boundary layer flowing in the x-direction, say. The boundary layer may be roughly divided into two regions:

1) An inner region whose thickness is usually of the order of 10 per cent of the total boundary layer thickness. The most important characteristic of this region is that it is virtually in a state of energy equilibrium -- the local production of turbulent energy being very nearly equal to local viscous dissipation. Furthermore, almost all of the production of turbulent energy in the boundary layer takes place within this region. There is a slight excess of production over dissipation, the excess energy diffusing outwards. Because of the high turbulent intensities within this portion of the boundary layer, its response to disturbances is relatively rapid. Moreover, the shear stress in the inner layer is approximately constant and equal to the wall shear stress. It is a consequence of the features mentioned above that the motion within this inner region is effectively determined by the wall shear stress τ_o and the viscosity ν. Therefore, the velocity distribution u is given by

$$\frac{u}{u^*} = f\left(\frac{y u^*}{\nu}\right) \qquad (4.1-1)$$

where $u^* = \sqrt{\frac{\tau_o}{\rho}}$ and y is the distance from the surface. (4.1-1) is sometimes called the "law of the wall".

2) An outer region. Here the motion resembles that in a wake. Production of turbulent energy and dissipation are small. Turbulent intensities are low or, in other words, the size of the eddies is large. This region responds only very slowly to disturbances and the velocity distribution within this portion of the boundary layer depends on the history of the boundary layer up to that point. The flow in the outer region is independent of viscosity except in so far as viscosity determines the wall shear stress. It has been found that the effect of changing skin-friction coefficient is minimized if the velocity defect $\left(\frac{u-U}{u^*}\right)$ is plotted against $\frac{y}{\delta}$ where δ is a properly chosen length scale. Therefore, in the outer portion of the boundary layer we have

$$\left(\frac{u-U}{u^*}\right) = g\left(\frac{y}{\delta}\right) \qquad (4.1\text{-}2)$$

where the form of the function g depends primarily on the pressure history of the boundary layer.

At sufficiently high Reynolds numbers the two profile descriptions (4.1-1) and (4.1-2) overlap and then it can be shown (e.g. see Millikan (1938)) that for consistency (4.1-1) takes the explicit form

$$\frac{u}{u^*} = \frac{1}{K} \ln \frac{yu^*}{\nu} + c \qquad (4.1\text{-}3)$$

where K and c are absolute constants. Since (4.1-2) and (4.1-3) both hold we must have

$$\frac{U}{u^*} = \frac{1}{K} \ln \frac{\delta u^*}{\nu} + c + A \qquad (4.1\text{-}4)$$

where A is a number depending on the pressure history of the flow and on the definition of δ. In the region of overlap (4.1-2) takes the form

$$\left(\frac{u-U}{u^*}\right) = \frac{1}{K}\ln\left(\frac{y}{\delta}\right) - A \qquad (4.1\text{-}5)$$

Coles (1956) in a most thorough study found that he could achieve a good description of a variety of two-dimensional turbulent boundary layer profiles by assuming a defect law in the form

$$\left(\frac{u-U}{u^*}\right) = \frac{1}{K}\ln\left(\frac{y}{\delta}\right) + \frac{\Pi}{K}\left[w\left(\frac{y}{\delta}\right) - 2\right] \qquad (4.1\text{-}6)$$

where $w\left(\frac{y}{\delta}\right)$ was a "universal" function found empirically and Π was a parameter depending only on x. Coles further suggested a way of generalizing his method of profile description to handle skewed boundary layers. It is this suggestion that we wish to pursue with some modifications.

4.2 The Profile Assumptions

Consider a typical skewed boundary layer profile plotted in a hodograph plane and shown in Figure 2. There are almost always two distinct directions, I and II, associated with such a plot. These directions are indicated by the angles α and β in the figure. The first direction I has been found to be the direction of the wall shear stress and so α is the angle between $\underline{e}_1$ and the wall shear stress vector $\underline{\tau}_o$. It will be recognized that tan β is the quantity referred to as M in Sec. 3.

Let $\underline{e}_o$ and $\underline{e}_w$ be unit vectors in the directions I and II respectively. Let $\underline{q}$ be the boundary layer velocity vector and $\underline{Q}$ the free-stream velocity vector. $\underline{q}$ makes an angle φ with respect to $\underline{e}_1$. All angles are assumed to be positive when measured toward $\underline{e}_3$. This notation is illustrated in Figure 3.

We assume a vector defect law

$$\underset{\sim}{q} - \underset{\sim}{Q} = \underset{\sim}{e}_o \frac{u^*}{K} F_1(\eta) + \underset{\sim}{e}_w \frac{\Pi u^*}{K} F_2(\eta) \qquad (4.2\text{-}1)$$

where η, u^* and K are as defined before and F_1 and F_2 are supposed to be "universal" functions. Further, it is supposed that near the wall the logarithmic "law of the wall" holds so that

$$\underset{\sim}{q} = \underset{\sim}{e}_o \frac{u^*}{K}\left[\ln \frac{x_2 u^*}{\nu} + Kc\right] \qquad (4.2\text{-}2)$$

for small x_2.

Since (4.2-1) and (4.2-2) hold simultaneously, consistency requires that

$$\underset{\sim}{Q} = \underset{\sim}{e}_o \frac{u^*}{K}\left[\ln \frac{\delta u^*}{\nu} + Kc + m_1\right] + \underset{\sim}{e}_w \frac{\Pi u^*}{K} m_2 \qquad (4.2\text{-}3)$$

where m_1 and m_2 are essentially arbitrary numbers -- m_1 depending only on the definition of the length scale δ, and m_2 determining the value of Π associated with any particular profile other than the one for which $\Pi = 0$. Equation (4.2-3) is nothing more than a vector skin-friction relationship. It relates the length scales δ and $\frac{\nu}{u^*}$ to the velocity scale u^*.

Combining (4.2-2) and (4.2-3) we see that for small x_2 the defect law (4.2-1) has the form

$$\underset{\sim}{q} - \underset{\sim}{Q} = \underset{\sim}{e}_o \frac{u^*}{K}\left[\ln \eta - m_1\right] - \underset{\sim}{e}_w \frac{\Pi u^*}{K} m_2 \qquad (4.2\text{-}4)$$

Comparing (4.2-4) and (4.2-1) we see that as $x_2 \longrightarrow 0$

$$F_1(\eta) \longrightarrow \ln \eta - m_1$$

and
$$F_2(\eta) \longrightarrow - m_2 \; .$$

At this juncture some reference should again be made to the work of Coles (1956). In Coles' profiles the function $F_1(\eta)$ had the form

$$F_1(\eta) = \ln \eta$$

while the function $F_2(\eta)$ had the form

$$F_2(\eta) = w(\eta) - 2$$

Coles chose the function $w(\eta)$ in such a way that $x_2 = \delta$ represented the "edge" of the boundary layer and so his profiles are defined only in the range $0 < \eta \leqslant 1$. This suggests a criticism of the functions used by Coles since a boundary layer "edge" is quite fictitious. In addition, it is impossible, using the profiles devised by Coles, to satisfy the requirement that as the free-stream is approached $\frac{\partial q}{\partial x_2} \longrightarrow 0$. One further remark is in order. In Coles' work the direction of the base vector $\underline{e}_w$ is not that assumed here. In examining the data of Kuethe, McKee, and Curry (1949) and that of Johnston (1957) Coles found that his base vector $\underline{e}_w$ was nearly in the direction of the pressure gradient. If we are to be guided by past experience this result would seem to be little more than fortuitous since it has been quite well established that there is little connection between the shape of a profile and any local pressure gradient parameter.

As regards the constants K and C appearing in the "law of the wall" there is some disagreement concerning their values. However, Coles examined a large amount of data and in every case he found that a good fit could be obtained with the values

$$K = 0.4$$
$$C = 5.1$$

We will therefore accept these values without further debate.

4.3 Preliminary Examination of Data

Two sets of data will be considered -- that of Johnston (1957) and Gardow (1958). Since this data is presented in terms of a streamline coordinate system, it will be expedient at this point to consider the profile relationships for this special case.

Let us suppose: $x_1 \Longrightarrow$ x where x is the direction of the main-flow streamline; $x_3 \Longrightarrow$ z where z is the cross-flow direction, $x_2 \Longrightarrow$ y and the $u_i \Longrightarrow$ u, v, w respectively. Then Φ, the angle between $\underline{Q}$ and $\underline{e}_1$, is zero.

Forming the dot product of (4.2-1) with first $\underline{e}_1$ and then $\underline{e}_3$ we obtain

$$\frac{u}{Q} - 1 = \gamma \cos\alpha \; F_1(\eta) + \Pi\gamma \cos\beta \; F_2(\eta) \qquad (4.3\text{-}1a)$$

and

$$\frac{w}{Q} = \gamma \sin\alpha \; F_1(\eta) + \Pi\gamma \sin\beta \; F_2(\eta) \qquad (4.3\text{-}1b)$$

where $\gamma = \dfrac{u^*}{KQ}$.

Similarly, from (4.2-3) we get

$$1 = \gamma\cos\alpha\left[\ln\frac{\delta u^*}{\nu} + \kappa C + m_1\right] + \Pi\gamma\cos\beta\, m_2 \tag{4.3-2a}$$

$$0 = \gamma\sin\alpha\left[\ln\frac{\delta u^*}{\nu} + \kappa C + m_1\right] + \Pi\gamma\sin\beta\, m_2 \tag{4.3-2b}$$

We now introduce the following notation:

$$\left.\begin{aligned} i_1 &= -\int_0^\infty F_1(\eta)\,d\eta \\ i_2 &= -\int_0^\infty F_2(\eta)\,d\eta \\ i_{11} &= \int_0^\infty F_1^2(\eta)\,d\eta \\ i_{12} &= \int_0^\infty F_1(\eta)F_2(\eta)\,d\eta \\ i_{22} &= \int_0^\infty F_2^2(\eta)\,d\eta \end{aligned}\right\} \tag{4.3-3}$$

If we define the displacement thicknesses, δ_x^* and δ_z^* as follows:

$$\frac{\delta_x^*}{\delta} = \int_0^\infty \left(1 - \frac{u}{Q}\right) d\eta$$

and

$$\frac{\delta_z^*}{\delta} = -\int_0^\infty \frac{w}{Q}\, d\eta$$

we obtain with the aid of (4.3-1) and (4.3-3)

$$\frac{\delta_x^*}{\delta} = i_1\gamma\cos\alpha + i_2\Pi\gamma\cos\beta \tag{4.3-4a}$$

and $$\frac{\delta_z^*}{\delta} = i_1 \gamma \sin\alpha + i_2 \Pi \gamma \sin\beta \qquad (4.3\text{-}4b)$$

The five parameters α, β, γ, δ, and Π are assumed to be functions of x_1 and x_3 in general, but not of x_2. Of these five δ and Π are the most difficult to determine. Therefore, we eliminate δ and Π between the four equations (4.3-2 a, b) and (4.3-4 a, b). After some lengthy algebra the following two relationships are obtained

$$G_2 - G_1 = m_1 + \ln\left(\frac{m_2}{i_2}\right) + Kc + \ln K \qquad (4.3\text{-}5)$$

and

$$G_2 - G_3 = i_1\left(\frac{m_2}{i_2}\right) \qquad (4.3\text{-}6)$$

where

$$G_1 = \ln\left[\gamma\left(\frac{\delta_x^* Q}{\nu}\right)\left(1 - \frac{\delta_z^*}{\delta_x^*}\cot\alpha\right)\right]$$

$$G_2 = -\frac{1}{\gamma\cos\alpha}\left[\frac{\tan\beta}{\tan\alpha - \tan\beta}\right]$$

$$G_3 = -\frac{1}{\gamma\cos\alpha}\left[\frac{\frac{\delta_z^*}{\delta_x^*}\cot\alpha}{1 - \frac{\delta_z^*}{\delta_x^*}\cot\alpha}\right]$$

G_1, G_2, and G_3 can be determined for each profile by using the tabulated values of δ_x^*, δ_z^* and $\frac{Q}{\nu}$, measuring α and β from hodograph plots, and determining γ from a fit of equation (4.2-2) to the data near the surface.

Equations (4.3-5) and (4.3-6) state that if G_2 is plotted against G_1 and G_2 is plotted against G_3, the points should fall on straight lines having a slope of +1. The straight line in each case should be unique if for arbitrarily chosen values of m_1 and m_2, i_1 and i_2 are constants independent of flow conditions. If this is so, $F_1(\eta)$ and $F_2(\eta)$ have some small claim to "universality".

The data is presented in this way in Figures 4 and 5. Unfortunately the range of values for G_1, G_2, and G_3 covered by these experiments is small and the scatter is great, particularly in the data of Johnston. For these reasons the tests are inconclusive. About all one can say is that G_2 versus G_3 might better be described by a single straight line than G_2 versus G_1.

More detailed examination of the data is necessary but first we must have explicit formulations for $F_1(\eta)$ and $F_2(\eta)$.

4.4 The Profile Functions $F_1(\eta)$ and $F_2(\eta)$

In his original paper Coles pointed out that a separation profile was specified by the conditions $\Pi \rightarrow \infty$, $\gamma \rightarrow 0$ in such a way that the product $\Pi\gamma \longrightarrow 1/2$. The separation profile is then given from (4.1-6) as

$$\frac{u}{U} = \frac{1}{2} w(\eta)$$

Following this lead we suppose that the function $F_2(\eta)$ is essentially related to a two-dimensional separation profile.

In a latter paper Coles (1957) speculated that the two-dimensional boundary layer whose profile is everywhere described by the condition $\Pi = 0$ is a boundary layer for which $\frac{\delta u^*}{\nu}$ and γ were constants. This boundary layer is the one produced by flow between converging plane surfaces and is in a sense quite special since it is the only boundary layer which -- in the words of Townsend (1956) -- is exactly self-preserving. All other self-preserving or equilibrium boundary layers occur only if the Reynolds number is sufficiently high. Adopting Coles' speculations to our particular needs we therefore assume that the function $F_1(\eta)$ is essentially the velocity profile associated with the exactly self-preserving boundary layer.

We are now in a position to make use of some analysis by Townsend (1956). Townsend formulated a differential equation for the outer velocity profiles of equilibrium turbulent boundary layers assuming a constant eddy viscosity ν_T. Making use of a result found experimentally by Clauser (1956) we write

$$\nu_T = \bar{k} U \delta^* \tag{4.4-1}$$

where δ^* is the displacement thickness as usually defined, U is the free-stream velocity, and $\bar{k}$ is a constant.

From our assumed velocity profiles written for the special case of a two-dimensional boundary layer we have

$$\kappa\left(\frac{\delta^* U}{\delta u^*}\right) = i_1 + i_2 \Pi \tag{4.4-2}$$

so that as a result of combining (4.4-1), (4.4-2)

$$\left(\frac{\nu_T}{\delta u^*}\right) = \frac{\bar{k}}{\kappa}\left(i_1 + i_2 \Pi\right) \tag{4.4-3}$$

First we consider the exactly self-preserving boundary layer. As indicated in the discussion above we assume the velocity profile is given by

$$\kappa\left(\frac{u-U}{u^*}\right) = F_1(\eta)$$

Then, for the <u>outer portion</u> of $F_1(\eta)$ Townsend's equation gives

$$F_1 = C_1 e^{-k_1 \eta} \tag{4.4-4}$$

where we have transformed from Townsend's notation to ours and made use of (4.4-3) with $\Pi = 0$.

$$k_1^2 = \frac{\kappa^2}{\bar{k}\, i_1^2} \tag{4.4-5}$$

k_1, C_1, and i_1 are numbers to be determined.

As indicated in Sec. 4.2 the inner portion of $F_1(\eta)$ is given by

$$F_1(\eta) = \ln \eta - m_1 . \tag{4.4-6}$$

Assuming the inner and outer profiles join smoothly at some value of η, say η_0, we have

$$C_1 e^{-K_1\eta_0} = \ln \eta_0 - m_1 \tag{4.4-7}$$

and

$$C_1 e^{-K_1\eta_0} = -\frac{1}{K_1\eta_0} \tag{4.4-8}$$

Finally, remembering that

$$i_1 = -\int_0^\infty F_1(\eta)\,d\eta$$

we have the relation

$$i_1 = -\eta_0 \ln \eta_0 + (m_1+1)\eta_0 - \frac{C_1}{K_1} e^{-K_1\eta_0} \tag{4.4-9}$$

Substituting for i_1 from (4.4-5) equation (4.4-9) becomes

$$\frac{K}{\sqrt{K}} = -K_1\eta_0 \ln \eta_0 + (m_1+1) K_1\eta_0 - C_1 e^{-K_1\eta_0} \tag{4.4-10}$$

Equations (4.4-7), (4.4-8) and (4.4-10) are three equations for the three unknowns k_1, C_1, and η_0.

Substituting for C_1 from (4.4-8) into (4.4-7) and (4.4-10), and combining the results we finally obtain

$$K_1\eta_0 = \frac{1}{2}\left[\left(\frac{K}{\sqrt{K}} - 1\right) \pm \sqrt{\left(\frac{K}{\sqrt{K}} - 1\right)^2 - 4}\,\right] \tag{4.4-11}$$

Clauser (1956) found experimentally that $\bar{k} = 0.018$. This value of $\bar{k}$ together with $\kappa = 0.4$ does not permit a real solution of (4.4-11). However, with a slight adjustment so that $\bar{k} = 0.0178$ we can obtain a unique solution.

In order to proceed further numerically we must assume some value for m_1. For convenience we take $m_1 = -\kappa C = -2.04$. Then, after some calculation we obtain

$$\eta_0 = 0.0478$$

$$k_1 = 20.905$$

$$C_1 = -e$$

$$i_1 = 0.14351$$

The final form of $F_1(\eta)$ is therefore

$$\left.\begin{array}{ll} 0 < \eta \leq 0.0478 \; , & F_1(\eta) = \ln \eta + 2.04 \\ 0.0478 \leq \eta < \infty \; , & F_1(\eta) = -e^{(1-20.905\eta)} \end{array}\right\} \qquad (4.4\text{-}12)$$

We next consider the equilibrium boundary layer with incipient separation. This is a limiting case and has to be approached carefully. If we write in general

$$\kappa\left(\frac{u-U}{u^*}\right) = F(\eta)$$

we now suppose that as $u^* \longrightarrow 0$, $F(\eta) \longrightarrow F_2(\eta)$.

Townsends equation yields for the <u>outer</u> <u>portion</u> of F the result

$$F = C\,Hh_n(t) \qquad (4.4\text{-}13)$$

where
$$t = T\eta$$
$$T^2 = \frac{K^2}{\bar{k}(1+n)(i_1 + i_2\Pi)^2}$$

Again we have changed from his notation to ours and made use of (4.4-3). n is a parameter defined by Townsend and related to the external pressure distribution; it is a constant for any particular equilibrium boundary layer. $Hh_n(t)$ is a function given in terms of an infinite series as

$$Hh_n(t) = \frac{\sqrt{\frac{\pi}{2}}}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\left[1 + \frac{nt^2}{2!} + \frac{n(n-2)t^4}{4!} + \cdots\right]$$
$$- \frac{\sqrt{\frac{\pi}{2}}}{2^{\left(\frac{n-1}{2}\right)}\left(\frac{n-1}{2}\right)!}\left[t + \frac{(n-1)t^3}{3!} + \frac{(n-1)(n-3)t^5}{5!} + \cdots\right]$$

(There should be no confusion between the number π and the parameter Π.) For the Boundary layer with incipient separation Townsend shows that n = -1 and
$$Hh_{-1}(t) = e^{-\frac{1}{2}t^2}$$

Again the outer profile has to be matched to the inner one at some point $t = t_o$. We have already chosen $m_1 = -KC$. For convenience we take $m_2 = 1$. As shown in Sec. 4.2 the inner profile is

$$F(\eta) = \ln\eta + Kc - \Pi \tag{4.4-14}$$

Π = constant for any particular equilibrium boundary layer so that in the present situation it may be considered to depend on n.

Using (4.4-13) and (4.4-14) we insure a smooth transition between inner and outer profiles by satisfying the relations

$$CHh_n(t_o) = \ln t_o - \ln T + \kappa c - \pi \qquad (4.4\text{-}15)$$

$$-CHh_{n-1}(t_o) = \frac{1}{t_o} \qquad (4.4\text{-}16)$$

Also we have the relation

$$-\int_0^\infty F(\eta)\,d\eta = (i_1 + i_2\pi)$$

which gives us the additional equation

$$-T(i_1 + i_2\pi) = t_o \ln t_o - t_o(\ln T - \kappa c + 1 + \pi) + C\,Hh_{n+1}(t_o) \qquad (4.4\text{-}17)$$

Substituting for $(i_1 + i_2\pi)$ in terms of T, (4.4-17) becomes

$$\frac{K}{\sqrt{K(1+n)}} = -t_o \ln t_o + t_o(\ln T - \kappa c + 1 + \pi) - CHh_{n+1}(t_o) \qquad (4.4\text{-}18)$$

Substitute C from (4.4-16) into (4.4-15) and (4.4-18), and combine the results to give

$$\frac{K}{\sqrt{K(1+n)}}\, t_o Hh_{n-1}(t_o) - t_o Hh_n(t_o) = t_o^2 Hh_{n-1}(t_o) + Hh_{n+1}(t_o) \qquad (4.4\text{-}19)$$

Ultimately we will be interested in the limiting case $n \rightarrow -1$. For this condition it is easily seen that $Hh_n(t)$ and $Hh_{n-1}(t)$ have approximate forms

$$Hh_n(t) \approx \frac{\sqrt{\frac{\pi}{2}}}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\left[1 + \frac{nt^2}{2!} + \frac{n(n-2)t^4}{4!} + \cdots\right]$$

$$Hh_{n-1}(t) \approx -\frac{\sqrt{\frac{\pi}{2}}}{2^{\left(\frac{n-2}{2}\right)}\left(\frac{n-2}{2}\right)!}\left[t + \frac{(n-2)t^3}{3!} + \frac{(n-2)(n-4)t^5}{5!} + \cdots\right]$$

$Hh_{n+1}(t)$ does not simplify.

A brief examination of (4.4-19) shows that as $n \longrightarrow -1$, $t_o \longrightarrow 0$. Therefore, retaining terms of $O(t_o^2)$ and larger, we find from (4.4-19)

$$t_o^2 \approx -\frac{1}{2K}\sqrt{\frac{\bar{K}(1+n)}{2}}\,\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n+1}{2}\right)!} \tag{4.4-20}$$

With the aid of (4.4-20) we obtain from (4.4-16)

$$C \approx -K\,\frac{2^{\left(\frac{n+1}{2}\right)}\left(\frac{n+1}{2}\right)!}{\sqrt{\frac{\pi}{2}}\,\sqrt{\bar{K}(1+n)}} \tag{4.4-21}$$

Substituting (4.4-20) and (4.4-21) in (4.4-15) we obtain

$$-K\sqrt{\frac{2}{\bar{K}(1+n)}}\,\frac{\left(\frac{n+1}{2}\right)!}{\left(\frac{n}{2}\right)!} + \frac{1}{2} = \frac{1}{2}\ln\sqrt{(1+n)}$$

$$+\frac{1}{2}\ln\left[-\frac{1}{2K}\sqrt{\frac{\bar{K}}{2}}\,\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n+1}{2}\right)!}\right] - \ln T + KC - \Pi \tag{4.4-22}$$

For $(n+1) \to 0$, $\left|(n+1)^{-\frac{1}{2}}\right| \gg \left|\ln(n+1)^{\frac{1}{2}}\right|$ and (4.4-22) becomes

$$\ln T - \frac{1}{2}\ln\left[-\frac{1}{2K}\sqrt{\frac{\bar{K}}{2}}\,\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n+1}{2}\right)!}\right] + \frac{1}{2} - KC + \Pi$$

$$\approx K\sqrt{\frac{2}{\bar{K}(1+n)}}\,\frac{\left(\frac{n+1}{2}\right)!}{\left(\frac{n}{2}\right)!} \tag{4.4-23}$$

As $(n+1) \longrightarrow 0$, $\Pi \longrightarrow -C$ and

$$\ln T - \frac{1}{2}\ln\left[-\frac{1}{2K}\sqrt{\frac{\bar{K}}{2}}\,\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n+1}{2}\right)!}\right] + \frac{1}{2} - KC$$

$$\longrightarrow K\sqrt{\frac{2}{\bar{K}(1+n)}}\,\frac{\left(\frac{n+1}{2}\right)!}{\left(\frac{n}{2}\right)!} - K\,\frac{2^{\left(\frac{n+1}{2}\right)}}{\sqrt{\frac{\pi}{2}}}\,\frac{\left(\frac{n+1}{2}\right)!}{\sqrt{\bar{K}(1+n)}}$$

Therefore, for the limiting case of n = -1

$$\ln T - \frac{1}{2}\ln\left(\frac{1}{K}\sqrt{\frac{\bar{K}\pi}{2}}\right) + \frac{1}{2} - Kc = 0$$

or

$$T = \frac{1}{\sqrt{K}}\left(\frac{\bar{K}\pi}{2}\right)^{\frac{1}{4}} e^{Kc - \frac{1}{2}} \qquad (4.4\text{-}25)$$

Since $Hh_{-1}(t) = e^{-\frac{1}{2}t^2}$, $F_2(\eta) = -e^{-\frac{1}{2}T^2\eta^2}$.

With $\bar{k}$ = 0.0178 as before, T = 3.01590 and we obtain finally

$$F_2(\eta) = -e^{-4.54783\eta^2} \qquad (4.4\text{-}26)$$

Using this formulation i_2 = 0.41557.

We are now in a position to compare the theoretical and experimental forms of $F_1(\eta)$ and $F_2(\eta)$.

Solve for $F_1(\eta)$ and $F_2(\eta)$ from (4.3-1 a, b).

$$F_1(\eta) = \frac{\left(1 - \frac{u}{Q}\right)\sin\beta + \frac{w}{Q}\cos\beta}{\gamma\sin(\alpha - \beta)} \qquad (4.4\text{-}27)$$

$$F_2(\eta) = \frac{\left(\frac{u}{Q} - 1\right)\sin\alpha - \frac{w}{Q}\cos\alpha}{\Pi\gamma\sin(\alpha - \beta)} \qquad (4.4\text{-}28)$$

From (4.3-2 a, b)

$$\Pi = \frac{\sin\alpha}{m_2\gamma\sin(\alpha - \beta)} \qquad (4.4\text{-}29)$$

With the aid of (4.4-29), (4.4-28) becomes

$$F_2(\eta) = -m_2\left[\left(1-\frac{u}{Q}\right) + \frac{w}{Q}\cot\alpha\right] \qquad (4.4\text{-}30)$$

From (4.3-4 a, b)

$$\Pi = \frac{\frac{\delta_x^*}{\delta}\sin\alpha - \frac{\delta_z^*}{\delta}\cos\alpha}{i_2\,\gamma\sin(\alpha-\beta)} \qquad (4.4\text{-}31)$$

Equating (4.4-29) and (4.4-31), we obtain

$$\delta = \frac{m_2}{i_2}\left(\delta_x^* - \delta_z^*\cot\alpha\right) \qquad (4.4\text{-}32)$$

Experimental forms of $F_1(\eta)$ and $F_2(\eta)$ were calculated from the data using (4.4-27) and (4.4-30) with values of δ calculated from (4.4-32). These forms are compared with equations (4.4-12) and (4.4-26) in Figures 6 and 7 for the experiments of Johnston (1957) and in Figures 8 and 9 for the experiments of Gardow (1958). In general the agreement is satisfactory for the function $F_1(\eta)$. The experimental profiles $F_1(\eta)$ which deviate greatly from the theoretical curve arise from situations where there is a good deal of uncertainty in β. The most striking discrepancy occurs with regard to the function $F_2(\eta)$. The fit to Gardow's data is quite satisfactory. However, the profiles $F_2(\eta)$ determined from Johnston's experiments exhibit a form quite different from equation (4.4-26). This disagreement cannot be attributed to experimental uncertainties. It is of a more fundamental nature and we will return to the point again in the next section.

In any event by way of summary, use of the relations (4.4-12) and (4.4-26) leads to the following values of the quantities (4.4-3):

$$i_1 = 0.14351$$

$$i_2 = 0.41557$$

$$i_{11} = 0.26306$$

$$i_{12} = 0.14095$$

$$i_{22} = 0.29385$$

4.5 Comparison of Theoretical and Experimental Velocity Profiles

We are now in a position to attempt a fit to actual measured velocity profiles starting from the measured values of α, β, γ, δ_x^*, δ_z^*, and $\frac{Q}{\nu}$.

From equations (4.3-2 a, b) we obtain an expression for δ

$$\ln K\gamma\left(\frac{\delta Q}{\nu}\right) = -\frac{\sin\beta}{\gamma\sin(\alpha-\beta)} - KC - m_1 \qquad (4.5\text{-}1)$$

Π may be calculated from (4.4-29) and, finally, the theoretical profiles from (4.3-1 a,b). Typical results are shown in Figures 10 and 11 for Gardow's experiments and Figures 12 and 13 for Johnston's experiments.

In general, Gardow's measurements can be fitted quite successfully like the profiles in Figure 10. There are a few profiles, like the ones in Figure 11, where there is no clearly defined straight line relationship between the two velocity components in the outer portion of the boundary layer. These profiles usually occurred just after a separated region and in such cases the theoretical description is not so good. The profile shown in Figure 11 was the worst one encountered in this respect.

As regards Johnston's measurements, however, the story is very different. A glance at Figures 12 and 13 indicates that the theoretical profiles developed above are virtually incapable of describing Johnston's measurements. The fit is perhaps even worse than would have been anticipated from the differences in the function $F_2(\eta)$. The reason for this is that we have now used a δ calculated in a different way. Though of course they should agree, in Johnston's experiments the δ's calculated from equations (4.4-32) and (4.5-1) actually differ by a factor of 4.

In seeking to explain the difference in the quality of agreement obtainable we notice that the experiments of Gardow are very different from those of Johston in the following way:

Gardow made his measurements in a radial diffuser where the flow was essentially axially-symmetric. If one considers two neighboring main-flow streamline directions, it is apparent that the cross-flow serves to bring fluid from one environment to another. In Gardow's case these two environments are almost the same, since the two main-flow streamlines have had essentially the same experience.

On the other hand Johnston's experiments were made in a flow in which a jet confined between two plane walls was made to impinge on a back plate and spread out evenly to both sides. There is a plane of symmetry down the middle of the jet and his measurements were made on and in the neighborhood of that plane of symmetry. This is a very severe situation in one respect. Along the plane of symmetry the boundary layer is collateral, the adverse pressure gradient is intense and in fact stagnation pressure is approached. Only a little way removed from the plane of symmetry, however, the pressure gradient in the direction of a main-flow streamline is less severe. Therefore, in this situation the cross-flow serves to bring low-energy, degenerate fluid from the plane of symmetry which displaces the more energetic fluid away from the plane of symmetry thereby forming a thicker boundary layer than would otherwise be present.

It is well known that a velocity profile in a turbulent boundary layer depends on the history of the boundary layer upstream of that point. With a skewed boundary layer the situation is even more complicated. At any particular point on the surface as one proceeds out along a line normal to the wall the streamline direction varies from that of the limiting wall streamline to the direction of the main-stream velocity vector. The situation is illustrated in Figure 14 where the shaded region indicates that portion of the flow whose past history determines the velocity profile at the point under consideration. It is apparent that the fluid at any particular point has come from a range of different environments.

The experiments we have considered suggest that the proposed method of profile description is good if these environments do not differ considerably, as in Gardow's case. On the other hand, if the past histories involved vary to a great extent, as in Johnston's experiments, the theoretical profiles are not, apparently, sufficiently flexible to handle the situation.

It seems likely that in the extremely complicated flows encountered in turbomachinery, the conditions are closer to Johnston's case than to Gardow's. On the other hand, external flows over wings and projectiles can probably be described reasonably well with the theory developed above.

5. FINAL FORM OF THE COMPUTATION SCHEME

The velocity profiles that have been developed involve two predetermined functions $F_1(\eta)$ and $F_2(\eta)$ and five parameters -- α, β, γ, δ, and Π -- which are functions of the surface coordinates x_1 and x_3 only. Given the initial values of these parameters along a suitable line we can determine their values everywhere else by using the momentum integral relations -- (2.3-2 a, b), the auxiliary equation -- (3.2-4), and the vector skin-friction relationship -- (4.2-3). We will now put these equations in more suitable form.

Forming the dot product of (4.2-1) with $\underline{e}_1$ and $\underline{e}_3$, we obtain after division by Q

$$\left(\frac{U_1-u_1}{Q}\right) = -\gamma\cos\alpha\, F_1(\eta) - \Pi\gamma\cos\beta\, F_2(\eta) \tag{5-1a}$$

and

$$\left(\frac{U_3-u_3}{Q}\right) = -\gamma\sin\alpha\, F_1(\eta) - \Pi\gamma\sin\beta\, F_2(\eta) \tag{5-1b}$$

Similarly, from (4.2-3) we get

$$\cos\Phi = \gamma\cos\alpha\left(\ln\frac{\delta u^*}{\nu}\right) + \Pi\gamma\cos\beta \tag{5-2a}$$

$$\sin\Phi = \gamma\sin\alpha\left(\ln\frac{\delta u^*}{\nu}\right) + \Pi\gamma\sin\beta \tag{5-2b}$$

Define new parameters as follows

$$\left.\begin{array}{l} p_1 = \gamma\cos\alpha\ , \quad p_2 = \Pi\gamma\cos\beta\ , \quad p_3 = \gamma\sin\alpha \\ p_4 = \Pi\gamma\sin\beta\ , \quad p_5 = \ln\dfrac{\delta u^*}{\nu} \end{array}\right\} \tag{5-3}$$

With the aid of (5-3), (5.1 a, b) and (5-2 a, b) become

$$\left(\frac{U_1 - u_1}{Q}\right) = -p_1 F_1(\eta) - p_2 F_2(\eta) \tag{5-4a}$$

$$\left(\frac{U_3 - u_3}{Q}\right) = -p_3 F_1(\eta) - p_4 F_2(\eta) \tag{5-4b}$$

and

$$\cos\Phi = p_1 p_5 + p_2 \tag{5-5a}$$

$$\sin\Phi = p_3 p_5 + p_4 \tag{5-5b}$$

If (2.3-1) are evaluated using (5-4 a, b) we get after substituting for p_2 and p_4 from (5-5 a, b)

$$\left.\begin{aligned}
I_1 &= p_1(i_1 - i_2 p_5) + i_2 \cos\Phi \\
I_3 &= p_3(i_1 - i_2 p_5) + i_2 \sin\Phi \\
I_{11} &= p_1^2(i_{11} - 2i_{12}p_5 + i_{22}p_5^2) + 2p_1(i_{12} - i_{22}p_5)\cos\Phi \\
&\qquad + i_{22}\cos^2\Phi \\
I_{13} &= p_1 p_3(i_{11} - 2i_{12}p_5 + i_{22}p_5^2) \\
&\quad + (p_1\sin\Phi + p_3\cos\Phi)(i_{12} - i_{22}p_5) + i_{22}\sin\Phi\cos\Phi \\
I_{33} &= p_3^2(i_{11} - 2i_{12}p_5 + i_{22}p_5^2) + 2p_3(i_{12} - i_{22}p_5)\sin\Phi \\
&\qquad + i_{22}\sin^2\Phi
\end{aligned}\right\} \tag{5-6}$$

Further, since

$$\frac{\delta u^*}{\nu} = \kappa\,\gamma\left(\frac{\delta Q}{\nu}\right)$$

and

$$\gamma = \left(p_1^2 + p_3^2\right)^{\frac{1}{2}}$$

we have

$$p_5 = \ln\kappa + \tfrac{1}{2}\ln\left(p_1^2 + p_3^2\right) + \ln\frac{\delta Q}{\nu}$$

which leads to

$$\frac{1}{\delta}\frac{\partial\delta}{\partial x_i} = \frac{\partial p_5}{\partial x_i} - \left(\frac{p_1}{p_1^2+p_3^2}\right)\frac{\partial p_1}{\partial x_i} - \left(\frac{p_3}{p_1^2+p_3^2}\right)\frac{\partial p_3}{\partial x_i} - \frac{1}{Q}\frac{\partial Q}{\partial x_i}\,. \qquad (5\text{-}7)$$

Also in terms of this notation

$$\frac{\tau_{01}}{\delta\rho Q^2} = \kappa^3 p_1\left(p_1^2 + p_3^2\right) e^{-p_5}\left(\frac{Q}{\nu}\right) \qquad (5\text{-}8a)$$

and

$$\frac{\tau_{03}}{\delta\rho Q^2} = \kappa^3 p_3\left(p_1^2 + p_3^2\right) e^{-p_5}\left(\frac{Q}{\nu}\right) \qquad (5\text{-}8b)$$

Finally, the quantity M appearing in the auxiliary equation (3.2-4) becomes since $M = \tan\beta$

$$M = \frac{p_4}{p_2} = \frac{\sin\Phi - p_3 p_5}{\cos\Phi - p_1 p_5} \qquad (5\text{-}9)$$

Using the relations (5-6), (5-7), (5-8 a, b), and (5-9), equations (2.3-2 a, b) and (3.2-4) become with the abreviations R_i indicated on the next page.

$$R_1 = i_{11} - 2i_{12}p_5 + i_{22}p_5^2$$

$$R_2 = (2i_{12} - i_1) - (2i_{22} - i_2)p_5$$

$$R_3 = i_{22} - i_2$$

$$R_4 = (i_{12} - i_1) - (i_{22} - i_2)p_5$$

$$R_5 = i_{12} - i_{22}p_5$$

$$R_6 = (i_{11} - 2i_{12}) - 2(i_{12} - i_{22})p_5 + i_{22}p_5^2$$

$$R_7 = (2i_{12} - 2i_{22} - i_1 + i_2) - (2i_{22} - i_2)p_5$$

$$R_8 = (i_{12} - i_{22} - i_1 + i_2) - (i_{22} - i_2)p_5$$

$$R_9 = (i_{12} - i_{22}) - i_{22}p_5$$

$$R_{10} = i_{22} - 2i_2$$

$$R_{11} = 2i_{22} - 3i_2$$

$$\frac{1}{h_1}\frac{\partial p_1}{\partial x_1}\left[\frac{R_1 p_1(p_1^2+2p_3^2) + R_2 p_3^2\cos\Phi - R_3 p_1\cos^2\Phi}{(p_1^2+p_3^2)}\right] + \frac{1}{h_3}\frac{\partial p_1}{\partial x_3}\left[\frac{R_1 p_3^3 + R_4 p_3^2\sin\Phi - R_5 p_1 p_3\cos\Phi - R_3 p_1 \sin\Phi\cos\Phi}{(p_1^2+p_3^2)}\right]$$

$$-\frac{1}{h_1}\frac{\partial p_3}{\partial x_1}\left[\frac{R_1 p_1^2 p_3 + R_2 p_1 p_3\cos\Phi + R_3 p_3\cos^2\Phi}{(p_1^2+p_3^2)}\right] + \frac{1}{h_3}\frac{\partial p_3}{\partial x_3}\left[\frac{R_1 p_1^3 - R_4 p_1 p_3\sin\Phi + R_5 p_1^2\cos\Phi - R_3 p_3\sin\Phi\cos\Phi}{(p_1^2+p_3^2)}\right]$$

$$+\frac{1}{h_1}\frac{\partial p_5}{\partial x_1}\left[R_6 p_1^2 + R_7 p_1\cos\Phi + R_3\cos^2\Phi\right] + \frac{1}{h_3}\frac{\partial p_5}{\partial x_3}\left[R_6 p_1 p_3 + R_8 p_1\sin\Phi + R_9 p_3\cos\Phi + R_3\sin\Phi\cos\Phi\right]$$

$$+\frac{1}{Qh_1}\frac{\partial Q}{\partial x_1}\left[R_1 p_1^2 + 2R_4 p_1\cos\Phi + R_{10}\cos^2\Phi\right] + \frac{1}{Qh_3}\frac{\partial Q}{\partial x_3}\left[R_1 p_1 p_3 + R_4 p_1\sin\Phi + R_4 p_3\cos\Phi + R_{10}\sin\Phi\cos\Phi\right]$$

$$-\frac{1}{h_1}\frac{\partial \Phi}{\partial x_1}\left[2R_4 p_1\sin\Phi + R_{11}\sin\Phi\cos\Phi\right] + \frac{1}{h_3}\frac{\partial \Phi}{\partial x_3}\left[R_4 p_1\cos\Phi - R_4 p_3\sin\Phi + R_3\cos^2\Phi - R_{10}\sin^2\Phi\right]$$

$$+\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1}\left[R_1(p_1^2-p_3^2) + R_2 p_1\cos\Phi - 2R_4 p_3\sin\Phi + R_3\cos^2\Phi - R_{10}\sin^2\Phi\right] + \kappa^3 p_1(p_1^2+p_3^2)e^{-p_5}\left[\frac{Q}{\nu}\right]$$

$$+\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3}\left[2R_1 p_1 p_3 + R_2 p_3\cos\Phi + 2R_4 p_1\sin\Phi + R_{11}\sin\Phi\cos\Phi\right] = 0 \quad \text{———} \quad (5\text{-}10a)$$

$$
\frac{1}{h_1}\frac{\partial p_1}{\partial x_1}\left[\frac{R_1 p_3^3 + R_5 p_3^2 \sin\Phi - R_4 p_1 p_3 \cos\Phi - R_3 p_1 \sin\Phi\cos\Phi}{(p_1^2+p_3^2)}\right] - \frac{1}{h_3}\frac{\partial p_1}{\partial x_3}\left[\frac{R_1 p_1 p_3^2 + R_2 p_1 p_3 \sin\Phi + R_3 p_1 \sin^2\Phi}{(p_1^2+p_3^2)}\right]
$$

$$
+\frac{1}{h_1}\frac{\partial p_3}{\partial x_1}\left[\frac{R_1 p_1^3 + R_4 p_1^2 \cos\Phi - R_5 p_1 p_3 \sin\Phi - R_3 p_3 \sin\Phi\cos\Phi}{(p_1^2+p_3^2)}\right] + \frac{1}{h_3}\frac{\partial p_3}{\partial x_3}\left[\frac{R_1 p_3(2p_1^2+p_3^2) + R_2 p_1^2 \sin\Phi - R_3 p_3 \sin^2\Phi}{(p_1^2+p_3^2)}\right]
$$

$$
+\frac{1}{h_1}\frac{\partial p_5}{\partial x_1}\left[R_6 p_1 p_3 + R_9 p_1 \sin\Phi + R_8 p_3 \cos\Phi + R_3 \sin\Phi\cos\Phi\right] + \frac{1}{h_3}\frac{\partial p_5}{\partial x_3}\left[R_6 p_3^2 + R_7 p_3 \sin\Phi + R_3 \sin^2\Phi\right]
$$

$$
+\frac{1}{Qh_1}\frac{\partial Q}{\partial x_1}\left[R_1 p_1 p_3 + R_4 p_1 \sin\Phi + R_4 p_3 \cos\Phi + R_{10} \sin\Phi\cos\Phi\right] + \frac{1}{Qh_3}\frac{\partial Q}{\partial x_3}\left[R_1 p_3^2 + 2R_4 p_3 \sin\Phi + R_{10} \sin^2\Phi\right]
$$

$$
+\frac{1}{h_1}\frac{\partial \Phi}{\partial x_1}\left[R_4 p_1 \cos\Phi - R_4 p_3 \sin\Phi + R_{10} \cos^2\Phi - R_3 \sin^2\Phi\right] + \frac{1}{h_3}\frac{\partial \Phi}{\partial x_3}\left[2R_4 p_3 \cos\Phi + R_{11} \sin\Phi\cos\Phi\right]
$$

$$
+\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1}\left[2R_1 p_1 p_3 + R_2 p_1 \sin\Phi + 2R_4 p_3 \cos\Phi + R_{11} \sin\Phi\cos\Phi\right] + \kappa^3 p_3 (p_1^2+p_3^2) e^{-p_5}\left[\frac{Q}{\nu}\right]
$$

$$
+\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3}\left[R_1(p_3^2-p_1^2) + R_2 p_3 \sin\Phi - 2R_4 p_1 \cos\Phi + R_3 \sin^2\Phi - R_{10} \cos^2\Phi\right] = 0 \text{———} (5\text{-}10b)
$$

$$
\begin{aligned}
&\frac{1}{h_1}\frac{\partial p_1}{\partial x_1}\Big[p_5\cos\Phi(\sin\Phi - p_3p_5)\Big] - \frac{1}{h_1}\frac{\partial p_3}{\partial x_1}\Big[p_5\cos\Phi(\cos\Phi - p_1p_5)\Big] + \frac{1}{h_1}\frac{\partial p_5}{\partial x_1}\Big[\cos\Phi(p_1\sin\Phi - p_3\cos\Phi)\Big] \\
&+\frac{1}{h_3}\frac{\partial p_1}{\partial x_3}\Big[p_5\sin\Phi(\sin\Phi - p_3p_5)\Big] - \frac{1}{h_3}\frac{\partial p_3}{\partial x_3}\Big[p_5\sin\Phi(\cos\Phi - p_1p_5)\Big] + \frac{1}{h_3}\frac{\partial p_5}{\partial x_3}\Big[\sin\Phi(p_1\sin\Phi - p_3\cos\Phi)\Big] \\
&-\frac{1}{Qh_1}\frac{\partial Q}{\partial x_1}\Big[(\cos\Phi - p_1p_5)(p_1p_5\sin\Phi - p_3p_5\cos\Phi)\Big] + \frac{1}{h_1}\frac{\partial \Phi}{\partial x_1}\Big[(2\cos\Phi - p_1p_5)(1 - p_1p_5\cos\Phi - p_3p_5\sin\Phi)\Big] \\
&-\frac{1}{Qh_3}\frac{\partial Q}{\partial x_3}\Big[(\sin\Phi - p_3p_5)(p_1p_5\sin\Phi - p_3p_5\cos\Phi)\Big] + \frac{1}{h_3}\frac{\partial \Phi}{\partial x_3}\Big[(2\sin\Phi - p_3p_5)(1 - p_1p_5\cos\Phi - p_3p_5\sin\Phi)\Big] \\
&+\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1}\Big[p_5^2\big[(p_1^2 + 2p_3^2)\sin\Phi + p_1p_3\cos\Phi\big] - p_5\big[3p_1\sin\Phi\cos\Phi + p_3(1 + 3\sin^2\Phi)\big] + 2\sin\Phi\Big] \\
&-\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3}\Big[p_5^2\big[(2p_1^2 + p_3^2)\cos\Phi + p_1p_3\sin\Phi\big] - p_5\big[3p_3\sin\Phi\cos\Phi + p_1(1 + 3\cos^2\Phi)\big] + 2\cos\Phi\Big] = 0 \qquad (5\text{-}11)
\end{aligned}
$$

Equations (5-10 a, b) are the x_1 - and x_3 - momentum integral equations respectively, and (5-11) is the auxiliary equation. Knowing the pressure distribution and having suitable initial values of p_1, p_3, p_5, the equations may be solved numerically to yield p_1, p_3, p_5 at all later stations. p_2 and p_4 may be found from (5-5 a, b). Although the theory is now complete, the calculation in practical cases will present a formidable problem. Some simplifications may on occasion be possible, however. For example, it may be that the Reynolds number $\frac{\delta u^*}{\nu}$ varies but little over the range of interest, in which case p_5 could be assumed to be constant as a first approximation. Then again, in situations involving only small cross-flows, α and β will be small. Under such conditions

$$p_1 \approx \gamma \quad , \quad p_3 \approx \gamma\alpha$$

and terms of order α^2 may be ignored.

6. SUMMARY

Following a suggestion of Coles (1956) a theoretical means of describing velocity profiles in skewed turbulent boundary layers has been devised. The theoretical velocity distribution involves two "universal", analytical functions and five parameters which are functions of the surface coordinates only. This method of profile description has proved to be quite successful in those instances in which gradients of flow quantities normal to a main-flow streamline are not too large. In situations where the past history of the fluid varies considerably, the theoretical profiles are quite inadequate. It is possible that more flexibility can be built into the profiles, but in doing so their essential simplicity should be retained if the description is to remain tractable from a practical point of view.

The five parameters are connected by a vector skin-friction relationship and their values everywhere in a boundary layer may be calculated using the momentum integral equations and an auxiliary equation if sufficient initial information and complete knowledge of the pressure distribution is given. Though in theory the computation scheme is completely defined, its practical application will in general be lengthy and tedious.

One further remark must be made. The auxiliary equation is based on the empirical observation that if a velocity profile is plotted in a hodograph plane, the outer portion is almost always linear. This is an essentially three-dimensional idea and hence the computation scheme will not handle two-dimensional boundary layers even though the method of profile description is probably quite adequate for the two-dimensional case -- e.g. see Coles (1956). A two-dimensional profile depends on three parameters rather than five, these three parameters being connected by a scalar skin-friction relationship. However, of the three equations in the computation scheme only one remains independent for a two-dimensional boundary layer, so that one additional equation is needed. It may be possible, however, to handle the case of a boundary layer which is initially collateral but which is in a three-dimensional environment so that skewing will occur, through most probably this latter situation has little practical importance.

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LIST OF FIGURES

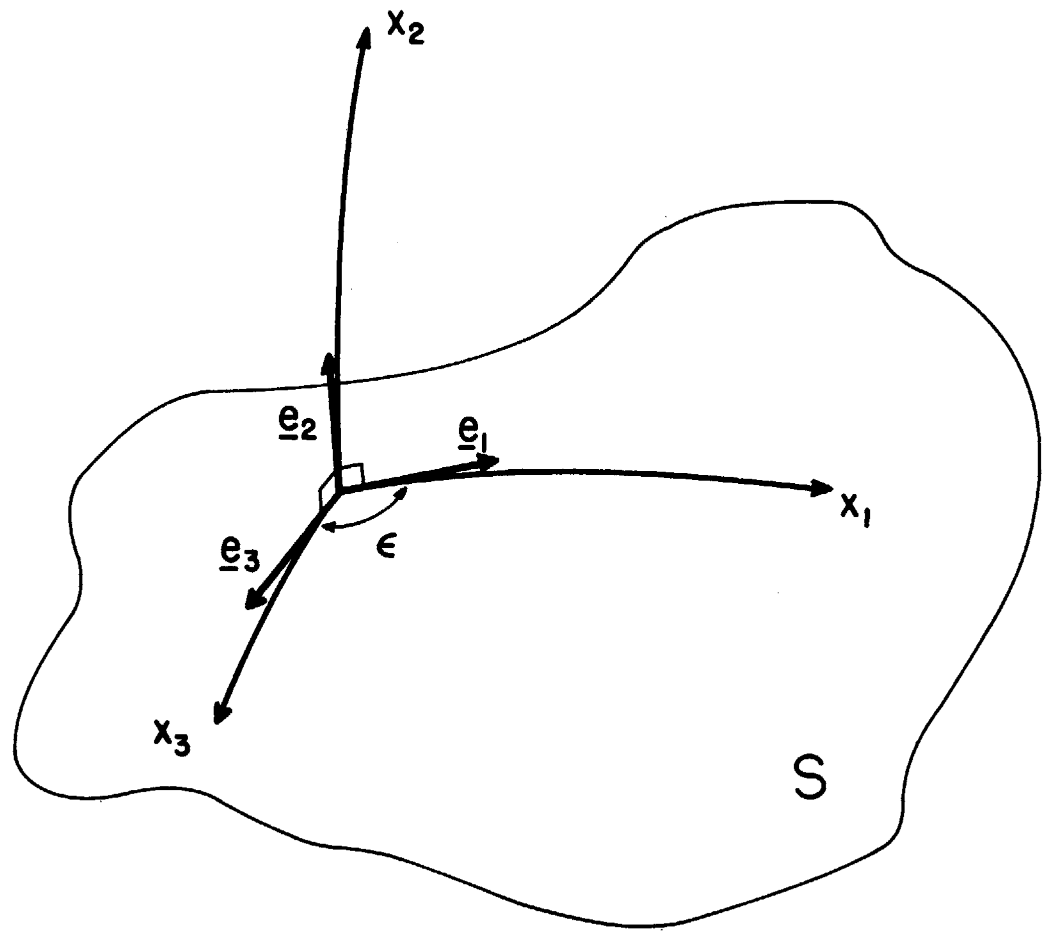


Fig. 1 Main features of the coordinate system.

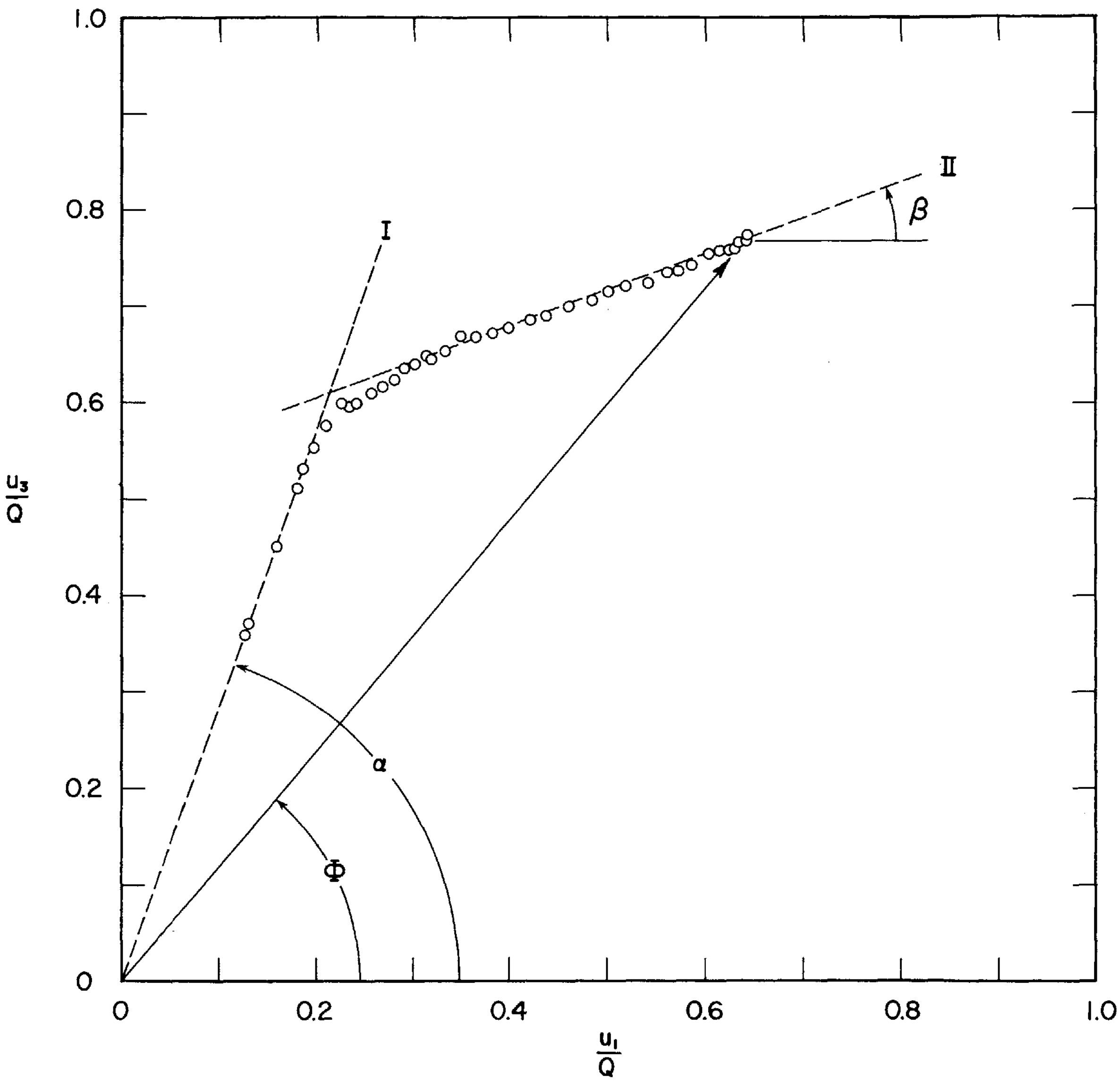


Fig. 2 Typical velocity profile in a hodograph plane.

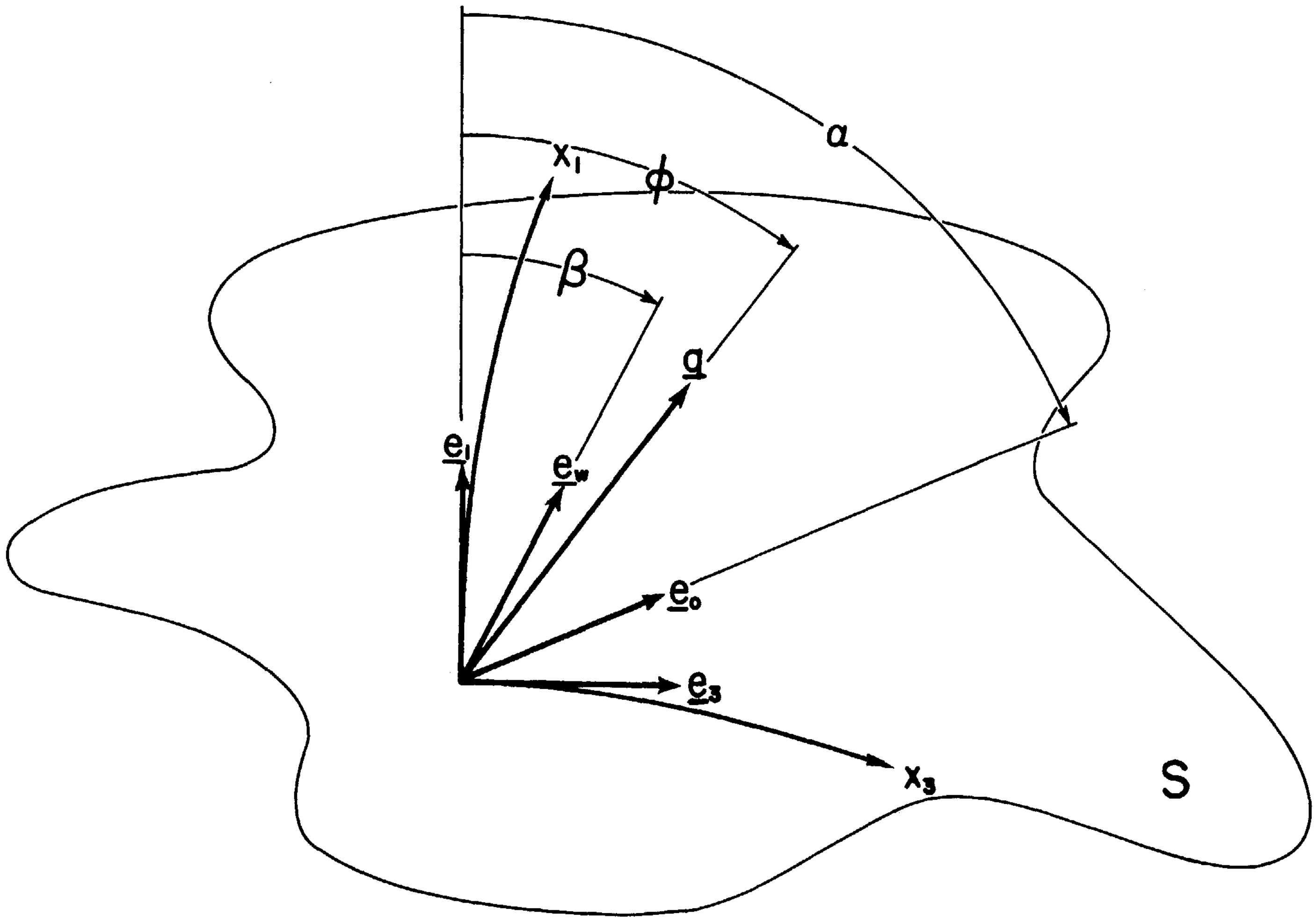


Fig. 3 Notation used in description of velocity profiles.

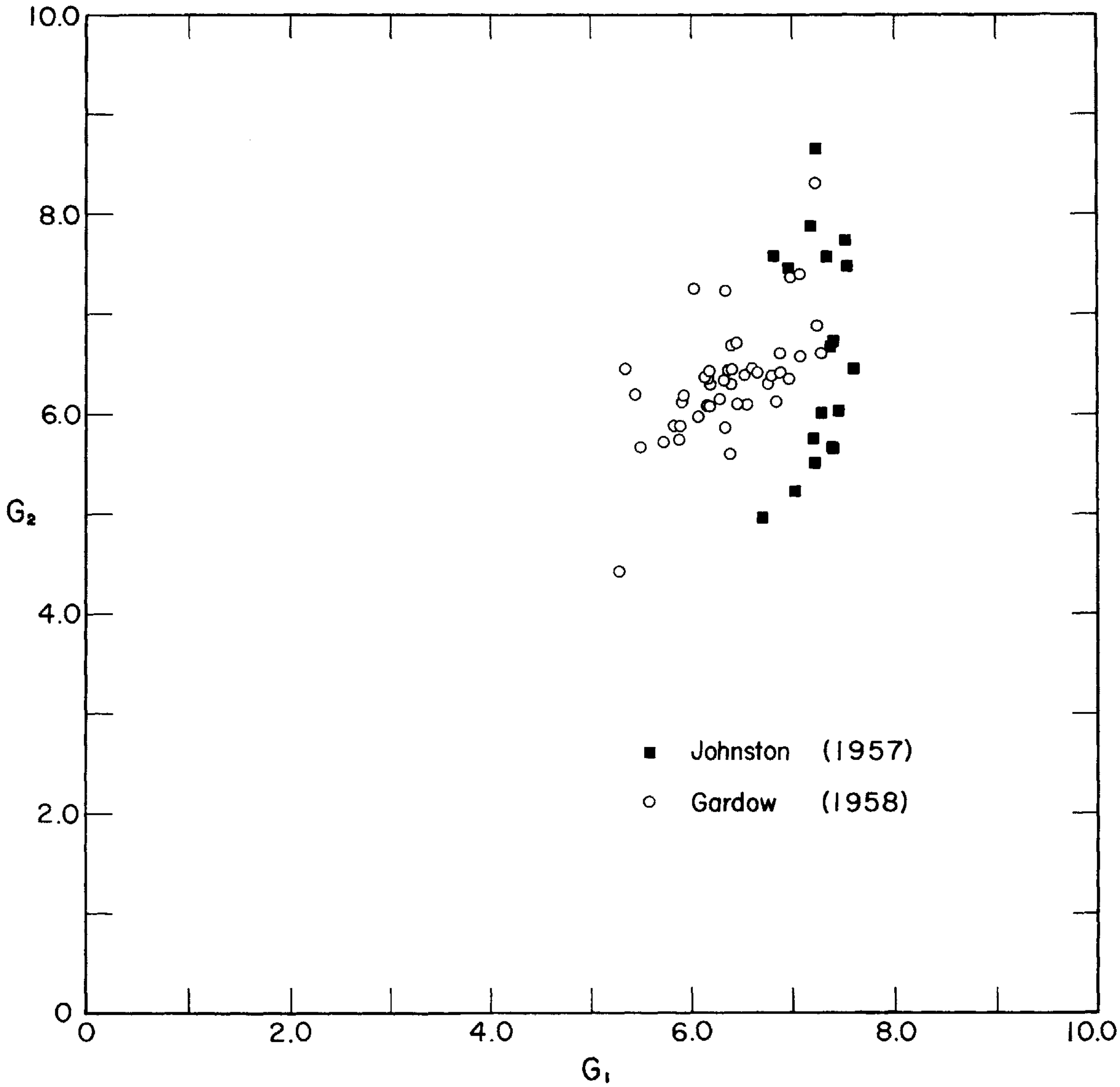


Fig. 4 Examination of the "universality" of $F_1(\eta)$ and $F_2(\eta)$.

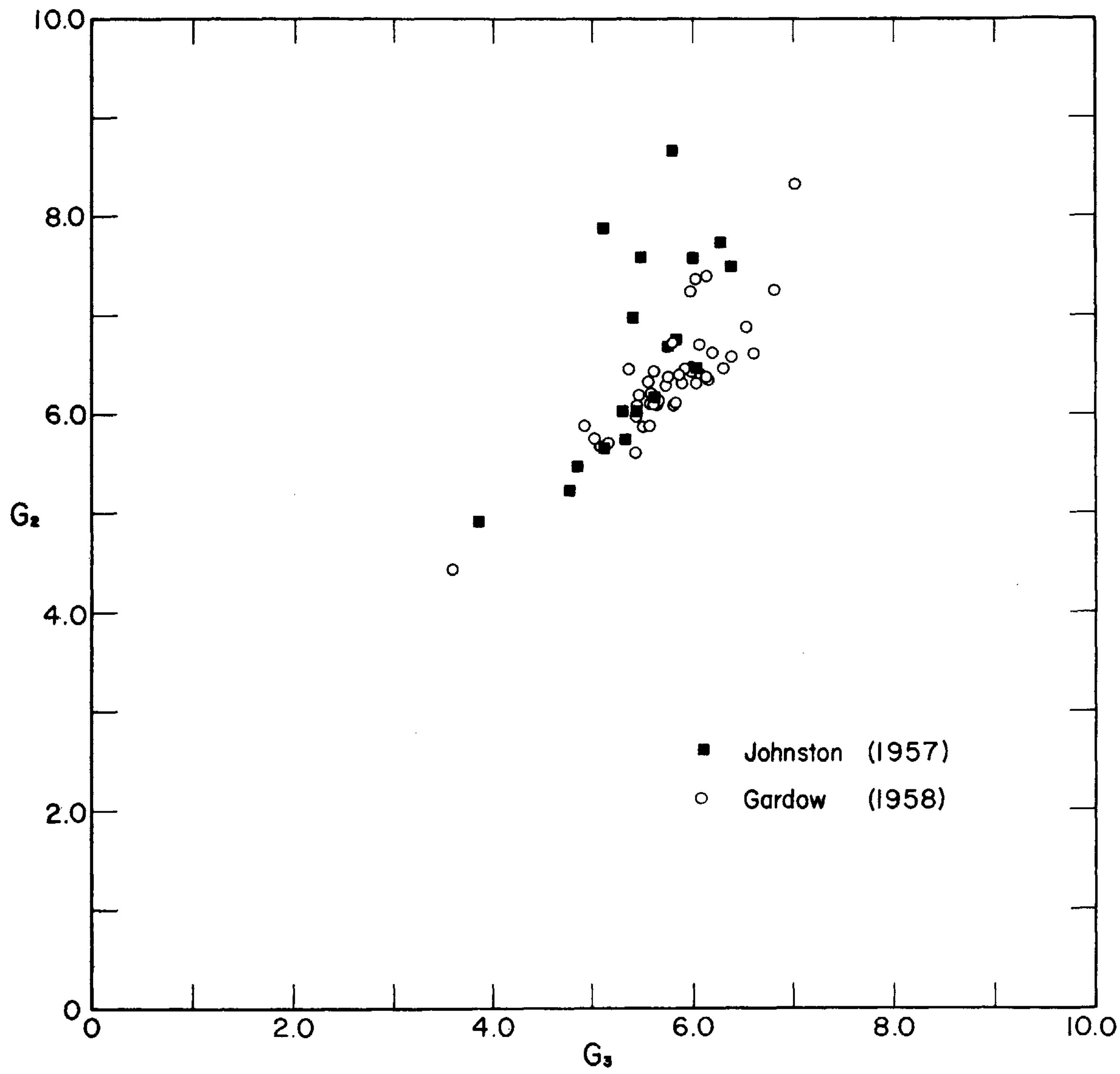


Fig. 5 Examination of the "universality" of $F_1(\eta)$ and $F_2(\eta)$.

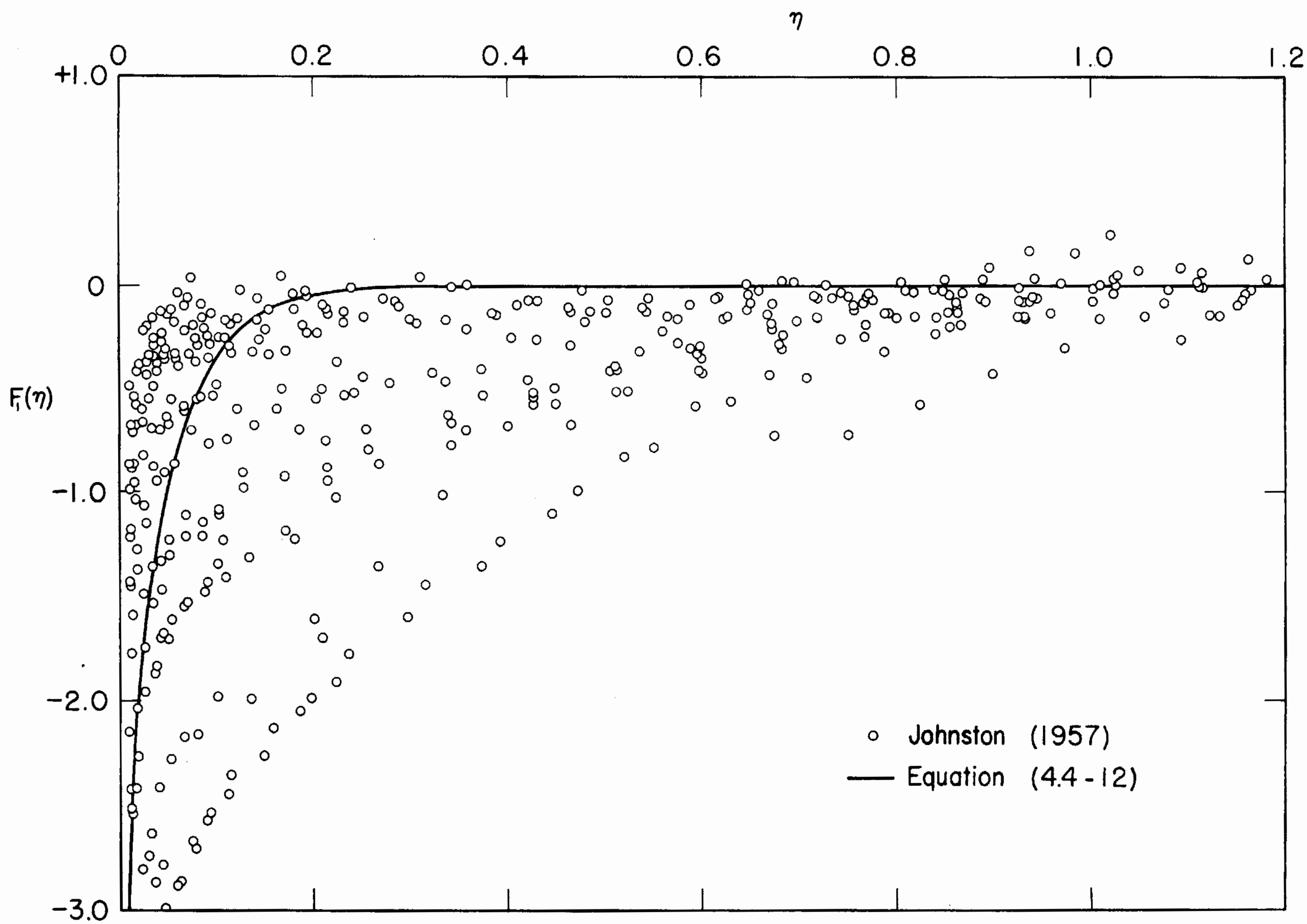


Fig. 6 The function $F_1(\eta)$.

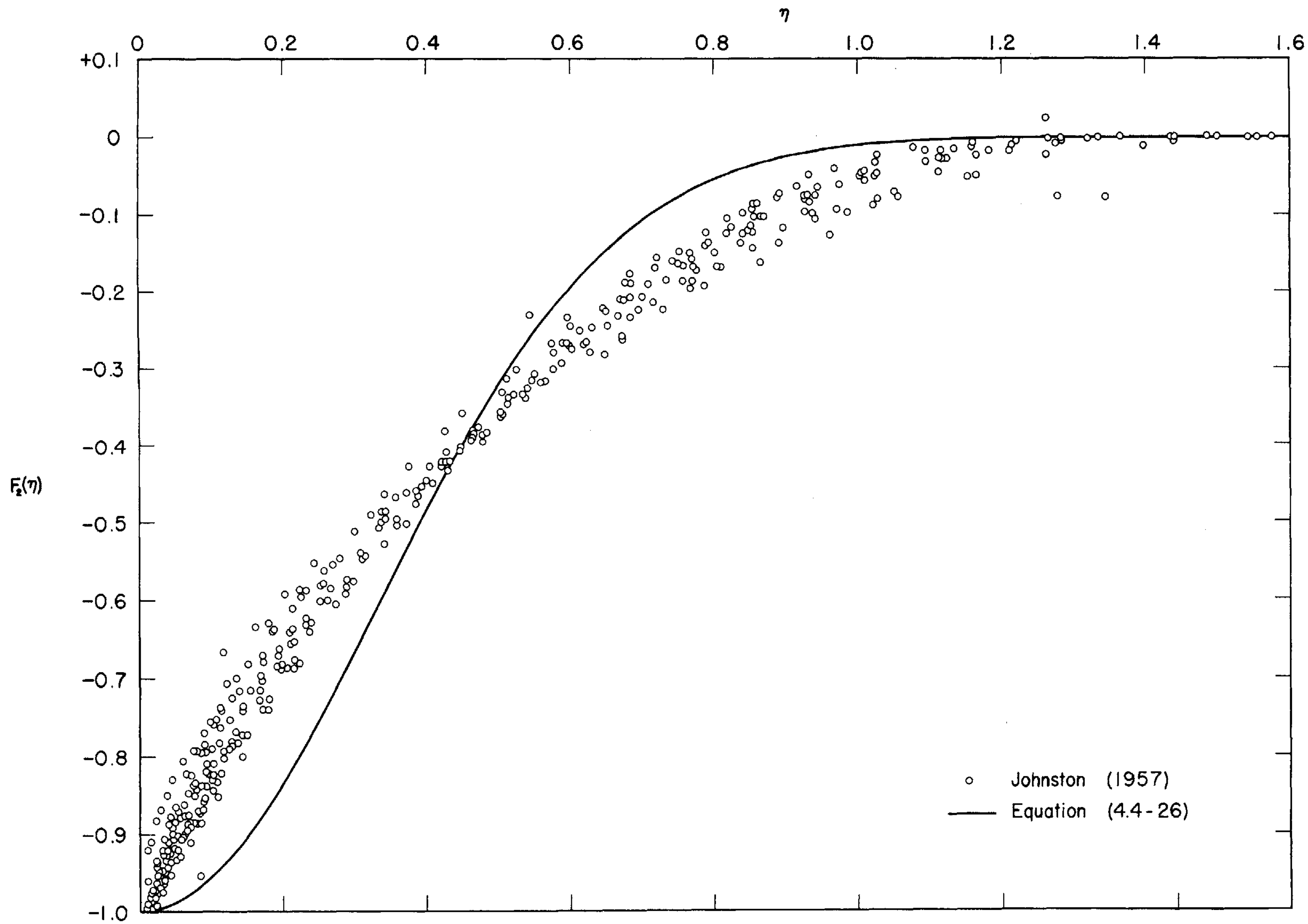


Fig. 7 The function $F_2(\eta)$.

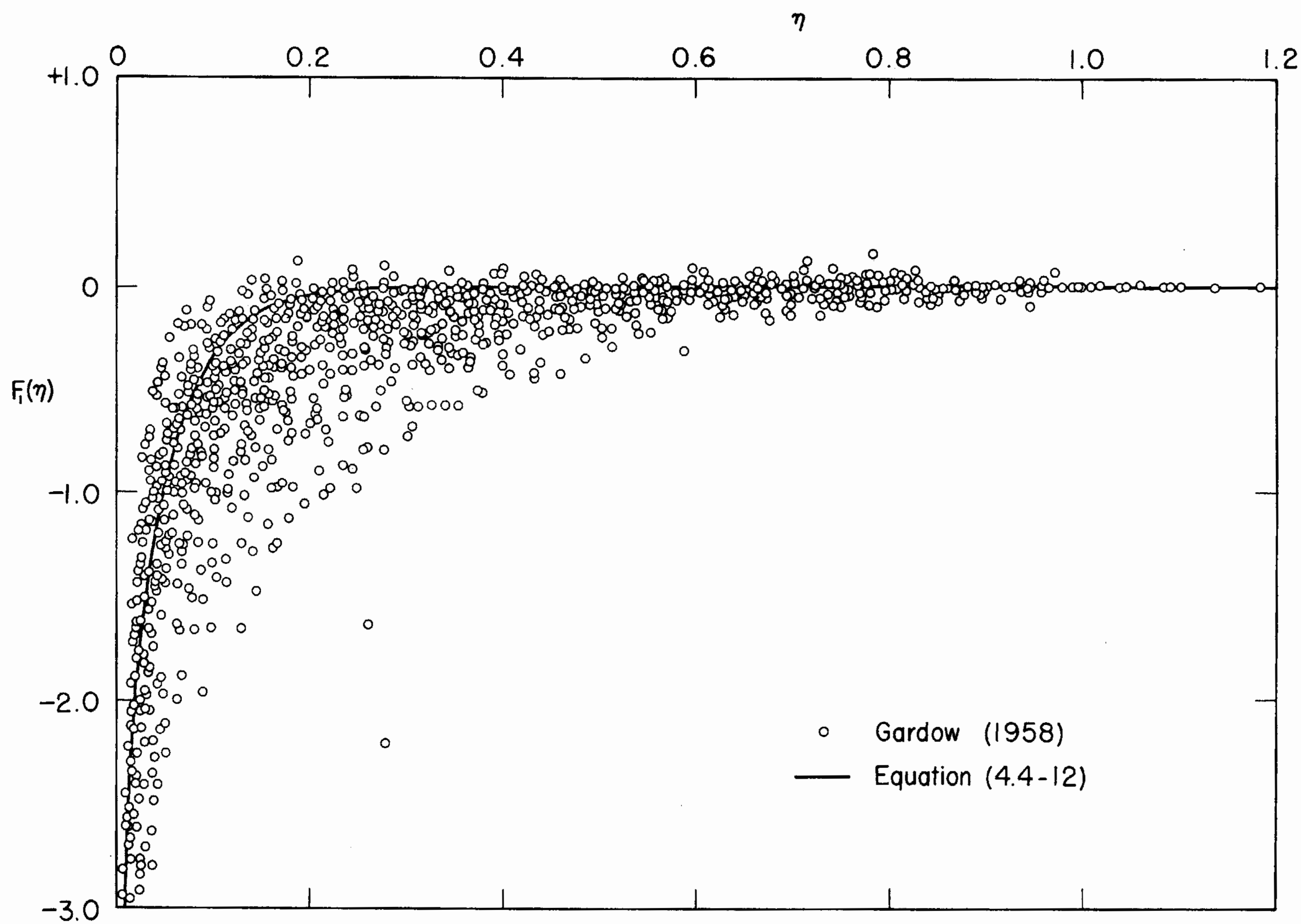


Fig. 8 The function $F_1(\eta)$.

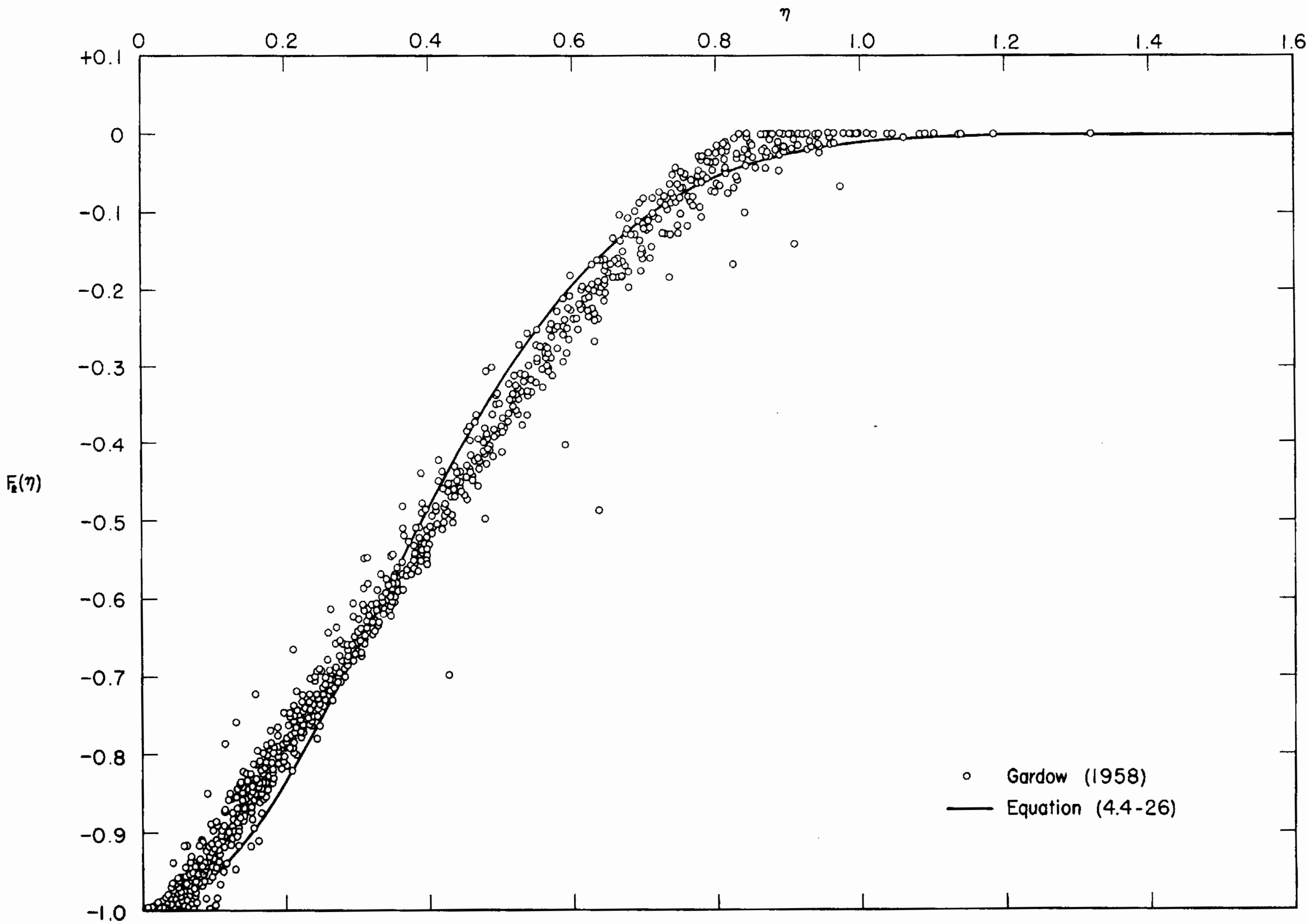
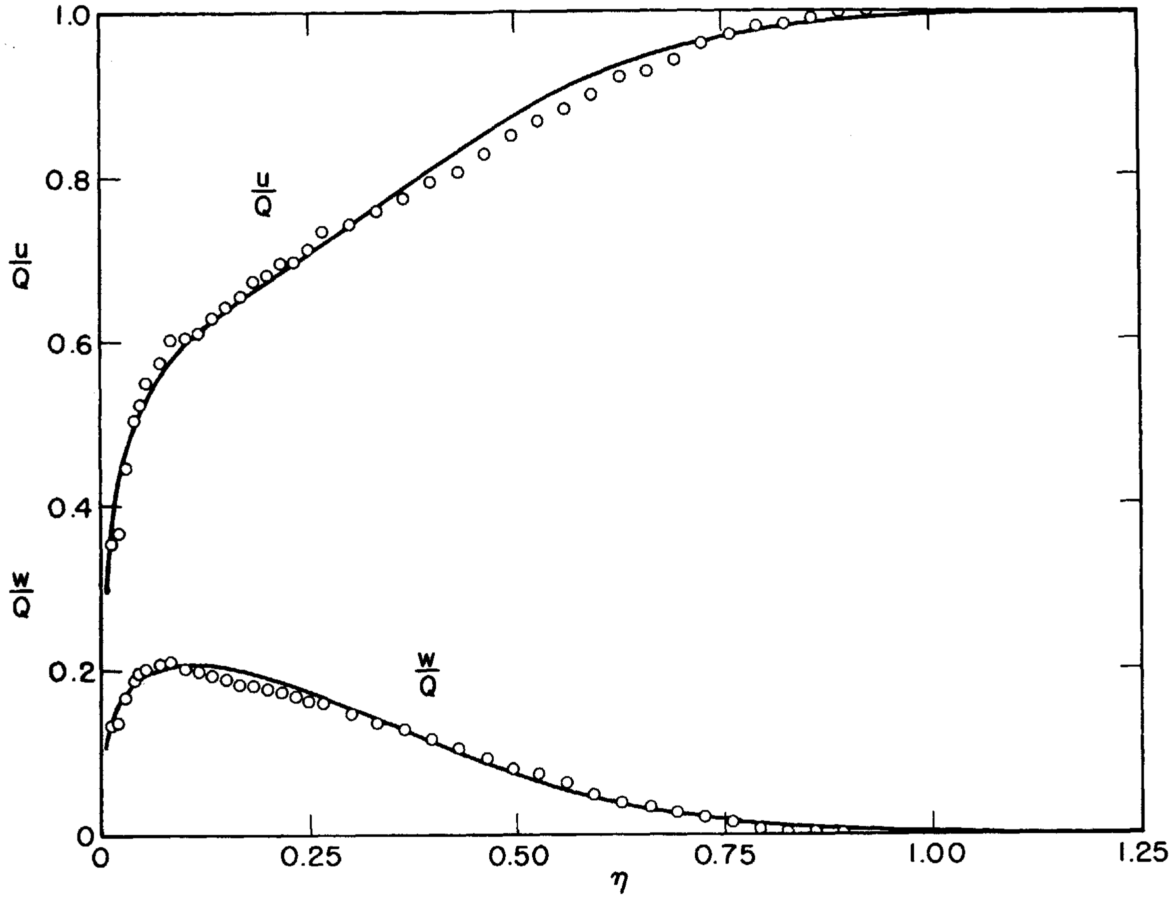


Fig. 9 The function $F_2(\eta)$.



○ Gardow (1958) —— Equations (4.3 - 1a,b)

α = 20.6°, β = -30.8°, γ = 0.1061, Π = 4.243, δ = 0.606 in.

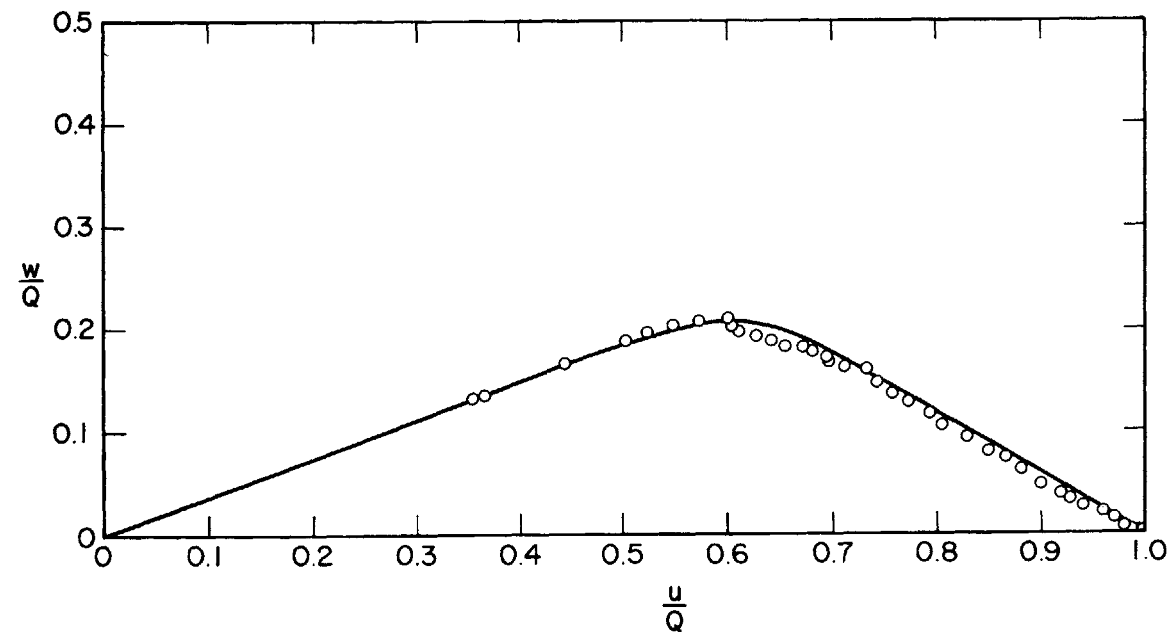
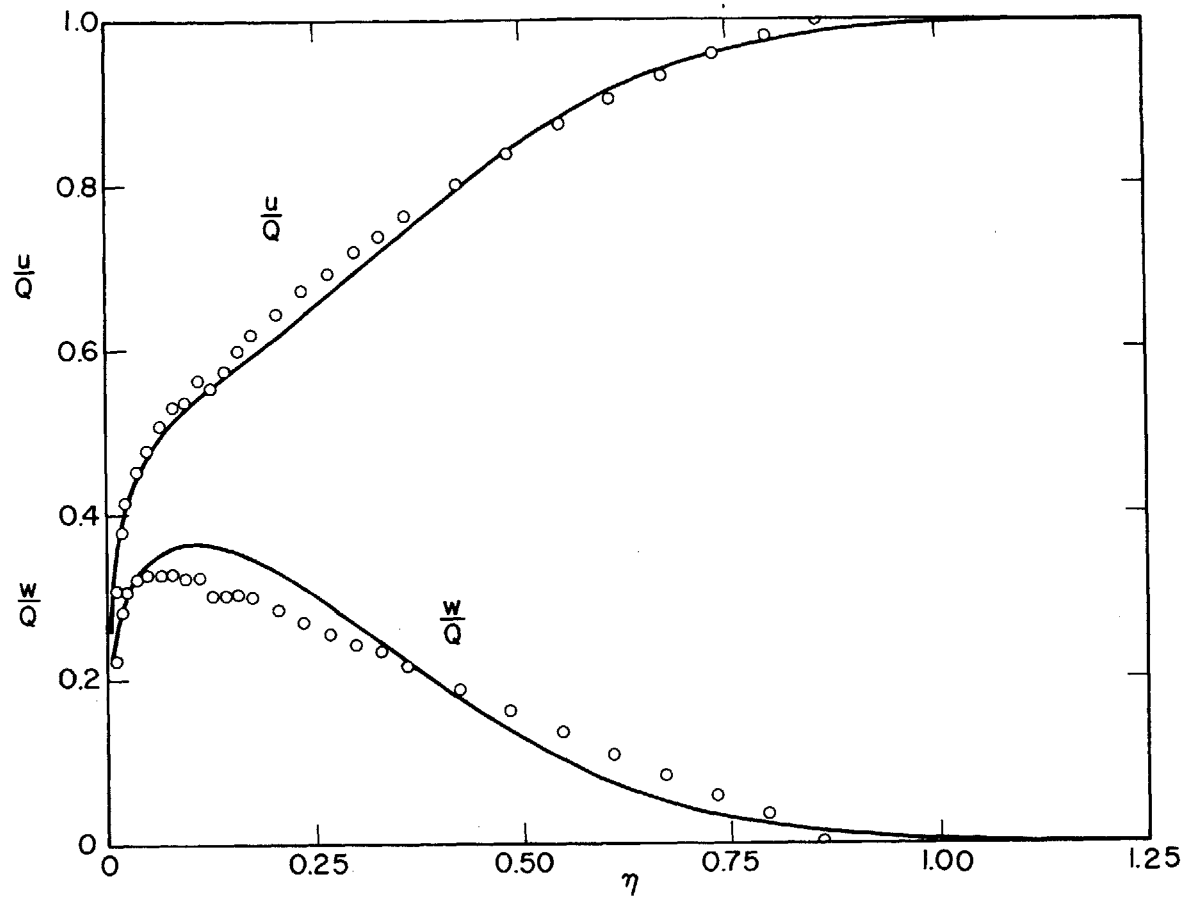


Fig. 10 Comparison of Gardow's profile A - 51.6 - 3 with theory.



○ Gardow (1958) —— Equations (4.3 - 1a,b)

α = 36.6°, β = − 41.0°, γ = 0.0930, Π = 6.564, δ = 1.602 in.

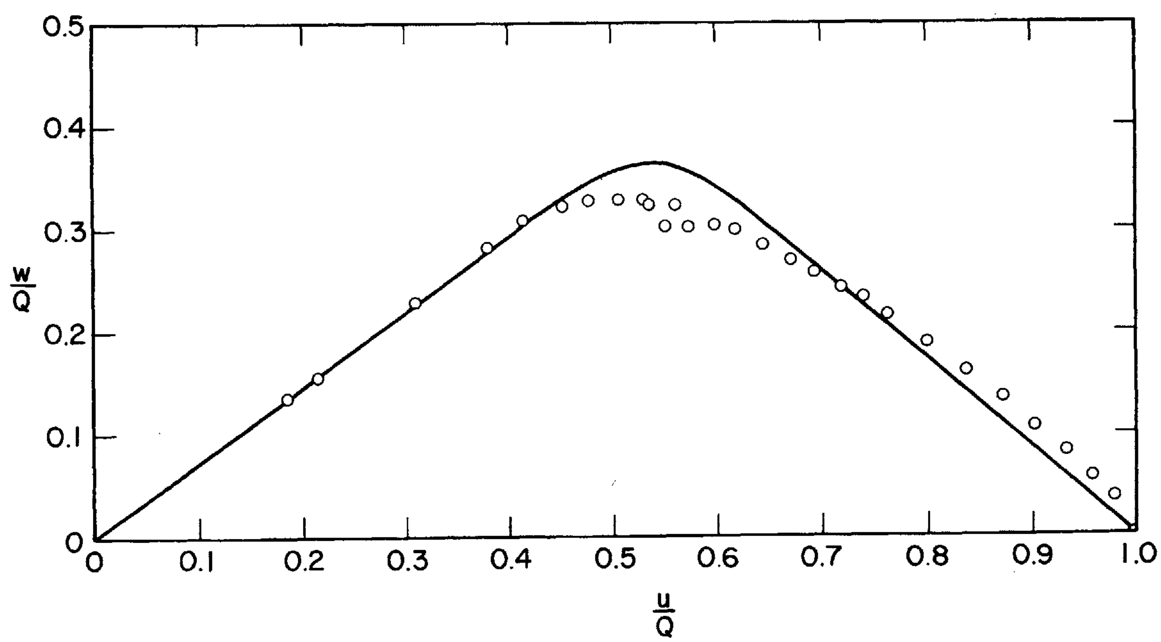


Fig. 11 Comparison of Gardow's profile B - 59.6 - 6 with theory.

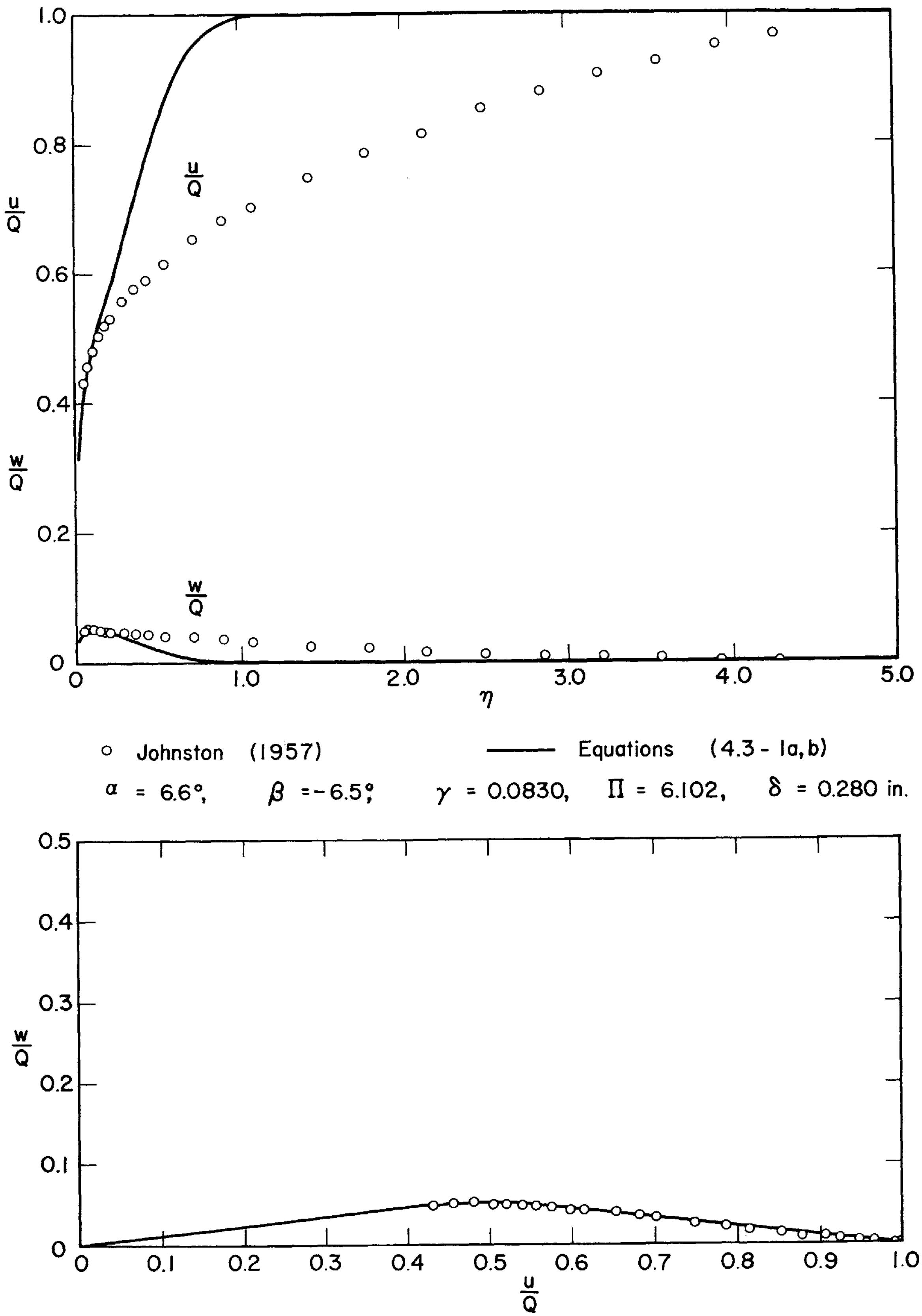


Fig. 12 Comparison of Johnston's profile F - 4 with theory.

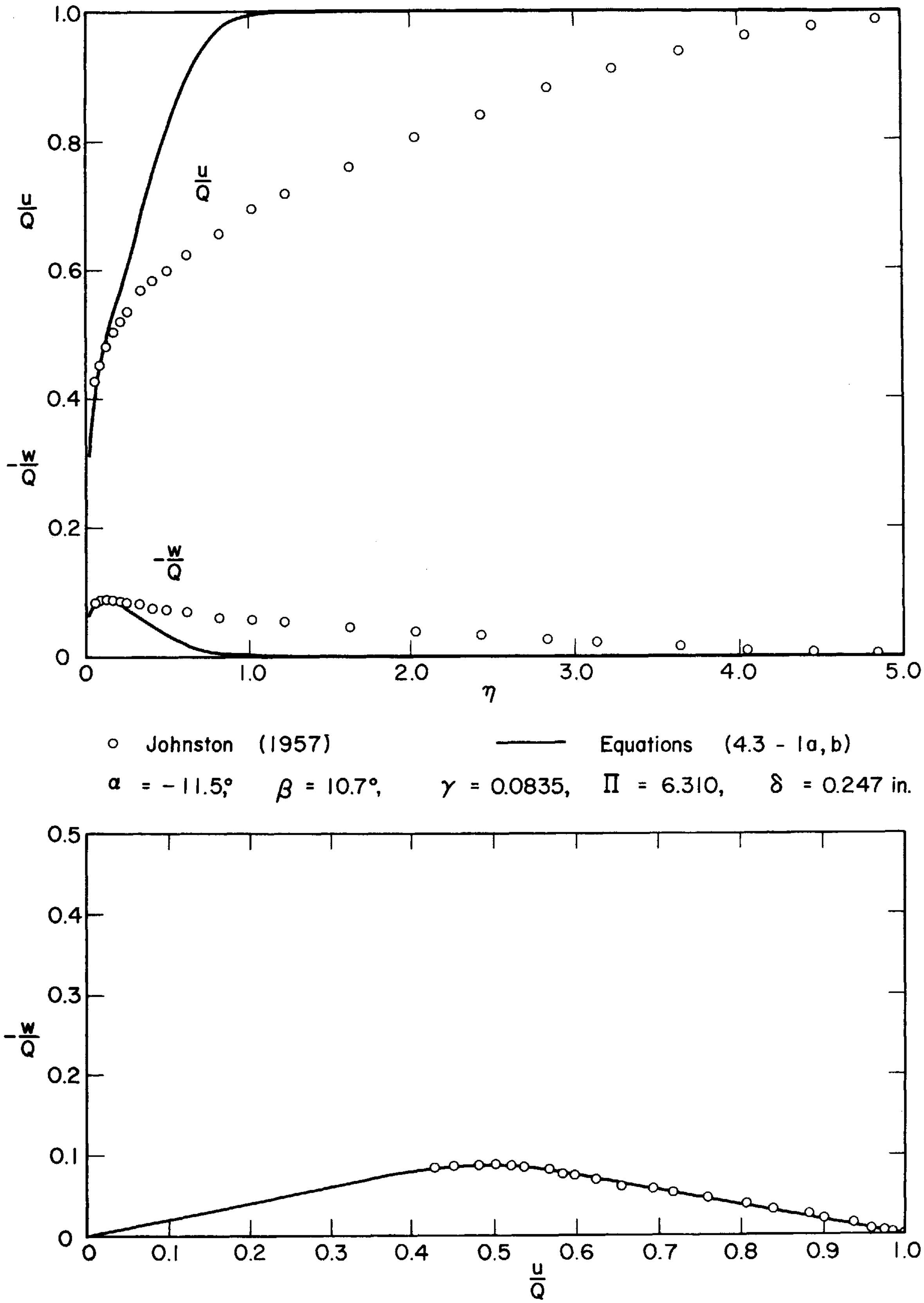


Fig. 13 Comparison of Johnston's profile A - X6 with theory.

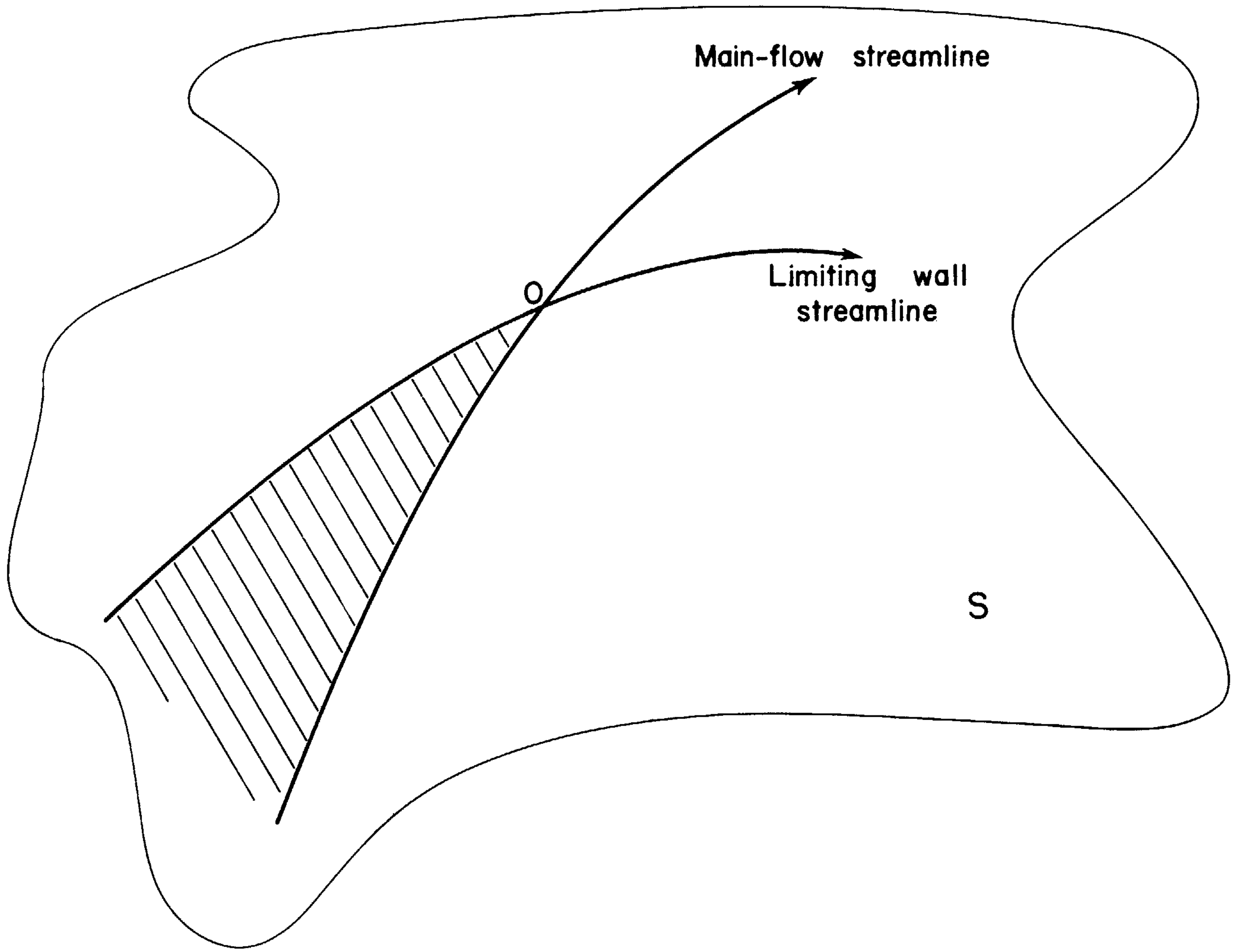


Fig. 14 Illustration of the manner in which streamlines from different environments come together.