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ALGEBRAIC K -THEORY WITH COEFFICIENTS OF CYCLIC QUOTIENT SINGULARITIES

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ABSTRACT. In this short note, by combining the work of Amiot-Iyama-Reiten and Thanhoffer de Völcsey-Van den Bergh on Cohen-Macaulay modules with the previous work of the author on orbit categories, we compute the (nonconnective) algebraic K -theory with coefficients of cyclic quotient singularities.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let k be an algebraically closed field of characteristic zero. Given an integer $d \geq 2$, consider the associated polynomial ring $S := k[t_1, \dots, t_d]$. Let G be a cyclic subgroup of $\mathrm{SL}(d, k)$ generated by $\mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$, where ζ is a primitive n^{th} root of unit and a_1, \dots, a_d are integers satisfying the following conditions: we have $0 < a_j < n$ and $\mathrm{gcd}(a_j, n) = 1$ for every $1 \leq j \leq d$; we have $a_1 + \dots + a_d = n$. The group G acts naturally on S and the invariant ring $R := S^G$ is a Gorenstein isolated singularity of Krull dimension d . For example, when $d = 2$, the ring R identifies with the Kleinian singularity $k[u, v, w]/(u^n + vw)$ of type A_{n-1} .

The affine k -scheme $X := \mathrm{Spec}(R)$ is singular. Following Orlov [3, 4], we can then consider the associated dg category of singularities $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sing}}(X)$; also known as matrix factorizations or maximal Cohen-Macaulay modules. Roughly speaking, this dg category encodes all the crucial information concerning the isolated singularity of X .

Let us denote by (Q, ρ) the quiver with relations defined by the following steps:

- (s1) consider the quiver with vertices $\mathbb{Z}/n\mathbb{Z}$ and with arrows $x_j^i: i \rightarrow i + a_j$, where $i \in \mathbb{Z}/n\mathbb{Z}$ and $1 \leq j \leq d$. The relations ρ are given by $x_{j'}^{i+a_j} x_j^i = x_j^{i+a_{j'}} x_{j'}^i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$ and $1 \leq j, j' \leq d$.
- (s2) remove from (s1) all arrows $x_j^i: i \rightarrow i'$ with $i > i'$;
- (s3) remove from (s2) the vertex 0.

Consider the matrix $(n-1) \times (n-1)$ matrix C such that C_{ij} equals the number of arrows in Q from j to i (counted modulo the relations). Let us write M for the matrix $(-1)^{d-1} C(C^{-1})^T - \mathrm{Id}$ and $M: \bigoplus_{r=1}^{n-1} \mathbb{Z}/l^\nu \rightarrow \bigoplus_{r=1}^{n-1} \mathbb{Z}/l^\nu$ for the associated (matrix) homomorphism, where l^ν is a (fixed) prime power.

Theorem 1.1. *We have the following computation:*

$$\mathbb{K}_i(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sing}}(X); \mathbb{Z}/l^\nu) \simeq \begin{cases} \text{cokernel of } M & \text{if } i \geq 0 \text{ even} \\ \text{kernel of } M & \text{if } i \geq 0 \text{ odd} \\ 0 & \text{if } i < 0. \end{cases}$$

Thanks to Theorem 1.1, the computation of the (nonconnective) algebraic K -theory with coefficients of the cyclic quotient singularities reduces to the computation of (co)kernels of explicit matrix homomorphisms! To the best of the author's

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knowledge, these computations are new in the literature. In the particular case of Kleinian singularities of type A_n they were originally established in [7, §3]

Corollary 1.2. (i) *If there exists a prime power l^ν and an even (resp. odd) integer $j \geq 0$ such that $\mathbb{K}_j(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) \neq 0$, then for every even (resp. odd) integer $i \geq 0$ at least one of the groups $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), \mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$ is non-zero.*

(ii) *If there exists a prime power l^ν such that $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) = 0$ for every $i \geq 0$, then the groups $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), i \geq 0$, are uniquely l^ν -divisible.*

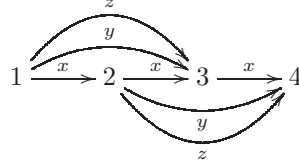
Proof. Combine the universal coefficients sequence (see [7, §5])

$$0 \longrightarrow \mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \longrightarrow \mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) \longrightarrow {}_l \mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)) \longrightarrow 0$$

with the computation of Theorem 1.1. \square

2. EXAMPLES

A low dimensional example. When $d = 3$, $n = 5$, $a_1 = 1$, and $a_2 = a_3 = 2$, the above three steps (s1)-(s3) lead to the following quiver

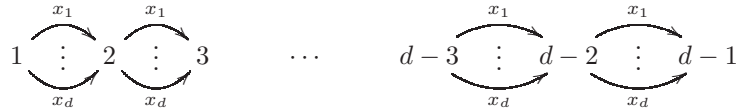


with relations $xy = yx$, $yz = zy$, and $zx = xz$. Consequently, we obtain the matrix

$$M = \begin{pmatrix} 0 & -1 & -3 & -3 \\ 1 & -1 & -4 & -6 \\ 3 & -2 & -10 & -13 \\ 3 & 0 & -11 & -19 \end{pmatrix}.$$

Since $\det(M) = 26$, we have $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) = 0$ whenever $l \neq 2, 13$. In the remaining two cases, a computation shows that $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l$ for every $i \geq 0$. Thanks to Corollary 1.2, this implies that for every $i \geq 0$ at least one of the groups $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), \mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$ is non-trivial. Moreover, the groups $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), i \geq 0$, are uniquely l -divisible for every prime $l \neq 2, 13$.

A family of examples. When $n = d \geq 3$ and $a_1 = \dots = a_d = 1$, the above three steps (s1)-(s3) lead to the following quiver



with relations $x_j x_i = x_i x_j$. In the case where d is odd, we obtain the matrix

$$M_{ij} = \begin{cases} -\sum_{r=0}^{i-1} \binom{d}{r} \binom{d}{(j-i)+r} & \text{if } i < j \\ -\sum_{r=1}^{i-1} \binom{d}{r}^2 & \text{if } i = j \\ -\sum_{r=1}^{j-1} \left(\binom{d}{(i-j)+r} \binom{d}{r} + \binom{d}{i-j} \right) & \text{if } i > j, \end{cases}$$

where $\left(\!\!\left(\right)\!\!\right)$ stands for the multicomination¹ symbol. Similarly, in the case where d is even, we obtain the matrix

$$M_{ij} = \begin{cases} \sum_{r=0}^{i-1} \binom{d}{r} \left(\!\!\left(\begin{smallmatrix} d \\ (j-i)+r \end{smallmatrix}\right)\!\!\right) & \text{if } i < j \\ -2 + \sum_{r=1}^{i-1} \binom{d}{r}^2 & \text{if } i = j \\ \sum_{r=1}^{j-1} \left(\!\!\left(\begin{smallmatrix} d \\ (i-j)+r \end{smallmatrix}\right)\!\!\right) \binom{d}{r} - \left(\!\!\left(\begin{smallmatrix} d \\ i-j \end{smallmatrix}\right)\!\!\right) & \text{if } i > j. \end{cases}$$

Whenever d is a prime number, all the multicombinations

$$\left(\!\!\left(\begin{smallmatrix} d \\ r \end{smallmatrix}\right)\!\!\right) = \binom{d+r-1}{r} = \frac{(d+r-1) \cdots d(d-1)!}{r!(d-1)!} \quad 0 \leq r \leq d-2$$

are multiples of d . This implies that the homomorphism $M: \oplus_{r=1}^{d-1} \mathbb{Z}/d \rightarrow \oplus_{r=1}^{d-1} \mathbb{Z}/d$ is zero, and consequently that $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/d) \simeq \oplus_{r=1}^{d-1} \mathbb{Z}/d$ for every $i \geq 0$. These isomorphisms are a far reaching generalization of the particular case $d = 3$ originally established in [7, Prop. 3.4]. Thanks to Corollary 1.2(i), we hence conclude that for every $i \geq 0$ at least one of the groups $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), \mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$ is non-trivial.

3. PROOF OF THEOREM 1.1

Let A be a finite dimensional k -algebra of finite global dimension. We write $\mathcal{D}^b(A)$ for the bounded derived category of (right) A -modules and $\mathcal{D}_{\text{dg}}^b(A)$ for the canonical dg enhancement of $\mathcal{D}^b(A)$. Consider the following dg functors

$$\tau^{-1}\Sigma^d: \mathcal{D}_{\text{dg}}^b(A) \longrightarrow \mathcal{D}_{\text{dg}}^b(A) \quad d \geq 0,$$

where τ stands for the Auslander-Reiten translation. Following Keller [2, §7.2], we can consider the associated dg orbit category $\mathcal{C}_A^{(d)} := \mathcal{D}_{\text{dg}}^b(A)/(\tau^{-1}\Sigma^d)^{\mathbb{Z}}$. Similarly to [7, Thm. 2.5] (consult [6, §2]), we have a distinguished triangle of spectra

$$\bigoplus_{r=1}^v \mathbb{K}(k; \mathbb{Z}/l^\nu) \xrightarrow{(-1)^d \Phi_A - \text{Id}} \bigoplus_{r=1}^v \mathbb{K}(k; \mathbb{Z}/l^\nu) \rightarrow \mathbb{K}(\mathcal{C}_A^{(d)}; \mathbb{Z}/l^\nu) \rightarrow \bigoplus_{r=1}^v \Sigma \mathbb{K}(k; \mathbb{Z}/l^\nu),$$

where v stands for the number of simple (right) A -modules and Φ_A for the inverse of the Coxeter matrix of A . Consider the (matrix) homomorphism

$$(3.1) \quad (-1)^d \Phi_A - \text{Id}: \bigoplus_{r=1}^v \mathbb{Z}/l^\nu \longrightarrow \bigoplus_{r=1}^v \mathbb{Z}/l^\nu.$$

As proved by Suslin in [5, Cor. 3.13], we have $\mathbb{K}_i(k; \mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l^\nu$ when $i \geq 0$ is even and $\mathbb{K}_i(k; \mathbb{Z}/l^\nu) = 0$ otherwise. Consequently, making use of the long exact sequence of algebraic K -theory groups with coefficients associated to the above distinguished triangle of spectra, we obtain the following computations:

$$\mathbb{K}_i(\mathcal{C}_A^{(d)}; \mathbb{Z}/l^\nu) \simeq \begin{cases} \text{cokernel of (3.1)} & \text{if } i \geq 0 \text{ even} \\ \text{kernel of (3.1)} & \text{if } i \geq 0 \text{ odd} \\ 0 & \text{if } i < 0. \end{cases}$$

Consider also the following dg functors

$$S^{-1}\Sigma^d: \mathcal{D}_{\text{dg}}^b(A) \longrightarrow \mathcal{D}_{\text{dg}}^b(A) \quad d \geq 0,$$

¹Also known as the *multisubset* symbol.

where S stands for the Serre dg functor. The associated dg orbit category $\mathcal{C}_d(A) := \mathcal{D}_{\text{dg}}^b(A)/(S^{-1}\Sigma^d)^{\mathbb{Z}}$ is usually called the *generalized d -cluster dg category of A* ; see [1, §1.3] and the references therein. Since $S^{-1}\Sigma = \tau^{-1}$, we have $\mathcal{C}_d(A) \simeq \mathcal{C}_A^{(d-1)}$.

Now, let us take for A the k -algebra $kQ/\langle \rho \rangle$ associated to the quiver with relations (Q, ρ) . As proved independently by Amiot-Iyama-Reiten [1, §5] and Thanhoffer de Völcsey-Van den Bergh [8], we have $\mathcal{D}_{\text{dg}}^{\text{sing}}(X) \simeq \mathcal{C}_{d-1}(A)$. Consequently, it remains then only to show that the homomorphism (3.1), with d replaced by $d-2$, agrees with the homomorphism M associated to the matrix $M := (-1)^{d-1}C(C^{-1})^T - \text{Id}$. On the one hand, the number of simple (right) A -modules agrees with the number of vertices of the quiver Q . This implies that $v = n-1$. On the other hand, the inverse of the Coxeter matrix of A can be expressed as $-C(C^{-1})^T$, where C_{ij} equals the number of arrows in Q from j to i (counted modulo the relations). This implies that $(-1)^{d-2}\Phi_A - \text{Id} = (-1)^{d-1}C(C^{-1})^T - \text{Id} = M$, and hence concludes the proof.

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