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Orthogonality Conditions and Asymptotic Stability in the Stefan Problem with Surface Tension

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Abstract

We prove nonlinear asymptotic stability of steady spheres in the two-phase Stefan problem with surface tension. Our method relies on the introduction of appropriate orthogonality conditions in conjunction with a high-order energy method.

1. Introduction

We are interested in the question of long-time nonlinear stability of steady state solutions to the two-phase Stefan problem with surface tension, one of the best known parabolic free boundary problems. It is a simple model of phase transitions in liquid–solid systems.

Let $\Omega \subset \mathbb{R}^n$ denote a C^1-domain that contains a liquid and a solid separated by an interface Γ. As melting or freezing takes place, the boundary moves and we are naturally led to a free boundary problem. Define the solid phase $\Omega^-(t)$ as a region encircled by $\Gamma(t)$ and define the liquid phase $\Omega^+(t) := \Omega \setminus \overline{\Omega^-}$. The unknowns are the location of the interface $\{\Gamma(t);\ t \geqq 0\}$ and the temperature function $v : [0, T] \times \Omega \to \mathbb{R}$. Let Γ_0 be the initial position of the free boundary and $v_0 : \Omega \to \mathbb{R}$ be the initial temperature. We denote the normal velocity of Γ by V and normalize it to be *positive if* Γ *is locally expanding* $\Omega^+(t)$. Furthermore, we denote the mean curvature of Γ by κ. With these notations, (v, Γ) satisfies the following free boundary value problem:

$$\partial_t v - \Delta v = 0 \quad \text{in } \Omega \setminus \Gamma. \tag{1}$$

$$v = \kappa \quad \text{on } \Gamma, \tag{2}$$

$$V = [v_n]^+_- \quad \text{on } \Gamma, \tag{3}$$

$$v_n = 0 \quad \text{on } \partial\Omega, \tag{4}$$

$$v(0, \cdot) = v_0; \quad \Gamma(0) = \Gamma_0. \tag{5}$$

Given v, we write v^+ and v^- for the restriction of v to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. With this notation $[v_n]_-^+$ stands for the jump of the normal derivatives across the interface $\Gamma(t)$, namely $[v_n]_-^+ := v_n^+ - v_n^-$, where n stands for the unit normal on the hypersurface $\Gamma(t)$ with respect to $\Omega^+(t)$. In (4) $\partial\Omega$ stands for the outer fixed boundary of Ω. Two basic identities related to the above problem are the "mass" conservation law:

$$\partial_t\left[\int_\Omega v(t,x)\,\mathrm{d}x + |\Omega^-(t)|\right] = 0, \tag{6}$$

and the energy dissipation law:

$$\partial_t\left[\frac{1}{2}\int_\Omega v^2\,\mathrm{d}x + |\Gamma(t)|\right] + \int_\Omega |\nabla v|^2\,\mathrm{d}x = 0.$$

Here $|\Omega^-(t)|$ and $|\Gamma(t)|$ stand for the Lebesgue volume of $\Omega^-(t)$ and the surface area of $\Gamma(t)$ respectively. Steady states of the above problem consist of static spheres and they form an $(n+1)$-dimensional family $\mathscr{F}$:

$$\mathscr{F} := \left\{\Sigma(R,\mathbf{a})\big|\ \ \mathbf{a}\in\mathbb{R}^n,\ R\in\mathbb{R}_+\right\}, \tag{7}$$

where for any $\mathbf{a} = (a^1,\dots,a^n)\in\mathbb{R}^n$, $R\in\mathbb{R}_+$ the pair

$$\Sigma(R,\mathbf{a}) := ((n-1)/R,\ S_R(\mathbf{a}))$$

is a time-independent solution of the problem (1)–(4), if $S_R(\mathbf{a})\subset\Omega$. Let us parametrize the moving boundary Γ as a graph over a given steady state $\Sigma(R,\mathbf{a})\in\mathscr{F}$: we introduce the radius function $r : [0,\infty[\times\mathbb{S}^{n-1}\to\mathbb{R}$ and the parametrization $\phi : [0,\infty[\times\mathbb{S}^{n-1}\to\Gamma$ such that

$$\phi(t,\xi) = \mathbf{a} + r(t,\xi)\xi, \quad \xi\in\mathbb{S}^{n-1} \tag{8}$$

and define the perturbation $(u,\ f) := (v-(n-1)/R,\ r-1)$. To $(u,\ f)$ we associate a high-order energy norm given by

$$||(u,\ f)(t)|| = \sum_{k=0}^{N}\left\{||u||^2_{W^{N-k,\infty}W^{2k,2}([0,t]\times\Omega)} + ||\nabla_g f||^2_{W^{N-k,\infty}W^{2k+1,2}([0,t]\times\mathbb{S}^{n-1})}\right\}.$$

Our main result is the following:

Theorem 1. *Assume that* $\zeta_R > 0$, *where* $\zeta_R := \frac{1}{|\Omega|} - \frac{n-1}{|S_R|R^2}$ *(that is the linear stability criterion holds). Then there exists* $\varepsilon > 0$, *such that if initially*

$$||(u,\ f)(0)|| < \varepsilon,$$

where $(u,\ f)(t) := (v(t,\cdot) - (n-1)/R, r(t,\cdot) - R)$ *is the perturbation from the steady state* $\Sigma(R,\mathbf{a})$, *then there exists a global-in-time unique solution* $(v,\Gamma)(t)$ *to the problem* (1)–(5). *Moreover, there exists a pair* $(\bar{R},\bar{\mathbf{a}})$ *close to* $(R,\mathbf{a})$ *such that* (v,Γ) *converges exponentially fast to* $\Sigma(\bar{R},\bar{\mathbf{a}})\in\mathscr{F}$. *More precisely, if we parametrize* $(v,\Gamma)(t)$ *as a perturbation of* $\Sigma(\bar{R},\bar{\mathbf{a}})$, *by setting* $\Gamma(t) = \left\{\bar{\mathbf{a}} + \bar{r}(t,\xi)\xi,\quad \xi\in\mathbb{S}^{n-1}\right\}$ *and define* $(\bar{u},\ \bar{f})(t) = (v(t,\cdot) - (n-1)/\bar{R},\ \bar{r}(t,\cdot) - \bar{R})$ *then there exist constants* C_1 *and* C_2 *such that*

$$||(\bar{u},\ \bar{f})(t)|| \leqq C_1 \mathrm{e}^{-C_2 t}, \quad t\geqq 0.$$

Remark. By choosing N large enough in the definition of the norm $||\cdot||$, we can make the solution as smooth as desired in a classical sense. We have not aimed for finding the minimal N; rather, for the sake of clarity and conciseness of the estimates, we allow ourselves the flexibility of keeping N sufficiently large.

Remark. Note that the time derivatives occurring in the expression $||(u, f)(0)||$ are implicitly given by terms with only spatial derivatives, via the equations (1) and (3).

1.1. Notation

On the unit sphere $\mathbb{S}^{n-1}$, the Riemannian gradient with respect to the standard metric is denoted by ∇_g and the Laplace–Beltrami operator by Δ_g. For a given function h the kth time derivative is interchangeably denoted by $\partial_{t^k} h$ or h_{t^k}. When writing various norms of the functions defined on the unit sphere $\mathbb{S}^{n-1}$, we drop the domain from the notation, for example $||f||_{L^2} := ||f||_{L^2(\mathbb{S}^{n-1})}$. A ball (sphere) of radius R centered at a point $\mathbf{a} \in \mathbb{R}^n$ will be denoted by $B_R(\mathbf{a})$ ($S_R(\mathbf{a})$). The spherical harmonics on the unit sphere are important tools in our analysis. For a given function $h \in L^2(\mathbb{S}^{n-1})$, we introduce the spherical harmonics decomposition for h

$$h = \sum_{k=0}^{\infty} \sum_{i=1}^{m(k)} h_{k,i} s_{k,i}.$$

Here for each $k \in \mathbb{N}_0$, $s_{k,i}$, $i = 1, \dots, m(k)$ stand for the spherical harmonics of degree k. They are defined as restrictions of homogeneous polynomials of degree k in n variables onto the unit sphere. The set $\bigcup_{k=0}^{\infty} \bigcup_{i=1}^{m(k)} \{s_{k,i}\}$ forms an orthonormal basis on $\mathbb{S}^{n-1}$ with respect to L^2-product, thus justifying the above expansion. For more details, we refer the reader to [24]. The first $n+1$ spherical harmonics $s_{0,1}, s_{1,1}, \dots, s_{1,n}$ will be denoted by $s_0, s_1, \dots, s_n$. For $i = 1, \dots, n$ we define the first momenta of h $\mathbb{P}_1 h_i$ as well as the mean $\mathbb{P}_0 h$ by setting

$$\mathbb{P}_1 h_i := \int_{\mathbb{S}^{n-1}} h s_i \, \mathrm{d}\xi; \quad \mathbb{P}_0 h = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} h \, \mathrm{d}\xi. \tag{9}$$

We denote $\mathbb{P}_{2+} h := h - \mathbb{P}_0 h - \sum_{k=1}^{n} \mathbb{P}_1 h_i$. For a given function $\mathscr{U}$ we interchangeably denote $\partial_{x^i} \mathscr{U} = \mathscr{U}_{x^i} = \mathscr{U}_i$, and similarly for mixed partial derivatives. Einstein's summation convention is employed and we sum over repeated indices. A generic constant appearing in various estimates is denoted by C and it may change from line to line. The unit normal to a given surface Γ is denoted by n_Γ and the index is dropped if the surface in question is clear from the context.

1.2. Previous Work

The Stefan problem has been studied in a variety of mathematical literature over the past century (see for instance [30]). If (2) is replaced by the boundary condition $v = 0$ on Γ, the resulting problem is called the *classical* Stefan problem. It has been shown that the classical Stefan problem admits unique global weak solutions in several dimensions [12,13,20]. The references to the regularity of weak solutions of the two-phase classical Stefan problem are, among others, [2–4,9]. Local classical solutions are established in [19] and [23].

If the diffusion equation (1) is replaced by the elliptic equation $\Delta u = 0$, then the resulting problem is called the Mullins–Sekerka problem (also known as the quasi-stationary Stefan problem or the Hele-Shaw problem with surface tension). Global existence for the two-phase quasi-stationary Stefan problem close to a sphere in two dimensions has been obtained in [5,8], and in arbitrary dimensions in [11]. Global stability for the one-phase quasi-stationary Stefan problem is established in [15]. Local-in-time solutions in parabolic Hölder spaces in arbitrary dimensions are established in [6].

As to the Stefan problem with surface tension (also known as the Stefan problem with Gibbs-Thomson correction), global weak existence theory (without uniqueness) is developed in [1,21,27]. In [14] the authors consider the Stefan problem with small surface tension, that is $\sigma \ll 1$, whereby (2) is replaced by $v = \sigma\kappa$. Local existence for the Stefan problem is studied in [26]. In [10] the authors prove a local existence and uniqueness result under a smallness assumption on the initial datum close to flat hypersurfaces. Linear stability and instability results for spheres are contained in [25].

The first global-in-time nonlinear stability result for the flat steady hypersurfaces was given in [17]. Some of the references for the Stefan problem with surface tension and kinetic undercooling effects are [7,26,28,29].

1.3. Motivation, Methods and Plan of the Paper

To explain the linear stability criterion $\zeta_R > 0$ and motivate our result, let us look at the linearization around a fixed steady state $\Sigma(1, \mathbf{a}) \in \mathscr{F}$:

$$\partial_t(u, f) = \mathscr{L}(u, f), \tag{10}$$

where $\mathscr{L}(u, f) = (\Delta u, -[\partial_n u]_-^+)$, $u = -(n-1)f - \Delta_g f$ on $\mathbb{S}^{n-1}$. This linearization is easily obtained if we parametrize the moving boundary Γ as a graph over $S_1(\mathbf{a})$: with the radius function $r = 1 + f$ and the parametrization ϕ as in (8), the perturbation (u, f) takes the form $(u,\ f) = (v - (n-1),\ R-1)$. It is readily checked that up to the first order $\kappa \circ \phi = (n-1) - (n-1)f - \Delta_g f$ and $V \circ \phi = f_t$. Associated to the linear problem (10) is the energy dissipation identity (cf. [18]):

$$\partial_t \left\{ \int_\Omega u^2 + \int_{\mathbb{S}^{n-1}} \{|\nabla_g f|^2 - (n-1)f^2\} \right\} = -2\int_\Omega |\nabla u|^2.$$

Note that it is not even clear whether the above energy is positive definite, due to the presence of the negative definite term $-\int_{\mathbb{S}^{n-1}}(n-1)f^2$. Indeed, problem (10) may allow for a strictly positive eigenvalue under certain assumptions on the relative size of the domain Ω and the steady state $\Sigma(R, \mathbf{a})$. More precisely, to any $\Sigma(R, \mathbf{a}) \in \mathscr{F}$ we associate the stability parameter ζ_R:

$$\zeta_R := \frac{1}{|\Omega|} - \frac{n-1}{|S_R|R^2},$$

where $|\Omega|$ and $|S_R|$ stand for the Lebesgue volume of Ω and the surface area of S_R respectively. It turns out that the sign of ζ_R is critical to the stability properties of $\Sigma(R, \mathbf{a})$: if $\zeta_R > 0$ the solution is *linearly* stable and if $\zeta_R < 0$ it is *linearly* unstable [18,25]. The full *nonlinear* instability in the case $\zeta_R < 0$, under the additional assumption that Ω is a perfect ball of a given radius R_* is proved in [18]. A closer look at the *instability* proof in [18] shows that the proof itself is insensitive to the specific shape of Ω and works for general domains. On the other hand, in the stability regime ($\zeta_R > 0$) the situation is more complicated. To explain this, note that the linearized problem (10) has $(n+1)$ non-decaying solutions $\{\sigma_i\}_{i=0,\dots,n}$, being exactly the $(n+1)$ eigenvectors spanning the null-space of $\mathscr{L}$:

$$\sigma_0 := (n-1, -1), \quad \sigma_i = (0, s_i) \quad i = 1, \dots, n,$$

where s_i, $i = 1, \dots, n$, are the spherical harmonics defined in Section 1.1. The existence of σ_i-s, $i = 0, \dots, n$, encodes the $(n+1)$-dimensionality of the set $\mathscr{F}$. We expect the perturbation of $\Sigma(R, \mathbf{a})$ to converge to a nearby asymptotic state $\Sigma(\bar{R}, \bar{\mathbf{a}})$. On the other hand, note that the first modes $\mathbb{P}_1 f_i = \int_{\mathbb{S}^{n-1}} f s_i \, \mathrm{d}\xi \, s_i$ ($i = 1, \dots, n$) in the spherical-harmonics expansion of f are precisely cancelled by the expression $Z(h) := \int_{\mathbb{S}^{n-1}} \{|\nabla_g h|^2 - (n-1)h^2\}$ and they cannot be a priori controlled by the energy. This feature of the problem is another manifestation of the non-trivial null-space structure of the linearized operator $\mathscr{L}$. It causes a major difficulty in controlling the first momenta $\mathbb{P}_1 f_i$, $i = 1, \dots, n$, thus introducing additional analytic difficulties.

In this paper, as stated in Theorem 1, we will prove the *nonlinear asymptotic stability* of the steady state $\Sigma(R, \mathbf{a}) \in \mathscr{F}$ in the stability regime $\zeta_R > 0$ for *general* domains Ω. Our method has two basic ingredients. We start by introducing a set of geometrically motivated orthogonality conditions, that allow us to "mod out" an inherent degeneracy related to the existence of the non-trivial null space of the linearized operator. In analytical terms, we express (v, Γ) in a set of new, tubular coordinates. The second ingredient is the high-order energy method developed in [17,18] as a part of the program to investigate stability and instability of steady states in phase transition phenomena. The goal is to prove an energy estimate of the form

$$\mathscr{E}(t) + \int_0^t \mathscr{D}(\tau) \, \mathrm{d}\tau \leqq \mathscr{E}(0) + \left(\delta + C \sup_{0 \leqq \tau \leqq t} \sqrt{\mathscr{E}(\tau)} \right) \int_0^t \mathscr{D}(\tau) \, \mathrm{d}\tau, \qquad (11)$$

where $\mathscr{E}$ and $\mathscr{D}$ are the *energy* and *dissipation* naturally associated to the problem (see (48) and (49)). If $\mathscr{E}$ and $\delta > 0$ are small, we can absorb the right-most term above into left-hand side, to obtain an a priori estimate

$$\mathscr{E}(t) + \frac{1}{2} \int_0^t \mathscr{D}(\tau) \, \mathrm{d}\tau \leqq \mathscr{E}(0),$$

thus recovering the smallness assumption on $\mathscr{E}$ if $\mathscr{E}(0)$ is small. Through an iteration scheme and a continuity argument, this reasoning is made rigorous: proof of (11) is based on a series of energy estimates for the error terms. However, while in the previous works [17] and [18] the boundary conditions (2) and (3) honored the energy method fully, by introducing the tubular coordinates new error terms of (only) quadratic order emerge (Lemma 2). The precise structure of the linearized curvature operator is exploited to get cancellation for such terms and to finally close the estimates (Theorem 2). To obtain global existence for the solution (v, Γ), we use a Poincaré-inequality type bound $\mathscr{E} \leqq C\mathscr{D}$. It allows us to prove some decay of the solution on a bounded time interval. Together with a suitable smallness assumption on the initial data, this decay can be bootstrapped to yield a global-in-time existence result.

We wish to point out that our method can also be used to prove asymptotic stability of steady spheres for the Mullins–Sekerka problem in arbitrary dimensions.

The plan of the paper is as follows. In Section 2 we heuristically motivate and then introduce the orthogonality conditions. We then derive the modulation equations and reformulate the Stefan problem in the set of new, tubular coordinates. In Section 3, the Stefan problem is reformulated on a fixed domain and the high-order energy is introduced. In Section 4 we prove a Poincaré type estimate (Lemma 3) and the positive-definiteness of the energy (Lemma 5). With these preparations, energy estimates are performed and local existence theorem is proved in Section 5. Finally, the proof of the main result (Theorem 1) is presented in Section 6.

2. Orthogonality Conditions: Evolution Problem

To facilitate the analysis, without loss of generality, we assume a few simplifications. We will perturb away from the steady state $\Sigma(1, \mathbf{0})$ (assuming $S_1(\mathbf{0}) \subset \Omega$). Furthermore, note that the mass conservation law (6) necessarily determines the radius of the final asymptotic state. Recalling the parametrization (8) of $\Gamma(t)$, (6) takes the form $\partial_t M(v, \Gamma)(t) = 0$, where

$$M(v, \Gamma)(t) := \int_\Omega v(t, x)\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}} \frac{r(t, \xi)^n}{n}\,\mathrm{d}\xi.$$

We will henceforth assume that initially

$$M(v_0, \Gamma_0) = M(\Sigma(1, \mathbf{0})). \tag{12}$$

Condition (12) forces the asymptotic steady state to have the radius $R_{asymp} = 1$ due to the conservation of $M(v, \Gamma)(t)$. This assumption constraints our stability analysis to the "manifold" of steady states $\mathscr{G} \subset \mathscr{F}$ (defined in (7)) consisting of the elements $\Sigma(\mathbf{a}) := \Sigma(1, \mathbf{a})$ of the fixed radius $R = 1$:

$$\mathscr{G} := \left\{\Sigma(\mathbf{a}) \middle| \;\; \mathbf{a} \in \Omega, \quad S_1(\mathbf{a}) \subset \Omega\right\}.$$

Finally, we assume $\int_{\mathbb{S}^{n-1}} f_0 s_i = 0$, $i = 1, \ldots, n$ where the initial surface Γ_0 is parametrized by $(1 + f_0(\xi))\xi$, $\xi \in \mathbb{S}^{n-1}$. Otherwise, translate the steady state to a nearby one, so that the condition is satisfied.

2.1. Heuristics

To motivate the analysis of the present work, in the following we provide some geometric heuristics for the choice of the above mentioned orthogonality conditions. Namely, let us think of $\mathscr{G}$ as a submanifold of the subspace $\mathscr{H}$ of $L^2(\Omega) \times L^2(\mathbb{S}^{n-1};\ \Omega)$ of the functions of the form $(u,\ \mathbf{a} + R(\xi)\xi)$:

$$\mathscr{H} := \left\{(u, \mathbf{a} + r(\xi)\xi),\ \ \mathbf{a} \in \mathbb{R}^n,\ u : \Omega \to \mathbb{R},\ \ r : \mathbb{S}^{n-1} \to \mathbb{R}\right\} \subset L^2(\Omega) \\ \times L^2(\mathbb{S}^{n-1};\ \Omega).$$

Thus, for $\mathbf{a} \in \mathbb{R}^n$, $\Sigma(\mathbf{a})(x, \xi) = (n-1, \mathbf{a} + \xi) \in \mathscr{G}$. The non-decaying solutions $\{\sigma_i\}_{i=1,\ldots,n}$ of (10) correspond to the infinitesimal changes in the center coordinate parameters a_i, $i = 1, \ldots, n$. Motivated by this observation, we expect the solution to the full nonlinear problem to decompose into a component tangential to $\mathscr{G}$ and the dissipating part that belongs to a plane *transversal* to $\mathscr{G}$ in $\mathscr{H}$. In fact, we shall demand that this plane is exactly the fiber $\mathscr{G}^\perp$, L^2-orthogonal to $\mathscr{G}$ in $\mathscr{H}$. In other words, we choose the tubular coordinates $(\mathbf{a}(t), u(t,\cdot), f(t,\cdot))$ in a neighborhood of the steady state manifold $\mathscr{G}$:

$$(v(t), \Gamma(t)) = \Sigma_{\mathbf{a}(t)} + (u(t), f(t,\xi)\xi), \tag{13}$$

such that

$$(u(t,\cdot), f(t,\cdot)\,\iota(\cdot)) \in \mathscr{G}^\perp, \tag{14}$$

where $\iota : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is the inclusion operator. Condition (14) is precisely the *orthogonality* condition. To obtain an analytic expression for (14), we note that the tangent space $T_{\mathbf{a}(t)}\mathscr{G}$ is spanned by the set $\{\Sigma_i(x,\xi) = \partial_{a^i}\Sigma_{\mathbf{a}(t)}\big|_{a^i = a^i(t)}\}_{i=1,\ldots,n} = \{(0, e_i)\}_{i=1,\ldots,n}$. Thus, condition (14) implies that for any $i = 1, \ldots, n$:

$$0 = \big((u, f\,\iota), (0, e_i)\big)_{L^2} = \int_{\mathbb{S}^{n-1}} f(t,\xi)\xi \cdot e_i \,\mathrm{d}\xi = \int_{\mathbb{S}^{n-1}} f(t,\xi) s_i \,\mathrm{d}\xi. \tag{15}$$

Here e_i stands for the ith unit vector in $\mathbb{R}^n$. We have provided a geometric picture that renders the right orthogonality condition (15), thus drawing a parallel to the work of Friesecke and Pego [16] on the stability of solitons in Fermi–Pasta–Ulam lattices. There, the authors introduce suitable orthogonality conditions exploiting the symplectic structure of the linearized problem. Their perturbed solution belongs to the fiber *symplectically* orthogonal to the soliton state manifold.

2.2. Modulation Equations: The Evolution Problem

A natural question raised by the form of the orthogonality condition (15) is to find the modulation equation (that is evolution equation) for the vector $\mathbf{a}(t)$. This is not straightforward because $\mathbf{a}(t)$ does not explicitly appear in (15), thus preventing us from differentiating in time directly. To resolve this difficulty, for any $i = 1, \ldots, n$, let us define the *momentum test functions* $p_i(t,\cdot) \in C^\infty(\Omega)$: $p_i(x,t) = x^i - a^i(t)$, where $x = (x^1, \ldots, x^n)$. Assume for the moment that

$(v,\Gamma)(t)$ is a classical solution of the Stefan problem (1)–(5). We then multiply the equation (1) by the momentum test functions, use the integration by parts once and the boundary condition $[\partial_n v]_-^+ = V_\Gamma$. As a result, for $i=1,\dots,n$, we obtain the identity

$$\int_{\Gamma(t)} V_\Gamma p_i = \int_\Omega u_t p_i + \int_\Omega \nabla u \cdot \nabla p_i. \tag{16}$$

Note that the normal velocity expressed in local coordinates takes the form

$$V_\Gamma \circ \phi(t,\xi) = -\frac{r_t r}{|g|} + \dot{\mathbf{a}} \cdot \boldsymbol{n}_\Gamma \circ \phi, \qquad |g| = \sqrt{r^2 + |\nabla_g r|^2}$$

and the mean curvature κ takes the form:

$$\kappa \circ \phi(t,\xi) = \frac{n-1}{|g|} - \frac{1}{R}\nabla_g \cdot \frac{\nabla_g R}{|g|} = (n-1) - (n-1)f - \Delta_g f + N(f),$$

where $N(f)$ stands for the quadratic nonlinear remainder. The volume element $\mathrm{d}S(\Gamma)$ takes the form $r^{n-2}|g|\,\mathrm{d}\xi$ in the local coordinates on the sphere, so we obtain

$$\begin{aligned}\int_{\Gamma(t)} V_\Gamma p_i \,\mathrm{d}S(\Gamma) &= -\int_{\mathbb{S}^{n-1}} r^n r_t \xi^i + \dot{\mathbf{a}} \cdot \int_{\Gamma(t)} p_i \boldsymbol{n}_\Gamma = -\int_{\mathbb{S}^{n-1}} r^n r_t \xi^i \\ &\quad + \dot{\mathbf{a}} \cdot \int_{\Omega^-(t)} \nabla p_i \\ &= -\int_{\mathbb{S}^{n-1}} r^n r_t \xi^i + \dot{a}^i(t)|\Omega^-(t)|,\end{aligned}$$

where we used the Stokes theorem in the second equality and $\nabla p_i = e_i$ in the last. Plugging this back into (16), we conclude

$$|\Omega^-(t)|\dot{a}^i(t) = \int_{\mathbb{S}^{n-1}} r^n r_t \xi^i + \int_\Omega u_t p_i + \int_\Omega u_{x^i}. \tag{17}$$

Thus, if (v,Γ) is the solution of the problem with the above choice of coordinate description of Γ, then a fortiori, the moving center components $a^i(t)$ satisfy the differential equation (17). Written in terms of f, for any $i=1,\dots,n$, the first term on right-hand side of (17) takes the form:

$$\int_{\mathbb{S}^{n-1}} r^n r_t s_i \,\mathrm{d}\xi = \int_{\mathbb{S}^{n-1}} (1+f)^n f_t s_i \,\mathrm{d}\xi = \sum_{k=1}^n \int_{\mathbb{S}^{n-1}} \binom{n}{k} f^k f_t s_i,$$

where in the second equality we used the orthogonality condition (16), implying in particular $\int_{\mathbb{S}^{n-1}} f_t s_i = 0, i = 1,\dots n$. The unknowns to be solved for, are the perturbation (u,f) and the moving center $\mathbf{a}$. Setting $\mathbf{p} = (p_1,\dots,p_n)$, the Stefan problem with surface tension (1)–(5) in the tubular coordinates $(u(t,\cdot), f(t,\cdot), \mathbf{a}(t))$ takes the form

$$\partial_t u - \Delta u = 0 \quad \text{in} \quad \Omega(t). \tag{18}$$

$$u = \kappa - (n-1) \quad \text{on} \quad \Gamma(t), \tag{19}$$

$$[u_n]_-^+ \circ \phi = -\frac{r_t r}{|g|} + \dot{\mathbf{a}} \cdot \boldsymbol{n}_\Gamma \circ \phi. \quad \text{on} \quad \mathbb{S}^{n-1}, \tag{20}$$

$$u_n = 0 \quad \text{on} \quad \partial\Omega, \tag{21}$$

$$\dot{\mathbf{a}}(t) = \frac{1}{|\Omega^-(t)|} \sum_{k=1}^{n} \int_{\mathbb{S}^{n-1}} \binom{n}{k} f^k f_t \xi + \frac{1}{|\Omega^-(t)|} \int_{\Omega(t)} u_t \mathbf{p} + \frac{1}{|\Omega^-(t)|} \int_{\Omega(t)} \nabla u. \tag{22}$$

$$\int_{\mathbb{S}^{n-1}} f\xi = \mathbf{0}. \tag{23}$$

$$u(0,\cdot) = u_0; \quad \Gamma(0) = \Gamma_0; \quad \mathbf{a}(0) = \mathbf{0}. \tag{24}$$

$$M(1+u_0, \Gamma_0) = M(\Sigma(1, \mathbf{0})). \tag{25}$$

The idea is to exploit dissipative properties of the problem (18)–(21) to obtain a time-decay estimate for (u, f). Plugging that into (22), we hope to obtain a decay estimate for $\dot{\mathbf{a}}$. By bootstrapping this procedure, we will "drive" the solution to its asymptotic equilibrium.

3. Fixing the Domain and the Energy

We first describe the pull back of the problem (18)–(25) onto the fixed domain $\Omega \setminus \mathbb{S}^{n-1}$ and then define the high-order energies $\mathscr{E}$ and $\mathscr{D}$, deriving in particular the corresponding energy identities. From this point onwards, we will denote $\mathscr{S} := \mathbb{S}^{n-1}$. Define the following change of variables:

$$\Theta(t,x) = \pi(t,x)(x - \mathbf{a}(t)),$$

where π is a smooth scalar-valued function with the following property:

$$\pi(t,x) = \begin{cases} \frac{1}{r\left(\frac{\mathbf{x}-\mathbf{a}(t)}{|\mathbf{x}-\mathbf{a}(t)|}\right)}, & |\mathbf{x}-\mathbf{a}(t)| - 1 \leqq d, \\ 1, & |\mathbf{x}-\mathbf{a}(t)| - 1 \geqq 2d \end{cases} \qquad d \quad \text{is small.} \tag{26}$$

For any $\mathbf{x} \in \Gamma$, we may write $\mathbf{x} = \mathbf{a}(t) + r\left(\frac{\mathbf{x}-\mathbf{a}(t)}{|\mathbf{x}-\mathbf{a}(t)|}\right) \frac{x-\mathbf{a}(t)}{|x-\mathbf{a}(t)|} = \mathbf{a}(t) + r(\xi)\xi$, by the definition of Γ. Hence

$$\begin{aligned} \Theta(t,\mathbf{x}) &= \pi\left(t, \mathbf{a}(t) + r(\xi)\frac{\mathbf{x}-\mathbf{a}(t)}{|\mathbf{x}-\mathbf{a}(t)|}\right) r(\xi) \frac{\mathbf{x}-\mathbf{a}(t)}{|\mathbf{x}-\mathbf{a}(t)|} \\ &= \frac{1}{r(\xi)} r(\xi) \frac{\mathbf{x}-\mathbf{a}(t)}{|\mathbf{x}-\mathbf{a}(t)|} = \frac{\mathbf{x}-\mathbf{a}(t)}{|\mathbf{x}-\mathbf{a}(t)|} \in \mathscr{S}. \end{aligned}$$

Thus Θ does the job for us: $\Theta : \Omega \to \Omega \setminus \mathscr{S}$. Note that this map is natural from the point of view of our *geometric* assumption that the evolving interfaces Γ are, in fact, graphs over the unit sphere $\mathscr{S}$.

The inverse map $x(\bar{x})$ to the above change of variables Θ is given by

$$x = \mathbf{a}(t) + \rho(t, \bar{x})\bar{x}.$$

To convince ourselves that the function $\rho(t, \cdot) : \Omega \setminus \mathscr{S} \to \mathbb{R}$ is well defined, we observe that ρ has to satisfy the relation

$$\pi(t, \mathbf{a}(t) + \rho(t, \bar{x})\bar{x})\rho(\bar{x}) - 1 = 0.$$

The existence of such a ρ can be established by the implicit function theorem, applied to the equation $F(t, s, \bar{x}) = 0$, where $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is given by $F(t, s, v) = \pi(t, \mathbf{a} + sv)s - 1$. Namely, $F_s(t, s, v) = (\nabla\pi \cdot v)s + \pi > 0$ since $\pi \approx 1$ and $\nabla\pi = O(\nabla_g f)$ is small. We define $w : \Omega \setminus \mathscr{S} \to \mathbb{R}$ by setting

$$w(t, \bar{x}) := u(t, \mathbf{a}(t) + \rho(t, \bar{x})\bar{x}). \tag{27}$$

The heat operator $\partial_t - \Delta$ on the domain Ω will transform into a more complicated operator in the new coordinates as stated in the following lemma:

Lemma 1. *The push forward of the operator $\partial_t - \Delta$ with respect to the map Θ : $\Omega \to \mathscr{S}$ reads:*

$$(\partial_t - \Delta)^{\#} u = w_t - a_{ij} w_{\bar{x}^i \bar{x}^j} - b_i w_{\bar{x}^i},$$

where

$$a_{ij} := \pi^2 \delta_{ij} + 2\pi x^j \pi_{x^i} + x^i x^j |\nabla\pi|^2, \quad b_i := 2\pi_{x^i} + \Delta\pi x^i - \pi_t x^i. \tag{28}$$

Furthermore,

$$[w_n]_-^+ = \frac{r^2}{|g|}\Big([u_n]_-^+\Big) \circ \phi \tag{29}$$

The proof of this lemma is a straightforward calculation and is presented in the appendix. In conclusion, on the fixed domain $\Omega \setminus \mathscr{S}$, equations (18)–(20) take the form

$$w_t - a_{ij} w_{ij} - b_i w_i = 0 \quad \text{in } \Omega, \tag{30}$$

$$w = \kappa \circ \phi - (n - 1) \quad \text{on } \mathscr{S}, \tag{31}$$

$$[w_n]_-^+ = \frac{r^2}{|g|}\left(-\frac{r_t r}{|g|} + \dot{\mathbf{a}} \cdot n_\Gamma \circ \phi\right) \quad \text{on } \mathscr{S}. \tag{32}$$

The boundary condition (21) reads $\partial_n w = 0$ on $\partial\Omega$. The modulation equation (22) is easily expressed in the fixed coordinates

$$\dot{\mathbf{a}}(t) = \frac{1}{|\Omega^-(t)|}\Big(\sum_{k=1}^n \int_{\mathbb{S}^{n-1}} \binom{n}{k} f^k f_t \xi + \int_\Omega (u_t \mathbf{p}) \circ \Theta |\det D\Theta^{-1}|$$
$$+ \int_\Omega \nabla u \circ \Theta |\det D\Theta^{-1}|\Big), \tag{33}$$

where formulas (A.74) and (A.75) are used to express $u_t \circ \Theta$ and $\nabla u \circ \Theta$ in terms of the function $w = u \circ \Theta$. The orthogonality condition (23) retains its form and the initial conditions take the form:

$$w(0, \cdot) = w_0 := u_0 \circ \Theta; \quad f(0, \cdot) = f_0; \quad \mathbf{a}(0) = \mathbf{0}. \tag{34}$$

3.1. The Model Problem

Let $\mu : \Omega \to \mathbb{R}_+$ be a smooth non-negative cut-off function such that $\mu = 0$ close to $\mathscr{S}$ and $\mu = 1$ close to $\partial\Omega$ and the origin.[1] For any $i = 1, \ldots, n$ let us define a differential operator

$$D^i = \mu \partial_{x^i} + (1 - \mu)\partial_{\xi^i},$$

where ∂_{ξ^i} is the partial tangential differentiation operator defined in Cartesian coordinates through

$$\partial_{\xi^i} u = \partial_{x^i} u - \frac{x^i}{|x|} \nabla u \cdot \frac{x}{|x|}.$$

For given functions $v, w \in H^1(\Omega)$ a simple integration by parts shows

$$\int_\Omega D^i v w = - \int_\Omega v D^i w - \int_\Omega v w \nu(\mu) + \int_{\partial\Omega} v w n^i,$$

where $\nu(\mu) := (\partial_{x^i} - \partial_{\xi^i})\mu = \frac{x^i}{|x|} \nabla \mu \cdot \frac{x}{|x|}$, thus giving us the integration-by-parts formula for the operator D^i. For a given multi-index $m = (m_1, \ldots, m_n)$ and a non-negative natural number $s \in \mathbb{N}_0$ we define

$$D_s^m := \partial_t^s \partial^m.$$

If we denote $\mathscr{L} := a_{ij} \partial_{x^i x^j} + b_i \partial_{x^i}$ the second order elliptic operator appearing in Lemma 1, then we shall define the commutator

$$[D_s^m, \mathscr{L}]u := D_s^m \mathscr{L} u - \mathscr{L} D_s^m u.$$

We formulate the following model problem, which is then used to derive the high-order energy identities. Corresponding to the equations (18)–(20) we analyze the equations:

$$\mathscr{U}_t - a_{ij} \mathscr{U}_{ij} - b_i \mathscr{U}_i = \mathscr{A} \quad \text{in } \Omega, \tag{35}$$

$$\mathscr{U} \circ \phi = -(n-1)\chi - \Delta_g \chi + \mathscr{B} \quad \text{on } \mathscr{S}, \tag{36}$$

$$[\partial_n \mathscr{U}]_-^+ \circ \phi = \frac{r^2}{|g|} \left(-\frac{\chi_t r}{|g|} + \alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi + \mathscr{C} \right) \quad \text{on } \mathscr{S}. \tag{37}$$

Lemma 2. *The following energy identities hold:*

(1)

$$\frac{1}{2}\partial_t \int_\Omega \rho \mathscr{U}^2 + \frac{1}{2}\partial_t \int_{\mathscr{S}} \left\{ |\nabla_g \chi|^2 - (n-1)\chi^2 \right\} + \int_\Omega \rho \nabla \mathscr{U}^t A \nabla \mathscr{U}$$
$$= \int_\Omega P + \int_{\mathscr{S}} Q + \int_{\partial\Omega} \mathscr{U} \mathscr{U}_n; \tag{38}$$

[1] By “close” we mean in an open neighborhood of prescribed positive thickness.

(2)

$$\int_\Omega \rho \mathscr{U}_t^2 + \frac{1}{2}\partial_t \int_\Omega \rho |D\mathscr{U}|^2 + \partial_t \int_\Omega \rho \nabla \mathscr{U}^t A \nabla \mathscr{U} + \int_\Omega \rho a_{ij} D_k \mathscr{U}_i D_k \mathscr{U}_j$$
$$+ \int_{\mathscr{S}} \{|\nabla_g \chi_t|^2 - (n-1)|\chi_t|^2\} + \frac{1}{2}\partial_t \int_{\mathscr{S}} \{|\nabla_g^2 \chi|^2 - (n-1)|\nabla_g \chi|^2\}$$
$$= \int_\Omega S + \int_{\mathscr{S}} T + \int_{\partial\Omega} \big[-(\mathscr{U}_t + \Delta \mathscr{U})\mathscr{U}_n + \nabla \mathscr{U} \cdot \nabla \mathscr{U}_n \big], \tag{39}$$

where

$$P := -(a_{ij}\rho)_{x^j} \mathscr{U}_i \mathscr{U} + (\mathscr{A} + b_i \mathscr{U}_i)\rho \mathscr{U};$$
$$Q := \underline{\frac{|g|}{r}\alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi\, \mathscr{U}} + \chi_t \mathscr{B} - \frac{|g|}{r}\mathscr{C}\mathscr{U}, \tag{40}$$

$$S := \big(\mathscr{A} + b_i \mathscr{U}_i\big)\big(\mathscr{U}_t - \rho D_i D_i \mathscr{U}\big) + \frac{1}{2}\rho_t |D\mathscr{U}|^2 + (\rho a_{ij})_t \mathscr{U}_i \mathscr{U}_j$$
$$-(a_{ij}\rho)_j \mathscr{U}_t \mathscr{U}_i + (a_{ij}\rho)_j \mathscr{U}_i D_k D_k \mathscr{U} - D_k(a_{ij}\rho)\mathscr{U}_i D_k \mathscr{U}_j$$
$$-\underline{\rho D_i \mathscr{U} \mathscr{U}_t \nu(\mu)} - \underline{a_{ij}\rho \mathscr{U}_i D_k \mathscr{U}_j \nu(\mu)}; \tag{41}$$

$$T := -\chi_t \mathscr{B}_t + \frac{|g|}{r}\mathscr{C}\mathscr{U}_t - \chi_t \Delta_g \mathscr{B} + \frac{|g|}{r}\mathscr{C}\Delta_g \mathscr{U}$$
$$+\underline{\frac{r}{|g|}\alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi\, \mathscr{U}_t} - \underline{\frac{r}{|g|}\alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi\, \Delta_g \mathscr{U}}. \tag{42}$$

Proof. We multiply (35) by $\rho\mathscr{U}$ and integrate over Ω. Integrating by parts, we obtain

$$\frac{1}{2}\partial_t \int_\Omega \rho \mathscr{U}^2 + \int_\Omega \rho \nabla \mathscr{U}^t A \nabla \mathscr{U} - \int_{\mathscr{S}} a_{ij}[\mathscr{U}_i] n^j \rho \mathscr{U}$$
$$= \frac{1}{2}\int_\Omega \rho_t \mathscr{U}^2 - \int_\Omega (a_{ij}\rho)_{x^j} \mathscr{U}_i \mathscr{U} + \int_\Omega \big(\mathscr{A} + b_i \mathscr{U}_i\big)\rho \mathscr{U} + \int_{\partial\Omega} \mathscr{U}\mathscr{U}_n.$$

Note that

$$-\int_{\mathscr{S}} a_{ij}[\mathscr{U}_i] n^j \rho \mathscr{U} = -\int_{\mathscr{S}} a_{ij} n^i n^j r [\mathscr{U}_n] \mathscr{U}$$
$$= -\int_{\mathscr{S}} \frac{|g|^2}{r^4} r \frac{r^2}{|g|}\Big(-\frac{\chi_t r}{|g|} + \alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi + \mathscr{C}\Big)\Big(-(n-1)\chi - \Delta_g \chi + \mathscr{B}\Big)$$
$$= \frac{1}{2}\partial_t \int_{\mathscr{S}} \{|\nabla_g \chi|^2 - (n-1)\chi^2\} - \underline{\int_{\mathscr{S}} \frac{|g|}{r}\alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi\, \mathscr{U}} + \int_{\mathscr{S}} -\chi_t \mathscr{B}$$
$$+ \frac{|g|}{r}\mathscr{C}\mathscr{U}. \tag{43}$$

This proves the first identity. As to the second identity, we multiply (35) by $\rho\big(\mathscr{U}_t - D_i D_i \mathscr{U}\big)$, integrate over Ω and use integration by parts:

$$\int_\Omega \rho \mathscr{U}_t^2 + \frac{1}{2}\partial_t \int_\Omega \rho |D\mathscr{U}|^2 + \partial_t \int_\Omega \rho \nabla \mathscr{U}^t A \nabla \mathscr{U} + \int_\Omega \rho a_{ij} D_k \mathscr{U}_i D_k \mathscr{U}_j$$
$$- \int_{\mathscr{S}} a_{ij} n^i n^j r[\mathscr{U}_n]\mathscr{U}_t + \int_{\mathscr{S}} a_{ij} n^i n^j r[\mathscr{U}_n]_-^+ \Delta_g \mathscr{U} = \int_\Omega \big(\mathscr{A} + b_i \mathscr{U}_i\big)$$
$$\big(\mathscr{U}_t - \rho D_i D_i \mathscr{U}\big) + \frac{1}{2}\int_\Omega \rho_t |D\mathscr{U}|^2 - \underline{\int_\Omega \rho D_i \mathscr{U} \mathscr{U}_t \nu(\mu)}$$
$$+ \int_\Omega (\rho a_{ij})_t \mathscr{U}_i \mathscr{U}_j - \int_\Omega (a_{ij}\rho)_j \mathscr{U}_t \mathscr{U}_i + \int_\Omega (a_{ij}\rho)_j \mathscr{U}_i D_k D_k \mathscr{U}$$
$$- \int_\Omega D_k(a_{ij}\rho)\mathscr{U}_i D_k \mathscr{U}_j - \underline{\int_\Omega a_{ij}\rho \mathscr{U}_i D_k \mathscr{U}_j \nu(\mu)}$$
$$+ \int_{\partial\Omega} \big[- (\mathscr{U}_t + \Delta \mathscr{U})\mathscr{U}_n + \nabla \mathscr{U} \cdot \nabla \mathscr{U}_n \big]. \tag{44}$$

Let us extract the energy contribution from the integrals over $\mathscr{S}$ in the above identity.

$$- \int_{\mathscr{S}} a_{ij} n^i n^j r[\mathscr{U}_n]\mathscr{U}_t = - \int_{\mathscr{S}} \frac{|g|^2}{r^4} r \frac{r^2}{|g|} \left(- \frac{\chi_t r}{|g|} + \alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi + \mathscr{C} \right)$$
$$\big(- (n-1)\chi_t - \Delta_g \chi_t + \mathscr{B}_t \big)$$
$$= \int_{\mathscr{S}} \big\{ |\nabla_g \chi_t|^2 - (n-1)|\chi_t|^2 \big\} - \underline{\int_{\mathscr{S}} \frac{r}{|g|}\alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi\, \mathscr{U}_t} + \int_{\mathscr{S}} \chi_t \mathscr{B}_t$$
$$- \frac{|g|}{r}\mathscr{C}\mathscr{U}_t.$$

Similarly,

$$\int_{\mathscr{S}} a_{ij} n^i n^j r[\mathscr{U}_n]_-^+ \Delta_{\mathscr{B}} \mathscr{U} = \int_{\mathscr{S}} \frac{|g|^2}{r^4} r \frac{r^2}{|g|} \left(- \frac{\chi_t r}{|g|} + \alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi + \mathscr{C} \right)$$
$$\big(- (n-1)\Delta_g \chi - \Delta_g^2 \chi + \Delta_g \mathscr{B} \big)$$
$$= \frac{1}{2}\partial_t \int_{\mathscr{S}} \big\{ |\nabla_g^2 \chi|^2 - (n-1)|\nabla_g \chi|^2 \big\} + \underline{\int_{\mathscr{S}} \frac{r}{|g|}\alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi\, \Delta_g \mathscr{U}}$$
$$+ \int_{\mathscr{S}} \chi_t \Delta_g \mathscr{B} - \frac{|g|}{r}\mathscr{C}\Delta_g \mathscr{U}.$$

From the two previous identities we obtain the expression (42) for the error term T. □

3.2. The Energy

We introduce a parameter $\gamma > 1$ to be fixed later. Motivated by Lemma 2, for given $\mathscr{U}$, χ we define the auxiliary energy

$$\overline{\mathscr{E}}(\mathscr{U},\chi) := \frac{\gamma}{2}\int_\Omega \rho\mathscr{U}^2 + \frac{1}{2}\int_\Omega \rho|D\mathscr{U}|^2 + \int_\Omega \rho\nabla\mathscr{U}^t A\nabla\mathscr{U} \\ + \frac{\gamma}{2}\int_{\mathscr{S}}\left\{|\nabla_g\chi|^2-(n-1)\chi^2\right\} + \frac{1}{2}\int_{\mathscr{S}}\left\{|\nabla_g^2\chi|^2-(n-1)|\nabla_g\chi|^2\right\} \tag{45}$$

and the auxiliary dissipation

$$\overline{\mathscr{D}}(\mathscr{U},\chi) := \gamma\int_\Omega \rho\nabla\mathscr{U}^t A\nabla\mathscr{U} + \int_\Omega \rho\mathscr{U}_t^2 + \int_\Omega \rho a_{ij}D_k\mathscr{U}_i D_k\mathscr{U}_j \\ + \int_{\mathscr{S}}\left\{|\nabla_g\chi_t|^2-(n-1)|\chi_t|^2\right\}. \tag{46}$$

Summing the two previous expression and using Lemma 2, we obtain

$$\overline{\mathscr{E}}(\mathscr{U},\chi)(t) + \int_0^t \overline{\mathscr{D}}(\mathscr{U},\chi)(s)\,\mathrm{d}s \\ = \overline{\mathscr{E}}(\mathscr{U},\chi)(0) + \int_0^t\int_\Omega\left\{\gamma P + S\right\} + \int_0^t\int_{\mathscr{S}}\left\{\gamma Q + T\right\} \\ + \int_0^t\int_{\partial\Omega}\left[\mathscr{U}_n(\gamma\mathscr{U}-\mathscr{U}_t-\Delta\mathscr{U}) + \nabla\mathscr{U}\cdot\nabla\mathscr{U}_n\right]. \tag{47}$$

We define the total energy and dissipation by setting

$$\mathscr{E}(w,f) = \sum_{|m|+2s\leqq 2N}\overline{\mathscr{E}}(D_s^m w, D_s^m f), \tag{48}$$

$$\mathscr{D}(w,f) = \sum_{|m|+2s\leqq 2N}\overline{\mathscr{D}}(D_s^m w, D_s^m f). \tag{49}$$

We shall often write $\mathscr{E}(t) := \mathscr{E}(w(t), f(t))$ and $\mathscr{D}(t) := \mathscr{D}(w(t), f(t))$.

4. A Priori Estimates

Lemma 3 (Poincaré type estimate). *There exists a positive constant β such that on the time interval of existence of a solution to the Stefan problem, the following estimate holds:*

$$||f||_{L^2} + ||w||_{L^2(\Omega)} \leqq \beta||\nabla w||_{L^2(\Omega)}.$$

Proof. To prove the lemma, it is much more instructive to work with the solution u on the moving domain $\Omega(t)$ and later convert it into a result for w. Recall the expansion: $u \circ \phi = -(n-1)f - \Delta_g f + N(f)$. Fix a spherical harmonic $s_{k,i}$ of degree k, where $k \geqq 2$ and $1 \leqq i \leqq N(k)$ (recall the notation introduced in the introduction). Multiply both sides of the above relation by $s_{k,i}$ and integrate over $\mathscr{S}$. Observing that $\int_{\mathscr{S}}(-(n-1)f - \Delta_g f)s_{k,i} = (n-1)(k^2-1)f_{k,i}$, we get

$$f_{k,i} = \frac{1}{(n-1)(k^2-1)}\int_{\mathscr{S}} u \circ \phi \, s_{k,i} - \frac{1}{(n-1)(k^2-1)}\int_{\mathscr{S}} N(f)\, s_{k,i}. \tag{50}$$

Note that

$$\Big|\int_{\mathscr{S}} u \circ \phi \, s_{k,i}\Big| = \Big|\int_{\mathscr{S}}\Big(u\circ\phi - \frac{1}{|\mathscr{S}|}\int_{\mathscr{S}} u\circ\phi\Big)\, s_{k,i}\Big|$$
$$\leqq C||u\circ\phi - \frac{1}{|\mathscr{S}|}\int_{\mathscr{S}} u\circ\phi||_{L^2} \leqq C||\nabla u||_{L^2(\Omega^\pm)},$$

where we used the Sobolev inequality in the last estimate above. The last inequality, together with (50), immediately implies

$$|f_{k,i}| \leqq C\sqrt{\mathscr{D}} + C\Big|\int_{\mathscr{S}} N(f)\, s_{k,i}\, \mathrm{d}\xi\Big|, \quad k \geqq 2. \tag{51}$$

In order to estimate $\int_{\mathscr{S}} f$, we observe that

$$u \circ \phi r^{n-2}|g| = -(n-1)f - \Delta_g f + q(f),$$

where q stands for the nonlinear remainder with a leading order quadratic term. Integrating the above equation over $\mathscr{S}$, we find

$$\int_{\Gamma(t)} u = -(n-1)\int_{\mathscr{S}} f + \int_{\mathscr{S}} q(f). \tag{52}$$

Multiplying the conservation law $\int_\Omega u + \int_{\mathscr{S}} f + \sum_{k=2}^n \binom{n}{k}\int_{\mathscr{S}} f^k/n = 0$ by $\frac{1}{|\Omega|}$ and (52) by $\frac{1}{|\mathscr{S}|}$ and subtracting the two equations, we obtain

$$\Big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma|}\Big)\int_{\mathscr{S}} f = -\Big(\frac{1}{|\Omega|}\int_\Omega u - \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} u\Big)$$
$$-\frac{1}{n|\Omega|}\sum_{k=2}^n \binom{n}{k}\int_{\mathscr{S}} f^k + \frac{1}{|\mathscr{S}|}\int_{\mathscr{S}} q(f). \tag{53}$$

Note that $\zeta - \big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma(t)|}\big) = \frac{n-1}{|\Gamma(t)|} - \frac{n-1}{\mathscr{S}}$ and hence $|\zeta - \big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma(t)|}\big)| \leqq C||f||_{H^1}$, which for $||f||_{H^1}$ small enough implies $|\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma(t)|}| \geqq \zeta/2 > 0$. Hence, upon dividing (53) by $K_1 := \frac{1}{|\mathscr{S}|}\big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma(t)|}\big)$, we conclude

$$\int_{\mathscr{S}} f = -\frac{1}{K_1}\Big(\frac{1}{|\Omega|}\int_\Omega u - \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} u\Big) - \frac{1}{nK_1|\Omega|}\sum_{k=2}^n\binom{n}{k}\int_{\mathscr{S}} f^k$$
$$+\frac{1}{K_1|\Gamma(t)|}\int_{\mathscr{S}} q(f). \tag{54}$$

From the mean value theorem, we deduce

$$\left|\frac{1}{|\Omega|}\int_\Omega u - \frac{1}{|\Gamma(t)|}\int_\Gamma u\right| \leqq C||\nabla u||_{L^2(\Omega^\pm)}.$$

Thus, from (54) and the previous inequality, we have

$$\left|\int_{\mathscr{S}} f\right| \leqq C||\nabla u||_{L^2(\Omega^\pm)} + C||f||_{L^2}^2. \tag{55}$$

Summing (55) and (51) and keeping in mind that $\mathbb{P}_1 f = 0$, we obtain

$$\left|\int_{\mathscr{S}} f\right| + \sum_{k=1}^{\infty}\sum_{i=1}^{N(k)} |f_{k,i}| \leqq C||\nabla u||_{L^2(\Omega)} + C||f||_{L^2}^2 + C\sum_{k=2}^{\infty}\sum_{i}\left|\int_{\mathscr{S}} N(f)s_{k,i}\right|$$
$$\leqq C||\nabla u||_{L^2(\Omega)} + C||f||_{L^2}^2 + C||N(f)||_{L^2} \leqq C||\nabla u||_{L^2(\Omega)} + C||f||_{L^2}^2.$$

Smallness of $||f||_{L^2}$ then implies

$$\sum_{k=0}^{\infty}\sum_{i=1}^{N(k)} |f_{k,i}| \leqq C||\nabla u||_{L^2(\Omega(t))}.$$

Changing variables ($x \to \Theta(x)$, that is $u \to w$) and using the smallness of $||\nabla_g f||_{L^2}$ we get the bound $||\nabla u||_{L^2(\Omega^\pm)} \leqq C||\nabla w||_{L^2(\Omega)}$. Combined with previous estimate we conclude

$$||f||_{L^2} \leqq C||\nabla w||_{L^2(\Omega(t))}. \tag{56}$$

Finally, from the previous inequality and the conservation law (6), we immediately deduce

$$\left|\int_\Omega u\right| \leqq C||\nabla u||_{L^2(\Omega(t))} \leqq C||\nabla w||_{L^2(\Omega(t))}.$$

Changing variables and using the smallness of $||\nabla_g f||_{L^2}$, we obtain

$$\left|\int_\Omega w\right| \leqq C||\nabla w||_{L^2(\Omega^\pm)}, \tag{57}$$

and this finishes the proof of the lemma. □

In the new coordinates, the mass conservation law (6) takes a different form. The following lemma expresses this conservation in a way that will be useful for proving the positive definitiveness of the energy expressions $\mathscr{E}$ and $\mathscr{D}$.

Lemma 4. (Mass conservation law) *The following identity holds:*

$$\int_\Omega \rho w + \int_{\mathscr{S}} f = -\int_{\mathscr{S}}\sum_{k=2}^{n}\binom{n}{k}\frac{f^k}{n} + \int_\Omega wg(\nabla\rho, f), \tag{58}$$

where g is a bounded smooth function with $g(\mathbf{0}, 0) = 0$.

Proof. Integrating the equation (18) on a moving domain $\Omega\setminus\Gamma(t)$, using the Stokes formula and the boundary condition (20), we obtain:

$$\partial_t \int_{\Omega(t)} u + \partial_t \int_{\mathscr{S}} \frac{(1+f)^n}{n} = 0.$$

The assumed initial condition (25) implies

$$\int_{\Omega(t)} u + \sum_{k=1}^{n} \binom{n}{k} f^k = 0.$$

To express the integral over $\Omega(t)$ above as an integral over $\Omega \setminus \mathscr{S}$, it remains to understand the Jacobian $|\det D\Theta^{-1}|$, where $\Theta : \Omega(t) \to \Omega \setminus \mathscr{S}$ is the change of variables map defined in the line before (26) and $\Theta^{-1} : \bar{x} \to \mathbf{a}(t)+\rho(t,\bar{x})\bar{x}$. Thus, $(D\Theta^{-1})_{ij} = \rho\delta_{ij} + \rho_{\bar{x}^i}\bar{x}^j$, where δ_{ij} denotes the Kronecker delta. This implies $\det D\Theta^{-1} = \rho^n + q_1(\nabla\rho) = \rho + (\rho-1)q_2(\rho) + q_1(\nabla\rho)$, for some polynomials q_1 and q_2 such that $q_i(0) = 0$, $i = 1, 2$. From here, we easily infer the lemma. Since $\rho - 1 = f$ close to $\mathscr{S}$ and ρ is smooth and equal to 0 close to the boundary $\partial\Omega$, the claim of the lemma follows. □

Lemma 5. (Positivity of the energy) *Under the smallness assumption on* $||D_s^m f||_{L^2} + ||D_s^m w||_{L^2}$*, for* $|m|+2s \leqq N$*, the energy quantities* $\mathscr{E}$ *and* $\mathscr{D}$ *are positive definite.*

Proof. For a given function $\omega : \mathscr{S} \to \mathbb{R}$, let us abbreviate $Z(\omega) = \int_{\mathscr{S}} |\nabla_g\omega|^2 - (n-1)\omega^2$. If $\chi = D_s^m f$, with $|\mu| \geqq 1$, then, by Wirtinger's inequality, we immediately see

$$Z(\chi) \geqq C||\mathbb{P}_{2+}\chi||_{L^2}^2$$

and analogous inequalities hold for $Z(\chi_t)$ and $Z(\nabla\chi)$. If however, χ is of the form $\chi = \partial_{t^s} f$, then we must exploit the conservation law (58). Note that

$$\int_\Omega \rho w_{t^s}^2 + Z(f_{t^s}) = \frac{1}{|\Omega|_\rho}\Big(\int_\Omega \rho w_{t^s}\Big)^2 + \int_\Omega \rho\Big(w_{t^s} - \frac{1}{|\Omega|_\rho}\int_\Omega \rho w_{t^s}\Big)^2 - \frac{(n-1)}{|\mathscr{S}|}\Big(\int_{\mathscr{S}} f_{t^s}\Big)^2 + Z(\mathbb{P}f_{t^s}), \tag{59}$$

where $|\Omega|_\rho = \int_\Omega \rho\, \mathrm{d}x$. From Lemma 4, we obtain

$$\int_\Omega \rho w_{t^s} + \int_{\mathscr{S}} f_{t^s} = G(w, f),$$

where $G(w,f) = \int_\Omega \{-\partial_{t^s}(\rho w) + \rho w_{t^s}\} - \int_{\mathscr{S}} \sum_{k=2}^n \binom{n}{k} f_{t^s}^k/n + \int_\Omega \partial_{t^s}(wg(\nabla\rho, f))$. Thus,

$$\frac{1}{|\Omega|_\rho}\Big(\int_\Omega \rho w_{t^s}\Big)^2 - \frac{(n-1)}{|\mathscr{S}|}\Big(\int_{\mathscr{S}} f_{t^s}\Big)^2 = \Big(\frac{1}{|\Omega|_\rho} - \frac{n-1}{|\mathscr{S}|}\Big)\Big(\int_\Omega \rho w_{t^s}\Big)^2 - G(w,f)\Big(G(w,f) - 2\int_\Omega \rho w_{t^s}\Big).$$

Note however

$$\frac{1}{|\Omega|_\rho} - \frac{n-1}{|\mathscr{S}|} = \zeta + \frac{1}{|\Omega|} - \frac{1}{|\Omega|_\rho} = \zeta + h(f),$$

where h is a smooth bounded function, $h(0) = 0$. Going back to (59), we conclude

$$\int_\Omega \rho w_{t^s}^2 + Z(f_{t^s}) = \zeta\Big(\int_\Omega \rho w_{t^s}\Big)^2 + \int_\Omega \rho\Big(w_{t^s} - \frac{1}{|\Omega|_\rho}\int_\Omega \rho w_{t^s}\Big)^2 + Z(\mathbb{P} f_{t^s}) + R(w_{t^s}, f_{t^s}),$$

where the remainder $R(w_{t^s}, f_{t^s})$ is a cubic nonlinearity given by

$$R(w_{t^s}, f_{t^s}) := -G(w, f)\Big(G(w, f) - 2\int_\Omega \rho w_{t^s}\Big) + h(f)\Big(\int_\Omega \rho w_{t^s}\Big)^2.$$

Under a smallness assumption on w and f it is easy to see that $\big|R(w_{t^s}, f_{t^s})\big| \leqq \lambda\big(||w_{t^s}||^2_{L^2(\Omega)} + ||f_{t^s}||^2_{L^2}\big)$, for a small constant λ, and thus there exists a positive constant $M \geqq 1$ such that

$$\frac{1}{M}\big(||w_{t^s}||^2_{L^2(\Omega)} + ||f_{t^s}||^2_{L^2}\big) \leqq \int_\Omega \rho w_{t^s}^2 + Z(f_{t^s}) \leqq M\big(||w_{t^s}||^2_{L^2(\Omega)} + ||f_{t^s}||^2_{L^2}\big).$$

This concludes the proof of the lemma. □

The following lemma states that the norms defined by $\mathscr{E}$ and $\mathscr{D}$ are equivalent to the corresponding parabolic Sobolev norms. The main new ingredient is that *all* the derivatives of w are bounded. In particular, derivatives of w in the normal direction are bounded by the energy quantities $\mathscr{E}$ and $\mathscr{D}$, although they are not a priori contained in their definitions (48) and (49).

Lemma 6. *The following norms are equivalent*:

$$\mathscr{E}(w, f)(t) \approx \sum_{k=0}^{N} \big\{||w||^2_{W^{N-k,\infty}W^{2k,2}([0,t]\times\Omega)} + ||\nabla_g f||^2_{W^{N-k,\infty}W^{2k+1,2}([0,t]\times\mathbb{S}^{n-1})}\big\}.$$

$$\mathscr{D}(w, f)(t) \approx \sum_{k=0}^{N} \big\{||\nabla w||^2_{W^{N-k,2}W^{2k+1,2}([0,t]\times\Omega)} + ||w_t||^2_{W^{N-k,2}W^{2k,2}([0,t]\times\Omega)} + ||f_t||^2_{W^{N-k,2}W^{2k+1,2}([0,t]\times\Omega)}\big\},$$

where $A \approx B$ means that there is a constant $c > 1$ such that $\frac{A}{c} \leqq B \leqq cA$.

Proof. We only sketch the proof as it is analogous to the proof of Lemma 3.5 in [17]. What we need to prove is that for any triple of indices (μ, r, s) satisfying $|\mu| + r + 2s \leqq 2N$ we have

$$\|D_\xi^\mu \partial_n^r \partial_{t^s} w\|^2_{L^2(\Omega)} \leqq C\mathscr{E}(w, f).$$

The claim is evidently true for $r = 0, 1$ following from the definition of $\mathscr{E}$. We then proceed by induction on the number of normal derivatives r. Inductive step uses the basic equation (30). It allows to express w_{nn} as the sum of the multiples of w_t, $D^{ij} w$ and $D^i w_n$ in the region of Ω close to the unit sphere $\mathscr{S}$, thus allowing to complete the proof in the case $r = 2$. By successively differentiating the equation (30) and expressing the term with the highest number of normal derivatives in terms of terms with either *fewer* normal derivatives or *purely tangential* derivatives, we inductively complete the argument. □

5. Local Existence

Theorem 2. *There exist small positive constants E_1, m_1 such that for any $E_0 \leqq E_1$, $m_0 \leqq m_1$ and positive constants T^*, C^*, such that if*

$$\mathscr{E}(w_0, f_0) \leqq E_0; \quad |\mathbf{a}(0)| \leqq m_0,$$

then there exists a unique solution to the Stefan problem (18)–(25) *on the time interval* $[0, T^*]$ *such that for any* $0 \leqq s \leqq t < T^*$:

$$\sup_{s \leqq \tau \leqq t} \mathscr{E}(\tau) + \int_s^t \mathscr{D}(\tau)\,\mathrm{d}\tau \leqq \mathscr{E}(s) + \left(\frac{1}{4} + \tilde{C} \sup_{s \leqq \tau \leqq t} \sqrt{\mathscr{E}(\tau)}\right) \int_0^t \mathscr{D}(\tau)\,\mathrm{d}\tau; \tag{60}$$

$$\sup_{0 \leqq \tau \leqq T^*} |a(\tau)| \leqq 2m_0. \tag{61}$$

Moreover, $\mathscr{E}$ is continuous on $[0, T^*[$ *and* $\sup_{s \leqq \tau \leqq t} \mathscr{E}(\tau) + \frac{1}{2}\int_s^t \mathscr{D}(\tau)\,\mathrm{d}\tau \leqq E_0$.

Proof. We first sketch the proof of the a priori bound (60) assuming that the solution already exists. We explain in detail how to bound the hardest quadratic-in-order terms. For all the other (cubic) terms we use rather standard energy estimates, that can be found in the corresponding local existence proofs in [17] (or [18]). To prove the estimate (60), we must bound the right-hand side of the energy identity (47) where we plug in $\mathscr{U} = D_s^m w$ and $\chi = D_s^m f$ for all $|m| + 2s \leqq 2N$. All the terms in the definitions of P, Q, S, and T (see (40)–(42)) are trilinear except for the underlined terms, which are only quadratic in the order of nonlinearity and it is a priori not clear how to bound them.

There are two types of the quadratic error terms: the first type arises for technical reasons due to the introduction of the cut-off function μ while fixing the domain in Section 3. These are the two underlined terms in the expression (41) for S. For $\mathscr{U} = D_s^m w$ and $\chi = D_s^m f$, $|m| + 2s \leqq 2N$, by Young's inequality, we have

$$\begin{aligned}\Big|\int_0^t \int_\Omega \rho D_i \mathscr{U}\,\mathscr{U}_t \nu(\mu)\Big| &\leqq \frac{1}{4}\int_0^t \int_\Omega \rho \mathscr{U}_t^2 + \|\nu(\mu)\|_{L^\infty(\Omega)}^2 \int_0^t \int_\Omega \rho |D\mathscr{U}|^2 \\ &\leqq \frac{1}{4}\int_0^t \int_\Omega \rho \mathscr{U}_t^2 + C^* \int_0^t \int_\Omega \rho \nabla \mathscr{U}^t A \nabla \mathscr{U}. \end{aligned} \tag{62}$$

Similarly, we estimate the integral of the second underlined term in the expression (41):

$$\Big|\int_0^t\int_\Omega a_{ij}\rho\mathscr{U}_i D_k\mathscr{U}_j\nu(\mu)\Big| \leqq \frac{1}{4}\int_0^t\int_\Omega a_{ij}\rho D_k\mathscr{U}_j D_k\mathscr{U}_i + C^*\int_0^t\int_\Omega \rho\nabla\mathscr{U}^t A\nabla\mathscr{U}, \tag{63}$$

where we possibly enlarge the constant C^*. The size of the constant C^* dictates the choice of γ in the definition of the energy (see Section 3.2): set $\gamma = 3C^*$. We can thus absorb the right-most terms in the estimates (62) and (63) into the γ-dependent term in the definition (46) of $\mathscr{D}$.

The second type of quadratic error term is completely *intrinsic* and arises due to the *presence of the moving center coordinate* $\mathbf{a}(t)$ in the parametrization of the moving surface $\Gamma(t)$. There are three of these terms, underlined in the expressions (40) and (42) for Q and T respectively. Note that, if $|m| \geqq 1$, then $\alpha(t) = D_s^m\mathbf{a}(t) = 0$, since $\mathbf{a}$ depends only on t. Thus, we are concerned only with estimating expressions containing $\alpha(t)$ of the form $\alpha(t) = \partial_t^{k+1}\mathbf{a}(t) = \mathbf{a}^{(k+1)}(t)$, where $0 \leqq k \leqq N-1$. The underlined expression in the formula for Q thus takes the form

$$\int_0^t\int_{\mathscr{S}} \frac{r}{|g|}\mathbf{a}^{(k+1)}(t)\cdot \boldsymbol{n}_\Gamma\circ\phi\,\partial_t^k w.$$

The difficulty is immediately clear: since the expression $\frac{r}{|g|}\boldsymbol{n}_\Gamma\circ\phi$ is a quantity of order 1 for small f, the whole integral is only of a quadratic order and it is hence unclear how to bound it by $\sqrt{\mathscr{E}}\int_0^t\mathscr{D}$. In the local coordinates on the sphere:

$$\boldsymbol{n}_\Gamma\circ\phi = \frac{r}{|g|}\xi - \frac{\nabla_g f}{|g|},$$

where we keep in mind that ξ is the unit normal of the unit sphere $\mathscr{S}$. From here,

$$\frac{r}{|g|}\boldsymbol{n}_\Gamma\circ\phi = \xi + \frac{r^2-|g|^2}{|g|^2}\xi - \frac{r\nabla_g f}{|g|^2}.$$

Using this identity,

$$\begin{aligned}&\int_0^t\int_{\mathscr{S}} \frac{r}{|g|}\mathbf{a}^{(k+1)}(t)\cdot\boldsymbol{n}_\Gamma\circ\phi\,\partial_t^k w\\ &\quad=\int_0^t\int_{\mathscr{S}} \mathbf{a}^{(k+1)}(t)\cdot\Big[\xi+\frac{r^2-|g|^2}{|g|^2}\xi-\frac{r\nabla_g f}{|g|^2}\Big]\partial_t^k(-(n-1)f-\Delta_g f+N(f)).\end{aligned}$$

The key observation is $\xi\in\mathrm{Null}(\Delta_g+(n-1)\mathscr{I})$ where $\mathscr{I}$ stands for the identity operator. Hence

$$\int_{\mathscr{S}}\mathbf{a}^{(k+1)}(t)\cdot\xi\,\partial_{t^k}(\Delta_g+(n-1)\mathscr{I})f = \mathbf{a}^{(k+1)}(t)\cdot\int_{\mathscr{S}}(\Delta_g+(n-1)\mathscr{I})\xi\,\partial_{t^k}f = 0.$$

Using the previous two identities, we eliminate the purely quadratic contribution:

$$\begin{aligned}
&\int_0^t \int_{\mathscr{S}} \mathbf{a}^{(k+1)}(t) \cdot \left[\xi + \frac{r^2 - |g|^2}{|g|^2} \xi - \frac{r \nabla_g f}{|g|^2} \right] \partial_{t^k} (-(n-1) f - \Delta_g f + N(f)) \\
&= \int_0^t \int_{\mathscr{S}} \mathbf{a}^{(k+1)}(t) \cdot \left[\frac{r^2 - |g|^2}{|g|^2} \xi - \frac{r \nabla_g f}{|g|^2} \right] (-(n-1) f_{t^k} - \Delta_g f_{t^k}) \\
&\quad + \int_0^t \int_{\mathscr{S}} \mathbf{a}^{(k+1)}(t) \cdot \left[\frac{r^2}{|g|^2} \xi - \frac{r \nabla_g f}{|g|^2} \right] \partial_{t^k} (N(f)).
\end{aligned}$$

Recall that $N(f)$ is a *quadratic* nonlinearity and therefore the two integrands on the right-hand side above have a manifestly trilinear structure and are thus easy to estimate. Namely, applying the differential operator ∂_{t^k} to (22), it is easy to deduce the bound $\left|\mathbf{a}^{(k+1)}(t)\right| \leqq C\sqrt{\mathscr{D}}$ if we know that $|\mathbf{a}(\tau)|$ is bounded. To estimate $|\mathbf{a}(\tau)|$ we use (22). Recalling $\mathbf{p} = \mathbf{x} - \mathbf{a}$, and assuming smallness of $\mathscr{E}$, we obtain

$$|\mathbf{a}(T)| \leqq |\mathbf{a}(0)| + T \sup_{0 \leqq \tau \leqq T} |\dot{\mathbf{a}}(\tau)| \leqq |\mathbf{a}(0)| + CT\sqrt{\mathscr{E}} + CT \sup_{0 \leqq \tau \leqq T} |\mathbf{a}(\tau)| \|u_t\|_{L^2(\Omega)}.$$

Therefore

$$\sup_{0 \leqq \tau \leqq T} |\mathbf{a}(\tau)| (1 - CT\sqrt{\mathscr{E}}) \leqq |\mathbf{a}(0)| + CT\sqrt{\mathscr{E}},$$

thus implying $\sup_{0 \leqq \tau \leqq T} |\mathbf{a}(\tau)| \leqq 2|\mathbf{a}(0)| \leqq 2m_0$ for appropriately small $\mathscr{E}$ and some finite T. Integration by parts, Lemma 3 and standard energy estimates then imply

$$\left| \int_0^t \int_{\mathscr{S}} \frac{r}{|g|} \mathbf{a}^{(k+1)}(t) \cdot \boldsymbol{n}_\Gamma \circ \phi \, \partial_t^k w \right| \leqq C\sqrt{\mathscr{E}} \int_0^t \mathscr{D}. \tag{64}$$

The same idea as in the proof of (64) works for the remaining two underlined integrals appearing in $\int_0^t \int_{\mathscr{S}} T$ (see (42)), so we finally conclude (for $\alpha = \mathbf{a}^{(k+1)}(t)$ and $\mathscr{U} = w_{t^{k+1}}$)

$$\left| \int_0^t \int_{\mathscr{S}} \frac{r}{|g|} \alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi \, \mathscr{U}_t \right| + \left| \int_0^t \int_{\mathscr{S}} \frac{r}{|g|} \alpha(t) \cdot \boldsymbol{n}_\Gamma \circ \phi \, \Delta_g \mathscr{U} \right| \leqq C\sqrt{\mathscr{E}} \int_0^t \mathscr{D}. \tag{65}$$

The construction of the solution (w, f) follows identically the construction scheme for the local solution from [17]. We briefly summarize the main steps: we set-up an iteration scheme, which generates a sequence of iterates $\{(w^m, f^m)\}_{m \in \mathbb{N}}$ solving a sequence of linear parabolic problems. As in [17], such an iteration is well defined, but it breaks the natural energy setting due to the lack of exact cancellations in the presence of cross-terms. We design the elliptic regularization

$$-\frac{f_t r}{|g|} - \varepsilon \frac{\Delta_g f_t}{r^{n-2}|g|} = [w_n]_-^+ \circ \phi$$

to overcome this difficulty. For a fixed ε, we use it to prove that $\{(w^m, f^m)\}_{m \in \mathbb{N}}$ is a Cauchy sequence in the energy space. Upon passing to the limit $m \to \infty$ we obtain a solution existing on an a priori ε-dependent time interval $[0, T^{\varepsilon}]$. But, in the limit $m \to \infty$ the dangerous cross-terms vanish. Since the elliptic regularization honors the energy structure, we obtain an energy bound analogous to (60) with ε-*independent* coefficients. By continuity, the solution exists on an ε-independent time interval $[0, T]$. Finally, we pass to the limit as $\varepsilon \to 0$ to obtain the solution of the original problem. □

6. Global Existence and Asymptotic Stability

The proof of the main result Theorem 1, is an immediate consequence of the following theorem and Lemma 6 about the equivalence of norms.

Theorem 3. *There exist small positive constants E and m, such that if*

$$\mathscr{E}(w_0, f_0) \leqq E; \quad |\mathbf{a}(0)| \leqq m,$$

then there exists a unique global-in-time solution to the Stefan problem (18)–(25) *converging asymptotically to some steady state solution $\Sigma(\bar{\mathbf{a}})$, where*

$$\mathbf{a}(t) \to \bar{\mathbf{a}} \quad as \quad t \to \infty.$$

Moreover, there are constants $c_1, c_2 > 0$ such that the following exponential decay estimate holds:

$$\mathscr{E}(w(t), f(t)) \leqq c_1 \mathrm{e}^{-c_2 t}.$$

Before we prove the theorem, we will prove an important auxiliary estimate, necessary for the proof of Theorem 3.

Lemma 7. *Let $[0, \mathfrak{t}]$ be an interval of existence of solution to the Stefan problem* (18)–(25) *for which the estimates* (60) *and* (61) *hold. Then there exist constants $\alpha, \delta \in \mathbb{R}_+$ such that*

$$\mathscr{E}(t) \leqq \frac{4\beta\mathscr{E}(0)}{t}\mathrm{e}^{-\frac{\alpha t}{2}}; \quad t \in [0, \mathfrak{t}[$$

and

$$|\dot{\mathbf{a}}(t)| \leqq \frac{c_{\beta\delta}\sqrt{\mathscr{E}(0)}}{\sqrt{t}}\mathrm{e}^{-\frac{\alpha t}{4}}; \quad t \in [0, \mathfrak{t}[, \tag{66}$$

where $\beta > 0$ is given by Lemma 3 *and $c_{\beta\delta} := 2\sqrt{\beta}\delta$.*

Proof. As in [22, p. 135], we define $V(s) := \int_s^{\mathfrak{t}} \mathscr{E}(\tau)\,\mathrm{d}\tau$. From (60) and Lemma 3, we conclude that there exists a constant $\alpha > 0$ such that

$$\sup_{s \leqq \tau \leqq t} \mathscr{E}(\tau) + \alpha \int_s^t \mathscr{E}(\tau)\,\mathrm{d}\tau \leqq \mathscr{E}(s). \tag{67}$$

Thus $\alpha V(s) \leqq \mathscr{E}(s)$. On the other hand, $V'(s) = -\mathscr{E}(s) \leqq -\alpha V(s)$, which, in turn, implies: $V(s) \leqq V(0)\mathrm{e}^{-\alpha s}$. Integrate (67) over the interval $[t/2, t]$ with respect to s, to obtain:

$$\frac{t}{2} \sup_{s \leqq \tau \leqq t} \mathscr{E}(\tau) \leqq V(\frac{t}{2}) \leqq V(0)\mathrm{e}^{-\frac{\alpha t}{2}},$$

implying

$$\mathscr{E}(t) \leqq \frac{2V(0)}{t}\mathrm{e}^{-\frac{\alpha t}{2}}.$$

Note that for E_0 and m_0 small enough ($\mathscr{E}(0) < \frac{1}{32\tilde{C}^2}$) in Theorem 2, by continuity and the inequality (60), we obtain the bound $\tilde{C} \sup_{0 \leqq s < \mathfrak{t}} \sqrt{\mathscr{E}(s)} \leqq \frac{1}{4}$. Plugging back into (60) and absorbing the right-most term into left-hand side, we obtain $\int_0^t \mathscr{D}(\tau)\,\mathrm{d}\tau \leqq 2\mathscr{E}(0)$ for all $t \in [0, \mathfrak{t}[$. Hence, by Lemma 3 $V(0) = \int_0^{\mathfrak{t}} \mathscr{E}(\tau)\,\mathrm{d}\tau \leqq \beta \int_0^{\mathfrak{t}} \mathscr{D}(\tau)\,\mathrm{d}\tau \leqq 2\beta\mathscr{E}(0)$ and we obtain

$$\mathscr{E}(t) \leqq \frac{4\beta\mathscr{E}(0)}{t}\mathrm{e}^{-\frac{\alpha t}{2}}, \tag{68}$$

thus proving the first claim of the lemma. From (22) and Jensen's inequality, we conclude that, for some constant $\delta > 0$ on the time interval $[0, \mathfrak{t}]$, the following inequality holds:

$$\begin{aligned}
|\dot{\mathbf{a}}| &\leqq \frac{\delta}{4} \sum_{k=1}^{n} ||f^k||_{L^2} ||f_t||_{L^2} + \frac{\delta}{4} ||w_t||_{L^2(\Omega)} + \frac{\delta}{4} ||\nabla w||_{L^2(\Omega)} \\
&\leqq \frac{\delta}{4} \left(2 + \sum_{k=1}^{n-1} (2\mathscr{E}(0)^k) \right) \sqrt{\mathscr{E}(w, f)} \\
&\leqq \delta \sqrt{\mathscr{E}(w, f)}.
\end{aligned}$$

By (68) we obtain

$$|\dot{\mathbf{a}}(t)| \leqq \frac{2\sqrt{\beta}\delta\sqrt{\mathscr{E}(0)}}{\sqrt{t}}\mathrm{e}^{-\frac{\alpha t}{4}} = \frac{c_{\beta\delta}\sqrt{\mathscr{E}(0)}}{\sqrt{t}}\mathrm{e}^{-\frac{\alpha t}{4}}$$

for any $t \in [0, \mathfrak{t}]$ and this finishes the proof of the lemma. □

Proof of Theorem 3. Let $m = \frac{m_1}{4}$ (m_1 is given by Theorem 2) and E be such that $c_{\beta\delta} c_\alpha \sqrt{E} \leqq \frac{m_1}{4}$, whereby $c_\alpha := \max_{y \geqq T^*} \frac{\sqrt{y}\mathrm{e}^{-\frac{\alpha y}{4}}}{(1-\mathrm{e}^{-\frac{\alpha T^*}{4}})}$. Define

$$\mathscr{T} := \sup_{\mathfrak{t}} \left\{ \sup_{0 \leqq \tau \leqq \mathfrak{t}} |\mathbf{a}(\tau)| \leqq 2m \;\; \& \;\; (60) \;\; \text{holds for any} \;\; s, t \in [0, \mathfrak{t}] \right\}. \tag{69}$$

Assume $\mathscr{T} < \infty$. With $T^* > 0$ given by Theorem 2, choose a unique $L \in \mathbb{N}$ such that

$$\mathscr{T} \in [LT^*, (L+1)T^*[.$$

Integrating over $[kT^*, (k+1)T^*]$ with $1 \leqq k \leqq L-1$, we obtain

$$\begin{aligned}|\mathbf{a}((k+1)T^*)| &\leqq |\mathbf{a}(kT^*)| + \int_{kT^*}^{(k+1)T^*} |\dot{\mathbf{a}}(\tau)|\,\mathrm{d}\tau \\ &\leqq |\mathbf{a}(kT^*)| + c_{\beta\delta}\sqrt{\mathscr{E}(0)}\int_{kT^*}^{(k+1)T^*} \frac{\mathrm{e}^{-\frac{\alpha\tau}{4}}}{\sqrt{\tau}}\,\mathrm{d}\tau \\ &\leqq |\mathbf{a}(kT^*)| + \frac{c_{\beta\delta}\sqrt{E(0)}}{\sqrt{kT^*}}\int_{kT^*}^{(k+1)T^*} \mathrm{e}^{-\frac{\alpha\tau}{4}}\,\mathrm{d}\tau \\ &\leqq |\mathbf{a}(kT^*)| + \frac{c_{\beta\delta}\sqrt{\mathscr{E}(0)}}{\sqrt{kT^*}}T^*\mathrm{e}^{-\frac{\alpha kT^*}{4}} \\ &\leqq |\mathbf{a}(kT^*)| + c_{\beta\delta}\sqrt{\mathscr{E}(0)}\sqrt{T^*}\mathrm{e}^{-\frac{\alpha kT^*}{4}}.\end{aligned}$$

Summing over $k = 1, \ldots, L-1$, we obtain

$$\begin{aligned}|\mathbf{a}(LT^*)| &\leqq |\mathbf{a}(T^*)| + c_{\beta\delta}\sqrt{\mathscr{E}(0)}\sqrt{T^*}\sum_{k=1}^{L-1}\mathrm{e}^{-\frac{\alpha kT^*}{4}} = |\mathbf{a}(T^*)| + \frac{c_{\beta\delta}\sqrt{\mathscr{E}(0)}\sqrt{T^*}}{1-\mathrm{e}^{-\frac{\alpha T^*}{4}}} \\ &\quad\times\left(\mathrm{e}^{-\frac{\alpha T^*}{4}} - \mathrm{e}^{-\frac{\alpha LT^*}{4}}\right) \\ &\leqq 2m + \frac{c_{\beta\delta}\sqrt{T^*}}{1-\mathrm{e}^{-\frac{\alpha T^*}{4}}}\mathrm{e}^{-\frac{\alpha T^*}{4}}\sqrt{E} \leqq 2m + c_{\beta\delta}c_\alpha\sqrt{E} \leqq \frac{3}{4}m_1,\end{aligned}$$

where the last inequality follows from the choice of m and E above. If we denote $T^1 := LT^* + \frac{\mathscr{T}-LT^*}{2}$ we obtain

$$\begin{aligned}|\mathbf{a}(T^1)| &\leqq |\mathbf{a}(LT^*)| + \int_{LT^*}^{T^1}|\dot{\mathbf{a}}(\tau)|\,\mathrm{d}\tau \leqq \frac{3m_1}{4} + c_{\beta\delta}\sqrt{\mathscr{E}(0)}\sqrt{T^*}\mathrm{e}^{-\frac{\alpha LT^*}{4}} \\ &\leqq \frac{3m_1}{4} + c_{\beta\delta}c_\alpha\sqrt{E} \leqq \frac{3m_1}{4} + \frac{m_1}{4} = m_1.\end{aligned}$$

Since $\mathscr{E}(T^1) \leqq E \leqq E_1$ (with E possibly smaller, see (68)) and $|\mathbf{a}(T^1)| \leqq m_1$, we can extend the solution (w, f) starting at $t_0 = T^1$ to the time interval $[0, T^1 + T^*[$, by Theorem 2 (local existence theorem). Same theorem guarantees the estimate

$$||\mathbf{a}||_{L^\infty[0,T^1+T^*]} \leqq 2m_1$$

as well as the validity of the energy estimate (60) on the time-interval $[0, T^1 + T^*]$. Since we assumed $\mathscr{T} < \infty$, we obtain $T^1 + T^* > \mathscr{T}$ and this, together with the continuity of $\mathscr{E}(w, f)$ and $\mathbf{a}$ contradicts the maximality of $\mathscr{T}$. It is now evident that the decay estimates (66) and (68) from Lemma 7 hold for all $t \in [0, \infty[$. This implies the decay claim of Theorem 3 as well as the existence of an asymptotic center $\bar{\mathbf{a}} \in \Omega$ such that $\mathbf{a}(t) \to \bar{\mathbf{a}}$, since $\dot{\mathbf{a}}(t) \to 0$ as $t \to \infty$. □

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Appendix A. Proof of Lemma 1

Note that (26) implies

$$\bar{x}_{x^i} = \pi e_i + \pi_{x^i}(x - \mathbf{a}). \tag{A.70}$$

From $\pi(x) = 1/\rho(\bar{x})$, we obtain $\pi_{x^i} = -(\nabla\rho \cdot \bar{x}_{x^i})/\rho^2$, which in turn, combined with (A.70), after an elementary calculation implies

$$\pi_{x^i} = -\frac{\rho_{\bar{x}^i}}{\rho^2\Psi}, \tag{A.71}$$

where

$$\Psi = \rho + \nabla\rho \cdot \bar{x}.$$

After further differentiating (A.71) with respect to x^i and using the relation (A.70), we arrive at

$$\Delta\pi = \frac{-\Delta\rho}{\rho^3\Psi} + \frac{\rho_{\bar{x}^i\bar{x}^j}\rho_{\bar{x}^i}\bar{x}^j(1-\rho^2)}{\rho^5\Psi^2} + \frac{-\rho_{\bar{x}^i\bar{x}^j}\bar{x}^i\bar{x}^j|\nabla\rho|^2 + 2|\nabla\rho|^2(\Psi+\rho)}{\rho^3\Psi^3}.$$

Similarly it is not hard to see that $\pi_t = (-\rho_t + \rho\nabla\rho \cdot \dot{\mathbf{a}})/\rho\Psi$. In order to evaluate the Laplacian in new coordinates, by (27) we first write

$$u(t,x) = w(t, \pi(t,x)x). \tag{A.72}$$

Applying Δ_x to the left-hand side, we obtain

$$\Delta u(t,x) = \pi^2\Delta_{\bar{x}}w + \left(2\pi x^j\pi_{x^i} + x^i x^j|\nabla\pi|^2\right)w_{\bar{x}^i\bar{x}^j} + \left(2\pi_{x^i} + \Delta\pi x^i\right)w_{\bar{x}^i}. \tag{A.73}$$

Applying ∂_t to the left-hand side of (A.72), we obtain

$$u_t(t,x) = w_t + \pi_t\nabla_{\bar{x}}w \cdot x. \tag{A.74}$$

Since $(\partial_t - \Delta)u = 0$, we conclude from (A.73) and (A.74)

$$(\partial_t - \Delta_{\bar{x}})w = a_{ij}w_{\bar{x}^i\bar{x}^i} + b_i w_{\bar{x}^i},$$

where a_{ij} and b_i are given by (28). Furthermore, using (A.71), it is easy to see that

$$a_{ij} = \frac{1}{\rho^2}\delta_{ij} - 2\frac{\rho_{\bar{x}^i}\bar{x}^j}{\rho^2\Psi} + \frac{\bar{x}^i\bar{x}^j|\nabla\rho|^2}{\rho^2\Psi^2}.$$

We now turn to the proof of (29). Note that

$$u_{x^i} = \nabla_{\bar{x}}w \cdot \bar{x}_{x^i} = \nabla_{\bar{x}}w \cdot \left(\pi e_i + \pi_{x^i}(x - \mathbf{a}(t))\right) = \nabla_{\bar{x}}w \cdot \left(\frac{e_i}{\rho} - \frac{\rho_{\bar{x}^i}}{\rho^2\Psi}\rho\bar{x}\right).$$

From here, we infer the formula

$$\nabla u(x) = \frac{1}{\rho}\nabla w(\bar{x}) - \frac{\nabla\rho(\bar{x})}{\rho(\bar{x})\Psi(\bar{x})}\bar{x} \cdot \nabla w(\bar{x}). \tag{A.75}$$

From the above formula, we obtain

$$[\nabla u]_-^+ \circ \phi(\xi) = \frac{1}{\rho(\xi)}[\nabla w]_-^+(\xi) - \frac{\nabla_g \rho(\xi)}{\rho(\xi)\Psi(\xi)}\xi \cdot [\nabla w]_-^+(\xi). \tag{A.76}$$

Note that

$$\xi \cdot [\nabla w]_-^+(\xi) = \xi \cdot \left([\nabla_g w]_-^+ + [w_n]_-^+ n_{\mathscr{S}}\right) = [w_n]_-^+,$$

since $[\nabla_g w]_-^+ = 0$ (due to the fact that $u^+|_{\mathscr{S}} = u^-|_{\mathscr{S}}$) and $\xi \cdot n_{\mathscr{S}} = |n_{\mathscr{S}}|^2 = 1$ ($n_{\mathscr{S}}$ stands for the unit normal on $\mathscr{S}$). Observe further that $\rho(\xi) = r(\xi)$. Furthermore, since $\nabla\rho(\xi)\cdot\xi = \nabla_g \rho \cdot \xi = 0$, we also have $\Psi(\xi) = \rho(\xi) = r(\xi)$. Form these observations and from (A.76) we obtain the formula

$$[\nabla u]_-^+ \circ \phi = \frac{1}{\rho}[w_n]_-^+ n_{\mathscr{S}} - \frac{\nabla_g \rho}{\rho\Psi}[w_n]_-^+ = \frac{1}{r}[w_n]_-^+ n_{\mathscr{S}} - \frac{\nabla_g r}{r^2}[w_n]_-^+.$$

It is straightforward to see that $n_\Gamma^i \circ \phi = \frac{r}{|g|} n_{\mathscr{S}}^i - \frac{\nabla_g f \cdot \nabla_g \xi^i}{|g|}$. Hence

$$\begin{aligned}\left([\nabla u]_-^+ \cdot n_\Gamma\right) \circ \phi &= \left(\frac{1}{r}[w_n]_-^+ n_{\mathscr{S}}^i - \frac{\nabla_g f \cdot \nabla_g \xi^i}{r^2}[w_n]_-^+\right) \cdot \left(\frac{r}{|g|} n_{\mathscr{S}}^i - \frac{\nabla_g f \cdot \nabla_g \xi^i}{|g|}\right)\\ &= \frac{1}{|g|}[w_n]_-^+ + \frac{|\nabla_g f|^2}{|g| r^2}[w_n]_-^+ = \frac{|g|}{r^2}[w_n]_-^+,\end{aligned}$$

and this proves (29). □

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