ON THE THEORY OF SHEAR FLOW

by

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ABSTRACT

In this report are collected together some introductory notes on the theory of the three-dimensional, steady flow of an inviscid fluid. The equations of motion are discussed and transformed and expressions for the streamwise or secondary vorticity derived both for incompressible and compressible flow. A special form of the Clebsch Transformation is used to show that three linearizing approximations are possible, depending on the order of magnitude of the stagnation pressure gradient and the disturbance to the upstream flow.

The equations for each approximation are developed in turn and are applied to some elementary examples of the flow through actuator discs, over slender bodies and thin airfoils--isolated and in cascade, the flow in curved channels, and about thick struts, airfoils and through cascades of large deflection.

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In this report are collected together some notes on the theory of secondary flow which it is hoped will serve as an introduction to the topic. Much of the work was done while the author was a visitor at the Gas Turbine Laboratory, from the Engineering Laboratory at the University of Cambridge. The comments and suggestions of many colleagues are gratefully acknowledged, but in particular the author wishes to thank Professor E. S. Taylor, Director of the Gas Turbine Laboratory, for his encouragement and assistance, and Donald R. Kotansky and Malcolm Rowe who, during their periods of research at M.I.T. and Cambridge respectively, stimulated and criticized the work and obtained many useful numerical solutions, some of which are reproduced in the text.

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NOMENCLATURE

a	Various constants, radius of cylinder
b	Scalar function constant on streamline various constants
c	Chord various constants
e	Exponential function
f	Scalar function
g	Scalar function
h	Enthalpy, scale factor
i	$\sqrt{-1}$
$\underline{i}$	Unit vector (x direction)
$\underline{j}$	Unit vector (y direction)
k	Ratio of specific beats, eigenvalue, parameter in Fourier transform, $2a/\sqrt{\psi^2 + 4a^2}$
$\underline{k}$	Unit vector (z direction)
ℓ	Distance between walls
m	Source of doublet strength integer, parameter
n	Integer, coordinate normal to streamline, exponent
$\underline{n}$	Unit vector normal to streamline
p	Pressure
q	Velocity
r	Radius, polar cylindrical coordinate
s	Entropy, distance along streamline, spacing between blades in cascade, parameter, $\sqrt{x^2 + z^2}$
$\underline{s}$	Unit vector along streamline
t	Time, drift function
u	Velocity (x direction, r direction), elliptic function

v	Velocity (y direction, θ direction)
w	Velocity (z direction), complex potential
x, y, z	Cartesian coordinates
x	sech ϕ
z	$x + iy$
sn, cn, dn	Elliptic functions
sgn	$\operatorname{sgn} x = +1 \quad x > 0$ $\operatorname{sgn} x = -1 \quad x < 0$
A	Various constants, area $- U' \int_{-\infty}^{\infty} \frac{v}{U} dx$ Vector function
B	Various constants
C_D	Drag coefficient
C_L	Lift coefficient
C_p	Specific heat at constant pressure
D_i	Induced drag
E	Elliptic integral of second kind
F	Body force, function, elliptic integral of first kind
G	Function
H	Heaviside's function
K	$\sqrt{k_n^2 + \lambda_n^2}$, complete elliptic integral of first kind
$\underline{\overline{K}}$	Modified Bessel's function of second kind
L	Lift, scale
M	Mach number
N	Integer
O	Order
P	Pressure gradient, pressure function
R	Radius

$\overline{R}$	Gas constant
T	Absolute temperature function $(\phi - t)$ kinetic energy
U	Velocity in x direction (upstream), scalar function identifying Bernoulli surfaces
V	Velocity
W	Reduced velocity V/b, complex potential, velocity relative to rotating axes
X	Darwin's drift function, Cartesian coordinate, duct dimension
Y	Cartesian coordinate, duct dimension
Z	X + iY
α	Flow angle, bend angle, parameter, angle of attack, angle between principal normal to streamline and normal to Bernoulli surface
α_0	Angle of attack on airfoil
α_1	Inlet flow angle to cascade
α_2	Outlet flow angle from cascade
α_m	Vector mean angle for cascade flow
β	Flow angle (subscript 1 inlet, subscript 2 outlet), parameter
γ	Circulation, parametric angle
Γ	Circulation, gamma function
δ	Dirac's delta function
ε	Small parameter, deflection
ζ	$\xi + i\eta$, complex coordinate, vorticity component (z direction)
η	Cartesian coordinate, vorticity component (θ direction)
θ	Angular coordinate, angle, flow angle
κ	Lift coefficient slope $d\,C_L/d\alpha$
λ	Eigenvalue, scale
μ	Scalar function
ν	Kinematic viscosity

ξ	Cartesian coordinate, vorticity component (x direction)
π	Conventional
ρ	Density
σ	Scalar function
τ	Scalar function
ϕ	Scalar function, velocity potential
χ	$\frac{p}{\rho} + \frac{1}{2} V^2 + \psi$
ψ	Stream function, body force potential logarithmic differential of gamma function
Ω	Vorticity
ω	Angular velocity
∇	Nabla or del

Subscripts

$-$	Vector quantity
0	Stagnation quantity, first order term $O(1)$
1	Second order term $O(\epsilon)$
2	Third order term $O(\epsilon^2)$
I, II	Different scalar functions
x,y,z	Differentiation with respect to x,y,z
z	z component
r	Differentiation, r component
s	Streamwise component, surface value
o	Outer radius, stagnation streamline
i	Inner radius
θ	θ component
m	Integer, maximum value, parameter
n	Integer, normal component

Subscripts (continued)

T	Trefftz plane
$\pm\infty$	Value at infinity

Superscripts

*	Reduced value ($\phi = b\,\phi^*$) value at which M = 1, two dimensional value, value at pole of order two
'	Differentiation, $s' = s \cos \alpha_m$, scalar function ϕ', $\theta' = \theta - \frac{\pi}{2}$
-	Complex conjugate, Fourier transform, mean value

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SECTION 1 - INTRODUCTION

In recent years the problems of fluid flow in three dimensions have received considerable study. The work has been prompted by interest in the flow over swept wings, in the flow in bent ducts, in cascades of turbines and compressor blades and in three-dimensional boundary layers. As in two-dimensional flow, it is possible in many instances to separate the flow field into an inviscid region and a boundary layer region. For example, some of the principal features of the flow in a bent duct can be deduced without considering the effect of viscosity which is confined to a thin layer at the wall. Naturally any such statement requires qualification, since the thickness of the viscous layer grows as the flow passes round the bend and is affected by Reynolds' number and pressure gradients. Nevertheless the development of inviscid solutions seems to have as much practical importance in three-dimensional flow as it has had in two-dimensional and axially symmetric flow.

Inviscid flows divide naturally into irrotational and rotational flows although the former may be regarded as a special case of the latter. The techniques for obtaining irrotational solutions by the use of a velocity potential are well established. The effect of vorticity or rotation on the motion of a fluid has been the subject of study from the earliest period of classical hydrodynamics. The kinematical theories of rotational flow have been comprehensively reviewed by Truesdell(1) whose treatise contains much valuable historical material. Some of the early work is summarised by Lamb(2). A brief review of more recent theoretical and experimental work is given by Thwaites(3).

Except in trivial cases inviscid flows with vorticity are complicated. The bending and stretching of the vortex filaments which occurs when a shear flow (that is, a flow in which the vortex filaments are normal to the streamlines) passes along a bent duct, or around obstacles immersed in the flow, produce a distribution of vorticity and an accompanying velocity distribution which has so far defied exact analysis.

In order to trace the development of the various approximate treatments of the problem consider the disturbance of a shear flow in which the streamlines are parallel but the velocity is not uniform. The shear, that is the velocity gradient normal to the streamlines, may be large or small:

the disturbance may be large as in the flow round a bent duct or over a blunt obstacle or it may be small as in the flow over a slender body.

When the shear is small the flow may be regarded as a potential flow, the primary flow, on which is superimposed a small rotational flow, the secondary flow. Squire and Winter[4] based an approximate solution to the shear flow in a bent duct on the assumption that the vortex filaments were transported by the primary flow. They found that the principal feature of the flow was an increase in the streamwise component of vorticity from zero before the disturbance to surprisingly large values in the bend. It is possible to determine exact expressions for the change in this streamwise component of vorticity or secondary vorticity in incompressible and compressible flow and in flow with density gradients under the action of gravity (Hawthorne[5][6]), as well as in rotating flow (Smith[7]). The expressions are given in terms of the velocity field. Several interesting results have been obtained which throw light on phenomena occurring in bends in the flow round struts and through cascades.

This type of approximation has been called the "secondary flow" approximation by Lighthill[8].

In another approximation it is assumed that the shear is large but the disturbance is small as in the flow over a slender body. In these circumstances it is possible to linearize the equations of motion as shown by Lighthill[9], who obtained solutions for the flow over a small source. Even for such an example, solutions over the whole field around the source are difficult to obtain although the solution by Honda[10][58] for thin airfoils in a shear flow is perhaps the most complete (see also von Karman and Tsien[11]).

When both the shear and the disturbances are small the disturbance becomes irrotational. Such solutions are common in the analysis of flows through screens and actuator disks representing rows of compressor and turbine blades. Lighthill[8] has compared the secondary flow approximations and the large shear, small disturbance solutions and shown how they tend to the same limit in this latter simplest type of approximation.

The above introductory discussion relates to flows which may be parallel upstream but become genuinely three-dimensional when disturbed. There has been considerable work on the purely two-dimensional shear flow problem including exact solutions for the special example of uniform vorticity, Taylor[12], Tsien[13]. For more realistic shear profiles approximations are necessary. A discussion will be found in Sears[14].

Three-dimensional boundary layer theory has been reviewed by Moore[15]. The literature on the subject is extensive and the methods of solution are extensions of two-dimensional methods[16].

The main object of these notes is to provide a review of the more recent work on inviscid three-dimensional shear flows, to clarify the methods of approximation and provide a starting point for the solution of specific problems. The notes contain some of the results reported in Reference (21) but are intended to be an extension and development of that earlier work.

SECTION 2 - THE EQUATIONS OF MOTION AND THEIR TRANSFORMATION

2.1 - The Equations of Motion

The various forms of the equations of motion will depend on the simplifying assumptions made in any particular part of the analysis. In the case of an arbitrary, inviscid fluid passing through reversible processes, Euler's equation is

$$\frac{D\underline{V}}{Dt} = \underline{F} - \frac{1}{\rho}\,\text{grad}\ p \qquad (2\text{-}1)$$

Where $\underline{V}$ represents the velocity, p and ρ the pressure and density, $\underline{F}$ the body force per unit mass and $\frac{D}{DT}$ denotes differentiation following the motion of the fluid, i. e.

$$\frac{D\underline{A}}{Dt} = \frac{\partial \underline{A}}{\partial t} + (\underline{V} \cdot \nabla)\ \underline{A}$$

The equation of continuity is:

$$\frac{\partial \rho}{\partial t} + \text{div}\ \rho\ \underline{V} = 0 \qquad (2\text{-}2)$$

When the fluid is of uniform density but the viscosity, ν, is not neglected equation (2-1) is replaced by the Navier-Stokes equation in the form

$$\frac{D\underline{V}}{Dt} = \underline{F} - \frac{1}{\rho}\,\text{grad}\ p + \nu\ \nabla^2\ \underline{V} \qquad (2\text{-}3)$$

In terms of the velocity $\underline{W}$ relative to axes rotating with angular velocity $\underline{\omega}$, Euler's equation becomes

$$\frac{\partial \underline{W}}{\partial t} + (\underline{W} \cdot \nabla)\ \underline{W} + 2\ \underline{\omega} \times \underline{W} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) = \underline{F} - \frac{1}{\rho}\,\text{grad}\ p \qquad (2\text{-}4)$$

where $\underline{r}$ is the radius vector from the axis of rotation.

Some common transformations of the equations of motion involve the introduction of the vorticity $\underline{\Omega}$ = curl $\underline{V}$. We note that div $\underline{\Omega}$ = div curl $\underline{V} \equiv 0$.

The substitution of the identity

$$(\underline{V} \cdot \nabla)\ \underline{V} \equiv \frac{1}{2}\,\text{grad}\ \underline{V} \cdot \underline{V} - \underline{V} \times \text{curl}\ \underline{V} \qquad (2\text{-}5)$$

in equation (2-1) yields

$$\frac{\partial \underline{V}}{\partial t} - (\underline{V} \times \underline{\Omega}) = \underline{F} - \frac{1}{\rho} \text{grad } p - \text{grad } \frac{1}{2} V^2 \tag{2-6}$$

By taking the curl of both sides of equation (2-6) using equation (2-2) and rearranging the terms we obtain the Helmholtz equation*.

$$\frac{D}{Dt} \frac{(\Omega)}{(\rho)} = (\frac{\Omega}{\rho} \cdot \nabla) \underline{V} + \frac{1}{\rho} \text{curl } (\underline{F} - \frac{1}{\rho} \text{grad } p) \tag{2-7}$$

When the body force has a potential, and flow is steady and the density of the fluid is uniform

$$\underline{V} \times \underline{\Omega} = \text{grad } \chi \tag{2-8}$$

where $$\chi = \frac{p}{\rho} + \frac{1}{2} V^2 + \psi \tag{2-9}$$

and $$\underline{F} = - \text{grad } \psi \tag{2-10}$$

For the simplest case, viz. that of steady flow of an inviscid fluid of uniform density in the absence of body forces $\chi = p_o/\rho$ and

$$\underline{V} \times \underline{\Omega} = \text{grad } \frac{p_o}{\rho} \tag{2-11}$$

where p_o is the stagnation pressure.

It is evident from equation (2-8) that the vortex filaments and streamlines lie in surfaces of constant χ (or p_o in the simpler case of equation (2-11)). Such surfaces have been called Lamb or Bernoulli surfaces. The effects of compressibility will be examined in Section 2.4

2.2 - Steady, Inviscid, Incompressible Flow

We may distinguish three types of such flows

(1) Irrotational Flow

The simplest flow is one in which $\underline{\Omega} = 0$ everywhere and hence we may write $\underline{V} = \text{grad } \phi$ and the continuity condition yields Laplace's equation $\nabla^2\phi = 0$. This is the commonest ideal flow in classical hydrodynamics and numerous solutions have been obtained.

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* *See Lamb [2] Chapter VII for a fuller presentation of this introductory theoretical matter.*

(2) Beltrami Flow

When the stagnation pressure p_o is constant everywhere but $\underline{\Omega} \neq 0$ we have

$$\underline{V} \times \underline{\Omega} = 0 \tag{2-12}$$

i.e. the vortex filaments lie along the streamlines. Physical examples occur in the flow (somewhat idealised it is true) behind airfoils of finite span or in the annular duct behind rows of turbine or compressor blades which are shedding circulation or trailing vorticity into the stream.

Equation (2-12) is satisfied by writing

$$\underline{\Omega} = \mu \underline{V} \tag{2-13}$$

where μ is a scalar. Since div $\underline{\Omega} = 0$ and div $\underline{V} = 0$, we find by taking the divergence of both sides of the above equation (2-13) that

$$\text{div } \mu \underline{V} \equiv \mu \text{ div } \underline{V} + (\underline{V} \cdot \nabla)\mu = 0$$

so that $$(\underline{V} \cdot \nabla)\mu = 0 \tag{2-14}$$

i.e. μ is constant along a streamline. It may be shown that μ is the circulation about the stream tube through which unit volume of fluid flows per unit time.

Furthermore taking the curl of both sides of equation (2-13) we obtain

$$\text{curl curl } \underline{V} = \text{curl } \mu \underline{V}$$

which after expansion becomes $$-\nabla^2 \underline{V} = \mu \underline{\Omega} + \text{grad } \mu \times \underline{V}$$

so that $$\nabla^2 \underline{V} = -\mu^2 \underline{V} + \underline{V} \times \text{grad } \mu \tag{2-15}$$

In special cases and in certain kinds of rotating flow μ may be constant

so that $$\nabla^2 \underline{V} = -\mu^2 \underline{V} \tag{2-16}$$

Such flows may have wave type solutions. Equation (2-12) shows that in a Beltrami flow equation (2-5) reduces to

$$\frac{1}{2} \text{grad } V^2 = (\underline{V} \cdot \nabla)\underline{V}$$

Hence if there is some direction in which grad V = 0 then the component of acceleration $(\underline{V} \cdot \nabla)\underline{V}$ in that direction is also zero.

A Beltrami flow always remains a Beltrami flow, since in the flow of a frictionless fluid of constant and uniform density p_o remains constant. An irrotational flow is a special case of a Beltrami flow.

(3) General Rotational Flow

This is the more general flow in which grad $p_o/\rho \neq 0$. However, since

$$\underline{V} \cdot \text{grad } p_o/\rho = \underline{V} \cdot \underline{V} \times \underline{\Omega} \equiv 0 \tag{2-17}$$

we find that p_o is constant along a streamline. We may visualize the flow field as consisting of closely packed Bernoulli surfaces of constant p_o and of infinite extent in which streamlines and vortex filaments are interwoven. The vortex filaments move with the fluid like lines of fluid particles marked with dye, drifting and bending with the flow in their own particular Bernoulli surface. Experimentally the distortion of the Bernoulli surfaces is easy to follow with a pitot tube, provided viscous effects are small.

Helmholtz' equation (2-7) for this incompressible steady flow in which the body forces have a potential is

$$(\underline{V} \cdot \nabla)\, \underline{\Omega} = (\underline{\Omega} \cdot \nabla)\, \underline{V} \tag{2-18}$$

In a two-dimensional plane flow the vorticity is perpendicular to the velocity so that the right hand side of equation (2-18) is zero and hence the vorticity is constant along a streamline.

For any general type of rotational flow it may be shown that the velocity may be written

$$\underline{V} = \text{grad } \phi + \text{curl } \underline{A} \tag{2-19}$$

where div $\underline{A} = 0$ and $\underline{A}$ is therefore a vector potential.

The vorticity is then given by

$$\underline{\Omega} = - \nabla^2 \underline{A} \tag{2-20}$$

and the continuity condition by

$$\nabla^2 \phi = 0 \tag{2-21}$$

This form of solution is useful in certain cases, e.g. in plane two-dimensional flow when the result

$$-\nabla^2 \psi = \Omega = f(\psi) \tag{2-22}$$

is obtained, where Ψ is the stream function.

In axi-symmetric flow (see Thwaites[3])

$$- [\psi_{rr} - \frac{1}{r} \psi_r + \psi_{xx}] = - \eta \, r = rv \frac{drv}{d\psi} - \frac{r^2}{\rho} \frac{dp_o}{d\psi} \tag{2-23}$$

where v and η are the circumferential components of the velocity and vorticity respectively, ψ is the Stokes stream function (a function of r and x) and p_o and rv are functions of ψ. When p_o is constant throughout the flow, we have an example of Beltrami flow.

Another transformation is due to Clebsch[17]. It may be shown that the velocity may be written as

$$\underline{V} = \text{grad } \phi + \sigma \text{ grad } \tau \tag{2-24}$$

where σ and τ are scalar functions. Since the vorticity is given by

$$\underline{\Omega} = \text{grad } \sigma \times \text{grad } \tau \tag{2-25}$$

it is evident that the intersection of any two surfaces $\sigma(x, y, z) =$ constant and $\tau(x, y, z) =$ constant is a vortex filament. It has been shown in reference (2) page 248 that there are an infinite number of ways of selecting the functions σ and τ.

Now it may be verified from equation (2-11) that $\underline{\Omega} \cdot \text{grad } p_o/\rho = 0$, so that as an example of the use of Clebsch's transformation which has useful physical meaning we may choose $\sigma = p_o/\rho$, so that one of the two families of surfaces is the family of Bernoulli surfaces, $p_o/\rho =$ constant. Then equations (2-25) and (2-11) yield

$$\begin{aligned} \underline{\Omega} &= (\underline{V} \times \underline{\Omega}) \times \text{grad } \tau \\ &= (\underline{V} \cdot \text{grad } \tau)\, \underline{\Omega} - (\underline{\Omega} \cdot \text{grad } \tau)\, \underline{V} \\ &= (\underline{V} \cdot \text{grad } \tau)\, \underline{\Omega} \end{aligned} \tag{2-26}$$

Since $(\underline{\Omega} \cdot \text{grad } \tau) = 0$ because the surfaces $\tau =$ constant also contain the vortex filaments..

From equation (2-26) we deduce that

$$\underline{V} \cdot \text{grad } \tau = 1$$

or

$$V \frac{\partial \tau}{\partial s} = 1 \tag{2-27}$$

where s is the distance along a streamline. If at any instant one of the surfaces $\tau =$ constant is identified, then at a time t later the fluid particles in that surface will have drifted a distance s downstream such that

$$t = \int \frac{ds}{V} \tag{2-28}$$

where the integral is taken along a streamline. Because the circulation associated with a fluid particle remains the same, the surfaces t = constant, as equations (2-27) and (2-28) show, will coincide with the required surfaces τ = constant*. The drift function τ has been determined for some simple flows by Darwin[18] and used by Lighthill[8] to compute the distortion of the vortex lines in weakly sheared flows past obstacles such as airfoils and spheres.

This result may be extended to the case of the unsteady flow of an inviscid compressible fluid, provided the body forces have a potential and the density ρ is a constant or a function of p only, by writing

$$\sigma = \int \frac{dp}{\rho} + \frac{1}{2} V^2 + \psi + \frac{\partial \phi}{\partial t} + \frac{\partial \tau}{\partial t} \tag{2-29}$$

where σ, τ and ϕ are functions of (x, y, z, t).

By making use of the result quoted by Lamb[2] page 249, we find that

$$\frac{D\sigma}{Dt} = 0 \qquad \text{and} \qquad \frac{D\tau}{Dt} = 1 \tag{2-30}$$

from which we conclude that the surfaces σ = constant move with the fluid, and the previous result for τ is generalized.

Returning now to the simpler case of steady flow of a fluid of uniform density in the absence of body forces we see that the velocity may be written either as

$$\underline{V} = \text{grad}\ \phi_I + \frac{p_o}{\rho}\ \text{grad}\ t \tag{2-31}$$

or as

$$\underline{V} = \text{grad}\ \phi_{II} - t\ \text{grad}\frac{p_o}{\rho} \tag{2-32}$$

From the second of these equations we deduce that streamlines and lines of constant ϕ_{II} form an orthogonal net on any Bernoulli surface and $V = \frac{\partial \phi_{II}}{\partial s}$.

By taking the dot product between $\underline{V}$ and the terms on both sides of equation (2-31) we deduce that

$$\frac{p}{\rho} = - V \frac{\partial \phi_I}{\partial s} + \frac{1}{2} V^2 \tag{2-33}$$

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* *Truesdell[1] Section 102 gives an account of earlier work leading to a result similar to that derived here.*

Arbitrary constants may be added to the functions $\frac{p_o}{\rho}$, t and p in equations (2-31) to (2-33) without affecting their form.

The application of the continuity condition yields the results

$$\nabla^2 \phi_I + \frac{p_o}{\rho} \nabla^2 t + \text{grad} \frac{p_o}{\rho} \cdot \text{grad } t = 0 \qquad (2\text{-}34)$$

and

$$\nabla^2 \phi_{II} - t \nabla^2 \frac{p_o}{\rho} - \text{grad} \frac{p_o}{\rho} \cdot \text{grad } t = 0 \qquad (2\text{-}35)$$

The manipulation of these equations to obtain what at best are likely to be only approximate solutions to any shear flow problem will be discussed in later sections of these notes.

We note from this discussion of Clebsch's transformation that if we can find a function τ such that $\underline{\Omega} \cdot \text{grad } \tau = 0$, i.e. if we can locate the surfaces containing the vortex filaments, then as the velocity may be written as the sum of the gradient of a potential and a vector, σ grad τ, some simplifications of the equations of motion will be possible. For instance, if there is no component of vorticity in the x-direction, then the surfaces containing the vortex filaments are normal to the x-axis and

$$\underline{\Omega} \cdot \text{grad } x = \underline{i}\, \Omega = 0$$

and the velocity may be written as

$$\underline{V} = \text{grad } \phi + \underline{i}\, A \qquad (2\text{-}36)$$

where A is a scalar function and $\underline{i}$ is the unit vector in the x-direction.

(4) Complex Lamellar Motion

In a complex lamellar motion

$$\underline{V} \cdot \underline{\Omega} = 0 \qquad \Omega \neq 0 \qquad (2\text{-}37)$$

We may satisfy this condition by writing

$$\underline{V} = f \text{ grad } g \qquad (2\text{-}38)$$

where f and g are scalars.

Now a plane shear flow is complex-lamellar as is an axi-symmetric shear flow without swirl. Steady complex lamellar flows are flows without secondary vorticity (commonly defined as the component of vorticity in the direction of flow). It has been shown [5] that shear flows in which the streamlines are geodesics on the Bernoulli surfaces develop no secondary vorticity and are therefore complex-lamellar.

2.3 Secondary Vorticity

Secondary vorticity is defined as the component of vorticity in the direction of flow. The simplest example of secondary vorticity occurs in the Beltrami flow ($\underline{V} \times \underline{\Omega} = 0$) in which the only component of vorticity is in the direction of flow. The presence of streamwise components of vorticity is associated with secondary motions or circulations transverse to the main flow. For instance, in the Beltrami flow represented by the trailing vortex sheet downstream of an aerofoil of finite span the vortex sheet may be said to induce secondary motions which cause the transverse displacement of the vortex sheet described as "wrapping up".

A complex lamellar flow ($\underline{V} \cdot \underline{\Omega} = 0$) on the other hand has no secondary vorticity or circulation, and should flow downstream comparatively free from twisting and spiralling motions.

Although a Beltrami flow remains a Beltrami flow, it is of fundamental importance to recognize that a flow which is complex lamellar upstream may not remain complex lamellar; in other words it may develop a secondary vorticity component as a result of flowing past obstacles or round bends.

We can demonstrate this for the steady flow of an incompressible fluid by writing the vector identity

$$\underline{V} \times (\underline{V} \times \underline{\Omega}) \equiv (\underline{V} \cdot \underline{\Omega})\, \underline{V} - (\underline{V} \cdot \underline{V})\, \underline{\Omega} \qquad (2\text{-}39)$$

and taking the divergence of both sides.

First we note that

$$\text{div}\, \underline{V} \times (\underline{V} \times \underline{\Omega}) \equiv (\underline{V} \times \underline{\Omega}) \cdot \text{curl}\, \underline{V} - \underline{V} \cdot \text{curl}\, (\underline{V} \times \underline{\Omega}) = 0.$$

Since the first triple product on the right hand side contains two identical vectors and

$$\text{curl}\, (\underline{V} \times \underline{\Omega}) = \text{curl}\, (\text{grad}\, \frac{p_o}{\rho}) = 0.$$

Hence equation (2-39) becomes

$$\operatorname{div} (\underline{V} \cdot \underline{\Omega})\, \underline{V} = \operatorname{div} (\underline{V} \cdot \underline{V})\, \underline{\Omega}$$

or $$(\underline{V} \cdot \underline{\Omega}) \operatorname{div} \underline{V} + (\underline{V} \cdot \nabla)(\underline{V} \cdot \underline{\Omega}) = (\underline{V} \cdot \underline{V}) \operatorname{div} \underline{\Omega} + (\underline{\Omega} \cdot \nabla)(\underline{V} \cdot \underline{V})$$

which, by virtue of the continuity condition $\operatorname{div} \underline{V} = 0$ and of the result, $\operatorname{div} \underline{\Omega} = \operatorname{div} \operatorname{curl} \underline{V} \equiv 0$, yields the result[5]

$$\underline{V} \cdot \operatorname{grad} (\underline{V} \cdot \underline{\Omega}) = \underline{\Omega} \cdot \operatorname{grad} V^2 . \tag{2-40}$$

Substituting for grad V from the identity, equation (2-5), we obtain the alternative form

$$\underline{V} \cdot \operatorname{grad} (\underline{V} \cdot \underline{\Omega}) = 2\underline{\Omega} \cdot (\underline{V} \cdot \nabla)\underline{V} . \tag{2-41}$$

Hence $(\underline{V} \cdot \underline{\Omega})$ will vary along a streamline if there is a component of acceleration or grad V in the direction of the vorticity, $\underline{\Omega}$. Hence if the vorticity lies in the direction in which the acceleration or grad $V^2 = 0$, as for instance in two-dimensional or plane flow, no secondary vorticity will develop.

The change from a flow which is complex-lamellar upstream to one in which there is a strong secondary vorticity occurs in the flow round bends. James Thomson[19] showed experimentally in 1877 that a spiralling flow could be observed in a curved water channel, the secondary motion at the bottom being inward towards the center of curvature of the bent channel and at the top outward. The secondary flow was attributed to the effect of the centrifugal pressure gradient in the main flow acting on the relatively stagnant fluid in the boundary layer on the bottom wall of the channel. Similar experimental observations of the secondary flow in channels and bends have been reported from time to time. The first attempt to predict such secondary flows was by Dean[20] who analysed the laminar flow in pipe bends of large curvature at low Reynolds number. The first interpretation of experimental results in terms of the generation of secondary vorticity in an inviscid flow was given by Squire and Winter[4].

In a long straight channel or duct upstream of a bend the flow is parallel but the velocity decreases towards the walls as a result of viscous effects. As the vorticity is normal to the flow velocity, the flow is complex-lamellar. Squire and Winter assumed that although viscous effects were responsible for establishing the velocity profile approaching the bend, they could be neglected in the flow in the bend. Equation (2-41) then directs attention to the product of the vorticity and the component of acceleration in the direction of the vorticity. To illustrate the application of equation (2-41) we consider the flow in a broad rectangular straight channel in which except near the vertical side walls the vorticity is horizontal and normal to the flow direction, Figure 2.1. As the flow enters a bend of mean radius R a component of centrifugal acceleration ($- V^2/R$) develops, lying in a horizontal plane, and therefore in the direction of the vorticity.

Hence equation (2-41) becomes

$$V \frac{\partial}{\partial s} (\underline{V} \cdot \underline{\Omega}) = - 2\Omega \frac{V^2}{R} . \qquad (2\text{-}42)$$

If ds is an elementary arc of the streamline,

$$\frac{1}{R} = \frac{d\theta}{ds} \qquad (2\text{-}43)$$

where θ is the bend angle, Figure 2.1.

Now for small values of θ we may assume that V and Ω remain constant so that the integral of equation (2-42) may be written

$$\underline{V} \cdot \underline{\Omega} = - 2\Omega V\theta$$

or since the secondary vorticity, $\Omega_s = 0$ at $\theta = 0$

$$\Omega_s = - 2\Omega\theta . \qquad (2\text{-}44)$$

This simple result, derived first by Squire and Winter[4], has been found to be of great value in interpreting the phenomenon of secondary flow in bends. Further developments of this approach together with a discussion of the approximations used in the analysis is deferred to Section 6. In this section we continue with the development of expressions for the secondary vorticity.

The identity, equation (2-39), may be written in the form

$$\underline{\Omega} \equiv \frac{\underline{V} \cdot \underline{\Omega}}{\underline{V} \cdot \underline{V}} \underline{V} + \frac{(\underline{V} \times \underline{\Omega}) \times \underline{V}}{(\underline{V} \cdot \underline{V})} \tag{2-45}$$

thus resolving the vorticity into two components. One component, the first term, is in the direction of flow and is therefore the secondary vorticity, Ω_s. The other component, Ω_n, is normal to the flow, but lies in the Bernoulli surfaces containing the vortex lines and streamlines. On taking the divergence of equation (2-45) the left hand side becomes zero and the right hand side

$$\frac{\Omega_s}{\rho V} \operatorname{div} \rho\underline{V} + \rho\underline{V} \cdot \operatorname{grad} \frac{\Omega_s}{\rho V} + (\underline{V} \times \underline{\Omega}) \times \underline{V} \cdot \operatorname{grad} (1/V^2)$$
$$+ \left[\underline{V} \cdot \operatorname{curl} \underline{V} \times \underline{\Omega} - \underline{V} \times \underline{\Omega} \cdot \operatorname{curl} \underline{V} \right] (1/V^2) = 0 \quad . \tag{2-46}$$

We note that $\Omega_s/\rho V$ is the circulation around a stream tube along which unit mass is flowing in unit time and that equation (2-13) is a special case of equation (2-45). In Beltrami flows described by equation (2-13) $\Omega_s/\rho V$ - the secondary circulation - is constant along a streamline.

When the fluid is inviscid and of uniform density and the body forces have a potential the first term in equation (2-46) is zero by virtue of the condition of continuity. The last term is also zero since $\underline{V} \times \underline{\Omega} = \operatorname{grad} \chi$ (equation 2-8)) and $\underline{V} \times \underline{\Omega} \cdot \operatorname{curl} \underline{V} = \underline{V} \times \underline{\Omega} \cdot \underline{\Omega} = 0$.

Hence equation becomes

$$\rho\underline{V} \cdot \operatorname{grad} (\Omega_s/\rho V) = - \{\underline{V} \times (\underline{V} \times \Omega) \cdot \operatorname{grad} V^2\} (1/V^4)$$
$$= - 2 \{\underline{V} \times (\underline{V} \times \underline{\Omega}) \cdot (\underline{V} \cdot \nabla)\underline{V} \} (1/V^4) \tag{2-47}$$

where the vector identity equation (2-5) has been used.

The triple product on the right hand side of equation (2-47) is proportional to the product of

(i) the velocity

(ii) a Vector ($\underline{V} \times \Omega$) normal to the Bernoulli surface of constant χ, or p_o. This vector is normal to the velocity vector.

(iii) the component of the vector representing the acceleration, which is normal to the velocity but lies in the Bernoulli surface.

The acceleration may be resolved into two components, one along the streamline and the other along the principal normal to the streamline. The acceleration along the principal normal towards the centre of curvature is V^2/R where $1/R$ is the curvature of the streamline. If α is the angle between the direction of the principal normal and the normal to the Bernoulli surface, then the component of the acceleration which enters into the triple product is $\sin \alpha \, V^2/R$.

Equation (2-47) therefore becomes

$$\rho \underline{V} \cdot \text{grad}\,(\Omega_s/\rho V) = -2 \,|\, \text{grad}\,\chi \,|\, \sin \alpha/(RV). \qquad (2\text{-}48)$$

Now $(\sin \alpha)/R = 1/R_g$, the geodesic curvature of the streamline in the Bernoulli surface. The sign of the change in secondary circulation depends on the direction in which the streamline is turning away from the plane containing the vectors $\underline{V}$ and $\underline{V} \times \underline{\Omega}$. If the streamline turns towards the direction of the vortex lines then the geodesic curvature, or $\sin \alpha$, is negative; if away, $\sin \alpha$ is positive, the positive direction of the vorticity being determined by the usual right hand screw rule. When the streamline is a geodesic on the Bernoulli surface, the geodesic curvature is zero and no change of secondary circulation occurs along the streamline.

Integrating equation (2-48) along a streamline

$$(\Omega_s/\rho V)_2 - (\Omega_s/\rho V)_1 = -2 \int_1^2 \{ \,|\text{grad}\,\chi\,|\, \sin \alpha/\rho V^2 \}\, d\theta \qquad (2\text{-}49)$$

where $d\theta$ is defined by equation (2-43).

In the example of flow round a bend discussed above, $\alpha = \Pi/2$ and $|\text{grad}\,\chi| = V\,\Omega$. Making these assumptions in equation (2-49) and assuming that V is constant along a streamline we obtain the result given in equation (2-44), viz: -

$$\Omega_s = -2\Omega\theta. \qquad (2\text{-}44)$$

Other derivations of the result given by equation (2-44) have been presented (see References 5, 6, 22, 24, 25, 26, 27).

2.4 The Effect of Compressibility

We now extend the foregoing results to ideal fluids in which not only viscous effects but also the diffusion of heat and material are negligible. The latter requirements are necessary to enable the concept of a shear flow to be extended to flows with non-uniform densities and temperatures. For instance, if the fluid is incompressible, but of non-uniform density, it will be assumed that each particle of fluid retains its initial density, ρ, throughout the flow, i.e.:

$$\frac{D\rho}{Dt} = 0. \tag{2-50}$$

Similarly, if there is no heat transfer or work due to shearing forces, the First Law of Thermodynamics leads to the result that the change of internal energy plus the work done is zero, whence

$$\frac{\partial h_o}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t} + \underline{V} \cdot \text{grad } h_o = 0 \tag{2-51}$$

where $h_o = h + \frac{1}{2} V^2$ ($\equiv$ stagnation enthalpy). We may extend this equation to the case when there are body forces $\underline{F}$ with a potential ψ such that

$$F = - \text{grad } \psi$$

and obtain the result

$$\frac{\partial (h_o + \psi)}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t} + \underline{V} \cdot \text{grad } (h_o + \psi) = 0. \tag{2-52}$$

In reversible, adiabatic flow, by the Second Law of Thermodynamics, the entropy of a fluid particle retains its initial value, i.e.

$$\frac{Ds}{Dt} = 0. \tag{2-53}$$

We now limit the discussion to pure substances, i.e. substances whose thermodynamic state is determined by two independent properties. For pure substances

$$T ds = dh - (1/\rho)\, dp$$

or

$$T \text{ grad } s = \text{grad } h - (1/\rho) \text{ grad } p. \tag{2-54}$$

Stagnation conditions are the conditions which would be obtained if a fluid particle were brought to rest reversibly and adiabatically, i.e. isentropically, hence $s_o = s$ and

$$T_o \text{ grad } s = \text{grad } h_o - (1/\rho_o) \text{ grad } p_o \tag{2-55}$$

where the subscript o denotes stagnation conditions.

On applying equation (2-55) to equation (2-6), we obtain the result when the body forces have a potential

$$\begin{aligned}\frac{\partial V}{\partial t} - \underline{V} \times \underline{\Omega} &= - \text{grad } (h_o + \psi) + T \text{ grad } s \\ &= - \text{grad } \psi + (\frac{T}{T_o} - 1) \text{ grad } h_o - \frac{T}{\rho_o T_o} \text{ grad } p_o \qquad (2\text{-}56) \\ &= - \text{grad } \psi - (T_o - T) \text{ grad } s - (1/\rho_o) \text{ grad } p_o \,.\end{aligned}$$

Steady Flow

In the first instance we shall apply these results to obtain an equation for the growth of secondary circulation in steady flow which is more general than equation (2-49). In steady flow $h_o + \psi$ and s remain constant along a streamline, equations (2-52) and (2-53). In the absence of body forces, the thermodynamic state at the stagnation condition is constant along the streamline, i.e. p_o, ρ_o, h_o, etc. are constant.

We start with equation (2-46) and note that

$$\underline{V} \times \underline{\Omega} = \text{grad}(h_o + \psi) - T \text{ grad } s \,,$$

so that curl $(\underline{V} \times \underline{\Omega}) = -$ grad T x grad s.

Hence equation (2-46) becomes, after some simplification,

$$\begin{aligned}\rho\underline{V} \cdot \text{grad } (\Omega_s/\rho V) &= \underline{V} \times \text{grad}(h_o + \psi) \cdot \text{grad } (1/V^2) \\ &\quad - \underline{V} \times \text{grad } s \cdot \text{grad}(T/V^2), \qquad (2\text{-}57)\end{aligned}$$

or $$\rho\underline{V} \cdot \text{grad } (\Omega_s/\rho V) = \underline{V} \times \text{grad } \psi \cdot \text{grad } 1/V^2$$

$$+ \underline{V} \times \frac{\text{grad } h_o}{T_o} \cdot \{\text{grad } \frac{T_o - T}{V2} - \frac{1}{V2} \text{grad } T_o\} + \underline{V} \times \frac{\text{grad } p_o}{\rho_o T_o} \cdot \text{grad}(T/V2). \tag{2-58}$$

This equation is derived in Reference (6) and applied to a range of general problems. Expressions are also given by Loos[27].

Perfect Gas in Steady Flow, in the Absence of Body Forces

We now discuss the application of these results to the example of a perfect gas in the absence of body forces.

When the fluid is a perfect gas $p = \rho \bar{R} T$, $h = C_p T$ and $T_o - T = V^2/2C_p$ where $\bar{R}$ is the gas constant and C_p the specific heat at constant pressure.

The problem of the flow of perfect gases when h_o, p_o, s and ρ_o vary from one streamline to another has been examined by Munk(28), Prim(29), Nemenyi and Prim(30), Hayes(31), Truesdell(32) and others. A summary is given on pages 41-43, Reference (33).

We proceed as follows: -

Let b be a scalar quantity which is constant along a streamline but which may vary from one streamline to another.

Let $\underline{W} = \underline{V}/b$, then (2-59)

$$\rho_o \, b^2 \, (\underline{W} \cdot \nabla) \, \underline{W} = - \frac{\rho_o}{\rho} \operatorname{grad} p, \tag{2-60}$$

and

$$\operatorname{div} (\rho/\rho_o) \, \underline{W} = 0 \, . \tag{2-61}$$

satisfy the equations of motion and continuity.

Now for a perfect gas

$$(\rho/\rho_o) = (p/p_o)^{1/k}$$

where k is the ratio of specific heats. Hence flows in which the distribution of p and p_o are the same will have the same distribution of $\underline{W}$ provided that

$$\rho_o \, b^2 = f(p, \, p_o) \, .$$

If $b^2 \propto T_o$, then $\rho_o \, b^2 \propto p_o$ and the condition is satisfied.

These considerations led Munk and Prim(34) to the substitution principle for the shear flows of a perfect gas, namely that <u>any flows with the same distribution of p_o have the same geometrical pattern of streamlines regardless of the distribution of T_o</u>.

We may illustrate this principle by applying equation (2-58) to the flow of perfect gases. The first term on the right hand side of equation (2-58) disappears, in the absence of body forces, and the second term equates to zero for a perfect gas, so that the equation becomes

$$\rho \underline{V} \cdot \operatorname{grad} (\Omega_s/\rho V) = (1/\rho_o \, T_o) \, \underline{V} \times \operatorname{grad} p_o \cdot \operatorname{grad}(T/V^2) \, . \tag{2-62}$$

<u>Hence it follows that secondary circulation change only occurs when there is</u>

a gradient of stagnation pressure and is not affected by the presence of stagnation temperature gradients.

Homentropic Flows of a Perfect Gas

Homentropic flows are flows in which the entropy, s, is constant throughout the flow. For such flows grad $h_o = (1/\rho_o)$ grad p_o from which we deduce that h_o, T_o, p_o and ρ_o are not independent, in fact T_o is related to p_o by the isentropic condition, vis.

$$T_o/(p_o)^{\frac{k-1}{k}} = \text{constant}. \tag{2-63}$$

We also find that grad $h = (1/\rho)$ grad p from which we deduce that ρ is a function of pressure only in a homentropic flow. The simplification offered by the homentropic condition may be used in the derivation of results which may then be generalised to non-homentropic flows by the use of Munk and Prim's substitution principle.

For instance, in a homentropic flow equation (2-56) becomes

$$\underline{V} \times \underline{\Omega} = \text{grad } h_o .$$

All the conditions leading to equations (2-47) and (2-49) are satisfied (with h_o being substituted for X), and, since grad $h_o = (1/\rho_o)$ grad p_o, equation (2-49) may be written: -

$$(\Omega_s/\rho V)_2 - (\Omega_s/\rho V)_1 = -2 \int_1^2 \left\{ |\text{ grad } p_o| \sin\alpha/\rho_o\, \rho\, V^2 \right\} d\theta \tag{2-64}$$

where the integral is taken along a streamline.

Now introducing the reduced velocity $\underline{W} = \underline{V}/b$ where b^2 is proportional to T_o and is therefore constant along a streamline and noting that ρ_o is also constant along a streamline

$$\left(\frac{\Omega_s/b}{(\rho/\rho_o)\, W}\right)_2 - \left(\frac{\Omega_s/b}{(\rho/\rho_o)\, W}\right)_1 = -2 \int_1^2 \frac{\bar{R}T_o\ |\text{ grad } \log_e p_o|}{b^2(\rho/\rho_o)\, W^2} \sin\alpha\, d\theta . \tag{2-65}$$

Various choices for b^2 may be made such as $b^2 = 2C_pT_o$ (Munk and Prim's) or $b^2 = k\bar{R}T_o/(k+1) = k\bar{R}T^* = V^{*2}$ where the superscript * denotes the value at the point in isentropic flow from stagnation temperature T_o at which the Mach number is unity.

The above equation then yields the reduced secondary vorticity (Ω_s/b) for any flow with variable T_o and p_o and is not confined to homentropic flows.

Tank Flow: p_o constant

Vazsonyi[35] has described flows in which p_o is everywhere constant, but T_o varies, as a tank flow, i.e. the flow produced when a non-uniformly heated perfect gas passes from slow movement in a large container into a ducting system.

Equation (2-56) shows that

$$\underline{V} \times \underline{\Omega} = \frac{1}{2} V^2 \text{ grad } (\log_e T_o). \tag{2-66}$$

As there is no gradient of stagnation pressure, there is no component of vorticity in the direction of flow, i.e. tank flows are complex-lamellar. The vorticity is normal to the velocity and lies in the surface of constant T_o with the magnitude

$$\Omega = \frac{1}{2} V \mid \text{grad}(\log_e T_o) \mid . \tag{2-67}$$

Tank flows are geometrically similar to and derivable from irrotational flows. In terms of the reduced velocity tank flows are characterised by the reduced irrotationality condition $\quad \text{curl } \underline{W} = 0$. (2-68)

The Reduced Helmholtz Equation

Now for a perfect gas in the absence of body forces (equation (2-56))

$$\underline{V} \times \underline{\Omega} = \frac{1}{2} V^2 \text{ grad } (\log_e T_o) + \bar{R}\, T \text{ grad } (\log_e p_o). \tag{2-69}$$

Now $\quad \underline{\Omega} = \text{curl } b\, \underline{W}$

$$= b \text{ curl } \underline{W} + \text{grad}\, b \times \underline{W} . \tag{2-70}$$

Hence $\quad \underline{V} \times \underline{\Omega} = b^2\, \underline{W} \times \text{curl } \underline{W} + b\, \underline{W} \times (\text{grad } b \times \underline{W})$

$$= b^2\, \underline{W} \times \text{curl } \underline{W} + b\, W^2 \text{ grad } b . \tag{2-71}$$

Substituting for $\underline{V} \times \underline{\Omega}$ in equation (2-69) we obtain

$$\underline{W} \times \text{curl } \underline{W} = \frac{1}{2} W^2 \text{ grad } (\log_e T_o) + \frac{\bar{R}T}{b^2} \text{ grad } (\log_e p_o) - \frac{1}{2} W^2 \text{ grad}(\log_e b^2)$$

$$= \frac{\bar{R}T}{b^2} \text{ grad}(\log_e p_o) , \tag{2-72}$$

since $\quad b^2 \propto T_o$.

We then obtain the result that

$$\text{curl} \left\{ (T_o/T)\, \underline{W} \times \text{curl}\, \underline{W} \right\} = 0 \tag{2-73}$$

which was first obtained by Crocco[36] for isoenergetic flows and later shown to be valid for this more general flow of perfect gases by Hicks, et al[37].

To obtain the reduced Helmholtz equation we expand equation (2-73) formally

$$(\text{curl}\, \underline{W} \cdot \nabla)\, (T_o/T)\, \underline{W} \;-(\, (T_o/T)\, \underline{W} \cdot \nabla)\, \text{curl}\, \underline{W} + (T_o/T)\, \underline{W}\, \text{div curl}\, \underline{W}$$

$$- \text{curl}\, \underline{W}\, \text{div}\, (T_o/T)\, \underline{W} = 0. \tag{2-74}$$

The third term in the above equation div curl $\underline{W} \equiv 0$, and the fourth term becomes

$$\text{curl}\, \underline{W}\, \text{div}\, (p_o/p)\, (\rho/\rho_o)\underline{W} = (p_o/p)\, \text{curl}\, \underline{W}\, \text{div}\, (\rho/\rho_o)\, \underline{W}$$

$$+ (\rho/\rho_o)\, \text{curl}\, \underline{W}\, (\underline{W} \cdot \nabla\, (p_o/p)). \tag{2-75}$$

Now the first term on the right hand side is zero since div $(\rho/\rho_o)\underline{W} = 0$ by continuity.

Hence equation (2-75) becomes

$$(\text{curl}\, \underline{W} \cdot \nabla)\, (T_o/T)\underline{W} - ((T_o/T)\underline{W} \cdot \nabla)\, \text{curl}\, \underline{W}$$

$$- (p/p_o\, \text{curl}\, \underline{W}\, ((T_o/T)\underline{W} \cdot \nabla\, (p_o/p)) = 0\,,$$

or $$\left\{ (p_o/p)\, \text{curl}\, \underline{W} \cdot \nabla \right\} (T_o/T)\underline{W} = \left\{ (T_o/T)\underline{W} \cdot \nabla \right\} (p_o/p)\, \text{curl}\, \underline{W}\,. \tag{2-76}$$

We note that (T_o/T) and (p_o/p) are functions of W.

<u>Application of the Clebsch Transformation</u>

Consider first the homentropic flow of a perfect gas in the absence of body forces. Let

$$V = \text{grad}\, \phi + h_o\, \text{grad}\, \tau\,. \tag{2-77}$$

Then by the method of equation (2-26) we find the result (2-27) and τ and t are identical, i.e. the surfaces τ = constant drift with the fluid.

Using the form of equation (2-32)

$$\begin{aligned} \underline{V} &= \text{grad}\, \phi - t\, \text{grad}\, h_o \\ &= \text{grad}\, \phi - (t/\rho_o)\, \text{grad}\, p_o \\ &= \text{grad}\, \phi - \overline{R}\, T_o\, t\, \text{grad}\, (\log_e p_o)\,. \end{aligned} \tag{2-78}$$

Dividing the equation by b and writing $\phi = b\phi^*$, we obtain

$$\underline{W} = \text{grad } \phi^* + \frac{1}{2}\phi^* \text{ grad}(\log_e b^2) - (RT_o/b^2) \text{ bt grad}(\log_e p_o). \quad (2\text{-}79)$$

Let $b^2 = 2 C_p T_o$, then for homentropic flow

$$\underline{W} = \text{grad } \phi^* + \frac{k-1}{2k}(\phi^* - bt) \text{ grad}(\log_e p_o). \quad (2\text{-}80)$$

We now demonstrate that this equation also applies to non-homentropic flows and is therefore a reduced form of equation (2-79)

$$\text{curl } \underline{W} = \frac{k-1}{2k}\left\{\text{grad } (\phi^* - bt) \times \text{grad}(\log_e p_o)\right\}. \quad (2\text{-}81)$$

Hence $\underline{W} \times \text{curl } \underline{W} = \frac{k-1}{2k}\Big\{(\underline{W} \cdot \text{grad } \log_e p_o) \text{ grad}(\phi^* - bt)$

$$- (\underline{W} \cdot \text{grad}(\phi^* - bt)) \text{ grad } \log_e p_o\Big\}. \quad (2\text{-}82)$$

The first term on the right hand side is zero since p_o is constant on a streamline. In order to examine the second term we note that

$$\underline{W} \cdot \underline{W} = \underline{W} \cdot \text{grad } \phi^* \quad (2\text{-}83)$$

by substituting for $\underline{W}$ from equation (2-80) and retaining the non-zero term. Hence equation (2-82) becomes

$$\underline{W} \times \text{curl } \underline{W} = \frac{k-1}{2k}\left[\frac{V}{b}\frac{\partial bt}{\partial s} - \frac{V^2}{b^2}\right] \text{grad } \log_e p_o$$

$$= \frac{k-1}{2k}(1 - (V^2/b^2)) \text{ grad } \log_e p_o ,$$

which is identical with the reduced equation (2-72) when $b^2 = 2 C_p T_o$.

The continuity equation (2-61) may be written

$$\text{div } \underline{W} + (1/\overline{k-1})\, \underline{W} \cdot \text{grad } \log_e (1 - W^2) = 0 \quad (2\text{-}84)$$

or, substituting for $\underline{W}$ from equation (2-80) and (2-83)

$$\nabla^2 \phi^* + \frac{k-1}{2k}\left[(\phi^* - bt)\nabla^2 \log_e p_o + \text{grad}(\phi^* - bt) \cdot \text{grad } \log_e p_o\right]$$

$$= \frac{2}{k-1} \frac{(\underline{W} \cdot \text{grad } W)}{(1 - W^2)} . \quad (2\text{-}85)$$

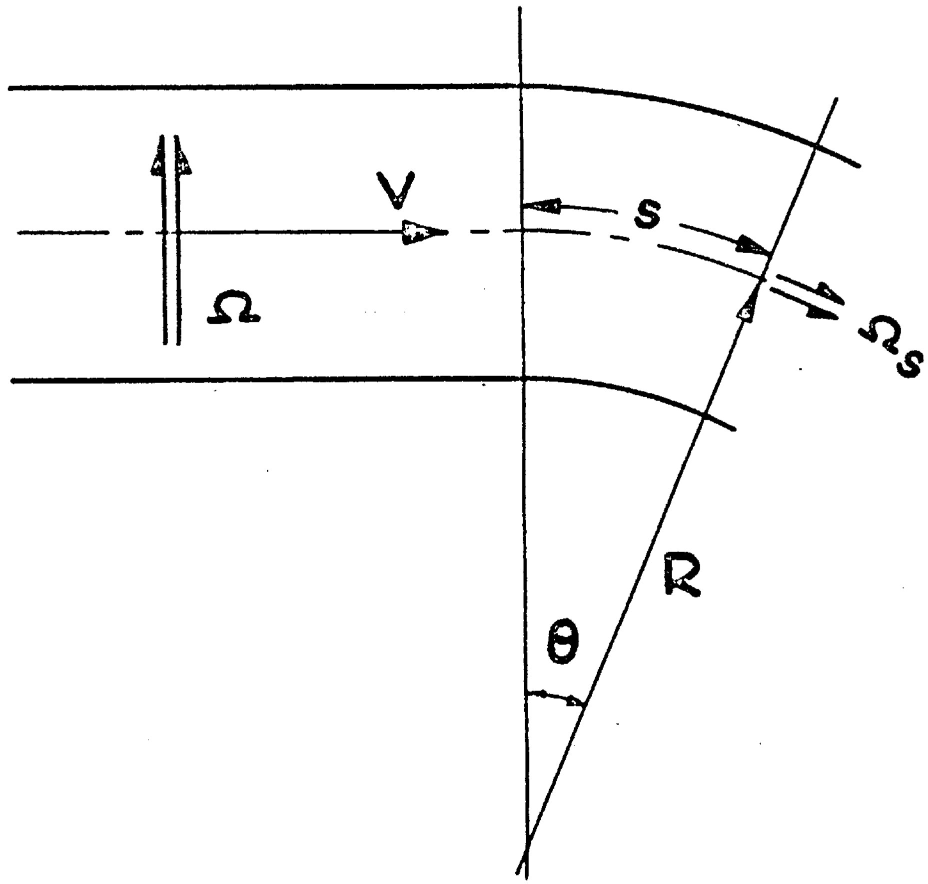


FIG. 2.1 FLOW ROUND BEND

SECTION 3 - APPROXIMATE METHODS OF SOLUTION

3.1 Introduction

With very few exceptions exact solutions of shear flow problems are unobtainable. Attempts to obtain approximate solutions follow a few different basic approaches depending on the nature of the flow and the type of approximation initially assumed. Their validity is open to some doubt since the question of convergence is inevitably raised. However, experimental results appear to justify the approximations in several cases.

To examine the nature of the approximations we assume, for simplicity, that the fluid is incompressible, inviscid and there are no body forces. We start with equation (2-32) which may be written

$$\underline{V} = \text{grad}\ \phi - t\ \text{grad}\ p_o/\rho . \qquad (3\text{-}1)$$

Now it is evident from the derivation of these equations that p_o/ρ and t may be modified by arbitrary constants. For p_o/ρ we may use the constant $p_{-\infty}/\rho$ where $p_{-\infty}$ is an upstream static pressure which, for a parallel shear flow upstream, is everywhere constant. We then write $(p_o - p_{-\infty})/\rho = \frac{1}{2} U^2$ where U is the upstream velocity which now serves to identify the Bernoulli surfaces throughout the flow. Then equation (3-1) becomes

$$\underline{V} = \text{grad}\ \phi - Ut\ \text{grad}\ U , \qquad (3\text{-}2)$$

In this equation U is constant on a Bernoulli surface, but grad U is not. Grad U may be thought of as a function of the distance between adjacent Bernoulli surfaces and therefore subject to change as the Bernoulli surfaces distort.

We shall use equations (3-1) or (3-2)* and equation (2-80) for compressible flow in the following discussion[21].

3.2 Small Shear, Small Disturbances

We imagine a flow in which grad p_o/ρ or grad U is everywhere small and in which the departure of the flow from some simple character is also small.

* *Equation (3.2) is suitable for use with parallel upstream flows, equation (3.1) is for more general use.*

One example of such a flow is a weakly sheared parallel flow passing over a thin airfoil or approaching a screen of low resistance. Another is a weakly rotational swirling flow in an annulus passing downstream from a cascade of turbine or compressor blades.

If we first write

$$\text{grad}\,\frac{p_o}{\rho} = \underline{P}_0 + \varepsilon\,\underline{P}_1 + \varepsilon^2\,\underline{P}_2 + \ldots, \tag{3-3}$$

$$t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots, \tag{3-4}$$

where ε is a small quantity. Then when the shear is small, $\underline{P}_0 = 0$ and $\underline{P}_1$ is the function grad p_o/ρ for the undisturbed stream. As the disturbances are of order ε, the disturbance of grad p_o/ρ will be of order ε^2. The function t for the undisturbed stream is of order unity, and its perturbation by the small disturbance will be of order ε or less. Hence t_0 represents the function t for the undisturbed stream.

On substituting the expressions for t and grad p_o/ρ in equation (3-1) we obtain

$$\underline{V} = \text{grad}\,\phi - \varepsilon t_0\,\underline{P}_1 - \varepsilon^2(t_1\,\underline{P}_1 + t_0\,\underline{P}_2) \tag{3-5}$$

and its divergence

$$\nabla^2\,\phi = \varepsilon\,\text{div}\,(t_0\,\underline{P}_1) + \varepsilon^2\,\text{div}\,(t_1\,\underline{P}_1 + t_0\,\underline{P}_2). \tag{3-6}$$

This last equation enables terms in ϕ up to order ε to be evaluated, because t_0 and $\underline{P}_1$ are known from the upstream flow. The expression obtained for ϕ will consist of two parts, a rotational component, ϕ_0, representing the solution for the original undisturbed shear flow ($\nabla^2\,\phi_0 = \text{div}\,t_0\,\underline{P}_1$) and an irrotational component whose velocity potential, ϕ_1, satisfies Laplace's equation, and which when added to the rotational component, satisfies the boundary conditions for the complete flow. Hence to the order ε the flow due to the disturbance is an irrotational, potential flow.

To the same order of approximation grad p_o/ρ retains its upstream value throughout the field of flow.

To apply the approximation to the steady compressible flow of a perfect gas in the absence of body forces we examine equation (2.80) in which grad $(\log_e p_o)$ is of order ε. We write

$$\phi_1^* = \phi_0^* + \phi_1^*$$

where ϕ_0^* is the value of ϕ^*in the undisturbed flow and ϕ_1^*, of order ε, is the term for the disturbance. Hence the reduced velocity to order ε becomes

$$\underline{W} = \text{grad}(\phi_0^* + \phi_1^*) + \left[(\phi_0^* - b_0 t_0) \text{ grad } (\log_e p_o)\right] \frac{k-1}{2k}$$

where b_0 represents the value of b for the undisturbed flow. Subtracting the terms in ϕ_0^* representing the undisturbed flow we find for the reduced disturbance velocity

$$\underline{W}_1 = \text{grad } \phi_1^*,$$

i.e. the reduced disturbance flow is irrotational.

We use equation (2.70) to determine the true vorticity

$$\underline{\Omega} = b \text{ curl } \underline{W} + \text{grad } b \times \underline{W}$$

$$= \text{grad } b_0 \times \underline{W}_0 + \varepsilon(b_0 \text{ curl } \underline{W}_0 + \text{grad } b_0 \times \underline{W}_1 + \text{grad } b_1 \times \underline{W}_0). \quad (3\text{-}7)$$

The first two vectors on the right hand side represent the vorticity in the undisturbed flow. If the undisturbed flow is complex lamellar, i.e. $\underline{W}_0 \cdot \text{curl } \underline{W}_0 = 0$, then both these vorticity vectors are normal to the velocity. The third and fourth vectors on the right hand side represent vorticity terms appearing in the disturbed flow (b_1 is the component of b due to the disturbance and since the disturbance is small, is of order ε). Hence the disturbed flow is not an irrotational flow, but a 'tank' flow whose streamlines have the same geometrical configuration as an irrotational flow, i.e. a flow with p_o and T_o constant. Note that, although the reduced vorticity is small, the true vorticity may be large, depending on the order of grad b_0 in the first term on the right hand side of equation (3.7)

The continuity equation (2·84) may be written

$$\text{Div } (\underline{W}_0 + \underline{W}_1) = \frac{1}{k-1} \frac{(\underline{W}_0 + \underline{W}_1) \cdot \text{grad } W^2}{1 - W^2}$$

$$= \frac{1}{k-1} \frac{(\underline{W}_0 + \underline{W}_1) \cdot \text{grad}\,(W_0^{\,2} + 2\underline{W}_0 \cdot \underline{W}_1)}{1 - (W_0^{\,2} + 2\underline{W}_0 \cdot \underline{W}_1)} \tag{3-8}$$

since $W^2 = \underline{W} \cdot \underline{W} = W_0^{\,2} + 2\underline{W}_0 \cdot \underline{W}_1 + W_1^{\,2}$ and the last term is of order ε^2. Subtracting the continuity equation for the undisturbed flow and rearranging terms, retaining only those of order ε, we obtain

$$\text{Div}\,\underline{W}_1 = \frac{2}{k-1} \frac{1}{1 - W_0^{\,2}} \Big\{\underline{W}_0 \cdot \text{grad}\,(\underline{W}_0 \cdot \underline{W}_1) + \underline{W}_1 \cdot \text{grad}\,\frac{W_0^{\,2}}{2}$$

$$+ \frac{\underline{W}_0 \cdot \underline{W}_1}{1 - W_0^{\,2}} \underline{W}_0 \cdot \text{grad}\,W_0^{\,2} \Big\} \tag{3-9}$$

For this approximation div $\underline{W}_1 = \nabla^2 \phi_1^{\,*}$. Some of the terms on the right hand side may become negligible for certain flow regimes. For instance, when the undisturbed flow is a parallel stream, the third term is zero and the second term may be neglected because grad W_0 as well as $\underline{W}_1$ is of order ε.

3.3 Large Shear, Small Disturbance.

When the shear is large $\underline{P}_0$ and t_0 in equations (3-3) and (3-4) are the functions grad p_0/ρ and t for the undisturbed stream. The term $\underline{P}_1$ of order ε represents the departure of grad p_0/ρ from its upstream value. The expression for the velocity is

$$\underline{V} = \text{grad}\,\phi - \underline{P}_0\, t_0 - \varepsilon(\underline{P}_1\, t_0 + \underline{P}_0\, t_1) \tag{3-10}$$

whence

$$\nabla^2 \phi = \text{div}\,\underline{P}_0\, t_0 + \varepsilon\, \text{div}(\underline{P}_1\, t_0 + \underline{P}_0\, t_1) \tag{3-11}$$

Hence ϕ has a component of order unity representing the undisturbed flow, which may be readily evaluated because $\underline{P}_0$ and t_0 are known. The determination of the component of order ε representing the perturbed flow requires the evaluation of $\underline{P}_1$ and t_1 both of which are linear in ϕ. Equation (3-11) therefore yields a linear expression for the component of ϕ of order ε. In this case the disturbed flow is rotational.

The extension of this result to compressible flow by means of equation (2-80) does not alter the above conclusion.

The equation for the reduced velocity becomes

$$\underline{W} = \text{grad}\,\phi_0^{\,*} + (\phi_0^{\,*} - b_0 t_0)\,\underline{P}_0^{\,*} + \varepsilon\,\Big\{\text{grad}\,\phi_1^{\,*} + (\phi_0^{\,*} - b_0 t_0)\,\underline{P}_1^{\,*}$$

$$+ (\phi_1^{\,*} - (b_0 t_1 + b_1 t_0))\,\underline{P}_0^{\,*}\Big\} \tag{3-12}$$

where $\frac{k-1}{2k}$ grad $\log_e p_o = \underline{P}_o^* + \varepsilon \underline{P}_1^*$ and $\underline{P}_1^*$ is the perturbation due to the disturbance.

The solution requires the derivation of expressions for $\underline{P}_1^*$, b_1, and t_1 which are linear in ϕ_1^* to order ε. The continuity equation is given by equation (3-9) where $\underline{W}_1$ is the bracketted term on the right hand side of equation (3-12).

3.4 Small Shear, Large Disturbance

In this case $\underline{P}_o = 0$ and the equations (3-5) and (3-6) are derived. However, t_o is no longer the function t for the undisturbed stream only. Let us identify the Bernoulli surfaces in terms of a function U(x, y, z) where $p_o/\rho = \frac{1}{2} U^2$ + constant. We have already noted that, when the upstream flow is parallel, U is the upstream velocity, if the constant chosen is $p_{-\infty}/\rho$

We may write

$$U = U_o + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots \tag{3-13}$$

where U_o is a constant, and because the shear is small of order ε, the upstream velocity is $U_o + \varepsilon U_1$.

Equation (3-2) may then be written

$$\underline{V} = \text{grad}\ \phi - \varepsilon U_o t_o \text{ grad } U_1 - O(\varepsilon^2) \tag{3-14}$$

Let

$$\phi = U_o(\phi_o + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots) \tag{3-15}$$

then

$$\underline{V} = U_o \text{ grad } \varepsilon_o + \varepsilon(U_o \text{ grad } \phi_1 - U_o t_o \cdot \text{grad } U_1) + O(\varepsilon^2) \tag{3-16}$$

and the continuity equation yields

$$U_o \nabla^2 \phi_o + \varepsilon(U_o \nabla^2 \phi_1 - U_o t_o \nabla^2 U_1 - \text{grad } U_o t_o \text{ grad } U_1) + O(\varepsilon^2) = 0 \tag{3-17}$$

Separating the terms of like order,

$$\nabla^2 \phi_o = 0 \tag{3-18}$$

and

$$\nabla^2 \phi_1 = t_o \nabla^2 U_1 + \text{grad } t_o \cdot \text{grad } U_1 \tag{3-19}$$

Hence the flow may be divided into two parts.

1) A primary flow of order unity which is an irrotational flow with velocity potential $U_0\ \phi_0$, satisfying the boundary conditions on the walls and on any body immersed in the flow. The values of t_0 and grad U_1 may then be obtained, because to the order of approximation assumed the vortex filaments and Bernoulli surfaces are convected by the primary flow.

2) A secondary flow of order ε given by the equation

$$\underline{V} = U_0 \text{ grad } \phi_1 - U_0\, t_0 \text{ grad } U_1$$

where ϕ_1 is determined from equation (3-17). The secondary flow is a rotational flow and must also satisfy the appropriate boundary conditions.

This approximation has been referred to as the secondary flow approximation. It is important to note that in this approximation the Bernoulli surfaces retain the position in space given by the primary flow. Any disturbance of the surfaces as a result of convection by the secondary flow is given by the term U_2 and is of order ε^2.

To obtain the equivalent result in compressible flow we write $\underline{P}_0^* = 0$ in equation (3-12) since $\text{grad}(\log_e p_0)$ is small. Then

$$\underline{W} = \text{grad } \phi_0^* + \varepsilon \{\text{grad } \phi_1^* + (\phi^* - bt)\, \underline{P}_1^*\}$$

and the same conclusions as before may be drawn in terms of reduced velocity, i.e. the primary flow is a tank flow.

3.5 Exact Solutions.

The only exact solutions available are for two dimensional or axi-symmetric flows. The two-dimensional solutions are somewhat restrictive in that a particular vorticity distribution or a constant vorticity is assumed. Examples of such solutions are given by Tsien(13), James(38), Jones(39) and Murray and Mitchell(40).

More interesting exact solutions for axisymmetrical flow deal with the actuator disc representation of flow through compressors and turbines. Examples are given by Bragg & Hawthorne(41) and by Hawthorne & Horlock(42). These examples provide useful checks of the accuracy of the approximate methods.

SECTION 4 - SMALL SHEAR, SMALL DISTURBANCE FLOWS

4.1 Introduction

The result that the disturbed reduced velocity field is irrotational makes many of the solutions for the small shear, small disturbance approximation fairly trivial. One category of flows for which this approximation has been found to be useful is in the flow through actuator discs representing screens or rows of compressor and turbine blades. Another example is in the flow behind weakly curved shocks. In this section we shall first consider the flow over slender bodies such as airfoils, we shall then recapitulate some of the work on actuator discs. The use of this approximation in compressible flow will also be discussed.

4.2 Flow Over Bodies

As the disturbance is small the body must be slender or thin and there can be no stagnation points.

Let us consider first a small body immersed in the weakly sheared parallel flow of an incompressible fluid. In the cartesian coordinate system shown in Fig. 4.1, which also shows the velocities $U(y) + u$, v, w, the requirement that the irrotational disturbance representing the flow about the body, located at the origin, gives the correct boundary condition is that on the surface

$$\frac{\partial \phi_1}{\partial n} + \underline{i} \cdot \underline{n}\, U = 0 \qquad (4\text{-}1)$$

where ϕ_1 is the potential of the disturbance and $\underline{i}$ and $\underline{n}$ are unit vectors in the direction of the x axis and the normal to the surface of the body respectively. We may write

$$U(y) = U(o) + U'(o)y + U''(o)\, y^2/2! + - - - \qquad (4\text{-}2)$$

where the prime, ', denotes differentiation with respect to y. Now since the shear, U', is small and the body dimensions must also be small, the second term in equation (4-2) is of order ϵ^2 and the only term appearing in equation (4 1) is U(0). Hence ϕ_1 is the potential of the disturbance flow which satisfies the boundary conditions in a uniform approaching stream of velocity U(0) and the problem reduces to the familiar problem in potential flow.

The pressure distribution may be obtained from the first Euler equation which to order ε becomes

$$U \frac{\partial u}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

or

$$\frac{p - p_{-\infty}}{\rho} = - U\phi_{1x} \,. \qquad (4\text{-}3)$$

The only term in the expansion of U which is not negligible in this equation is U(0) so that the pressure distribution is the same as if the body were immersed in a uniform stream of velocity U(0).

A thin airfoil with its spanwise direction stretching to infinity along the y-axis may experience a large variation in y, if the airfoil span and shear layer is large enough, even if the shear is small. To satisfy the boundary conditions at large y we write

$$\phi_1(x, y, z) = \frac{U(y)}{U(0)} \phi_0(x, z) \qquad (4\text{-}4)$$

where ϕ_0 is the disturbance potential of the airfoil in a uniform stream of velocity U(0).

We note that

$$\begin{aligned} \nabla^2\phi_1 &= \frac{U(y)}{U(0)} (\phi_{0xx} + \phi_{0zz}) + \frac{U''(y)}{U(0)} \phi_0 \\ &= \frac{U(y)}{U(0)} (\phi_{0xx} + \phi_{0zz}) = 0 \,, \end{aligned} \qquad (4\text{-}5)$$

since the term omitted is of order ε^2.

The pressure, from equation (4.3), is

$$\frac{p - p_{-\infty}}{\rho} = - \frac{U^2(y)}{U(0)} \phi_{0x} \,. \qquad (4\text{-}6)$$

So that the lift coefficient based on the local velocity is constant. The spanwise velocity is given by

$$\begin{aligned} v = \phi_{1y} &= \phi_0 \frac{U'(y)}{U(0)} \\ &= O(\varepsilon^2) \,. \end{aligned} \qquad (4\text{-}7)$$

from which we deduce that the strength of the trailing vortex sheet is negligible, and there are no induced velocities and therefore the local angle of attack and lift coefficient remain constant.

The above results are not affected if U is a function of both y and z, provided $\partial U/\partial z$ is of the same small order as $\partial U/\partial y$.

When the fluid is a perfect gas and the flow is compressible we may express the reduced velocity $\underline{W}$ in terms of the components

$$U^*(y, z) + u^*, v^*, w^*$$

where the superscript * denotes a component of the velocity reduced by division by b. We have shown in Section 3.2 that u^*, v^*, w^* are the components of the gradient of a potential ϕ_1^*. To satisfy the boundary condition at the surface of this body we adapt equation (4-1) to give

$$b \frac{\partial \phi_1^*}{\partial n} + \underline{i} \cdot \underline{n}\, U = 0$$

or

$$\frac{\partial \phi_1^*}{\partial n} + \underline{i} \cdot \underline{n}\, U^* = 0 . \tag{4-8}$$

It follows by similar arguments to those following equations (4 1) and (4 2) that the problem reduces to that of finding the velocity potential of the compressible flow round the same body when the reduced velocity upstream is $U^*(0, 0)$ (as before the body is assumed to be placed at the origin).

An expression for ϕ_1^* may be obtained from the continuity equations (2-84) or (3-8)

$$\nabla^2 \phi_1^* = \frac{1}{k-1} \frac{\underline{W} \cdot \text{grad } W^2}{1 - W^2} . \tag{4-9}$$

Now $$W^2 = U^{*2} + 2U^* u^* \tag{4-10}$$

and grad W^2 is of order ε. Hence omitting all terms of order ε^2 and less,

$$\nabla^2 \phi_1^* = \frac{2}{k-1} \frac{U^{*2}}{1-U^{*2}} \frac{\partial^2 \phi_1^*}{\partial x^2}$$

$$= M^2 \phi^*_{1xx} \tag{4-11}$$

where $M(y, z)$ is the Mach number of the undisturbed parallel stream. The limitations on the use of equation (4-11) are similar to those in irrotational flow, and the approximation fails near $M = 1$.

As b is constant along a streamline we may write the first Euler equa- in the form

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = - b^2\, U^* \frac{\partial u^*}{\partial x}$$

whence
$$\frac{p - p_{-\infty}}{\frac{1}{2}\rho\, U^2} = - \frac{2u^*}{U^*} = - \frac{2\phi^*_{1x}}{U^*}. \qquad (4\text{-}12)$$

Hence a slender body placed at the origin produces a disturbance which is the same as that produced by the body in a uniform flow with upstream Mach number M(0, 0). It also has the same pressure distribution.

The result for the airfoil extending along the y-axis is similar to that obtained for incompressible flow. In the transonic region the solution at this order of approximation becomes trivial as in irrotational flow.

4.3 Actuator Discs

The actuator plane is a mathematical model which has been used for many problems such as the flow through screens, flame fronts, compressor and turbine blading, propellers, fans, radiators, heat exchangers, etc. The actuator disc or plane is a plane of discontinuity across which the flow parameters change by a finite amount. It may be used to represent a screen or a row of blades across which there are finite changes of pressure and flow direction. Shock waves and flame fronts are similar planes or surfaces of discontinuity of density and pressure.

As a first example consider the steady, weakly sheared flow of an incompressible fluid through a wire mesh screen (Collar [43], Taylor and Batchelor[44], Davis[45], Elder[46], Bonneville and Harper[47]). Let the flow be parallel far upstream of the screen (see Fig. 4.2 for notation). Let the velocity upstream be $U_1(y, z) + u_1$, v_1, w_1 and downstream $U_2(y,z) + u_2$, v_2, w_2 where the velocities u, v, w vanish far from the screen. Now if $\partial U_1/\partial y$, $\partial U_1/\partial z$ are small, it is reasonable to suppose that the disturbance velocities are also small, and hence, as shown in Section 3.2, that the disturbance velocities are the gradients of a potential. The upstream and downstream disturbance potentials ϕ_1 and ϕ_2 must satisfy Laplace's equation and the boundary conditions far from the screen, on the walls of the duct and at the screen

The boundary condition far from the disc can be satisfied by letting ϕ decay exponentially with distance from the screen. Thus for a rectangular duct an expression of the form

$$\phi = \sum_{m=1,2,..}^{\infty} \sum_{n=1,2,..}^{\infty} A_{m,n} \exp(\pm k_{m,n} x) \cos \alpha_m y \cos \beta_n z \qquad (4\text{-}13)$$

where $k^2_{m,n} = \alpha^2_m + \beta^2_n$, satisfies Laplace's equation and the boundary conditions far from the screen (with the appropriate sign in the exponential). For the boundary condition at the walls we note that

$$\phi_y = 0 \quad \text{at } y = 0 \quad \text{and} \quad y = m\pi/\alpha_m,$$

$$\phi_z = 0 \quad \text{at } z = 0 \quad \text{and} \quad z = n\pi/\beta_n,$$

where m and n are integers. So that if Y and Z are the dimensions of the rectangular duct $\alpha_m = m\pi/Y$, $\beta_m = n\pi/Z$.

The conditions at the remaining boundary, the screen, and the integrated continuity condition are required to determine the constants $A_{1m,n}$, $A_{2m,n}$ and the required far downstream velocity profile $U_2(y, z)$. At the screen the continuity condition is, for $x = 0$

$$U_1 + \phi_{1x} = U_2 + \phi_{2x} \qquad (4\text{-}14)$$

or

$$U_2 - U_1 = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} k_{m,n}(A_{1m,n} + A_{2m,n}) \cos \alpha_m y \cos \beta_n z \, , \qquad (4\text{-}15)$$

The values of $A_{1m,n}$ and $A_{2m,n}$ are then obtained from the equation for the pressure loss through the screen and the relationships which determine the change in angle of flow caused by the screen. An arbitrary constant in the result is determined from the integrated continuity condition

$$\int_0^Y \int_0^Z (U_2 - U_1) \, dy \, dz = 0 \, . \qquad (4\text{-}16)$$

When the fluid is a compressible, perfect gas reduced velocities and potentials may be used. For the above example we solve equation (4-11) to obtain for the reduced velocity potential

$$\phi^* = \sum \sum A_{1m,n} \exp(\pm k_{m,n} \sqrt{1 - M^2}\, x) \cos \alpha_m y \cos \beta_n z , \qquad (4.17)$$

i.e. an expression similar to equation (4.13) except for the term $\sqrt{1 - M^2}$ where M is the Mach number far away from the actuator plane, which is a function of y and z. However since derivatives of $\sqrt{1 - M^2}$ are of order ε terms containing these derivatives may be neglected in the expressions for ϕ^*_y, ϕ^*_z and the values of α_m, β_n are the same as before.

The expression for continuity across the screen is

$$b_1(\rho_1 + \Delta\rho_1)\ (U^*_1 + \phi^*_{1x}) = b_2(\rho_2 + \Delta\rho_2)\ (U^*_2 + \phi^*_{2x}) \qquad (4\text{-}18)$$

where ρ_1 is the density far upstream and $\Delta\rho$ is the change in density between the screen and $x = \pm \infty$. Now to the order of approximation used here

$$\frac{\Delta\rho}{\rho} = \frac{1}{k-1}\frac{\Delta T}{T} = -\frac{2}{k-1}\frac{Uu}{2C_p T} . \qquad (4\text{-}19)$$

Substituting in equation and neglecting terms of order ε^2

$$b_1\ \rho_1(U^*_1 + (1-M_1^2)\ \phi^*_{1x}) = b_2\ \rho_2(U^*_2 + (1-M_2^2)\ \phi^*_{2x}) \qquad (4\text{-}20)$$

where the term ϕ^* is evaluated at the screen.

Rows of compressor and turbine blades may be represented by actuator discs. The actuator disc may be considered as the limiting case of a blade row consisting of an infinite number of blades of infinitesimal chord or axial width. A representation of this model together with the nomenclature and coordinate system to be used is shown in Fig. 4.3.

The actuator disc technique borrowed from propeller theory was first used to represent the flow through fans, compressors and turbines by Ruden[48] and Merchant[49]. The work of several authors is summarised in Chapter XI of Thwaites[3] and by Marble in Section C of Reference (50). Theory and experiment are compared in Reference (51) and the effect of compressibility discussed by Horlock[52] and Hawthorne & Ringrose[53]. This discussion is confined to examples where the flow is weakly sheared, i.e. the gradient of stagnation pressure is small; the solution for large gradients and exact solutions are discussed elsewhere.

In the coordinate system r, θ, z, Fig. 4.3, let the reduced velocities be u_1^*, $v_0^* + v_1^*$, $w_0^* + w_1^*$, where subscript 0 is used to denote the undisturbed flow and the subscript 1 denotes the disturbance velocity which has a potential ϕ_1^*.

The continuity equation (3-8) becomes

$$\frac{1}{r}\frac{\partial r u_1^*}{\partial r} + \frac{\partial v_1^*}{r\partial\theta} + \frac{\partial w_1^*}{\partial z}$$

$$= \frac{2}{k-1}\left[\frac{1}{1-(v_0^{*2}+w_0^{*2})}\right]\left\{\left(\frac{v_0^*\partial}{r\partial\theta} + w_0^*\frac{\partial}{\partial z}\right)(v_0^* v_1^* + w_0^* w_1^*)\right.$$

$$+ \left(u_1^*\frac{\partial}{\partial r} + v_1^*\frac{\partial}{r\partial\theta} + w_1^*\frac{\partial}{\partial z}\right)\frac{1}{2}(v_0^{*2} + w_0^{*2})$$

$$\left. + \frac{v_0^* v_1^* + w_0^* w_1^*}{1-(v_0^{*2}+w_0^{*2})}\left(v_0^*\frac{\partial}{r\partial\theta} + w_0^*\frac{\partial}{\partial z}\right)(v_0^{*2} + w_0^{*2})\right\} \qquad (4\text{-}21)$$

The reduced vorticity components in the r, θ, z direction are respectively

$$\frac{\partial w^*}{r\partial\theta} - \frac{\partial v^*}{\partial z};$$

$$\frac{\partial u^*}{\partial z} - \frac{\partial w^*}{\partial r} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial r v^*}{\partial r} - \frac{\partial u^*}{r\partial\theta} \qquad (4\text{-}22)$$

Let us assume that the reduced flow is axisymmetric then all derivatives with respect to θ and derivatives of v_0^* and w_0^* with respect to z vanish.

Simplifying equation (4-21), we obtain

$$\frac{1}{r}\frac{\partial r u_1^*}{\partial r} + \frac{\partial w_1^*}{\partial z} = \frac{2}{k-1}\left[\frac{1}{1-(v_0^{*2}+w_0^{*2})}\right]\left\{w_0^* v_0^*\frac{\partial v_1^*}{\partial z} + w_0^{*2}\frac{\partial w_1^*}{\partial z}\right.$$

$$\left. + u_1^* v_0^*\frac{\partial (r v_0^*/r)}{\partial r} + u_1^* w_0^*\frac{\partial w_0^*}{\partial r}\right\} \qquad (4\text{-}23)$$

Now the reduced vorticity in the undisturbed flow is of order ϵ, hence $\partial w_0^*/\partial r$ and $\partial(r v_0^*)/\partial r$ are both of this same order. Furthermore as the reduced disturbance flow is irrotational $\partial v_1^*/\partial z = 0$ and $\partial(r v_1^*)/\partial r = 0$ from which we conclude that $v_1^* = 0$, i.e. ϕ_1^* is not a function of θ. Retaining only the terms of order ϵ equation (4-23) becomes

$$\frac{1}{r}\frac{\partial r u_1^*}{\partial r} + \frac{\partial w_1^*}{\partial z} = \frac{2}{k-1}\left[\frac{1}{1-(v_0^{*2}+w_0^{*2})}\right]\left\{w_0^{*2}\frac{\partial w_1^*}{\partial z} - \frac{v_0^{*2}}{r}\,u_1^*\right\}$$

or

$$\frac{\partial^2 \phi_1^*}{\partial r^2} + \frac{1+M_\theta^2}{r}\,\frac{\partial \phi_1^*}{\partial r} + (1-M_z^2)\,\frac{\partial^2 \phi_1^*}{\partial z^2} = 0 \qquad (4\text{-}24)$$

where $M_\theta(r)$ and $M_z(r)$ are the Mach numbers for v_0 and w_0 in the undisturbed flow.

The expression for continuity across the actuator disc is similar to that given in equation (4-20).

$$b_1\rho_1(w_{0_1}^* + (1-M_{z_1}^2)\phi_{11_z}^*) = b_2\rho_2(w_{0_2}^* + (1-M_{z_2}^2)\phi_{12_z}^*) \qquad (4\text{-}25)$$

and if no radial force is exerted by the blades

$$b_1\phi_{11_r}^* = b_2\,\phi_{12_r}^* \qquad (4\text{-}26)$$

where in both expressions the terms in ϕ_1^* are evaluated at the disc.

Techniques for solving equation (4-24) may be found in References (50) and (53). It should be noted that equation (4-24) is applicable whether the undisturbed flow is irrotational or weakly sheared and the solutions given in Reference (53) are in fact for the case in which the undisturbed flow is irrotational but compressible (free vortex flow in a turbine stage) and the disturbance arises because of the density variation in the flow.

The quoted references also show how the technique may be applied to adjacent actuator discs to simulate rows of compressor blades and spaced screens.

The two illustrations of parallel flow and swirling flow differ principally in that the incompressible flow result viz. $\nabla^2\phi_1 = 0$ for all undisturbed flow conditions is modified by compressibility to the differing equations (4-11) and (4-24).

Other examples to which the actuator disc technique may be applied include the representation of flame fronts[54] and shocks.

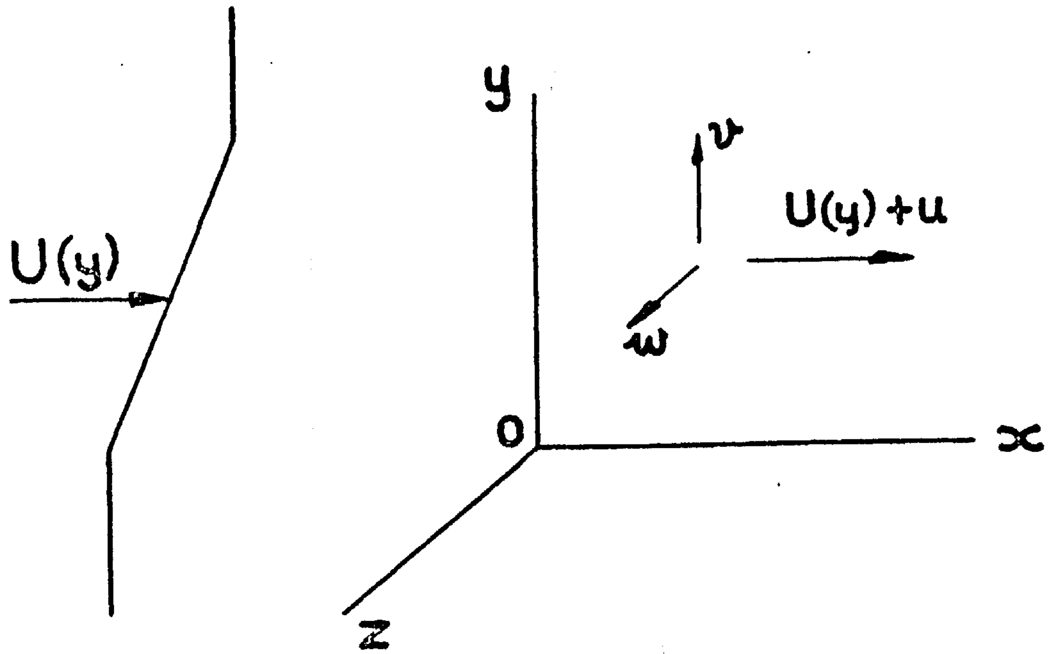


FIG. 4.1 COORDINATE SYSTEM

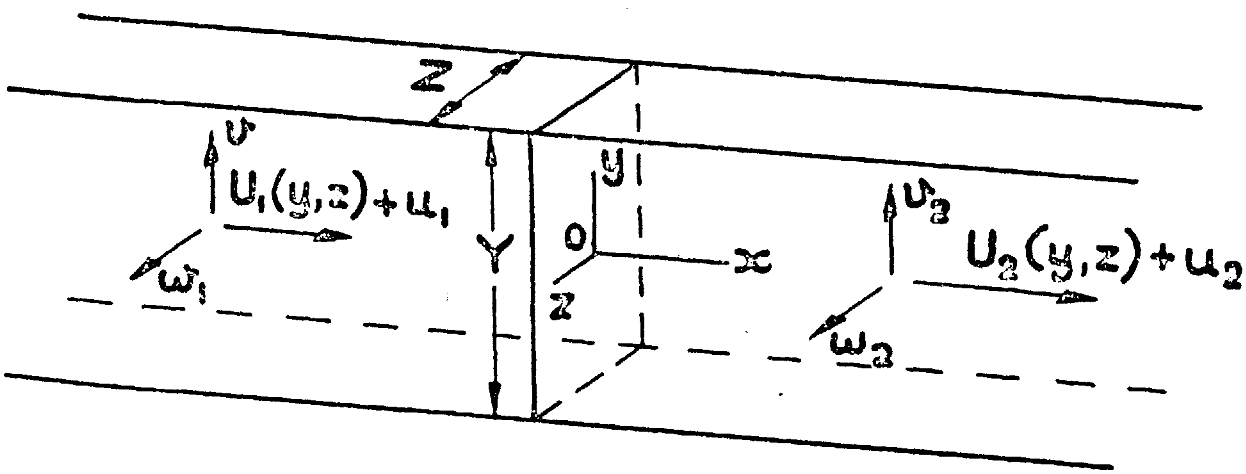


FIG. 4.2 FLOW THROUGH AN ACTUATOR PLANE

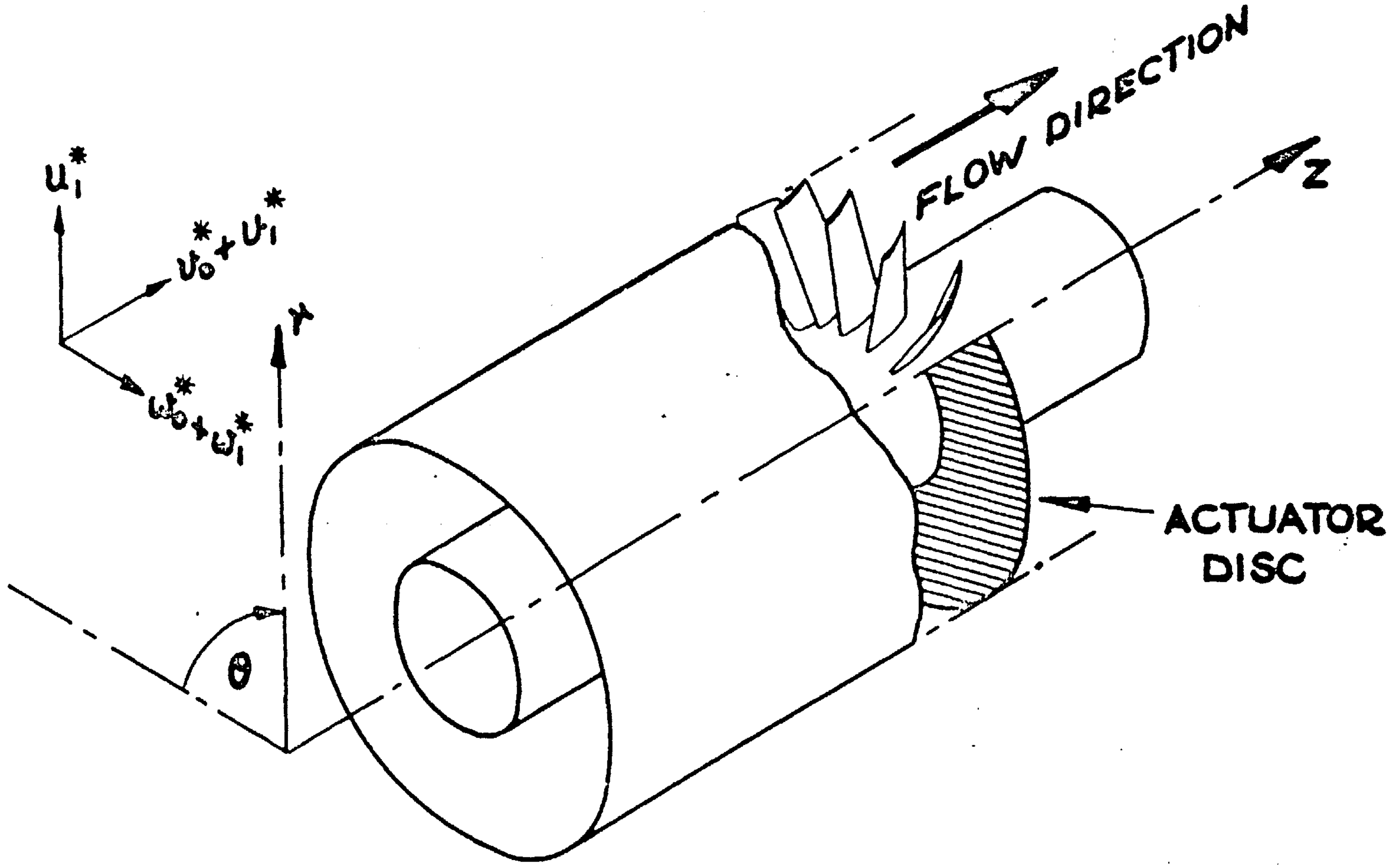


FIG. 4.3 COORDINATE SYSTEM FOR THE FLOW THROUGH AN ACTUATOR DISC

SECTION 5. SMALL DISTURBANCES TO A PARALLEL SHEAR FLOW

5.1 General Theory

Consider an inviscid, incompressible fluid with velocity components $U_{(y)} + u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ in the directions x, y, z, respectively, of a Cartesian co-ordinate system, Figure 4.1. The flow will be assumed to be steady and there will be no body forces. Upstream at infinity, $x = -\infty$, u, v, w will be assumed to vanish, and in the remainder of the flow field they will be small quantities such that their squares and products may be neglected. The other boundary conditions will be considered later.

The equations of motion, neglecting second order terms, become

$$Uu_x + vU' + p_x/\rho = 0 \tag{5-1}$$

$$Uv_x + p_y/\rho = 0 \tag{5-2}$$

$$Uw_x + p_z/\rho = 0 \tag{5-3}$$

where $U' = dU/dy$ and subscripts x, y, z denote partial differentiation with respect to x, y, z respectively.

The equation of continuity is

$$u_x + v_y + w_z = 0 \,. \tag{5-4}$$

Eliminating the pressure, p, from equation (5-2) and (5-3) we obtain

$$\frac{\partial}{\partial x} \{ (Uw)_y - (Uv)_z \} = 0 \,. \tag{5.5}$$

Now the term within the brackets is zero at $x = -\infty$, hence it is zero everywhere within the flow field. If we denote the disturbance velocity u, v, w by the vector $\underline{v}$, we conclude from equation (2-36) that the vector $\underline{v}$ may be represented by

$$U \underline{v} = \text{grad } U\phi + \underline{i} UA \tag{5-6}$$

where $\underline{i}$ is the unit vector in the x-direction.

Eliminating p in turn from equations (5-1) and (5-2) and equations (5-1) and (5-3) we obtain

$$\frac{\partial}{\partial x}\left\{ (Uv)_x - (Uu)_y \right\} = \frac{\partial}{\partial y}(U'v) \tag{5-7}$$

$$\frac{\partial}{\partial x}\left\{ (Uu)_z - (Uw)_x \right\} = -\frac{\partial}{\partial z}(U'v) . \tag{5-8}$$

The velocity $U(y)$ must not approach zero anywhere otherwise the condition that $\underline{v}/U$ is small will be violated.

Now $$\text{curl } U\underline{v} = \underline{j}\,(UA)_z - \underline{k}\,(UA)_y \tag{5-9}$$

where $\underline{j}$ and $\underline{k}$ are unit vectors in the y and z directions respectively.

Substituting for curl $U\underline{v}$ in equations (5-7) and (5-8) we obtain

$$\frac{\partial^2 UA}{\partial x \partial y} = -\frac{\partial}{\partial y}(U'v)$$

$$\frac{\partial^2 UA}{\partial x \partial z} = -\frac{\partial}{\partial z}(U'v) .$$

The integration for the boundary condition that all disturbances vanish far upstream, leads to the result

$$A = -U' \int_{-\infty}^{x} \frac{v}{U}\,dx . \tag{5-11}$$

Now $-v$ is the downwash velocity due to the disturbance and the downwash displacement, that is the displacement downwards in the $-y$ direction, of a fluid particle as it travels from $x = -\infty$ to x, to the order of approximation assumed here, is given by

$$-\int_{-\infty}^{x} \frac{v}{U}\,dx .$$

Now at $x = -\infty$ where the flow is parallel the pressure $p = p_{-\infty}$ is uniform but the stagnation pressure $p_{o-\infty}$ is a function of y given by

$$p_{o-\infty} = p_{-\infty} + \frac{1}{2}\rho U^2 . \tag{5-12}$$

Elsewhere in the flow field the linearizing approximation gives

$$p_o = p + \frac{1}{2}\rho U^2 + \rho U u . \tag{5-13}$$

Subtracting equation (5-12) from equation (5-13)

$$\frac{p_o - p_{o-\infty}}{\rho} = \frac{p - p_{-\infty}}{\rho} + Uu . \tag{5-14}$$

We find $(p - p_{-\infty})$ by integrating equation (5-1) from $x = -\infty$ to x,

$$\frac{p - p_{-\infty}}{\rho} = -Uu - U' \int_{-\infty}^{x} v\,dx = -Uu + UA . \tag{5-15}$$

On adding equations (5-14) and (5-15) we obtain

$$\frac{p_o - p_{o-\infty}}{\rho} = UA . \tag{5-16}$$

We conclude that the function A gives a measure of the downwash or the downwards motion of the Bernoulli surfaces.

The velocity component in the x-direction is obtained from equation (5-6)

$$Uu = \frac{\partial U\phi}{\partial x} + UA . \tag{5-17}$$

Subtracting this equation (5-17) from equation (5-15) we obtain a useful relation for the pressure, viz.

$$\frac{p - p_{-\infty}}{\rho} = - U\phi_x , \tag{5-18}$$

The velocity components obtained from equation (5-6) are:

$$u = \phi_x + A , \tag{5-19}$$

$$v = \phi_y + \frac{U'}{U} \phi = \frac{1}{U} \frac{\partial U\phi}{\partial y} , \tag{5-20}$$

$$w = \phi_z . \tag{5-21}$$

Substituting for v from equation (5.20) in equation (5 11) we obtain

$$A = - \frac{U'}{U^2} \int_{-\infty}^{x} \frac{\partial U\phi}{\partial y} dx . \tag{5 22}$$

Substituting these results for u, v, w and A in the continuity equation (4-4) we obtain

$$\begin{aligned} \text{div } \underline{v} &= \nabla^2 \phi + \left| \frac{U''}{U} - 2 \left(\frac{U'}{U}\right)^2 \right| \phi \\ &= \nabla^2 \phi - \phi U \left(\frac{1}{U}\right)'' \\ &= 0 , \end{aligned} \tag{5-23}$$

where $\left(\frac{1}{U}\right)'' = \frac{d^2}{dy^2} \left(\frac{1}{U}\right)$.

Another form which will be used extensively is

$$U \text{ div } \underline{v} = \nabla^2 U\phi - 2 \frac{U'}{U} \frac{\partial U\phi}{\partial y} = 0 , \tag{5-24}$$

If we differentiate this equation with respect to y and divide by U we obtain

$$\frac{1}{U} \frac{\partial}{\partial y} (U \text{ div } \underline{v}) = \frac{1}{U} \nabla^2 Uv - \frac{2}{U} \frac{\partial U'v}{\partial y} = \nabla^2 v - \frac{U''}{U} v = 0 , \tag{5-25}$$

where equation (5-20) has been used to eliminate ϕ, to give the form used by

Lighthill[9].

Another form is obtained by differentiating equation (5-23) with respect to z,

$$\frac{\partial}{\partial z} \operatorname{div} \underline{v} = \nabla^2 w - w\, U(\tfrac{1}{U})'' = 0. \tag{5-26}$$

The continuity equation can also be written

$$\operatorname{div} \underline{v} = U \operatorname{div} (U^{-2} \operatorname{grad} U\phi) = 0. \tag{5-27}$$

If we differentiate equation (5 24) with respect to x we obtain

$$\frac{\partial}{\partial x} U \operatorname{div} \underline{v} = \nabla^2\, U\, \phi_x - 2\, \frac{U'}{U}\; \frac{\partial U\phi_x}{\partial y}$$

$$= \nabla^2\, p - 2\frac{U'}{U}\, p_y = 0\,, \tag{5-28}$$

which is the form used by Kármán and Tsien[11] and Honda[10] in their analysis of shear flow over airfoils.

Consider the expression

$$\operatorname{div} (\underline{v}/U) = (1/U)\operatorname{div} \underline{v} - (U'/U^2)\cdot v$$

which, using equation (5.11), may be written

$$\operatorname{div} (\underline{v}/U) - \frac{\partial(\Lambda/U)}{\partial x} = (1/U)\, \operatorname{div} \underline{v}$$

$$= 0,$$

or

$$\frac{\partial}{\partial y}\, (\frac{v}{U}) + \frac{\partial}{\partial z}\, (\frac{w}{U}) = \frac{\partial}{\partial x}\, (\frac{A}{U} - \frac{u}{U})$$

$$= \frac{\partial}{\partial x}\; \frac{p - p_{-\infty}}{\rho U^2}\,, \tag{5-20}$$

in which the last step is the result of substitution from equation (5.15). When the pressure is, or is assumed to be uniform, as in the Trefftz plane assumption, the right hand side vanishes. Kármán and Tsien[11] have shown that, when the right hand side is zero, the equation expresses the condition that the cross-sectional areas of the stream tubes remain constant.

The components of vorticity of the total flow may be obtained from equation (5-9)

$$\xi = -\frac{U'}{U}\, \phi_z = \frac{U'}{U}\, w$$

$$\eta = A_z = \frac{p_{oz}}{\rho U} = -\frac{U'}{U^2} \int_{-\infty}^{x} (U\phi_z)_y\, dx$$

$$\begin{aligned}\zeta &= \frac{U'}{U}\phi_x - A_y - U' \\ &= \frac{U'}{U}u - \frac{(UA)_y}{U} - U' \\ &= \frac{U'}{U}u - \frac{1}{\rho U}\frac{\partial p_o}{\partial y} - U'.\end{aligned} \qquad (5\text{-}30)$$

5.2 Exact Solutions

Inspection of equation (5 24) shows that

$$U\phi = f(x, z) \qquad (5\text{-}31)$$

will be a solution if f satisfies the equation

$$U \operatorname{div} \underline{v} = f_{xx} + f_{zz} = 0, \qquad (5\text{-}32)$$

i.e. if f is the potential of a plane two-dimensional flow. Substitution of this expression for $U\phi$ in equations (5 19) to (5 22) shows that in this flow v = 0 and A = 0 everywhere, and there is no downwash or displacement of the Bernoulli surfaces. The flow streamlines are then contained in the planes y = constant, the streamline configuration is the same as that in the two-dimensional flow with velocity potential f(x, z) but the velocity in each plane is inversely proportional to U(y), i.e.

$$\begin{aligned}Uu &= f_x \\ Uw &= f_z\end{aligned} \qquad (5\text{-}33)$$

A thin airfoil of constant chord extending in the spanwise direction along the y=axis would therefore give this type of flow if its thickness, camber and angle of attack were inversely proportional to U^2. We postpone discussion of the details of the flow about such bodies to a subsequent section.

There are two other exact theoretical solutions. One occurs when the shear distribution has the unrealistic value satisfying $(1/U)'' = 0$, or $1/U = ay + b$. Then equation (5-23) becomes

$$\operatorname{div} \underline{v} = \nabla^2\phi \qquad (5\text{-}34)$$

and ϕ is the potential of an irrotational flow with the same distribution of sources and sinks as the shear flow. The velocity distributions are not the same since they are obtained for the shear flow by substituting for ϕ in equations (5-19) to (5-21).

If we take the example of a source at the origin of strength m

$$\phi = \frac{m}{4\pi r} \tag{5-35}$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \text{we find that}$$

$$w = \frac{m}{4\pi}\frac{z}{r^3}$$

$$v = \frac{m}{4\pi}\frac{y}{r^3} - \frac{U'(y)}{U_{(y)}}\frac{m}{4\pi}\int_{-\infty}^{x}\left(\frac{y}{r^3} - \frac{U'(y)}{U_{(y)}}\frac{1}{r}\right)dx. \tag{5-36}$$

As Lighthill[9] found the last term is infinite, possibly because of the unrealistic shear distribution.

If U = ay + b, U'' = 0 and we may write the equation (5-25) as

$$\nabla^2 v = \frac{1}{U}\frac{\partial}{\partial y}(U \operatorname{div} \underline{v}) = \frac{\partial}{\partial y}(\operatorname{div} \underline{v}) + \frac{U'}{U}\operatorname{div} \underline{v} \tag{5-37}$$

Hence for a source of strength m at the origin

$$v = \frac{m}{4\pi}\frac{y}{r^3} - \frac{U'(0)}{U(0)}\frac{m}{4\pi}\frac{1}{r},$$

$$w = \frac{mz}{4\pi r^3} - \frac{U'}{U(0)\,U(y)}\frac{mz}{4\pi r}$$

$$u = \frac{m}{4\pi}\left(\frac{x}{r^3} - \frac{U'^2}{U(0)\,U(y)}\frac{x}{r}\right) - \frac{U'}{U(y)}\frac{m}{4\pi}\int_{-\infty}^{x}\left(\frac{y}{r^3} - \frac{U'}{U(0)}\frac{1}{r}\right)dx \tag{5-38}$$

These are the results obtained by Lighthill[9] who worked from equation (5-25) and who noted the difficulty with the last integral.

We note that as $r \to 0$ both examples tend to the limiting values,

$$w = \frac{mz}{4\pi r^3}$$

$$v = \frac{m}{4\pi}\left(\frac{y}{r^3} - \frac{U'(0)}{U(0)}\frac{1}{r}\right)$$

$$u = \frac{m}{4\pi}\left\{\frac{x}{r^3} - \frac{U'(0)}{U(0)}\frac{y}{y^2 + z^2}\left(1 + \frac{x}{r}\right)\right\}. \tag{5-39}$$

We shall show later that, whatever the shear flow distribution, the above equations give that part of the velocity distribution which tends to infinity as the source is approached.

5.3 Asymptotic Solutions Close to a Singularity

The behaviour of the solution close to a singularity (the "in-field" solution) has been discussed by Lighthill[9] for the example of a point source located at the origin. This behaviour may be explored by assuming that the scale of distance from the singularity is small compared with the scale of the variation of velocity in the upstream shear flow*.

Consider first a continuous distribution of sources along the y-axis whose strength $m(y)$ is a function of y. Let the scale for variation in $U(y)$ and $m(y)$ be of the same order, say L. Let us consider the flow at distances from the y-axis of order λ, where $\lambda/L = \varepsilon << 1$. We normalize the scales in the equation of motion by dividing x and z coordinates by λ and the y coordinate by L. The equation of continuity (equation (5-23)) becomes

$$\phi_{xx} + \phi_{zz} + \varepsilon^2(\phi_{yy} - \phi U(\tfrac{1}{U})'') = 0. \qquad (5\text{-}40)$$

Writing

$$\phi = \phi_o + \varepsilon\phi_1 + \varepsilon^2 \phi_2 + - - - \qquad (5\text{-}41)$$

and equating terms of the same order we obtain

$$\phi_{oxx} + \phi_{ozz} = 0 \qquad (5\text{-}42)$$

$$\phi_{2xx} + \phi_{2zz} = \phi_o\, U(\tfrac{1}{U})'' - \phi_{oyy} \qquad (5\text{-}43)$$

The solution to the first of these equations which satisfies the source flow condition is

$$\phi_o = \frac{m(y)}{2\pi} \log_e s \qquad (5\text{-}44)$$

Where

$$s^2 = x^2 + z^2.$$

From the second equation we obtain

$$\phi_2 = (m(y)\, U(\tfrac{1}{U})'' - m''(y))\, \frac{s^2}{8\pi}\, (\log_e s - 1). \qquad (5\text{-}45)$$

The equation (5-44) gives the term which approaches infinity as $s \to 0$, and is the leading term in the expansion of ϕ, equation (5-41), for small s. It is the potential of a source of strength $m(y)$ in plane irrotational flow. It yields the same value of the velocity w and pressure p (equation

- - - - - - - - - - -

* *This method was suggested to the author by Dr. T. B. Brooke Benjamin.*

(5-18)) as for the plane potential flow with velocity U(y) about a source of strength m(y).

The velocity v close to the y-axis may also be determined from the leading term. It approaches

$$v = \frac{1}{U}\frac{\partial U\phi}{\partial y} = \left(m'(y) + m(y)\,\frac{U'(y)}{U(y)}\right)\frac{1}{2\pi}\log_e s\,. \tag{5-46}$$

However the values of u, A and p_o are not obtainable without a knowledge of ϕ throughout the flow field.

This result may be extended to doublets and to source and sink distributions representing slender struts extending along the y-axis. In thin airfoil theory the thickness of the strut is determined by the value of w/U at z = 0. As long as the chord of the strut is small compared to the scale of the shear layer the "in-field" solution (equation (5-42) will be controlling and the singularity distribution for a given strut shape will be the same as in a plane potential flow with upstream velocity U(y). We obtain a strut of uniform thickness at all values of y by letting the magnitude of the singularity distribution vary as U(y). In these circumstances the surface pressure coefficient based on the upstream velocity U(y) will be independent of y.

When there is an abrupt discontinuity in the strength of the source distribution along the y-axis, as for instance when a point source is placed at the origin, it is no longer possible to consider that the variation of the shear potential function ϕ in the y-direction has the same scale as the upstream shear variation. It must in fact be of the same order λ as the distance from the singularity. Hence foran isolated singularity equation (5-40) becomes

$$\phi_{xx} + \phi_{yy} + \phi_{zz} - \epsilon^2\,\phi\,U\left(\frac{1}{U}\right)'' = 0 \tag{5-47}$$

Hence

$$\nabla^2\,\phi_o = 0 \tag{5-48}$$

$$\nabla^2\,\phi_2 = \epsilon^2\,\phi_o\,U\left(\frac{1}{U}\right)'' \tag{5-49}$$

etc.

It is now easy to demonstrate Lighthill's result[9] that for a source of strength m at the origin the leading term as $r \to 0$ is

$$\phi_o = \frac{m}{4\pi r}. \tag{5-50}$$

Furthermore the consideration of the orders of magnitude of the two scales leads to the result

$$\begin{aligned} v &= \phi_y + \varepsilon\, \phi \frac{U'}{U} \\ &= \phi_y + \varepsilon\, \phi \left\{\frac{U'(o)}{U(o)} + \varepsilon\, y \left| \frac{d}{dy}\left(\frac{U'}{U}\right) \right|_{y=o} + O(\varepsilon^2) + - - -\right\}, \end{aligned} \tag{5-51}$$

from which the results of equation (5-38) follow, although the value of u cannot be determined from the in-field value of ϕ only.

The pressure distribution around a small slender body may, however, be determined from the in-field value of ϕ by virtue of equation (5-18). The value of ϕ has to be chosen so that the boundary conditions on the body are satisfied. For a slender body the slopes are w/U and v/U respectively and the velocity u does not appear in the formulation of the boundary conditions. On the other hand the function ϕ is not the same as in an irrotational flow because of the additional term appearing in the expression for v, equation (5-51). For a body centered at the origin it may be shown from equation (5-20) that

$$\phi_o = \frac{U(o)}{U}\, \phi^* \tag{5-52}$$

satisfies the same boundary conditions on a slender body as the potential function ϕ^* does in a body immersed in an irrotational flow of velocity U(o). From equation (5-18) we find that the pressure is given by

$$\frac{p - p_{-\infty}}{\rho} = - U\phi_{ox} = - U(o)\ \phi^*_x\ ,$$

i.e. the same as on the body immersed in the irrotational flow. The continuity equation $\nabla^2\phi_o = 0$ is satisfied because $\nabla^2\phi^* = 0$ is satisfied and the derivatives of U with respect to y are of negligible order when the body is small compared to the thickness of the shear layer.

Of greater interest is the asymptotic solution for a thin airfoil with chord extending along the x-axis and span along the y-axis. The lift may be expressed as $L = -\oint p\, dx$. Substituting for p from equation (5-18) we obtain

$$L = \rho U \left| \phi(c, y, 0^+) - \phi(c, y, 0^-) \right| \tag{5-53}$$

where the airfoil stretches from x = 0 to x = c and the term in the bracket is the change in ϕ which occurs across the streamline leaving the trailing edge. The change in spanwise velocity, v, across this streamline is given by

$$\begin{aligned} & v(c, y, 0^+) - v(c, y, 0^-) \\ &= \frac{1}{U}\frac{\partial}{\partial y} U \left| \phi(c, y, 0^+) - \phi(c, y, 0^-) \right| \\ &= \frac{1}{\rho U}\frac{\partial L}{\partial y}. \end{aligned} \tag{5-54}$$

This then is the strength of the trailing vortex sheet.

When the airfoil is small the in-field value of ϕ tends to ϕ_o the solution of equation (5-42), i.e. the potential of a plane irrotational flow. But the solution for ϕ_o is indeterminate without some knowledge of the rest of the flow from which the upwash or downwash at the airfoil may be obtained. There is some similarity between this problem and the familiar problem of irrotational flow over wings of finite span. The differences arise because velocities are induced not only by the trailing vortex sheet, but by the secondary vorticity distributed throughout the shear flow. The limiting case of a small chord airfoil is a lifting line, and the lifting line theory will be presented later. As the ratio of chord to shear layer thickness increases the second order terms, equation (5-43), begin to change the value of ϕ, and the pressure distribution round the airfoil is no longer the same as that round the airfoil in plane potential flow with the same lift coefficient (based on the local value of U(y)).

In this flow the circulation about the surface of the airfoil is not necessarily related to the lift in some simple fashion.

For

$$\begin{aligned} \Gamma &= \oint u\, dx \\ &= \left| \phi(c, y, 0^+) - \phi(c, y, 0^-) \right| + \int_0^c \left(A(x,y,0^+) - A(x,y,0^-)\right) dx \end{aligned}$$

in which equation (5-19) for u has been used.

Hence $\rho U\Gamma = L + \rho U \oint A \, dx.$ (5-55)

Now when the chord is small compared with the shear layer thickness, because the term in A contains gradients with respect to y, A becomes of order ε^2, and hence in the in-field $L \to \rho U\Gamma$.

Except for the airfoil with lift the in-field solutions bear a close resemblance to those obtained in Section 4.

5.4 Lifting Line Theory

The theory of the shear flow about an airfoil represented by a concentrated lifting line has been given by Kármán and Tsien[11]. Apart from the assumption that the airfoil can be treated as a singularity, the basic assumption of the theory is that the pressure everywhere far downstream ($x = \infty$) is uniform and equal to the ambient pressure. This assumption is basic to the Trefftz plane concept in the theory of wings of finite span. Kármán and Tsien show that the velocities v and w at $x = 0$ are half those at $x = \infty$, they develop expressions for lift and induced drag and show that the minimum induced drag occurs when the induced downwash angle is constant along the span. Although some measurements of the lift on airfoils in a shear flow have been reported by Mendelsohn and Polhamus[55] and by Mair[56], the first systematic attempt to perform experiments and to obtain numerical results from the lifting line theory has been made only recently by Kotansky[57].

The theory assumes that the velocities and pressure are continuous everywhere except at $x = 0$, $z = \varepsilon$ where, as $\varepsilon \to 0$

$$\int_{-\infty}^{+\infty} p \, dx = -\frac{1}{2} L(y) \qquad \text{for } z = +0,$$

$$\int_{-\infty}^{+\infty} p \, dx = \frac{1}{2} L(y) \qquad \text{for } z = -0. \qquad (5\text{-}56)$$

Substituting for the pressure from equation (5-18) and integrating we find that in the limit as $\varepsilon \to 0$

$$\rho\phi = (L/2U) \, H(x) \qquad \text{for } z = +0$$

$$\rho\phi = -(L/2U) \, H(x) \qquad \text{for } z = -0. \qquad (5\text{-}57)$$

where H(x) is the generalised function defined by

$$H(x) = 0 \qquad \text{for } x < 0$$

$$H(x) = 1 \qquad \text{for } x > 0.$$

We note that ϕ and $p - p_{-\infty}$ are both zero at $z = 0$.

At $x = -\infty$, $(p - p_{-\infty})$, ϕ and grad ϕ are zero. At $x = +\infty$ we assume that $(p - p_{-\infty})$ and therefore ϕ_x are zero. The other boundary conditions depend on whether the airfoil and flow field are infinite as in Kármán and Tsien[11] or bounded by walls at $y = \pm Y$ as in Honda[10] or additionally bounded by walls at $z = \pm Z$ as in Kotansky[57]. We shall assume that the boundary conditions in z are simple and symmetrical. Then there will be a symmetry in the flow such that

$$\phi(x, y, z) = -\phi(x, y, -z). \tag{5-58}$$

As the change in ϕ from $(-\infty, y, \pm 0)$ to $(+\infty, y, \pm 0)$ is a step change at $x = 0$ and as this boundary condition determines the main features of the flow, then we may expect ϕ to change from zero at $x = -\infty$ to $\phi(+\infty, y, z)$ at $x = +\infty$ so that half the total change will have occurred at $x = 0$. Hence we deduce that $v(0, y, z) = \frac{1}{2} v(\infty, y, z)$

and $w(0, y, z) = \frac{1}{2} w(\infty, y, z)$. (5-59)

Kármán and Tsien[11] have given a more mathematical proof of this result. The x component of velocity is given by

$$u = \phi_x + A \tag{5-19}$$

or

$$u = \phi_x - \frac{U'}{U^2} \int_{-\infty}^{x} \frac{\partial U\phi}{\partial y}\, dx . \tag{5-60}$$

The first term tends to zero as $x \to \infty$ but the second becomes infinite, because for an airfoil ϕ is not zero at $x = \infty$. This difficulty arises because A is a convection term describing the distortion of the Bernoulli surfaces. A small velocity acting for an infinite time will produce a large change. The difficulty is that any large convection of the surfaces of constant U is incorrectly estimated by the theory as the velocity v, and hence the displacements, are assumed to be small. This may be more readily understood if the linearisation of equation (5-1) is examined, for the term vu_y is naturally omitted in view of the assumption that both u and v are small. This assumption fails when u becomes large owing to a large convection. Fortunately the estimation of u in the Trefftz plane is not required either to obtain the downwash at the lifting line or the induced drag. Nor does the component of velocity u appear in the equation of continuity as $x \to \infty$, for reference to equation (5-29) shows that this becomes

$$\frac{\partial}{\partial y}\left(\frac{v}{U}\right) + \frac{\partial}{\partial z}\left(\frac{w}{U}\right) = 0 \tag{5-61}$$

The next step in the solution is to determine the value of w in the Trefftz plane. For this purpose one of the forms of the equation of continuity (equations (5-23) to (5-28)) is required. Kármán and Tsien used equation (5-28). It is perhaps simpler to use equation (5-25), viz.

$$v_{Tyy} + v_{Tzz} = (U''/U)\ v_T \tag{5-62}$$

where the subscript T denotes the Trefftz plane value and the boundary conditions at z = 0 are given by equations (5-57) and (5-54), i.e. at z = + 0

$$v_T = \frac{1}{2\rho U}\frac{dL}{dy} \tag{5-63}$$

since the strength of the trailing vortex sheet is the same from x = 0 to $x = \infty$. The other boundary conditions depend on the extent of the flow. If it is unlimited, $v \to 0$ as $|y| \to \infty$ or $|z| \to \infty$.

If the airfoil has chord c, geometrical angle of attack α and slope $\kappa = d\,C_L/d\alpha$ we may write

$$L(y) = \frac{1}{2}\,\rho\,U^2\,c\,\kappa\left(\alpha + \frac{1}{2}\,\frac{w_T(y,\,0)}{U}\right). \tag{5-64}$$

Substituting in equation (5-63) for z = +0

$$v_T(y,0) = \frac{1}{4}\,c\,\kappa\,\left\{\,2\,\alpha\,U' + \frac{1}{2U}\,\frac{\partial}{\partial y}\,(U\,w_T(y,0)\right\}$$

or from equation (5-5)

$$v_T = \frac{1}{2}\,c\,\kappa\,\left\{\,\alpha\,U' + \frac{1}{4}\left|\frac{\partial v_T}{\partial z}\right|_{z\,=\,+0}\right\}. \tag{5-65}$$

Now equations (5-62), (5-65) and the remaining boundary conditions are sufficient for the determination of v_T and hence the lift distribution, and any other results required. The convenience of using equation (5-62) is that it is generally possible to approximate to any shear flow profile by dividing it into regions in which U''/U is either zero or a constant[57]. At the boundaries of each region v and p are necessarily continuous.

5.5 Single lifting line spanning a duct

Consider the flow about a lifting line bounded by walls at $y = \pm \ell/2$ with an approach velocity of constant gradient U'

$$U(y) = \frac{U_1 + U_2}{2} + U' \cdot y. \tag{5.66}$$

Then equation (5.62) leads to solutions of the form

$$v_T = \sum_{n=1,}^{\infty} F_n(y)\, G_n(z)$$

which when the boundary conditions at $y = \pm \ell/2$ and $z = \pm \infty$ are satisfied may be written for $z > 0$

$$v_T = \sum_{n=1,3}^{\infty} A_n \cos \frac{n\pi}{\ell} y \exp\left(- \frac{n\pi}{\ell} z\right). \tag{5.67}$$

Substituting this expression in equation (5.65) for $z = +0$

$$A_n \{1 + \frac{1}{8}\frac{n\pi}{\ell} c\,\kappa\} \cos \frac{n\pi y}{\ell} = \frac{1}{2} c\,\kappa\,\alpha\,U'$$

whence
$$A_n = (-1)^{\frac{n-1}{2}} \frac{2\,c\,\kappa\,\alpha\,U'}{n\pi\,\{1 + \frac{1}{8}\frac{n\pi}{\ell} c\,\kappa\}}. \tag{5.68}$$

Substituting for v_T in equation (5.61) we find that

$$w_T(y, +z) = - \sum_{n=1,3}^{\infty} \frac{\ell}{n\pi} A_n\, e^{-\frac{n\pi}{\ell} z} \left\{\frac{U'}{U} \cos \frac{n\pi}{\ell} y + \frac{n\pi}{\ell} \sin \frac{n\pi y}{\ell}\right\} \tag{5.69}$$

Hence

$$\frac{\Delta C_L}{C_L} = \frac{1}{2} \frac{w_T(y, 0)}{U\,\alpha}$$

$$= -2 \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (c/\ell)}{n\,\{1 + \frac{1}{4} n\pi^2 (c/\ell)\}} \left\{ \frac{U_2 - U_1}{U} \sin \frac{n\pi y}{\ell} \right.$$

$$\left. + \frac{1}{n\pi} \left(\frac{U_2 - U_1}{U}\right)^2 \cos \frac{n\pi}{\ell} y \right\} - - - \tag{5.70}$$

where κ has been given the conventional value 2π. Fig. 5.1 shows some results[57].

5.6 Cascades of lifting lines

The lifting line theory can be applied to cascades of small deflection. Consider an infinite cascade of airfoils represented by lifting lines and shown diagrammatically in Fig. 5.2. Let the x axis be the direction of the vector mean of the velocities far upstream and far downstream from the cascade in a potential two-dimensional flow. Let U(y) be the x component of velocity far upstream and $w_1(y)$ be the z component far upstream, where w_1 is small and $\tan\beta_1 = w_1/U$ = constant $\doteq \beta_1$. The axes Ox', Oz' make an angle α_m, the cascade angle with the axes Ox, Oz.

We again note that by symmetry half the total change in ϕ from $x' = -\infty$ to $x' = +\infty$ will occur at $x' = 0$, i.e. in the plane of the cascade.

Hence $$w(o, y, z) = \frac{1}{2}(w_T(y, z) + w_1(y)). \qquad (5\text{-}71)$$

Equation (5-64) for the lift becomes

$$L(y) = \frac{1}{2}\rho U^2 c\,\kappa\left(\alpha_o + \frac{w(0, y, 0)}{U}\right)$$

$$= \frac{1}{2}\rho U^2 c\,\kappa\left(\alpha_o + \frac{w_T(y, 0) + w_1(y)}{2U}\right) \qquad (5\text{-}72)$$

where α_o the angle of attack in uniform flow is $(\alpha_m - \alpha_{mo})$ where α_{mo} is the value of α_m for zero lift.

We may determine the value of κ approximately as follows. The relation between lift coefficient and circulation for a cascade in uniform flow is given by

$$C_L = \kappa\alpha_o = \frac{2}{c}\frac{\Gamma}{U}.$$

Now the circulation in terms of the vector mean velocity U and the inlet and outlet angles α_1 and α_2 which the flow makes with the axis Ox' is

$$\Gamma = s\,U\cos\alpha_m(\tan\alpha_1 - \tan\alpha_2),$$

also $$\tan\alpha_m = \frac{1}{2}(\tan\alpha_1 + \tan\alpha_2).$$

Hence $$\kappa\alpha_o = 4\frac{s}{c}\cos\alpha_m(\tan\alpha_1 - \tan\alpha_m)$$

$$= 4\frac{s}{c}\cos\alpha_m(\tan\alpha_m - \tan\alpha_2).$$

Now when the deflections are small

$$\tan\alpha_1 - \tan\alpha_m = (\alpha_1 - \alpha_m)\sec^2\alpha_m = \beta_1\sec^2\alpha_m$$

$$= \tan\alpha_m - \tan\alpha_2$$

$$= (\alpha_m - \alpha_2)\sec^2\alpha_m = \beta_2\sec^2\alpha_m.$$

Hence $$\kappa\alpha_o = 4\frac{s}{c}\beta_1\sec\alpha_m \qquad (5\text{-}73)$$

When the cascade is closely pitched the value of α_2 remains approximately constant as the angle of attack is varied so that $\alpha_o = \beta_2 = \beta_1$ and we obtain the result

$$\kappa \doteq 4 \frac{s}{c} \sec \alpha_m \; . \tag{5-74}$$

Weinig[92] suggests that this expression is satisfactory if $\frac{s}{c} \leq 0.7$. He gives more exact results for a range of cascades.

Now by integrating equation (5-5) from $x = -\infty$ to $x = +\infty$ we obtain

$$\frac{\partial U w_T}{\partial y} - \frac{\partial U v_T}{\partial z} = \frac{dUw_1}{dy} = 2UU' \; \beta_1 . \tag{5-75}$$

Applying this result to equations (5-72) and (5-63),

$$v_T(y, +o) = \frac{1}{2} c\kappa \left\{ U'(\alpha_o + \beta_1) + \frac{1}{4} \left| \frac{\partial}{\partial z} v_T(y, z) \right|_{z=+o} \right\} \tag{5-76}$$

Now the cascade flow repeats itself periodically with z, so that at $x = +\infty$ and $z = s \cos \alpha_m - \varepsilon$, where $\varepsilon \to 0$

$$v_T(y, s \cos \alpha_m - \varepsilon) = - v_T(y, +o) \tag{5-77a}$$

and

$$w_T(y, s \cos \alpha_m - \varepsilon) = w_T(y, +o) \quad . \tag{5-77b}$$

The latter condition applied to equation (5-75) leads to the result

$$v_{Tz}(y, s \cos \alpha_m - \varepsilon) = v_{Tz}(y, +o) \quad . \tag{5-77c}$$

There is also a continuity condition

$$\int_o^{s'} v_T \, dz = 0. \tag{5-77d}$$

For a cascade bounded by walls at $y = \pm \ell/2$ the remaining boundary conditions are

$$v_T(\pm \ell/2, z) = 0 \tag{5-77e}$$

If the upstream velocity is the simple linear function of y given by the equation (5-66), the solution of equation (5-62) which is valid in the region $z = 0$ to $z = s' = s \cos \alpha_m$ and which satisfies the conditions of equations (5-77) is

$$v_T = \sum_{n=1,3}^{\infty} B_n \cos \frac{n\pi y}{\ell} \sinh \frac{n\pi}{\ell} (z - s'/2). \tag{5-78}$$

Substituting for v_T in equation (5-76) and evaluating the coefficients, B_n, in the usual manner we obtain

$$B_n = \frac{(-1)^{(n+1)/2} \; 2 \; c\kappa \; U'(\alpha_o + \beta_1)}{n\pi \left\{ \sinh \frac{n\pi s'}{2\ell} + \frac{c\kappa}{8} \frac{n\pi}{\ell} \cosh \frac{n\pi s'}{2\ell} \right\}} \tag{5-79}$$

The velocity w_T may be determined from equation (5-75)

$$\frac{\partial U(w_T - w_1)}{\partial y} = \frac{\partial U v_T}{\partial z} = \sum_{1,3}^{\infty} \frac{n\pi}{\ell} B_n \; U \cos \frac{n\pi y}{\ell} \cosh \frac{n\pi}{\ell} (z - \frac{s'}{2}),$$

so that
$$U(w_T - w_1) = \sum_{1,3}^{\infty} B_n \cosh \frac{n\pi}{\ell} (z - \frac{s'}{2}) G_n(y) + B_o(z) \tag{5-80}$$

where
$$G_n(y) = U \sin \frac{n\pi y}{\ell} + \frac{\ell U'}{n\pi} \cos n\pi \frac{y}{\ell} \tag{5-81}$$

and $B_o(z)$ is an arbitrary function of z.

Now the pressure far downstream is independent of x so that equation (5-61) applies. We use it in the form,

$$\frac{\partial}{\partial y} (\frac{v_T}{U}) + \frac{\partial}{\partial z} (\frac{w_T - w_1}{U}) = 0 , \tag{5-82}$$

noting that $w_1/U = \beta_1$, a constant. On inserting the expressions for v_T and $(w_T - w_1)$ from equations (5-78) and (5-80) in equation (5-82) we find that $B_o'(z) = 0$ and hence $B_o(z) = B_o$ = constant.

From the expressions for v_T and w_T given by equations (5-78) and (5-80) we may now derive the expression for ϕ_T by using equations (5-20) and (5-21).

We find that
$$U\phi_T = - \frac{\Delta p}{\rho} x + B_o(z - \frac{s'}{2}) + \sum_{1,3}^{\infty} B_n \frac{\ell}{n\pi} \sinh \frac{n\pi}{\ell}(z - \frac{s'}{2}) G_n(y), \tag{5-83}$$

The first term in this expression arises from equation (5-18) where $(p_{\infty} - p_{-\infty}) = \Delta p$, the pressure rise across the cascade.

The velocity in the x direction is given by equation (5-19) and is
$$u_T = - \frac{\Delta p}{\rho U} + A_T \quad .$$

As pointed out in the discussion of equation (5-60) A_T will become infinite. Honda[58] has shown that the average value of A_T across the span remains finite. Along the walls, $y = \pm \ell/2$, where v is zero, A_T is also zero.

Now from equation (5-53) for $x > 0$
$$\begin{aligned} L &= \rho U [\phi(x, y, +o) - \phi(x, y, -o)] \\ &= \rho U [\phi_T(x, y, +o) - \phi_T(x + s \sin \alpha_m, y, s')] \end{aligned}$$

where the term $s \sin \alpha_m$ is introduced to allow for the displacement of similar points in the flow about the airfoils. Equation (5-83) then leads to the result

$$\begin{aligned} \frac{L}{\rho} &= \frac{\Delta p}{\rho} s' \tan \alpha_m - B_o s' - 2 \sum_{1,3}^{\infty} B_n \frac{\ell}{n\pi} \sinh \frac{n\pi s'}{2\ell} G_n(y) \\ &= \frac{\Delta p}{\rho} s' \tan \alpha_m - \int_o^{s'} U(w_T - w_1)\, dz \quad . \end{aligned} \tag{5-84}$$

In a uniform flow the lift L^* which gives the same pressure rise is given from elementary cascade theory by

$$L^* \sin \alpha_m = s\Delta p$$

or
$$L^* = s'\Delta p/(\sin \alpha_m \cos \alpha_m).$$

Subtracting this lift from the value given in equation (5-84) we obtain

$$(L - L^*)/\rho = -\frac{\Delta p}{\rho} s' \cos\alpha_m - B_o s' - 2\sum_{1,3}^{\infty} B_n \frac{\ell}{n\pi} \sinh\frac{n\pi s'}{2\ell} G_n(y).$$

Now the difference between L and L* is due to the existence of the velocity gradient U', the effect of which is found in the series terms. When U' vanishes, $L = L^*$ and $B_n = 0$, therefore

$$\frac{\Delta p}{\rho} = -B_o \tan\alpha_m \quad , \tag{5-85}$$

and

$$L/\rho U^2 = -s' B_o \sec^2\alpha_m/U^2 - 2\sum_{1,3}^{\infty} B_n \frac{\ell}{n\pi} \sinh\frac{n\pi s'}{2\ell} G_n(y)/U^2 \tag{5-86}$$

The lift is also given by equation (5-72) and on substituting for $w_T(y,o)$ from equation (5-80) we obtain

$$L/\rho U^2 = \frac{1}{2} c\kappa \left\{ \alpha_o + \beta_1 + \frac{B_o}{2U^2} + \frac{1}{2}\sum_{1,3}^{\infty} B_n \cosh\frac{n\pi s'}{2\ell} G_n(y)/U^2 \right\} \tag{5-87}$$

Before eliminating the lift from equations (5-86) and (5-87) we integrate both equations from $y = -\ell/2$ to $y = +\ell/2$. There is an advantage in this because it may readily be shown that

$$\int_{-\ell/2}^{+\ell/2} \frac{G_n(y)}{U^2} \, dy = 0.$$

We then obtain

$$(s' \sec^2\alpha_m + \frac{c\kappa}{4})B_o \int_{-\ell/2}^{+\ell/2} \frac{dy}{U^2} = -\frac{\ell}{2} c\kappa(\alpha_o + \beta_1) \quad ,$$

or on substituting for κ from equation (5-73) and integrating

$$B_o = -2U_1U_2\beta_1 \tag{5-88}$$

so that the pressure rise is

$$\Delta p = 2\rho U_1 U_2 \tan\alpha_m \quad . \tag{5-89}$$

The variation of lift coefficient is given by,

$$\frac{\Delta C_L}{C_L} = \frac{L}{\frac{1}{2}\rho U^2 c\kappa\alpha_o} - 1 \quad ,$$

where ΔC_L represents the increase from the two-dimensional value. The most rapid convergence is obtained by using equation (5-87) giving

$$\frac{\Delta C_L}{C_L} = \frac{U_1 U_2}{U^2} - 1 + 8\left(1 + \frac{\beta_1}{\alpha_o}\right)\sum_{1,3}^{\infty} \frac{(-1)^{(n-1)/2}\left[\frac{U_2 - U_1}{U} \sin\frac{n\pi y}{\ell} + \frac{1}{n\pi}\left(\frac{U_2 - U_1}{U}\right)^2 \cos\frac{n\pi y}{\ell}\right]}{n^2\pi^2\left(1 + \frac{c\kappa}{8}\frac{n\pi}{\ell}\coth\frac{n\pi s'}{2\ell}\right)} \tag{5-90}$$

The increase in outlet angle is given by

$$\Delta\beta_2 = \frac{\overline{w}_T + w_1}{U}$$

where $\overline{w}_T$ is the mean value of w_T, i.e.

$$\overline{w}_T = \int_0^{s'} w_T \, d(z/s') \ .$$

After some simplification we obtain

$$\frac{\Delta\beta_2}{2\beta_1} = (1 - \frac{U_1 U_2}{U^2})$$

$$- 8 \sec^2 \alpha_m (1 + \frac{\beta_1}{\alpha_0}) \sum_{1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (\frac{U_2-U_1}{U} \sin\frac{n\pi y}{\ell} + \frac{1}{n\pi}(\frac{U_2-U_1}{U})^2 \cos\frac{n\pi y}{\ell})}{n^2\pi^2 (1 + \frac{c\kappa}{8} \frac{n\pi}{\ell} \coth \frac{n\pi s'}{2\ell})} \tag{5-91}$$

The induced drag is given by

$$D_i = \frac{1}{2} \frac{w_T(y,o) + w_1}{U} L(y) = - \frac{\Delta C_L}{C_L} L(y) \tag{5-92}$$

Results for other upstream velocity profiles may be obtained with the help of Honda's(58) and Kotansky's(57) work.

5.7 Lifting Line Solutions in Terms of the Pressure.

In this section we illustrate some other approaches to the solution of lifting line and thin airfoil problems. Both Karman and Tsien(11) and Honda(10)(58) start from the continuity equation (equation (5-28)) in the form

$$\frac{\partial}{\partial x} U \operatorname{div} \underline{v} = \nabla^2 p - 2 \frac{U'}{U} \frac{\partial p}{\partial y} \tag{5-28}$$

Now by analogy with the potential flow about a line doublet it is plausible to assume that the pressure field about a lifting line along the y axis of strength $L(y)$ (equation (5-56)) may be expressed in the form

$$\nabla^2 p - 2 \frac{U'}{U} \frac{\partial p}{\partial y} = - L(y) \, \delta(x) \, \delta'(z) \ , \tag{5-93}$$

where $\delta(x)$ and $\delta(z)$ are Dirac's delta functions and $\frac{d}{dz} (\delta(z)) = \delta'(z)$.

The validity of this formulation may be demonstrated by noting that in the "in-field" (section 5.3) equation (5-93) becomes

$$p_{xx} + p_{zz} = - L(y) \, \delta(x) \, \delta'(z).$$

The equation is the same as that for the two-dimensional flow about a doublet, so that the in-field solution is

$$p = - \frac{L(y)}{2\pi} \frac{z}{r^2} \ ,$$

where $r^2 = x^2 + z^2$.

If $z/x = \tan\theta$, $z dx = - r^2 \, d\theta$ so that as $\varepsilon \to 0$

for $z = +0$

$$\int_{-\varepsilon}^{+\varepsilon} p dx = \frac{L(y)}{2\pi} \int_{\pi}^{0} d\theta = -\frac{1}{2} L(y) \quad ,$$

and for $z = -0$

$$\int_{-\varepsilon}^{+\varepsilon} p dx = \frac{L(y)}{2\pi} \int_{\pi}^{2\pi} d\theta = \frac{1}{2} L(y) \quad .$$

This result is demonstrably equivalent to that presented in equation (5-56) because $p = 0$ at $z = 0$ except near the singularity.

The solution of equation (5-93) proceeds by writing

$$p(x,y,z) = \sum_{n=0}^{\infty} G_n(y)\, P_n(x,z) \quad , \tag{5-94}$$

and

$$L(y) = -\sum_{n=0}^{\infty} A_n\, G_n(y) \quad . \tag{5-95}$$

On separation of the variables we obtain the results

$$G_n'' - 2\frac{U'}{U} G_n' = -\lambda_n^2 G_n \quad ,$$

or

$$\frac{d}{dy}\left(\frac{G_n'}{U^2}\right) = -\lambda_n^2 \frac{G_n}{U^2} \tag{5-96}$$

and

$$\frac{\partial^2 P_n}{\partial x^2} + \frac{\partial^2 P_n}{\partial z^2} - \lambda_n^2 P_n = A_n\, \delta(x)\, \delta'(z) \quad . \tag{5-97}$$

A powerful method for treating equation (5-97) is to use the Fourier transform of P_n where

$$P_n^*(x,k_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik_n z} P_n(x,z)\, dz \quad ,$$

and

$$P_n(x,z) = \int_{-\infty}^{+\infty} e^{ik_n z} P_n^*(x,k_n)\, dk_n \quad .$$

Equation (5-97) then becomes

$$\frac{\partial^2 P_n^*}{\partial x^2} - K_n^2 P_n = \frac{ik_n}{2\pi} A_n\, \delta(x) \quad , \tag{5-98}$$

where $K_n^2 = k_n^2 + \lambda_n^2$.

$$[\int_{-\infty}^{+\infty} e^{-ikz}\,\delta'(z)\,dz = [e^{-ikz}\,\delta(z)]_{-\infty}^{+\infty} + ik\int_{-\infty}^{+\infty} e^{-ikz}\,\delta(z)\,dz = ik \cdots$$

see reference (86)].

The solution of equation (5-98) is

$$P_n^{*} = \frac{ik_n A_n}{4\pi K_n}\, e^{-K_n |x|} \tag{5-99}$$

where K_n is always positive.

From equation (5-3)

$$U\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{1}{\rho}\sum_{n=0}^{\infty} G_n \frac{\partial P_n}{\partial z}$$

or

$$\rho U\, w(x,y,z) = -\sum_{n=0}^{\infty} G_n \int_{-\infty}^{x} \frac{\partial P_n}{\partial z}\,dx \tag{5-100}$$

At $x = 0$

$$\int_{-\infty}^{0} \frac{\partial P_n}{\partial z}\,dx = \int_{-\infty}^{+\infty} ik_n\, e^{ik_n z}\,dk_n \int_{-\infty}^{0} P_n^{*}\,dx$$

$$= \int_{-\infty}^{+\infty} \frac{k_n^2 A_n\, e^{ik_n z}}{4\pi(k_n^2 + \lambda_n^2)}\,dk_n$$

This integral may be readily evaluated by contour integration and hence an expression for $w(0,y,0)$ determined. We find that

$$4\rho U w(0,y,0) = \sum_{n=0}^{\infty} \lambda_n A_n G_n(y)$$

Inserting this result and the expression for $L(y)$ equation (5-95) in equation (5-72) we obtain

$$\sum_{n=0}^{\infty} A_n G_n(y)(1 + \tfrac{1}{8}\lambda_n c\kappa) = -\tfrac{1}{2}\rho U^2 c\kappa\ \alpha_0 \ . \tag{5-101}$$

Now the solution for G_n equation (5-96) depends on the shear profile and boundary conditions. At a wall $y = c$, equation (5-2) shows that p_y and therefore $G_n'(c) = 0$. The boundary conditions enable λ_n to be obtained as a series of eigenvalues. It may also be shown[(10)(57)] that the functions G_n are orthogonal. The zeroth term G_0 corresponds to $\lambda_n = 0$ and equation (5-96) shows that G_0 = constant. With walls at $y = \pm \ell/2$ integration of equation (5-96) shows that for $n \neq 0$

$$\int_{-\ell/2}^{+\ell/2} \frac{G_n}{U^2}\, dy = 0.$$

Dividing both sides of equation (5-101) and (5-95) by U^2 and integrating between $y = \pm \ell/2$ we find that

$$A_0\, G_0 \int_{-\ell/2}^{+\ell/2} \frac{dy}{U^2} = -\frac{1}{2}\rho c \kappa \alpha_0 = -\int_{-\ell/2}^{+\ell/2} \frac{L(y)}{U^2}\, dy \quad . \tag{5-102}$$

For a simple shear profile with walls at $y = \pm\ \ell/2$, $G_n(y)$ is given by equation (5-81). Equation (5-101) then gives the same results as those obtained in section 5.5.

Expressions for $G_n(y)$ for other upstream profiles are given by Kotansky[57].

The left hand side of equation (5-97) may be written

$$\frac{\partial^2 P_n}{\partial r^2} + \frac{1}{r}\frac{\partial P_n}{\partial r} - \lambda_n^{\ 2} P_n \quad .$$

Honda[10] points out that, when this expression is zero, the solution is a modified Bessel's function of order zero. The fundamental solution which satisfies the condition that $P_n \to 0$ as $r \to \infty$ and that there is a source type of singularity at $r = 0$ is

$$P_n = \overline{K}_0(\lambda_n r) \quad ,$$

where $\overline{K}_0$ is the modified Bessel's function of the second kind of order zero.

Now

$$\lim_{r \to 0} \overline{K}_0(\lambda_n r) = -\log_e(\lambda_n r)$$
$$= -\log_e r$$

This is evidently the in-field solution, so that by application of the arguments used earlier in this section we obtain for the lifting line solution

$$P_n = -\frac{A_n}{2\pi}\frac{\partial}{\partial z}\ \overline{K}_0(\lambda_n r)$$

$$= \frac{A_n}{2\pi}\frac{\lambda_n z}{r}\ \overline{K}_1(\lambda_n r) \tag{5-103}$$

When $n = 0$, $\lambda_n = 0$ and equation (5-97) shows that

$$P_0 = \frac{A_0}{2\pi}\frac{z}{r^2} \tag{5-104}$$

This is the starting point for Honda's[10] analysis of the shear flow about a thin airfoil.

The pressure field about a cascade of lifting lines, Fig. 5.2, is given by

$$\nabla^2 p - 2\frac{U'}{U}\frac{\partial p}{\partial y} = -L(y)\sum_{m=-\infty}^{+\infty} \delta(x - m\, s \sin \alpha_m)\, \delta'(z - m\, s \cos \alpha_m) \tag{5-105}$$

Equation (5-94), (5-95), (5-96) are still applicable, but equation (5-100) now becomes

$$\rho U(w(x,y,z) - w_1(y)) = -\sum_{n=0}^{\infty} G_n(y)\int_{-\infty}^{x}\frac{\partial P_n}{\partial z}\,dx \quad . \tag{5-106}$$

It is convenient to write

$$P_n = \int_{m=-\infty}^{+\infty} P_{nm} \quad .$$

Then by the same procedure as for the isolated airfoil

$$\int_{-\infty}^{0}\frac{\partial P_{nm}}{\partial z}\,dx = \int_{-\infty}^{+\infty}\frac{k_n^2 A_n}{4\pi K_n}\, e^{-ik_n(z-ms')}\,dk_n \int_{-\infty}^{0} e^{-k_n|x - m\, s \sin \alpha_m|}\,dx \quad . \tag{5-107}$$

Care is required in the evaluation of the integral with respect to x. We find that

$$\int_{-\infty}^{0} e^{-K|x - ms \sin \alpha_m|}\,dx = \frac{e^{-K_n ms \sin \alpha_m}}{K_n} \qquad \text{for } m \geq 0$$

$$= \frac{2}{K_n} - \frac{e^{K_n ms \sin \alpha_m}}{K_n} \qquad \text{for } m < 0$$

Equation (5-107) is now solved for z = 0 by contour integration. When m > 0 we integrate below the real axis of k_n and when m < 0 above the real axis. When m = 0 the result is the same as that obtained for the isolated airfoil. On summing results for all the terms we obtain

$$\int_{-\infty}^{0}\frac{\partial P_n}{\partial z}\,dx = -\frac{\lambda_n A_n}{4}\left(1 + \sum_{m=1}^{\infty} e^{-\lambda_n ms'} + \sum_{m=-1}^{-\infty} e^{\lambda_n ms'}\right)$$

$$= -\frac{\lambda_n A_n}{4}\left(1 + 2\sum_{m=1}^{\infty} e^{-\lambda_n ms'}\right)$$

$$= -\frac{\lambda_n A_n}{4}\coth\left(\frac{1}{2}\lambda_n s'\right) \quad , \tag{5-108}$$

Where the series is summed by using the well known result that for $x^2 < 1$

$$(1 + x)^{-1} = 1 + x + x^2 + \cdots \quad .$$

The limiting value of this expression for n = 0 requires special attention. When $\lambda_n = 0$ the expression for P_0 equivalent to equation (5-97) becomes that for a plane two dimensional in which the pressure field G_0P_0 is that about a cascade of lifting lines of strength or lift equal to $- A_0G_0$. Now as shown on page 52 for two dimensional flow through a cascade the lift may be related to the flow angles by writing

$$- A_0 G_0 = \rho U \Gamma = \rho s U^2 \cos {}_m(\tan \alpha_1 - \tan \alpha_2)$$

$$= 2\rho s U^2 \cos \alpha_m(\tan \alpha_1 - \tan \alpha_m)$$

$$= 2\rho s U^2 \sec \alpha_m(\alpha_1 - \alpha_m).$$

Hence the increase in the value of w from far upstream to the cascade is given by

$$U(\alpha_m - \alpha_1) = \frac{A_0 G_0 \cos \alpha_m}{2\rho s U} .$$

Therefore we deduce that for n = 0

$$\int_{-\infty}^{0} \frac{\partial P_0}{\partial z} dx = - \frac{A_0 \cos \alpha_m}{2s} \tag{5-109}$$

We note that this is not the same result as that which would be obtained by formally determining the limit of equation (5-108) as $\lambda_n \to 0$.

Equations (5-108) and (5-109) may be combined in equation (5-106) to give

$$U(w(0,y,0) - w_1(y)) = \frac{G_0 A_0 \cos \alpha_m}{2s} + \sum_{n=1}^{\infty} \frac{G_n A_n \lambda_n}{4} \coth \frac{1}{2} \lambda_n s' \tag{5-110}$$

The expression for the lift equation (5-72) therefore becomes

$$- \sum_{n=0}^{\infty} A_n G_n(y) = \frac{1}{2} \rho U^2 c\kappa \left(\alpha_0 + \beta_1 + \frac{G_0 A_0 \cos \alpha_m}{2\rho s U^2}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{G_n A_n{}_n}{4 \rho U^2} \coth (\tfrac{1}{2} \lambda_n s')\right) \tag{5-111}$$

On dividing by U^2 and integrating from $y = - \ell/2$ to $y = + \ell/2$ where there are walls we find that

$$A_0 G_0(1 + \frac{c}{4s} \cos \alpha_m) \int_{-\ell/2}^{+\ell/2} \frac{dy}{U^2} = - \frac{1}{2} \rho(\alpha_0 + \beta_1) c \kappa \ell$$

or substituting for κ from equation (5-73)

$$\frac{A_0\, G_0 \cos\alpha_m}{2\,\rho s} \int_{-\ell/2}^{+\ell/2} \frac{d(y/\ell)}{U^2} = -\beta_1 \qquad (5\text{-}112)$$

It may be shown that equations (5-111) and (5-112) lead to the results obtained in section 5.6.

5.8 Thin Airfoils and cascades.

The methods of section 5.7 may be extended to the flow about thin airfoils by representing the airfoil as a line of lifting lines extending from $x = 0$ to $x = c$ along the x axis. The right hand side of equation (5-97) then becomes

$$\int_0^c A_n(t)\, \delta(x - t)\, \delta'(z)\, dt.$$

Honda[10][58] starts from the fundamental solution given in equation (5-103). The work is too extensive to be summarized here, and reference should be made to the original papers. It is of interest to examine the conditions in which the lifting line theory of the previous paragraphs begin to break down so that the need for using Honda's more exact theory may be determined. We start from the expression for the lift $L = -\oint p\, dx$ and substitute for the pressure from equation (5-15)

$$L = \rho U \int_0^c (u(x,y,0^+) - u(x,y,0^-))\, dx \qquad (5\text{-}113)$$

$$- \rho U \int_0^c (A(x,y,0^+) - A(x,y,0^-))\, dx$$

$$= L_0 + \rho U' \int_0^c dx \int_{-\infty}^{x} [v(x,y,0^+) - v(x,y,0^-)]\, dx$$

where equation (5-11) for A has been used and L_0 is the lift determined from the lifting line theory for which in the in-field $L \to \rho U\Gamma$.

Now by symmetry the value of v at $z = 0$ and $x < 0$ is zero and v increases gradually from $x = 0$ to c along the surface of the airfoil until at the trailing edge it reaches v_T, the Trefftz plane value. To obtain an estimate of the effect let us assume that v is linear in x. Then

$$L - L_0 = \frac{c^2}{6} \rho U'(v(c,y,0^+) - v(c,y,0^-)) = \frac{c^2}{6} \rho U'(v_T^{\,+} - v_T^{\,-}) \qquad (5\text{-}114)$$

By substitution from equation (5-54) and assuming that $L - L_0$ is small we find that

$$L - L_0 = \frac{c^2}{6}\, \frac{U'}{U}\, \frac{dL_0}{dy} \quad . \qquad (5\text{-}115)$$

Now

$$L_0 = \frac{1}{2} \rho U^2 c C_L (1 + \frac{\Delta C_L}{C_L}) \quad , \qquad (5\text{-}116)$$

hence

$$\frac{L - L_0}{L_0} = \frac{c^2}{6} \frac{U'}{U} \left[2 \frac{U'}{U} + \frac{\frac{d}{dy} (\frac{\Delta C_L}{C_L})}{1 + \frac{\Delta C_L}{C_L}} \right]$$

$$= \frac{c^2}{6} \frac{U'}{U} [2 \frac{U'}{U} + \frac{d}{dy} (\frac{\Delta C_L}{C_L})] \quad , \qquad (5\text{-}117)$$

since $\frac{\Delta C_L}{C_L}$ is small compared to unity.

For the example considered in paragraph 5.5 the term in $\Delta C_L/C_L$ is, for small n, nearly linear in (c/ℓ) so that approximately

$$\frac{L - L_0}{L_0} = \frac{1}{3} (\frac{c}{\ell})^2 (\frac{U_2 - U_1}{U})^2 \qquad (5\text{-}118)$$

which gives a maximum fractional correction of about $(c/\ell)^2/3$ when $(U_2/U_1) = 2$.

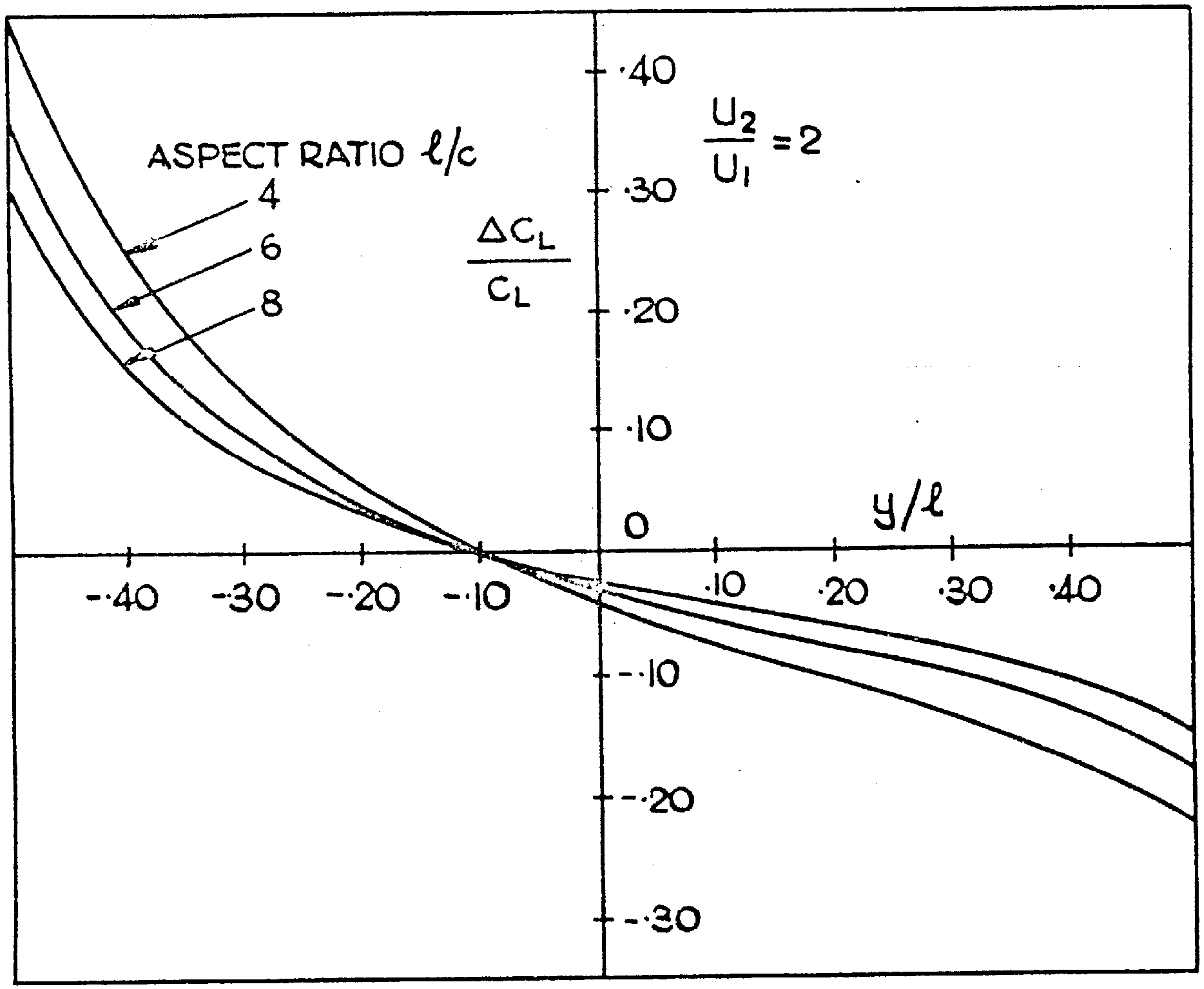


FIG. 5.1 CHANGE IN LIFT COEFFICIENT FOR ISOLATED AIRFOIL SPANNING A DUCT.

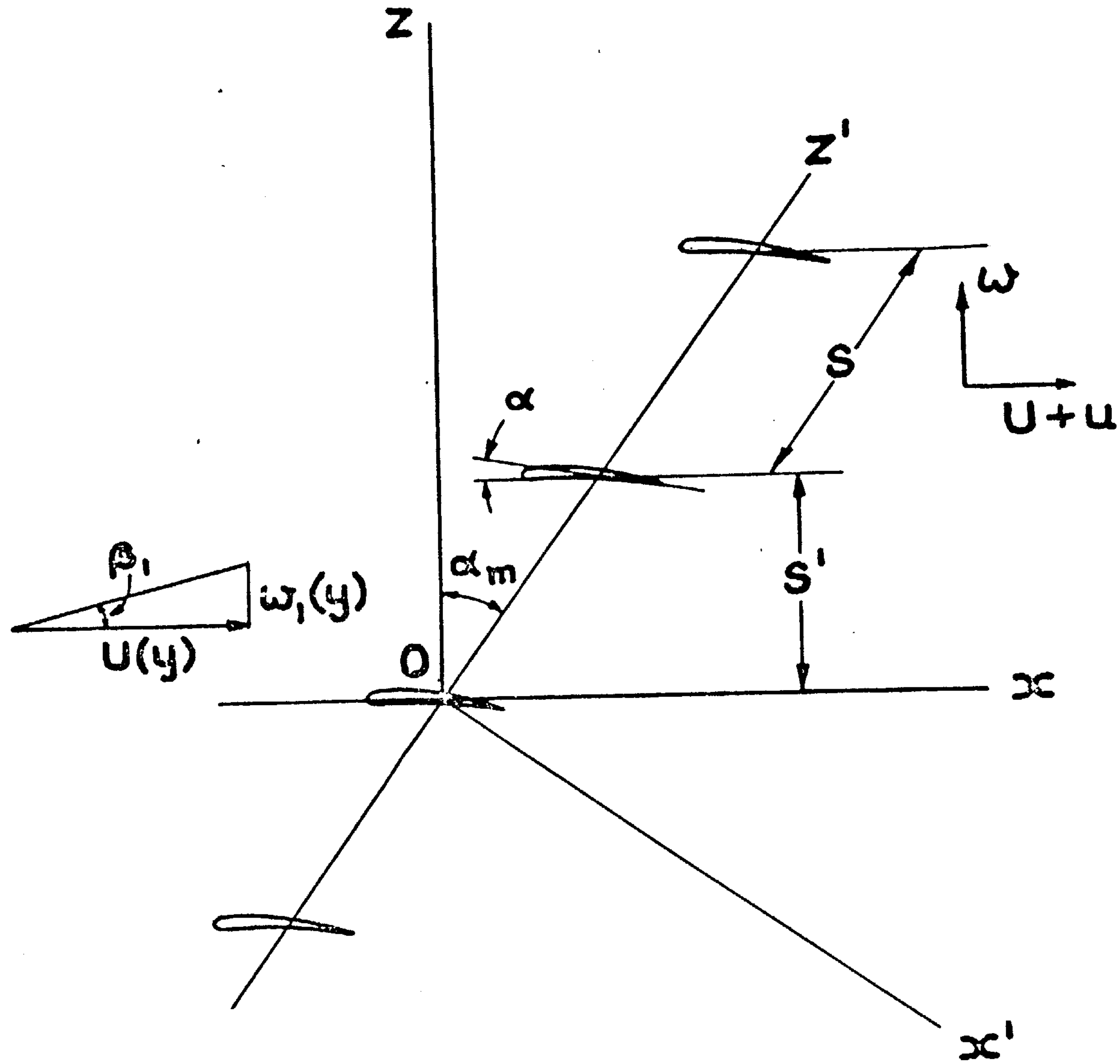


FIG. 5.2 COORDINATE SYSTEM FOR A CASCADE OF LIFTING LINES

SECTION 6 THE SECONDARY FLOW APPROXIMATION

6.1 Introduction.

When the flow has a small shear, or gradient of stagnation pressure, and the disturbance is large the analysis in section 3.4 shows that the vorticity is convected by an irrotational (or reduced irrotational) primary flow. The shear velocities and secondary flows are induced by the stretched and bent vortex filaments.

Squire and Winter[4] first determined the distortion of the vorticity by the primary flow in a bend. They showed that the streamwise component of vorticity, Ω_s, underwent the most change. Expressions for this component have been given in section 2.3 and may be obtained from equations such as (2-49) and (2-65) by assuming that the streamlines and velocities in the integrals are those of an irrotational or a reduced irrotational flow. Secondary vorticities based on this approximation have been obtained for bends*, for cascades**, for the flow in turbo-machines[76-79], about struts and airfoils[22][80], and for spheres[81][90]. If the primary flow becomes parallel far downstream, then the secondary vorticity approaches a limiting value and the far downstream secondary velocities may be fairly readily estimated. The determination of secondary velocity throughout the entire field of flow is more difficult. Lighthill[8][82][83] using the drift or t function, equation (2-28), calculated the vortex fields about a cylinder and a sphere and then derived the induced velocities for the simple shear case, i.e. when in the parallel flow upstream the velocity is linear in y.

When the upstream flow is parallel the basic equation for the velocity in the incompressible version of this approximation is equation (3-16)

$$\underline{V} = U_0 \text{ grad } \phi_0 + \left(U_0 \text{ grad } \phi_1 - U_0 t_0 \text{ grad } U_1\right) \qquad (3\text{-}16)$$

The scalar ϕ_0 is the potential function for an irrotational flow with unit velocity upstream which satisfies the boundary conditions. The bracketted term is the velocity induced by the vorticity transported by the primary potential flow. It must therefore contain a term which gives the upstream shear velocity U_1. We can generate such a term by dividing ϕ_1 into two parts, i.e., let

$$\phi_1 = \frac{U - U_0}{U_0} \phi_0 + \frac{\phi'}{U_0}$$

so that

* See references (4), (5), (59-62), (67), (68).

** See references (63-66), (69), (70-75).

$$\underline{V} = U \text{ grad } \phi_o + (\phi_o - t_o) \text{ grad } U + \text{grad } \phi' \tag{6-1}$$

where the opportunity is taken to write $U = U_o + U_1$ and t_o is now the drift function in the flow described by the potential ϕ_o with unit velocity upstream.

The continuity condition obtained by taking the divergence of equation (6-1) becomes

$$U\nabla^2\phi_o + \text{grad } U \cdot \text{grad } \phi_o + (\phi_o - t_o)\nabla^2 U + \text{grad}(\phi_o - t_o) \cdot \text{grad } U + \nabla^2\phi' = 0 \tag{6-2}$$

Now the first term is zero because ϕ_o is the potential of the unit primary flow, and the second term is zero because the vectors involved are perpendicular, hence

$$\nabla^2\phi' = -(\phi_o - t_o)\nabla^2 U + \text{grad } t_o \cdot \text{grad } U \tag{6-3}$$

Equations (6-1), (6-3) and the application of the boundary conditions, e.g. $\underline{V} = 0$ normal to solid walls, enable complete solutions to be obtained.

A class of special cases may be identified when the upstream velocity U is linear in y (or z), i.e. grad U is constant, for then $\nabla^2 U = 0$ and equation (6-3) becomes

$$\nabla^2\phi' = \text{grad } t_o \cdot \text{grad } U \tag{6-4}$$

If grad U is constant in magnitude and direction throughout a field of flow which stretches to infinity in all directions we have the special case described by Lighthill as *simple shear*. Lighthill's[(8)(81)] solution for the sphere is in fact a solution of equations (6-4) and (6-1) for simple shear flow about a sphere.

Another set of special cases occurs when the primary flow potential, ϕ_o, may be identified as the potential of a two-dimensional flow in the x, y plane and the gradient of the upstream velocity U is in the z direction, for then the term grad t_o · grad U vanishes from equation (6-3) and the equation becomes

$$\nabla^2\phi' = -(\phi_o - t_o)\nabla^2 U \tag{6-5}$$

Examples are the flows about cylinders, airfoils or cascades of airfoils, whose axes lie in the z direction and for which the upstream velocity U is a function of z only. This set of special cases will be designated as *plane primary flows*. Equation (6-1) becomes

$$\underline{V} = U \text{ grad } \phi_o + \underline{k}\,(\phi_o - t_o)U' + \text{grad } \phi' \tag{6-6}$$

where $\underline{k}$ is the unit vector in the z direction and $U' = dU/dz$.

A further restriction of the plane primary flows to cases when U is a *linear* function of z yields for equation (6-5) the result

$$\nabla^2\phi' = 0$$

For such *constant shear* plane primary flows the boundary conditions on the walls of the cylinder, etc., are already satisfied by the first two terms on the right hand side of equation (6-6). The potential ϕ' is then the potential of a three-dimensional flow which must also satisfy the boundary conditions at the walls of the cylinder, etc.

For the case of *simple shear* and plane primary flow, e.g. for a cylinder extending to infinity along the z axis with U linear in z, the solution, as first shown by Lighthill[8], is simply $\phi' = 0$ and the velocity becomes

$$\underline{V} = U \text{ grad } \phi_o + \underline{k}(\phi_o - t_o) U' \tag{6-7}$$

with components $U\phi_{ox}$, $U\phi_{oy}$ and $(\phi_o - t_o)U'$ in the x, y and z directions, respectively. The flow is therefore a plane potential flow proportional to U (and therefore varying linearly with z) in the x, y plane, plus a flow in the z direction whose velocity depends on position in the x, y plane but is independent of z. Since U' is $O(\epsilon)$ the contribution to the pressure variation from the velocity component in the z direction, $(\phi_o - t_o)U'$, is $O(\epsilon^2)$, so that to order ϵ the pressure distribution in each x, y plane is the same as that in a two-dimensional rotational flow with upstream velocity U.

Another class of primary flows is those which are axisymmetrical, such as flow about a body of revolution, e.g. a sphere.

The first step in the analysis of a secondary flow problem by the method discussed above is the evaluation of the function $(\phi_o - t_o)$. The section therefore starts with illustrations of methods for determining the drift function t (the subscript o is no longer required since higher approximations t_1, etc., are not used). Then follows a description of procedures for obtaining the vorticity components. The determination of the velocities far downstream is the next topic followed by a discussion of the more difficult problem of estimating the velocities throughout the field of flow. Most attention is given to examples with plane primary flows, and, as in Section 5, the effects of compressibility are neglected. The section concludes with discussion of the flow round isolated airfoils and cascades.

6.2 The Drift Function for Plane Primary Flows

When the primary flow is a plane two-dimensional flow, such as the flow about a strut or through a bent duct of rectangular cross-section, potential and stream functions may be defined. If the fluid is incompressible, a complex potential may be written:

$$w = \phi + i\psi = f(z), \tag{6-8}$$

where $z = x + iy$ and x, y are the rectangular co-ordinates.

The drift function t is given by equation (2-28), viz.

$$t = \int \frac{ds}{V}, \tag{2-28}$$

where the integral is taken along a stream line.

This equation may be written

$$t = \int_{\psi} \frac{ds}{\partial\phi/\partial s} = \int_{\psi} \frac{d\phi}{(\partial\phi/\partial s)^2} = \text{real part of} \int \frac{dw}{\frac{d\overline{w}}{d\overline{z}} \cdot \frac{dw}{dz}}, \tag{6-9}$$

where $\overline{w}$ and $\overline{z}$ are the complex conjugates of w and z, respectively.

In some cases it may be possible to solve for t from equation (6-8) directly. For instance for the flow through an aperture (Milne-Thompson[84] p. 144)

$$z = c \cosh w$$

and $\frac{dz}{dw} \cdot \frac{d\overline{z}}{d\overline{w}} = c^2 \ (\cosh 2\phi - \cos 2\psi)$.

Hence $\quad t = c^2 \ (\frac{1}{2} \sinh 2\phi - \phi \cos 2\psi) + f(\psi)$,

where $f(\psi)$ is arbitrary.

The flow in a corner of included angle π/n is given by

$$w \propto z^n$$

or $\quad z = c\, n\, w^{1/n}$.

Hence
$$t = c^2 \int (\phi^2 + \psi^2)^{1/n-1} d\phi. \tag{6-10}$$

Along the walls $\theta = 0$ and $\theta = \pi/n$ forming the corner $\psi = 0$ and

$$t = c^2 \ \phi^{\frac{2}{n}-1}/(\tfrac{2}{n} - 1) + \text{const.} \tag{6-11}$$

Now at $\theta = \pi/2n$, $\phi = 0$ by symmetry. Hence $t \to \infty$ at the stagnation point in the corner unless

$$\frac{2}{n} - 1 > 0$$

or
$$\frac{\pi}{n} > \frac{\pi}{2}, \tag{6-12}$$

i.e., the included angle formed by the walls must be obtuse.

This result will be of value later because it shows that for angled bends in ducts whose angle of bend is *smaller* than a right angle and for bodies with wedge-shaped leading and trailing edges t remains finite. Whereas with bends of a right angle or larger and with blunt-nosed bodies t becomes infinite at the stagnation point. A proof of this result is also given in reference (22).

6.2.1 Angled Bends.

An angled bend in a channel of width π may be mapped by the Schwartz-Christoffel transformation from the physical z plane to the top half of the ζ plane, Fig. 6.1.

When the bend angle is α, i.e. the included angle is $\pi - \alpha$, the transformation is

$$\frac{dz}{d\zeta} = \frac{-2e^{i\alpha}}{(\zeta^2 - 1)\zeta^{\alpha/\pi}} . \tag{6-13}$$

The complex potential is

$$w = \ell n \frac{\zeta + 1}{\zeta - 1} , \tag{6-14}$$

where the uniform velocity far upstream is unity.

The velocity is given by

$$\frac{dw}{dz} = u - iv = e^{-i\alpha}\zeta^{\alpha/\pi} = e^{-i\alpha}\left(\frac{e^w + 1}{e^w - 1}\right)^{\alpha/\pi} \tag{6-15}$$

Hence

$$q^2 = u^2 + v^2 = \frac{dw}{dz} \cdot \frac{d\bar{w}}{d\bar{z}} = \left[\frac{e^{2\phi} + 2e^{\phi}\cos\psi + 1}{e^{2\phi} - 2e^{\phi}\cos\psi + 1}\right]^{\alpha/\pi} \tag{6-16}$$

We may write

$$1/q^2 = \left[\frac{\cosh\phi - \cos\psi}{\cosh\phi + \cos\psi}\right]^{\alpha/\pi} = \left[\frac{1 - kx}{1 + kx}\right]^{\alpha/\pi} \tag{6-17}$$

where $k = \cos\psi$, $x = \text{sech}\,\phi$, i.e. $x \leq 1$ for all ϕ. Now along ADC $\psi = 0$ and along ABC $\psi = -\pi$ (by inspection of the flow in the ζ plane). The line BD is by symmetry $\phi = 0$. At A, where $\phi = -\infty$, we shall make lines of constant t coincide with lines of constant ϕ, so that the particle lines of constant t represent vortex filaments normal to the flow.

Now $\quad dx = -\,\text{sech}\,\phi\tanh\phi\, d\phi$

or

$$d\phi = -\frac{(\text{sgn}\,\phi)\,dx}{x\sqrt{1 - x^2}} \tag{6-18}$$

where $\text{sgn}\,\phi = +1$ for $\phi > 0$ and $\text{sgn}\,\phi = -1$ for $\phi < 0$. Hence

$$t - \phi = \int_{-\infty}\left(\frac{1}{q^2} - 1\right) d\phi; \quad \psi = \text{const.}$$

$$= -\int_0^x \frac{\text{sgn}\,\phi}{x\sqrt{1 - x^2}}\left\{\left(\frac{1 - kx}{1 + kx}\right)^{\alpha/\pi} - 1\right\} dx \tag{6-19}$$

Along the dividing streamline, $\psi = -\pi/2$, we see from equation (6-16) that $q = 1$ and hence from equation (6-19) $t - \phi = 0$ for all values of ϕ.

Right-angled bend

When $\alpha/\pi = 1/2$ the bend is right-angled and equation (6-19) becomes

$$t - \phi = -\int_0^x \frac{\text{sgn}\,\phi}{x\sqrt{1 - x^2}}\left(\frac{1 - kx}{\sqrt{1 - k^2x^2}} - 1\right) dx , \tag{6-20}$$

which is soluble in terms of elliptic functions. For $-\infty < \phi < 0$

$$\phi - t = \log_e \left(\frac{\sqrt{1 - x^2} + \sqrt{1 - k^2x^2}}{\sqrt{1 - x^2} + 1} \right) + kF\,(\sin^{-1} x, k) \ ,$$

where $F(\sin^{-1} x, k)$ is the elliptic integral of the first kind

$$F = \int_0^x \frac{dx}{\sqrt{1 - x^2}\,\sqrt{1 - k^2x^2}}$$

The value at $\phi = 0$ on the diagonal BD is

$$\phi - t = \log_e \sin \psi + \cos \psi K(\cos \psi),$$

where K is the complete elliptic integral of the first kind. At the corner D, $\phi - t = \log_e 4$, but at B, where there is a stagnation point in the right-angled bend, the expression becomes infinite. It remains infinite along the wall BC.

When $0 < \phi < + \infty$,

$$\phi - t = 2(\log_e \sin \psi + \cos \psi K(\cos \psi)) - \log_e \left(\frac{\sqrt{1 - x^2} + \sqrt{1 - k^2x^2}}{\sqrt{1 - x^2} + 1} \right) \tag{6-23}$$

At $\phi = + \infty$ the value of $\phi - t$ will be twice the value at $\phi = 0$.

From the equation (6-15)

$$\frac{dw}{dz} = - i \sqrt{\tanh \frac{w}{2}} \ . \tag{6-24}$$

Hence integrating

$$\frac{\pi}{2} (1 + i) + \frac{iz}{2} = \tan^{-1} \sqrt{\tanh \frac{w}{2}} - \tanh^{-1} \sqrt{\tanh \frac{w}{2}}, \tag{6-25}$$

from which streamlines and potential lines may be located in the physical plane. Fig. 6.2 shows the results for a right angled bend and Fig. 6.3 for a bend of 45° angle.(87)

<u>6.2.2 Gradual Bends</u>

The gradual bends most commonly considered are ducts whose centre lines and walls are circular arcs. After a short transition region at the inlet the potential flow becomes a free-vortex. If the mean radius of the bend is R and the radius of any streamline is r then the velocity q is given by

$$rq = R,$$

where the upstream velocity is unity. Note that R is the logarithmic mean of the bend inner and outer radii, so that continuity is satisfied. Hence

$$t = \int_0^\theta \frac{r d\theta}{q}$$

or

$$t = \frac{r^2\theta}{R}, \tag{6-27}$$

where θ is the angle of the bend in the cylindrical co-ordinate system (r, θ, z).

The potential function is given by

$$q = \frac{\partial \phi}{r \partial \theta}$$

or

$$\phi = R\theta. \tag{6-28}$$

whence

$$\phi - t = R\theta(1 - \frac{r^2}{R^2}). \tag{6-29}$$

The above analysis neglects the effect of inlet and outlet transition regions.

We may examine this effect approximately as follows. Consider a straight duct leading to a bend whose outside walls are arcs of circles of radii r_o and r_i. The irrotational flow in the straight pipe approaching the bend may be expressed as

$$w = \phi + i\psi = z + \sum_{n=1}^{\infty} b_n e^{k_n z} \tag{6-30}$$

where $z = x + iy$. The boundary conditions of uniform flow with unit velocity at $x = -\infty$ are satisfied if k_n is positive. The boundary conditions on the walls are satisfied by writing $k_n = n\pi/(r_o - r_i)$.

The flow in the bend may be expressed as

$$w = i\, R\, \ell n(\zeta/r_i) + \sum_{n=1}^{\infty} c_n \exp(-ik_n\, R\, \ell n\, (\zeta/r_i)), \tag{6-31}$$

where $\zeta = r\, e^{-i\theta}$ and $R = (r_o - r_i)/\ell n(r_o/r_i)$. This equation satisfies the boundary conditions on the walls giving $\psi(r_i) = 0$ and $\psi(r_o) = r_o - r_i$ as does equation (6-30). It also gives a free vortex flow at large θ, when the exponential terms become vanishingly small.

The values of the constants b_n and c_n may be obtained by matching the values of ϕ and ψ, or the velocity components obtained from equations (6-30) and (6-31), at the bend inlet plane $x = 0$, $\theta = 0$.

Eichenberger[85] assumed that all the adjustment occurs in the bend, i.e. that $b_n = 0$ and found by matching stream functions that

$$c_n = \frac{r_i}{n\pi} \frac{\left((-1)^n \frac{r_o}{r_i} - 1\right)}{1 + \left(\frac{n\pi}{\ell n(r_o/r_i)}\right)^2}$$

Owing to the influence of the squared term the values of nc_n decrease rapidly with increasing n.

If we assume that all the adjustment occurs upstream of the bend, i.e. that $c_n = 0$, then by matching velocities at x, $\theta = 0$ we find

$$\sum k_n\, b_n \cos k_n\, y = -1 + \frac{R}{r_i + y},$$

or multiplting both sides by $\cos k_n y$ and integrating between $y = 0$ and $(r_0 - r_i)$

$$b_n = \frac{2}{n\pi} \int_0^{r_0 - r_i} \frac{R \cos k_n y}{r_i + y} \, dy \quad .$$

Examination of the integral shows that b_n contains terms which vary as $(1/n)^3$ and terms of higher powers than the cube so that it also decays rapidly as n increases.

When the bend angle is α, flow adjustment occurs both at inlet and at outlet from the bend. The flow is symmetrical about $\theta = \alpha/2$. Instead of equation (6-31) we may write

$$w = iR\ln(\zeta/r_i) + \sum_{n=1}^{\infty} A_n \sinh\{k_n R(i\ln(\frac{\zeta}{r_i}) - \frac{\alpha}{2})\} \quad . \tag{6-32}$$

This equation satisfies the boundary conditions for ψ on the walls and the condition of symmetry.

The flow in the straight duct downstream of the bend is given by

$$w = z + \sum_{n=1}^{\infty} c_n e^{-k_n z} \quad , \tag{6-33}$$

which satisfies the condition that the flow is uniform at $x = +\infty$.

As a first approximation we neglect terms other than the first in the series and match ϕ on the inner and outer walls at $\theta = 0$ and α. Then

$$b_1 = - A_1 \sinh(k_1 R\alpha/2) = - c_1 . \tag{6-34}$$

If we match velocities at $\theta = 0$ and α, we find for the inner wall

$$k_1 A_1 = - \frac{R - r_i}{R \cosh k_1 R\alpha/2 + r_0 \sinh k_1 R\alpha/2} \quad ,$$

and for the outer wall

$$k_1 A_1 = - \frac{r_0 - R}{R \cosh k_1 R \alpha/2 + r_0 \sinh k_1 R\alpha/2} \quad .$$

Both conditions cannot be satisfied simultaneously, so that some average is required. The harmonic mean yields

$$- k_1 A_1 \sinh k_1 R\alpha/2 = \frac{2}{1 + \coth k_1 R\alpha/2} \frac{(r_0 - R)(R - r_i)}{R(r_0 - r_i)} \quad ,$$

When the ratio bend radius to duct width is large, R approaches the mean radius so that approximately

$$- \frac{A_1}{R} \sinh k_1 R\alpha/2 = \frac{2}{1 + \coth k_1 R\alpha/2} \frac{(r_0 - r_i)^2}{4\pi R^2} \quad , \tag{6-35}$$

where the substitution $k_1 = \pi/(r_o - r_i)$ has been made.

In the straight duct approaching the bend the velocity on the inner and outer walls is given approximately by

$$\frac{\partial\phi}{\partial x} = 1 \pm k_1 \, b_1 \, e^{k_1 x},$$

where the positive sign is taken for the inner wall. Hence at $x = 0$ we find that

$$\phi - t = \int_{-\infty}^{0} \left(\frac{\partial\phi}{\partial x} - \frac{1}{\partial\phi/\partial x}\right)dx = \pm 2b_1. \tag{6-36}$$

In the curved part of the bend the velocity is given by

$$\frac{\partial\phi}{r\partial\theta} = \frac{R}{r} \pm \frac{R}{r} k_1 A_1 \cosh k_1 R(\theta - \alpha/2) \tag{6-37}$$

and

$$\phi - t = \int_{0}^{\alpha} \left(\frac{\partial\phi}{r\partial\theta} - \frac{1}{\partial\phi/r\partial\theta}\right) r d\theta.$$

For bends of large radius compared to width, we may write

$$\frac{1}{\partial\phi/r\partial\theta} = \frac{r}{R}\,(1 - (\pm 1)\; k_1 \; A_1 \; \cosh k_1 R(\theta - \tfrac{\alpha}{2})),$$

so that

$$\phi - t = \int_{0}^{\alpha} \{R(1 - \frac{r^2}{R^2}) \pm R(1 + \frac{r^2}{R^2})\; k_1 \; A_1 \; \cosh k_1 R(\theta - \alpha/2)\}\; d\theta$$

$$= R\alpha(1 - \frac{r^2}{R^2}) \pm 2(1 + \frac{r^2}{R^2})\; A_1 \; \sinh k_1 R\alpha/2. \tag{6-38}$$

By a similar process we find that in the straight duct downstream the integral from $x = 0$ to $x = +\infty$ in equation (6-36) yields

$$\phi - t = -\,(\pm 1)\; 2c_1 = \pm 2b_1. \tag{6-39}$$

Adding the results of equations (6-36), (6-38) and (6-39) we find that far downstream of the bend

$$\frac{\phi - t}{R} = (1 - \frac{r^2}{R^2})\; \{\alpha - (\pm 1) 2\, \frac{A_1}{R} \sinh k_1 R\alpha/2\},$$

or substituting from equation (6-35)

$$\frac{\phi - t}{R} = (1 - \frac{r^2}{R^2}) \left\{\alpha \pm \frac{2}{1 + \coth k_1 R\alpha/2}\, \frac{(r_o - r_i)^2}{2\pi R^2}\right\}. \tag{6-40}$$

This result is to be compared with the result obtained in equation (6-29) in which the effect of the inlet and outlet transition regions is neglected.

The results of a numerical analysis of the flow round a 180° bend with $r_o = 3.5$ and $r_i = 2.5$ performed by M. Rowe[87] are shown in Fig. 6.4.

The flow in the bend was transformed to the flow in a parallel channel by the method given on page 173 of Thwaites[3] and results for ϕ and ψ computed by the iterative process described in the reference. The value of t was obtained by numerical integration. For this example the correction term in the braces in equation (6-40) only amounts to 0.6%, and the values of $\phi - t$ far downstream of 2.75 and -3.59 (at $r = r_i$ and r_o respectively) obtained from equation (6-40) agree well with the computed results.

For a large range of values of α and $(r_o - r_i)/R$ the effect of the inlet and outlet transition regions on the downstream values of $\phi - t$ may be neglected without appreciable error.

6.2.3 The Circular Cylinder

To illustrate another method of determining the drift function, t, consider the flow about a circular cylinder of radius a.

Let us suppose that in the flow upstream of the cylinder an infinite line of fluid at right angles to the motion is marked at some one instant so that the particles are made visible and their subsequent motion may be followed. Such a mark might be made in water by a line of dye of a line of bubbles produced by electrolysis. The time taken for a particle to travel from a position x_o far upstream to x is given by

$$t = \int_{x_o}^{x} \frac{dx}{\partial\phi/\partial x} \tag{6-41}$$

where ϕ is the potential function for a two-dimensional flow about the cylinder and the integral is taken along a streamline.

During this time a particle on the same marked fluid line but an infinite distance from the cylinder, i.e. at $y = \pm\infty$ will reach the point given by

$$x_\infty = x_o + t,$$

if the velocity at infinity is unity.

The relative displacement in the x direction of the two particles on the same fluid line is then

$$X = x_\infty - x = t - (x - x_o) = \int_{x_o}^{x} \left(\frac{1}{\partial\phi/\partial x} - 1\right) dx. \tag{6-42}$$

Note that the sign of X is chosen so that in motion from left to right X will be positive, if $x_\infty > x$.

Now as pointed out in Section 3.1, t may be modified by an arbitrary constant. Let us do this in such a way that x_o is eliminated from equation (6-42), i.e. write $t = t' - x_o$. Physically the substitution means that instead of measuring the time t from the moment the fluid passes the line $x = x_o$, we now measure the time t' from the moment when the fluid line *would* have passed the line $x = 0$, if it had remained undisturbed by the cylinder

(Lighthill[8]). On this basis we redefine the drift function t by adding x_o to the right hand side of equation (6-41) so that

$$t = x + \int_{x_o}^{x} \left(\frac{1}{\partial\phi/\partial x} - 1\right) dx = x + X, \tag{6-43}$$

where $x_o \to -\infty$, and the integral is taken along a streamline.

Darwin[18] called the relative displacement, X, the *drift* and used the following method to calculate it.

Suppose that the whole system is given a unit velocity in the negative direction of x so that the cylinder moves at unit velocity and the fluid is at rest at infinity. Now the relative displacement or drift remains the same when a steady velocity is superimposed on the system, so that, by tracing the motion of a particle of fluid as the cylinder moves past it, we can find X which is the displacement of the fluid particle with reference to the fluid at rest at infinity.

Consider the instant when the centre of the moving cylinder is at the origin of a co-ordinate system fixed in space. The complex potential of the flow is at that instant

$$w = \frac{a^2}{z} \tag{6-44}$$

and the velocity is

$$\frac{dw}{dz} = -\frac{a^2}{r^2}e^{-i2\theta} \tag{6-45}$$

This is the absolute velocity at a fixed point in space whose coordinates are r and θ with respect to the moving axes at the instant of time considered. The slope of the path of a particle of fluid disturbed by the cylinder is then

$$\frac{dy}{dx} = -\frac{dy}{dX} = \tan 2\theta, \tag{6-46}$$

where the co-ordinate y of the particle is the same as the co-ordinate on the streamline moving past the fixed cylinder, since the displacement y is not affected by the superimposed velocity. The stream function in the flow past the cylinder is

$$\psi = y\left(1 - \frac{a^2}{r^2}\right) = y\left(1 - \frac{a^2\sin^2\theta}{y^2}\right),$$

so that

$$2y = \psi + \sqrt{\psi^2 + 4a^2\sin^2\theta}, \tag{6-47}$$

and

$$dy = \frac{a^2\sin 2\theta\, d\theta}{\sqrt{\psi^2 + 4a^2\sin^2\theta}} \tag{6-48}$$

Substituting for dy in equation (6-46) we find that the rate of increase of drift X with θ as the particle travels along a streamline is given by

$$dX = -\frac{a^2 \cos 2\theta \, d\theta}{\sqrt{\psi^2 + 4a^2 \sin^2 \theta}} \tag{6-49}$$

To determine X we integrate from far upstream where $\theta = \pi$ to θ. Or, if we write $\theta' = \quad - \pi/2$, over the range from $\theta' = \pi/2$ far upstream to $\theta' = -\pi/2$ far downstream, we find

$$X = \frac{1}{2} ka \int_{\pi/2}^{\theta'} \frac{\cos 2\theta' \, d\theta'}{\sqrt{1 - k^2 \sin^2 \theta'}} \quad , \tag{6-50}$$

where

$$k = 2a/\sqrt{\psi^2 + 4a^2} \; .$$

Equation (6-50) may be solved in terms of elliptic functions by writing $\mathrm{sn}u = \sin \theta'$ where

$$u = \int_0^{\theta'} \frac{d\theta'}{\sqrt{1 - k^2 \sin^2 \theta'}}$$

Then

$$X = \frac{ka}{2} \int_{\theta'=\pi/2}^{\theta'} \frac{d(\mathrm{sn}u \cdot \mathrm{cn}u)}{du} = \frac{a}{k} \left[E(\theta') - (1 - \frac{k^2}{2}) u(\theta') \right]_{\pi/2}^{\theta'} , \tag{6-51}$$

where the whole motion ranges from

$$u = K, \; \theta' = \pi/2, \; \theta = \pi,$$

to $\quad u = -K, \; \theta' = -\pi/2, \; \theta = 0,$

where K is the complete integral of the first kind,

$$K = \int_0^{\pi/2} \frac{d\theta'}{\sqrt{1 - k^2 \sin^2 \theta'}}$$

and $\quad E(\theta') = \int_0^{\theta'} \sqrt{1 - k^2 \sin^2 \theta'} \, d\theta' \quad .$

Equation (6-47) may be written

$$y = \frac{a}{k} (k' + \mathrm{dn}\, u) \; , \tag{6-52}$$

where $k'^2 = 1 - k^2$.

Writing $x = r \cos \theta = y \cot \theta = -y \tan \theta'$, we find that

$$\begin{aligned} t &= x + X \\ &= X - \frac{a}{k} \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u} (k' + \mathrm{dn}\, u). \end{aligned} \tag{6-53}$$

Now

$$\begin{aligned} \phi - t &= \phi - x - X \\ &= \frac{a^2 \cos \theta}{r} - X \\ &= -\left\{ X + ak \frac{\mathrm{sn}\, u \; \mathrm{cn}\, u}{k' + \mathrm{dn}\, u} \right\} . \end{aligned} \tag{6-54}$$

Equations (6-51), (6-52) and (6-54) are given by Darwin[18] with changes of sign consequent on the different direction of motion used here. Darwin also points out that

$$\int_{-\infty}^{+\infty} X_\infty \, dy = \pi a^2, \tag{6-55}$$

where X_∞, the total drift, is the value of X far downstream. The integral is the *drift-volume* (an area in two dimensions) or the volume enclosed between the final and initial dyed surfaces on passage of the body through a plane of dyed particles initially normal to the direction of motion. Darwin[18] shows that the drift-volume is equal to the volume corresponding to the *hydrodynamic mass*. Lines of constant t and ϕ - t for the flow about a circular cylinder are shown in Figures 6.5 and 6.6 (Rowe[87], Lighthill[8]).

6.2.4 Other Flows

Flows about several other bodies have been considered by Rowe[87]. Strut shapes for which t and ϕ - t have been computed include a symmetrical Joukowsky profile, an ellipse, a bi-convex or lens shaped strut consisting of two circular arcs, and a bicusped profile. The stream and potential functions were obtained either by conformal transformation from the flow about a circular cylinder or from known results. Computations for t were performed on a computer and the output from the computer fed directly to a plotting table on which diagrams of the type shown in Figures 6.5 and 6.6 could be obtained directly.

The work was extended to include a Joukowsky airfoil with lift and a cascade derived from Merchant and Collar's[88] transformations.* The behavior of the functions t and ϕ - t for a body with circulation will be considered in Section 6.6.

In the flow through enlargements and contractions and cascades the primary flow velocity, q_∞, far downstream will not be the same as the velocity far upstream, which has been assumed to be unity.

Hence far downstream

$$\frac{\partial(\phi - t)}{\partial x} = \frac{\partial\phi}{\partial x} - \frac{1}{\partial\phi/\partial x} = q_\infty - \frac{1}{q_\infty} \quad . \tag{6-56}$$

Therefore ϕ - t increases continuously as the flow passes downstream and the spanwise component of velocity (ϕ - t)U' also increases. The generation of this spanwise velocity is the result of a spanwise pressure gradient far downstream which, unless affected by the presence of walls, will produce an infinite spanwise velocity when acting on the fluid for an infinite time. The result can be adjusted to suit more realistic downstream boundary conditions by adding a velocity grad ϕ'. This problem is considered in Sections 6.4.1 and 6.6.2.

* See Figures 6.14, 6.15, 6.16, 6.17.

6.3 Secondary Vorticity for Plane Primary Flows

For plane primary flows the components of vorticity derived from equation (2-25),

$$\underline{\Omega} = - \text{grad } t \times \text{grad } po/\rho \quad , \tag{2-25}$$

are to order (ε) given by

$$\Omega_z = 0$$

$$\Omega_n = U' \frac{\partial t}{\partial s} = \frac{U'}{q} = \frac{UU'}{V}$$

$$\Omega_s = - U' \frac{\partial t}{\partial n} = - U'q \frac{\partial t}{\partial \psi} \quad , \tag{6-57}$$

where in the last set of equations t and ψ are the drift and stream functions for the plane flow with unit upstream velocity, $\underline{s}$ and $\underline{n}$ are unit vectors lying in the plane along and normal to the directions of the velocity, V = Uq where q is the velocity in the plane flow with unit upstream velocity and U' is the gradient of the upstream velocity normal to the plane.

As $\quad t = \int^{\phi}_{\psi=\text{const}} \frac{d\phi}{q^2} \quad ,$

$$\frac{\partial t}{\partial \psi} = - 2 \int^{\phi} \frac{d\phi}{q^3} \left(\frac{\partial q}{\partial \psi}\right) = - 2 \int \frac{ds}{q^3} \left(\frac{\partial q}{\partial n}\right)$$

$$= - 2 \int \frac{d\theta}{q^2} \left(\frac{R}{q} \frac{\partial q}{\partial n}\right) \quad . \tag{6-58}$$

By differentiating Bernoulli's equation,
$p + \frac{1}{2} \rho V^2 = p_o$, we obtain

$$\frac{1}{\rho} \frac{\partial p}{\partial n} + V\frac{\partial V}{\partial n} = \frac{1}{\rho} \frac{\partial p_o}{\partial n} = 0 \quad ,$$

since the stagnation pressure is constant in a Bernoulli plane. Hence

$$V\frac{\partial V}{\partial n} = - \frac{1}{\rho} \frac{\partial p}{\partial n} = \frac{V^2}{R} \quad , \tag{6-59}$$

where R is the radius of curvature of the streamline*. Therefore,

$$\frac{R}{q} \frac{\partial q}{\partial n} = 1 \quad ,$$

$$\frac{\partial t}{\partial \psi} = - 2 \int \frac{d\theta}{q^2}$$

and $\quad \Omega_s = 2U'q \int_{\psi=\text{const}} \frac{d\theta}{q^2}$

or

$$\frac{\Omega_s}{U'} = 2\ UV \int_{\psi=\text{const}} \frac{d\theta}{V^2} \ . \tag{6-60}$$

Squire and Winter[4] point out that for the orthogonal coordinate system ϕ, ψ, and z the scale factors h_1, h_2, h_3 are given by $h_1 = \partial s/\partial \phi$, $h_2 = \partial n/\partial \psi$,

* $\underline{s} \times \underline{n} = \underline{k}$ is the unit vector in the z direction, and $Rd\theta = ds$, so that when R is positive $\underline{n}$ is directed towards the center of curvature of the streamline.

$h_3 = 1$,
hence $h_1 = h_2 = h = 1/q$,
and

$$\frac{\partial(h\Omega s)}{\partial\phi} = U'h^4 \frac{\partial}{\partial\psi}\left(\frac{1}{h^2}\right). \tag{6-61}$$

Equation (6-60) may also be obtained from equation (2-49) which gives the growth of secondary vorticity along a streamline, viz:-

$$[\Omega_s/\rho V]_1^2 = -2\int_1^2 \{|\text{grad } po/\rho| \quad \sin\alpha/\rho V^2\}\, d\theta \tag{2-49}$$

As grad po/ρ is of order ε, the remaining terms on the right hand side must be of order unity and are therefore to be determined from the primary flow. For plane primary flows $\alpha = -\pi/2$ and grad $po/\rho = UU'$ are both constant along a streamline, so that by inspection equation (2-49) reduces to equation (6-60). The evaluation of Ω_s from equation (6-60) can be performed by straightforward numerical computation from known values of the complex potential of plane flows. Values for the flow about a circular cylinder are shown in Fig. 6.7 (Hawthorne[22]).

At stagnation points where q approaches zero the value of Ω_s approaches infinity so that in the flow about a circular cylinder the value of Ω_s on the stagnation streamline reaches infinity both at the leading edge and at the trailing edge. The effects at the leading and trailing stagnation points do not cancel because the changes of θ in the neighbourhood of both stagnation points are of the same sign.

At sharp corners in the flow where q approaches infinity the contribution to Ω_s approaches zero.

6.3.1 The Inverse Hodograph

The transformation into the inverse hodograph plane is given by

$$Z = dz/dw = e^{i\theta}/q$$

When the streamlines are mapped in the Z- plane the radius from the origin to a point on a streamline is inversely proportional to the velocity at the point. As the tip of the radius from the origin moves along a streamline, the radius sweeps over an area which is given by

$$A = \frac{1}{2}\int_{\psi=\text{const}} d\theta/q^2, \tag{6-63}$$

substituting in equation (6-60) we find that

$$\Omega_s = 4U'qA. \tag{6-64}$$

To illustrate the application of this transformation, we apply it to the flow in an angled bend (Section 6.21). From equation (6-15) we find that

$$Z = dz/dw = e^{i\alpha} \zeta^{-\alpha/\pi}. \tag{6-65}$$

On eliminating ζ from equation (6-14) and equation (6-65) we find that the complex potential is given by

$$w = \ln \frac{1 - Z^{\pi/\alpha}}{1 + Z^{\pi/\alpha}}. \tag{6-66}$$

The inverse hodograph is shown in Fig. 6.1. The corner D is at the origin, B is at infinity and A and C are at unit distances from the origin. By writing $2i\psi = w - \bar{w}$ we find after some simplification that

$$\psi = \tan^{-1} \frac{2q^{\pi/\alpha} \sin \frac{\pi\theta}{\alpha}}{1 - q^{2\pi/\alpha}}, \tag{6-67}$$

so that

$$\left(\frac{1}{q}\right)^{\pi/\alpha} = \cot\psi \sin\frac{\pi\theta}{\alpha} + \sqrt{1 + \cot^2\psi \sin^2\frac{\pi\theta}{\alpha}}. \tag{6-68}$$

Inspection of equation (6-67) shows that ψ is negative for $q < 1$ and is zero along AD and DC, Fig. 6.1. At $q = 1$, $\psi = -\pi/2$ for all θ. Hence for $q > 1$ ψ remains $< -\pi/2$ and becomes $-\pi$ along AB and BC. Hence the area swept out by the radius from the origin along the wall ADC, $\psi = 0$, is zero and the area along the wall ABC, $\psi = -\pi$, is infinite.

We conclude that Ω_s is everywhere zero along wall ADC. It is zero along wall AB but infinite along wall BC. Furthermore we conclude from equation (6-57) that $\partial t/\partial\psi$ is also zero along walls AD, DC and AB. Hence the lines of constant t and $\phi - t$ approach these walls at the same angle as the potential lines, i.e. normally. Along wall BC $\partial t/\partial\psi$ is infinite and the lines of constant t and $\phi - t$ approach the wall tangentially. For right angled bends, $\alpha = \pi/2$,

$$\int_0^{\theta} d\theta/q^2 = \int_0^{\theta} \{\cot\psi \sin 2\theta + \sqrt{1 + \cot^2\psi \sin^2\theta}\}\, d\theta$$

$$= \frac{1}{2}\cot\psi\,(1 - \cos 2\theta) + \operatorname{cosec}\psi \int_0^{\theta} \sqrt{1 - \cos^2\psi \cos^2 2\theta}\; d\theta$$

$$= \frac{1}{2}\cot\psi\,(1 - \cos 2\theta) - \frac{1}{2}\operatorname{cosec}\psi \int_{\pi/2-2\theta}^{\pi/2} \sqrt{1 - \cos^2\psi \sin^2\phi}\; d\phi, \tag{6-69}$$

where $\phi = \frac{\pi}{2} - 2\theta$

The last term in the equation is an elliptic integral of the second kind. Along the dividing streamline where $\psi = -\pi/2$, $q = 1$ the value of

Ω_s is given by

$$\Omega_s = 2 U' \theta.$$

Fig. 6.8(a) shows diagrammatically the inverse hodograph for the flow about the top half of a circular cylinder in the physical plane. The streamlines cover the entire plane to the right of $X = \frac{1}{2}$ and form closed curves beginning and ending at $X = 1$, $Y = 0$. The complex potential for the flow in the Z plane is given by

$$w = a\sqrt{\frac{Z}{Z - 1}} + a\sqrt{\frac{Z - 1}{Z}}$$

where a is the radius of the circle. The areas swept out by the radius from the origin increase towards infinity as ψ approaches zero and hence Ω_s approaches infinity as the stagnation streamline is approached.

A profile for which Ω_s is to remain small must have curves of constant ψ which enclose small areas on the Z plane. A circle of radius a with its center at Z(+1, 0) is an example of a stagnation streamline, $\psi = 0$, which would lead to small values of Ω_s when all the flow is mapped inside the circle, Fig. 6.8(b). The complex potential for such a flow[22] is

$$w = \sqrt{(Z - 1)/a} + \sqrt{a/(Z - 1)}$$

The transformation is

$$\frac{dz}{dZ} = Z \frac{dw}{dZ},$$

from which it may be shown[22] that the profile is bicusped with coordinates

$$\frac{x}{c} = \cos \frac{\gamma}{2} \left(\frac{1}{2} - \frac{y_m}{c} \sin^2 \frac{\gamma}{2}\right),$$

$$y/y_m = \sin^3 \gamma/2,$$

where c is the chord, y_m is half the thickness and γ is the angle shown in Fig. 6.8(b) and may be regarded as a parameter.

Expressions for and numerical examples of the surface pressure coefficient and downstream secondary vorticity are given in reference[22].

6.3.2 The Downstream Secondary Vorticity.

It has been shown that the value of $\Omega_{s\infty}$, the value of the secondary vorticity far downstream, may be obtained from the total area enclosed by the curve ψ = constant on the inverse hodograph plane.

For a right-angled bend for instance the value of Ω_s is obtained by writing $\theta = \pi/2$ in equation (6.69), so that

$$\Omega_{s\infty} = - 2 U' \{\cot \psi - \operatorname{cosec} \psi\, E(\cos \psi)\}, \tag{6-70}$$

where E is the complete elliptic integral of the second kind, whose value is unity at $\psi = 0$ and $-\pi$, and $\pi/2$ at $\psi = -\pi/2$.

For bent ducts whose walls are circular arcs the velocities on the inner and outer walls are given approximately by equation (6.37). The secondary vorticity on the walls far downstream of such a bend may then be obtained by substitution in equation (6.60) and gives approximately

$$\Omega_{s\infty} = 2\,U' \frac{r^2}{R^2} \int_0^{\alpha} \{1 - (\pm 1)\, 2k_1\, A_1 \cosh k_1\, R(\theta - \tfrac{\alpha}{2})\}\, d\theta,$$

where the positive sign is taken for the inner radius and the negative sign for the outer radius. On integration and substitution from equation (6-35) the expression becomes

$$\Omega_{s\infty} = 2\,U' \frac{r^2}{R^2} \left\{ \alpha \pm \frac{2}{1 + \coth k_1 R\alpha/2} \; \frac{(r_o - r_i)^2}{\pi R^2} \right\}. \qquad (6\text{-}71)$$

As in section 6.22 we conclude that for a large range of values of α and $(r_o - r_i)/R$ the correction for the effect of the inlet and outlet transitions may be neglected.

Values of $\Omega_{s\infty}/U'$ obtained from equations (6-70) and (6-71) are shown in Fig. 6.9. The curves for the circular bend are drawn for ratios of the arithmetic mean radius to bend width of 1 and 3.

Values of the downstream vorticity for several strut shapes are shown in Fig. 6.10 which is reporduced from reference[22]. For all shapes with stagnation points at the leading or trailing edges $\Omega_{s\infty}$ approaches infinity. For the bicusped shape described in Section 6.31 $\Omega_{s\infty}$ remains finite throughout the flow.

The secondary *circulation* about a streamtube is the product of Ω_s and the cross-sectional area of the tube. For the bundle of tubes between ψ_1 and ψ_2 whose cross-section is of unit height in the z or spanwise direction the secondary circulation is given by

$$\Gamma_s = \int_{\psi_1}^{\psi_2} \Omega_s\, dn,$$

and on substitution for Ω_s from equation (6-57) we find that

$$\Gamma_s/U' = -\,[t]_{\psi_1}^{\psi_2}. \qquad (6\text{-}72)$$

For a strut the total secondary circulation far downstream between y = 0 (the dividing streamline) and y = ∞ may be deduced from equation (6-43) and is

$$\Gamma_{s\infty}/U' = -\int_0^{\infty} X_\infty\, dy = -\,X_{0\infty},$$

where $X_{O\infty}$ is the total drift on the dividing streamline. Now an infinite value of $X_{O\infty}$ implies infinite spanwise velocities on the dividing streamline and an infinite value for $\Gamma_{S\infty}$. Hence for corners which are right-angled or acute and for bodies with blunt leading edges $\Gamma_{S\infty}$ is infinite because t becomes infinite on the bounding or dividing streamline. For bodies with wedge shaped leading edges or corners with included angles between $\pi/2$ and π, although $\Omega_{S\infty}$ becomes infinite on the bounding streamline because of the stagnation point, the values of $\Gamma_{S\infty}$, t and the secondary velocities remain finite.

These results are summarised in the following table:

Table I Corner and leading edge effects

Included angle of corner	$\leq \frac{\pi}{2}$	$> \frac{\pi}{2}$	Gradual
Leading edge	Blunt	Wedge	Cusped
$\phi - t$, $\Gamma_{S\infty}$	Infinite	Finite	Finite
$\Omega_{S\infty}$ on bounding Streamline	Infinite	Infinite	Finite

6.4 Secondary Velocities Far Downstream for Plane Primary Flows.

After passing over a body or round a bend the primary flow streamlines eventually become parallel and the secondary vorticity ceases to change. When the primary flow far downstream has the same velocity as far upstream, the only difference between downstream and upstream flows is a consequence of the persistence of a secondary vorticity component $\Omega_{S\infty}$ in the downstream flow. [When the primary flow velocity downstream is q_∞, equation (6.57) shows that the vorticity normal to the flow becomes U'/q_∞ and the spanwise pressure gradients and flows thereby introduced have to be taken into account (section 6.24).]

One method of calculating the secondary flow far downstream is, therefore, to calculate the velocities induced by the parallel secondary vortex lines. Far downstream the only velocities induced by the secondary vorticity will lie in a plane normal to the primary flow, so that the problem for solution is essentially a two-dimensional one.

If the secondary velocity components are v, w in a plane yz normal to the primary flow far downstream, we may write

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \Omega_{S\infty}. \qquad (6\text{-}73)$$

The component u in the primary stream direction x is zero so that continuity for the secondary velocity is satisfied by defining a stream function $\psi(y,z)$ such that

$$v = \psi_z, \; w = -\psi_y.$$

Equation (6-73) may then be written

$$\psi_{yy} + \psi_{zz} = -\Omega_{S\infty}. \qquad (6\text{-}74)$$

The secondary flow may then be determined from a solution of this Poisson's equation which satisfies the boundary conditions on the walls.

The total kinetic energy associated with the secondary motion is

$$T = \frac{1}{2}\rho\iint(\psi_y{}^2 + \psi_z{}^2)\,dy\,dz,$$

where the integrals are taken over the whole yz plane affected by the flow. By integration by parts and by setting $\psi = 0$ on the boundaries of the flow we find that

$$T = \frac{1}{2}\rho\iint\psi\Omega_{S\infty}\,dy\,dz. \qquad (6\text{-}75)$$

This energy may be regarded as a source of induced drag.

The secondary velocities can also be obtained from equation (6-6).

$$\underline{V} = U\,\text{grad}\,\phi + \underline{k}\,(\phi - t)U' + \text{grad}\,\phi', \qquad (6\text{-}6)$$

from which we deduce that far downstream

$$\Omega_{S\infty} = \text{curl } \underline{k} \, (\phi - t)U' = -U' \frac{\partial t}{\partial y}$$

which could also be obtained from equation (6-57). There may be instances for which a solution based on equation (6-6) is easier to obtain than one based on equation (6-74). We have already noted in equation (6-7) the simplicity of the simple shear case for which ϕ' is zero. Another class of examples is when the upstream velocity U is linear in z or is made up of regions in each of which the velocity is linear in z. In any such region equation (6-5) becomes

$$\nabla^2\phi' = -(\phi - t)U'' = 0.$$

Across the boundary $z = a$ between two such regions in which $U' = U'_1$ for $z < a$ and $U' = U'_2$ for $z > a$ we may write

$$U' = U'_1 + H(z - a)\,(U'_2 - U'_1),$$

where $H(z - a)$, equal to 1 for $z > a$ and 0 for $z < a$, is Heaviside's unit function(86). Then

$$\begin{aligned}\nabla^2 \phi' &= -(\phi - t)\,(U'_2 - U'_1)\frac{d}{dz} H(z - a)\\ &= (\phi - t)\,(U'_1 - U'_2)\,\delta(z - a),\end{aligned}$$

where δ is Dirac's delta function. The potential ϕ' is therefore that due to a sheet of sources of strength $(\phi - t)(U'_1 - U'_2)$ situated at $z = a$. In this way expressions for ϕ' may be obtained to satisfy differing upstream conditions of constant shear regions.

In particular the value of ϕ' for the flow between walls at $z = 0$ and ℓ, when the value of U' is constant, may be shown to be that due to sheets of sources at $z = 0, \pm 2n\ell$, where $n = 1, 2, 3$ etc., of strengths $-2(\phi - t)U'$ and sheets of sinks of the same strength at $z = \ell, \ell \pm 2n\ell$.

Far downstream where the flow is two-dimensional the complex potential for such an array of sources and sinks may be obtained as follows. The complex potential of rows of point sources of equal strength m at $z = 0$, $\pm 2n\ell$ is (Lamb(2) p.71)

$$W' = \frac{m}{2\pi} \ln \sinh \frac{\pi Z}{2\ell}$$

where $Z = y + iz$.

For the array described above when the flow extends from $y = -\infty$ to $+\infty$, as for instance downstream from a strut,

$$W' = \int_{-\infty}^{+\infty} \frac{U'T(\eta)}{\pi} \quad \ell n \; \frac{\sinh \; (Z - \eta)/2\ell}{\sinh \pi(Z - \eta - i\ell)/2\ell} \; d\eta$$

where $T(\eta) = -(\phi - t)(\eta)$.

Therefore,

$$W' = \int_{+\infty}^{-\infty} \frac{U'T(\eta)}{\pi} \; \ell n(i \tanh \pi(Z - \eta)/2\ell) \; d\eta \tag{6-76}$$

and

$$\frac{dW'}{dZ} = v - iw$$

$$= \int_{+\infty}^{-\infty} \frac{U'T(\eta)}{\ell} \; \operatorname{cosech} \frac{\pi}{\ell} (Z - \eta) \; d\eta \tag{6-77}$$

In the following sections these methods of calculating downstream secondary flows will be illustrated with examples. Airfoils and cascades will be considered in section 6.6

6.4.1 Rectangular bends

In the straight duct downstream from a bend let the walls be at $z = 0$ and ℓ and $y = 0$ and 1.

We may write equation (6-74) in the form

$$\psi_{yy} = \psi_{zz} = -\left(\frac{\Omega_{s\infty}}{U'}\right) \; U', \tag{6-78}$$

where $(\Omega_{s\infty}/U')$ is a function of y only for plane primary flows and U' is a function of z.

For bends in which ℓ is large it is convenient to express $(\Omega_{s\infty}/U')$ by a Fourier sine series

$$\frac{\Omega_{s\infty}}{U'} = \sum_{n=1}^{\infty} c_n \sin n\pi y. \tag{6-79}$$

Then write $\psi = \Sigma \; \psi_n \; (z) \sin n\pi y$,

so that

$$\psi_n'' - n^2\pi^2\psi_n = -c_n U'.$$

By the method of variation of parameters and with a choice of limits to make $\psi_n = 0$ at $z = 0$ we find for the solution

$$\psi_n = a_n \sinh n\pi z - \frac{c_n}{n\pi} \int_0^z \frac{dU(t)}{dt} \; \sinh n\pi(z - t) \; dt,$$

where U(t) is the upstream velocity with the parameter t replacing z.

The constant a_n is determined from the condition that $\psi = 0$ at $z = \ell$.

Hence

$$a_n = \frac{c_n}{n\pi} \int_0^\ell U' \; \frac{\sinh n\pi(\ell - t)}{\sinh n\pi\ell} \; dt.$$

After some rearrangement the expression for ψ_n becomes

$$\psi_n(z) = \frac{c_n}{n\pi \sinh n\pi\ell} \left[\sinh n\pi z \int_z^\ell U' \sinh n\pi(\ell - t)\, dt + \sinh n\pi(\ell - z) \int_0^z U' \sinh n\pi t\, dt\right], \tag{6-80}$$

and

$$\psi'_n(z) = \frac{-c_n}{\sinh n\pi\ell} \left[\cosh n\pi(\ell - z) \int_0^z U' \sinh n\pi t\, dt - \cosh n\pi z \int_z^\ell U' \sinh n\pi(\ell - t)\, dt\right]. \tag{6-81}$$

When $\ell < 1$ a more rapidly convergent series may be obtained by writing

$$U' = \sum_{n=1}^{\infty} b_n \sin n\pi\frac{z}{\ell}, \tag{6-82}$$

and

$$\psi = \sum_{n=1}^{\infty} \psi_n(y) \sin \frac{n\pi z}{\ell} .$$

By an analogous process we find that

$$\psi_n(y) = \frac{-\ell b_n}{n\pi \sinh n\pi/\ell} \left[\sinh (n\pi y/\ell) \int_y^1 \left(\frac{\Omega_{s\infty}}{U'}\right) \sinh \frac{n\pi}{\ell}(1 - t)\, dt + \sinh \frac{n\pi(1 - y)}{\ell} \int_0^y \left(\frac{\Omega_{s\infty}}{U'}\right) \sinh \frac{n\pi t}{\ell}\, dt\right]. \tag{6-83}$$

When U is not constant in any plane z = constant, as for instance when there are boundary layers on the walls $y = 0$ and 1 approaching the bend, the primary flow may still be regarded as a plane primary flow, but the variation of U' with y will have to be allowed for in the solution of the Poisson's equation.

For bent ducts whose walls are circular arcs the value of $\Omega_{s\infty}$ may be obtained from equation (6-71) and for a large ratio of mean radius to duct width is approximately

$$\frac{\Omega_{s\infty}}{U'} = 2\alpha \frac{r^2}{R^2}, \tag{6-84}$$

or when $R >> 1$

$$\frac{\Omega_{s\infty}}{U'} = 2\alpha. \tag{6-85}$$

In this latter instance the value of c_n in equation (6-79) is given by

$$c_n = 8\alpha/n\pi \tag{6-86}$$

for n odd. It is zero for n even.

To use the more exact equation (6-84) it is necessary to relate the value of $\Omega_{s\infty}$ to the appropriate streamline in the downstream flow. From equations (6-30) and 6-31) we note that

$$y = R\ln r/r_i ,$$

where a value of $r_o - r_i = 1$ is to be assumed. Substituting for r in equation (6-84) we find that

$$\frac{\Omega_{s\infty}}{U'} = 2\alpha \frac{r_i^{\,2}}{R^2} e^{2y/R} \tag{6-87}$$

and

$$c_n = \frac{4n \ \pi\alpha r_i^{\,2}}{4 + n^2\pi^2R^2} \left(1 - (-1)^n e^{2/R}\right). \tag{6-88}$$

When the cross-sectional area of the bent channel is changed so that the flow passes through an enlargement or contraction as well as a bend, the vorticity normal to the flow far downstream becomes U'/q_∞, equation (6-57). Other components of velocity u and w may therefore be superimposed on the flow. They are related to the vorticity by the equation

$$\frac{\partial}{\partial z} (q_\infty U + u) - \frac{\partial w}{\partial x} = U'/q_\infty ,$$

or

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{1 - q_\infty^{\,2}}{q_\infty} U' \tag{6-89}$$

In section 6-24 it was shown, equation (6-56) that the spanwise velocity component for the simple shear case is

$$U'(\phi - t) = \frac{q_\infty^{\,2} - 1}{q_\infty} U' x \quad .$$

This result corresponds to the assumption that $u = 0$ in equation (6-89). As suggested in section (6-24) the result can be modified by adding a velocity grad ϕ'.

If we set

$$\phi' = x\, U \frac{1 - q_\infty^{\,2}}{q_\infty} \tag{6-90}$$

then we reduce the spanwise velocity to zero and find

$$u = U\left(\frac{1 - q_\infty^{\,2}}{q_\infty}\right) \tag{6-91}$$

which is equivalent to setting $w = 0$ in equation (6-89).

For flows between walls it is more realistic to assume that the flow velocities are independent of x far downstream so that equation (6-89) becomes

$$\frac{du}{dz} = \frac{1 - q_\infty^2}{q_\infty} U'.$$

We must also satisfy the continuity for the total flow in the duct by writing

$$\int_0^\ell U\,dz = \frac{1}{q_\infty} \int_0^\ell (q_\infty U + u)\,dz,$$

so that

$$u = \frac{1 - q_\infty^2}{q_\infty} \{U - \int_0^\ell U\,d(z/\ell)\} \qquad (6\text{-}92)$$

As U departs by order (ε) from its average value, u is also of order (ε).

6.4.2 Struts

When a strut is bounded by walls at $z = 0$ and ℓ the value of $(\Omega_{s\infty}/U') = g(y)$ falls from a maximum at $y = 0$ to zero at $y = \infty$. Furthermore by symmetry $g(y) = -g(-y)$. It is appropriate therefore to apply the Fourier sine transform to equation (6-74) to obtain

$$\overline{\psi}_{zz} - k^2\,\overline{\psi} = -U'\,\overline{g}, \qquad (6\text{-}93)$$

where

$$\overline{g}(k) = \frac{2}{\pi} \int_0^\infty g(y) \sin ky\,dy$$

and $g(y) = \int_0^\infty \overline{g}(k) \sin ky\,dk$

with similar expressions for $\overline{\psi}(k,z)$. We note that the boundary conditions for ψ at $y = 0$ and ∞ are satisfied by this representation.

A solution for equation (6-93) satisfying the boundary conditions at $z = 0$ and ℓ is given by equation (6-80) in which $n\pi$ is replaced by k and c_n by $\overline{g}$. The difficulty of inverting the transform will depend on the character of g(y).

For the flow past a bicusped profile* it has been shown in reference (22) that it is a good approximation to write

$$g(y) = A\,e^{-my} \qquad (6\text{-}94)$$

where

$$A = 36\pi\left(\frac{y_{m/c}}{1 + y_m/c}\right)^2$$

and

$$m = \frac{15\pi}{4c(1 + y_m/c)}$$

where y_m is half the maximum thickness and c is the chord of the strut.

* i.e. the profile described in section 6.3.1.

Then

$$\bar{g}(k) = \frac{2Ak}{\pi(m^2 + k^2)},$$

which when substituted in equation (6-80) with appropriate changes yields

$$\bar{\psi}(k,z) = \frac{2A}{\pi(m^2 + k^2)} \left| \frac{\sinh kz}{\sinh k\ell} \int_z^\ell U' \sinh k(\ell - t)\, dt + \frac{\sinh k(\ell - z)}{\sinh k\ell} \int_0^z U' \sinh kt\, dt \right| \qquad (6\text{-}95)$$

As $|k| \to \infty$ it is found by inspection that $\bar{\psi}(k,z)$ tends to zero uniformly. Above the real axis of k there is a pole at $k = + im$, and there are singularities at $\sinh k\ell = 0$. As $|k|$ approaches zero $\bar{\psi}(k,z)$ remains finite. Now

$$\psi(y,z) = \int_0^\infty \bar{\psi}(k,z) \sin ky\, dk,$$

and because $\bar{\psi}(k,z)$ is odd in k we may write

$$\psi(y,z) = \frac{1}{2i} \int_{-\infty}^{+\infty} \bar{\psi}(k,z)\, e^{iky}\, dk,$$

where the integral will be evaluated by contour integration in the upper half of the k plane.

The contribution due to the pole at $z = + im$ is

$$\psi_o = \frac{Ae^{-my}}{m \sin m\ell} \left[\sin mz \int_z^\ell U' \sin m(\ell - t)\, dt + \sin m(\ell - z) \int_0^z U' \sin mt\, dt \right] \qquad (6\text{-}96)$$

The singularities at $\sinh k\ell = 0$ turn out to be simple poles at $k = i\frac{n\pi}{\ell}$ provided $m \neq \frac{n\pi}{\ell}$, where n = 1, 2, 3 etc. The contributions due to these poles are

$$\psi_n = \frac{2Ae^{-n\pi y/\ell}}{m^2 - \frac{n^2\pi^2}{\ell^2}} \sin(n\pi z/\ell) \int_0^1 U' \sin(n\pi t/\ell) d(t/\ell). \qquad (6\text{-}97)$$

When U' = constant the result for the stream function becomes

$$\psi = \frac{AU'}{m^2} \left\{ e^{-my}(\cos mz - 1 + \tan \frac{m\ell}{2} \sin mz) + 4 \sum_{n=1,3}^{\infty} \frac{e^{-n\pi y/\ell} \sin n\pi z/\ell}{n\pi(1 - n^2\pi^2/\ell^2 m^2)} \right\}. \qquad (6\text{-}98)$$

Halfway between the walls on the dividing streamline i.e. at $y = 0$, $z = \ell/2$ the downwash velocity is given by

$$- \frac{AU'}{m} \left\{ (1 - \sec m\ell/2) + 4 \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n+1}{2}}}{\ell m(1 - n^2\pi^2/\ell^2 m^2)} \right\} \quad (6\text{-}99)$$

When $m = N\pi/\ell$, $\overline{\psi}(k,z)$ has a pole of order two at $k = + iN\pi/\ell$, and the contribution to [illegible] from this term is

$$\psi^* \ldots \frac{\ldots y/\ell}{N\pi} \Big[(y \sin N\pi z/\ell - z \cos N\pi z/\ell) \int_0^\ell U' \sin N\pi t/\ell \, dt$$

$$\ldots \ell \sin N\pi z/\ell \, (\int^\ell \ldots \cos N\pi t/ \ldots \int_0^\ell U' \, t/\ell \cos N\pi t/\ell \, dt)$$

$$+ \ell \cos N\pi z/\ell \int^{?} \ldots \sin N\pi t/\ell \ldots (\ell/2N\pi) \sin N\pi z/2 \int_0^\ell U' \sin N\pi t/\ell \, dt \Big] \quad (6\text{-}100)$$

The complete solution for $\psi(y,z)$ is then obtained by adding to equation (6-100) the remaining terms of the series given by equation (6-97). When U' is constant equation (6-100) becomes for N even

$$\psi^*(y,z) = \frac{AU' e^{-N\pi y/\ell}}{N^2\pi^2/\ell^2} (\cos \frac{N\pi z}{\ell} - 1)$$

and for N odd

$$\psi^*(y,z) = \frac{AU' e^{-N\pi y/\ell}}{N^2\pi^2/\ell^2} \left\{ \cos \frac{N\pi z}{\ell} - 1 + \frac{2}{N\pi} \left[(1 + \frac{N\pi y}{\ell}) \sin N\pi \frac{z}{\ell} - \frac{N\pi z}{\ell} \cos \frac{N\pi z}{\ell} \right] \right\} .$$

When U' = constant equation (6-95) becomes

$$\overline{\psi}(k,z) = \frac{2AU'}{\pi k(m^2 + k^2)} \left[(1 - \cosh kz) + \tanh k \frac{\ell}{2} \sinh kz \right]$$

Now the velocity $w = - \psi_y$.

$$= - \int_0^\infty k \, \overline{\psi} \cos ky \, dy.$$

$$= - \frac{1}{2} \int_{-\infty}^{+\infty} k \, \overline{\psi} e^{iky} \, dy.$$

Hence at $y = 0$ and $z = \ell/2$ the velocity becomes for $m = N\pi/\ell$ and U' = const

$$w = - \frac{AU'}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \operatorname{sech} k\ell/2}{(k^2 + \frac{N^2\pi^2}{\ell^2})} \, dk. \quad (6\text{-}101)$$

The first term makes the contribution $\frac{-AU'\ell}{\pi N}$. To evaluate the second term we note that

$$\frac{\operatorname{sech} k\ell/2}{k^2 + N^2\pi^2/\ell^2}$$

has simple poles at $k = i\, n\pi/\ell$ where n is an odd integer not equal to N, and another simple pole at $k = i\, N\pi/\ell$ when N is even. When N is odd the pole at $k = i\, N\pi/\ell$ is of order two.

The result for N even is

$$w = -\frac{AU'\ell}{\pi N}\left\{1 - (-1)^{N/2} + \sum_{n=1,3}^{\infty} \frac{4(-1)^{\frac{n+1}{2}}}{\pi N(1 - n^2/N^2)}\right\}, \quad (6\text{-}102)$$

as can be obtained from equation (6-98).

When N is odd

$$w = -\frac{AU'\ell}{\pi N}\left\{1 + \frac{(-1)^{\frac{N+1}{2}}}{\pi N} + \sum_{1,3}^{\infty} \frac{4(-1)^{\frac{n+1}{2}}}{\pi N(1 - n^2/N^2)}\right\} \quad (6\text{-}103)$$

where the term $n = N$ is ommitted from the series.

The velocity at $y = 0$ and $z = \ell/2$ may also be calculated from equations (6-6) and (6-77). We obtain $(\phi - t)$ from equations (6-94) and (6-57)

$$(\phi - t) = -\frac{A}{m} e^{-my},$$

hence

$$T(\eta) = \frac{A}{m} e^{-m|\eta|}.$$

Equation (6-77) yields at $y = 0$ and $z = \ell/2$

$$v - iw = -\frac{iU'}{\pi} \int_{-\infty}^{+\infty} \frac{T(\eta)}{\cosh \pi\eta/\ell}\, d(\pi\eta/\ell)$$

or

$$w = \frac{2AU'}{m\pi} \int_0^{\infty} \frac{e^{-m\eta}}{\cosh \frac{\pi\eta}{\ell}}\, d\left(\frac{\pi\eta}{\ell}\right) \quad (6\text{-}104)$$

To which must be added the term in $\underline{k}$ $(\phi - t)$ in equation (6-6) to give a total value for w at $y - 0$, $z = \ell/2$

$$w = -\frac{AU'}{m}\left(1 - \frac{2}{\pi}\int_0^{\infty} \frac{e^{-m\eta}}{\cosh \frac{\pi\eta}{\ell}}\, d\left(\frac{\pi\eta}{\ell}\right)\right) \quad (6\text{-}105)$$

The integral in equation (6-104) is a Laplace transform. Its value is

$$\int_0^{\infty} e^{-st} \operatorname{sech} t\, dt = \frac{1}{2}\left\{\psi\left(\frac{s+3}{4}\right) - \psi\left(\frac{s+1}{4}\right)\right\},$$

where $s = \frac{m\ell}{\pi}$ and ψ is the logarithmic derivative of the gamma function, i.e. $\psi(x) = \Gamma'(x)/\Gamma(x)$.

6.5 Secondary Velocities Throughout the Field of Flow.

The estimation of the secondary velocities throughout the field of flow is difficult except for the simplest of cases, such as in simple shear flow (section 6.1). One such example is the flow round circular arc bends for which the ratio of mean bend radius to duct width is large enough for equation (6-29) to apply and ϕ and t are linear in θ.

In these conditions the continuity equation (6-3),

$$\nabla^2 \phi' = -(\phi - t)\nabla^2 U + \text{grad } t . \text{ grad } U, \qquad (6\text{-}3)$$

shows that ϕ' also will be linear in θ. Hence the solution to equation (6-3) becomes a two dimensional problem similar to that described in section 6.4.1.

A formal solution in terms of a stream function is developed from equations (2-19) and (2-20). We write

$$\underline{V} = \text{grad } \phi + \text{curl } \underline{\psi}$$

where div $\underline{\psi} = 0$. Then

$$\text{curl } \underline{V} = -\nabla^2 \underline{\psi} = -\text{ grad } t \times \text{grad } \frac{po}{\rho} .$$

Now if the components of $\underline{\psi}$ in the cylindrical polar coordinate system r, θ, z are ψ_r, ψ_θ, ψ_z then

$$\frac{\partial^2 \psi_z}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_z}{\partial r} + \frac{\partial^2 \psi_z}{r^2 \partial \theta^2} + \frac{\partial^2 \psi_z}{\partial z^2} = -\Omega_z ,$$

$$\frac{\partial^2 \psi_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_\theta}{\partial r} + \frac{\partial^2 \psi_\theta}{r^2 \partial \theta^2} + \frac{\partial^2 \psi_\theta}{\partial z^2} - \frac{\psi_\theta}{r^2} = -\Omega_\theta ,$$

$$\frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_r}{\partial r} + \frac{\partial^2 \psi_r}{r^2 \partial \theta^2} - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial \psi_\theta}{\partial \theta} = -\Omega_r ,$$

and

$$\frac{1}{r}\frac{\partial r\psi_r}{\partial r} + \frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta} + \frac{\partial \psi_z}{\partial z} = 0 . \qquad (6\text{-}106)$$

For the simple circular arc bends described in sections 6.2.2 and 6.4.1, $\Omega_z = 0$, Ω_θ is linear in θ and Ω_r is not a function of θ. Hence ψ_z vanishes, ψ_r is invariant with θ and ψ_θ is linear in θ.

The velocities are, therefore, given by

$$V_r = -\frac{\partial \psi_\theta}{\partial z} , \quad V_z = \frac{1}{r}\frac{\partial r\psi_\theta}{\partial r}$$

and

$$V = \frac{\partial \psi_r}{\partial z} .$$

The velocities in the r, z plane may be expressed in terms of a stream function, ψ_0, which may be obtained from the solution of equation (6-106) and which satisfies the boundary conditions on the walls of the duct. When the bend radius is large the equation reduces to equation (6-74) with $\Omega_{s\infty}$ replaced by Ω_s.

The above solutions assume that the bend flow conditions are established. They cannot be applied within the transition regions at inlet and outlet from the bend where Ω_s no longer varies linearly with θ.

The determination of the velocity throughout the field of flow except for simple shear and the simple bend flow described above is difficult and no satisfactory solutions have yet been reported. In a constant shear flow with walls at $z = 0$ and ℓ the boundary conditions on the walls can be satisfied by the use of sheets of sources and sinks as described in section 6.4. But the solution for ϕ' must also satisfy the boundary conditions on the other walls. For instance in the flow about a circular cylinder the walls at $z = 0$ and ℓ may be represented by source sheets of strength $\pm 2(\phi - t)\, U'$, but as there is no simple way of reflecting point sources in a long cylinder, the satisfaction of the boundary condition for ϕ' on the surface of the cylinder is a matter of substantial difficulty. The problem is no easier with bodies of other shapes or with cascades.

6.6 Airfoils

For an airfoil whose span extends to infinity along the z axis in a simple shear flow the solution given in equation (6-7) applies. The only secondary velocity is in the spanwise direction and as the discussion following that equation shows the pressure distribution and lift vary as U^2.

Across the stagnation streamline there is a change in spanwise velocities, $(\phi - t)\, U'$, indicating the presence of a trailing vortex sheet of strength

$$[(\phi_+ - \phi_-) - (t_+ - t_-)]\, U',$$

where the subscripts + and - denote values of quantities close to and on opposite sides of the stagnation streamline in the primary flow leaving the trailing edge of the airfoil. In this nomenclature the lift acts in the direction of y increasing and the subscript + denotes values for which y is greater than zero.

Now the potential jump across the trailing stagnation streamline is related to the circulation by

$$\gamma = \phi_+ - \phi_- ,$$

where γ is the circulation round the airfoil for a potential flow with unit upstream velocity. We note that in the convention assumed here the positive direction for γ is <u>opposite</u> to the positive direction of a bound vortex lying along the z axis.

If we consider a fluid particle travelling along the stagnation streamline but dividing at the leading edge into one particle passing along the suction or upper surface ($y > 0$) and the other passing along the pressure or lower surface ($y < 0$), we find that owing to the higher velocity on the upper surface compared to the lower surface, the upper particle will reach the trailing edge in a time t_+ which is different from the time t_- taken for the lower particle to reach the trailing edge. This difference in time may be written as

$$t_+ - t_- = \oint \frac{ds}{q_s} \tag{6-107}$$

where q_s is the velocity on the surface of the airfoil and the integral is taken completely round the surface of the airfoil in the same direction as the circulation. Along the trailing stagnation streamline downstream there is no further change in $(t_+ - t_-)$.

Hence the strength of the trailing vortex sheet may be written

$$[(\phi_+ - \phi_-) - (t_+ - t_-)]\, U' = U'(\gamma - \oint \frac{ds}{q_s}) \tag{6-108}$$

The first term $U'\gamma$ is the circulation shed off the airfoil, the second term may be regarded as the contribution to the trailing vortex sheet from the vortex filaments originally transverse to the flow which, in flowing about the airfoil, become stretched along the stagnation streamline leaving the trailing edge. This component has been designated the <u>trailing filament circulation</u>[70]. Its sign is generally such that $(t_+ - t_-) < 0$.

It is of interest to compare this result with that obtained in the large shear, small disturbance approximation where the strength of the vortex sheet is given by equation (5-54) which in the notation of this section is written

$$w_+ - w_- = \frac{1}{\rho U} \frac{dL}{dz} \tag{5-54}$$

If U' is constant for $-\infty < z < +\infty$, the Trefftz plane velocities v_T and w_T are also independent of z. For the lifting line approximation we may write

$$L = \rho U^2 \gamma \ .$$

Hence equation (5-54) becomes

$$w_+ - w_- = 2U' \gamma \ ,$$

which when compared with equation (6-108) leads to the result

$$\oint \frac{ds}{q_s} = -\gamma .$$

This approximation relation may be verified by noting that for a thin airfoil $q_s = 1 + (q_s - 1)$, where $(q_s - 1)$ is small compared with unity. Then

$$\oint \frac{ds}{q_s} = \int_0^c [(1 - (q_{s+} - 1)) - (1 - (q_{s-} - 1))]\, ds$$

$$= - \int_0^c (q_{s+} - q_{s-})\, ds$$

$$= - \gamma \; ,$$

where q_{s+} and q_{s-} are the velocity on upper and lower surfaces respectively.

The net distributed secondary circulation taken far downstream over a slice in the yz plane of unit height in z direction and stretching from $y = -\infty$ to $+\infty$ is shown in reference (22) to be $U'(\gamma + \oint \frac{ds}{q_s})$ and the total circulation enclosed in the slice is $2\, U'\, \gamma$.

Wilson[80] has shown that for a thin, circular arc, airfoil at zero incidence

$$\oint \frac{ds}{q_s} = - \frac{4\pi \sin\beta}{\cos^4 \beta} \tag{6-109}$$

where the camber is given by $1/2 \tan \beta$, and the circulation round the airfoil by $4\pi \sin \beta$. Figures 6.14 and 6.15 show t and $(\phi - t)$ lines in the flow about a Joukowsky airfoil[87].

6.6.1 The Airfoil between Walls.

Suppose that the flow about an airfoil is bounded by walls at the spanwise positions $z = 0$ and ℓ and it is assumed for the present that U' = constant in the approaching flow. Let us also suppose that the technique described in section 6.5 of replacing the walls by sheets of sources and sinks and determining the value of ϕ' which satisfies the boundary conditions on the airfoil has been mastered. It will then be possible to determine the component of velocity grad ϕ' on the surface of the airfoil due to the presence of walls. It is more than likely that this component of velocity will not vanish at the trailing edge of the airfoil and will therefore violate the Kutta condition there. In order to re-establish the Kutta condition it will be necessary to alter the circulation about the airfoil from $U\gamma$ to $U\gamma + \Delta\Gamma$ where $\Delta\Gamma$ is of order ε and is a function of z.

It is of prime importance to be able to estimate the value of $\Delta\Gamma$ for isolated airfoils and airfoils in cascade, but owing to the difficulties of determining the secondary velocities throughout the field of flow discussed in section 6.5 the problem has not yet been rigorously solved.

An approximate method which has been used for cascades will now be illustrated for a simple airfoil. The approximation is based on the fact that although it is too difficult to calculate the secondary flow close to the airfoil, it is relatively easy to obtain the secondary velocities far downstream. It is then assumed that, when there is no secondary velocity in the xy plane normal to the trailing edge stagnation streamline in the primary flow, the Kutta condition at the trailing edge will be satisfied. In effect this condition implies that the trailing edge is extended by a thin wall which follows the trailing edge streamline downstream until the far downstream flow is fully established. The secondary flow is contained in regions on either side of this imaginary wall (or between the walls from adjacent blades in a cascade).

The calculation of the secondary flow now follows the methods described in section 6.4.2. The change in circulation is given by

$$\Delta\Gamma = -\int_0^\infty v_1 \, dy - \int_{-\infty}^0 v_2 \, dy$$

where v_1 and v_2 are the velocities in the y direction in the regions $y > 0$ and $y < 0$ respectively, and the integration is performed far downstream.

For the special case when $U' =$ const and the secondary vorticity is of the form $\Omega_{c\infty} = AU'e^{-my}$, we obtain from equation (6-98) the following expression for $\Delta\Gamma$

$$\frac{\Delta\Gamma}{\ell^2 U'} = \frac{A_2}{\ell^2 m_2^2} \left\{\sin m_2\left(\frac{\ell}{2} - z\right) \sec \frac{\ell m_2}{2} + \sum_{1,3}^{\infty} \frac{4 \cos n\pi \frac{z}{\ell}}{n\pi(1 - n^2\pi^2/\ell^2 m_2^2)}\right\}$$

$$- \frac{A_1}{\ell^2 m_1^2} \left\{\sin m_1\left(\frac{\ell}{2} - z\right) \sec \frac{\ell m_1}{2} + \sum_{1,3}^{\infty} \frac{4 \cos n\pi \frac{z}{\ell}}{n\pi(1 - n^2\pi^2/\ell^2 m_1^2)}\right\} \qquad (6\text{-}110)$$

where $\ell m/\pi$ is not an odd integer and where subscripts 1 and 2 refer to the upper and lower surfaces of the airfoil respectively.

The asymmetry of $\Delta\Gamma$ about $z = \ell/2$ is obvious from physical consideration as well as from the solution. For more general distributions of U' the equations and methods outlined in section 6.4.2 may be used.

The velocity distribution obtained in equation (6-77) for the sheets of sources and sinks representing the walls leads to the following result for the strength of the trailing vortex sheet due to the walls alone

$$\frac{d\Delta\Gamma}{dz} = - \operatorname{Im} \frac{2U'}{\ell} \int_0^\infty (T_1(\eta) - T_2(\eta)) \operatorname{cosech} - (iz - \eta)\, d\eta \quad .$$

$$= \frac{2U'}{\ell} \int_0^\infty \frac{(T_1 - T_2) \cosh \frac{\pi}{\ell} \eta \sin \frac{\pi}{\ell} z}{\sinh^2 \pi \frac{\eta}{\ell} \cos^2 \frac{\pi z}{\ell} - \cosh^2 \frac{\pi\eta}{\ell} \sin^2 \frac{\pi z}{\ell}} d\eta \qquad (6\text{-}111)$$

As this equation applies when $U' =$ constant, the value of $\Delta\Gamma$ at $z = \ell/2$ is zero.

6.2.2 Cascades

Once more we start with the simple shear problem. Let the airfoils of infinite span be set along the y' axis, Fig. 6.11. The inlet and outlet angles of the potential flow through the cascade are α_1 and α_2, and for unit upstream velocity the downstream velocity is $q_\infty = \cos \alpha_1/\cos \alpha_2$. In the flow downstream there are vorticities, U'/q_∞, and $\Omega_{s\infty}$ normal to and along the streamlines of the potential flow.

The distributed secondary circulation in a slice of unit height in the z direction, normal to the primary flow and reading from the upper side of one stagnation streamline to the lower side of the next along PQ, Fig. 6.11, is given by

$$\int_0^1 \Omega_s \, dy, \qquad (6\text{-}112)$$

in which a unit length for PQ is assumed. Now in section 6.6 we show that for simple shear the strength of the trailing vortex sheet is given by

$$w_+ - w_- = U'(\phi_+ - t_+) - U'(\phi_- - t_-)$$

$$= U'(\gamma - \oint \frac{ds}{q_s}) \qquad (6\text{-}113)$$

by equation (6-108). In this case we note that if $q_\infty \neq 1$ there is a span-wise velocity due to the enlargement effect given by, equation (6-56),

$$w = \frac{q_\infty^2 - 1}{q_\infty} U'x$$

which increases with x but has no effect on the strength of the trailing vortex sheet. The circulation around the airfoil, i.e. about ABCD in Fig. 6.11 is the two dimensional value in the simple shear case, i.e.

$$U\gamma = U \frac{\cos \alpha_1}{\cos \alpha_2} (\tan \alpha_1 - \tan \alpha_2) \quad . \qquad (6\text{-}114)$$

Now if as shown in section 6.4.1 we modify the downstream flow by introducing the potential

$$\phi' = x\,U\,\frac{1 - q_\infty^{2}}{q_\infty} \tag{6-90}$$

to cancel out the infinitely growing spanwise velocity, we introduce a velocity

$$u = U\,\frac{1 - q_\infty^{2}}{q_\infty} = U\left(\frac{\cos\alpha_2}{\cos\alpha_1} - \frac{\cos\alpha_1}{\cos\alpha_2}\right) . \tag{6-115}$$

The velocity u adds a term to the circulation along CD given by,

$$\Delta\Gamma_1 = -u\,\frac{\sin\alpha_2}{\cos\alpha_2} = u\tan\alpha_2\left(\frac{\cos\alpha_1}{\cos\alpha_2} - \frac{\cos\alpha_2}{\cos\alpha_1}\right)$$

which would appear to increase the strength of the trailing vortex sheet by the term

$$\frac{d\Delta\Gamma_1}{dz} = U'\tan\alpha_2\left(\frac{\cos\alpha_1}{\cos\alpha_2} - \frac{\cos\alpha_2}{\cos\alpha_1}\right)$$

On the other hand the addition of the velocities due to the ϕ' term far downstream plainly does not alter the strength of the trailing vortex sheet. To reconcile these results we note that it is not possible to introduce a potential ϕ' which only modifies the flow far downstream. In order to modify the flow far downstream we have in fact to specify a value of ϕ' throughout the flow which varies continuously from zero far upstream to the value given in equation (6-90) far downstream. Although flow represented by grad ϕ' is irrotational it may, as discussed in section 6.6.1, affect the Kutta condition at the trailing edges of the airfoils, in which case the circulation round each airfoil will be altered to re-establish the Kutta condition. As no other vorticity can be introduced into the flow it is plausible to assume that within the limits of the secondary flow approximation this change of circulation is equal and opposite to $\Delta\Gamma_1$.

As a simple demonstration of the foregoing conclusion, consider the flow about a cascade of airfoils with the inlet velocity AB shown diagrammatically in Fig. 6.12. The outlet velocity in two dimensional flow is AC. Suppose that in a three dimensional flow the inlet velocity remains the same but there are effects in the remainder of the flow which far downstream would produce an additional velocity in the direction of AC shown by the vector CD. At first sight it appears that the circulation about the cascade which was originally proportional to BC is now proportional to HD.

Let us suppose that in the close neighborhood of the airfoils the additional velocity CE is just one half CD, i.e. CE = 1/2 CD, Fig. 6.12. Assuming that there is no vorticity normal to the plane of the figure, there can be change in the component of velocity ND (the "tangential" component) between the far downstream position and the position just downstream of the cascade, so that this tangential component is MD'. Similarly there is no change between LB far upstream and MH' the tangential velocity just ahead of the cascade. However, the outlet angle from the cascade is then given by the vector AD' which is not in the same direction as for the potential flow. To satisfy the Kutta condition it is necessary to add a velocity shown by the vector D'E to the flow just downstream from the cascade. This effect can be produced by altering the circulation round the airfoils. An alteration of the circulation by an amount sufficient to produce a velocity D'E just downstream from the çascade will also produce an equal and opposite velocity H'F just upstream (BF is drawn parallel to CE). Hence, the modified circulation is proportional to FE which is equal to BC the circulation about the airfoil in two dimensional flow. Hence, we conclude that the circulation introduced far downstream by the additional velocity CD is balanced by the equal and opposite circulation (D'E + FH') required to satisfy the Kutta condition at the trailing edge, in such a way that the total circulation is the same as that in two dimensional flow. Far downstream with the circulation proportional to HD" = BC the velocity vector becomes AD" and it is evident that there is a change in flow angle.

The above illustration can be generalized for additional velocities which are not in the direction AC. Fig. 6.13 shows a diagram in which the additional velocity far downstream is given by the vector CD. In the neighborhood of the airfoils the additional velocity is by supposition CE = 1/2 CD. The airfoils then see a flow with an inlet velocity AF (BF is parallel to CE) and an outlet velocity AE. So far, because FE = BC, the circulation is the same as in a two-dimensional flow, the difference between HD and FE being due to the components D'E and FH'. The component D'E is not, however, sufficient in this case to make the direction of the velocity AE the same as the outlet direction AC. An additional component EE' is evidently required to satisfy this approximation to the Kutta condition. Immediately upstream of the cascade the change of circulation required to produce EE' produces an equal and opposite velocity component FF'. Consequently the circulation is now proportional to F'E' and is substantially changed from its two-dimensional value. Since by geometry GD = 2EE' the change in circulation from the two-dimensional value (BC) is proportional to DG. When CD lies along AC, GD becomes zero and the circulation about the cascade is not changed from its two-dimensional value. We, therefore, neglect the effect on the circulation of velocities superimposed on a two-dimensional cascade flow in the direction of the outlet velocity vector but not those normal to it.

In the simple shear case, we, therefore, conclude that the circulation is the same as in two-dimensional flow. There is, however, a change in outlet angle due to the velocity u. The new outlet angle α_2' is given by

$$\tan \alpha_2' = \frac{U \cos \alpha_1 \tan \alpha_2}{U \cos \alpha_1 + u \cos \alpha_2} \quad . \tag{6-116}$$

When the cascade is placed between walls components of secondary velocity v normal to the primary flow direction appear throughout the flow. To calculate the effect of this secondary flow on the circulation from the component of velocity v, Fig. 6.11 we make the approximation that the Kutta condition requires no flow across the trailing vortex sheet far downstream, as described in section 6.6.1. The methods for calculating the flow in rectangular bends described in section 6.4.1 may then be applied. The change of circulation is given by

$$\begin{aligned} \Delta\Gamma &= -\bar{v} = -\int_0^1 v dy \quad . \qquad (6\text{-}117) \\ &= -\int_0^1 \psi_z \, dy \quad , \end{aligned}$$

or in terms of the series expression,

$$\psi = \sum_{n=1}^{\infty} \psi_n(z) \sin n\pi y,$$

used in section 6.4.1,

$$\Delta\Gamma = -\bar{v} = -\frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{\psi_n'}{n} \tag{6-118}$$

where only the terms for n odd appear in the series.

The average axial velocity far downstream is now

$$U \cos \alpha_1 + u \cos \alpha_2 - \bar{v} \sin \alpha_2$$

so that the average flow angle α_2' far downstream is given by

$$\tan \alpha_2' = \frac{U \cos \alpha_1 \tan \alpha_2 - \Delta\Gamma \cos \alpha_2}{U \cos \alpha_1 + u \cos \alpha_2 - \bar{v} \sin \alpha_2} \quad ,$$

which to order (ε) becomes

$$\tan \alpha_2' = \tan \alpha_2 - \frac{u}{U} \frac{\sin \alpha_2}{\cos \alpha_1} + \frac{\bar{v}}{U \cos \alpha_1 \cos \alpha_2} \quad . \tag{6-119}$$

The change in outlet angle is therefore

$$\Delta\alpha_2 = \frac{\bar{v}}{U}\frac{\cos\alpha_2}{\cos\alpha_1} - \frac{1}{2}\frac{u}{U}\sin 2\alpha_2. \tag{6-120}$$

To illustrate these general results consider the specific problem of a cascade with walls at $z = 0$ and ℓ and an upstream velocity $U = U_1 + (U_2 - U_1)\,z/\ell$ so that U' is constant. If the spacing between the airfoils is s, we assume that $s \cos\alpha_2 = 1$ so that the ratio of cascade span to spacing is $\ell \cos\alpha_2$.

From equations (6-118) and (6-81) we find after some simplification that

$$\bar{v} = -2U' \sum_{n=1,3}^{\infty} \frac{c_n \sinh n\pi(z - \ell/2)}{n^2\pi^2 \cosh n\pi\ell/2} \tag{6-121}$$

From equation (6-92) the value of u becomes

$$u = U\left(\frac{\cos\alpha_2}{\cos\alpha_1} - \frac{\cos\alpha_1}{\cos\alpha_2}\right)\left(1 - \frac{U_2 + U_1}{2U}\right). \tag{6-122}$$

The value of c_n in equation (6-121) has to be determined from a knowledge of $\Omega_{s\infty}$. If the cascade were regarded as a simple bend of large radius and deflection $\varepsilon = \alpha_1 - \alpha_2$ we may use equation (6-85) for $\Omega_{s\infty}/U'$ with $\alpha = -\varepsilon$. The negative sign is required because the cascade deflection shown in Fig. 6.11 is negative. Substituting in equation (6-86) we obtain the result

$$c_n = -8\varepsilon/n\pi$$

which when substituted in equation (6-121) gives the result

$$\bar{v} = 16\varepsilon U' \sum_{1,3}^{\infty} \frac{\sinh n\pi(z - \ell/2)}{n^3\pi^3 \cosh n\pi\ell/2}. \tag{6-123}$$

Alternatively we may assume a mean value for $\Omega_{s\infty}$ writing

$$-2\varepsilon = \int_0^1 \left(\frac{\Omega_{s\infty}}{U'}\right) dy . \tag{6-124}$$

At $z = 0$

$$\bar{v} = \frac{16\varepsilon U'}{\pi^3} \sum_{1,3} \frac{\tanh n\pi\ell/2}{n^3}.$$

The series converges very rapidly and for $\ell > 2$ the error in neglecting all other terms except the first is sma-l.

When $\alpha_2 = -\alpha_1$, i.e. for an impulse turbine cascade, $u = 0$. The change in outlet angle at $z = 0$ is approximately for $\ell > 2$

$$\frac{\Delta\alpha_2}{\varepsilon} = \frac{16}{\ell\pi^3}\left(\frac{U_2}{U_1} - 1\right) \tag{6-125}$$

Where in this case a positive value for $\Delta\alpha_2$ implies an apparent reduction of deflection. The ratio of cascade span to spacing in the notation used here is $\ell \cos \alpha_2$.

The lines of constant t and (ϕ - t) in the flow about a cascade of airfoils derived by the use of Merchant and Collar's[88] transformation.

7.0 Conclusion

The object of these notes has been to set out the principles which underlie the theoretical study of shear flows and the methods by which solutions to some of the problems may be approached. The applications of the theory which have been discussed are limited to a small number of simple examples and very few numerical solutions have been included. Certain ommissions are obvious. The work by Honda on thin airfoils[10] and cascades of thin airfoils[58] has been only mentioned. The use of the secondary flow approximation for axisymmetric primary flows by Lighthill[8][82] and Hall[90] in the flow about spheres has not been developed. For those interested in flow in turbo-machinery there is clearly a need for the rigorous application of these methods to the problems of shear flow in annular cascades and rotating blade rows, and an assessment of the results and analyses of many authors of which references (76 - 79) constitute only a few examples.

The methods described are inevitably limited by the approximations. As Joy's[89] experimental results (also reported in reference (5)) for a simple rectangular bend have shown the linearising procedure eliminates many of the interesting non-linear effects which are found in practice. Attempts to calculate the non-linear behaviour of the flow in bends are just beginning[87].

A recent article[91] expands some of these remarks in a critical survey of the success and limitations of secondary flow theory.

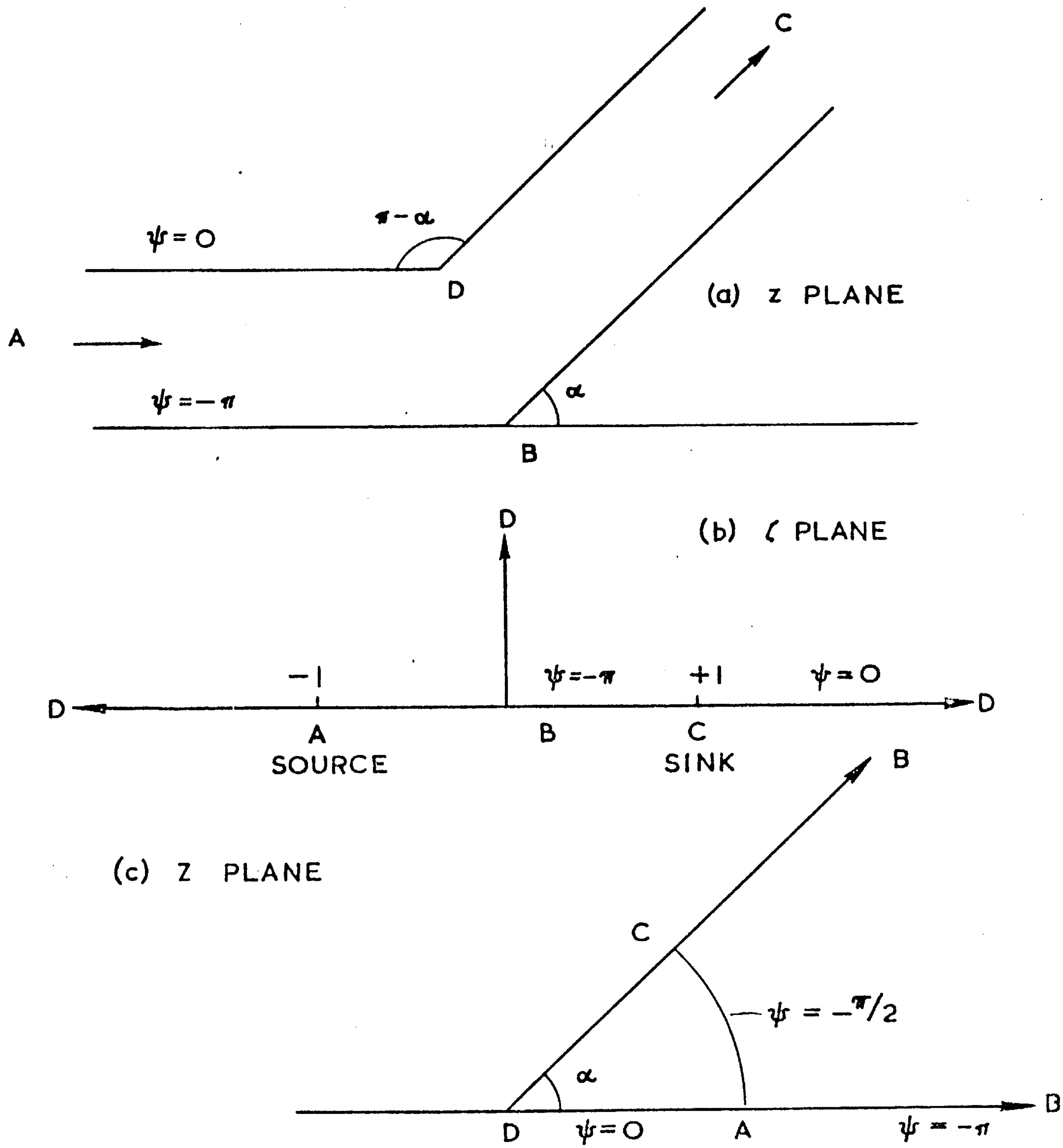


FIG 6·1 FLOW IN ANGLED BEND

(a) PHYSICAL PLANE

(b) SCHWARTZ CHRISTOFFEL TRANSFORMATION

(c) INVERSE HODOGRAPH

FIG 6·2

LINES OF CONSTANT

$\phi - t$ (SOLID LINES)

t (BROKEN LINES)

FOR THE FLOW IN A 90°

BEND

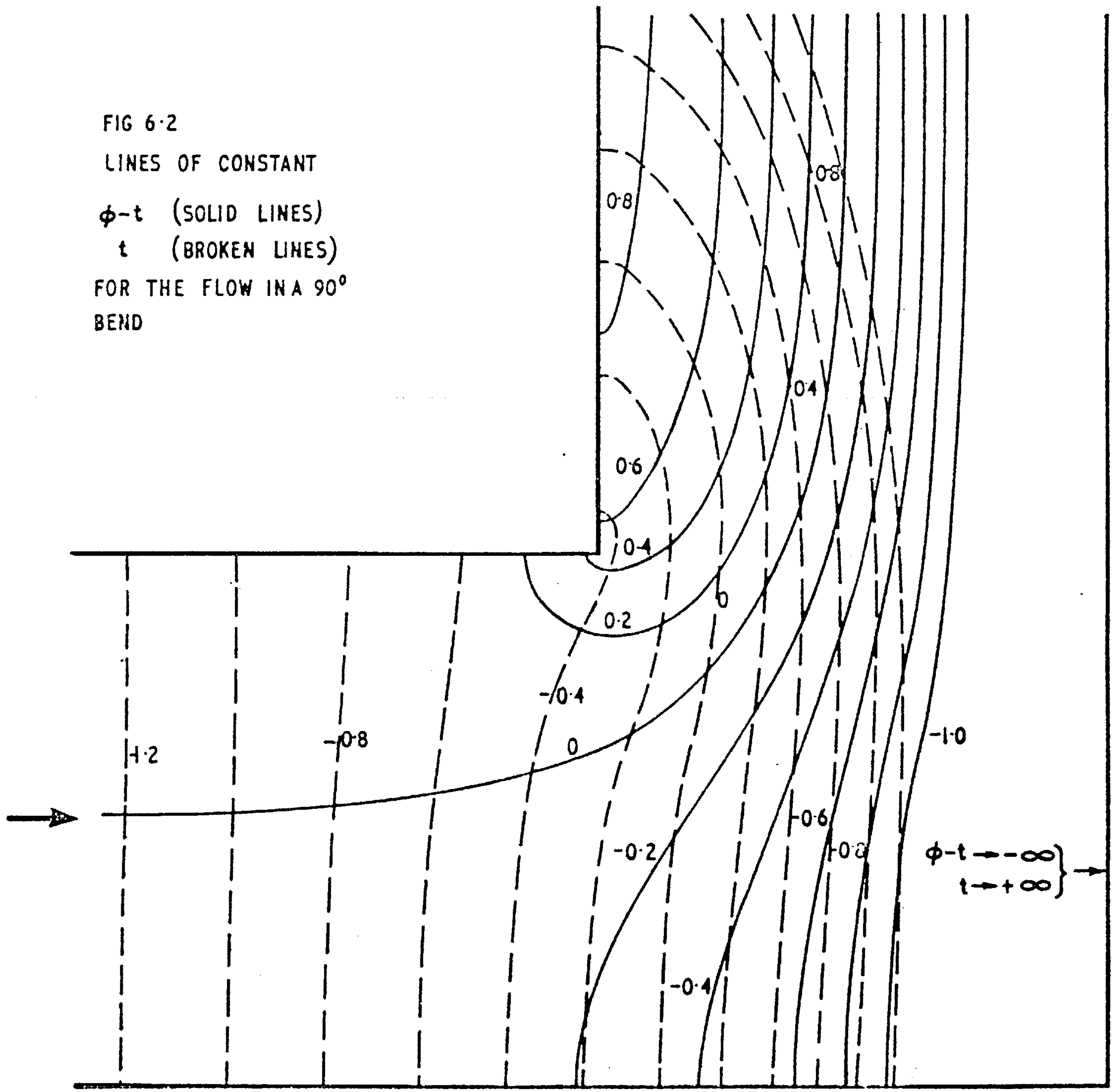
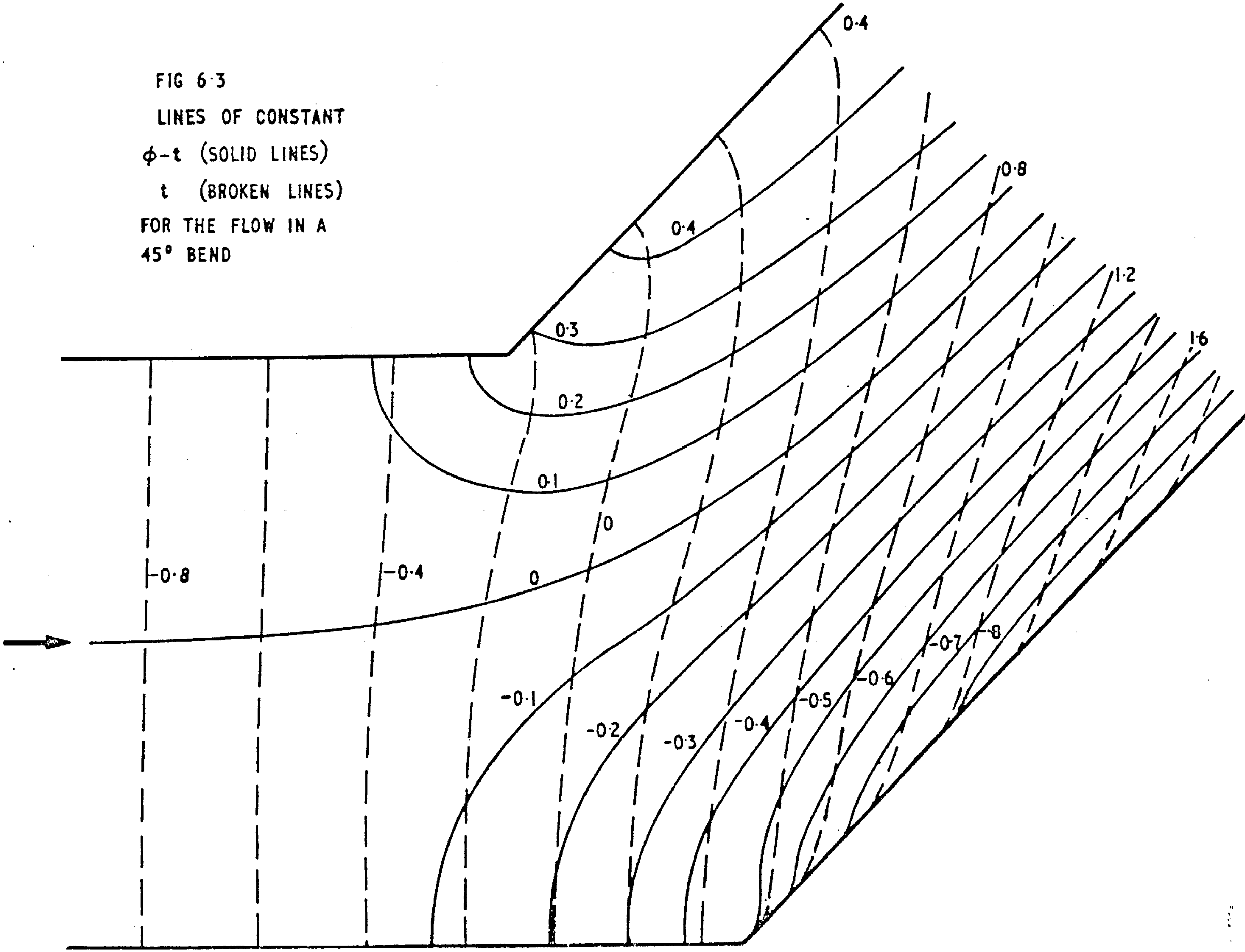


FIG 6·3
LINES OF CONSTANT
$\phi - t$ (SOLID LINES)
t (BROKEN LINES)
FOR THE FLOW IN A
45° BEND



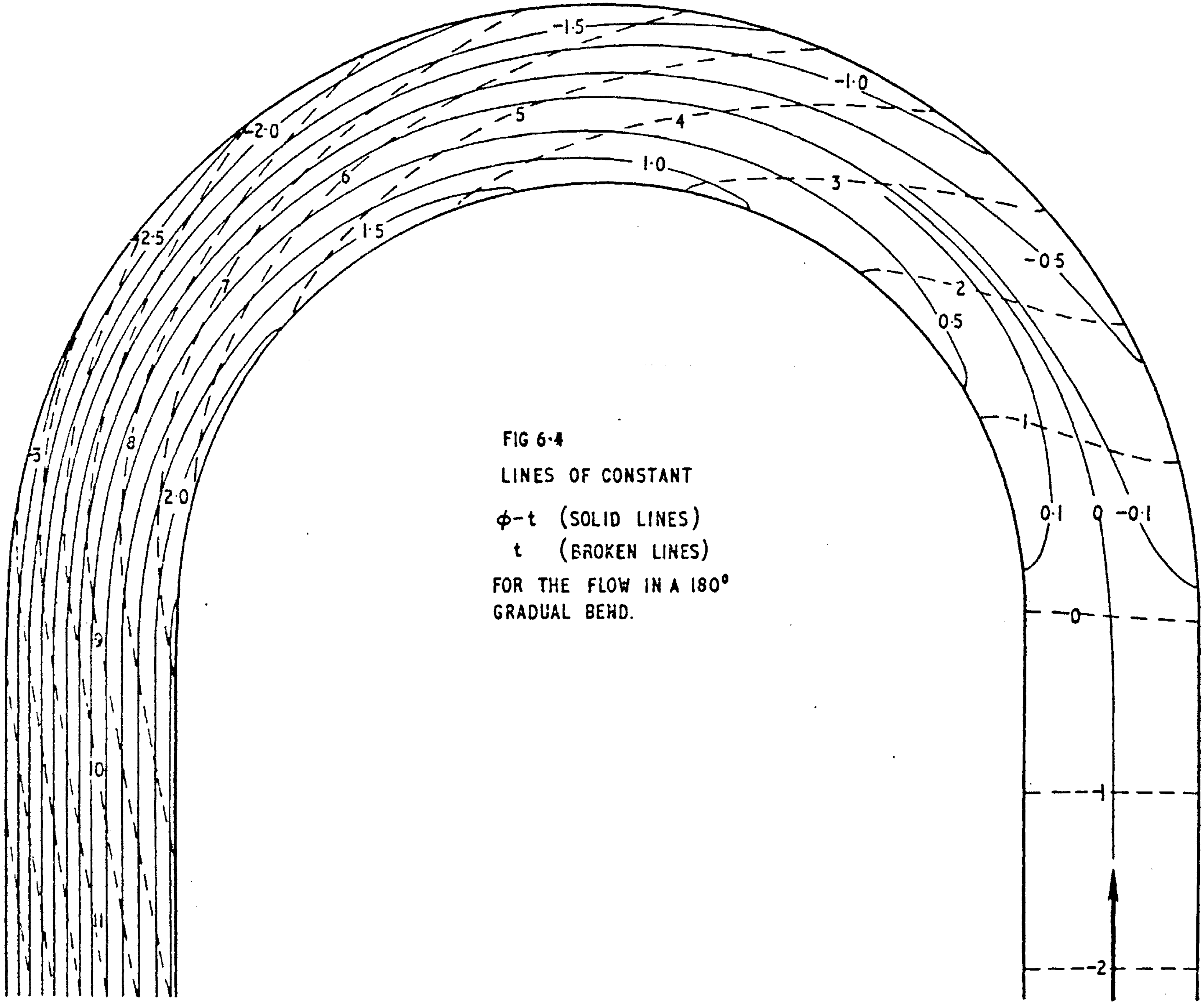


FIG 6·4

LINES OF CONSTANT

$\phi - t$ (SOLID LINES)

t (BROKEN LINES)

FOR THE FLOW IN A 180° GRADUAL BEND.

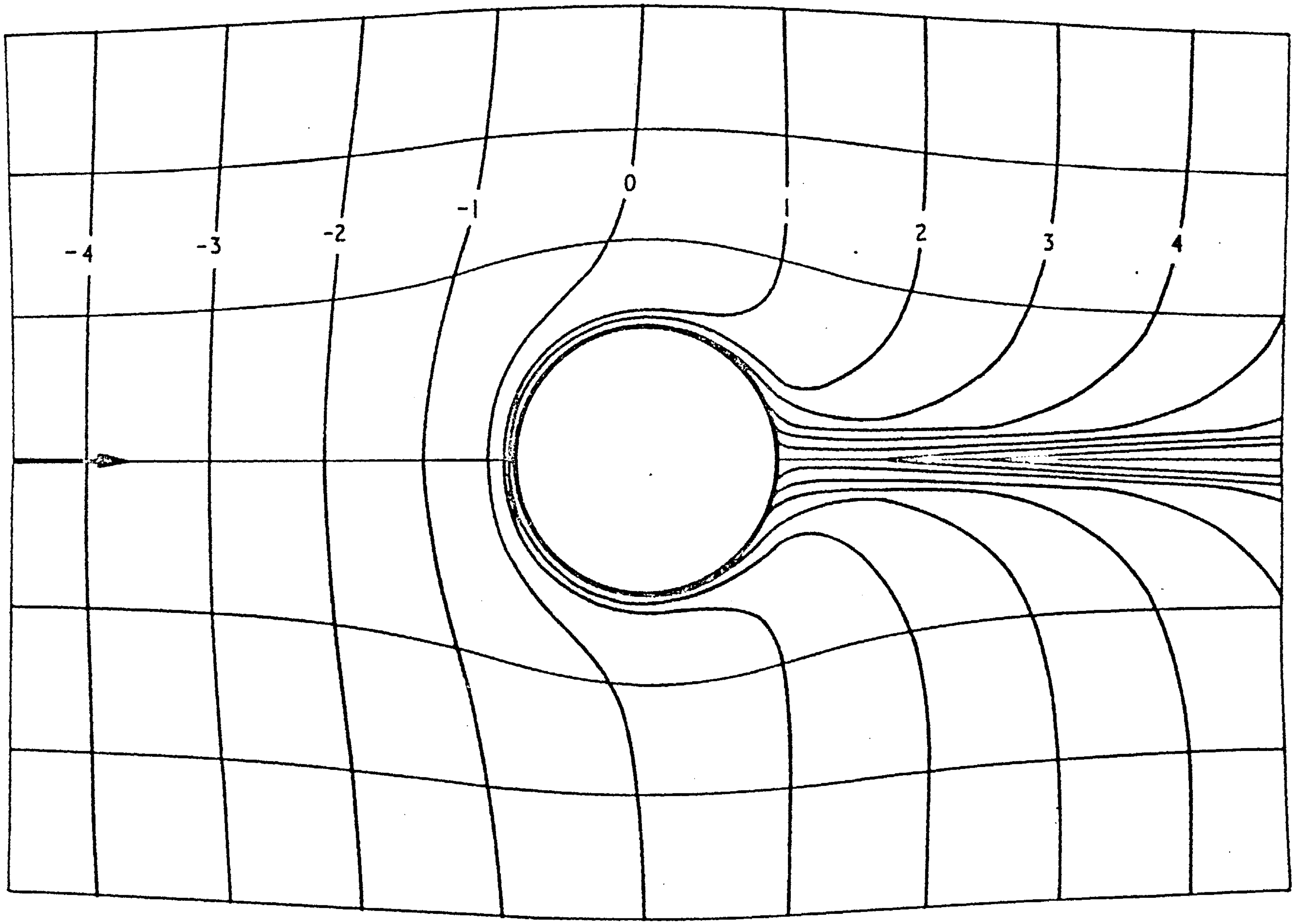


FIG 6·5
LINES OF CONSTANT t FOR THE
FLOW PAST A CIRCULAR CYLINDER

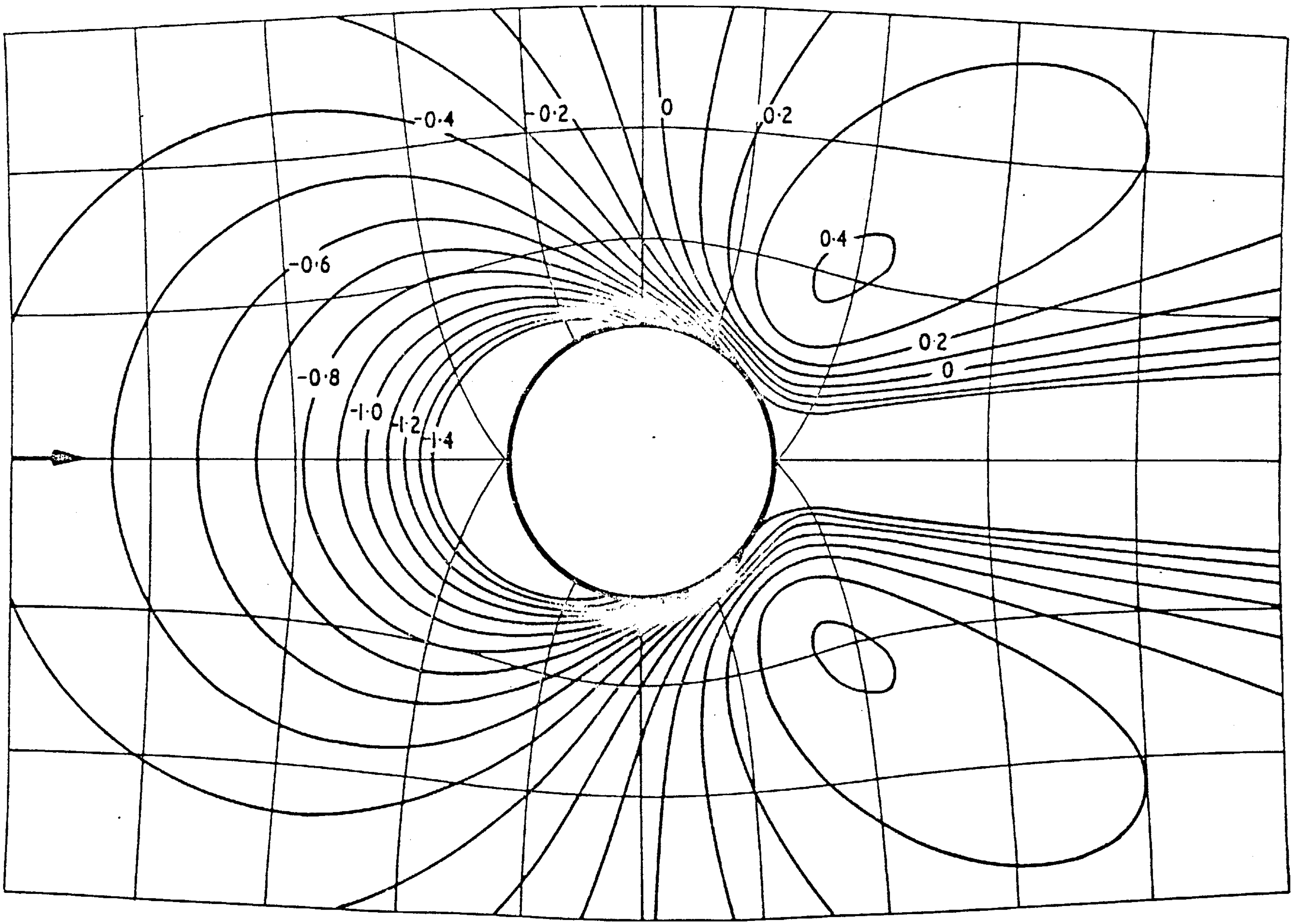


FIG 6·6

LINES OF CONSTANT $\phi - t$ FOR THE FLOW PAST A CIRCULAR CYLINDER

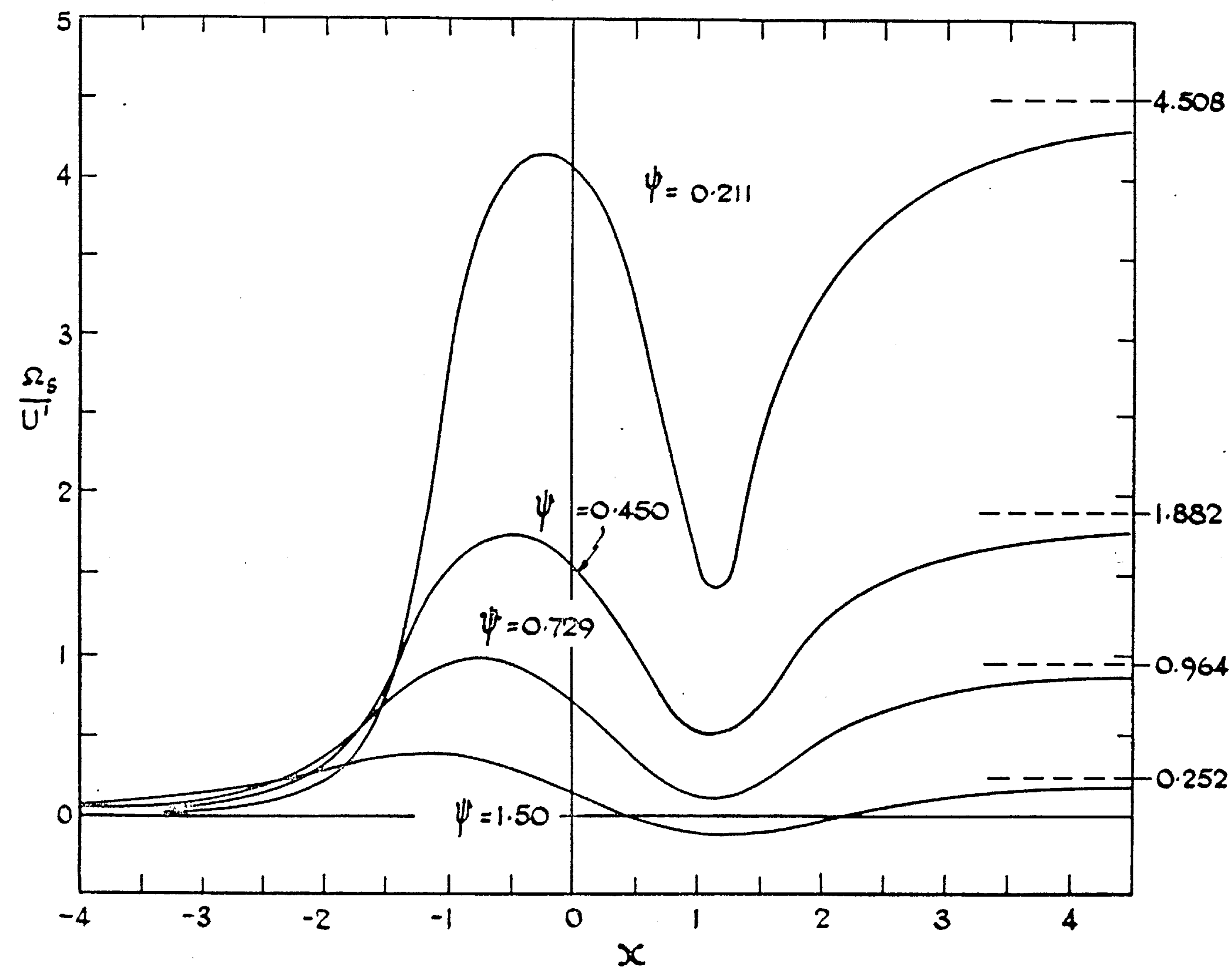


FIG. 6·7 VALUES OF Ω_S IN THE FLOW ABOUT A CIRCULAR CYLINDER.

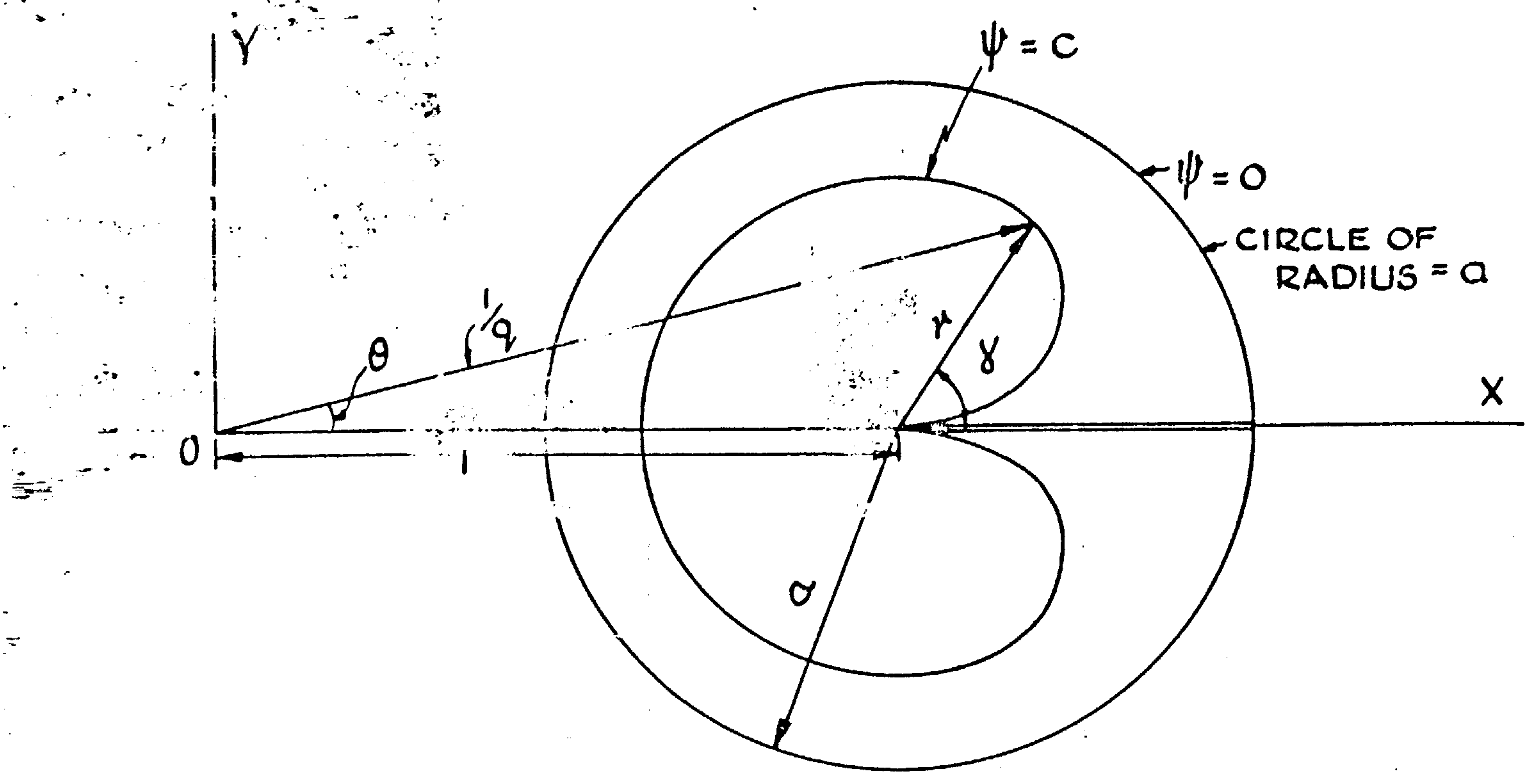


FIG. 6·8 (b) A CIRCLE IN THE INVERSE HODOGRAPH PLANE

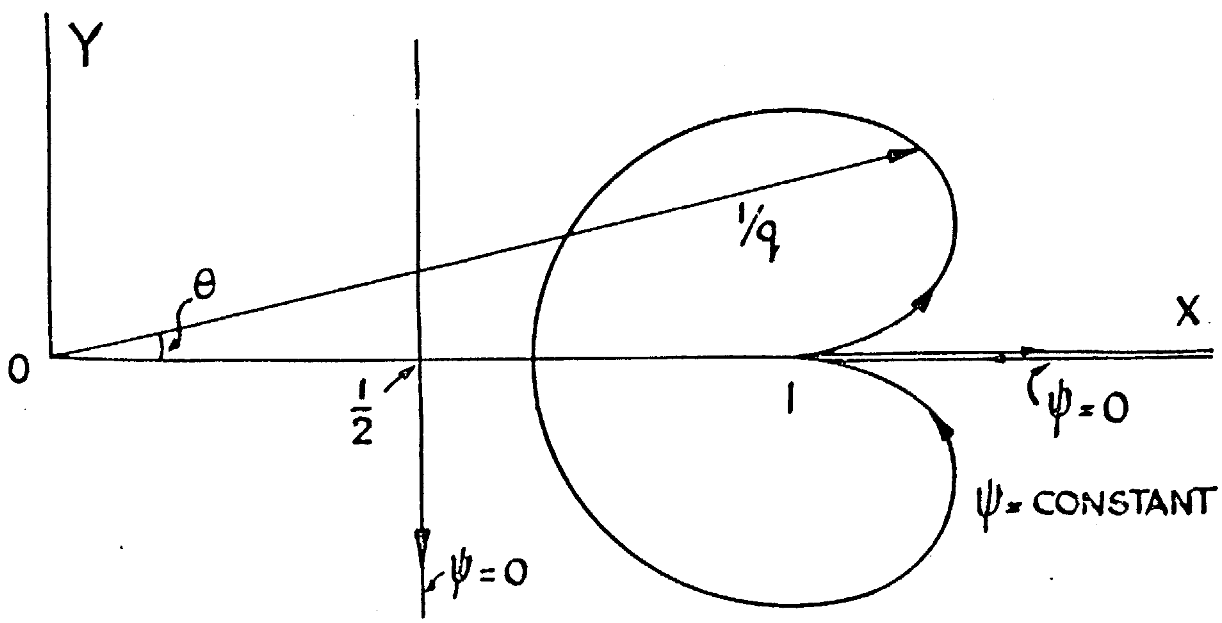


FIG. 6·8 (a) INVERSE HODOGRAPH FOR THE FLOW ABOUT A CIRCLE.

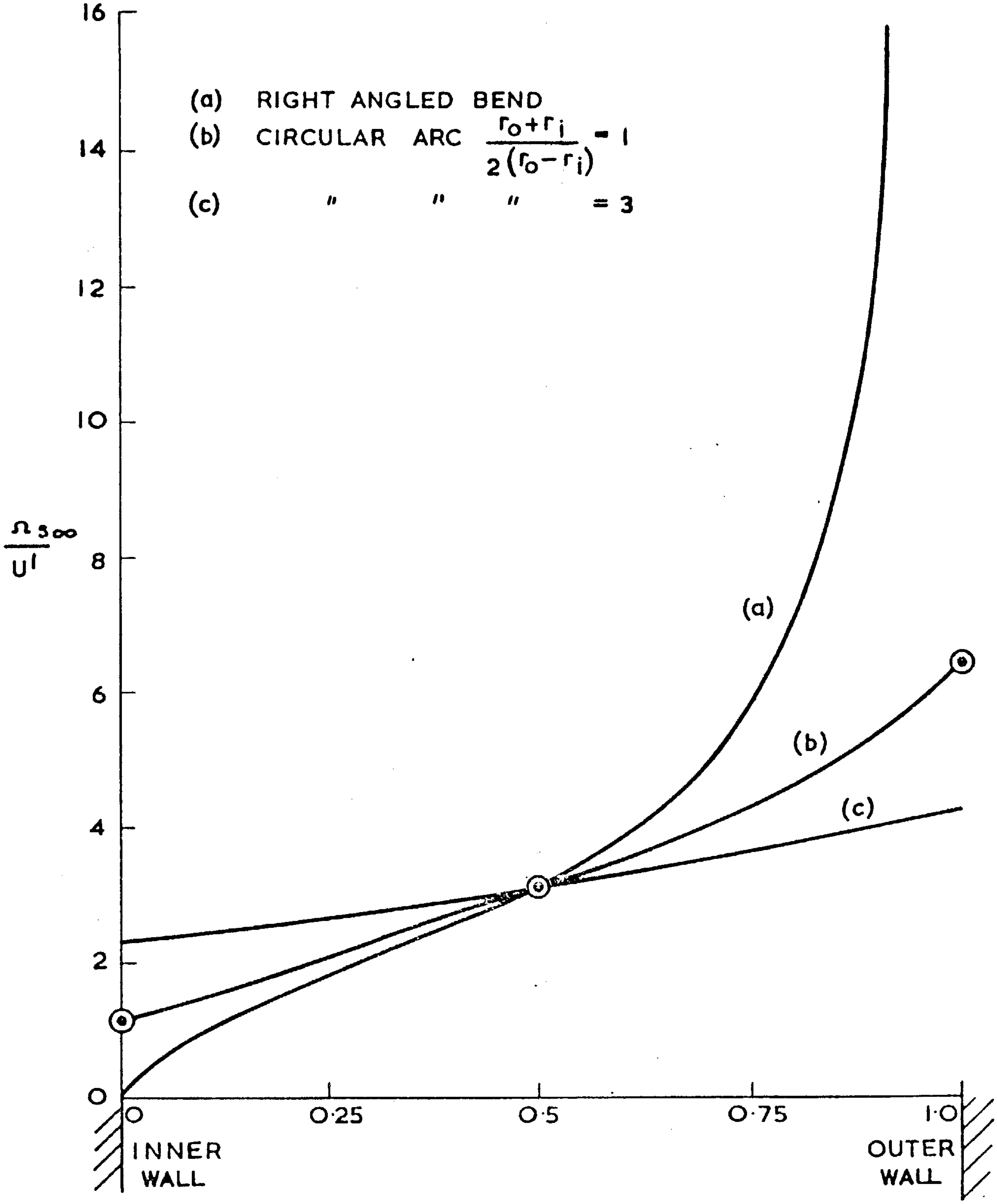


FIG 6·9 VALUES OF DOWNSTREAM VORTICITY FOR 90° BEND
(a) RIGHT-ANGLED
(b)(c) CIRCULAR ARC-MEAN RADIUS / WIDTH = 1 AND 3

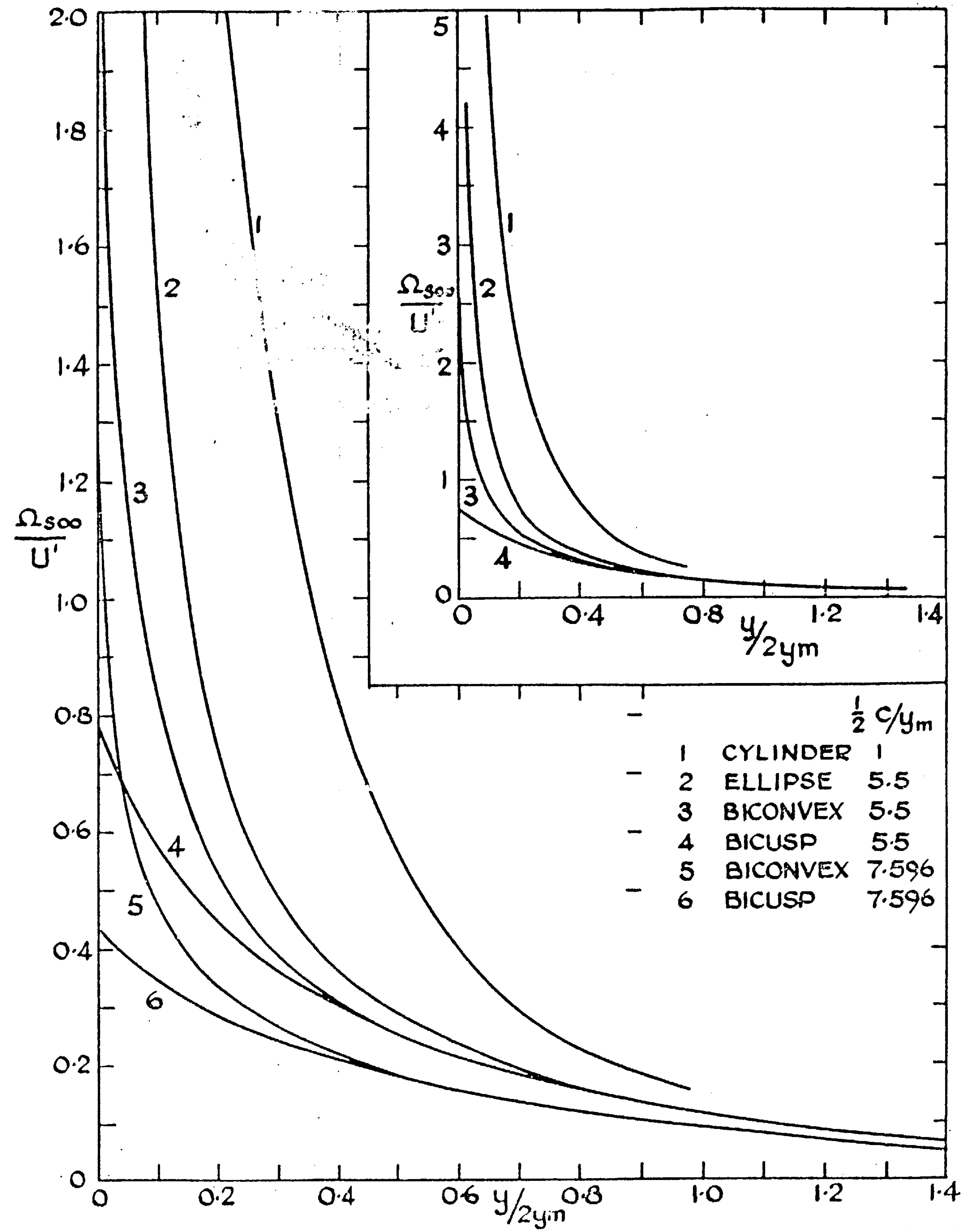


FIG. 6.10 DOWNSTREAM VORTICITY FOR VARIOUS STRUTS

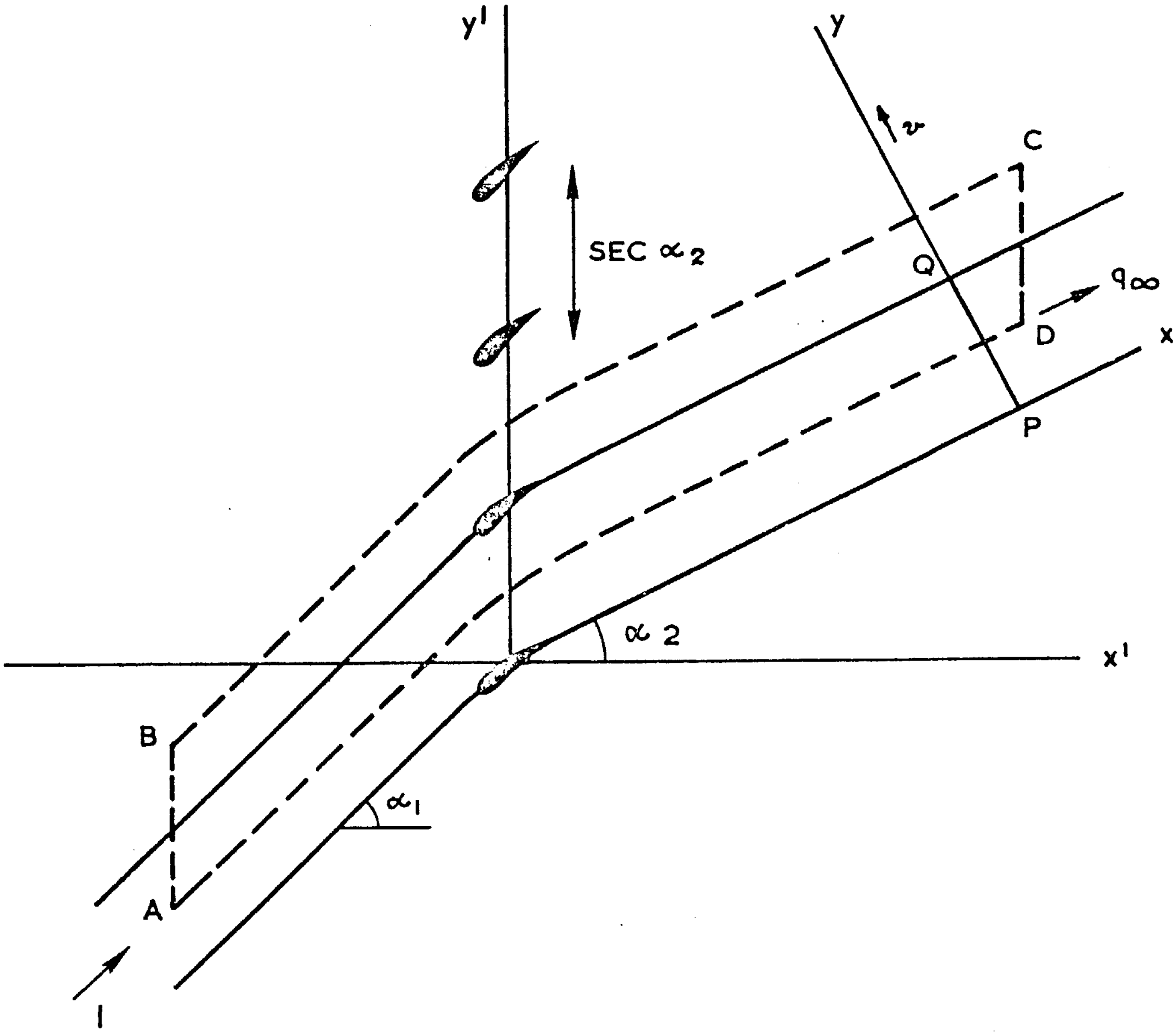


FIG 6·11 FLOW THROUGH A CASCADE

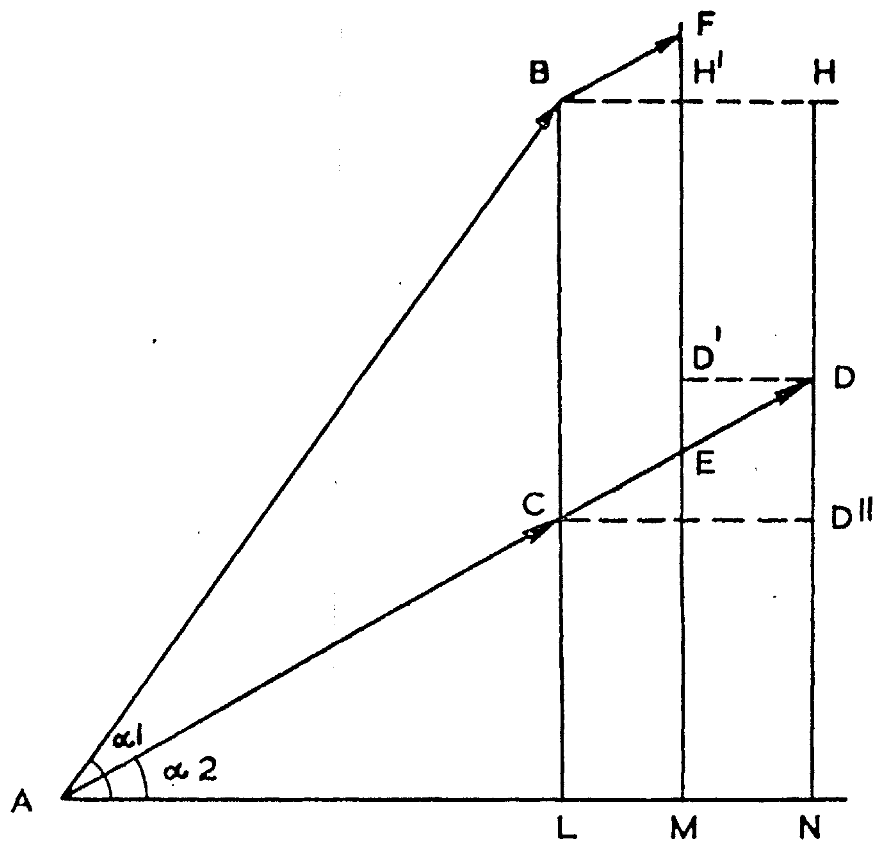


FIG 6·12 VELOCITY TRIANGLE FOR CASCADE FLOW WITH SUPERIMPOSED VELOCITIES IN THE DIRECTION OF THE OUTLET FLOW

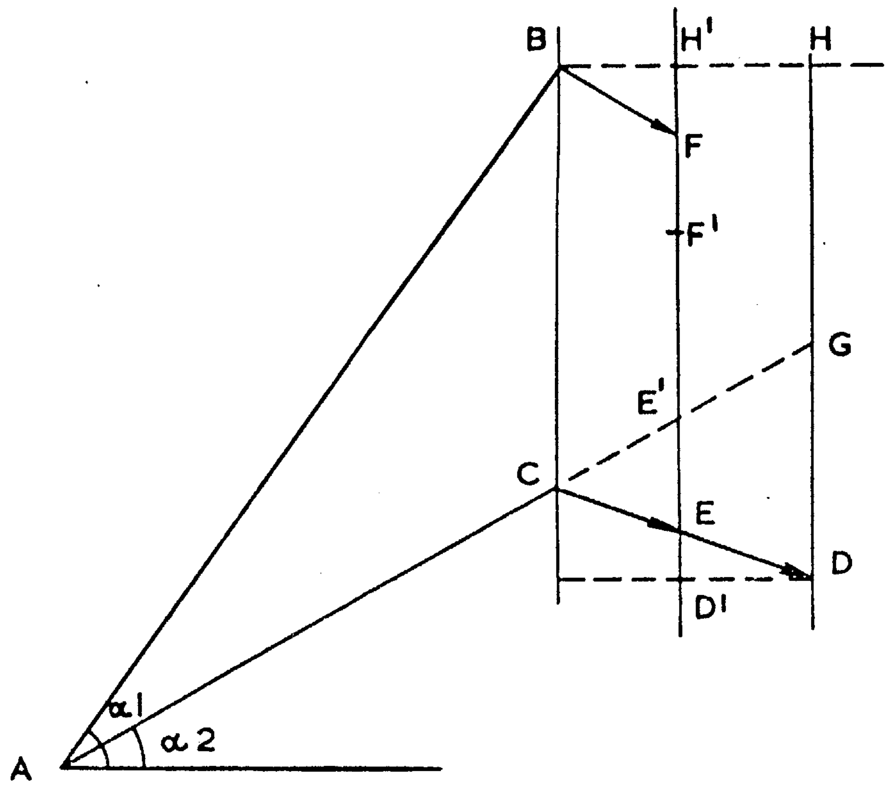


FIG 6·13 VELOCITY TRIANGLES FOR CASCADE FLOW WITH SUPERIMPOSED VELOCITIES IN A GENERAL DIRECTION

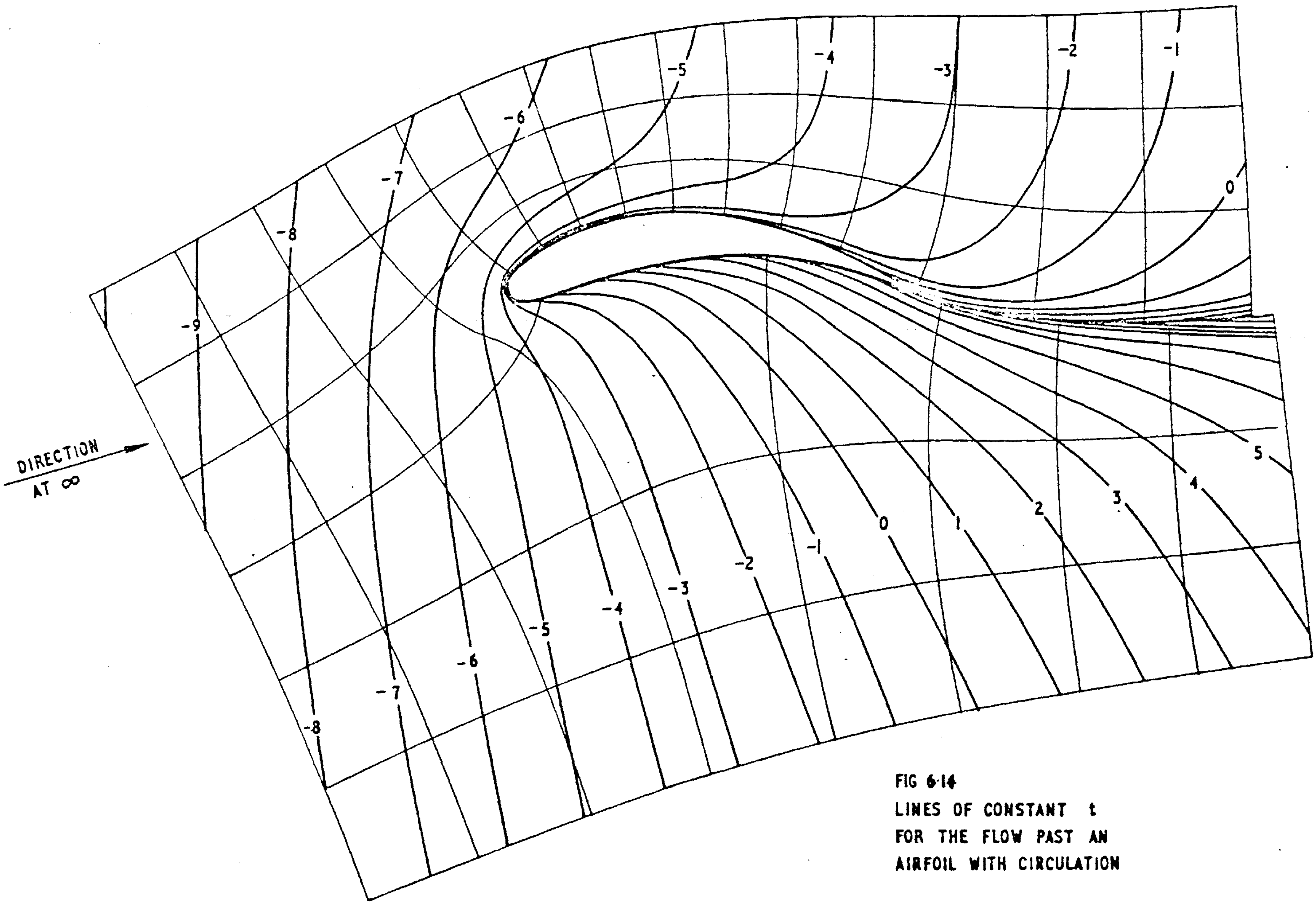


FIG 6·14
LINES OF CONSTANT t
FOR THE FLOW PAST AN
AIRFOIL WITH CIRCULATION

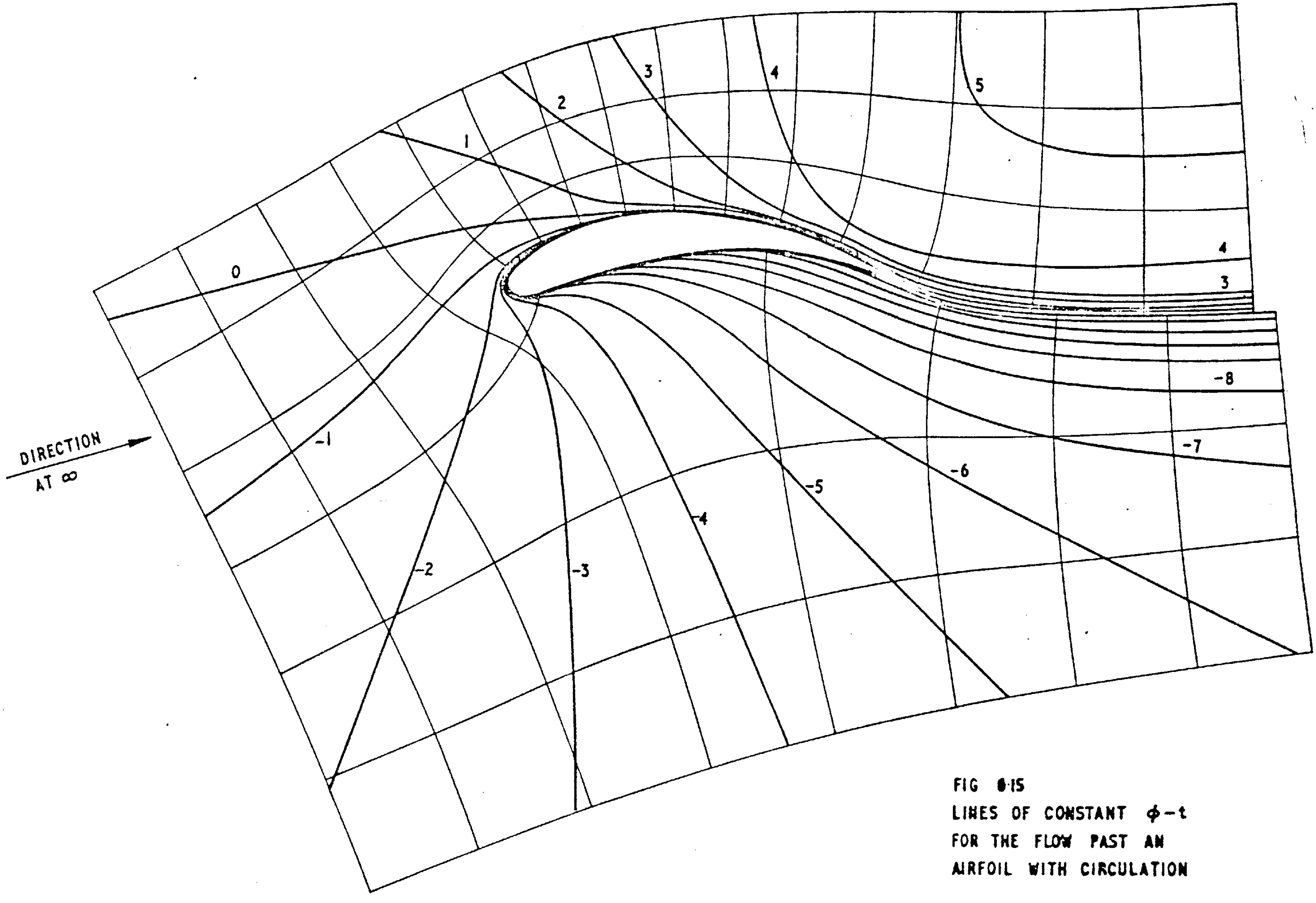


FIG 6·15
LINES OF CONSTANT $\phi - t$
FOR THE FLOW PAST AN
AIRFOIL WITH CIRCULATION

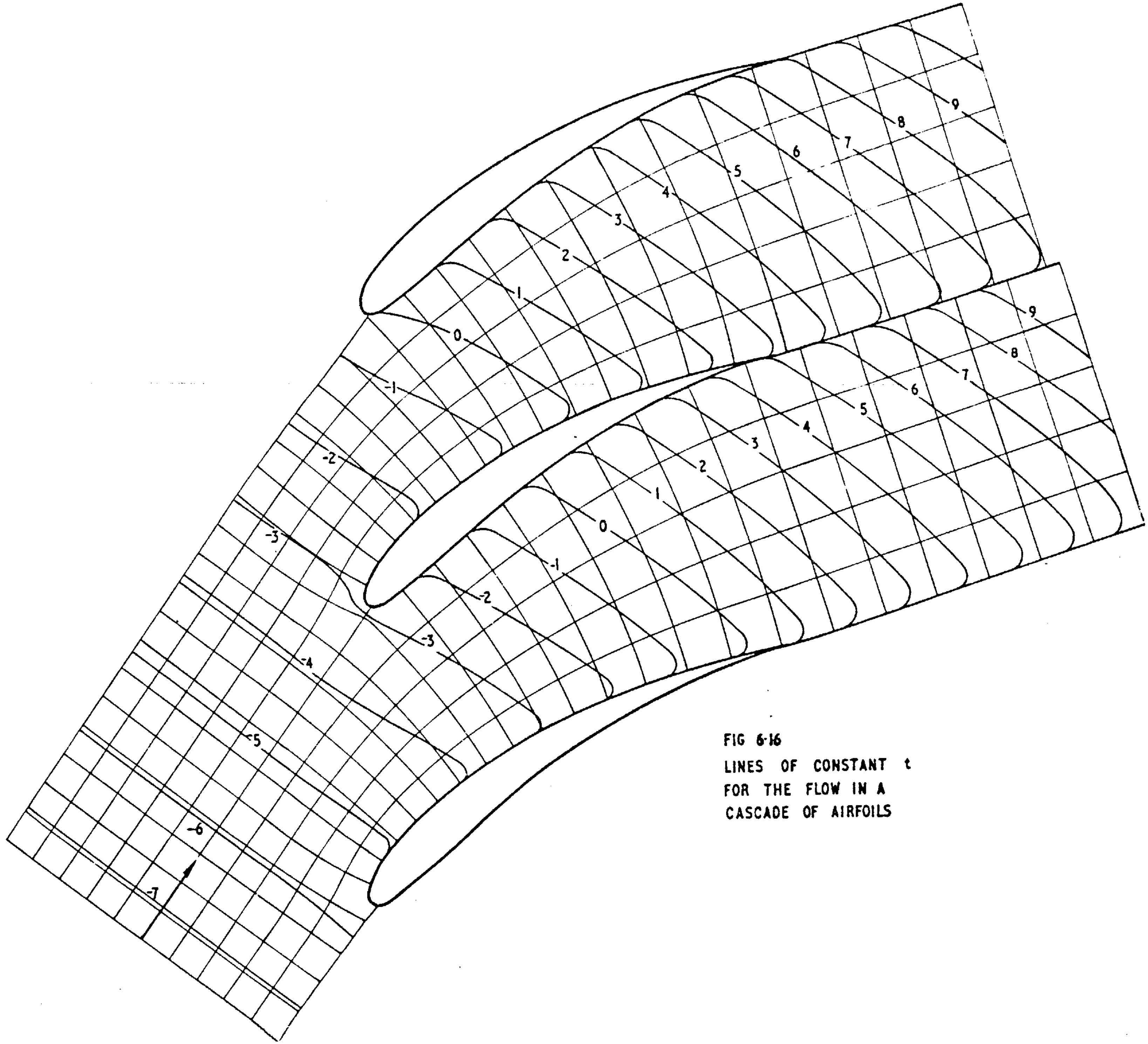


FIG 6·16
LINES OF CONSTANT t
FOR THE FLOW IN A
CASCADE OF AIRFOILS

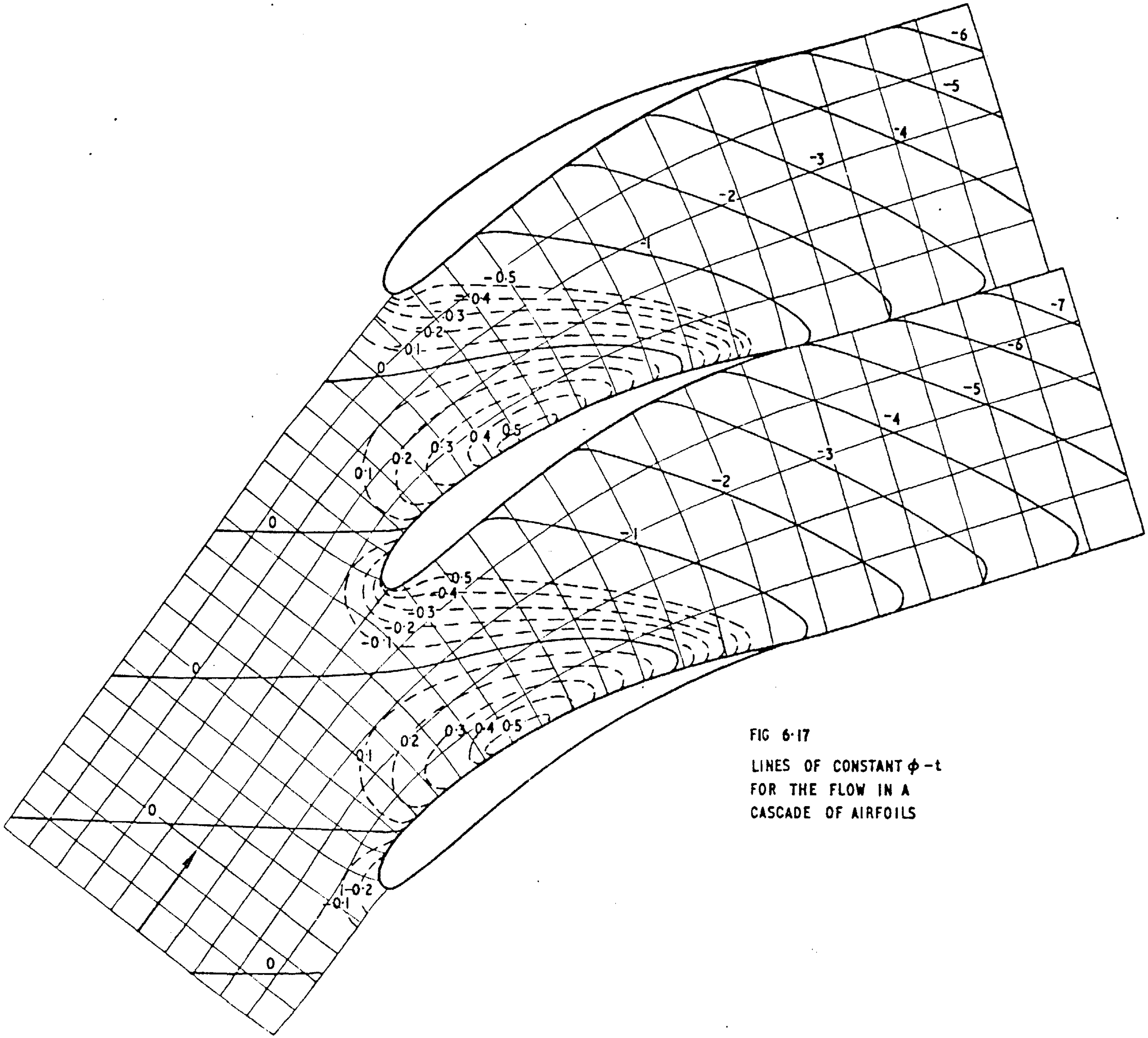


FIG 6·17

LINES OF CONSTANT $\phi - t$
FOR THE FLOW IN A
CASCADE OF AIRFOILS

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ON THE THEORY OF SHEAR FLOW:

Supplement and Corrigenda to

Gas Turbine Laboratory Report No. 88

by

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ABSTRACT

This report contains corrections to a report on the theory of shear flow (Gas Turbine Laboratory Report No. 88)

A section on secondary flow in cascades has been rewritten and expanded.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

As in G.T.L. Report No. 88

Additional subscript

T.E. Trailing Edge

SECTION 1 - INTRODUCTION

In October 1966 a collection of notes on the theory of secondary flow was published as a Gas Turbine Laboratory Report (No. 88) under the title "On the Theory of Shear Flow". As a result of experience with the use of these notes both for teaching and research various errors were discovered. Most of the errors are relatively minor and are listed in Section 2 of this report as Corrections to G.T.L. Report No. 88. The last section 6.2.2* of G.T.L. Report No. 88 contained more serious errors as well as being unsatisfactorily presented. This section has been completely rewritten as Section 3 of this report.

Experience has also shown that Sections 5.6, 5.7 and 5.8 of G.T.L. Report No. 88 can be better presented, but as these sections contain no errors, except the minor ones listed in Section 2 below, revision is postponed until the work can be made more complete by extending it to cascades of thin airfoils.

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*pages 96-101

SECTION 2 - CORRIGENDA TO REPORT No. 88

Page No.	Line	Correction
4	9	Delete " $\frac{D}{DT}$ " substitute " $\frac{D}{Dt}$ "
27	23	Delete " grad ε_o " substitute " grad ϕ_o "
40	9	Rewrite last line in equation (5-28) $- \frac{\partial}{\partial x} \rho U \operatorname{div} \underline{v} = \nabla p - 2 \frac{U'}{U} p_y = 0$
51	2	Additional reference for subject matter included in Section 5.6. S-1. Railly J. W., "Lift Distribution and Secondary Flow for a Cascade in Non-uniform Flow", Proc. Ninth International Congress of Applied Mechanics, Vol. 11 p 320 - 330, Brussels University, 1957. Se also reference S.3.
54	1	Delete "cos α_m" substitute "cot α_m"
54	19	Equation (5-89) should read " $\Delta p = 2\rho U_1 U_2 \beta_1 \tan \alpha_m$ " i.e. insert " β_1 ".
56	2nd line from bot. of page	Superscript * to be added to second P_n
59	11	Delete " $\int_{m=-\infty}^{+\infty} P_{nm}$ " substitute " $\sum_{m=-\infty}^{+\infty} P_{nm}$ "
59	13	Right-hand side of equation (5-107) should read: $-\int_{-\infty}^{+\infty} \frac{k_n^2 A_n}{4\pi K_n} e^{ik_n(z-ms')} dk_n \int_{-\infty}^{o} e^{-K_n \lvert x - m s \sin \alpha_m \rvert} dx$
59	17	Capital " K " in first integral should have subscript " n "
60	7	Insert " α " after cos.

Page No.	Line	Correction
60	eq. (5-111)	Insert λ in last term to read $\sum_{n=1}^{\infty} \frac{G_n A_n \lambda_n}{4 \rho U^2} \coth (\frac{1}{2} \lambda_n s')$
62	After last line	Add "It should be noted that this method of correction is only very approximate. More accurate solutions cast some doubt on its validity."
64	Lines 15-24	Delete paragraph "A class of special cases ... flow about a sphere." and substitute: " If *far upstream* in a flow which stretches to infinity in all directions grad U is constant in magnitude and direction, we have the special case described by Lighthill as *simple shear*. Lighthill's(8)(81) solution for the sphere is a solution of equations (6-3) and (6-1) for simple shear flow about a sphere. "
65	18	Delete "rotational" substitute " irrotational ".
67	17	Delete " $x \leq 1$ " substitute " $x \geq 1$ ".
67	28	After " $\psi = - \pi/2$ " insert " $k = 0$ ".
68	19	Add " The width of the channel is unity in these figures. "
74	3	Insert " θ " to read " write $\theta' = \theta - \pi/2$, over the... "
80	Bot. line	Rewrite as " $\frac{\Gamma_{s\infty}}{U'} = - \int_0^{\infty} (x + X_{\infty})\, dy = X_{0\infty}$ "
81	Table I	Add another line " Drag Infinite Finite Finite "
84	Eq. (6-78)	Write as " $\psi_{yy} + \psi_{zz} = - (\frac{\Omega_{s\infty}}{U'})\, U'$ "
91	21	Insert " $+ \frac{\partial^2 \psi_r}{\partial z^2}$ " before equals sign in third of eq. (6-106)
91	3rd fr. bottom	Add subscript " θ " to " V "
96 - 100		Delete Section 6.2.2 Cascades and substitute new section given below. Note: Figures 6.11, 6.12 and 6.13 are also to be deleted.

SECTION 3 - REPLACEMENT FOR SECTION 6.2.2 (CASCADES, PAGES 96 - 101) IN GAS TURBINE LABORATORY REPORT NO. 88

Once more we start with the simple shear problem. Let the airfoils of infinite span be set along the y' axis, Fig. S 6-11. The inlet and outlet angles of the potential flow through the cascade are α_1 and α_2, the upstream velocity is unity and the downstream velocity is $q_\infty = \cos\alpha_1/\cos\alpha_2$. In the flow downstream there are vorticities, U'/q_∞ and $\Omega_{s\infty}$ normal to and along the streamlines of the potential flow, except that along the streamlines leaving the trailing edge there is an additional component due to the circulation shed from the blades.

The strength of the trailing vortex sheet, as shown in section 6.6, is given by

$$w_+ - w_- = U'[\phi_+ - \phi_- - (t_+ - t_-)] = U'(\gamma - \oint \frac{ds}{q_s}),$$

by equation (6-108).

Here $U'\gamma$ is the shed circulation and the other term is the trailing filament circulation resulting from the stretching of the vortex filaments as they pass round the airfoil.

To determine the total secondary circulation far downstream we consider the circulation in a section along DE in Fig. S 6-11, which is of unit height in the z direction and normal to the far downstream primary flow. The only components of circulatory flow about the section are the spanwise components $(\phi - t)U'$ at D and E, so that the total secondary circulation enclosed in section DE is

$$\Gamma_{s\infty} = U'(\phi_E - \phi_D - (t_E - t_D)) \tag{6-112}$$

Now $\phi_E - \phi_D$ is the same as the change in potential across the streamline leaving the trailing edge so that

$$\phi_E - \phi_D = \gamma$$

If the control surface ABCEFD is chosen so that AB, normal to the upstream flow, represents a vortex filament at time t = 0 and if AD and CF lie along similar streamlines in adjacent passages so that $t_F - t_C = t_D - t_A$, then

$$\begin{aligned} t_E - t_D &= (t_E - t_B) - (t_D - t_A) \\ &= (t_E - t_F) - (t_B - t_C) \\ &= - EF/q_\infty + BC/1 \\ &= AC \sin\alpha_1 - DE \tan\alpha_2/q_\infty \end{aligned}$$

We choose DE = 1 so that AC = sec α_2 and note that, by continuity, $\cos\alpha_1 = q_\infty \cos\alpha_2$, so that

$$t_E - t_D = \frac{\sin\alpha_1}{\cos\alpha_2} - \frac{\sin\alpha_2}{\cos\alpha_1}$$

On substituting in equation (6-112) we obtain

$$\Gamma_{s\infty} = U'(\gamma + \frac{\sin\alpha_2}{\cos\alpha_1} - \frac{\sin\alpha_1}{\cos\alpha_2}) \tag{6-113}$$

Now $\gamma = AC \sin\alpha_1 - DF\ q_\infty \sin\alpha_2$

$$= \frac{\cos\alpha_1}{\cos\alpha_2} (\tan\alpha_1 - \tan\alpha_2)$$

so that

$$\Gamma_{s\infty}/U' = \frac{\sin\alpha_2}{\cos\alpha_1}(1 - \frac{\cos^2\alpha_1}{\cos^2\alpha_2}) = \frac{1 - q_\infty^2}{q_\infty} \tan\alpha_2 \tag{6-114}$$

Since DE = 1, $\Gamma_{s\infty}$ is the circulation in a unit area normal to the flow direction and is therefore the averaged streamwise vorticity far downstream. By subtracting from it the vorticity in the trailing vortex sheet,

$U'(\gamma - \oint \frac{ds}{q_s})$, we obtain the average distributed secondary vorticity in the flow passing between the blades, viz

$$\overline{\Omega}_{s\infty} = \int_0^1 \Omega_{s\infty}\, dy = U'(\oint \frac{ds}{q_s} + \frac{\sin\alpha_2}{\cos\alpha_1} - \frac{\sin\alpha_1}{\cos\alpha_2}) \qquad (6\text{-}115)$$

Exact values of $\oint \frac{ds}{q_s}$ for airfoils in cascade have not been determined, but the following approximations have been suggested. For thin airfoils the analysis of section 6.6 gives the result $\oint \frac{ds}{q_s} = -\gamma$. As a better approximation for cascades Smith(78) suggests $\oint \frac{ds}{q_s} = -\gamma/q_m^2$, where q_m is the vector mean of the inlet and outlet velocities, i.e. $q_m = \cos\alpha_1/\cos\alpha_m$ and $\tan\alpha_m = \frac{1}{2}(\tan\alpha_1 + \tan\alpha_2)$.

Other writers have applied Squire and Winter's(4) formula for the secondary vorticity in a bent channel to the flow in the passages between the blades, viz

$$\overline{\Omega}_{s\infty}/U' = 2(\alpha_2 - \alpha_1)$$

We may refine this latter result by applying equation (6-60) to the mid-passage streamline passing through the cascade, where $\theta = \alpha - \alpha_1$ and we assume that $q\cos\alpha = \cos\alpha_1$. Then

$$\Omega_{s\infty}/U' = 2\, q_\infty \int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{q^2}$$

$$= \frac{2}{\cos\alpha_1\cos\alpha_2}\int_{\alpha_1}^{\alpha_2} \cos^2\alpha\; d\alpha$$

$$= (\frac{\alpha_2 - \alpha_1}{\cos\alpha_1\cos\alpha_2} + \frac{\sin\alpha_2}{\cos\alpha_1} - \frac{\sin\alpha_1}{\cos\alpha_2})$$

The above results for the simple shear case are modified when U is no longer linear in z and when the flow is bounded by walls because the term $\nabla\phi'$ in equation (6-1) is no longer zero. The scalar ϕ' must satisfy

$$\nabla^2\phi' = -(\phi - t)\,U''$$

derived from equation (6-5). $\nabla\phi'$ must satisfy the boundary conditions on the airfoils, must reduce the flow normal to the walls at $z = 0$ and ℓ to zero and must satisfy the Kutta condition on the airfoils. The satisfaction of this latter condition will in general involve a change of circulation about the blades of, say, $\Delta\Gamma(z)$ which is $O(\varepsilon)$. Hence the strength of the trailing vortex sheet and the magnitude of $\Gamma_{s\infty}$ will both be increased by $d\,\Delta\Gamma/dz$, but the distributed secondary vorticity, equation (6-115), will be unaffected.

The difficulty of determining ϕ' is so great that it has not yet been attempted. The most recent work is that of Gomi[S.2], who uses a cascade of flat plates to give an approximation to the Kutta condition for the real cascade. The approximate methods described below are less precise but easier to apply.

As in the flow round a bend with enlargement or contraction both the normal and the streamwise components of vorticity will induce velocities and cause changes in the circulation and flow angles. The circulation and outlet angle changes may be written

$$\Delta\Gamma = \Delta\Gamma_n + \Delta\Gamma_s$$

and

$$\Delta\alpha_2 = \Delta\alpha_{2_n} + \Delta\alpha_{2_s}$$

where subscripts n and s refer to the effects calculated by consideration of the normal and streamwise components of vorticity respectively.

The velocities induced by the distributed secondary vorticity within the blade passages are similar to those obtained in a bent rectangular channel. We have seen in section 6.5 that in a curved channel with a free vortex primary flow the flow pattern in the channel after any bend angle θ is the same as that far downstream in the Trefftz plane. If this result were applicable to the flow in the blade passages of a cascade, we should expect to obtain a solution for the stream function of the secondary flow at the trailing edge of the cascade by writing $\psi_{yy} + \psi_{zz} = - \Omega_{sT.E.}$ By considering a rectangular channel at DE, Fig. S.6-11 and writing $\psi_{yy} + \psi_{zz} = - \overline{\Omega}_{s\infty}$, we obtain a secondary flow which does not cross the "walls" formed by the primary flow streamlines leaving the stagnation points at the trailing edges. The only difference between the two stream functions will be due to the small difference between $\Omega_{s_{T.E.}}$ and $\Omega_{s\infty}$. The expression for the stream function and secondary flow velocities are the same as those given in section 6.4.1 for a rectangular bend. The change in the outlet angle averaged over the gap between the blades is given by

$$\Delta\alpha_{2_s} = \overline{v}/Uq_\infty$$

$$= \frac{2 \cos \alpha_2}{U \cos \alpha_1} \sum_{n=1,3}^{\infty} \psi_n'(z)/n\pi \qquad (6\text{-}116)$$

where $\psi_n'(z)$ is given by equation (6-8) with

$$c_n = \frac{4}{n\pi} \left(\frac{\overline{\Omega}_{s\infty}}{U'}\right)$$

The change in circulation is

$$\Delta\Gamma_s = - \overline{v}$$

$$= - 2 \sum_{n=1,3}^{\infty} \psi_n'(z)/n\pi \qquad (6\text{-}117)$$

To estimate the effect of the normal vorticity component we proceed as in section 6.4.1 by introducing a component $\nabla\phi'$ which far downstream suppresses the velocity $w = x\ (\phi - t)\ U'$ and introduces the velocity given by equation (6-92),

$$u = \frac{1 - q_\infty{}^2}{q_\infty} (U - Um) \tag{6-118}$$

where $Um = \int_0^\ell U\ d(z/\ell)$ is the average velocity.

The introduction of this component $\nabla\phi'$ which introduces axial and tangential velocities far downstream must also alter the velocities at the airfoils and alter the circulation round them. To estimate the change in circulation we assume, as in actuator disc theory, that half the total change in axial velocity occurs at the trailing edge and that the flow angle at the trailing edge is the same as that in two-dimensional flow, namely α_2. Referring to Fig. S.6-12 if AL is the inlet axial velocity and AN the outlet value, the axial velocity at the trailing edge is AM, where LM = MN. Hence the velocity leaving the trailing edge is AE, where $E\hat{A}M = \alpha_2$. As there is no component of vorticity Ω_z, the tangential velocity ME at the trailing edge is the same as that far downstream so that ME = ND. The velocity far downstream is therefore AD which can be resolved into a component AG in the primary flow direction and a component GD normal to it. Hence CG = u and GD is the additional velocity required far downstream, if the Kutta condition on the blades is to be satisfied. From the geometry of Fig. S.6-12

$$CG = LM \sec\alpha_2 + MN \cos\alpha_2 = MN\ (\cos\alpha_2 + \sec\alpha_2),$$

and $\quad MN \sin\alpha_2 = GD$

so that $\quad GD = CG \tan\alpha_2/(1 + \sec^2\alpha_2).$

Substituting for CG = u from equation (6-118) we have

$$GD = (U - Um)\,\frac{1 - q_\infty^2}{q_\infty}\,\frac{\tan\alpha_2}{2 + \tan^2\alpha_2} \tag{6-119}$$

The change in outlet angle is

$$\Delta\alpha_{2_n} = -\,GD/Uq_\infty$$

$$= \left(\frac{Um}{U} - 1\right)\left(\frac{\cos^2\alpha_2}{\cos^2\alpha_1} - 1\right)\frac{\tan\alpha_2}{2 + \tan^2\alpha_2} \tag{6-120}$$

The change in circulation is proportional to the difference between BC, the two-dimensional value, and EF. From Fig. S.6-12 we deduce that

$$\Delta\Gamma_n = -\,(LM\tan\alpha_2)\,s = -\,MN\tan\alpha_2\sec\alpha_2$$

$$= -\,(U\text{-}Um)\,\frac{1 - q_\infty^2}{q_\infty}\,\frac{\tan\alpha_2\sec^2\alpha_2}{2 + \tan^2\alpha_2}$$

$$= -\,(U\text{-}Um)\,\frac{1 - q_\infty^2}{q_\infty}\,\frac{\tan\alpha_2}{1 + \cos^2\alpha_2} \tag{6-121}$$

If we now consider the total streamwise component of circulation passing through the section DE Fig. S.6-11 we find that to the total circulation $\Gamma_{s\infty}$ for the simple shear case given in equation (6-114) we must add the components of shed circulation given by $d\Delta\Gamma_s/dz$ and $d\Delta\Gamma_n/dz$. As a result of the changes introduced to take account of the walls and the need to satisfy the Kutta condition the spanwise contributions to the circulatory flow about the section DE are equal and opposite (the spanwise velocity x U' (ϕ - t) has been suppressed). The effective contributions to the circulation result from the velocities *along* DE, namely $\bar{v}$ and - GD. We should therefore expect that

$$\Gamma_{s\infty} + \frac{d}{dz}(\Delta\Gamma_n + \Delta\Gamma_s) = -\frac{d}{dz}(\bar{v} - GD) \qquad (6\text{-}122)$$

Substitution of the results from equations (6-114), (6-117), (6-119) and (6-121) substantiate this conclusion.

We further note that as the spacing between the blades approaches zero, the effect of the secondary flow in the blade passages becomes vanishingly small and $\bar{v}$ and $\Delta\Gamma_s$ also approach zero. This limiting case corresponds to that of the actuator disc. It may be shown even for finite blade spacing that the expresions derived for GD, $\Delta\alpha_{2_n}$ and $\Delta\Gamma_n$ equation (6-119), (6-120) and (6-121) are in fact identical with those obtained by treating the row of blades as an actuator disc. For this reason these terms are sometimes referred to as being due to the *actuator disc effect* or to the spanwise *displacement* of the Bernoulli surfaces in their flow from far upstream to far downstream of the actuator disc. The terms $\bar{v}$, $\Delta\alpha_{2_s}$ and $\Delta\Gamma_s$ are due to the *secondary flow effect.* In particular $\Delta\alpha_{2_s}$ may be regarded as the change of the effective outlet angle at the trailing edge of the blade from its two-dimensional value.

Fig. S.6-13 illustrates the results of the application of this theory to the case of the flow through a cascade. The upstream velocity U(y) is linear in y and $U(\ell) = 2\,U(o)$, where the walls are at $y = o$ and ℓ. The angles $\alpha_1 = 55°$, $\alpha_2 = 30°$, $s/c = 1$. The full lines on the curve for $\Delta\alpha/(\alpha_1 - \alpha_2)$ versus y/ℓ show for different values of ℓ/s the variation of the flow angle far downstream. When $\ell/s \to \infty$ $s \to o$ and $\Delta\alpha_{2_s} \to o$, so that the curve for $\ell/s = \infty$ shows the actuator disc effect only, i.e. $\Delta\alpha = \Delta\alpha_{2_n}$. The difference between this curve and the curves for $\ell/s = 1$ and 3 show the effects of $\Delta\alpha_{2_s}$. In these calculations it was assumed that $\bar{\Omega}s_\infty = 2(\alpha_2 - \alpha_1)$.

For the sake of comparison calculations for the lifting line theory equation (5-91) have also been included with $U_2/U_1 = 2$, $\alpha_m = 55°$, and $\kappa = 4 \frac{s}{c} \sec \alpha_m$. The results are shown by the broken lines. The curve for $\ell/s = \infty$ is not the same as in the case of the secondary flow approximation, because in the latter case the shear is assumed to be small whereas for the lifting line case the shear is large. The two sets of curves show qualitatively similar behaviour.

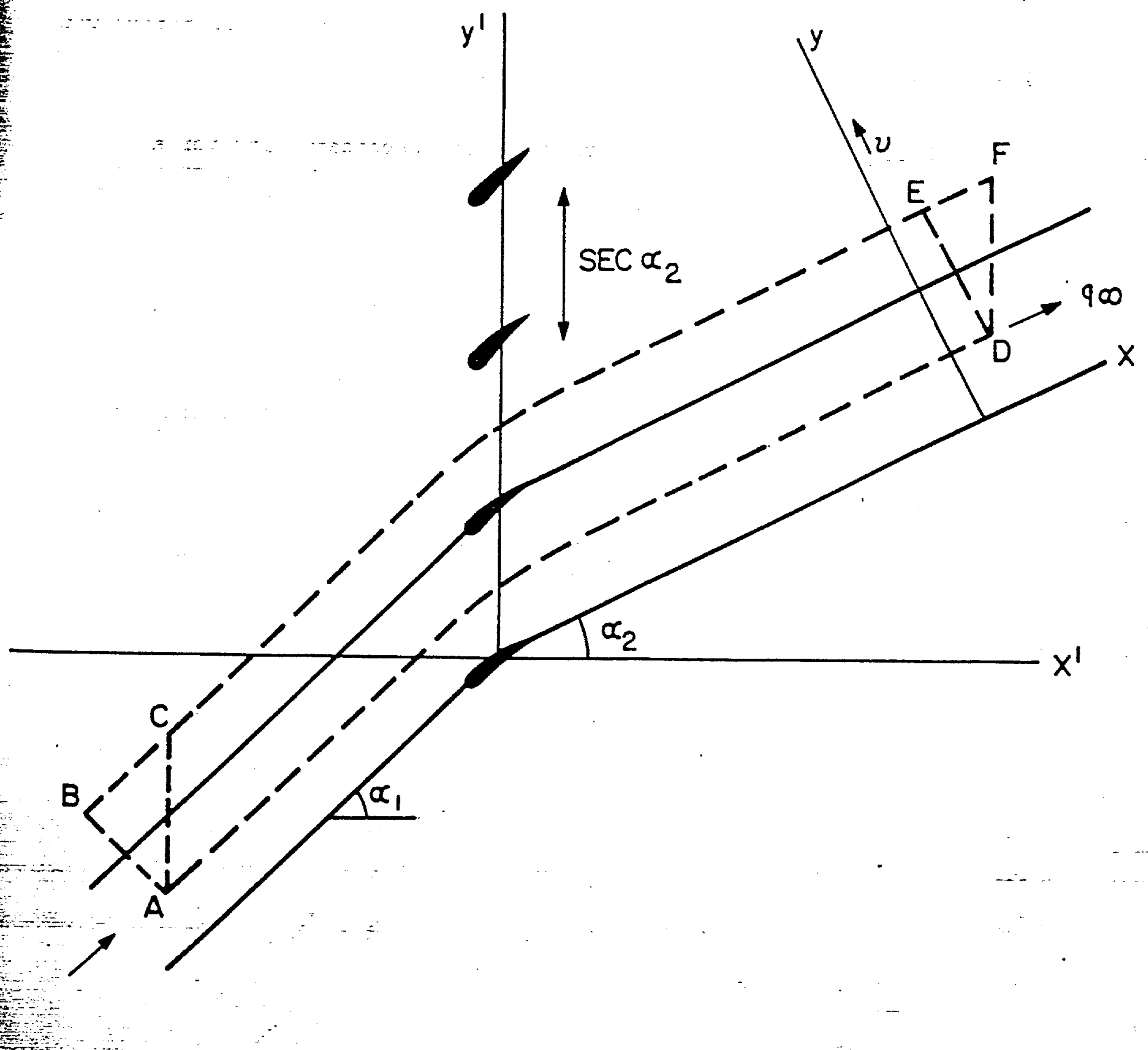

FIG. S 6.11 FLOW THROUGH A CASCADE

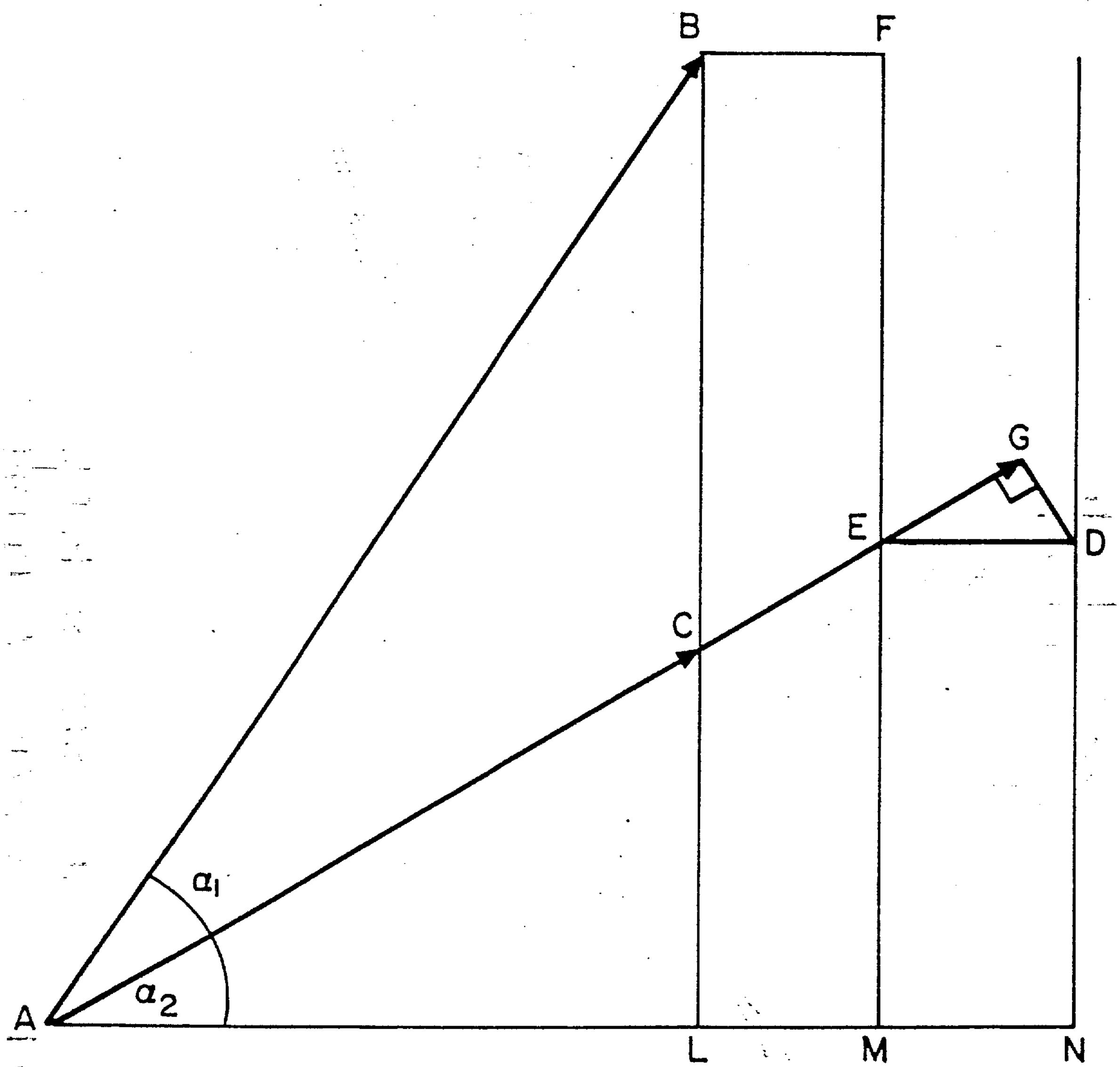

FIG. S 6.12 VELOCITY TRIANGLE FOR CASCADE FLOW

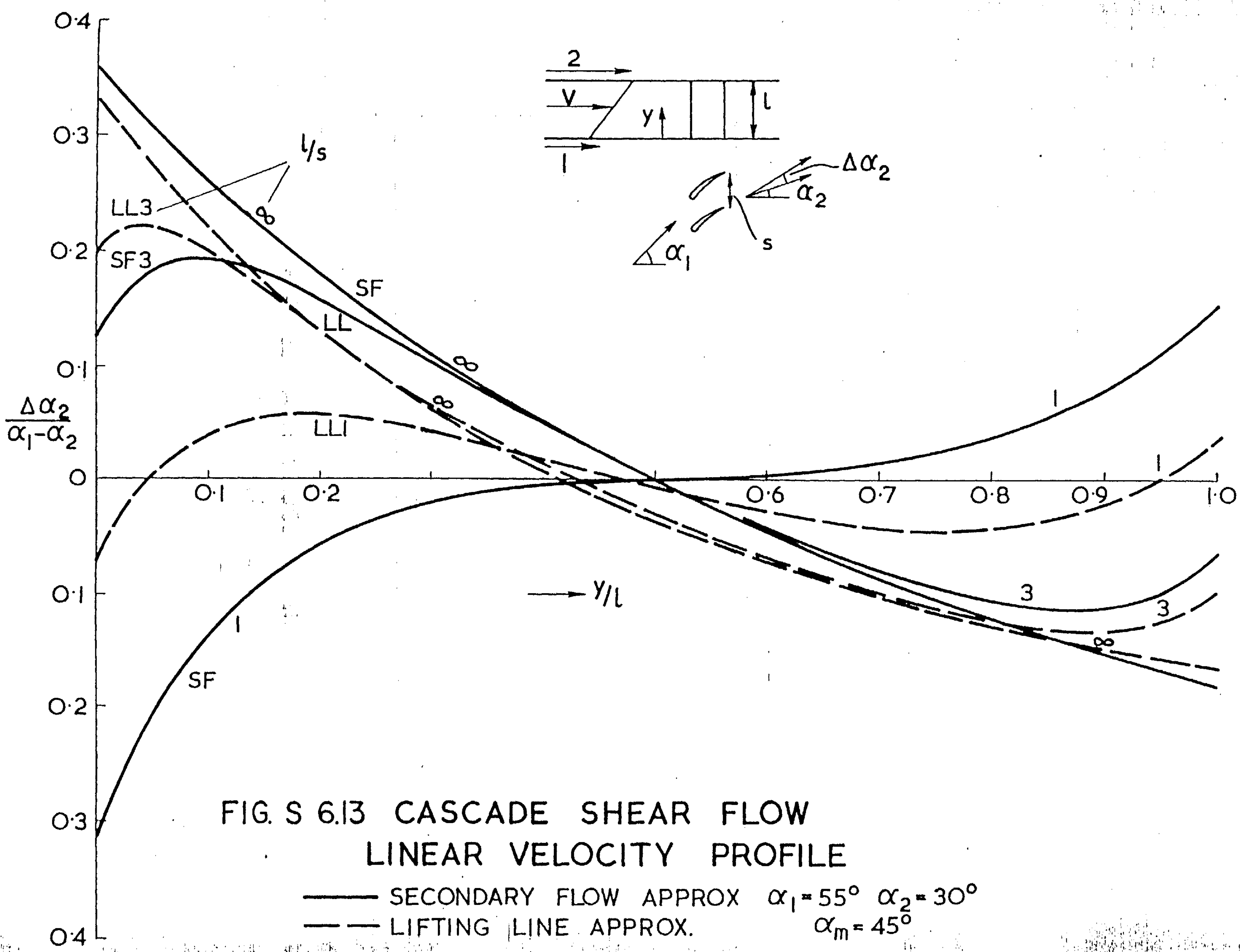

FIG. S 6.13 CASCADE SHEAR FLOW

LINEAR VELOCITY PROFILE

—— SECONDARY FLOW APPROX $\alpha_1 = 55°$ $\alpha_2 = 30°$

— — LIFTING LINE APPROX. $\alpha_m = 45°$

REFERENCES

References quoted with simple numbers, e.g. (65), will be found in G.T.L. Report No. 88. New references are quoted in the text of this report with prefix S.

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S.2 Gomi, M., "Theory on the Secondary Flows through Cascades", Bulletin Japan. Soc. Mech. Engrs. Vol X, No. 37, p. 86-99, Feb. 1967

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