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Witt vectors and K -theory of automorphisms via noncommutative motives

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Abstract. We prove that the functor ring-of-rational-Witt-vectors $W_0(-)$ becomes co-representable in the category of noncommutative motives. As an application, we obtain an immediate extension of $W_0(-)$ from commutative rings to schemes. Then, making use of the theory of noncommutative motives, we classify all natural transformations of the functor K -theory-of-automorphisms.

1. Introduction and statement of results

The theory of *noncommutative motives*, envisioned by Kontsevich [9] in his seminal talk, was initiated in a series of articles [4, 5, 10, 11, 15, 16]. Among other applications, this theory allowed the first conceptual characterization of algebraic K -theory since Quillen's pioneering work, a streamlined construction of all the higher Chern characters, a universal characterization of Drinfeld's DG quotient, etc; see the survey [17]. Here we further the study of noncommutative motives by developing its interactions with Witt vectors and K -theory of automorphisms.

Witt vectors

Witt vectors were introduced in the thirties by Witt [21] in his work on algebraic number theory. Given a commutative ring A , the *Witt ring* $W(A)$ of A is the abelian group of all power series of the form $1 + a_1t + a_2t^2 + \dots$, with $a_i \in A$, endowed with the multiplication $*$ determined by the equality $(1 - a_1t) * (1 - a_2t) = (1 - a_1a_2t)$. The *rational* Witt ring $W_0(A)$ of A consists of the elements of the form

$$\left\{ \frac{1 + a_1t + \dots + a_it^i + \dots + a_nt^n}{1 + b_1t + \dots + b_jt^j + \dots + b_mt^m} \mid a_i, b_j \in A \right\} \subset W(A);$$

consult [7] for details. Although widely used in several branches of mathematics the rings $W(R)$ and $W_0(R)$ remain rather mysterious. Our co-representability Theorem 1.2 below offers a new viewpoint on the subject.

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Recall from [16] the construction of the *universal additive invariant*

$$\mathcal{U} : \text{dgcats} \longrightarrow \mathbf{Hmo}_0.$$

Roughly speaking, \mathcal{U} is the universal functor (defined on dg categories) with values in an idempotent complete additive category which inverts Morita equivalences and satisfies additivity; see Sect. 4. Examples of additive invariants include algebraic K -theory, Hochschild homology, cyclic homology (and all its variants), and even topological Hochschild homology. Because of its universal property, which is a reminiscence of motives, \mathbf{Hmo}_0 is called the *category of noncommutative motives*.

The tensor product of rings extends naturally to dg categories and, as proved in Proposition 4.3, induces in a universal way a symmetric monoidal structure on \mathbf{Hmo}_0 making the functor \mathcal{U} symmetric monoidal. The ring homomorphisms

$$\Delta : \mathbb{Z}[r] \xrightarrow{r \mapsto r \otimes r} \mathbb{Z}[r] \otimes \mathbb{Z}[r] \qquad \epsilon : \mathbb{Z}[r] \xrightarrow{r=1} \mathbb{Z} \tag{1.1}$$

endow the ring of polynomials $\mathbb{Z}[r]$ with a co-unital co-associative co-algebra structure. Since \mathcal{U} is symmetric monoidal the noncommutative motive $\mathcal{U}(\mathbb{Z}[r])$ becomes then a co-unital co-associative co-monoid in \mathbf{Hmo}_0 . As a consequence, given any unital associative monoid $\mathcal{O} \in \mathbf{Hmo}_0$, the symmetric monoidal structure of \mathbf{Hmo}_0 gives rise to a unital associative *convolution* ring structure on $\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r]), \mathcal{O})$. Our first main result is the following:

Theorem 1.2. *The natural homomorphisms $\mathbb{Z} \rightarrow \mathbb{Z}[r]$ and $\mathbb{Z}[r] \xrightarrow{r=0} \mathbb{Z}$ give rise to a direct sum decomposition $\mathcal{U}(\mathbb{Z}[r]) \simeq \mathcal{U}(\mathbb{Z}) \oplus \mathcal{W}_0$ of co-unital co-associative co-monoids in \mathbf{Hmo}_0 . For every (flat) commutative ring A we have a ring isomorphism*

$$\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(A)) \simeq W_0(A). \tag{1.3}$$

Note that the isomorphism (1.3) furnish us a conceptual characterization of the ring of rational Witt vectors since $\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(A))$ is defined solely in terms of precise universal properties. Roughly speaking, the noncommutative motive associated to the “affine line” decomposes into a “point” plus a “formal variable” which co-represents the ring of rational Witt vectors.

As an application, Theorem 1.2 allows us to immediately extend the functor $W_0(-)$ from commutative rings to schemes. Let X be a quasi-compact and quasi-separated scheme. Recall from [12] that the derived category of perfect complexes of \mathcal{O}_X -modules admits a (unique) differential graded enhancement $\mathcal{D}_{\text{perf}}^{\text{dg}}(X)$, under which the bi-exact functor $(E^\bullet, F^\bullet) \mapsto E^\bullet \otimes_{\mathcal{O}_X}^{\mathbb{L}} F^\bullet$ lifts to a dg functor $\mathcal{D}_{\text{perf}}^{\text{dg}}(X) \otimes \mathcal{D}_{\text{perf}}^{\text{dg}}(X) \rightarrow \mathcal{D}_{\text{perf}}^{\text{dg}}(X)$. The noncommutative motive $\mathcal{U}(X) := \mathcal{U}(\mathcal{D}_{\text{perf}}^{\text{dg}}(X))$ becomes then a unital associative monoid and so the abelian group $\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(X))$ carries a unital associative convolution ring structure. It is then natural to call $\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(X))$ the *rational Witt ring of X* . To the best of the author’s knowledge this invariant of schemes is new in the literature.

K-theory of automorphisms

Given a ring A , let $\mathbf{P}(A)$ be the category of finitely generated projective A -modules and $\text{Aut}(\mathbf{P}(A))$ the category of automorphisms. This latter category inherits from $\mathbf{P}(A)$ a natural exact structure in the sense of Quillen and so following [1, 2] one can consider its

Grothendieck group. This construction extends naturally to dg categories giving rise to a well-defined functor $K_0\text{Aut} : \text{dgc}at \rightarrow \text{Ab}$ with values in abelian groups; see Sect. 3. Moreover, when A is commutative the tensor product over A endows $K_0\text{Aut}(A)$ with a ring structure.

Theorem 1.4. *The functor $K_0\text{Aut}$ becomes co-representable in Hmo_0 by the ring $\mathbb{Z}[r, r^{-1}]$ of Laurent polynomials, i.e. for every dg category \mathcal{A} there is a group isomorphism*

$$\text{Hom}_{\text{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r, r^{-1}]), \mathcal{U}(\mathcal{A})) \simeq K_0\text{Aut}(\mathcal{A}). \tag{1.5}$$

Moreover, $\mathbb{Z}[r, r^{-1}]$ inherits naturally from $\mathbb{Z}[r]$ a co-unital co-associative co-algebra structure such that for every (flat) commutative ring A we have a ring isomorphism

$$\text{Hom}_{\text{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r, r^{-1}]), \mathcal{U}(A)) \simeq K_0\text{Aut}(A). \tag{1.6}$$

Informally speaking, Theorem 1.4 shows us that $K_0\text{Aut}$ becomes co-represented by the noncommutative motive associated to the ‘‘punctured affine line’’.

Classically the K -theory of automorphisms comes equipped with several natural transformations such as the Frobenius (F_n) and the Verschiebung (V_n) operations; consult [1, 2]. A fundamental problem in the field is then the classification of all the natural transformations of $K_0\text{Aut}$. Making use of noncommutative motives and the above co-representability Theorem 1.4 we solve this problem as follows:

Theorem 1.7. *There is a natural isomorphism of abelian groups*

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \simeq K_0\text{Aut}(\mathbb{Z}[r, r^{-1}]), \tag{1.8}$$

where Nat stands for the abelian group of natural transformations. Moreover, the quotient of $K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])$ by the relations

$$\left\{ [(M, \alpha)] + [(M, \beta)] = [(M, \alpha\beta)] \mid M \in \mathbf{P}(\mathbb{Z}[r, r^{-1}]), \alpha, \beta \text{ automorphisms} \right\} \tag{1.9}$$

identifies with $\{-1, 1\} \times \mathbb{Z}$. Furthermore, under the quotient homomorphism

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \simeq K_0\text{Aut}(\mathbb{Z}[r, r^{-1}]) \twoheadrightarrow \{-1, 1\} \times \mathbb{Z}, \tag{1.10}$$

the Frobenius operation F_n corresponds to $(1, n)$ and the Verschiebung operation V_n to $((-1)^{n+1}, 1)$.

Theorem 1.7 shows us that all the information concerning a natural transformation of $K_0\text{Aut}$ can be completely encoded into an element of $K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])$. Moreover, by imposing the relations (1.9) this data reduces simply to a parity plus an integer. The Frobenius (resp. Verschiebung) operations become then the simplest ones with respect to the parity (resp. to the integer).

2. Differential graded categories

A differential graded (=dg) category is a category enriched over (unbounded) cochain complexes of abelian groups in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = d(f) \circ g + (-1)^{\text{deg}(f)} f \circ d(g)$. For a survey article, consult Keller’s ICM address [8]. The category of dg categories will be denoted by $\text{dgc}at$. Given a ring A , we will still denote by A the dg category with a single object and A as the dg ring of endomorphisms (concentrated in degree zero).

Dg (bi-)modules

Let \mathcal{A} be a dg category. Recall from [8, §3] the construction of the category $\mathcal{C}(\mathcal{A})$ of \mathcal{A} -modules. We will denote by $\mathcal{D}(\mathcal{A})$ the *derived category* of \mathcal{A} , i.e. the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Recall from [8, §2.3] that the tensor product endows $\mathbf{dgc}at$ with a symmetric monoidal structure with \otimes -unit \mathbb{Z} . Finally, recall that a \mathcal{A} - \mathcal{B} -bimodule is simply a dg functor $\mathcal{A}^{op} \otimes \mathcal{B} \rightarrow \mathcal{C}_{dg}(\mathbb{Z})$ with values in the dg category of complexes of abelian groups.

Morita equivalences

A dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called a *Morita equivalence* if the derived extension of scalars functor $\mathbb{L}F_! : \mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ is an equivalence of (triangulated) categories; see [8, §4.6]. Recall from [8, Thm. 4.10] that $\mathbf{dgc}at$ carries a Quillen model structure, whose weak equivalences are the *Morita equivalences*. We will denote by \mathbf{Hmo} the homotopy category hence obtained. The tensor product of dg categories can be naturally derived $- \otimes^{\mathbb{L}} -$, thus giving rise to a symmetric monoidal structure on \mathbf{Hmo} . Given dg categories \mathcal{A} and \mathcal{B} , let $\mathbf{rep}(\mathcal{A}, \mathcal{B})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{op} \otimes^{\mathbb{L}} \mathcal{B})$ spanned by the cofibrant \mathcal{A} - \mathcal{B} -bimodules X such that for every object $x \in \mathcal{A}$ the associated \mathcal{B} -module $X(x, -)$ becomes compact (see [13, Def. 4.2.7]) in $\mathcal{D}(\mathcal{B})$.

3. K-theory of automorphisms of dg categories

Given a dg category \mathcal{A} , let $\mathbf{perf}(\mathcal{A})$ be the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of those \mathcal{A} -modules which are cofibrant in the projective model structure and which become compact in $\mathcal{D}(\mathcal{A})$. As explained in [6, Example 3.5], $\mathbf{perf}(\mathcal{A})$ carries a Waldhausen structure [19] and so by passing to the Grothendieck group of the associated category $\mathbf{Aut}(\mathbf{perf}(\mathcal{A}))$ of automorphisms we obtain the following composed functor

$$K_0\mathbf{Aut} : \mathbf{dgc}at \xrightarrow{\mathbf{perf}} \{\mathbf{Wald}\} \xrightarrow{\mathbf{Aut}} \{\mathbf{Wald}\} \xrightarrow{K_0} \mathbf{Ab} \quad \mathcal{A} \mapsto K_0\mathbf{Aut}(\mathbf{perf}(\mathcal{A})).$$

Proposition 3.1. (Agreement) *For every ring A , the exact functor $\mathbf{P}(A) \rightarrow \mathbf{perf}(A)$ (mapping a A -module to the associated complex of A -modules concentrated in degree zero) gives rise to an abelian group isomorphism*

$$K_0\mathbf{Aut}(\mathbf{P}(A)) \xrightarrow{\sim} K_0\mathbf{Aut}(\mathbf{perf}(A)). \tag{3.2}$$

Proof. The following assignment

$$(M^\bullet, \alpha) \mapsto \sum_{n \in \mathbb{Z}} (-1)^n [(H^n(M^\bullet), H^n(\alpha))]$$

gives rise to the inverse to (3.2). □

4. Noncommutative motives

For a survey article on noncommutative motives we invite the reader to consult [17]. Recall from [16] the construction of the category \mathbf{Hmo}_0 of *noncommutative motives*. It is defined as the pseudo-abelian envelope of the category whose objects are the dg categories and whose abelian groups of morphisms are given by

$$\mathrm{Hom}_{\mathbf{Hmo}_0}(\mathcal{A}, \mathcal{B}) := K_0\mathrm{rep}(\mathcal{A}, \mathcal{B}).$$

Composition is induced by the derived tensor product of bimodules. The category \mathbf{Hmo}_0 is additive, idempotent complete, and there is a natural functor

$$\mathcal{U} : \mathrm{dgc}at \longrightarrow \mathbf{Hmo}_0, \tag{4.1}$$

sending a dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to the class in the Grothendieck group $K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$ of the bimodule in $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ naturally associated to F . Recall also that a functor $\mathrm{dgc}at \rightarrow \mathbf{D}$, with values in an idempotent complete additive category, is called an *additive invariant* if it inverts Morita equivalences and satisfies additivity. As explained in [17, §4], the above functor (4.1) is the *universal additive invariant*, i.e. given any idempotent complete additive category \mathbf{D} we have an induced equivalence of categories

$$\mathcal{U}^* : \mathrm{Fun}_{\mathrm{add}}(\mathbf{Hmo}_0, \mathbf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathbf{A}}(\mathrm{dgc}at, \mathbf{D}), \tag{4.2}$$

where $\mathrm{Fun}_{\mathrm{add}}(\mathbf{Hmo}_0, \mathbf{D})$ denotes the category of additive functors and $\mathrm{Fun}_{\mathbf{A}}(\mathrm{dgc}at, \mathbf{D})$ the category of additive invariants.

Now, recall from [16, Lemma 4.3] that the derived tensor product $-\otimes^{\mathbb{L}}-$ on \mathbf{Hmo} (see Sect. 2) induces a symmetric monoidal structure on \mathbf{Hmo}_0 making the functor \mathcal{U} symmetric monoidal. This symmetric monoidal structure is characterized by the following universal property.

Proposition 4.3. *Given any idempotent complete symmetric monoidal additive category \mathbf{D} , the above equivalence (4.2) admits the following monoidal refinement*

$$\mathcal{U}^* : \mathrm{Fun}_{\mathrm{add}}^{\otimes}(\mathbf{Hmo}_0, \mathbf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathbf{A}}^{\otimes}(\mathrm{dgc}at, \mathbf{D}),$$

where $\mathrm{Fun}_{\mathrm{add}}^{\otimes}(\mathbf{Hmo}_0, \mathbf{D})$ denotes the category of symmetric monoidal additive functors and $\mathrm{Fun}_{\mathbf{A}}^{\otimes}(\mathrm{dgc}at, \mathbf{D})$ the category of additive invariants which are moreover symmetric monoidal.

Proof. Let E be an object of $\mathrm{Fun}_{\mathrm{add}}^{\otimes}(\mathbf{Hmo}_0, \mathbf{D})$. Since \mathcal{U} is symmetric monoidal the composite $E \circ \mathcal{U}$ is also symmetric monoidal and so by equivalence (4.2) we conclude that it belongs to $\mathrm{Fun}_{\mathbf{A}}^{\otimes}(\mathrm{dgc}at, \mathbf{D})$. Now, let H be an object of $\mathrm{Fun}_{\mathbf{A}}^{\otimes}(\mathrm{dgc}at, \mathbf{D})$. By equivalence (4.2) it factors uniquely through \mathcal{U} giving rise to an additive invariant $\overline{H} : \mathbf{Hmo}_0 \rightarrow \mathbf{D}$. By construction of \mathbf{Hmo}_0 the functor \overline{H} remains symmetric monoidal and so it belongs to $\mathrm{Fun}_{\mathrm{add}}^{\otimes}(\mathbf{Hmo}_0, \mathbf{D})$. This achieves the proof. \square

5. Proofs

Proof of Theorem 1.2

Let us start by showing that the natural ring homomorphisms $\iota : \mathbb{Z} \rightarrow \mathbb{Z}[r]$ and $\pi : \mathbb{Z}[r] \xrightarrow{r=0} \mathbb{Z}$ give rise to a direct sum decomposition $\mathcal{U}(\mathbb{Z}[r]) \simeq \mathcal{U}(\mathbb{Z}) \oplus \mathcal{W}_0$ of co-unital co-associative

co-monoids in \mathbf{Hmo}_0 . Since $\pi \circ \iota = \text{id}_{\mathbb{Z}}$ the composite $\iota \circ \pi$ is an idempotent endomorphism of $\mathbb{Z}[r]$ and hence $\mathcal{U}(\iota \circ \pi)$ is an idempotent endomorphism of $\mathcal{U}(\mathbb{Z}[r])$. By construction the category \mathbf{Hmo}_0 is idempotent complete and so we obtain a direct sum decomposition $\mathcal{U}(\mathbb{Z}) \oplus \mathcal{W}_0$. The noncommutative motive \mathcal{W}_0 identifies then with the kernel of the map $\mathcal{U}(\pi) : \mathcal{U}(\mathbb{Z}[r]) \rightarrow \mathcal{U}(\mathbb{Z})$. The following commutative diagram

$$\begin{CD} \mathbb{Z}[r] @>\Delta>> \mathbb{Z}[r] \otimes \mathbb{Z}[r] \\ @V\pi VV @VV\pi \otimes \pi V \\ \mathbb{Z} @>1 \mapsto 1 \otimes 1>> \mathbb{Z} \otimes \mathbb{Z} \end{CD}$$

allows us then to conclude that the co-monoid structure on $\mathcal{U}(\mathbb{Z}[r])$ restricts to \mathcal{W}_0 . Its co-unit is given by the composition $\mathcal{W}_0 \hookrightarrow \mathcal{U}(\mathbb{Z}[r]) \xrightarrow{\mathcal{U}(\epsilon)} \mathcal{U}(\mathbb{Z})$.

Let us now construct the ring isomorphism (1.3). Recall from [18, Thm. 1.1] that for every ring A we have a natural isomorphism

$$\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r]), \mathcal{U}(A)) \simeq K_0\text{End}(A), \tag{5.1}$$

where $K_0\text{End}(A)$ denotes the Grothendieck group of endomorphisms of A . We start by proving that (5.1) is a ring isomorphism whenever A is commutative. The isomorphism (5.1) is obtained by composing

$$\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r]), \mathcal{U}(A)) := K_0\text{rep}(\mathbb{Z}[r], A) \xrightarrow{\sim} K_0\text{End}(\text{perf}(A)) \tag{5.2}$$

with $K_0\text{End}(\text{perf}(A)) \simeq K_0\text{End}(\mathbf{P}(A))$. The tensor product of perfect A -modules induces naturally a ring structure on $K_0\text{End}(\text{perf}(A))$ making the isomorphism $K_0\text{End}(\text{perf}(A)) \simeq K_0\text{End}(\mathbf{P}(A))$ into a ring isomorphism. Hence, it suffices to show that under the above group isomorphism (5.2) the convolution multiplication (on the left-hand-side) identifies with the one given by the tensor product (on the right-hand-side). Let $X, Y \in \text{rep}(\mathbb{Z}[r], A)$. Since the underlying abelian group of $\mathbb{Z}[r]$ is torsionfree, [20, Corollary 3.1.5] implies that the ring $\mathbb{Z}[r]$ (and hence the dg category $\mathbb{Z}[r]$) is flat. As a consequence, $\mathbb{Z}[r] \otimes^{\mathbb{L}} A \simeq \mathbb{Z}[r] \otimes A$ and so $\text{rep}(\mathbb{Z}[r], A)$ identifies with the full triangulated subcategory of $\mathcal{D}(\mathbb{Z}[r]^{\text{op}} \otimes A)$ spanned by those $\mathbb{Z}[r]$ - A -bimodules X such that the A -module $X(-, *)$ belongs to $\text{perf}(A)$, where $*$ is the unique object of $\mathbb{Z}[r]$. Let us denote by $[(X, \alpha)]$ and $[(Y, \beta)]$ the images of $[X]$ and $[Y]$ under the isomorphism (5.2), where α and β are the endomorphisms associated to the left action of $\mathbb{Z}[r]$ on X and Y . The multiplication of $[(X, \alpha)]$ with $[(Y, \beta)]$ is then the element

$$[(X \otimes_A Y), \alpha \otimes_A \beta] \in K_0\text{End}(\text{perf}(A)). \tag{5.3}$$

Note that by definition of the category $\text{perf}(A)$ there is no need to derive the tensor product over A . Now, let us analyze the convolution multiplication. Since by hypothesis A is commutative, its multiplication $m : A \otimes A \rightarrow A$ is a ring homomorphism. Hence, we have a well-defined dg functor $m : A \otimes A \rightarrow A$. Let us denote by ${}_m A$ the associated bimodule and by $[_m A]$ its class in $K_0\text{rep}(A \otimes A, A)$. Note that since by hypothesis A is flat, the bimodule ${}_m A$ belongs to $\text{rep}(A \otimes A, A)$. Similarly, let us denote by ${}_{\Delta}(\mathbb{Z}[r] \otimes \mathbb{Z}[r])$ the bimodule associated to Δ [see (1.1)] and by $[\Delta(\mathbb{Z}[r] \otimes \mathbb{Z}[r])]$ its class in $K_0\text{rep}(\mathbb{Z}[r], \mathbb{Z}[r] \otimes \mathbb{Z}[r])$. Note that the (derived) tensor product of X with Y is an element of $\text{rep}(\mathbb{Z}[r] \otimes \mathbb{Z}[r], A \otimes A)$. The convolution multiplication of $[X]$ with $[Y]$ is then the following composition

$$\mathbb{Z}[r] \xrightarrow{[\Delta(\mathbb{Z}[r] \otimes \mathbb{Z}[r])]} \mathbb{Z}[r] \otimes \mathbb{Z}[r] \xrightarrow{[X \otimes Y]} A \otimes A \xrightarrow{[_m A]} A$$

in the category \mathbf{Hmo}_0 . By definition of \mathbf{Hmo}_0 this composition corresponds to the class

$$[\Delta(\mathbb{Z}[r] \otimes \mathbb{Z}[r]) \otimes_{\mathbb{Z}[r] \otimes \mathbb{Z}[r]} (X \otimes Y) \otimes_{A \otimes A} mA],$$

which naturally identifies with $[X \otimes_A Y]$. Since the left action of $\mathbb{Z}[r]$ on $X \otimes_A Y$ is the diagonal one, we conclude finally that the image of $[X \otimes_A Y]$ under the above isomorphism (5.2) is precisely the element (5.3). This shows that (5.1) is a ring isomorphism.

Now, note that the exact functors $\mathbf{P}(A) \rightarrow \text{End}(\mathbf{P}(A)), M \mapsto (M, 0)$, and $\text{End}(\mathbf{P}(A)) \rightarrow \mathbf{P}(A), (M, \alpha) \mapsto M$, induce a splitting $K_0\text{End}(\mathbf{P}(A)) \simeq K_0(A) \oplus \tilde{K}_0\text{End}(\mathbf{P}(A))$ of abelian groups which is moreover compatible with the ring structure. In order to simplify the exposition let us denote the ring $\tilde{K}_0\text{End}(\mathbf{P}(A))$ by $\tilde{K}_0\text{End}(A)$. The map

$$\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r]), \mathcal{U}(A)) \simeq K_0\text{End}(A) \xrightarrow{\mathcal{U}(i)^*} K_0(A) \simeq \text{Hom}_{\mathbf{Hmo}_0}(\mathcal{U}(\mathbb{Z}), \mathcal{U}(A))$$

induced by $\mathcal{U}(i)$ identifies with the homomorphism

$$K_0\text{rep}(\mathbb{Z}[r], A) \longrightarrow K_0\text{rep}(\mathbb{Z}, A) \quad [(M, \alpha)] \mapsto [M].$$

As a consequence, the abelian group $\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(A))$ identifies with the kernel of $\mathcal{U}(i)^*$. This kernel is by construction $\tilde{K}_0\text{End}(A)$ and so we obtain the following abelian group isomorphism

$$\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(A)) \simeq \tilde{K}_0\text{End}(A). \tag{5.4}$$

From the above ring isomorphism (5.1) and the decomposition of $K_0\text{End}(A)$ one concludes that (5.4) preserves moreover the ring structure.

Now, recall from [2] that we have the following ring isomorphism

$$\tilde{K}_0\text{End}(A) \xrightarrow{\sim} W_0(A) \quad (M, \alpha) \mapsto \det(\text{Id} + \alpha t).$$

By combining it with (5.4) we obtain then the searched ring isomorphism (1.3).

Proof of Theorem 1.4

Recall that by construction we have

$$\text{Hom}_{\mathbf{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r, r^{-1}]), \mathcal{U}(\mathcal{A})) := K_0\text{rep}(\mathbb{Z}[r, r^{-1}], \mathcal{A})$$

Similarly to the proof of Theorem 1.2, the ring $\mathbb{Z}[r, r^{-1}]$ is flat and so $\text{rep}(\mathbb{Z}[r, r^{-1}], \mathcal{A})$ identifies with the full triangulated subcategory of $\mathcal{D}(\mathbb{Z}[r, r^{-1}]^{\text{op}} \otimes \mathcal{A})$ spanned by the $\mathbb{Z}[r]$ - \mathcal{A} -bimodules X such that the \mathcal{A} -module $X(-, *)$ belongs to $\text{perf}(\mathcal{A})$. Such a bimodule consists precisely on the same data as an object of $\text{Aut}(\text{perf}(\mathcal{A}))$. Hence, we have an equivalence $\text{rep}(\mathbb{Z}[r, r^{-1}], \mathcal{A}) \simeq \text{Aut}(\text{perf}(\mathcal{A}))[w^{-1}]$, where w denotes the class of quasi-isomorphisms. Since $K_0\text{Aut}(\text{perf}(\mathcal{A})) \simeq K_0\text{Aut}(\text{perf}(\mathcal{A}))[w^{-1}]$ we obtain then the searched abelian group isomorphism (1.5).

When $\mathcal{A} = A$, with A a commutative ring, an argument analogous to the one used in the proof of Theorem 1.2 (with $\mathbb{Z}[r]$ replaced by $\mathbb{Z}[r, r^{-1}]$) shows us that (1.5) reduces to the ring isomorphism (1.6). This concludes the proof.

Proof of Theorem 1.7

Let us start by constructing the isomorphism (1.8). The co-representability of the functor $K_0\text{Aut}$ in the category \mathbf{Hmo}_0 of noncommutative motives (see Theorem 1.4) implies that $K_0\text{Aut}$ is an additive invariant of dg categories. Hence, $K_0\text{Aut}$ belongs to the category $\text{Fun}_{\mathbf{A}}(\text{dgc}at, \mathbf{Ab})$. Using the equivalence (4.2) we obtain then an abelian group isomorphism

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \xrightarrow{\sim} \text{Nat}(\overline{K_0\text{Aut}}, \overline{K_0\text{Aut}}) \quad \eta \mapsto \bar{\eta}. \tag{5.5}$$

By Theorem 1.4 the additive functor $\overline{K_0\text{Aut}}$ is co-represented in \mathbf{Hmo}_0 by $\mathcal{U}(\mathbb{Z}[r, r^{-1}])$ and so using the enriched Yoneda lemma [3, Thm. 8.3.5] one obtains the following abelian group isomorphism

$$\text{Nat}(\overline{K_0\text{Aut}}, \overline{K_0\text{Aut}}) \xrightarrow{\sim} \overline{K_0\text{Aut}}\left(\mathcal{U}(\mathbb{Z}[r, r^{-1}])\right) \quad \bar{\eta} \mapsto \bar{\eta}(\mathbb{Z}[r, r^{-1}])\left([\text{id}_{\mathbb{Z}[r, r^{-1}]}\right]).$$

By construction of \mathbf{Hmo}_0 the class $[\text{id}_{\mathbb{Z}[r, r^{-1}]}$ identifies with $[(\mathbb{Z}[r, r^{-1}], \cdot r)]$. Hence, by combining this latter isomorphism with (5.5) we obtain the isomorphism

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \xrightarrow{\sim} K_0\text{Aut}(\mathbb{Z}[r, r^{-1}]), \tag{5.6}$$

sending a natural transformation η to $\eta(\mathbb{Z}[r, r^{-1}])([\mathbb{Z}[r, r^{-1}], \cdot r])$. This is the searched isomorphism (1.8).

Let us now prove the remaining claims of the Theorem. As explained in [14, Def. 3.1.6], the quotient group $K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])/(1.9)$ identifies with the K_1 -group of the category $\mathbf{P}(\mathbb{Z}[r, r^{-1}])$, which by [14, Thm. 3.1.7] is isomorphic to $K_1(\mathbb{Z}[r, r^{-1}])$. Since \mathbb{Z} is a regular ring we have by Bass-Heller-Swan (see [14, Cor. 3.2.20]) an isomorphism $K_1(\mathbb{Z}[r, r^{-1}]) \simeq K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z})$. Finally, since $K_1(\mathbb{Z}) \simeq \{-1, 1\}$ (see [14, Cor. 2.3.3]) and $K_0(\mathbb{Z}) \simeq \mathbb{Z}$, we conclude that

$$K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])/(1.9) \simeq \{-1, 1\} \times \mathbb{Z}.$$

Now, let us show that under the quotient homomorphism (1.10) the Frobenius operation F_n corresponds to $(1, n)$ and that the Verschiebung operation V_n corresponds to $((-1)^{n+1}, 1)$. We start with the Frobenius operation

$$F_n : K_0\text{Aut} \Rightarrow K_0\text{Aut} \quad [(M, \alpha)] \mapsto [(M, \alpha^n)].$$

Its image under the above canonical map (5.6) is the element $[(\mathbb{Z}[r, r^{-1}], \cdot r^n)]$. It suffices then to understand the image of this element under the homomorphism

$$K_0\text{Aut}(\mathbb{Z}[r, r^{-1}]) \twoheadrightarrow K_1\left(\mathbf{P}(\mathbb{Z}[r, r^{-1}])\right) \xrightarrow{\Phi} K_1(\mathbf{P}(\mathbb{Z})) \oplus K_0(\mathbb{Z}) \xrightarrow{\sim} \{-1, 1\} \times \mathbb{Z}, \tag{5.7}$$

where the right-hand-side isomorphism is induced by the determinant and rank assignments. In order to simplify the exposition we will make no notational distinction between the elements of $K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])$ and $K_1(\mathbf{P}(\mathbb{Z}[r, r^{-1}]))$. The relations (1.9) imply that the image of $[(\mathbb{Z}[r, r^{-1}], \cdot r^n)]$ under (5.7) identifies with n -times the image of $[(\mathbb{Z}[r, r^{-1}], \cdot r)]$ (corresponding to the Frobenius operation F_1). As explained in [14, Prop. 3.2.18], the inverse of the isomorphism Φ is given by

$$([(M, \alpha)], [M']) \mapsto \left[(\mathbb{Z}[r, r^{-1}] \otimes M, \text{id} \otimes \alpha) \right] + \left[(\mathbb{Z}[r, r^{-1}] \otimes M', r \otimes \text{id}) \right]. \tag{5.8}$$

Hence, we observe that the element $([\mathbb{Z}, \text{id}], [\mathbb{Z}])$ corresponds to $([\mathbb{Z}[r, r^{-1}], \text{id}]) + ([\mathbb{Z}[r, r^{-1}], \cdot r])$. Since $([\mathbb{Z}[r, r^{-1}], \text{id}])$ is the trivial element of $K_1(\mathbf{P}(\mathbb{Z}[r, r^{-1}]))$ we conclude that the image of $([\mathbb{Z}[r, r^{-1}], \cdot r])$ under (5.7) is $(1, 1)$. By the previous arguments the image of $([\mathbb{Z}[r, r^{-1}], \cdot r^n])$ is then $(1, n)$.

Let us now study the Verschiebung operation

$$V_n : K_0\text{Aut} \Rightarrow K_0\text{Aut} \quad [(M, \alpha)] \mapsto [(M^\oplus, V_n(\alpha))],$$

where

$$V_n(\alpha) := \begin{bmatrix} 0 & \cdots & \cdots & 0 & \alpha \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(n \times n)}.$$

Its image under (5.6) is the element $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, V_n(\cdot r)])$ and so it suffices to study the image of this element under (5.7). The matrix equality $V_n(\alpha) = W_n \cdot I_n(\alpha)$, where

$$W_n := \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(n \times n)} \quad I_n(\alpha) := \begin{bmatrix} 1 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 0 & \cdots & 0 & 0 & \alpha \end{bmatrix}_{(n \times n)},$$

combined with the relations (1.9) implies that the image of $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, V_n(\cdot r)])$ under the homomorphism (5.7) agrees with the sum of the images of $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, W_n])$ and $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, I_n(\alpha)])$. Under the assignment (5.8), $([\mathbb{Z}^{\oplus n}, W_n], [0])$ corresponds to $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, W_n])$. Since the determinant of the matrix W_n equals $(-1)^{n+1}$, we then conclude that the image of $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, W_n])$ under (5.7) is exactly $((-1)^{n+1}, 0)$. In what concerns the element $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, I_n(\alpha)])$ we have the following short exact sequence

$$0 \longrightarrow ([\mathbb{Z}[r, r^{-1}]^{\oplus(n-1)}, \text{Id}]) \longrightarrow ([\mathbb{Z}[r, r^{-1}]^{\oplus n}, I_n(\alpha)]) \longrightarrow ([\mathbb{Z}[r, r^{-1}], \cdot r]) \longrightarrow 0,$$

where the first map corresponds to the upper-left-block matrix inclusion. Since $([\mathbb{Z}[r, r^{-1}]^{\oplus(n-1)}, \text{Id}])$ is the trivial element of $K_1(\mathbf{P}(\mathbb{Z}[r, r^{-1}]))$ we conclude that $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, I_n(\alpha)])$ agrees with $([\mathbb{Z}[r, r^{-1}], \cdot r])$. As explained above the image of $([\mathbb{Z}[r, r^{-1}], \cdot r])$ under (5.7) is $(1, 1)$. Hence, by adding $((-1)^{n+1}, 0)$ with $(1, 1)$ we conclude finally that the image of $([\mathbb{Z}[r, r^{-1}]^{\oplus n}, V_n(\alpha)])$ under (5.7) is $((-1)^{n+1}, 1)$. This achieves the proof.

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