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A variational characterization of the catenoid

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Abstract In this note, we use a result of Osserman and Schiffer (Arch. Rational Mech. Anal. 58:285–307, 1975) to give a variational characterization of the catenoid. Namely, we show that subsets of the catenoid minimize area within a geometrically natural class of minimal annuli. To the best of our knowledge, this fact has gone unremarked upon in the literature. As an application of the techniques, we give a sharp condition on the lengths of a pair of connected, simple closed curves σ_1 and σ_2 lying in parallel planes that precludes the existence of a connected minimal surface Σ with $\partial\Sigma = \sigma_1 \cup \sigma_2$.

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1 Introduction

Recall that the catenoid is the minimal surface of revolution given by

$$Cat = \{x_1^2 + x_2^2 = \cosh^2 x_3\} \subset \mathbb{R}^3.$$

The catenoid was discovered by Euler in 1744 and is one of the classic examples in the theory of minimal surfaces and, more broadly, in the calculus of variations. For instance, a sequence of homothetic blow-downs of Cat provides the simplest model of the failure of smooth convergence for a sequence of minimal surfaces. Due to the invariance of the minimal surface equation under rigid motions and homotheties of \mathbb{R}^3 , Cat sits within a six

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dimensional family of catenoids which we henceforth denote by \mathcal{C} . In other words, $C \in \mathcal{C}$ if C can be obtained from Cat by a rigid motion composed with a homothety.

The catenoid (or rather \mathcal{C}) has been characterized in many ways. We list some notable results: In the spirit of Euler, O. Bonnet, in the mid-nineteenth century, showed that the catenoid and plane are the only minimal surfaces of revolution. More recently, in [15], Schoen showed that the catenoid is the unique complete, minimal surface with finite total curvature and two embedded ends. In a similar vein, but by very different techniques, López and Ros showed that the catenoid and plane are the unique complete, embedded minimal surfaces of finite total curvature and genus-zero [9]. Additionally, building on work of Fischer-Colbrie [6], López and Ros characterized the catenoid as the unique complete, embedded minimal surface in \mathbb{R}^3 of Morse index one—see [8]. Finally, we note that in [14] pieces of the catenoid are shown to be the only minimal annuli in a slab that meet the boundary of the slab in a constant contact angle—a fact that will be relevant in this note. It bears mentioning that work of Collin [4] and Colding and Minicozzi [3] allows one to replace the geometric assumption of finite total curvature in [9] and [15] with the weaker topological assumption that the surfaces—with more than one end are embedded and of finite topology—that is, diffeomorphic to finitely punctured compact surfaces.

In this note, we characterize the catenoid as the unique minimal surface which minimizes area within a geometrically natural class of minimal annuli. This turns out to be a simple consequence of the proof due to Osserman and Schiffer [13] of the isoperimetric inequality for minimal annuli in \mathbb{R}^3 .

Let us now describe in what sense the catenoid minimizes area. Fix two parallel planes $P_{\pm} \subset \mathbb{R}^3$ with $P_+ \neq P_-$ and denote by Ω the open slab between them. We remark that for any plane $P \subset \Omega$, P must be parallel to P_- . Let us denote by $\mathcal{M}(\Omega)$ the class of smooth minimal surfaces spanning P_+ and P_- . That is, $\Sigma \in \mathcal{M}(\Omega)$ if $\Sigma \subset \Omega$ may be parameterized by a conformal, harmonic immersion $\mathbf{F} : M \rightarrow \mathbb{R}^3$ so that $b\Sigma := \overline{\Sigma} \setminus \Sigma \subset \partial\Omega = P_+ \cup P_-$. Here M is an open orientable surface. Notice that an element $\Sigma \in \mathcal{M}(\Omega)$ may have arbitrarily bad behavior as one approaches $\partial\Omega$; however, if Σ has the structure of a surface with boundary then $\partial\Sigma = b\Sigma$. The class $\mathcal{M}(\Omega)$ is too broad for our methods and so we will restrict attention to the subclass $\mathcal{A}(\Omega) \subset \mathcal{M}(\Omega)$ consisting of embedded annuli. Precisely, $\Sigma \in \mathcal{A}(\Omega)$ if, in addition to lying in $\mathcal{M}(\Omega)$, Σ may be injectively parameterized by an annulus; i.e. there is a smooth embedding $\mathbf{F} : (0, 1) \times \mathbb{S} \rightarrow \mathbb{R}^3$ with image Σ . It bears mentioning that in [11], by using global analysis techniques, Meeks and White have shown that the subset of $\mathcal{A}(\Omega)$ consisting of surfaces with $b\Sigma$ a pair of $C^{2,\alpha}$ (planar) convex curves has the structure of a contractible Banach manifold.

Recall that given a pair of connected, simple closed curves $\sigma_+ \subset P_+$ and $\sigma_- \subset P_-$ there need not exist $\Sigma \in \mathcal{M}(\Omega)$ with $\partial\Sigma = \sigma_+ \cup \sigma_-$. Indeed, for σ_{\pm} of sufficiently small length relative to the height of the slab, the existence of such a surface would violate the isoperimetric inequality for minimal surfaces. In Theorem 4 of [13], Osserman and Schiffer give a sharp condition on the lengths of the σ_{\pm} that precludes the existence of a minimal annulus spanning the σ_{\pm} . In Sect. 4, we refine their result and show that their condition actually precludes the existence of *any* connected minimal surface spanning the curves and prove, in addition, an interesting rigidity result.

We emphasize that the class of surfaces examined consists of *minimal* surfaces. If one were to consider classes of arbitrary surfaces spanning P_- and P_+ then the infimum of area would be zero—as can be seen by considering thin cylinders. Similarly, the surfaces in $\mathcal{M}(\Omega)$ are, in general, not stationary for area with respect to variations moving their boundary. In particular, we do not consider the free boundary problem as usually formulated for minimal surfaces.

We claim that for some $C \in \mathcal{C}$, $C \cap \Omega \in \mathcal{A}(\Omega)$ has less area than any other surface in $\mathcal{A}(\Omega)$. More precisely, denote by:

$$Cat_{MS} = \{x_1^2 + x_2^2 = \cosh^2 x_3\} \cap \{1 - x_3 \tanh x_3 > 0\} \subset Cat$$

the maximally symmetric marginally stable piece of the catenoid. That is Cat_{MS} is stable but any domain in Cat strictly containing Cat_{MS} is unstable. Recall a minimal surface is *stable* if no compactly supported infinitesimal deformation decreases area; it is *unstable* if there is a compactly supported infinitesimal deformation decreasing area. Marginally stable surfaces are on the boundary between these two classes. We show that Cat_{MS} provides the model for least area surfaces in $\mathcal{A}(\Omega)$.

Theorem 1.1 *Let P_- and P_+ be distinct parallel planes in \mathbb{R}^3 and let Ω be the open slab between them. Let $C_{MS} \in \mathcal{A}(\Omega)$ be the (unique up to translations parallel to P_{\pm}) minimal surface in $\mathcal{A}(\Omega)$ obtained from rigid motions and homotheties of Cat_{MS} . Then for any $\Sigma \in \mathcal{A}(\Omega)$:*

$$\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(C_{MS}) \tag{1.1}$$

with equality if and only if Σ is a translate of C_{MS} .

Here \mathcal{H}^k denotes k -dimensional Hausdorff measure. We remark that the two disks $D_{\pm} \subset P_{\pm}$ with $\partial D_+ \cup \partial D_- = \partial C_{MS}$ satisfy $\mathcal{H}^2(D_- \cup D_+) < \mathcal{H}^2(C_{MS})$. That is C_{MS} is not an area minimizer with respect to its boundary even though it does minimize area in the class of spanning minimal annuli $\mathcal{A}(\Omega)$.

The restriction to $\mathcal{A}(\Omega)$ in Theorem 1.1 rather than $\mathcal{M}(\Omega)$ is necessitated by our argument. However, it seems reasonable to believe that an area minimizer, $\Sigma_0 \in \mathcal{M}(\Omega)$ should have $b\Sigma_0$ rather nice—for instance consisting of a pair of convex planar curves. Hence, in light of the embeddedness results of Ekholm et al. [5] and a long standing conjecture of Meeks on the non-existence of postive genus surfaces in $\mathcal{M}(\Omega)$ bounded by convex curves (see Conjecture 3.10 of [10]), it is natural to conjecture:

Conjecture 1.2 *Let P_- and P_+ be distinct parallel planes in \mathbb{R}^3 and let Ω be the open slab between them. Let $C_{MS} \in \mathcal{M}(\Omega)$ be the (unique up to translations parallel to P_{\pm}) minimal surface in $\mathcal{M}(\Omega)$ obtained from rigid motions and homotheties of Cat_{MS} . Then for any $\Sigma \in \mathcal{M}(\Omega)$:*

$$\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(C_{MS}) \tag{1.2}$$

with equality if and only if Σ is a translate of C_{MS} .

Returning to the more restricted setting of $\mathcal{A}(\Omega)$, we note that Theorem 1.1 is a simple consequence of a more general area minimization property of the catenoid. Indeed, minimal annuli have a natural scale which may be computed as the length of the flux vector associated to the generator of the homology group—we refer to Sect. 2.1 for precise definitions. Normalizing with respect to this scale gives an area lower bound:

Theorem 1.3 *Let $P_- = \{x_3 = h_-\}$ and $P_+ = \{x_3 = h_+\}$ be distinct parallel planes in \mathbb{R}^3 with $h_- < h_+$ and let Ω be the open slab between them. Fix $\Sigma \in \mathcal{A}(\Omega)$. Let $P_0 = \{x_3 = h_0\} \subset \bar{\Omega}$ denote the plane that satisfies:*

$$\mathcal{H}^1(\Sigma \cap P_0) = \inf_{t \in (h_-, h_+)} \mathcal{H}^1(\Sigma_t).$$

Here $\Sigma_t = \Sigma \cap \{x_3 = t\}$ and $\mathcal{H}^1(\Sigma \cap P_+)$ is defined as $\liminf_{t \nearrow h_+} \mathcal{H}^1(\Sigma_t)$ and likewise for $\mathcal{H}^1(\Sigma \cap P_-)$. Let F_3 denote the vertical component of $\text{Flux}(\Sigma)$. If C is the vertical catenoid with $\text{Flux}(C) = (0, 0, F_3)$ and symmetric with respect to reflection through the plane P_0 then:

$$\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(C \cap \Omega) \tag{1.3}$$

with equality if and only if Σ is a translate of $C \cap \Omega$.

The proof of Theorem 1.3 will comprise the bulk of this note. Let us first use it to prove Theorem 1.1:

Proof By Theorem 1.3, the least area surface must be a piece of a catenoid. Up to a rescaling and rigid motion we may take $\Omega = \{-1 < x_3 < 1\}$ and so may restrict attention to subsets of vertical catenoids. The space of these catenoids are parameterized by $\lambda > 0$ and t where

$$\text{Cat}_{\lambda,t} = \lambda \text{Cat} + t\mathbf{e}_3.$$

Set

$$A(\lambda, t) = \mathcal{H}^2(\text{Cat}_{\lambda,t} \cap \Omega).$$

We will check that this value is minimized at λ_0, t_0 chosen so $\text{Cat}_{\lambda_0,t_0} \cap \Omega = \lambda_0 \text{Cat}_{MS}$.

Let us parameterize a scale λ vertical catenoid by

$$\mathbf{F}_\lambda(h, \theta) = \left(\lambda \cosh \frac{h}{\lambda} \cos \theta, \lambda \cosh \frac{h}{\lambda} \sin \theta, h \right).$$

Then we have by the area formula:

$$\begin{aligned} A(\lambda, t) &= \lambda \int_{-1-t}^{1-t} \int_0^{2\pi} \cosh^2 \frac{h}{\lambda} d\theta dh \\ &= 2\pi\lambda^2 \left(\frac{1}{4} \sinh \frac{2h}{\lambda} + \frac{h}{2\lambda} \right) \Big|_{h=-1-t}^{h=1-t} \\ &= \frac{\lambda^2\pi}{2} \left(\sinh \frac{2(1-t)}{\lambda} - \sinh \frac{2(-1-t)}{\lambda} \right) + 2\pi\lambda \\ &= \lambda^2\pi \cosh \frac{2t}{\lambda} \sinh \frac{2}{\lambda} + 2\pi\lambda. \end{aligned}$$

It is elementary to check that $A(\lambda, t) \rightarrow \infty$ as $|\ln \lambda, t| \rightarrow \infty$, so in order to find the least area catenoid we look for critical points of A . We expect these to occur for catenoids that are symmetric about the plane $\{x_3 = 0\}$. Indeed, $\partial_t A = 2\pi\lambda \sinh \frac{2t}{\lambda} \sinh \frac{2}{\lambda}$ and this equals zero if and only if $t = 0$ and hence $t_0 = 0$. Thus, we need only find λ so

$$0 = \partial_\lambda A(\lambda, 0) = 2\pi\lambda \sinh \frac{2}{\lambda} - 2\pi \cosh \frac{2}{\lambda} + 2\pi.$$

Using $\cosh x = \sqrt{1 + \sinh^2 x}$, this equation is equivalent to solving:

$$(\lambda^2 - 1) \sinh^2 \frac{2}{\lambda} + 2\lambda \sinh \frac{2}{\lambda} = 0$$

which has a unique solution $\lambda = \lambda_0 \approx 0.833$ determined by

$$\frac{2\lambda}{1 - \lambda^2} = \sinh \frac{2}{\lambda}. \tag{1.4}$$

Uniqueness follows from properties of the two functions in (1.4) on the domain $\lambda \in [0, 1) \cup (1, \infty)$. First, observe that the right hand side is always positive on this domain while the left hand function is only positive on $[0, 1)$. Second, $\sinh \frac{2}{\lambda}$ is strictly decreasing on $[0, 1)$ while $\frac{2\lambda}{1-\lambda^2}$ is strictly increasing. Finally, $\lim_{\lambda \searrow 0} \sinh \frac{2}{\lambda} - \frac{2\lambda}{1-\lambda^2} = \infty$ while $\lim_{\lambda \nearrow 1} \sinh \frac{2}{\lambda} - \frac{2\lambda}{1-\lambda^2} = -\infty$.

Lastly, we must verify that $Cat_{\lambda_0, t_0} \cap \Omega = \lambda_0 Cat_{MS}$. To that end we note that standard properties of hyperbolic functions give that

$$\sinh \frac{1}{\lambda_0} = \frac{\lambda_0}{\sqrt{1 - \lambda_0^2}}$$

and so

$$\tanh \frac{1}{\lambda_0} = \lambda_0.$$

Hence, $1 - \frac{1}{\lambda_0} \tanh \frac{1}{\lambda_0} = 0$. But this implies that $1 - \frac{1}{\lambda_0} \tanh \frac{x_3}{\lambda_0} > 0$ on $Cat_{\lambda_0, t_0} \cap \Omega$. That is, $\frac{1}{\lambda_0} (Cat_{\lambda_0, t_0} \cap \Omega) = Cat_{MS}$.

2 Convexity of the length of level sets

Before we proceed with the proof of Theorem 1.3 we recall some important definitions. We also state the result of Osserman and Schiffer regarding the convexity of the length of certain families of curves in minimal annuli in \mathbb{R}^3 .

2.1 The flux vector

For the purposes of this discussion we assume that $\Sigma \in \mathcal{A}(\Omega)$ for some Ω . Fix an orientation of Σ and let $\gamma \subset \Sigma$ be a simple C^1 closed curve in Σ on which we also fix an orientation. Our choices of orientation give rise to a conormal vector field in Σ along γ which we denote by ν . We always think of the vectors ν as vectors in \mathbb{R}^3 . The *flux* of γ is defined to be the vector:

$$Flux(\gamma) = \int_{\gamma} \nu \, d\mathcal{H}^1 \in \mathbb{R}^3. \tag{2.1}$$

As $\nu \cdot \mathbf{e}_i = \nu \cdot \nabla_{\Sigma} x_i$ and on a minimal surface $\Delta_{\Sigma} x_i = 0$, the divergence theorem implies that the flux of a curve depends only on its homology class. In particular, for a minimal annulus, Σ , we may associate a vector $Flux(\Sigma)$, by choosing γ so that $[\gamma]$ is a generator of $H_1(\Sigma)$ and setting $Flux(\Sigma) = Flux(\gamma)$; up to a reflection through the origin, $Flux(\Sigma)$ is independent of the choice of orientation of Σ and of γ . In the sequel, we will consider

$$F_3(\Sigma) = Flux(\Sigma) \cdot \mathbf{e}_3,$$

the vertical component of the flux of the minimal annulus Σ . We always choose orientations so that $F_3 \geq 0$.

An important property of the flux is that it sets a natural scale for a minimal annulus. Namely, suppose that Σ is a minimal annulus and $\Sigma' = \lambda \Sigma$ is the annulus obtained by homothetically scaling Σ by $\lambda > 0$. Then one computes $Flux(\Sigma') = \lambda Flux(\Sigma)$. In particular, the flux allows one to distinguish between catenoids of differing scales. A more subtle property of the flux is that it also helps to set a natural conformal scale for elements of $\mathcal{A}(\Omega)$.

More precisely consider $\Sigma \in \mathcal{A}(\Omega)$. By the uniformization theorem there is a conformal diffeomorphism ψ between Σ and a flat open cylinder, $(h_-, h_+) \times \mu\mathbb{S}$ where here (h_-, h_+) denotes a (possibly infinite) interval and $\mu\mathbb{S}$ denotes the circle of radius μ . Moreover, the ratio between $|h_+ - h_-|$ and μ is determined by Σ . We claim this ratio is actually determined only by Ω and $Flux(\Sigma)$.

In order to show this we first need the following fact:

Lemma 2.1 *Let $\Sigma \in \mathcal{A}(\Omega)$ and suppose that $\bar{\Sigma}$ has the structure of a surface with boundary and that $\partial\Sigma$ is smooth. Then for any plane $P \subset \bar{\Omega}$, P meets Σ transversally.*

Proof Using a rigid motion and homothety, we take $\Omega = \{-1 < x_3 < 1\}$. As $\partial\Sigma$ is smooth, standard boundary regularity results imply that Σ may be viewed as a smooth surface with boundary. We first show that Σ meets the planes $P_1 = \{x_3 = 1\}$ and $P_2 = \{x_3 = -1\}$ transversally. To that end we note that as $\partial\Sigma$ is smooth there is a uniform constant $r_0 > 0$ so that for any point $p \in \partial\Sigma$, the inner and outer osculating circles have radius greater than r_0 . Moreover, there is a uniform bound on the ratio between intrinsic and extrinsic distance between points of $\partial\Sigma$. Hence, there exists r_1 with $0 < r_1 < r_0$ so for any $p \in P_1 \cap \partial\Sigma$ the following holds: there are circles in P_1 denoted by $C_{in}(p)$ and $C_{out}(p)$, both of radius r_1 , such that $C_{in}(p)$ lies within $\partial\Sigma \cap P_1$ (thought of as a plane curve in P_1) while $C_{out}(p)$ lies outside $\partial\Sigma \cap P_1$. Moreover, both circles $C_{in}(p)$, $C_{out}(p)$ meet $\partial\Sigma$ only at p . A similar result holds for $p \in \partial\Sigma \cap P_2$. Without loss of generality we consider only $p \in \partial\Sigma \cap P_1$.

Now let $Cat^+ = Cat \cap \{x_3 \geq 0\}$. Denote by $Cat_{in}(p)$ the set obtained from Cat^+ by translations and homotheties so that $\partial Cat_{in}(p) = C_{in}(p)$ and let $Cat_{out}(p)$ be defined in an analogous manner; notice that both $Cat_{in}(p)$ and $Cat_{out}(p)$ are disjoint from Ω . Denote by $Cat'_{in}(p)$ the surfaces obtained by scaling $Cat_{in}(p)$ by $\frac{1}{2}$ about the center of $C_{in}(p)$ and define $Cat'_{out}(p)$ similarly. By the strict maximum principle and the definition of $\mathcal{A}(\Omega)$ we have that $\bar{\Sigma} \cap \partial\Omega = \partial\Sigma$. Hence, there is a $\delta > 0$ so that for all p both $Cat'_{in}(p)$ and $Cat'_{out}(p)$ can be translated along their axes by δ into Ω while remaining disjoint from Σ (that is translated in the direction $-\mathbf{e}_3$). Let us denote by $Cat''_{in}(p)$ and $Cat''_{out}(p)$ the surfaces resulting from this translation and let $\lambda Cat''_{in}(p)$ denote the result of scaling $Cat''_{in}(p)$ by $\lambda > 0$ about the center of $\partial Cat''_{in}(p)$ and similarly for $\lambda Cat''_{out}(p)$. As $\lambda \rightarrow 0$ both $\lambda Cat''_{in}(p)$ and $\lambda Cat''_{out}(p)$ converge to a plane $P''(p) \subset \Omega$ which must meet Σ as otherwise $\partial\Sigma \cap P_1 = \emptyset$ which is impossible by the convex hull property and our definition of $\mathcal{A}(\Omega)$. Hence, there are $0 < \lambda_{in}(p) < 1$ and $0 < \lambda_{out}(p) < 1$ so that, for $\lambda < \lambda_{in}(p)$, $\lambda Cat''_{in}(p)$ meets Σ but, for $\lambda > \lambda_{in}(p)$, $\lambda Cat''_{in}(p)$ is disjoint from Σ ; and the same for $\lambda_{out}(p)$ with respect to $\lambda Cat''_{out}(p)$. As a consequence, $\lambda_{in}(p) Cat''_{in}(p)$ is disjoint from $\Sigma \setminus \partial\Sigma$ but meets $\partial\Sigma$ and it must do so precisely at p and the same is true for $\lambda_{out}(p) Cat''_{out}(p)$. By the boundary maximum principle we then see that the normal to Σ at p cannot be orthogonal to P_1 and hence P_1 meets Σ transversally as do all planes $\{x_3 = t\}$ for $1 - \epsilon < t \leq 1$. A similar result holds for P_2 . Thus, for all planes $P \subset \Omega$ near P_1 or P_2 , $P \cap \Sigma$ consists of a single smooth simple closed curve.

Now let $f = x_3$ be the function whose level sets are planes in Ω . As Σ is minimal f is a harmonic function on Σ and so has no local maxima or minima. In particular, at any critical points of f the vector field ∇f has negative index. By our previous discussion f has no critical points near $\partial\Sigma$ and, moreover, ∇f is transverse to $\partial\Sigma$. As Σ is an annulus, the Hopf index theorem then implies that f has no critical points.

As a consequence, $Flux(\Sigma)$ and Ω determine the cylinder with which Σ is conformally equivalent:

Corollary 2.2 *Suppose that $\Omega = \{h_- < x_3 < h_+\}$ is an open slab. For any $\Sigma \in \mathcal{A}(\Omega)$ there is a conformal diffeomorphism*

$$\psi : \Sigma \rightarrow (h_-, h_+) \times \mu\mathbb{S}$$

given by $\psi(p) = (x_3(p), x_3^*(p))$. Moreover, $\mu = \frac{1}{2\pi} F_3(\Sigma) > 0$.

Proof Let J be the almost-complex structure on Σ arising from the metric and some choice of orientation. As x_3 is a harmonic function, dx_3 is a harmonic one form. Moreover, $dx_3^* = dx_3 \circ J$, the conjugate differential, is also harmonic. In general, dx_3^* , while closed, will never be exact. Indeed, for a general closed curve γ in Σ :

$$\text{Flux}(\gamma) \cdot \mathbf{e}_3 = \int_{\gamma} dx_3^*.$$

Thus, integrating dx_3^* gives a map $x_3^* : \Sigma \rightarrow \mathbb{R}/F_3\mathbb{Z}$ where $F_3 = F_3(\Sigma) \geq 0$. We will see that we must have $F_3 > 0$ and so $\mathbb{R}/F_3\mathbb{Z} = \mu\mathbb{S}$. As dx_3 and dx_3^* have the same length and are orthogonal, if we set $\psi = (x_3, x_3^*)$ then ψ is a conformal map.

By Sard’s theorem, for each $\epsilon > 0$ there is an $0 < \epsilon' < \epsilon$ so that both $\{x_3 = h_- + \epsilon'\}$ and $\{x_3 = h_+ - \epsilon'\}$ meet Σ transversally. Moreover, as $b\Sigma \subset \partial\Omega$, each of the finitely many components of $\{h_- + \epsilon' \leq x_3 \leq h_+ - \epsilon'\} \cap \Sigma$ has the structure of a surface with boundary. By the convex hull property of minimal surfaces there is exactly one such component $\Sigma_{\epsilon'}$ and it is an annulus. Lemma 2.1 implies that for $h_- + \epsilon' < t < h_+ - \epsilon'$, each plane $\{x_3 = t\}$ meets $\Sigma_{\epsilon'}$, and hence Σ , transversally. In particular, dx_3 does not vanish on $\Sigma_{\epsilon'}$ and hence ψ restricted to $\Sigma_{\epsilon'}$ is a local diffeomorphism. As each level set of $x_3 = t$ for $h_- + \epsilon' \leq t \leq h_+ - \epsilon'$ is connected and dx_3^* does not vanish, ψ is injective on these level sets and hence the restriction of ψ to $\Sigma_{\epsilon'}$ is injective. In addition, it is then clear that $F_3 \neq 0$. Taken together it follows that ψ restricts to a conformal diffeomorphism between $\Sigma_{\epsilon'}$ and $(h_- + \epsilon', h_+ - \epsilon') \times \mu\mathbb{S}$. As ϵ may be taken as small as we like, the result is shown.

2.2 Osserman and Schiffer’s result

We now record the convexity result of Osserman and Schiffer—Theorem 1 from [13]—that we will use. This result was a key step in their proof—also in [13]—of the sharp isoperimetric inequality for doubly connected minimal surfaces in \mathbb{R}^3 . We point out that the restriction to \mathbb{R}^3 comes from their use of the Weierstrass representation in order to prove the convexity result. Roughly speaking, Osserman and Schiffer show that when a minimal annulus $\Sigma \subset \mathbb{R}^3$ is conformally parametrized by an annulus A in the complex plane, then the length of the images in Σ of the circles foliating A satisfy a convexity condition that is sharp on catenoids and planar annuli. Precisely,

Lemma 2.3 *Let $A_{r,R} = \{z : r < |z| < R\} \subset \mathbb{C}$ and suppose that $\mathbf{F} : A_{r,R} \rightarrow \mathbb{R}^3$ is a conformal harmonic immersion (so in particular the image of \mathbf{F} is a minimal surface). If we let σ_{ρ} be the image of $|z| = \rho$ under \mathbf{F} and define:*

$$L(t) = \mathcal{H}^1(\sigma_{e^t}) \tag{2.2}$$

then

$$L''(t) \geq L(t) \tag{2.3}$$

with equality if and only if \mathbf{F} maps into a planar annulus or into a piece of a catenoid bounded by coaxial circles in parallel planes.

Rather than using Lemma 2.3 directly we use the following corollary:

Corollary 2.4 *Suppose that $\Omega = \{h_- < x_3 < h_+\}$ and $\Sigma \in \mathcal{A}(\Omega)$. Set $\Sigma_t = \Sigma \cap \{x_3 = t\}$. Then for $t \in (h_-, h_+)$:*

$$\frac{d^2}{dt^2} \mathcal{H}^1(\Sigma_t) \geq \frac{(2\pi)^2}{F_3(\Sigma)^2} \mathcal{H}^1(\Sigma_t)$$

with equality if and only if Σ is a piece of a vertical catenoid C .

Proof By Corollary 2.2, $F_3 \neq 0$ and there is a conformal diffeomorphism

$$\psi : (h_-, h_+) \times \mu\mathbb{S} \rightarrow \mathbb{R}^3$$

with image Σ and so that Σ_t is the image of (t, \cdot) under ψ . Here $\mu = \frac{1}{2\pi} F_3$. One verifies that the map:

$$\begin{aligned} G : (h_-, h_+) \times \mu\mathbb{S} &\rightarrow A_{R_-, R_+} \subset \mathbb{C} \\ (h, \theta) &\mapsto e^{\frac{h}{\mu} + i\frac{\theta}{\mu}} \end{aligned}$$

is a conformal diffeomorphism. Here $A_{R_-, R_+} = \{R_- < |z| < R_+\}$ with $R_- = e^{\frac{h_-}{\mu}}$ and $R_+ = e^{\frac{h_+}{\mu}}$. As a consequence, we obtain a conformal diffeomorphism:

$$\mathbf{F} = \psi \circ G^{-1}$$

as in Lemma 2.3. We check that $\Sigma_t = \mathbf{F} \left(|z| = e^{\frac{t}{\mu}} \right)$ and so

$$\mathcal{H}^1(\Sigma_t) = L \left(\frac{t}{\mu} \right).$$

The corollary then follows immediately from Lemma 2.3 and the fact that $\frac{1}{\mu^2} = \frac{(2\pi)^2}{F_3^2}$.

Remark 2.5 We give an alternate approach to Corollary 2.4 in Appendix 5. While this approach avoids the use of the Weierstrass representation and gives a sharper conclusion, it requires a certain geometric estimate that is still conjectural.

3 The area bound

In order to prove Theorem 1.3 we use Corollary 2.4 to obtain a bound for the lengths of level sets:

Proposition 3.1 *Let $P_- = \{x_3 = h_-\}$ and $P_+ = \{x_3 = h_+\}$ be distinct parallel planes in \mathbb{R}^3 with $h_- < h_+$ and let Ω be the open slab between them. Fix $\Sigma \in \mathcal{A}(\Omega)$. Let $P_0 = \{x_3 = h_0\} \subset \bar{\Omega}$ denote the plane that satisfies:*

$$\mathcal{H}^1(\Sigma \cap P_0) = \inf_{t \in (h_-, h_+)} \mathcal{H}^1(\Sigma_t).$$

Here $\Sigma_t = \Sigma \cap \{x_3 = t\}$ and $\mathcal{H}^1(\Sigma \cap P_+)$ is defined as $\liminf_{t \nearrow h_+} \mathcal{H}^1(\Sigma_t)$ and likewise for $\mathcal{H}^1(\Sigma \cap P_-)$. Let C denote the vertical catenoid with $\text{Flux}(C) = (0, 0, F_3(\Sigma))$, symmetric with respect to reflection through the plane P_0 . If $C_t = C \cap \{x_3 = t\}$ then for $t \in [h_-, h_+]$:

$$\mathcal{H}^1(\Sigma_t) \geq \mathcal{H}^1(C_t). \tag{3.1}$$

Equality can hold when $t \neq h_0$ if and only if Σ is a translate of $C \cap \Omega$.

Proof Set $L_\Sigma(t) = \mathcal{H}^1(\Sigma_t)$ for $t \in (h_-, h_+)$. By Lemma 2.1, $L_\Sigma(t)$ depends smoothly on t and by Corollary 2.4 one has:

$$\frac{d^2}{dt^2} L_\Sigma(t) \geq \frac{(2\pi)^2}{F_3(\Sigma)^2} L_\Sigma(t),$$

with equality if and only if Σ is piece of a catenoid. Notice that L_Σ is a convex function on (h_-, h_+) . By setting $L_\Sigma(h_-) = \lim_{t \searrow h_-} L_\Sigma(t)$ and $L_\Sigma(h_+) = \lim_{t \nearrow h_+} L_\Sigma(t)$, we may think of L_Σ as a function on $[h_-, h_+]$ but possibly taking the value ∞ at the end points. The convexity ensures these limits exist.

For C as in the statement of the theorem, $C \cap \Omega \in \mathcal{A}(\Omega)$. Set $L_C(t) = \mathcal{H}^1(C_t)$. As $\text{Flux}(C) \cdot \mathbf{e}_3 = F_3(\Sigma) = F_3$ by assumption, Corollary 2.4 implies:

$$\frac{d^2}{dt^2} L_C(t) = \frac{(2\pi)^2}{F_3^2} L_C(t).$$

Notice that L_C is smooth on $[h_-, h_+]$ and the symmetry about $P_0 = \{x_3 = h_0\}$ implies $L'_C(h_0) = 0$.

We claim that $L_\Sigma \geq L_C$ on $[h_-, h_+]$ with equality if and only if Σ is a horizontal translate of $C \cap \Omega$. To see this we distinguish between when $h_0 \in (h_-, h_+)$ and when h_0 is an endpoint. For any $t \in (h_-, h_+)$

$$F_3 = \left| \int_{\Sigma_t} \mathbf{e}_3 \cdot \nu ds \right| \leq \int_{\Sigma_t} ds = L_\Sigma(t)$$

with equality if and only if Σ_t is a geodesic in Σ . Similarly, if $t = h_-$ then

$$F_3 = \lim_{t \searrow h_-} \left| \int_{\Sigma_t} \mathbf{e}_3 \cdot \nu ds \right| \leq \lim_{t \searrow h_-} \int_{\Sigma_t} ds = L_\Sigma(h_-)$$

and the corresponding result holds when $t = h_+$. As C_{h_0} is a geodesic in C , $F_3 = F_3(C) = L_C(h_0)$. Hence, $L_C(h_0) \leq L_\Sigma(h_0)$.

Now assume that $h_0 \in (h_-, h_+)$. For $t \in (h_-, h_+)$, the choice of C and Corollary 2.4 ensure that $L_C(h_0) \leq L_\Sigma(h_0)$, $L'_\Sigma(h_0) = 0 = L'_C(h_0)$ and $\frac{d^2}{dt^2} (L_\Sigma(t) - L_C(t)) \geq \frac{(2\pi)^2}{F_3^2} (L_\Sigma(t) - L_C(t))$. An ODE comparison then implies that for all $t \in [h_-, h_+]$, $L_C(t) \leq L_\Sigma(t)$ with equality holding for any $t \neq h_0$ if and only if Σ is a piece of a catenoid and Σ_{h_0} is a geodesic in Σ . Thus, equality holds for any $t \neq h_0$, if and only if Σ is equal (up to a translation) to $C \cap \Omega$.

When $h_0 = h_-$ we argue as follows: For $\epsilon > 0$ small, set $L_{C,\epsilon}(t) = L_C(t - \epsilon)$ and restrict attention to $(h_- + \epsilon, h_+)$. Clearly, $L''_{C,\epsilon} = \frac{(2\pi)^2}{F_3^2} L_{C,\epsilon}$, $L'_{C,\epsilon}(h_- + \epsilon) = 0$ and $L_\Sigma(h_- + \epsilon) > L_\Sigma(h_-) \geq L_{C,\epsilon}(h_- + \epsilon)$. Moreover, as L_Σ is convex and has its minimum at h_- , $L'_\Sigma(h_- + \epsilon) > 0$. Hence by an ODE comparison, $L_\Sigma(t) > L_{C,\epsilon}(t)$ for $t \in [h_- + \epsilon, h_+)$. Letting $\epsilon \rightarrow 0$ implies $L_\Sigma(t) \geq L_C(t)$ for $t \in [h_-, h_+]$. Equality can hold if and only if Σ is equal (up to a translation) to $C \cap \Omega$. An identical argument applies when $h_0 = h_+$.

Remark 3.2 Proposition 3.1 fails if Σ were taken in the larger class of embedded elements of $\mathcal{M}(\Omega)$. Indeed, normalizing as in the proposition, it can be verified that (a suitably modified) version of Corollary 2.4 continues to hold for Σ at t so that $\{x_3 = t\}$ meets Σ transversally.

In particular, a modified version of Proposition 3.1 holds between critical values of x_3 . However, while the length of level sets is continuous across critical values of x_3 , the rate of change of the length of these level sets becomes infinite at a critical value. In particular, there is never convexity across critical levels.

Let us now use Proposition 3.1 to prove Theorem 1.3:

Proof We have verified $\mathcal{H}^1(C_t) \leq \mathcal{H}^1(\Sigma_t)$ with equality for all t if and only if Σ is equal (up to a translation) to $C \cap \Omega$. Fix $h_* \in (h_-, h_+)$ and define the function $A_{\Sigma, h_*}(t)$ on $[h_*, h_+)$ by

$$A_{\Sigma, h_*}(t) = \mathcal{H}^2(\Sigma \cap \{h_* \leq x_3 < t\})$$

and the function A_{C, h_*} similarly.

The co-area formula implies that

$$\frac{d}{dt} A_{\Sigma, h_*}(t) = \int_{\{x_3=t\} \cap \Sigma} \frac{1}{|\nabla_{\Sigma} x_3|}.$$

Applying the Cauchy-Schwarz inequality yields:

$$\frac{d}{dt} A_{\Sigma, h_*}(t) \geq \frac{\mathcal{H}^1(\Sigma_t)^2}{\int_{\Sigma_t} |\nabla_{\Sigma} x_3|} = \frac{\mathcal{H}^1(\Sigma_t)^2}{F_3}.$$

Notice that one has equality if and only if $\frac{1}{|\nabla_{\Sigma} x_3|}$ and $|\nabla_{\Sigma} x_3|$ are linearly dependent, in other words are both constant. This is readily checked to be the case on C and so using the above estimate for length:

$$\frac{d}{dt} A_{\Sigma, h_*}(t) \geq \frac{\mathcal{H}^1(C_t)^2}{F_3} = \frac{d}{dt} A_{C, h_*}(t).$$

Integrating implies $A_{\Sigma, h_*}(t) \geq A_{C, h_*}(t)$ for $t \in [h_*, h_+)$ with equality if and only if Σ is a horizontal translate of $C \cap \Omega$. Letting $h_* \rightarrow h_-$ proves the theorem.

4 Sharp non-existence result

As discussed in the introduction, given an open slab Ω with $\partial\Omega = P_+ \cup P_-$ and connected, simple closed curves $\sigma_{\pm} \subset P_{\pm}$ there need not be a surface $\Sigma \in \mathcal{M}(\Omega)$ with $\partial\Sigma = \sigma_+ \cup \sigma_-$. For instance, if the curves σ_{\pm} are too short relative to the height of the slab then there cannot be a connected minimal surface $\Sigma \in \mathcal{M}(\Omega)$ spanning σ_{\pm} . Indeed, the monotonicity formula gives a lower bound on the area of such a Σ in terms of the distance between the planes, while the isoperimetric inequality gives an upper bound in terms of the lengths of the curves (for surfaces in $\mathcal{M}(\Omega)$ with two boundary components the isoperimetric inequality with sharp constant is known to hold—see [16]). Alternatively, if the σ_{\pm} are well separated, barrier arguments can be used to rule out the existence of such Σ . Using Proposition 3.1, we are able to give a sharp condition (see also Theorem 6 of [13] for a related result):

Theorem 4.1 *Fix Ω an open slab with $\partial\Omega = P_+ \cup P_-$ the union of two parallel planes. Let $\sigma_{\pm} \subset P_{\pm}$ be a pair of connected simple closed curves. Let C_{MS} be the unique (up to translations parallel to P_{\pm}) minimal surface in $\mathcal{A}(\Omega)$ obtained via rigid motions and homotheties from Cat_{MS} . If we define $L_{crit}(\Omega) := \mathcal{H}^1(\partial C_{MS})$ and*

$$\mathcal{H}^1(\sigma_+ \cup \sigma_-) < L_{crit}(\Omega)$$

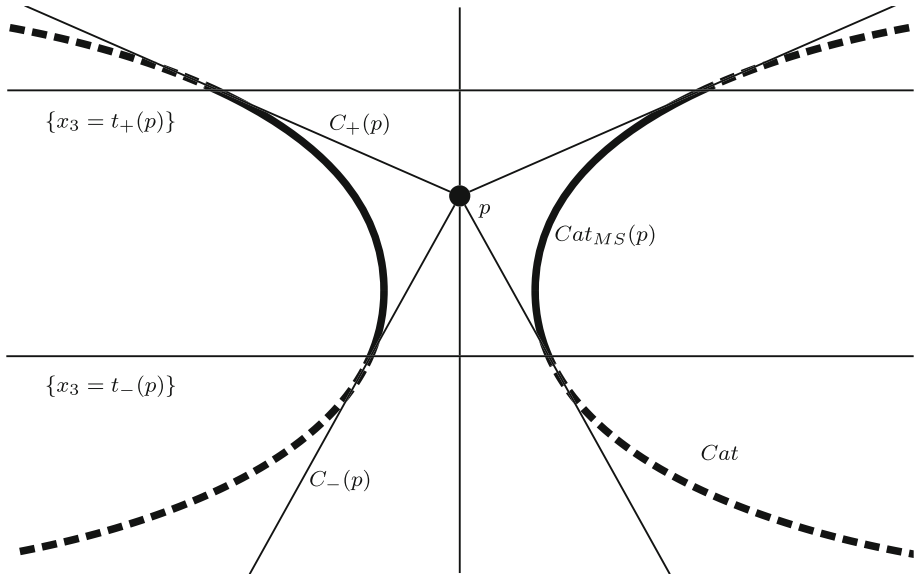


Fig. 1 The subset $Cat_{MS}(p)$ of Cat is indicated as well as the cones $C_+(p)$ and $C_-(p)$.

then there is no surface $\Sigma \in \mathcal{M}(\Omega)$ with $\partial \Sigma = \sigma_+ \cup \sigma_-$. Moreover, if $\Sigma \in \mathcal{M}(\Omega)$ is a smooth minimal surface with $\partial \Sigma = \sigma_- \cup \sigma_+$ and

$$\mathcal{H}^1(\partial \Sigma) = L_{crit}(\Omega)$$

then Σ is a translate of C_{MS} .

This theorem follows from a more general result. Namely, we will show the existence of a $\Sigma \in \mathcal{M}(\Omega)$ with $\partial \Sigma = \sigma_+ \cup \sigma_-$ is precluded if one boundary curve is too short as determined by a function of the length of the other boundary curve. Roughly speaking, the existence of such a Σ implies the existence of a vertical catenoid C so that $\mathcal{H}^1(C \cap P_{\pm}) = \mathcal{H}^1(\sigma_{\pm})$. We point out that Theorem 4 of [13] gives the same result when one considers only $\Sigma \in \mathcal{A}(\Omega)$. As in the case for area bounds, a marginally stable piece of a catenoid will serve as the model. However, here the marginally stable pieces are generally not obtained from rigid motions and homotheties of Cat_{MS} .

We begin by describing the general class of marginally stable pieces of Cat we will need. First note that the rotational symmetry and convexity of the function $\cosh t$ imply that for each point $p = ze_3$ on the x_3 -axis, there are unique cones over p that intersect Cat tangentially. Precisely, there exist values $t_+ = t_+(p) > 0$ and $t_- = t_-(p) < 0$ with $t_-(p) < z < t_+(p)$ and so that the cones $C_+(p)$ (resp. $C_-(p)$) over p of the curve $\gamma_{t_+(p)} = Cat \cap \{x_3 = t_+(p)\}$ (resp. $\gamma_{t_-(p)} = Cat \cap \{x_3 = t_-(p)\}$) meet Cat only at $\gamma_{t_+(p)}$ (resp. $\gamma_{t_-(p)}$). We observe also that t_+ is an increasing and continuous function of z with range $(0, \infty)$; similarly, t_- is increasing and continuous with range $(-\infty, 0)$. Notice that Cat is tangential to $C_{\pm}(p)$ at $\gamma_{\pm}(p)$. We refer the reader to Fig. 1.

Let $Cat_{MS}(p)$ be the bounded component of $Cat \setminus (C_+(p) \cup C_-(p))$. Clearly, $Cat_{MS} = Cat_{MS}(0)$ and as $p \rightarrow (0, 0, \infty)$, $Cat_{MS}(p)$ converges to $Cat \cap \{x_3 > 0\}$. We claim that $Cat_{MS}(p)$ is marginally stable for each p . This follows from the observation that for $\lambda > 0$ the surfaces $Cat_{MS}^{\lambda}(p) = p + \lambda(Cat_{MS}(p) - p)$ give a foliation of the component $\hat{C}(p)$

of $\mathbb{R}^3 \setminus (C_+(p) \cup C_-(p))$ containing $Cat_{MS}(p)$. Moreover, as Cat meets $\partial\hat{C}(p) := C(p)$ tangentially, the motion of $\partial Cat_{MS}^\lambda(p)$ at $\lambda = 1$ is tangential to $Cat_{MS}(p)$. As a consequence, the normal variation at $\lambda = 1$ gives a positive Jacobi field on $Cat_{MS}(p)$ vanishing on $\partial Cat_{MS}(p)$. One verifies that if $C_0 = Cat \cap \{h_- < x_3 < h_+\}$ is marginally stable then $C_0 = Cat_{MS}(p)$ for a unique value p .

These marginally stable pieces determine the function F_Ω , which is described in Lemma 4.3. In the proof of that lemma, we frequently reference Theorem 1.1 of [11]; for the readers convenience, we restate it here:

Theorem 4.2 *Let P_\pm be parallel planes and $\Omega \subset \mathbb{R}^3$ the open slab with $\partial\Omega = P_+ \cup P_-$. Let $\sigma_\pm \subset P_\pm$ be closed convex curves of class $C^{2,\alpha}$. Then $\sigma_+ \cup \sigma_-$ bound either*

1. *No minimal surface in Ω ;*
2. *Exactly one minimal surface, $\Sigma \subset \Omega$, which is a marginally stable annulus;*
3. *One strictly stable minimal annulus $\Sigma_S \subset \Omega$ and one index one minimal annulus $\Sigma_U \subset \Omega$ and possibly other minimal surfaces in Ω of higher genus.*

Using Theorem 4.2 and the marginally stable pieces described previously, we have:

Lemma 4.3 *Fix $\Omega_0 = \{-1 < x_3 < 1\}$ and $\partial\Omega_0 = P_+ \cup P_-$. There exist two functions $C_{MS} : \mathbb{R}^+ \rightarrow \mathcal{A}(\Omega_0)$ and $F_{\Omega_0} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ determined in the following way: For each $L_- > 0$ let $C_{MS}(L_-)$ denote the unique (up to translations parallel to P_\pm) marginally stable piece of a vertical catenoid with*

$$\partial C_{MS}(L_-) = \gamma_+ \cup \gamma_- \subset P_+ \cup P_-$$

and $\mathcal{H}^1(\gamma_-) = L_-$. Given $C_{MS}(L_-)$, define

$$F_{\Omega_0}(L_-) = \mathcal{H}^1(\gamma_+).$$

Furthermore, F_{Ω_0} has the following properties:

1. *If $L_+ \geq F_{\Omega_0}(L_-)$ then there is a vertical catenoid $C = C(L_-, L_+)$ so that if $\partial(C \cap \Omega_0) = \gamma_+ \cup \gamma_-$ then $\mathcal{H}^1(\gamma_\pm) = L_\pm$, while if $L_+ < F_{\Omega_0}(L_-)$ no such vertical catenoid exists.*
2. *If C is a vertical catenoid with $\partial(C \cap \Omega_0) = \gamma_+ \cup \gamma_-$ and $\mathcal{H}^1(\gamma_-) < L_-$ then $\mathcal{H}^1(\gamma_+) > F_{\Omega_0}(L_-)$.*
3. *If C is a vertical catenoid with $\partial(C \cap \Omega_0) = \gamma_+ \cup \gamma_-$, $\mathcal{H}^1(\gamma_-) = L_-$, and $\mathcal{H}^1(\gamma_+) = F_{\Omega_0}(L_-)$ then $C \cap \Omega_0$ is a translate of $C_{MS}(L_-)$.*

Remark 4.4 We note that for other slabs Ω it is straightforward to determine F_Ω in terms of F_{Ω_0} . Indeed, rigid motions leave the function invariant and $F_{\lambda\Omega}(L) = \lambda F_{\Omega}(\lambda^{-1}L)$.

Proof We claim that given any $L_- > 0$ there is a marginally stable piece of a vertical catenoid $C_{MS}(L_-)$ with $\partial C_{MS}(L_-) = \gamma_+ \cup \gamma_- \subset P_+ \cup P_-$ and $\mathcal{H}^1(\gamma_-) = L_-$. Rather than prove this by direct computation, we use global arguments. Set

$$C = \frac{L_-}{2\pi} Cat - \mathbf{e}_3.$$

Let $\Sigma_0 = C \cap \Omega_0$ and denote $\partial\Sigma_0 = \sigma_+^0 \cup \sigma_-^0$ so that $\mathcal{H}^1(\sigma_-^0) = L_-$. By domain monotonicity for eigenvalues, Σ_0 is strictly stable because $C \cap \{x_3 \geq -1\}$ is stable. Consider now the following smooth family of coaxial circles in P_\pm : for $t > 0$ set $\sigma_-^t = \sigma_-^0$ and $\sigma_+^t = t(\sigma_+^0 - \mathbf{e}_3) + \mathbf{e}_3$. By Theorem 4.2, there is a $1 > T_{crit} > 0$ so that for each $t \in (T_{crit}, 1]$

there are Σ_t , strictly stable minimal annuli smoothly depending on t , with $\partial \Sigma_t = \sigma_+^t \cup \sigma_-^t$ and for $t = T_{crit}$ there is a marginally stable annulus, $\Sigma_{T_{crit}}$, with $\partial \Sigma_{T_{crit}} = \sigma_-^{T_{crit}} \cup \sigma_+^{T_{crit}}$.

We claim that $C_{MS}(L_-) = \Sigma_{T_{crit}}$. To that end, let Σ be a piece of a vertical catenoid with $\partial \Sigma = \sigma_+ \cup \sigma_- \subset P_+ \cup P_-$ and $\mathcal{H}^1(\sigma_-) = L_-$ while $\mathcal{H}^1(\sigma_+) \leq \mathcal{H}^1(\sigma_+^{T_{crit}})$. Using Theorem 4.2 and a translation we may take Σ to be stable and coaxial with $\Sigma_{T_{crit}}$. Suppose $\mathcal{H}^1(\sigma_+) < \mathcal{H}^1(\sigma_+^{T_{crit}})$. Using Theorem 4.2, we observe Σ is a subset of the bounded component of $\Omega_0 \setminus \Sigma_{T_{crit}}$. Otherwise, $\Sigma \cap \Sigma_{T_{crit}}$ would contain two coaxial circles in parallel planes. Domain monotonicity would imply that these circles bounded two stable annuli.

Our previous discussion implies there exist p_0, λ_0 and h_0 so that $\Sigma_{T_{crit}} = \lambda_0(Cat_{MS}(p_0) - p_0) + p_0 + h_0e_3$. If $\hat{C}(p_0) + h_0e_3$ is the open cone containing $\Sigma_{T_{crit}}$ then by construction it is foliated by the surfaces $\lambda(Cat_{MS}(p_0) - p_0) + p_0 + h_0e_3$, all of which are tangent to $\partial \hat{C}(p_0) + h_0e_3$. As the vertex of $\hat{C}(p_0) + h_0e_3$ is contained within Ω_0 , $\hat{\Sigma} := \Sigma \cap (\hat{C}(p_0) + h_0e_3) \neq \emptyset$. Using the boundary maximum principle on $\hat{\Sigma}$, and the foliations just mentioned, there exists some λ_1 such that $\hat{\Sigma} = \lambda_1(Cat_{MS}(p_0) - p_0) + p_0 + h_0e_3$. In particular $\hat{\Sigma}$ is marginally stable. This is impossible as domain monotonicity would then imply Σ was strictly unstable. Hence, $\mathcal{H}^1(\sigma_+) = \mathcal{H}^1(\sigma_+^{T_{crit}})$ and so $\sigma_{\pm} = \sigma_{\pm}^{T_{crit}}$. However, as $\Sigma_{T_{crit}}$ is a marginally stable annulus, Theorem 4.2 implies $\Sigma = \Sigma_{T_{crit}}$ which verifies the claim and Item (3).

Clearly, Item (1) is an immediate consequence of the preceding argument together with a barrier argument (see the proof of Proposition 4.5 for a detailed argument). Furthermore, if Item (2) failed to hold for a vertical catenoid C' then it could be used as a barrier allowing one to construct a piece of a vertical catenoid violating Item (1).

As a consequence, we conclude a general proposition giving sharp conditions for the non-existence of minimal surfaces spanning a given pair of curves:

Proposition 4.5 *Fix Ω an open slab with $\partial \Omega = P_+ \cup P_-$ the union of two parallel planes and let $C_{MS} : \mathbb{R}^+ \rightarrow \mathcal{A}(\Omega)$, $F_{\Omega} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the functions given by Lemma 4.3. Let $\sigma_{\pm} \subset P_{\pm}$ be a pair of connected, simple closed curves. If*

$$\mathcal{H}^1(\sigma_+) < F_{\Omega}(\mathcal{H}^1(\sigma_-))$$

then there is no surface $\Sigma \in \mathcal{M}(\Omega)$ with $\partial \Sigma = \sigma_+ \cup \sigma_-$. Moreover, if $\Sigma \in \mathcal{M}(\Omega)$ is a smooth minimal surface with $\partial \Sigma = \sigma_+ \cup \sigma_-$ such that

$$\mathcal{H}^1(\sigma_+) = F_{\Omega}(\mathcal{H}^1(\sigma_-))$$

then Σ is a translate of $C_{MS}(\mathcal{H}^1(\sigma_-))$.

Remark 4.6 Let $\mathcal{M}_2(\Omega) \subset \mathcal{M}(\Omega)$ be the set of all $\Sigma \in \mathcal{M}(\Omega)$ with $b\Sigma = \partial \Sigma$ consisting of exactly two connected boundary components. If one considers $\Psi_{\Omega} : \mathcal{M}_2(\Omega) \rightarrow \mathbb{R}^2$ the map defined by

$$\Psi_{\Omega}(\Sigma) = (\mathcal{H}^1(\partial \Sigma \cap P_-), \mathcal{H}^1(\partial \Sigma \cap P_+))$$

then the proposition says that the image of Ψ_{Ω} is an unbounded region in the first quadrant of the plane whose boundary consists of the images of marginally stable pieces of catenoids—i.e. is the graph of the function F_{Ω} .

Proof Up to a rescaling and rigid motion we may take $\Omega = \{-1 < x_3 < 1\}$. Suppose that $\Sigma \in \mathcal{M}(\Omega)$ has the structure of a smooth manifold with boundary and that $\partial \Sigma$ is embedded and consists of two connected components $\sigma_{\pm} \subset P_{\pm}$. By assumption, the σ_{\pm} are connected, simple closed curves in P_+ and P_- . It will suffice to show that $\mathcal{H}^1(\sigma_+) \geq F_{\Omega}(\mathcal{H}^1(\sigma_-))$. Note

that Σ is allowed to be immersed and have arbitrary genus, however as Σ is oriented it may still be used as a barrier to construct an embedded annulus with the same boundary. Indeed, while $\Omega \setminus \Sigma$ may have more than 2 components, only one of these, Ω' , is unbounded. Clearly, σ_+ and σ_- are homotopic in $\tilde{\Omega}'$ but are not null homotopic in $\tilde{\Omega}'$. In particular, there is an annulus A in Ω' with $\partial A = \sigma_+ \cup \sigma_-$ but no disk D in Ω' with $\partial D = \sigma_+$ or $\partial D = \sigma_-$. Finally, we point out that $\tilde{\Omega}'$ is mean convex in the sense of Meeks and Yau [12]. As a consequence, by [12] there is an embedded minimal annulus $\Gamma \subset \Omega'$ with $\partial\Gamma = \sigma_+ \cup \sigma_-$.

By Proposition 3.1, there is a vertical catenoid C so that if we write $\partial(C \cap \Omega) = \gamma_+ \cup \gamma_-$ where $\gamma_{\pm} \subset P_{\pm}$ then $\mathcal{H}^1(\sigma_{\pm}) \geq \mathcal{H}^1(\gamma_{\pm})$. Moreover the inequality is strict for at least one pair of curves unless $C \cap \Omega$ and Γ agree up to a translation parallel to P_{\pm} . As $\mathcal{H}^1(\gamma_-) \leq \mathcal{H}^1(\sigma_-)$, by (2) of Lemma 4.3

$$\mathcal{H}^1(\sigma_+) \geq \mathcal{H}^1(\gamma_+) \geq F_{\Omega}(\mathcal{H}^1(\sigma_-)).$$

Finally, by Lemma 4.3 and Proposition 3.1 equality is only achieved if Γ is a horizontal translate of $C_{MS}(\mathcal{H}^1(\sigma_-))$. In this case, $\sigma_+ \cup \sigma_- = \partial\Gamma$ consist of coaxial circles in parallel planes. Hence, the proof of [15] implies the σ_{\pm} can bound only pieces of a catenoid; that is, Γ is a horizontal translate of $C_{MS}(\mathcal{H}^1(\sigma_-))$. Thus by Theorem 4.2, Σ is as well.

We now prove Theorem 4.1:

Proof Up to a rescaling and rigid motion we may take $\Omega = \{-1 < x_3 < 1\}$. By Proposition 4.5 we need only verify the theorem for vertical catenoids. The space of vertical catenoids is parameterized by $\lambda > 0$ and t where $Cat_{\lambda,t} = \lambda Cat + t\mathbf{e}_3$. Just as in the proof of Theorem 1.1 where we saw C_{MS} minimized area among all vertical catenoid pieces in Ω , we now show C_{MS} minimizes boundary length in this same class. Setting $L(\lambda, t) = \mathcal{H}^1(\partial(\Omega \cap Cat_{\lambda,t}))$, one computes

$$L(\lambda, t) = 2\pi\lambda \cosh \frac{1-t}{\lambda} + 2\pi\lambda \cosh \frac{-1-t}{\lambda}.$$

As $L(\lambda, t) \rightarrow \infty$ when $|\ln \lambda, t| \rightarrow \infty$, it suffices to find critical points of L . First observe that $\frac{\partial}{\partial t} L = -2\pi \sinh \frac{1-t}{\lambda} - 2\pi \sinh \frac{-1-t}{\lambda}$, which is zero only when $t = 0$. Thus one must only minimize $L(\lambda, 0) = 4\pi\lambda \cosh \frac{1}{\lambda}$. One verifies that the critical points of $L(\lambda, t)$ are of the form $(\lambda_0, 0)$ where λ_0 satisfies $\lambda_0 = \tanh \frac{1}{\lambda_0}$. Hence, as in Theorem 1.1, λ_0 is unique and $Cat_{\lambda_0,0} = C_{MS}$.

5 Conjectural approach

The proof of Lemma 2.3 depends in an essential manner on the Weierstrass representation. This has the disadvantage of obscuring some of the geometric meaning as well as restricting applications to minimal surfaces in \mathbb{R}^3 . For both these reasons it is fruitful to find a proof that avoids the use of complex analysis. In this appendix we present such an approach, albeit with an important caveat. Namely, we use a certain sharp eigenvalue estimate that is, to our knowledge, still conjectural. We feel justified in presenting this approach both for the reasons already mentioned and because the conjecture is geometrically natural and seems to have broader applications in spectral theory.

Conjecture 5.1 *Let σ be a smooth closed curve in \mathbb{R}^3 parameterized by arclength s . Denote by κ the geodesic curvature of σ . Then for any smooth function f on σ*

$$\mathcal{H}^1(\sigma)^2 \int_{\sigma} \left(\left| \frac{df}{ds} \right|^2 + \kappa^2 f^2 \right) ds \geq (2\pi)^2 \int_{\sigma} f^2 ds. \tag{5.1}$$

Remark 5.2 It is straightforward to verify that this inequality holds when σ is the round circle; in this case one has equality for the function $f = 1$.

This conjecture is termed the ‘‘Oval’s problem’’ and seems to have first appeared in the literature in [1]. In that paper, R. D. Benguria and M. Loss show that Conjecture 5.1 is related to conjectures about the one-dimensional Lieb-Thirring inequality. They also prove that (5.1) holds if one replaces $(2\pi)^2$ by $\frac{1}{2}(2\pi)^2$. One of the difficulties in proving the conjecture is the existence of a family of curves on which the putative best constant $(2\pi)^2$ is achieved. This family was constructed Benguria and Loss and consist of a one parameter family of ovals that contain the round circle and degenerate into a multiplicity two line segment. Burchard and Thomas show in [2] that in a neighborhood of this family the conjecture holds. For the general conjecture, the best constant so far achieved is $\approx 0.6(2\pi)^2$ in [7].

Using Conjecture 5.1, we show the following proposition which is a sharpening of Corollary 2.4. The argument completely avoids the use of the Weierstrass representation.

Proposition 5.3 *Fix Ω the open region between two parallel planes $P_1 = \{x_3 = h_1\}$ and $P_2 = \{x_3 = h_2\}$ in \mathbb{R}^3 where $h_1 < h_2$. Let $\Sigma \in \mathcal{A}(\Omega)$ and for $t \in (h_1, h_2)$ set $\Sigma_t = \Sigma \cap \{x_3 = t\}$. Then*

$$\frac{d^2}{dt^2} \mathcal{H}^1(\Sigma_t) \geq \frac{(2\pi)^2}{F_3^2} \mathcal{H}^1(\Sigma_t) + \int_{\Sigma_t} \frac{|A(v, E_2)|^2}{|\nabla x_3|^2} d\mathcal{H}^1.$$

Here A is the second fundamental form of Σ , and v, E_2 are a global orthonormal frame on Σ so that v is parallel to $\nabla_{\Sigma} x_3$.

Remark 5.4 The term $A(v, E_2)$ at a point $p \in \Sigma$ measures the rate of change at p of the ‘‘contact angle’’ between Σ and the plane $P = \{x_3 = x_3(p)\}$ along the curve $\Sigma \cap P$. Recall the contact angle at p is the angle between $\mathbf{n}(p)$, the normal to Σ at p , and the plane P . In particular, this term vanishes identically on a vertical catenoid. Indeed, the everywhere vanishing of such a term characterizes the vertical catenoid—see [14].

Before proving the proposition we do a slightly more general computation:

Lemma 5.5 *Consider Σ a minimal hypersurface in \mathbb{R}^{n+1} . Suppose that $\{x_{n+1} = t\}$ meets Σ transversely for all $-\epsilon < t < \epsilon$ and that the intersection Σ_t is a closed manifold. Then*

$$\frac{d^2}{dt^2} \mathcal{H}^{n-1}(\Sigma_t) = \int_{\Sigma_t} \left| \frac{1}{|\nabla_{\Sigma} x_{n+1}|} \right|^2 + \frac{(H_{\Sigma_t})^2 + \sum_i |\beta_i|^2 + (H_{\Sigma_t}^{\Sigma})^2 - |A_{\Sigma_t}^{\Sigma}|^2}{|\nabla_{\Sigma} x_{n+1}|^2}. \tag{5.2}$$

Here H_{Σ_t} is the mean curvature of Σ_t as a codimension two surface in \mathbb{R}^{n+1} and $H_{\Sigma_t}^{\Sigma}$ is the mean curvature of Σ_t as a hypersurface in Σ . Similarly, $A_{\Sigma_t}^{\Sigma}$ is the second fundamental form of Σ_t as a hypersurface in Σ . Finally, $\beta_i = A(v, E_i)$ where A is the second fundamental form of Σ , v is a vector field on Σ so E_2, \dots, E_n are an orthonormal frame on Σ_t and v is normal in Σ to Σ_t .

Proof The lemma will follow from the second variation formula for area. For $t \in (-\epsilon, \epsilon)$, let $\phi_t : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ denote a smooth family of C^1 diffeomorphisms of \mathbb{R}^{n+1} with $\phi_0(\mathbf{x}) = \mathbf{x}$

and ϕ_t equal to the identity outside of a compact set. Then we may write $\phi_t(\mathbf{x}) = \mathbf{x} + t\mathbf{X}(\mathbf{x}) + \frac{1}{2}t^2\mathbf{Z}(\mathbf{x}) + O(t^3)$ where \mathbf{X}, \mathbf{Z} are compactly supported vector fields. Fixing $M \subset \mathbb{R}^{n+1}$ a k -dimensional compact surface and letting $M_t = \phi_t(M)$ the second variation formula (see [17]) gives:

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^k(M_t) &= \int_M \operatorname{div}_M \mathbf{Z} + (\operatorname{div}_M \mathbf{X})^2 + \sum_{i=1}^k |(D_{\tau_i} \mathbf{X})^\perp|^2 \\ &\quad - \sum_{i,j=1}^k (\tau_i \cdot D_{\tau_j} \mathbf{X}) (\tau_j \cdot D_{\tau_i} \mathbf{X}). \end{aligned} \tag{5.3}$$

We claim the lemma is a simple consequence of this formula. Indeed, for fixed $t_0 \in (-\epsilon, \epsilon)$ let \mathbf{X}, \mathbf{Z} be vector fields normal to Σ_{t_0} given by

$$\begin{aligned} \mathbf{X} &= \frac{\nabla_\Sigma x_{n+1}}{|\nabla_\Sigma x_{n+1}|^2} & \text{and } \mathbf{Z} &= -\frac{A(v, \nu)}{|\nabla_\Sigma x_{n+1}|^2} \left(\mathbf{n} - \frac{\mathbf{n} \cdot \mathbf{e}_{n+1}}{|\nabla_\Sigma x_{n+1}|} \nu \right) \\ &= \frac{1}{|\nabla_\Sigma x_{n+1}|} \nu & &= \frac{-H_\Sigma - \mathbf{H}_{\Sigma_t} \cdot \mathbf{n}}{|\nabla_\Sigma x_{n+1}|^2} \left(\mathbf{n} - \frac{\mathbf{n} \cdot \mathbf{e}_{n+1}}{|\nabla_\Sigma x_{n+1}|} \nu \right) \\ & & &= -\frac{H_\Sigma + \mathbf{H}_{\Sigma_t} \cdot \mathbf{n}}{|\nabla_\Sigma x_{n+1}|^3} \mathbf{N}. \end{aligned}$$

Here \mathbf{n} is the normal to Σ , ν is the conormal to Σ_{t_0} in Σ and \mathbf{N} is the outward normal to Σ_{t_0} as a hypersurface in $\{x_{n+1} = t_0\}$. Using these vector fields, given a parameterization \mathbf{F}_0 of Σ_{t_0} if we set

$$\mathbf{F}(\cdot, t) = \mathbf{F}_0(\cdot) + (t - t_0)\mathbf{X}(\cdot) + \frac{1}{2}(t - t_0)^2\mathbf{Z}(\cdot) + \mathbf{G}(\cdot, t)$$

then $\mathbf{F}(\cdot, t)$ is a parameterization of Σ_t for t near t_0 with $\mathbf{G}(\cdot, t) = O(|t - t_0|^3)$. In particular, (5.2) will follow from (5.3) by using these vector fields.

It remains to evaluate the various terms in (5.3). We first compute:

$$\operatorname{div}_{\Sigma_0} \mathbf{Z} = -\mathbf{H}_{\Sigma_t} \cdot \mathbf{Z} = -\mathbf{H}_{\Sigma_t} \cdot \left(-\frac{H_\Sigma + \mathbf{H}_{\Sigma_t} \cdot \mathbf{n}}{|\nabla_\Sigma x_{n+1}|^3} \mathbf{N} \right) = \frac{H_{\Sigma_t}^2}{|\nabla_\Sigma x_{n+1}|^2}$$

where the last equality follows from the minimality of Σ . One also computes:

$$\begin{aligned} \sum_{i=2}^n |(D_{E_i} \mathbf{X})^\perp|^2 &= \left| \nabla_{\Sigma_0} \frac{1}{|\nabla_\Sigma x_{n+1}|} \right|^2 + \sum_{i=2}^n \frac{|A(E_i, \nu_0)|^2}{|\nabla_\Sigma x_{n+1}|^2}, \\ \sum_{i,j=2}^n (E_i \cdot D_{E_j} \mathbf{X}) (E_j \cdot D_{E_i} \mathbf{X}) &= \frac{|A_{\Sigma_t}^\Sigma|^2}{|\nabla_\Sigma x_{n+1}|^2}, \text{ and } (\operatorname{div}_{\Sigma_0} \mathbf{X})^2 = \left(H_{\Sigma_t}^\Sigma \frac{1}{|\nabla_\Sigma x_{n+1}|} \right)^2. \end{aligned}$$

Substituting these into (5.3) completes the proof.

We now show how Proposition 5.3 follows from Conjecture 5.1:

Proof Set $\Sigma_t = \Sigma \cap \{x_3 = t\}$. By Lemma 2.1 all the Σ_t are smooth curves. As Σ_t is a curve, $H_{\Sigma_t} = \kappa_{\Sigma_t}$, the geodesic curvature, and $|A_{\Sigma_t}^\Sigma|^2 = (H_{\Sigma_t}^\Sigma)^2$. Thus, Lemma 5.5 gives:

$$\frac{d^2}{dt^2} \mathcal{H}^1(\Sigma_t) = \int_{\Sigma_t} \left| \nabla_{\Sigma_t} \frac{1}{|\nabla_\Sigma x_3|} \right|^2 + \frac{\kappa_{\Sigma_t}^2 + |\beta_2|^2}{|\nabla_\Sigma x_3|^2} \geq \frac{(2\pi)^2}{\mathcal{H}^1(\Sigma_t)^2} \int_{\Sigma_t} \frac{1}{|\nabla_\Sigma x_3|^2} + \int_{\Sigma_t} \frac{|\beta_2|^2}{|\nabla_\Sigma x_3|^2}.$$

Here the inequality used Conjecture 5.1. Set

$$\alpha_t = \frac{1}{\mathcal{H}^1(\Sigma_t)} \int_{\Sigma_t} \frac{1}{|\nabla_{\Sigma} x_3|} \geq \frac{1}{\mathcal{H}^1(\Sigma_t)} \frac{(\mathcal{H}^1(\Sigma_t))^2}{\int_{\Sigma_t} |\nabla_{\Sigma} x_3|} = \frac{\mathcal{H}^1(\Sigma_t)}{F_3},$$

where the second inequality follows from the Cauchy-Schwarz inequality and the last equality uses the fact that $\mathbf{e}_3 \cdot \nu = \mathbf{e}_3 \cdot \frac{\nabla_{\Sigma} x_3}{|\nabla_{\Sigma} x_3|} = |\nabla_{\Sigma} x_3|$. Note that one has equality if and only if $|\nabla_{\Sigma} x_3|$ is constant on Σ_t . Then on Σ_t $\frac{1}{|\nabla_{\Sigma} x_3|} = \alpha_t + \psi$ where ψ is a smooth function on Σ_t with $\int_{\Sigma_t} \psi = 0$. Then one has:

$$\int_{\Sigma_t} \frac{1}{|\nabla_{\Sigma} x_3|^2} = \alpha_t^2 \mathcal{H}^1(\Sigma_t) + \int_{\Sigma_t} \psi^2 \geq \frac{(\mathcal{H}^1(\Sigma_t))^3}{F_3^2} + \int_{\Sigma_t} \psi^2.$$

Hence:

$$\frac{d^2}{dt^2} \mathcal{H}^1(\Sigma_t) \geq \frac{(2\pi)^2}{F_3^2} \mathcal{H}^1(\Sigma_t) + \int_{\Sigma_t} \frac{|\beta_2|^2}{|\nabla_{\Sigma} x_3|^2}.$$

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