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DOUBLE POISSON VERTEX ALGEBRAS AND NON-COMMUTATIVE HAMILTONIAN EQUATIONS

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ABSTRACT. We develop the formalism of double Poisson vertex algebras (local and non-local) aimed at the study of non-commutative Hamiltonian PDEs. This is a generalization of the theory of double Poisson algebras, developed by Van den Bergh, which is used in the study of Hamiltonian ODEs. We apply our theory of double Poisson vertex algebras to non-commutative KP and Gelfand-Dickey hierarchies. We also construct the related non-commutative de Rham and variational complexes.

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0. INTRODUCTION

First, we recall the notion of a Poisson vertex algebra. Given a unital commutative associative algebra \mathcal{V} with a derivation ∂ , a λ -bracket on \mathcal{V} is a linear map $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$, $a \otimes b \rightarrow \{a_\lambda b\}$, satisfying the following axioms:

$$\{\partial a_\lambda b\} = -\lambda \{a_\lambda b\}, \quad \{a_\lambda \partial b\} = (\lambda + \partial) \{a_\lambda b\} \quad (\text{sesquilinearity}), \quad (0.1)$$

$$\{a_\lambda bc\} = \{a_\lambda b\}c + \{a_\lambda c\}b \quad (\text{left Leibniz rule}), \quad (0.2)$$

$$\{ab_\lambda c\} = \{a_{\lambda+\partial c}\} \rightarrow b + \{b_{\lambda+\partial c}\} \rightarrow a \quad (\text{right Leibniz rule}). \quad (0.3)$$

The algebra \mathcal{V} , endowed with a λ -bracket satisfying

$$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\} \quad (\text{skewsymmetry}), \quad (0.4)$$

$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\} \quad (\text{Jacobi identity}), \quad (0.5)$$

is called a *Poisson vertex algebra* (PVA). (Of course, if $\partial = 0$, then (0.2)–(0.5) turn into axioms of a Poisson algebra if we let $\lambda = \mu = 0$.)

In the case when \mathcal{V} is the algebra of functions on the space of jets of a finite-dimensional manifold, the notion of a PVA is equivalent to that of a local Poisson bracket [BDSK09]. In order to accommodate non-local Poisson brackets one needs to introduce the non-local PVA, for which the λ -bracket takes values in $\mathcal{V}((\lambda^{-1}))$, the formal Laurent series in λ^{-1} (instead of polynomials in λ). The axioms (0.1)–(0.4) can be easily interpreted, but for the Jacobi identity (0.5) to make sense in the non-local case, one needs, in addition, the *admissibility* property of the λ -bracket:

$$\{a_\lambda \{b_\mu c\}\} \in \mathcal{V}[[\lambda^{-1}, \mu^{-1}, (\lambda + \mu)^{-1}]][[\lambda, \mu]], \quad \text{for all } a, b, c \in \mathcal{V}, \quad (0.6)$$

(see [DSK13] for details).

Local and non-local Poisson brackets play a fundamental role in the theory of integrable systems (see e.g. [FT86]). In recent years there have been attempts to develop a theory of integrable systems on non-commutative associative algebras (see e.g. [DF92, OS98, Kup00, MS00, ORS13]). An important advance in this direction was made by Van den Bergh, who in [VdB08] introduced the notion of a double Poisson algebra structure in a non-commutative associative algebra V . His basic idea was to consider a 2-fold bracket $\{\!\{ -, - \}\!\}$ on V , with values in $V \otimes V$. The Leibniz rules of a 2-fold bracket are almost identical to the usual Leibniz rules:

$$\{\!\{ a, bc \}\!\} = \{\!\{ a, b \}\!\}c + b\{\!\{ a, c \}\!\}, \quad (0.7)$$

$$\{\!\{ ab, c \}\!\} = \{\!\{ a, c \}\!\} \star_1 b + a \star_1 \{\!\{ b, c \}\!\}, \quad (0.8)$$

where $(a_1 \otimes a_2) \star_1 a_3 = a_1 a_3 \otimes a_2$, $a_1 \star_1 (a_2 \otimes a_3) = a_2 \otimes a_1 a_3$. The skewsymmetry axiom is

$$\{\!\{ a, b \}\!\} = -\{\!\{ b, a \}\!\}^\sigma, \quad (0.9)$$

where σ is the permutation of factors in $V \otimes V$, and the Jacobi identity is

$$\{\!\{ a, \{\!\{ b, c \}\!\} \}\!\}_L - \{\!\{ b, \{\!\{ a, c \}\!\} \}\!\}_R = \{\!\{ \{\!\{ a, b \}\!\}, c \}\!\}_L, \quad (0.10)$$

where we denote $\{\!\{ a_1, a_2 \otimes a_3 \}\!\}_L = \{\!\{ a_1, a_2 \}\!\} \otimes a_3$, $\{\!\{ a_1, a_2 \otimes a_3 \}\!\}_R = a_2 \otimes \{\!\{ a_1, a_3 \}\!\}$, $\{\!\{ a_1 \otimes a_2, a_3 \}\!\}_L = \{\!\{ a_1, a_3 \}\!\} \otimes_1 a_2$, and $(a \otimes b) \otimes_1 c = a \otimes c \otimes b$. (Formula (0.10) is slightly different from but equivalent to the Jacobi identity in [VdB08].)

An associative algebra V , endowed with a 2-fold bracket $\{\!\{ -, - \}\!\}$ satisfying axioms (0.9) and (0.10), is called a *double Poisson algebra*.

Given a double Poisson algebra V , Van den Bergh defines the following bracket on V with values in V :

$$\{a, b\} = m\{\!\!\{a, b\}\!\!\}, \quad (0.11)$$

where $m : V \otimes V \rightarrow V$ is the multiplication map. The bracket (0.11) still satisfies the left Leibniz rule, but does not satisfy, in general, other axioms of a Poisson bracket. However, this bracket induces well-defined linear maps

$$V/[V, V] \otimes V \rightarrow V \quad \text{and} \quad V/[V, V] \otimes V/[V, V] \rightarrow V/[V, V],$$

given by

$$\{\text{tr}(a), b\} = \{a, b\} \quad \text{and} \quad \{\text{tr}(a), \text{tr}(b)\} = \text{tr}\{a, b\}, \quad (0.12)$$

where $\text{tr} : V \rightarrow V/[V, V]$ is the quotient map and $[V, V]$ is the linear span of commutators $ab - ba$. These maps have the following properties, important for the theory of non-commutative Hamiltonian ODEs: the vector space $V/[V, V]$ is a Lie algebra and one has its representation on V by derivations, both defined by (0.12) (see Proposition 2.5).

Brackets (0.12) allow one to define the basic notions of a Hamiltonian theory. Given a *Hamiltonian function* $\text{tr}(h)$, $h \in V$, one defines the associated *Hamiltonian equation*

$$\frac{dx}{dt} = \{\text{tr}(h), x\}, \quad x \in V. \quad (0.13)$$

Two Hamiltonian functions $\text{tr}(f)$ and $\text{tr}(g)$ are said to be in involution if

$$\{\text{tr}(f), \text{tr}(g)\} = 0. \quad (0.14)$$

In the case of $V = R_\ell$, the algebra of non-commutative polynomials in ℓ variables x_1, \dots, x_ℓ , any 2-fold bracket can be written in a traditional form ($f, g \in R_\ell$):

$$\{\!\!\{f, g\}\!\!\} = \nabla g \bullet H \bullet (\nabla f)^\sigma, \quad (0.15)$$

where ∇f , the *gradient* of f , is the vector of 2-fold derivatives $\frac{\partial f}{\partial x_i} : R_\ell \rightarrow R_\ell \otimes R_\ell$, defined by $\frac{\partial f}{\partial x_i}(x_j) = \delta_{ij}(1 \otimes 1)$ (see [CBEG07]), $H = (H_{ij})_{i,j=1}^\ell$, where $H_{ij} = \{\!\!\{x_j, x_i\}\!\!\} \in V \otimes V$, and \bullet denotes the multiplication in $V \otimes V^{op}$. Moreover, the 2-fold bracket (0.15) is skewsymmetric if and only if skewsymmetry holds on any pair of generators x_i, x_j (equivalently, if the matrix H is skewadjoint: $(H^t)^\sigma = -H$), and, provided that H is skewadjoint, the Jacobi identity (0.10) holds if and only if it holds on any triple of generators x_i, x_j, x_k (see Theorem 2.8).

In the case of $V = R_\ell$ the Hamiltonian equation (0.13) becomes the following evolution ODE, where $x = (x_i)_{i=1}^\ell$:

$$\frac{dx}{dt} = m(H(\nabla h)^\sigma), \quad (0.16)$$

and the bracket $\{-, -\}$ on $R_\ell/[R_\ell, R_\ell]$ becomes:

$$\{\text{tr}(f), \text{tr}(g)\} = \text{tr}(m(\nabla g \bullet H \bullet (\nabla h)^\sigma)). \quad (0.17)$$

Developing the ideas of [MS00], we study in detail the Euler hierarchy, obtained via the Lenard-Magri scheme from the following pair of compatible double Poisson brackets on $R_2 = \mathbb{F}\langle x, y \rangle$, where y is central for both 2-fold brackets:

$$\{\!\!\{x, x\}\!\!\}_0 = 1 \otimes y - y \otimes 1, \quad \{\!\!\{x, x\}\!\!\}_1 = x \otimes y - y \otimes x. \quad (0.18)$$

Letting

$$h_0 = 1 \quad \text{and} \quad h_n = \frac{1}{n}(x + y)^n \quad \text{for } n \geq 1, \quad (0.19)$$

it is easy to establish the following recursion relations:

$$\{\text{tr}(h_0), x\}_0 = 0 \quad \text{and} \quad \{\text{tr}(h_n), x\}_1 = \{\text{tr}(h_{n+1}), x\}_0 \quad \text{for } n \in \mathbb{Z}_+.$$

Hence, by the Lenard-Magri scheme, all $\text{tr}(h_n)$ are in involution with respect to both brackets, and we obtain, by (0.16), the following compatible set of Hamiltonian ODEs ($m \in \mathbb{Z}_+$):

$$\frac{dx}{dt_m} = x(x+y)^m y - y(x+y)^m x, \quad \frac{dy}{dt_m} = 0, \quad (0.20)$$

for which all h_n , $n \in \mathbb{Z}_+$, are conserved densities. Thus, by definition in [MS00], the hierarchy of Hamiltonian ODEs (0.20) is integrable (see Section 2.6 for details).

We develop this theory in the natural generality of a (non-commutative) *algebra of ordinary differential functions*. This is a unital associative algebra V endowed with ℓ strongly commuting 2-fold derivations $\frac{\partial}{\partial x_i}$. We say that the 2-fold derivations D and E *strongly commute* if $D_L \circ E = E_R \circ D$, where $D_L(a_1 \otimes a_2) = (D_L a_1) \otimes a_2$ and $D_R(a_1 \otimes a_2) = a_1 \otimes (D_R a_2)$ (then D and E commute).

Formula (0.15) defines a 2-fold bracket on any algebra of ordinary differential functions V , and V is a double Poisson algebra if and only if the matrix H is skewadjoint and (0.10) holds on any ‘‘triple of generators’’ (see Theorem 2.8 for details).

The assumption of strong commutativity of the $\frac{\partial}{\partial x_i}$ is used in the proof of the Jacobi identity for the 2-fold bracket (0.15) and the proof that $d^2 = 0$ for the de Rham complex $\tilde{\Omega}(V)$, discussed below.

For an algebra of differential functions V we construct in Section 2.7 the de Rham complex $\tilde{\Omega}(V)$ as a free product of V and the algebra of non-commutative polynomials in the indeterminates dx_1, \dots, dx_ℓ . This is a superalgebra with the \mathbb{Z}_+ -grading given by $\deg(V) = 0$, $\deg(dx_i) = 1$, compatible with the parity, and with the de Rham differential d being the odd derivation such that $d(dx_i) = 0$ and $df = \sum_{i=1}^{\ell} m \left(\frac{\partial f}{\partial x_i} \otimes_1 dx_i \right)$ for $f \in \tilde{\Omega}^0(V) = V$.

The relevant to the theory of non-commutative evolution ODEs is the reduced complex

$$\Omega(V) = \tilde{\Omega}(V) / [\tilde{\Omega}(V), \tilde{\Omega}(V)]. \quad (0.21)$$

In Section 2.8 we give an explicit description of the space of k -forms $\Omega^k(V)$. In particular: $\Omega^0(V) = V/[V, V]$, $\Omega^1(V) = V^{\oplus \ell}$, $\Omega^2(V) = (\text{Mat}_{\ell \times \ell}(V \otimes V))_-$, where the subscript $-$ means that $(A_{ij})_{i,j=1}^{\ell} = -(A_{ji}^{\sigma})_{i,j=1}^{\ell}$. Under these identifications the formula for d becomes:

$$d(\text{tr } f) = m(\nabla f)^{\sigma}, \quad dF = \frac{1}{2}(J_F^t - J_F^{\sigma}), \quad (0.22)$$

where for $F = (F_i)_{i=1}^{\ell} \in V^{\oplus \ell}$ we define the 2-fold Jacobian matrix as $J_F = \left(\frac{\partial F_i}{\partial x_j} \right)_{i,j=1}^{\ell}$. We also prove that both complexes $(\tilde{\Omega}(R_\ell), d)$ and $(\Omega(R_\ell), d)$ are acyclic.

The theory of double Poisson algebras and non-commutative Hamiltonian ODEs, over an algebra of ordinary differential functions V , developed in Section 2 sets the stage for the theory of double PVAs and non-commutative Hamiltonian PDEs over an algebra of differential functions \mathcal{V} , which we develop in Section 3.

In a nutshell a double PVA is a cross between a double Poisson algebra and a PVA. Let \mathcal{V} be unital associative algebra with a derivation ∂ . We define a *2-fold λ -bracket* on \mathcal{V} as a map $\{\{-\lambda-\}\} : \mathcal{V} \otimes \mathcal{V} \rightarrow (\mathcal{V} \otimes \mathcal{V})[\lambda]$, satisfying sesquilinearity (3.5) and left and right Leibniz rules (3.6), similar to (0.1) and (0.2),(0.3) for PVA. A *double PVA* is the algebra \mathcal{V} , endowed with a 2-fold λ -bracket satisfying skewsymmetry (3.8) and Jacobi identity (3.9) (= cross between (0.4) and (0.8), and between (0.5) and (0.10), respectively). If $\partial = 0$, the axioms of a double PVA turn into the axioms of a double Poisson algebra if we let $\lambda = \mu = 0$.

A non-local double PVA is defined in the same way as a non-local PVA (but the one cannot put $\lambda = 0$).

For a (local) double PVA \mathcal{V} , we denote by \int the quotient map $\mathcal{V} \rightarrow \overline{\mathcal{V}} := \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$. Similarly to (0.11), define the map $\{-_\lambda -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$ by

$$\{a_\lambda b\} = m\{\{a_\lambda b\}\}. \quad (0.23)$$

As for the (local) PVA (see [BDSK09]), (0.23) induces well-defined maps

$$\overline{\mathcal{V}} \otimes \mathcal{V} \rightarrow \mathcal{V} \quad \text{and} \quad \overline{\mathcal{V}} \otimes \overline{\mathcal{V}} \rightarrow \overline{\mathcal{V}},$$

given by formulas, similar to (0.12):

$$\{\int a, b\} = \{a_\lambda b\}|_{\lambda=0} \quad \text{and} \quad \{\int a, \int b\} = \int \{a_\lambda b\}|_{\lambda=0}. \quad (0.24)$$

As before, these maps have properties important for the theory of non-commutative Hamiltonian PDEs: $\overline{\mathcal{V}}$ is a Lie algebra and one has its representation on \mathcal{V} by derivations, both defined by (0.24) (see Theorem 3.6). We have definitions of a Hamiltonian function $\int h \in \overline{\mathcal{V}}$, the corresponding Hamiltonian equation in \mathcal{V} , similar to (0.13), and involutiveness, similar to (0.14), where tr is replaced by \int .

We define an *algebra of (non-commutative) differential functions* as a unital associative algebra \mathcal{V} with a derivation ∂ and strongly commuting 2-fold derivations $\frac{\partial}{\partial u_i^{(n)}}$, $i = 1, \dots, \ell$, $n \in \mathbb{Z}_+$, such that the following two properties hold (cf. [BDSK09]):

$$\text{for each } f \in \mathcal{V}, \frac{\partial f}{\partial u_i^{(n)}} = 0 \text{ for all but finitely many } (i, n), \quad (0.25)$$

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}. \quad (0.26)$$

The most important example of an algebra of non-commutative differential functions is the algebra of non-commutative differential polynomials \mathcal{R}_ℓ in the indeterminates $u_i^{(n)}$, $i = 1, \dots, \ell$, $n \in \mathbb{Z}_+$, with the derivation ∂ defined by $\partial u_i^{(n)} = u_i^{(n+1)}$.

First, we prove a PVA analogue of Theorem 2.8. Namely, any 2-fold λ -bracket on \mathcal{R}_ℓ has the form (cf. (0.15)):

$$\{\{f_\lambda g\}\} = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} \bullet (\lambda + \partial)^n H_{ij}(\lambda + \partial)(-\lambda - \partial)^m \bullet \left(\frac{\partial f}{\partial u_i^{(m)}} \right)^\sigma, \quad (0.27)$$

where $H_{ij}(\lambda) = \{\{u_j \lambda u_i\}\} \in (\mathcal{R}_\ell \otimes \mathcal{R}_\ell)[\lambda]$. Formula (0.27) defines a 2-fold λ -bracket on any algebra of differential functions \mathcal{V} . This 2-fold λ -bracket is skewsymmetric if and only if $H(\partial) = (H_{ij}(\partial))_{i,j=1}^\ell$ is a skewadjoint differential operator over $\mathcal{V} \otimes \mathcal{V}$ and, provided that $H(\partial)$ is skewadjoint, the Jacobi identity holds if and only if it holds on any triple of generators (see Theorem 3.10).

Next, the non-commutative Hamiltonian PDE, associated to the matrix differential operator $H(\partial)$, defining the 2-fold λ -bracket, and to the Hamiltonian functional $\int h \in \overline{\mathcal{V}}$ is the following evolution PDE, where $u = (u_i)_{i=1}^\ell$ (cf. (0.16)):

$$\frac{du}{dt} = m(H(\partial) \bullet (\delta h)^\sigma), \quad (0.28)$$

where $\delta h = \left(\frac{\delta h}{\delta u_i} \right)_{i=1}^\ell \in (\mathcal{V} \otimes \mathcal{V})^{\oplus \ell}$ is the vector of 2-fold variational derivatives

$$\frac{\delta h}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial h}{\partial u_i^{(n)}}.$$

The bracket $\{-, -\}$ on $\bar{\mathcal{V}}$ is given by a formula, similar to (0.17):

$$\{f, g\} = f \mathfrak{m}(\delta g \bullet H(\partial) \bullet (\delta f)^\sigma). \quad (0.29)$$

Furthermore the de Rham complex $\tilde{\Omega}(\mathcal{V})$ over an algebra of differential functions \mathcal{V} is defined in the same way as the de Rham complex $\tilde{\Omega}(V)$ over an algebra of ordinary differential functions V . Moreover, the action of ∂ on \mathcal{V} naturally extends to an action on $\tilde{\Omega}(\mathcal{V})$, commuting with the de Rham differential δ (see Section 3.5). This allows us to define the *variational complex*

$$\Omega(\mathcal{V}) = \tilde{\Omega}(\mathcal{V}) / (\partial \tilde{\Omega}(\mathcal{V}) + [\tilde{\Omega}(\mathcal{V}), \tilde{\Omega}(\mathcal{V})]).$$

In Section 3.6 we give an explicit description of the space of the variational k -forms $\Omega^k(\mathcal{V})$. In particular,

$$\Omega^0(\mathcal{V}) = \mathcal{V} / (\partial \mathcal{V} + [\mathcal{V}, \mathcal{V}]), \quad \Omega^1(\mathcal{V}) = \mathcal{V}^{\oplus \ell}, \quad \Omega^2(\mathcal{V}) = (\text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\lambda])_-,$$

where the subscript $-$ means that $(A_{ij}(\lambda))_{i,j=1}^\ell = -(A_{ji}(-\lambda - \partial)^\sigma)_{i,j=1}^\ell$. Under these identifications, the formula for δ is similar to (0.22):

$$\delta(f) = \mathfrak{m}(\delta f)^\sigma, \quad (\delta F)(\partial) = \frac{1}{2}(D_F(\partial)^t - D_F(\partial)^{* \sigma}), \quad (0.30)$$

where $(D_F(\partial))_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n$ is the 2-fold Frechet derivative.

We also prove that both complexes $(\tilde{\Omega}(\mathcal{R}_\ell), \delta)$ and $(\Omega(\mathcal{R}_\ell), \delta)$ are acyclic. Hence, in particular, we obtain the description of the kernel and the image of the 2-fold variational derivative δ .

The main motivation for introducing double Poisson algebras was the observation that, given a unital finitely generated associative algebra V , each double Poisson algebra structure on V associates for each positive integer m a Poisson algebra structure on the algebra of polynomial functions V_m on the affine algebraic variety of m -dimensional representations of the algebra V [VdB08].

Note that V_m is isomorphic to the commutative associative algebra with generators $\{a_{ij} \mid a \in V, i, j = 1, \dots, m\}$, subject to the relations (3.51) in Section 3.7.

If \mathcal{V} is a (non-commutative) differential associative algebra with derivation ∂ , then \mathcal{V}_m is a commutative differential associative algebra with derivation defined on generators by $\partial(a_{ij}) = (\partial a)_{ij}$. Similarly to [VdB08], we show that, given a double PVA structure on \mathcal{V} , one can associate for each positive integer m a PVA structure on \mathcal{V}_m (see Theorem 3.22).

The simplest example of a double PVA is the following family of compatible double PVA structures on $\mathcal{R}_1 = \mathbb{F}\langle u, u', u'', \dots \rangle$:

$$\{\{u_\lambda u\}\} = 1 \otimes u - u \otimes 1 + c(1 \otimes 1)\lambda,$$

where $c \in \mathbb{F}$ is a parameter. In this case the associated PVA structure on $(\mathcal{R}_1)_m = \mathbb{F}[u_{ij}^{(n)} \mid i, j = 1, \dots, m, n \in \mathbb{Z}_+]$ is just the affine PVA for \mathfrak{gl}_m (see Section 3.8.1). The extension of this double PVA structure on \mathcal{R}_1 to the double PVA structure on $\mathcal{R}_2 = \mathbb{F}\langle u, v, u', v', \dots \rangle$, where v is central, produces the following hierarchy of non-commutative Hamiltonian PDE's ($n \in \mathbb{Z}_+$):

$$\frac{du}{dt_n} = v(u+v)^n u - u(u+v)^n v + c\partial(u+v)^{n+1}, \quad \frac{dv}{dt_n} = 0.$$

Our next example is a compatible pair of double PVA structures on the algebra \mathcal{R}_N (resp. \mathcal{R}_∞), associated to the “generic” differential (resp. pseudodifferential) operator with coefficients in \mathcal{R}_N (resp. \mathcal{R}_∞), similar to the PVA structures in the commutative case, considered in [DSKV14a] (see Section 3.8.2).

We apply for the latter example the theory of Dirac reduction for (non-local) double PVA developed in Section 4.2, similar to that for (non-local) PVA, developed in [DSKV14b]. As a result, we obtain compatible non-local double Poisson vertex algebra structures on $\mathcal{V} = \mathcal{R}_N$ and \mathcal{R}_∞ , such that the corresponding non-local PVA structures on \mathcal{V}_m coincide with those in [DSKV14a] (see Example 4.11).

Finally, in Section 5 we apply the Lenard-Magri scheme to the double PVA constructed in Section 3.8.2 to prove integrability of the non-commutative KP hierarchy and of non-commutative Gelfand-Dickey hierarchies. In particular, we prove integrability of the non-commutative KdV and Boussinesq hierarchies.

Throughout the paper we let \mathbb{F} be a field of characteristic 0, and, unless otherwise specified, we consider all vector spaces, tensor products, etc., over the field \mathbb{F} .

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1. PRELIMINARIES AND NOTATION FOR ASSOCIATIVE ALGEBRAS

Throughout this and the next section, we let V be an associative algebra over the field \mathbb{F} .

1.1. **‘‘Double’’ linear algebra.** Consider the associative product \bullet on $V \otimes V$ given, in Sweedler’s notation, by the formula

$$A \bullet B = A' B' \otimes B'' A'' . \quad (1.1)$$

(It is the product in $V \otimes V^{op}$.) The endomorphism σ of $V \otimes V$ obtained by exchanging the two factors:

$$(a \otimes b)^\sigma = b \otimes a , \quad (1.2)$$

is an antiautomorphism of the \bullet -product:

$$(A \bullet B)^\sigma = B^\sigma \bullet A^\sigma . \quad (1.3)$$

We define an inner product $(\cdot | \cdot)$ on the vector space $(V \otimes V)^{\oplus \ell}$ with values in the algebra $V \otimes V$, letting for $F = (F_i)_{i=1}^\ell$ and $G = (G_i)_{i=1}^\ell$ ($F_i, G_i \in V \otimes V$):

$$(F|G) = \sum_{i=1}^{\ell} F_i \bullet G_i^\sigma . \quad (1.4)$$

By (1.3), it satisfies the following symmetry property:

$$(G|F)^\sigma = (F|G) . \quad (1.5)$$

Let $H = (H_{ij})_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}(V \otimes V)$. We define its action on $(V \otimes V)^{\oplus \ell}$ by the following formula:

$$(HF)_i = \sum_{j=1}^{\ell} (H_{ij} \bullet F_j^\sigma)^\sigma . \quad (1.6)$$

In Sweedler’s notation, it is $(HF)_i = \sum_{j=1}^{\ell} F_j' H_{ij}'' \otimes H_{ij}' F_j''$. The *adjoint* of the matrix H is defined by

$$(H^\dagger)_{ij} = (H_{ji})^\sigma . \quad (1.7)$$

It is adjoint with respect to the inner product (1.4) in the usual sense:

$$(F|HG) = (H^\dagger F|G) , \quad F, G \in (V \otimes V)^{\oplus \ell} . \quad (1.8)$$

1.2. Trace map and reduction. Denote by $m : V \otimes V \rightarrow V$ (or, in general, $m : V^{\otimes n} \rightarrow V$) the multiplication map:

$$m(a \otimes b) = ab .$$

We define the following “reduction” map $(V \otimes V)^{\oplus \ell} \rightarrow V^{\oplus \ell}$ by

$$F \mapsto m(F^\sigma) \left(= (m(F_i^\sigma))_{i=1}^\ell \right) . \quad (1.9)$$

Furthermore, let $[V, V]$ be the subspace of V spanned by commutators $ab - ba$, for $a, b \in V$, and denote by

$$\text{tr} : V \rightarrow V/[V, V] \quad (1.10)$$

the canonical quotient map.

The inner product (1.4) induces a well defined inner product $(\cdot|\cdot) : V^{\oplus \ell} \times V^{\oplus \ell} \rightarrow V/[V, V]$ so that the following diagram is commutative:

$$\begin{array}{ccc} (V \otimes V)^{\oplus \ell} \times (V \otimes V)^{\oplus \ell} & \xrightarrow{(\cdot|\cdot)} & V \otimes V \\ \text{m} \circ \sigma \downarrow & & \downarrow \text{tr} \circ \text{m} \\ V^{\oplus \ell} \times V^{\oplus \ell} & \xrightarrow{\dots(\cdot|\cdot)\dots} & V/[V, V] \end{array}$$

It is given explicitly by

$$(F|G) = \sum_{i=1}^{\ell} \text{tr}(F_i G_i) , \quad F, G \in V^{\oplus \ell} . \quad (1.11)$$

This bilinear form is obviously symmetric.

Furthermore, the action (1.6) of the matrix $H \in \text{Mat}_{\ell \times \ell}(V \otimes V)$ induces an action on $V^{\oplus \ell}$ via the commutative diagram

$$\begin{array}{ccc} (V \otimes V)^{\oplus \ell} & \xrightarrow{H} & (V \otimes V)^{\oplus \ell} \\ \text{m} \circ \sigma \downarrow & & \downarrow \text{m} \circ \sigma \\ V^{\oplus \ell} & \xrightarrow{\dots H \dots} & V^{\oplus \ell} \end{array}$$

Explicitly, it is given, in Sweedler’s notation, by

$$(HF)_i = \sum_{j=1}^{\ell} H'_{ij} F_j H''_{ij} , \quad F \in V^{\oplus \ell} . \quad (1.12)$$

1.3. V -module structures of $V^{\otimes n}$. The space $V^{\otimes n}$ has the following outer V -bimodule structure:

$$a(b_1 \otimes \cdots \otimes b_n)c = (ab_1) \otimes b_2 \cdots \otimes b_{n-1} \otimes (b_n c) . \quad (1.13)$$

More generally, for every $i = 0, \dots, n-1$, we define the i -th left and right V -module structures of $V^{\otimes n}$ by

$$\begin{aligned} a \star_i (b_1 \otimes \cdots \otimes b_n) &= b_1 \otimes \cdots \otimes b_i \otimes ab_{i+1} \otimes \cdots \otimes b_n , \\ (a_1 \otimes \cdots \otimes a_n) \star_i b &= a_1 \otimes \cdots \otimes a_{n-i} b \otimes \cdots \otimes a_n . \end{aligned} \quad (1.14)$$

(The index denotes the number of “jumps”.) In particular, the bimodule structure (1.13) is given by $aBc = a \star_0 B \star_0 c$.

We use a similar notation for \otimes -product of an element of V and an element of $V^{\otimes n}$:

$$\begin{aligned} a \otimes_i (b_1 \otimes \cdots \otimes b_n) &= b_1 \otimes \cdots \otimes b_i \otimes a \otimes b_{i+1} \otimes \cdots \otimes b_n, \\ (a_1 \otimes \cdots \otimes a_n) \otimes_i b &= a_1 \otimes \cdots \otimes a_{n-i} \otimes b \otimes \cdots \otimes a_n. \end{aligned} \quad (1.15)$$

For example, the action (1.12) of $H \in \text{Mat}_{\ell \times \ell}(V \otimes V)$ on $V^{\oplus \ell}$ can be rewritten, using this notation, as $(HF)_i = \sum_{j=1}^{\ell} m(H_{ij} \otimes_1 F_j) (= \sum_{j=1}^{\ell} m(F_j \otimes_1 H_{ij}) = \sum_{j=1}^{\ell} m(H_{ij} \star_1 F_j) = \sum_{j=1}^{\ell} m(F_j \star_1 H_{ij}))$.

Note that, for $f \in V$ and $A, B \in V^{\otimes 2}$, we have

$$\begin{aligned} (fA) \bullet B &= f(A \bullet B), \quad (Af) \bullet B = (A \bullet B)f, \\ A \bullet (f \star_1 B) &= f \star_1 (A \bullet B), \quad A \bullet (B \star_1 f) = (A \bullet B) \star_1 f. \end{aligned} \quad (1.16)$$

We also will be using the *multiplication map* $V^{\otimes m} \times V^{\otimes n} \rightarrow V^{\otimes(m+n-1)}$, given by

$$(a_1 \otimes \cdots \otimes a_m)(b_1 \otimes \cdots \otimes b_n) = a_1 \otimes \cdots \otimes a_{m-1} \otimes (a_m b_1) \otimes b_2 \cdots \otimes b_n. \quad (1.17)$$

1.4. Some further notation. We define three possible left and right actions of the algebra $V^{\otimes 2}$ on $V^{\otimes 3}$, denoted by \bullet_i , $i = 1, 2, 3$, as follows (here the subscript indicates the place where no multiplication occurs)

$$\begin{aligned} (a \otimes b) \bullet_1 (x \otimes y \otimes z) &= x \otimes ay \otimes zb, \quad (x \otimes y \otimes z) \bullet_1 (a \otimes b) = x \otimes ya \otimes bz, \\ (a \otimes b) \bullet_2 (x \otimes y \otimes z) &= ax \otimes y \otimes zb, \quad (x \otimes y \otimes z) \bullet_2 (a \otimes b) = xa \otimes y \otimes bz, \\ (a \otimes b) \bullet_3 (x \otimes y \otimes z) &= ax \otimes yb \otimes z, \quad (x \otimes y \otimes z) \bullet_3 (a \otimes b) = xa \otimes by \otimes z. \end{aligned} \quad (1.18)$$

Lemma 1.1. (a) The \bullet_i left (and right) actions of $V^{\otimes 2}$ on $V^{\otimes 3}$ are indeed actions, i.e. they are associative with respect to the \bullet -product of $V^{\otimes 2}$:

$$A \bullet_i (B \bullet_i X) = (A \bullet B) \bullet_i X \quad \text{and} \quad (X \bullet_i A) \bullet_i B = X \bullet_i (A \bullet B),$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$.

(b) The left \bullet_i and the right \bullet_j actions commute for every $i, j = 1, 2, 3$:

$$A \bullet_i (X \bullet_j B) = (A \bullet_i X) \bullet_j B$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$.

(c) The \bullet_1 and \bullet_3 left (resp. right) actions of $V^{\otimes 2}$ on $V^{\otimes 3}$ commute:

$$A \bullet_1 (B \bullet_3 X) = B \bullet_3 (A \bullet_1 X) \quad \text{and} \quad (X \bullet_1 A) \bullet_3 B = (X \bullet_3 B) \bullet_1 A,$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$. (In general, the \bullet_i and \bullet_j left (resp. right) actions do NOT commute if $|i - j| = 1$.)

Proof. Straightforward. □

Lemma 1.2. For every $A \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$ we have

$$\begin{aligned} (A \bullet_1 X)^\sigma &= X^\sigma \bullet_2 A^\sigma, \quad (A \bullet_2 X)^\sigma = X^\sigma \bullet_3 A^\sigma, \quad (A \bullet_3 X)^\sigma = A \bullet_1 X^\sigma, \\ (A \bullet_1 X)^{\sigma^2} &= A \bullet_3 X^{\sigma^2}, \quad (A \bullet_2 X)^{\sigma^2} = X^{\sigma^2} \bullet_1 A^\sigma, \quad (A \bullet_3 X)^{\sigma^2} = X^{\sigma^2} \bullet_2 A^\sigma, \\ (X \bullet_1 A)^\sigma &= A^\sigma \bullet_2 X^\sigma, \quad (X \bullet_2 A)^\sigma = A^\sigma \bullet_3 X^\sigma, \quad (X \bullet_3 A)^\sigma = X^\sigma \bullet_1 A, \\ (X \bullet_1 A)^{\sigma^2} &= X^{\sigma^2} \bullet_3 A, \quad (X \bullet_2 A)^{\sigma^2} = A^\sigma \bullet_1 X^{\sigma^2}, \quad (X \bullet_3 A)^{\sigma^2} = A^\sigma \bullet_2 X^{\sigma^2}. \end{aligned} \quad (1.19)$$

Proof. Straightforward. □

1.5. n -fold derivations.

Definition 1.3. An n -fold derivation of V is a linear map $D : V \rightarrow V^{\otimes n}$ such that

$$D(ab) = (Da)b + a(Db), \quad a, b \in V, \quad (1.20)$$

where, in the RHS, we use the bimodule structure (1.13).

We can extend the n -fold derivation $D : V \rightarrow V^{\otimes n}$ to a map $D : V^{\otimes m} \rightarrow V^{\otimes(m+n-1)}$ for every $m \in \mathbb{Z}_+$, by

$$D(a_1 \otimes \cdots \otimes a_m) = \sum_{i=1}^m a_1 \otimes \cdots \otimes (Da_i) \otimes \cdots \otimes a_m, \quad (1.21)$$

where, in the RHS, we use the concatenation product in the tensor algebra over V . We can also extend the n -fold derivation D to a map $V^{\otimes m} \rightarrow V^{\otimes m+n-1}$ by applying D to only the i -th factor, and we denote the corresponding map by $D_{(i)}$:

$$D_{(i)}(a_1 \otimes \cdots \otimes a_m) = a_1 \otimes \cdots \otimes D(a_i) \otimes \cdots \otimes a_m. \quad (1.22)$$

We will use the special notation for the derivation on the leftmost and the rightmost factors:

$$D_L := D_{(1)} \quad \text{and} \quad D_R := D_{(m)}. \quad (1.23)$$

The symmetric group S_n acts on $V^{\otimes n}$ in the usual way: for $\tau \in S_n$, we have

$$(a_1 \otimes \cdots \otimes a_n)^\tau = a_{\tau^{-1}(1)} \otimes \cdots \otimes a_{\tau^{-1}(n)}. \quad (1.24)$$

For every n , we shall denote by σ the cyclic permutation $(1, \dots, n)$, so that

$$(a_1 \otimes \cdots \otimes a_n)^\sigma = a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}. \quad (1.25)$$

For convenience, we write here the formulas for the action of the whole cyclic group C_n on $A^{\otimes n}$ ($s = 1, \dots, n$):

$$(a_1 \otimes \cdots \otimes a_n)^{\sigma^s} = a_{n+1-s} \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{n-s}, \quad (1.26)$$

and on an n -tuple of indices $(i_1, \dots, i_n) \in \{1, \dots, \ell\}^n$:

$$(i_{\sigma^{-s}(1)}, \dots, i_{\sigma^{-s}(n)}) = (i_{n+1-s}, \dots, i_n, i_1, \dots, i_{n-s}). \quad (1.27)$$

Lemma 1.4. Let $D : V \rightarrow V^{\otimes n}$ be an n -fold derivation of V . For $A \in V^{\otimes m}$, we have

$$\begin{aligned} (D_{(i)}A)^\sigma &= D_{(i+1)}(A^\sigma) \quad \text{for } 1 \leq i \leq m-1, \\ (D_{(m)}A)^{\sigma^n} &= D_{(1)}(A^\sigma). \end{aligned} \quad (1.28)$$

Proof. Straightforward. \square

Lemma 1.5. If $D : V \rightarrow V^{\otimes n}$ is an n -fold derivation of V , then, for every $A \in V^{\otimes h}$ and $B \in V^{\otimes k}$ ($h, k \in \mathbb{Z}_+$), we have

$$D(AB) = (DA)B + A(DB), \quad (1.29)$$

where, in the RHS, we use the multiplication map (1.17).

Proof. Straightforward. \square

Given an m -fold derivation $D_1 : V \rightarrow V^{\otimes m}$ and an n -fold derivation $D_2 : V \rightarrow V^{\otimes n}$, we can compose them, using their extension (1.21), to get a map

$$D_1 \circ D_2 : V \rightarrow V^{\otimes m+n-1}, \quad (D_1 \circ D_2)(a) = D_1(D_2a). \quad (1.30)$$

Proposition 1.6. Let D_1 and D_2 be respectively an m -fold and an n -fold derivation of V . Then:

(a) The commutator $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is an $m+n-1$ -fold derivation of V .

(b) $(D_1)_L \circ D_2 - (D_2)_R \circ D_1$ is an $m + n - 1$ -fold derivation of V .

Proof. Part (a) follows from Lemma 1.5. Using the fact that D_2 is an n -fold derivation, and by the definition (1.23) of $(D_1)_L$, we have $(a, b \in V)$

$$\begin{aligned} ((D_1)_L \circ D_2)(ab) &= (D_1)_L(D_2(a)b) + (D_1)_L(aD_2(b)) \\ &= ((D_1)_L(D_2a))b + (D_1a)(D_2b) + a((D_1)_L(D_2b)), \end{aligned} \quad (1.31)$$

where in the second term we are using the multiplication in the tensor algebra given by (1.17). Similarly we get

$$((D_2)_R \circ D_1)(ab) = ((D_2)_R(D_1a))b + (D_1a)(D_2b) + a((D_2)_R(D_1b)). \quad (1.32)$$

Combining equations (1.31) and (1.32) we get part (b). \square

Remark 1.7. Suppose that the algebra V is counital (with counity $\epsilon \in V^*$), so that \mathbb{F} is an A -bimodule (using the counit). Then, by Proposition 1.6, we have a \mathbb{Z} -graded Lie algebra $\mathfrak{g} = \bigoplus_{n=-1}^{\infty} \mathfrak{g}_n$, where \mathfrak{g}_n is the space of $n + 1$ -fold derivations of V .

1.6. n -fold brackets.

Definition 1.8. An n -fold bracket on V is a linear map $\{\{-, \dots, -\} : A^{\otimes n} \rightarrow A^{\otimes n}$ satisfying the following Leibniz rules (using the notation (1.14)):

$$\{\{a_1, \dots, bc, \dots, a_n\}\} = b \star_i \{\{a_1, \dots, c, \dots, a_n\}\} + \{\{a_1, \dots, b, \dots, a_n\}\} \star_{n-i} c, \quad (1.33)$$

for all $i = 1 \dots, n$ (in the first term of the RHS we let $\star_n = \star_0$).

Note that, if $\{\{-, \dots, -\}$ is an n -fold bracket, then, for every $a_1, \dots, a_{n-1} \in V$, the map $\{\{a_1, \dots, a_{n-1}, -\} : V \rightarrow V^{\otimes n}$ is an n -fold derivation of V . Then, according to the notation (1.23), we let, for $a_1, \dots, a_{n-1}, b_1, \dots, b_m \in V$,

$$\begin{aligned} \{\{a_1, \dots, a_{n-1}, b_1 \otimes \dots \otimes b_m\}\}_L &= \{\{a_1, \dots, a_{n-1}, b_1\}\} \otimes b_2 \otimes \dots \otimes b_m, \\ \{\{a_1, \dots, a_{n-1}, b_1 \otimes \dots \otimes b_m\}\}_R &= b_1 \otimes \dots \otimes b_{m-1} \otimes \{\{a_1, \dots, a_{n-1}, b_m\}\}. \end{aligned} \quad (1.34)$$

As a special case of Lemma 1.4, we get the following result, which will be used later

Corollary 1.9. Given a 2-fold bracket $\{\{-, -\}$ on V , we have

$$\{\{a, \{\{b, c\}\}\}_R = \{\{a, \{\{b, c\}\}^\sigma\}\}_L^\sigma. \quad (1.35)$$

Proof. Equation (1.35) is a special case of equation (1.28) for $D = \{\{a, -\} : V \rightarrow V^{\otimes 2}$. \square

The following result will be used later as well.

Lemma 1.10. Let $\{\{-, -\}$ be a 2-fold bracket on the associative algebra V . For every $a \in V$ and $B, C \in V^{\otimes 2}$, we have

$$\{\{a, B \bullet C\}\}_L = B \bullet_2 \{\{a, C\}\}_L + \{\{a, B\}\}_L \bullet_1 C,$$

where in the LHS we use the product (1.1) while in the RHS we use notation (1.18).

Proof. Straightforward. \square

2. DOUBLE POISSON ALGEBRAS AND NON-COMMUTATIVE HAMILTONIAN ODES

2.1. Definition of double Poisson algebras.

Definition 2.1 ([VdB08]). A *double Poisson algebra* is an associative algebra V endowed with a 2-fold bracket $\{\{-, -\} : V \otimes V \rightarrow V \otimes V$ satisfying the following axioms:

- (i) skewsymmetry: $\{\{a, b\} = -\{\{b, a\}^\sigma$,
- (ii) Jacobi identity:

$$\{\{a, \{\{b, c\}\}_L\}_L + \{\{b, \{\{c, a\}\}_L^\sigma\}_L + \{\{c, \{\{a, b\}\}_L^\sigma\}_L = 0. \quad (2.1)$$

(Here we are using notation (1.34), i.e. we let $\{\{a, b \otimes c\}\}_L = \{\{a, b\} \otimes c \in V \otimes V \otimes V$.)

We can write explicitly the Leibniz rules for a double Poisson algebra bracket: equation (1.33) with $i = 2$ and 1 reads, respectively,

$$\{\{a, bc\}\} = \{\{a, b\}\}c + b\{\{a, c\}\}, \quad (2.2)$$

and

$$\{\{ab, c\}\} = \{\{a, c\}\} \star_1 b + a \star_1 \{\{b, c\}\}. \quad (2.3)$$

Note that equation (2.3) follows from (2.2) and the skewsymmetry axiom.

Remark 2.2. We can write the Jacobi identity (2.1) in an alternative way, analogous to the usual “representation theoretic form” of the Jacobi identity for Lie algebras. For this we extend the notation (1.34) of $\{\{-, -\}_L$ and $\{\{-, -\}_R$ to the case when we have an element of $V \otimes V$ in the first entry. We do this in analogy to the two terms in the RHS of (2.3): using notation (1.15), we let

$$\{\{a \otimes b, c\}\}_L := \{\{a, c\}\} \otimes_1 b, \quad \{\{a \otimes b, c\}\}_R := a \otimes_1 \{\{b, c\}\}. \quad (2.4)$$

With this notation, it is not hard to check, using skewsymmetry, that the Jacobi identity (2.1) can be equivalently written as follows:

$$\{\{a, \{\{b, c\}\}\}_L - \{\{b, \{\{a, c\}\}\}_R = \{\{\{\{a, b\}\}, c\}\}_L. \quad (2.5)$$

Lemma 2.3 ([VdB08]). *If $\{\{-, -\} : V \rightarrow V^{\otimes 2}$ is a 2-fold bracket satisfying the skewsymmetry axiom, then the LHS of the Jacobi identity (2.1)*

$$\{\{a, b, c\}\} := \{\{a, \{\{b, c\}\}\}_L + \{\{b, \{\{c, a\}\}\}_L^\sigma + \{\{c, \{\{a, b\}\}\}_L^\sigma \quad (2.6)$$

is a 3-fold bracket on V .

Proof. Using the Leibniz rules for the 2-fold bracket $\{\{-, -\}$, we have

$$\begin{aligned} \{\{a, \{\{b, cd\}\}\}_L &= \{\{a, \{\{b, c\}\}d\}_L + \{\{a, c\}\}\{\{b, d\}\}_L \\ &= \{\{a, \{\{b, c\}\}\}_L d + \{\{a, c\}\}\{\{b, d\}\} + c\{\{a, \{\{b, d\}\}\}_L, \end{aligned} \quad (2.7)$$

$$\begin{aligned} \{\{b, \{\{cd, a\}\}\}_L^\sigma &= (\{\{b, c \star_1 \{\{d, a\}\}\}_L + \{\{b, \{\{c, a\}\}\} \star_1 d\}_L)^\sigma \\ &= (c \star_2 \{\{b, \{\{d, a\}\}\}_L + \{\{b, \{\{c, a\}\}\}_L \star_1 d + \{\{c, a\}\}'\{\{b, d\}\}\{\{c, a\}\}'')^\sigma \\ &= c\{\{b, \{\{d, a\}\}\}_L^\sigma + \{\{b, \{\{c, a\}\}\}_L^\sigma d + \{\{c, a\}\}^\sigma \{\{b, d\}\}. \end{aligned} \quad (2.8)$$

Hereafter we use Sweedler’s notation: for $A \in V^{\otimes 2}$, we let $A = A' \otimes A''$ (omitting the sign of summation). Similarly,

$$\begin{aligned} \{\{cd, \{\{a, b\}\}\}_L^\sigma &= (c \star_1 \{\{d, \{\{a, b\}\}\}_L + \{\{c, \{\{a, b\}\}\}_L \star_2 d)^\sigma \\ &= c\{\{d, \{\{a, b\}\}\}_L^\sigma + \{\{c, \{\{a, b\}\}\}_L^\sigma d. \end{aligned} \quad (2.9)$$

Combining equations (2.7), (2.8) and (2.9), and using the skewsymmetry assumption on $\{\{-, -\}\}$, we get

$$\{\{a, b, cd\}\} = c\{\{a, b, d\}\} + \{\{a, b, c\}\}d, \quad (2.10)$$

i.e. the Leibniz rule holds on the third entry. The Leibniz rules (1.33) on the first and second entry follow from (2.10) and the following obvious identities ($a, b, c \in V$)

$$\{\{a, b, c\}\} = \{\{b, c, a\}\}^\sigma = \{\{c, a, b\}\}^{\sigma^2}. \quad (2.11)$$

□

2.2. The trace map and connection to Lie algebras. Let $\{\{-, -\}\}$ be a 2-fold bracket on the associative algebra V . By composing it with the multiplication map $m : V \otimes V \rightarrow V$, we obtain the corresponding map $\{-, -\} : V \otimes V \rightarrow V$:

$$\{a, b\} = m\{\{a, b\}\}. \quad (2.12)$$

It is immediate to check that this bracket satisfies the Leibniz rule on the second entry:

$$\{a, bc\} = b\{a, c\} + \{a, b\}c. \quad (2.13)$$

(While, in general, it does not satisfies the Leibniz rule on the first entry.) Hence, for every $a \in V$, the map $\{a, -\} : V \rightarrow V$ is a (1-fold) derivation, and we extend it to a map $\{a, -\} : V^{\otimes n} \rightarrow V^{\otimes n}$, for every $n \in \mathbb{Z}_+$, via (1.21).

Lemma 2.4 ([VdB08]). *If $\{\{-, -\}\}$ is a skewsymmetric 2-fold bracket on V , then the following identity holds in $V^{\otimes 2}$ ($a, b, c \in V$)*

$$\{a, \{\{b, c\}\}\} - \{\{a, b\}, c\} - \{\{b, \{a, c\}\}\} = (m \otimes 1)\{\{a, b, c\}\} - (1 \otimes m)\{\{b, a, c\}\}, \quad (2.14)$$

where $\{\{a, b, c\}\}$ is as in (2.6).

Proof. Let us compute the three terms appearing in the LHS of equation (2.14). Using the Leibniz rules we get, after a straightforward computation,

$$\begin{aligned} \{a, \{\{b, c\}\}\} &= (m \otimes 1)\{\{a, \{\{b, c\}\}\}_L\} + (1 \otimes m)\{\{a, \{\{b, c\}\}\}\}_R \\ &= (m \otimes 1)\{\{a, \{\{b, c\}\}\}\}_L - (1 \otimes m)\{\{a, \{\{c, b\}\}\}\}_L^\sigma. \end{aligned} \quad (2.15)$$

For the last equality we used Corollary 1.9 and the skewsymmetry assumption. With similar computations, we get

$$\{\{b, \{a, c\}\}\} = (1 \otimes m)\{\{b, \{\{a, c\}\}\}\}_L - (m \otimes 1)\{\{b, \{\{c, a\}\}\}\}_L^\sigma, \quad (2.16)$$

and

$$\{\{\{a, b\}, c\}\} = -\{\{c, \{a, b\}\}\}^\sigma = -(m \otimes 1)\{\{c, \{\{a, b\}\}\}\}_L^{\sigma^2} + (1 \otimes m)\{\{c, \{\{b, a\}\}\}\}_L^{\sigma^2}. \quad (2.17)$$

Combining equations (2.15), (2.17) and (2.16), and using the definition (2.6) of $\{\{a, b, c\}\}$, equation (2.14) follows, thus concluding the proof. □

Proposition 2.5 ([VdB08]). *(a) If $\{\{-, -\}\}$ is a 2-fold bracket on V , then the associated bracket $\{-, -\}$ defined in (2.12) induces well defined maps (which we denote by the same symbol) $\{-, -\} : V/[V, V] \otimes V \rightarrow V$, given by (recalling (1.10))*

$$\{\text{tr}(a), b\} := \{a, b\} \quad (2.18)$$

and $\{-, -\} : V/[V, V] \otimes V/[V, V] \rightarrow V/[V, V]$, given by

$$\{\text{tr}(a), \text{tr}(b)\} := \text{tr}\{a, b\}. \quad (2.19)$$

(b) If V is a double Poisson algebra, then $V/[V, V]$ is a Lie algebra with the bracket defined by (2.19), and (2.18) defines a representation of this Lie algebra by derivations of V .

Proof. First, we have, using Sweedler's notation,

$$\{ab, c\} = \{\{a, c\}'b\{a, c\}'' + \{\{b, c\}'a\{b, c\}'' = \{ba, c\}. \quad (2.20)$$

Moreover, by (2.13), we have

$$\{a, bc\} - \{a, cb\} = [\{a, b\}, c] + [b, \{a, c\}] \in [V, V]. \quad (2.21)$$

Equations (2.20) and (2.21) imply that both brackets (2.18) and (2.19) are well defined, proving (a).

The skewsymmetry for the bracket $\{-, -\}$ on $V/[V, V]$ follows immediately by the skewsymmetry axiom for the 2-fold bracket $\{\{-, -\}$. By the Jacobi identity axiom (in Definition 2.1), the RHS of equation (2.14) is identically zero. Applying the multiplication map m to its LHS, we obtain the Jacobi identity for the bracket $\{-, -\}$ of $V/[V, V]$, and, at the same time, the claim that (2.18) defines a representation of the Lie algebra $V/[V, V]$ on V . Finally, this Lie algebra action is by derivations of the associative algebra V , thanks to equation (2.13). \square

2.3. Double Poisson brackets on an algebra of (non-commutative) ordinary differential functions. Consider the algebra of non-commutative polynomials $R_\ell = \mathbb{F}\langle x_1, \dots, x_\ell \rangle$, i.e. the free unital associative algebra in the indeterminates x_1, \dots, x_ℓ . Let $\frac{\partial}{\partial x_i} : R_\ell \rightarrow R_\ell \otimes R_\ell$, $i = 1, \dots, \ell$, be the linear maps defined on monomials by

$$\frac{\partial}{\partial x_i}(x_{i_1} \dots x_{i_s}) = \sum_{k=1}^s \delta_{i_k, i} x_{i_1} \dots x_{i_{k-1}} \otimes x_{i_{k+1}} \dots x_{i_s}. \quad (2.22)$$

The partial derivatives $\frac{\partial}{\partial x_i}$ are clearly 2-fold derivations of R_ℓ . Furthermore, by Proposition 1.6, $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]$ is a 3-fold derivation. Since it is zero on generators x_k , $k = 1, \dots, \ell$, we obtain

$$\frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \circ \frac{\partial}{\partial x_i}. \quad (2.23)$$

Lemma 2.6. *We have the following equality of maps $R_\ell \rightarrow R_\ell^{\otimes 3}$ ($i, j \in \mathbb{Z}_+$):*

$$\left(\frac{\partial}{\partial x_i}\right)_L \circ \frac{\partial}{\partial x_j} = \left(\frac{\partial}{\partial x_j}\right)_R \circ \frac{\partial}{\partial x_i}. \quad (2.24)$$

Proof. By Proposition 1.6(b) it follows that

$$\left(\frac{\partial}{\partial x_i}\right)_L \circ \frac{\partial}{\partial x_j} - \left(\frac{\partial}{\partial x_j}\right)_R \circ \frac{\partial}{\partial x_i}$$

is a 3-fold derivation. The claim follows since this 3-fold derivation is obviously zero on the generators x_k , $k = 1, \dots, \ell$. \square

Note that equation (2.24) implies (2.23), by the definition (1.30) of the composition of 2-fold derivations. Hence, we will say that the 2-fold derivations $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ *strongly commute*.

Definition 2.7. An algebra of ordinary differential functions is a unital associative algebra V , endowed with ℓ strongly commuting 2-fold derivations $\frac{\partial}{\partial x_i} : V \rightarrow V \otimes V$, i.e. satisfying (2.24).

An example is the algebra $R_\ell = \mathbb{F}\langle x_1, \dots, x_\ell \rangle$ of non-commutative polynomials in the variables x_1, \dots, x_ℓ , endowed with the 2-fold derivations (2.22). Other examples can be obtained as extensions of R_ℓ by localization by one or more non-zero elements.

Theorem 2.8. (a) Any 2-fold bracket on R_ℓ has the following form ($f, g \in V$)

$$\{\{f, g\}\} = \sum_{i,j=1}^{\ell} \frac{\partial g}{\partial x_j} \bullet \{\{x_i, x_j\}\} \bullet \left(\frac{\partial f}{\partial x_i}\right)^\sigma, \quad (2.25)$$

where \bullet is as in (1.1).

(b) Let V be an algebra of ordinary differential functions, and let H be an $\ell \times \ell$ matrix with entries in $V \otimes V$. Denoting its entries by $H_{ji} = \{\{x_i, x_j\}\}$, $i, j = 1, \dots, \ell$, formula (2.25) defines a 2-fold bracket on V .

(c) Equation (2.25) defines a structure of double Poisson algebra on V if and only if the skewsymmetry axiom (i) and the Jacobi identity (ii) hold on the x_i 's, namely, for all $i, j, k = 1, \dots, \ell$, we have

$$\{\{x_i, x_j\}\} = -\{\{x_j, x_i\}\}^\sigma \quad (\text{equivalently, recalling (1.7), } H^\dagger = -H), \quad (2.26)$$

and

$$\{\{x_i, \{\{x_j, x_k\}\}\}_L + \{\{x_j, \{\{x_k, x_i\}\}\}_L + \{\{x_k, \{\{x_i, x_j\}\}\}_L = 0, \quad (2.27)$$

where each term is understood, in terms of the matrix H , via (2.25), for example

$$\{\{x_i, \{\{x_j, x_k\}\}\}_L = \sum_{h=1}^{\ell} \left(\left(\frac{\partial}{\partial x_h}\right)_L H_{kj} \right) \bullet_3 H_{hi}. \quad (2.28)$$

Proof. We prove that the RHS of (2.25) satisfies the Leibniz rules (2.2) and (2.3). Using the fact that partial derivatives are 2-fold derivations, the LHS of (2.2) is

$$\sum_{i,j=1}^{\ell} \left(\frac{\partial g}{\partial x_j} h\right) \bullet \{\{x_i, x_j\}\} \bullet \left(\frac{\partial f}{\partial x_i}\right)^\sigma + \sum_{i,j=1}^{\ell} \left(g \frac{\partial h}{\partial x_j}\right) \bullet \{\{x_i, x_j\}\} \bullet \left(\frac{\partial f}{\partial x_i}\right)^\sigma. \quad (2.29)$$

By (1.16) we have that (2.29) is the same as RHS of (2.2). The Leibniz rule (2.3) is proved in the same way using again (1.16). Since any 2-fold bracket is extended uniquely from the generators to R_ℓ via the Leibniz rules, this concludes the proof of (a) and (b).

The fact that the bracket given by equation (2.25) satisfies the skewsymmetry axiom is immediate from (1.3) and the assumption that $\{\{x_i, x_j\}\} = -\{\{x_j, x_i\}\}^\sigma$ for every $i, j = 1, \dots, \ell$.

To conclude the proof of part (c) we are left to prove Jacobi identity (2.1). First, for $f \in V$ and $G \in V^{\otimes 2}$ we have, by (2.25) and the definition of the \bullet_i actions,

$$\{\{f, G\}\}_L = \sum_{i,j=1}^{\ell} \left(\left(\frac{\partial}{\partial x_j}\right)_L G \right) \bullet_3 \left(\{\{x_i, x_j\}\} \bullet \left(\frac{\partial f}{\partial x_i}\right)^\sigma \right). \quad (2.30)$$

Then, by equation (2.25), Lemma 1.10, and equation (2.30), we get

$$\begin{aligned} & \{\{f, \{\{g, h\}\}\}\}_L \\ &= \sum_{i,j,h,k=1}^{\ell} \left(\frac{\partial}{\partial x_h}\right)_L \left(\frac{\partial h}{\partial x_k}\right) \bullet_3 \left(\{\{x_i, x_h\}\} \bullet \left(\frac{\partial f}{\partial x_i}\right)^\sigma \right) \bullet_1 \left(\{\{x_j, x_k\}\} \bullet \left(\frac{\partial g}{\partial x_j}\right)^\sigma \right) \end{aligned} \quad (2.31)$$

$$+ \sum_{i,j,k=1}^{\ell} \frac{\partial h}{\partial x_k} \bullet_2 \{\{x_i, \{\{x_j, x_k\}\}\}\}_L \bullet_3 \left(\frac{\partial f}{\partial x_i}\right)^\sigma \bullet_1 \left(\frac{\partial g}{\partial x_j}\right)^\sigma \quad (2.32)$$

$$+ \sum_{i,j,h,k=1}^{\ell} \left(\frac{\partial h}{\partial x_k} \bullet \{\{x_j, x_k\}\}\right) \bullet_2 \left(\frac{\partial}{\partial x_h}\right)_L \left(\frac{\partial g}{\partial x_j}\right)^\sigma \bullet_3 \left(\{\{x_i, x_h\}\} \bullet \left(\frac{\partial f}{\partial x_i}\right)^\sigma \right). \quad (2.33)$$

Note that in the RHS above we avoided using unnecessary parentheses, in view of Lemma 1.1. From the above equation, permuting f, g and h , applying the permutation σ , and using Lemma 1.2 several times, we get

$$\begin{aligned} & \{\{g, \{h, f\}\}\}_L^\sigma \\ &= \sum_{i,j,h,k=1}^{\ell} \left(\frac{\partial h}{\partial x_k} \bullet \{\{x_k, x_i\}\}^\sigma \right) \bullet_2 \left(\left(\frac{\partial}{\partial x_h} \right)_L \left(\frac{\partial f}{\partial x_i} \right) \right)^\sigma \bullet_1 \left(\{\{x_j, x_h\}\} \bullet \left(\frac{\partial g}{\partial x_j} \right)^\sigma \right) \end{aligned} \quad (2.34)$$

$$+ \sum_{i,j,k=1}^{\ell} \frac{\partial h}{\partial x_k} \bullet_2 \{\{x_j, \{x_k, x_i\}\}\}_L^\sigma \bullet_1 \left(\frac{\partial g}{\partial x_j} \right)^\sigma \bullet_3 \left(\frac{\partial f}{\partial x_i} \right)^\sigma \quad (2.35)$$

$$+ \sum_{i,j,h,k=1}^{\ell} \left(\left(\frac{\partial}{\partial x_h} \right)_L \left(\frac{\partial h}{\partial x_k} \right) \right)^\sigma \bullet_1 \left(\{\{x_j, x_h\}\} \bullet \left(\frac{\partial g}{\partial x_j} \right)^\sigma \right) \bullet_3 \left(\{\{x_k, x_i\}\}^\sigma \bullet \left(\frac{\partial f}{\partial x_i} \right)^\sigma \right). \quad (2.36)$$

and, similarly,

$$\begin{aligned} & \{\{h, \{f, g\}\}\}_L^{\sigma^2} \\ &= \sum_{i,j,h,k=1}^{\ell} \left(\frac{\partial h}{\partial x_k} \bullet \{\{x_k, x_h\}\}^\sigma \right) \bullet_2 \left(\left(\frac{\partial}{\partial x_h} \right)_L \left(\frac{\partial g}{\partial x_j} \right) \right)^\sigma \bullet_3 \left(\{\{x_i, x_j\}\} \bullet \left(\frac{\partial f}{\partial x_i} \right)^\sigma \right) \end{aligned} \quad (2.37)$$

$$+ \sum_{i,j,k=1}^{\ell} \frac{\partial h}{\partial x_k} \bullet_2 \{\{x_k, \{x_i, x_j\}\}\}_L^{\sigma^2} \bullet_3 \left(\frac{\partial f}{\partial x_i} \right)^\sigma \bullet_1 \left(\frac{\partial g}{\partial x_j} \right)^\sigma \quad (2.38)$$

$$+ \sum_{i,j,h,k=1}^{\ell} \left(\frac{\partial h}{\partial x_k} \bullet \{\{x_k, x_h\}\}^\sigma \right) \bullet_2 \left(\left(\frac{\partial}{\partial x_h} \right)_L \left(\frac{\partial f}{\partial x_i} \right) \right)^\sigma \bullet_1 \left(\{\{x_i, x_j\}\}^\sigma \bullet \left(\frac{\partial g}{\partial x_j} \right)^\sigma \right). \quad (2.39)$$

Since, by assumption, the Jacobi identity (2.1) holds on generators, we have that (2.32), (2.35) and (2.38) sum up to zero. Note that here we are using the fact that the \bullet_1 and \bullet_3 actions commute, by Lemma 1.1(c). Furthermore, by Lemma 1.4 and assumption (2.24), we have

$$\left(\left(\frac{\partial}{\partial x_k} \right)_L \left(\frac{\partial h}{\partial x_h} \right) \right)^\sigma = \left(\frac{\partial}{\partial x_k} \right)_R \left(\frac{\partial h}{\partial x_h} \right) = \left(\frac{\partial}{\partial x_h} \right)_L \left(\frac{\partial h}{\partial x_k} \right).$$

It follows that (2.31) and (2.36) are opposite to each other by the skewsymmetry on generators. Similarly, (2.33) and (2.37) (respectively (2.34) and (2.39)) are opposite to each other. This concludes the proof. \square

Definition 2.9. The matrix $H = (H_{ij})_{i,j=1}^{\ell} \in \text{Mat}_{\ell \times \ell}(V \otimes V)$ satisfying (2.26) and (2.27) is called a *Poisson structure* on the algebra of ordinary differential functions V .

Remark 2.10. Formula (2.25) for the 2-fold bracket of $f, g \in V$ can be written in a “traditional” form as

$$\{\{f, g\}\} = (\nabla g | H \nabla f), \quad (2.40)$$

where $\nabla f = \left(\frac{\partial f}{\partial x_i} \right)_{i=1}^{\ell} \in (V \otimes V)^{\oplus \ell}$, $H = (H_{ji})_{i,j=1}^{\ell} \in \text{Mat}_{\ell \times \ell}(V \otimes V)$ is the matrix with entries $H_{ji} = \{\{x_i, x_j\}\}$, its action on $(V \otimes V)^{\oplus \ell}$ is defined by (1.6), and $(\cdot | \cdot)$ is the inner product (1.4).

Remark 2.11. One can define a *double Lie algebra* as a vector space \mathfrak{g} endowed with a double bracket $\{\{-, -\}\} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ satisfying the skewsymmetry and Jacobi identity axioms from Definition 2.1. Then, by Theorem 2.8 the tensor algebra $\mathcal{T}(\mathfrak{g})$ is a double Poisson algebra, with the 2-fold bracket given by (2.25).

2.4. Evolution ODE and Hamiltonian ODE. Let V be an algebra of ordinary differential functions. An *evolution ODE* over V is an equation of the form

$$\frac{dx_i}{dt} = P_i \in V, \quad i = 1, \dots, \ell. \quad (2.41)$$

By the chain rule, a function $f \in V$ evolves, in virtue of (2.41), according to the equation

$$\frac{df}{dt} = \sum_{i=1}^{\ell} m \left(P_i \otimes_1 \frac{\partial f}{\partial x_i} \right). \quad (2.42)$$

It is immediate to check that the time derivative $\frac{d}{dt} : V \rightarrow V$ as given by (2.42) is a derivation of V . In particular, it induces a well defined map on $V/[V, V]$. An *integral of motion* for equation (2.41) is an element $\text{tr}(h) \in V/[V, V]$ constant in time, i.e. such that

$$\frac{d \text{tr} h}{dt} = \text{tr} \sum_{i=1}^{\ell} m \left(P_i \otimes_1 \frac{\partial h}{\partial x_i} \right) = 0. \quad (2.43)$$

Given $P \in V^\ell$, we define the corresponding *evolutionary vector field* X_P as the derivation of V given by the RHS of (2.42):

$$X_P(f) = \sum_{i=1}^{\ell} m \left(P_i \otimes_1 \frac{\partial f}{\partial x_i} \right). \quad (2.44)$$

Evolutionary vector fields form the Lie algebra $\text{Vect}(V)$, with commutator given by the usual formula:

$$[X_P, X_Q] = X_{[P, Q]}, \quad \text{where } [P, Q]_i = X_P(Q_i) - X_Q(P_i), \quad i = 1, \dots, \ell. \quad (2.45)$$

This can be easily checked using condition (2.24) on V .

Equation (2.41) is called *compatible* with another evolution ODE $\frac{dx_i}{d\tau} = Q_i$, $i = 1, \dots, \ell$, if the corresponding evolutionary vector fields commute: $[X_P, X_Q] = 0$.

A *Hamiltonian ODE* on an arbitrary double Poisson algebra V , associated to the *Hamiltonian function* $\text{tr} h \in V/[V, V]$ is an equation of the form

$$\frac{du}{dt} = \{\text{tr} h, u\}, \quad u \in V. \quad (2.46)$$

An *integral of motion* for the Hamiltonian ODE (2.46) is an element $\text{tr} f \in V/[V, V]$ such that $\{\text{tr} h, \text{tr} f\} = 0$. In this case, by Proposition 2.5, the derivations $X^h = \{\text{tr} h, -\}$ and $X^f = \{\text{tr} f, -\}$ (called Hamiltonian vector fields) commute, and equations $\frac{du}{dt} = \{\text{tr} f, u\}$ and (2.46) are *compatible*.

In the special case when V is an algebra of ordinary differential functions and the 2-fold bracket $\{\{-, -\}\}$ is given by a Poisson structure H on V via (2.25), where $\{\{x_i, x_j\}\} = H_{ji}$, the Hamiltonian ODE (2.46) becomes the following evolution ODE

$$\frac{dx_i}{dt} = m \sum_{j=1}^{\ell} H_{ij} \cdot \left(\frac{\partial h}{\partial x_j} \right)^\sigma. \quad (2.47)$$

Furthermore the Lie algebra bracket $\{-, -\}$ on $V/[V, V]$ associated to the double Poisson bracket (2.25), defined by Proposition 2.5, is:

$$\{\mathrm{tr}(f), \mathrm{tr}(g)\} = \mathrm{tr} \left(m \sum_{i,j=1}^{\ell} \frac{\partial g}{\partial x_j} \bullet H_{ji} \bullet \left(\frac{\partial f}{\partial x_i} \right)^{\sigma} \right), \quad f, g \in V. \quad (2.48)$$

In this case the notions of compatibility and of integrals of motion are consistent with those for general evolution ODEs, due to Theorem 2.8(c).

Remark 2.12. Recalling the definition (1.12) of the action of the matrix $H \in \mathrm{Mat}_{\ell \times \ell}(V \otimes V)$ on $V^{\oplus \ell}$ formula (2.47) can be written in the “traditional” form

$$\frac{dx}{dt} = Hdh, \quad (2.49)$$

and similarly formula (2.48) can be written as (see (2.40))

$$\{\mathrm{tr}(f), \mathrm{tr}(g)\} = (dg|Hdf), \quad (2.50)$$

where $(\cdot|\cdot)$ is the inner product (1.11) on $V^{\oplus \ell}$, and $df = m(\nabla f)^{\sigma}$ (cf. (1.9)). The latter notation is compatible with the theory of the reduced de Rham complex, cf. (2.70).

2.5. Lenard-Magri scheme for Hamiltonian ODE. Let $H \in \mathrm{Mat}_{\ell \times \ell}(V \otimes V)$. We define a bilinear form associated to H , $\langle -, - \rangle_H : V^{\oplus \ell} \otimes V^{\oplus \ell} \rightarrow V/[V, V]$ as follows

$$\langle F, G \rangle_H = (F|HG), \quad F, G \in V^{\oplus \ell}. \quad (2.51)$$

Here we are using notation (1.11) and (1.12). Recalling (1.8), we obtain that the form (2.51) is symmetric (resp. skewsymmetric) if H is selfadjoint (resp. skewadjoint) (see (1.7)).

Lemma 2.13. *Let $H_0, H_1 \in \mathrm{Mat}_{\ell \times \ell}(V \otimes V)$ be skewadjoint, and let $\{F_n\}_{n=0}^N \subset V^{\oplus \ell}$ be such that*

$$H_0 F^{n+1} = H_1 F^n \quad \text{for } n = 0, \dots, N-1.$$

Then, the following orthogonality relations hold:

$$\langle F^n, F^m \rangle_{H_0} = \langle F^n, F^m \rangle_{H_1} = 0 \quad \text{for } n, m = 0, \dots, N.$$

Proof. Same proof as in [Mag78] or Lemma 2.6 in [BDSK09]. \square

Corollary 2.14. *Let $\{\{-, -\}_0\}$ and $\{\{-, -\}_1\}$ be two skewsymmetric 2-fold brackets on an algebra of ordinary differential functions V given by the equation (2.25) for $H_0, H_1 \in \mathrm{Mat}_{\ell \times \ell}(V \otimes V)$ respectively. Let $\{h_n\}_{n=0}^N \subset V$ be such that*

$$\{\mathrm{tr}(h_n), u\}_0 = \{\mathrm{tr}(h_{n+1}), u\}_1 \quad \text{for } u \in V, n = 0, \dots, N-1. \quad (2.52)$$

Then we have

$$\{\mathrm{tr}(h_n), \mathrm{tr}(h_m)\}_0 = \{\mathrm{tr}(h_n), \mathrm{tr}(h_m)\}_1 = 0 \quad \text{for } n, m = 0, \dots, N.$$

(See (2.12), (2.18) and (2.19) for notation).

Proof. It follows from Lemma 2.13 and equation (2.50). \square

Remark 2.15. The analogues of Propositions 2.9 and 2.10 in [BDSK09] still hold in the present setup with the same proof.

Theorem 2.16. *Let $V = R_{\ell}$ be the algebra of non-commutative polynomials in ℓ variables. Let H_0, H_1 be compatible Poisson structures on V , and assume that K is a non-degenerate matrix. Let $h_0, h_1 \in V$ and $F \in V^{\ell}$ be such that*

$$H_1 dh_0 = H_0 dh_1, \quad H_1 dF = H_0 F, \quad (2.53)$$

where, as before, $dh = m(\nabla h)^\sigma$, and we are using the notation (1.12). Then, there exists $h_2 \in V$ such that $F = dh_2$.

Proof. First, one checks that the Jacobian of F_2 (defined after equation (0.22)), is self-adjoint, i.e. $J_F = J_F^\dagger$. The proof of this claim is long but straightforward, and the argument is the same as in the commutative case, cf. e.g. [Olv93, Lemma 7.25]. The proof uses non-degeneracy of the bilinear form (1.11), which follows from the well-known property of the algebra $V = R_\ell$ that for $a \in V$, $\text{tr } aV = 0$ implies $a = 0$. Then, one uses the exactness of the reduced de Rham complex, proved in Theorem 2.18 below, to conclude that the 1-form F is exact. \square

Using this theorem and Remark 2.15, one shows that the Lenard-Magri scheme works for $V = R_\ell$, i.e. the Lenard-Magri recursive relation in Corollary 2.14 for $N = 1$ can be extended to $N = \infty$, provided that the orthogonality condition, similar to that in [BDSK09, Proposition 2.9], holds.

2.6. Examples. It is claimed in [VdB08] that the only non-zero double Poisson brackets on $R_1 = \mathbb{F}[x]$ (up to automorphisms of R_1) are given by

$$\{\{x, x\}\} = x \otimes 1 - 1 \otimes x \quad \text{and} \quad \{\{x, x\}\} = x^2 \otimes x - x \otimes x^2.$$

The corresponding Lie algebra brackets on $R_1/[R_1, R_1]$ given by (2.48) are trivial. Hence, we do not get non-trivial Hamiltonian equations.

The simplest double Poisson bracket on $R_2 = \mathbb{F}\langle x, y \rangle$ has the following form:

$$\{\{x, x\}\} = p \otimes q - q \otimes p \quad \text{where } p, q \in R_2, \quad \{\{x, y\}\} = \{\{y, y\}\} = 0. \quad (2.54)$$

The double Poisson bracket given by (2.25) and (2.54) is obviously skewsymmetric and it satisfies Jacobi identity if and only if equation (2.27) holds for $x_i = x_j = x_k = x$ (see Theorem 2.8). Using equation (2.28), this condition can be rewritten as follows (using Sweedler's notation)

$$\begin{aligned} & \left(\frac{\partial p}{\partial x}\right)' p \otimes q \left(\frac{\partial p}{\partial x}\right)'' \otimes q - \left(\frac{\partial p}{\partial x}\right)' q \otimes p \left(\frac{\partial p}{\partial x}\right)'' \otimes q - \left(\frac{\partial q}{\partial x}\right)' p \otimes q \left(\frac{\partial q}{\partial x}\right)'' \otimes p \\ & + \left(\frac{\partial q}{\partial x}\right)' q \otimes p \left(\frac{\partial q}{\partial x}\right)'' \otimes p + \text{cyclic permutations} = 0. \end{aligned} \quad (2.55)$$

We found the following three types of solutions:

- (a) $p = p(y), q = q(y)$;
- (b) $p = x, q = q(y)$;
- (c) $p = x, q = xf(y)x$.

In fact, one can show that, up to automorphisms of R_2 , these are all solutions.

Replacing x by $x + 1$ in (b) we get the following compatible double Poisson structures (where y is central for both structures):

$$\{\{x, x\}\}_0 = 1 \otimes y - y \otimes 1 \quad \text{and} \quad \{\{x, x\}\}_1 = x \otimes y - y \otimes x.$$

Let

$$h_0 = 1 \quad \text{and} \quad h_n = \frac{1}{n}(x + y)^n \quad \text{for every integer } n \geq 1. \quad (2.56)$$

We have the following recursion relations

$$\{\text{tr}(h_0), x\}_0 = 0 \quad \text{and} \quad \{\text{tr}(h_n), x\}_1 = \{\text{tr}(h_{n+1}), x\}_0, \quad n \in \mathbb{Z}_+. \quad (2.57)$$

which can be proved using equation (2.48) and the fact that for $n \geq 1$

$$\frac{\partial h_n}{\partial x} = \frac{1}{n} \sum_{k=0}^{n-1} (x + y)^k \otimes (x + y)^{n-1-k}.$$

Hence, by Corollary 2.14, we get that all $\text{tr}(h_n)$ are in involution with respect to both brackets $\{-, -\}_{0,1}$ on $R_2/[R_2, R_2]$, given by equation (2.48). Furthermore all the $\text{tr}(h_n)$ are integrals of motion for the corresponding compatible Hamiltonian ODEs, which are computed using (2.47):

$$\frac{dx}{dt_n} = x(x+y)^n y - y(x+y)^n x, \quad \frac{dy}{dt_n} = 0, \quad n \in \mathbb{Z}_+. \quad (2.58)$$

Note that the algebra R_2 is \mathbb{Z}_+^2 -graded in the obvious way. The homogeneous components of the h_n 's are still in involution with respect to both brackets and the corresponding Hamiltonian vector fields still commute. Thus, for $n \in \mathbb{Z}_+$, equation (2.58) produces $n+1$ compatible Hamiltonian ODEs, and they are compatible for all n . For example, for $n=0$, equation (2.58) is

$$\frac{dx}{dt_0} = xy - yx, \quad \frac{dy}{dt_0} = 0.$$

The homogeneous components of equation (2.58) for $n=1$ produce the following two Hamiltonian ODEs:

$$\begin{aligned} \frac{dx}{dt_1} &= x^2 y - yx^2, & \frac{dy}{dt_1} &= 0, \\ \frac{dx}{dt_{\bar{1}}} &= xy^2 - y^2 x, & \frac{dy}{dt_{\bar{1}}} &= 0. \end{aligned}$$

As explained in [MS00] the first of the above equations is a higher symmetry of the *Euler equation* and they find infinitely many its conserved densities. For $n=2$, the homogeneous (2,2)-component of equation (2.58) produces the following equation which also was found in [MS00]:

$$\frac{dx}{dt_0} = x^2 y^2 + xyxy - yxyx - y^2 x^2, \quad \frac{dy}{dt_0} = 0.$$

Furthermore, let us consider on R_2 the following double Poisson structure (of type (c)):

$$\{\{x, x\}\} = x \otimes xyx - xyx \otimes x, \quad \{\{x, y\}\} = \{\{y, y\}\} = 0.$$

Let $h_0 = 1$ and

$$h_n = \frac{x^n}{n}, \quad \tilde{h}_n = x^n y, \quad \bar{h}_n = x(yx)^n \quad \text{for every integer } n \geq 1. \quad (2.59)$$

Using the fact that

$$\begin{aligned} \frac{\partial h_n}{\partial x} &= \frac{1}{n} \sum_{k=0}^{n-1} x^k \otimes x^{n-1-k}, & \frac{\partial \tilde{h}_n}{\partial x} &= \sum_{k=0}^{n-1} x^k \otimes x^{n-1-k} y, \\ \frac{\partial \bar{h}_n}{\partial x} &= 1 \otimes (yx)^n + \sum_{k=0}^{n-1} x(yx)^k y \otimes (yx)^{n-1-k}. \end{aligned}$$

and equation (2.48), it is straightforward to show that

$$\{\text{tr}(X_n), \text{tr}(Y_m)\} = 0 \quad \text{for every integers } n, m \geq 1,$$

where X and Y can be h , \tilde{h} or \bar{h} . Hence, we get the following integrable hierarchy of Hamiltonian ODEs:

$$\begin{aligned} \frac{dx}{dt_n} &= x^{n+1} yx - xyx^{n+1}, & \frac{dy}{dt_n} &= 0, \quad n \geq 1, \\ \frac{dx}{dt_{\tilde{n}}} &= \sum_{k=0}^{\tilde{n}-2} x^{\tilde{n}-k} yx^{k+1} yx - xyx^{k+1} yx^{\tilde{n}-k}, & \frac{dy}{dt_{\tilde{n}}} &= 0, \quad \tilde{n} \geq 1, \\ \frac{dx}{dt_{\bar{n}}} &= x(xy)^{\bar{n}+1} x - x(yx)^{\bar{n}+1} x, & \frac{dy}{dt_{\bar{n}}} &= 0, \quad \bar{n} \geq 1. \end{aligned} \quad (2.60)$$

The first equation of the hierarchy, corresponding to $n = 1$ (and the Hamiltonian density $h_1 = x$), is

$$\frac{dx}{dt_1} = x^2 y x - x y x^2, \quad \frac{dy}{dt_1} = 0.$$

This equation already appeared in [MS00], where its higher symmetries of degree $n + 2$ in the variable x and degree 1 in the variable y were found (which correspond to the first line in equation (2.60)).

2.7. Basic de Rham complex over an algebra of ordinary differential functions. Let V be an algebra of ordinary differential functions. We define the *basic de Rham complex* $\tilde{\Omega}(V)$ as the free product of the algebra V and the algebra $\mathbb{F}\langle dx_1, \dots, dx_\ell \rangle$ of non-commutative polynomials in ℓ variables dx_1, \dots, dx_ℓ . It is a \mathbb{Z}_+ -graded unital associative algebra, where $f \in V$ has degree 0 and the dx_i 's have degree 1. We consider it as a superalgebra, with superstructure compatible with the \mathbb{Z}_+ -grading. The subspace $\tilde{\Omega}^n(V)$ of degree n consists of linear combinations of elements of the form

$$\tilde{\omega} = f_1 dx_{i_1} f_2 dx_{i_2} \dots f_n dx_{i_n} f_{n+1}, \quad \text{where } f_1, \dots, f_{n+1} \in V. \quad (2.61)$$

In particular, $\tilde{\Omega}^0(V) = V$ and $\tilde{\Omega}^1(V) \simeq \bigoplus_{i=1}^{\ell} V dx_i V$.

Define the de Rham differential d on $\tilde{\Omega}(V)$ as the odd derivation of degree 1 on the superalgebra $\tilde{\Omega}(V)$ by letting

$$df = \sum_{i=1}^{\ell} \left(\frac{\partial f}{\partial x_i} \right)' dx_i \left(\frac{\partial f}{\partial x_i} \right)'' \in \tilde{\Omega}^1(V) \quad \text{for } f \in \tilde{\Omega}^0(V), \quad d(dx_i) = 0. \quad (2.62)$$

In order to check that d is a differential on $\tilde{\Omega}(V)$, i.e. $d^2 = 0$, it suffices to check that $d^2 f = 0$ for $f \in V$. Using Sweedler's notation and equation (2.62) we have

$$\begin{aligned} d^2 f &= \sum_{i,j=1}^{\ell} \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right)' \right)' dx_j \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right)'' \right)'' dx_i \left(\frac{\partial f}{\partial x_i} \right)'' \\ &\quad - \sum_{i,j=1}^{\ell} \left(\frac{\partial f}{\partial x_i} \right)' dx_i \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right)'' \right)' dx_j \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right)'' \right)'' \\ &= \sum_{i,j=1}^{\ell} m \left(dx_j \star_1 \left(\left(\frac{\partial}{\partial x_j} \right)_L \left(\frac{\partial f}{\partial x_i} \right) \right) \star_1 dx_i \right) \\ &\quad - \sum_{i,j=1}^{\ell} m \left(dx_i \star_1 \left(\left(\frac{\partial}{\partial x_j} \right)_R \left(\frac{\partial f}{\partial x_i} \right) \right) \star_1 dx_j \right) = 0. \end{aligned}$$

The last equality follows by condition (2.24) on V . So we can consider the corresponding cohomology complex $(\tilde{\Omega}(V), d)$.

Given an evolutionary vector field $X_P \in \text{Vect}(V)$ (cf. (2.44)), we define the corresponding *Lie derivative* $L_P : \tilde{\Omega}(V) \rightarrow \tilde{\Omega}(V)$ as the even derivation of degree 0 extending X_P from V , and such that $L_P(dx_i) = dP_i$, $i = 1, \dots, \ell$. In other words, if $\tilde{\omega}$ is as in (2.61), then, recalling (2.62), we have

$$\begin{aligned} L_P(\tilde{\omega}) &= \sum_{s=1}^{n+1} f_1 dx_{i_1} f_2 \dots dx_{i_{s-1}} X_P(f_s) dx_{i_s} \dots dx_{i_n} f_{n+1} \\ &\quad + \sum_{s=1}^n \sum_{i=1}^{\ell} f_1 dx_{i_1} f_2 \dots f_s \left(\frac{\partial P_{i_s}}{\partial x_i} \right)' dx_i \left(\frac{\partial P_{i_s}}{\partial x_i} \right)'' f_{s+1} \dots dx_{i_n} f_{n+1}. \end{aligned} \quad (2.63)$$

Next, we define the corresponding *contraction operator* $\iota_P : \tilde{\Omega}(V) \rightarrow \tilde{\Omega}(V)$ as the odd derivation of degree -1 defined on generators by $\iota_P(f) = 0$, for $f \in V$, and

$\iota_P(dx_i) = P_i$. In other words, if $\tilde{\omega}$ is as in (2.61), then

$$\iota_X(\tilde{\omega}) = \sum_{s=1}^n (-1)^{s+1} f_1 dx_{i_1} f_2 \dots dx_{i_{s-1}} f_s P_{i_s} f_{s+1} dx_{i_{s+1}} f_{s+2} \dots dx_{i_n} f_{n+1}. \quad (2.64)$$

Proposition 2.17. *The Lie derivatives (2.63) and the contraction operators (2.64) define a structure of a $\text{Vect}(V)$ -complex on $\tilde{\Omega}(V)$. In other words the following relations hold for $P, Q \in V^\ell$ (recalling (2.45)):*

- (a) $[\iota_P, \iota_Q] = 0$;
- (b) $[L_P, \iota_Q] = \iota_{[P, Q]}$;
- (c) $[L_P, L_Q] = L_{[P, Q]}$;
- (d) $L_P = [\iota_P, d] (= \iota_P d + d \iota_P)$ (Cartan's formula).

Proof. Both sides in each formula are derivations of the superalgebra $\tilde{\Omega}(V)$, and they are obviously equal on generators dx_i and $f \in V$. \square

Theorem 2.18. *The complex $(\tilde{\Omega}(R_\ell), d)$ is acyclic, i.e. $H^n(\tilde{\Omega}(R_\ell), d) = \delta_{n,0}\mathbb{F}$.*

Proof. Let $\Delta = (x_i)_{i=1}^\ell \in R_\ell^\ell$ (it corresponds to the degree derivation X_Δ of R_ℓ). Then L_Δ is a diagonalizable operator on $\tilde{\Omega}(R_\ell)$, with eigenvalues given by the total degree in the x_i 's and the dx_i 's. Moreover, $h = \iota_\Delta$ is a homotopy operator for the complex $(\tilde{\Omega}(R_\ell), d)$: by Cartan's formula we have $h(d\tilde{\omega}) + d(h\tilde{\omega}) = k\tilde{\omega}$, for every $\tilde{\omega} \in \tilde{\Omega}(R_\ell)$ of total degree k . In particular, if $k \neq 0$ and $d\tilde{\omega} = 0$, then $\tilde{\omega} = \frac{1}{k}d(h\tilde{\omega}) \in d\tilde{\Omega}(R_\ell)$. The claim follows. \square

Remark 2.19. Let $\Delta_i = (\delta_{ij}x_j)_{j=1}^\ell \in V^\ell$. Consider the filtration $V_0 \subset V_1 \subset \dots \subset V_\ell = V$, where

$$V_i = \left\{ f \in V \mid \frac{\partial f}{\partial x_j} = 0 \text{ for all } j > i \right\},$$

and extend it to a filtration of $\tilde{\Omega}(V)$, letting $\tilde{\Omega}_i^n(V)$ contain elements of the form (2.61), with $f_1, \dots, f_{n+1} \in V_i$ and $i_1, \dots, i_n \leq i$. Then, L_{Δ_i} preserves the space $\tilde{\Omega}_i^k(V)$, and $\iota_{\Delta_i}(\tilde{\Omega}_i^k(V)) \subset \tilde{\Omega}_i^{k-1}(V)$. Assume that $L_{\Delta_i} : \tilde{\Omega}_i^k(V) \rightarrow \tilde{\Omega}_i^k(V)$ is surjective and its kernel is $\tilde{\Omega}_{i-1}^k(V)$. This is true, for example, for the algebra R_ℓ of non-commutative polynomials in ℓ variables. Then, it is easy to check that the operator $h_i = L_{\Delta_i}^{-1} \circ \iota_{\Delta_i} : \tilde{\Omega}_i^k(V) \rightarrow \tilde{\Omega}_i^{k-1}(V)$, defined up to elements of $\tilde{\Omega}_{i-1}^{k-1}(V)$, is a local homotopy operator, in the sense that

$$(d \circ h_i + h_i \circ d)(\tilde{\omega}) - \tilde{\omega} \in \tilde{\Omega}_{i-1}^k(V) \text{ for every } \tilde{\omega} \in \tilde{\Omega}_i^k(V).$$

As a consequence, under the above assumptions the basic de Rham complex is acyclic.

2.8. Reduced de Rham complex. Obviously the subspace $[\tilde{\Omega}(V), \tilde{\Omega}(V)] \subset \tilde{\Omega}(V)$ is compatible with the \mathbb{Z}_+ -grading and it is preserved by d . Hence, we can consider the *reduced de Rham complex*

$$\Omega(V) = \tilde{\Omega}(V)/[\tilde{\Omega}(V), \tilde{\Omega}(V)] = \bigoplus_{n \in \mathbb{Z}_+} \Omega^n(V), \quad (2.65)$$

with the induced action of d . Since Lie derivatives L_P and contraction operators ι_P , for $P \in V^\ell$, are derivations of the superalgebra $\tilde{\Omega}(V)$, they preserve the commutator subspace $[\tilde{\Omega}(V), \tilde{\Omega}(V)]$. Hence, they induce well defined maps on the reduced de Rham complex $\Omega(V)$ satisfying all relations of Proposition 2.17.

It follows by the same argument as in the proof of Theorem 2.18 that the complex $(\Omega(R_\ell), d)$ is acyclic as well:

$$H^n(\Omega(R_\ell), d) = \delta_{n,0}\mathbb{F}. \quad (2.66)$$

We obviously have $\Omega^0(V) = V/[V, V]$. Elements of $\Omega^1(V)$ can be understood as 1-forms $\omega = \sum_{i=1}^{\ell} f_i dx_i$, and elements of $\Omega^n(V)$ as sort of cyclically skewsymmetric “products” of 1-forms. We provide now an alternative explicit description of the spaces $\Omega^n(V)$ for $n \geq 1$ as follows. We identify the space $\Omega^n(V)$ with the space $\Sigma^n(V)$ of arrays $(A_{i_1 \dots i_n})_{i_1, \dots, i_n=1}^{\ell}$ with entries $A_{i_1 \dots i_n} \in V^{\otimes n}$, satisfying the following skewsymmetry condition:

$$A_{i_1 \dots i_n} = -(-1)^n (A_{i_2 \dots i_n i_1})^{\sigma} \quad \text{for all } i_1, \dots, i_n \in \{1, \dots, \ell\}, \quad (2.67)$$

where σ denotes the action of the cyclic permutation on $V^{\otimes n}$ as in (1.25). To prove the isomorphism $\Omega^n(V) \simeq \Sigma^n(V)$ we write explicitly the maps in both directions. Given the coset $\omega = [\tilde{\omega}] \in \Omega^n(V)$, with $\tilde{\omega}$ as in (2.61), we map it to the array (recall (1.26) and (1.27)) $(A_{i_1 \dots i_n})_{i_1, \dots, i_n=1}^{\ell} \in \Sigma^n(V)$, with entries $A_{j_1 \dots j_n} = 0$, unless (j_1, \dots, j_n) is a cyclic permutation of (i_1, \dots, i_n) , and

$$A_{j_1 \dots j_n} = \frac{1}{n} (-1)^{s(n-s)} a^{s+1} \otimes \dots \otimes a^n \otimes a^{n+1} a^1 \otimes a^2 \otimes \dots \otimes a^s, \quad (2.68)$$

for $(j_1, \dots, j_n) = (i_{\sigma^s(1)}, \dots, i_{\sigma^s(n)})$. The inverse map $\Sigma^n(V) \rightarrow \Omega^n(V)$ is given by (in Sweedler’s notation):

$$(A_{i_1 \dots i_n})_{i_1, \dots, i_n=1}^{\ell} \mapsto \sum_{i_1, \dots, i_n=1}^{\ell} [A'_{i_1 \dots i_n} dx_{i_1} A''_{i_1 \dots i_n} dx_{i_2} \dots A'_{i_1 \dots i_n} dx_{i_n}]. \quad (2.69)$$

It is easy to check that the maps (2.68) and (2.69) are well defined, and that they are inverse to each other, thus proving that the space $\Omega^n(V)$ and the space of arrays $\Sigma^n(V)$ can be identified using these maps.

It is also not hard to find the formula for the differential $d : \Sigma^n(V) \rightarrow \Sigma^{n+1}(V)$ corresponding to the differential d of the reduced complex $\Omega(V)$ under this identification. For $n = 0$, we have (recalling (1.10))

$$d(\text{tr}(f)) = \left(m \left(\frac{\partial f}{\partial x_i} \right)^{\sigma} \right)_{i=1}^{\ell}. \quad (2.70)$$

For $A = (A_{i_1 \dots i_n})_{i_1, \dots, i_n=1}^{\ell} \in \Sigma^n(V)$, where $n \geq 1$, we have, using notation (1.22) and recalling (1.26) and (1.27),

$$(dA)_{i_1 \dots i_{n+1}} = \frac{1}{n+1} \sum_{s=1}^{n+1} \sum_{t=1}^n (-1)^{sn+t-1} \left(\left(\frac{\partial}{\partial x_{i_{\sigma^s(t)}}} \right)_{(t)} A_{i_{\sigma^s(1)} \dots i_{\sigma^s(n+1)}} \right)^{\sigma^s}, \quad (2.71)$$

where \dots means that we skip the index $i_{\sigma^s(t)}$. Thanks to Lemma 1.4, we can rewrite formula (2.71) as follows

$$\begin{aligned} & (dA)_{i_1 \dots i_{n+1}} \\ &= \frac{n}{n+1} \left(\sum_{s=1}^n (-1)^{s-1} \left(\frac{\partial}{\partial x_{i_s}} \right)_{(s)} A_{i_1 \dots i_{n+1}} + (-1)^k \left(\left(\frac{\partial}{\partial x_{i_{n+1}}} \right)_{(1)} A_{i_1 \dots i_n} \right)^{\sigma^n} \right). \end{aligned} \quad (2.72)$$

In particular, for $n = 1$, we have, for $F = (F_j)_{j=1}^{\ell} \in V^{\oplus \ell} = \Sigma^1(V)$,

$$(dF)_{ij} = \frac{1}{2} \left(\frac{\partial F_j}{\partial x_i} - \left(\frac{\partial F_i}{\partial x_j} \right)^{\sigma} \right). \quad (2.73)$$

For $n = 2$, let $A = (A_{ij})_{i,j=1}^{\ell} \in \Sigma^2(V)$, namely $A_{ij} \in V \otimes V$ and $(A_{ji})^{\sigma} = -A_{ij}$. We have

$$(dA)_{ijk} = \frac{2}{3} \left(\left(\frac{\partial}{\partial x_i} \right)_L A_{jk} - \left(\frac{\partial}{\partial x_j} \right)_R A_{ik} + \left(\left(\frac{\partial}{\partial x_k} \right)_L A_{ij} \right)^{\sigma^2} \right). \quad (2.74)$$

As an application of (2.66), we get the following

- Corollary 2.20.** (a) A 0-form $\text{tr}(f) \in \Omega^0(R_\ell)$ is closed if and only if $f \in \mathbb{F} + [R_\ell, R_\ell]$.
(b) A 1-form $F = (F_i)_{i=1}^\ell \in R_\ell^{\oplus \ell} = \Sigma^1(R_\ell)$ is closed if and only if there exists $\text{tr}(f) \in R_\ell/[R_\ell, R_\ell]$ such that $F_i = \text{m} \left(\frac{\partial f}{\partial x_i} \right)^\sigma$ for every $i = 1, \dots, \ell$.
(c) A 2-form $\alpha = (A_{ij})_{i,j=1}^\ell \in \Sigma^2(R_\ell)$ is closed if and only if there exists $F = (F_i)_{i=1}^\ell \in R_\ell^{\oplus \ell}$ such that

$$A_{ij} = \frac{1}{2} \left(\frac{\partial F_j}{\partial x_i} - \left(\frac{\partial F_i}{\partial x_j} \right)^\sigma \right),$$

for every $i, j = 1, \dots, \ell$.

Remark 2.21. For $F \in V^{\oplus \ell} = \Sigma^1(V)$, define the *Jacobian*

$$J_F = \left(\frac{\partial F_i}{\partial x_j} \right)_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}(V \otimes V).$$

Then equation (2.73) becomes:

$$dF = \frac{1}{2} (J_F^t - J_F^\sigma).$$

(Here, t stands for transpose.) Therefore, recalling (1.7), we see that $dF = 0$ if and only if $J_F = J_F^\dagger$.

3. DOUBLE POISSON VERTEX ALGEBRAS AND NON-COMMUTATIVE HAMILTONIAN PDES

3.1. Definition of double Poisson vertex algebra. By a *differential algebra* we mean a unital associative (not necessarily commutative) algebra \mathcal{V} over the field \mathbb{F} , endowed with a derivation $\partial \in \text{Der } \mathcal{V}$.

Definition 3.1. An n -fold λ -bracket on \mathcal{V} is a linear map

$$\{\{-\lambda_1 - \dots - \lambda_{n-1} -\}\} : \mathcal{V}^{\otimes n} \rightarrow \mathcal{V}^{\otimes n}[\lambda_1, \dots, \lambda_{n-1}]$$

which satisfies the following sesquilinearity conditions

$$\{\{a_{1\lambda_1} \cdots \lambda_{i-1}(\partial a_i)_{\lambda_i} \cdots a_{n-1\lambda_{n-1}} a_n\}\} = -\lambda_i \{\{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}} a_n\}\}, \quad (3.1)$$

for all $i = 1, \dots, n-1$, and

$$\{\{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}}(\partial a_n)\}\} = (\lambda_1 \cdots + \lambda_{n-1} + \partial) \{\{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}} a_n\}\}, \quad (3.2)$$

and the following Leibniz rules:

$$\begin{aligned} \{\{a_{1\lambda_1} \cdots b c_{\lambda_i} \cdots a_{n-1\lambda_{n-1}} a_n\}\} &= (e^{\partial \lambda_i} b) \star_i \{\{a_{1\lambda_1} \cdots c_{\lambda_i} \cdots a_{n-1\lambda_{n-1}} a_n\}\} \\ &+ \{\{a_{1\lambda_1} \cdots b_{\lambda_i + \partial} \cdots a_{n-1\lambda_{n-1}} a_n\}\} \rightarrow \star_{n-i} c, \end{aligned} \quad (3.3)$$

for all $i = 1, \dots, n$, where $\star_n = \star_0$. The notation on the RHS of (3.3) needs some explanations. The \star_i -products were introduced in (1.34). Given a polynomial $P(\lambda_1, \dots, \lambda_{n-1}) = \sum a_1 \otimes \cdots \otimes a_n \lambda_1^{k_1} \cdots \lambda_{n-1}^{k_{n-1}} \in \mathcal{V}^{\otimes n}[\lambda_1, \dots, \lambda_{n-1}]$, we denote

$$\begin{aligned} &(e^{\partial \lambda_i} f) \star_i P(\lambda_1, \dots, \lambda_i, \dots, \lambda_{n-1}) \\ &= \sum_{j=0}^{k_i} \sum \binom{k_i}{j} a_1 \otimes \cdots \otimes a_i \otimes (\partial^j f) a_{i+1} \otimes \cdots \otimes a_n \lambda_1^{k_1} \cdots \lambda_i^{k_i-j} \cdots \lambda_{n-1}^{k_{n-1}}, \end{aligned}$$

in other words, applying $e^{\partial\lambda_i}$ amounts to replacing λ_i by $\lambda_i + \partial$, and the parentheses mean that ∂ should be applied to f . Moreover, we denote

$$\begin{aligned} & P(\lambda_1, \dots, \lambda_i + \partial, \dots, \lambda_{n-1}) \rightarrow \star_{n-i} f \\ & = \sum_{j=0}^{k_i} \binom{k_i}{j} a_1 \otimes \dots \otimes a_i (\partial^j f) \otimes \dots \otimes a_n \lambda_1^{k_1} \dots \lambda_i^{k_i-j} \dots \lambda_{n-1}^{k_{n-1}}, \end{aligned} \quad (3.4)$$

in other words, the arrow means that ∂ is applied to f .

In the special case of 2-fold λ -brackets, the sesquilinearity conditions (3.1) and (3.2) are

$$\{\{\partial a_\lambda b\}\} = -\lambda \{\{a_\lambda b\}\}, \quad \{\{a_\lambda \partial b\}\} = (\lambda + \partial) \{\{a_\lambda b\}\}, \quad (3.5)$$

and the Leibniz rules (3.3) are

$$\begin{aligned} \{\{a_\lambda b c\}\} &= \{\{a_\lambda b\}\} c + b \{\{a_\lambda c\}\} \\ \{\{a b_\lambda c\}\} &= \{\{a_{\lambda+\partial} c\}\} \rightarrow \star_1 b + (e^{\partial\lambda} a) \star_1 \{\{b_\lambda c\}\}. \end{aligned} \quad (3.6)$$

In analogy with (1.34) and (2.4), we also let, for $a, b, c \in \mathcal{V}$,

$$\begin{aligned} \{\{a_\lambda (b \otimes c)\}\}_L &:= \{\{a_\lambda b\}\} \otimes c, \quad \{\{a_\lambda (b \otimes c)\}\}_R := b \otimes \{\{a_\lambda c\}\}, \\ \{\{(a \otimes b)_\lambda c\}\}_L &:= \{\{a_{\lambda+\partial} c\}\} \rightarrow \otimes_1 b, \\ \{\{(a \otimes b)_\lambda c\}\}_R &:= (e^{\partial\lambda} a) \otimes_1 \{\{b_\lambda c\}\} \left(= \{\{b_{\lambda+\partial} c\}\} \rightarrow \otimes_1 a \right), \end{aligned} \quad (3.7)$$

Definition 3.2. A *double Poisson vertex algebra* is a differential algebra \mathcal{V} , with derivation $\partial : \mathcal{V} \rightarrow \mathcal{V}$, endowed with a 2-fold λ -bracket $\{\{-\lambda-\}\} : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}$ satisfying the following axioms:

(i) skewsymmetry:

$$\{\{a_\lambda b\}\} = -\{\{b_{-\lambda-\partial} a\}\}^\sigma, \quad (3.8)$$

where $-\lambda - \partial$ in the RHS is moved to the left, acting on the coefficients.

(ii) Jacobi identity (cf. Remark 2.2)

$$\{\{a_\lambda \{\{b_\mu c\}\}\}\}_L - \{\{b_\mu \{\{a_\lambda c\}\}\}\}_R = \{\{\{\{a_\lambda b\}\}_{\lambda+\mu} c\}\}_L. \quad (3.9)$$

Lemma 3.3. (a) *The sesquilinearity relations*

$$\begin{aligned} \{\{\partial A_\lambda B\}\}_{L(\text{resp. } R)} &= -\lambda \{\{A_\lambda B\}\}_{L(\text{resp. } R)}, \\ \{\{A_\lambda \partial B\}\}_{L(\text{resp. } R)} &= (\lambda + \partial) \{\{A_\lambda B\}\}_{L(\text{resp. } R)}, \end{aligned}$$

hold if either A or B lies in $\mathcal{V} \otimes \mathcal{V}$, and the other one lies in \mathcal{V} .

(b) For $a, b \in \mathcal{V}$ and $A, B \in \mathcal{V} \otimes \mathcal{V}$, the following skewsymmetry relations hold

$$\begin{aligned} \{\{a_\lambda B\}\}_{L(\text{resp. } R)} &= -\{\{B_{-\lambda-\partial} a\}\}_{R(\text{resp. } L)}^{\sigma(\text{resp. } \sigma^2)}, \\ \{\{A_\lambda b\}\}_{L(\text{resp. } R)} &= -\{\{b_{-\lambda-\partial} A^\sigma\}\}_{R(\text{resp. } L)}^{\sigma(\text{resp. } \sigma^2)}. \end{aligned}$$

(c) For $a, b \in \mathcal{V}$ and $A, B \in \mathcal{V} \otimes \mathcal{V}$, we have

$$\{\{a_\lambda B\}\}_L^\sigma = \{\{a_\lambda B^\sigma\}\}_R, \quad \{\{A_\lambda b\}\}_L = \{\{A^\sigma_\lambda b\}\}_R.$$

Proof. Straightforward. \square

Lemma 3.4. Let $\{\{-\lambda-\}\}$ be a 2-fold λ -bracket on \mathcal{V} satisfying the skewsymmetry axiom (3.8). Then the linear map $\{\{-\lambda-\mu-\}\} : \mathcal{V}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}[\lambda, \mu]$ defined by $(a, b, c \in \mathcal{V})$:

$$\{\{a_\lambda b_\mu c\}\} := \{\{a_\lambda \{\{b_\mu c\}\}\}\}_L - \{\{b_\mu \{\{a_\lambda c\}\}\}\}_R - \{\{\{\{a_\lambda b\}\}_{\lambda+\mu} c\}\}_L, \quad (3.10)$$

is a 3-fold λ -bracket on \mathcal{V} .

Proof. Sesquilinearity properties (3.1) and (3.2) follow easily from Lemma 3.3(a) and (3.5). Let us prove that $\{\{-\lambda - \mu -\}$ satisfies the Leibniz rule (3.3) with $i = 3$. By (3.6) and (3.7), we have

$$\begin{aligned} \{a_\lambda \{b_\mu c d\}\}_L &= \{a_\lambda \{b_\mu c\}\}_L d + c \{a_\lambda \{b_\mu d\}\}_L + \{a_\lambda c\} \{b_\mu d\}, \\ \{b_\mu \{a_\lambda c d\}\}_R &= \{b_\mu \{a_\lambda c\}\}_R d + c \{b_\mu \{a_\lambda d\}\}_R + \{a_\lambda c\} \{b_\mu d\}, \\ \{\{a_\lambda b\}_{\lambda+\mu} c d\}_L &= \{\{a_\lambda b\}_{\lambda+\mu} c\}_L d + c \{\{a_\lambda b\}_{\lambda+\mu} d\}_L. \end{aligned} \quad (3.11)$$

Hence, equation (3.3) with $i = 3$ holds.

Next we prove that

$$\{a_\lambda b_\mu c\} = \{b_\mu c_{-\lambda-\mu-\partial} a\}^\sigma. \quad (3.12)$$

Equation (3.3) with $i = 1$ and 2 then follows from (3.12) and (3.3) with $i = 3$. By Lemma 3.3(b) and (c), we have

$$\begin{aligned} \{a_\lambda \{b_\mu c\}\}_L &= \{a_\lambda \{b_\mu c\}^\sigma\}_R^\sigma = -\{\{b_\mu c\}_{-\lambda-\partial} a\}_L^\sigma, \\ \{b_\mu \{a_\lambda c\}\}_R &= \{b_\mu \{a_\lambda c\}^\sigma\}_L^\sigma = -\{b_\mu \{c_{-\lambda-\partial} a\}\}_L^\sigma, \\ \{\{a_\lambda b\}_{\lambda+\mu} c\}_L &= -\{\{b_\mu a\}_{\lambda+\mu} c\}_L^\sigma = \{c_{-\lambda-\mu-\partial} \{b_\mu a\}\}_R^\sigma. \end{aligned} \quad (3.13)$$

Combining the three equations (3.13) we get (3.12). \square

3.2. The trace map and connection to Lie conformal algebras. Given a (not necessarily commutative) differential algebra \mathcal{V} , we denote by $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ the commutator subspace. For $f \in \mathcal{V}$, we let $\text{tr}(f) \in \mathcal{V}/[\mathcal{V}, \mathcal{V}]$ be the corresponding coset. Furthermore, for $f \in \mathcal{V}$, we also let $\int f$ be the coset of $f \in \mathcal{V}$ in the quotient space $\mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$.

Given a 2-fold λ -bracket $\{\{-\lambda -\}$ on \mathcal{V} we define the following map $\{-\lambda -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$ by

$$\{a_\lambda b\} = m\{a_\lambda b\} \quad (3.14)$$

(we extend the multiplication map on $\mathcal{V} \otimes \mathcal{V}$ to a multiplication map $m : (\mathcal{V} \otimes \mathcal{V})[\lambda] \rightarrow \mathcal{V}[\lambda]$ in the obvious way), and we also define the map $\{-, -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}$ by

$$\{a, b\} = m\{a_\lambda b\}|_{\lambda=0}. \quad (3.15)$$

Lemma 3.5. *If $\{\{-\lambda -\}$ is a skewsymmetric 2-fold λ -bracket on \mathcal{V} , then the following identity holds in $\mathcal{V}^{\otimes 2}[\lambda, \mu]$ ($a, b, c \in \mathcal{V}$):*

$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} - \{\{a_\lambda b\}_{\lambda+\mu} c\} = (m \otimes 1)\{a_\lambda b_\mu c\} - (1 \otimes m)\{b_\mu a_\lambda c\}. \quad (3.16)$$

where we set $\{a_\lambda b \otimes c\} = \{a_\lambda b\} \otimes c + b \otimes \{a_\lambda c\}$.

Proof. Let us compute the three terms of the LHS of (3.16). Using (3.7), we get, after a straightforward computation,

$$\begin{aligned} \{a_\lambda \{b_\mu c\}\} &= (m \otimes 1)\{a_\lambda \{b_\mu c\}\}_L + (1 \otimes m)\{a_\lambda \{b_\mu c\}\}_R, \\ \{b_\mu \{a_\lambda c\}\} &= (1 \otimes m)\{b_\mu \{a_\lambda c\}\}_L + (m \otimes 1)\{b_\mu \{a_\lambda c\}\}_R, \\ \{\{a_\lambda b\}_{\lambda+\mu} c\} &= (m \otimes 1)\{\{a_\lambda b\}_{\lambda+\mu} c\}_L + (1 \otimes m)\{\{a_\lambda b\}_{\lambda+\mu} c\}_R. \end{aligned}$$

By skewsymmetry and Lemma 3.3(c), we can replace the last term in the RHS of the third equation by $-(1 \otimes m)\{\{b_\mu a\}_{\lambda+\mu} c\}_L$. Hence, combining the three equations above, we get (3.16). \square

Theorem 3.6. *Let \mathcal{V} be a differential algebra, with derivation ∂ , endowed with a 2-fold λ -bracket $\{\{-\lambda-\}\}$, and let $\{-\lambda-\}$ and $\{-,-\}$ be defined as in (3.14) and (3.15).*

- (a) $\partial[\mathcal{V}, \mathcal{V}] \subset [\mathcal{V}, \mathcal{V}]$. Hence, we have a well defined induced map (denoted, by abuse of notation, by the same symbol) $\partial : \mathcal{V}/[\mathcal{V}, \mathcal{V}] \rightarrow \mathcal{V}/[\mathcal{V}, \mathcal{V}]$, given by $\partial(\text{tr } f) = \text{tr}(\partial f)$.
- (b) $\{[\mathcal{V}, \mathcal{V}]_\lambda \mathcal{V}\} = 0$, and $\{\mathcal{V}_\lambda[\mathcal{V}, \mathcal{V}]\} \subset [\mathcal{V}, \mathcal{V}] \otimes \mathbb{F}[\lambda]$. Hence, we have well defined induced maps (denoted, by abuse of notation, by the same symbol)

$$\{-\lambda-\} : \mathcal{V}/[\mathcal{V}, \mathcal{V}] \times \mathcal{V} \rightarrow \mathcal{V}[\lambda],$$

and

$$\{-\lambda-\} : \mathcal{V}/[\mathcal{V}, \mathcal{V}] \times \mathcal{V}/[\mathcal{V}, \mathcal{V}] \rightarrow \mathcal{V}/[\mathcal{V}, \mathcal{V}][\lambda],$$

given, respectively, by

$$\{\text{tr}(f)_\lambda g\} = \text{m}\{\{f_\lambda g\}\}, \quad (3.17)$$

and

$$\{\text{tr}(f)_\lambda \text{tr}(g)\} = \text{tr}(\text{m}\{\{f_\lambda g\}\}). \quad (3.18)$$

- (c) If the 2-fold λ -bracket is skewsymmetric, then so is the λ -bracket (3.18):

$$\{\text{tr}(f)_\lambda \text{tr}(g)\} = -\{\text{tr}(g)_{-\lambda-\partial} \text{tr}(f)\}.$$

- (d) If the 2-fold λ -bracket defines a structure of a double Poisson vertex algebra on \mathcal{V} , then the λ -bracket (3.18) satisfies the Jacobi identity ($f, g, h \in \mathcal{V}$)

$$\{\text{tr}(f)_\lambda \{\text{tr}(g)_\mu \text{tr}(h)\}\} - \{\text{tr}(g)_\mu \{\text{tr}(f)_\lambda \text{tr}(h)\}\} = \{\{\text{tr}(f)_\lambda \text{tr}(g)\}_{\lambda+\mu} \text{tr}(h)\},$$

thus defining a structure of a Lie conformal algebra on $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$. Furthermore, the λ -action of $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$ on \mathcal{V} , given by (3.17), defines a representation of the Lie conformal algebra $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$ given by conformal derivations of \mathcal{V} .

- (e) $\{[\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}, \mathcal{V}\} = 0$, and $\{\mathcal{V}, [\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}\} \subset ([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}) \otimes \mathbb{F}[\lambda]$. Hence, we have well defined induced brackets (denoted, by abuse of notation, by the same symbol)

$$\{-,-\} : \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}) \times \mathcal{V} \rightarrow \mathcal{V},$$

and

$$\{-,-\} : \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}) \times \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}) \rightarrow \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V}),$$

given, respectively, by

$$\{\int f, g\} := \text{m}\{\{f_\lambda g\}\}|_{\lambda=0}, \quad (3.19)$$

and

$$\{\int f, \int g\} := \int \text{m}\{\{f_\lambda g\}\}|_{\lambda=0}. \quad (3.20)$$

- (f) If the 2-fold λ -bracket is skewsymmetric, then so is the bracket (3.20).
- (g) If the 2-fold λ -bracket defines a structure of a double Poisson vertex algebra on \mathcal{V} , then the bracket (3.20) defines a structure of a Lie algebra on $\mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$. Furthermore, the action of $\mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$ on \mathcal{V} , given by (3.19), defines a representation of the Lie algebra $\mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$ by derivations of \mathcal{V} commuting with ∂ .

Proof. Part (a) is clear, since ∂ is a derivation of the associative product of \mathcal{V} . By the second Leibniz rule (3.6) we have, for $a, b, c \in \mathcal{V}$,

$$\{ab_\lambda c\} = \text{m}\{\{ab_\lambda c\}\} = \text{m}\left(\left(e^{\partial\lambda} a\right) \otimes_1 \{\{b_\lambda c\}\} + \{\{a_{\lambda+\partial} c\}\} \otimes_1 b\right).$$

Note that the expression in parenthesis in the RHS above is unchanged if we switch a and b (cf. the last identity in (3.7)). Hence, $\{ab_\lambda c\} = \{ba_\lambda c\}$. Furthermore, we have

$$\{a_\lambda bc\} = m\{\{a_\lambda bc\}\} = m(\{\{a_\lambda b\}c + b\{\{a_\lambda c\}\}) = \{a_\lambda b\}c + b\{a_\lambda c\}.$$

Namely, the λ -action $\{a_\lambda -\}$ is by derivations of the associative product of \mathcal{V} . But then

$$\{a_\lambda bc - cb\} = [b, \{a_\lambda c\}] + [\{a_\lambda b\}, c] \in [\mathcal{V}, \mathcal{V}] \otimes \mathbb{F}[\lambda].$$

This proves part (b). Part (c) is immediate. Part (d) is an immediate consequence of Lemma 3.5. Finally, parts (e), (f) and (g) can be proved in the same way, or by using the usual construction that associates to a Lie conformal algebra R the corresponding Lie algebra $R/\partial R$ and its representation on R . \square

3.3. Double Poisson vertex algebra structure on an algebra of (non-commutative) differential functions. Consider the algebra of non-commutative differential polynomials \mathcal{R}_ℓ in ℓ variables u_i , $i \in I = \{1, \dots, \ell\}$. It is the algebra of non-commutative polynomials in the indeterminates $u_i^{(n)}$,

$$\mathcal{R}_\ell = \mathbb{F}\langle u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+ \rangle,$$

endowed with a derivation ∂ , defined on generators by $\partial u_i^{(n)} = u_i^{(n+1)}$, and partial derivatives $\frac{\partial}{\partial u_i^{(n)}}$, for every $i \in I$ and $n \in \mathbb{Z}_+$, defined on monomials by

$$\frac{\partial}{\partial u_i^{(n)}}(u_{i_1}^{(n_1)} \dots u_{i_s}^{(n_s)}) = \sum_{k=1}^s \delta_{i_k, i} \delta_{n_k, n} u_{i_1}^{(n_1)} \dots x_{i_{k-1}}^{(n_{k-1})} \otimes u_{i_{k+1}}^{(n_{k+1})} \dots u_{i_s}^{(n_s)}, \quad (3.21)$$

which are commuting 2-fold derivations of \mathcal{R}_ℓ such that

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}, \quad (3.22)$$

where the RHS is zero for $n = 0$.

Lemma 3.7. *For any non-commutative differential polynomial $f \in \mathcal{R}_\ell$, and $i, j \in I$ and $n, m \in \mathbb{Z}_+$, the partial derivatives strongly commute, i.e. we have*

$$\left(\frac{\partial}{\partial u_i^{(m)}} \right)_L \frac{\partial f}{\partial u_j^{(n)}} = \left(\frac{\partial}{\partial u_j^{(n)}} \right)_R \frac{\partial f}{\partial u_i^{(m)}}.$$

Proof. Same as the proof of Lemma 2.6. \square

Definition 3.8. An algebra of differential functions in ℓ variables is a unital associative differential algebra \mathcal{V} , with derivation ∂ , endowed with strongly commuting 2-fold derivations $\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}$, $i \in I = \{1, \dots, \ell\}$, $n \in \mathbb{Z}_+$, such that (3.22)

holds and, for every $f \in \mathcal{V}$, we have $\frac{\partial f}{\partial u_i^{(n)}} = 0$ for all but finitely many choices of indices $(i, n) \in I \times \mathbb{Z}_+$.

An example of such an algebra is the algebra \mathcal{R}_ℓ , endowed with the 2-fold derivations defined in (3.21), or its localization by non-zero elements.

Lemma 3.9. *Let $\{\{-\lambda-\}\}$ be a 2-fold λ -bracket on a differential algebra \mathcal{V} . For every $a \in \mathcal{V}$ and $B, C \in \mathcal{V}^{\otimes 2}$, we have*

$$\begin{aligned} \{a_\lambda B \bullet C\}_L &= B \bullet_2 \{a_\lambda C\}_L + \{a_\lambda B\}_L \bullet_1 C, \\ \{a_\lambda B \bullet C\}_R &= B \bullet_2 \{a_\lambda C\}_R + \{a_\lambda B\}_R \bullet_3 C, \\ \{B \bullet C_\lambda a\}_L &= \{B_{\lambda+\partial} a\}_L \bullet_3 C + \{C_{\lambda+\partial} a\}_L \bullet_1 B^\sigma. \end{aligned} \quad (3.23)$$

In the last equation the arrow has a meaning similar to (3.4) (with \bullet product in place of \star product).

Proof. Straightforward. \square

Theorem 3.10. (a) Any 2-fold λ -bracket on \mathcal{R}_ℓ has the form ($f, g \in \mathcal{R}_\ell$):

$$\{\{f\lambda g\}\} = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} \bullet (\lambda + \partial)^n \{\{u_{i\lambda+\partial} u_j\}\} \rightarrow (-\lambda - \partial)^m \bullet \left(\frac{\partial f}{\partial u_i^{(m)}} \right)^\sigma. \quad (3.24)$$

where \bullet is as in (1.1).

- (b) Let \mathcal{V} be an algebra of differential functions in ℓ variables. Let $H(\lambda)$ be an $\ell \times \ell$ matrix with entries in $(\mathcal{V} \otimes \mathcal{V})[\lambda]$. We denote its entries by $H_{ij}(\lambda) = \{\{u_{j\lambda} u_i\}\}$, $i, j \in I$. Then formula (3.24) defines a 2-fold λ -bracket on \mathcal{V} .
- (c) Equation (3.24) defines a structure of double Poisson vertex algebra on \mathcal{V} if and only if the skewsymmetry axiom (i) and the Jacobi identity (ii) hold on the u_i 's, namely, for all $i, j, k \in I$, we have

$$\{\{u_{i\lambda} u_j\}\} = -\{\{u_{j-\lambda-\partial} u_i\}\}^\sigma, \quad (3.25)$$

and

$$\{\{u_{i\lambda} \{\{u_{j\mu} u_k\}\}\}\}_L - \{\{u_{j\mu} \{\{u_{i\lambda} u_k\}\}\}\}_R = \{\{\{u_{i\lambda} u_j\}\}_{\lambda+\mu} u_k\}\}_L, \quad (3.26)$$

where each term is understood, in terms of the matrix H , via (2.25), for example

$$\{\{u_{i\lambda} \{\{u_{j\mu} u_k\}\}\}\}_L = \sum_{i \in I, n \in \mathbb{Z}_+} \left(\left(\frac{\partial}{\partial u_h^{(n)}} \right)_L H_{kj}(\mu) \right) \bullet_3 (\lambda + \partial)^n H_{hi}(\lambda). \quad (3.27)$$

Proof. The proof is similar to the proof of Theorem 2.8 using Lemma 3.9. \square

Remark 3.11. For a matrix differential operator $H(\partial) = (H_{ij}(\partial))_{i,j=1}^\ell$, with entries $H_{ij}(\partial) \in (\mathcal{V} \otimes \mathcal{V})[\partial]$, we define its adjoint operator $H^\dagger(\partial)$, with entries $(H^\dagger)_{ij}(\partial) = ((H_{ji})^*(\partial))^\sigma$, where $A^*(\partial)$ denotes the formal adjoint of the differential operator $A(\partial)$. Then property (3.25) means that the matrix differential operator $H(\partial)$ is skewadjoint.

Definition 3.12. A matrix differential operator $H(\partial) = (H_{ij}(\partial))_{i,j=1}^\ell$, with entries $H_{ij}(\partial) \in (\mathcal{V} \otimes \mathcal{V})[\partial]$, satisfying (3.25) and (3.26) is called a *Poisson structure* on the algebra of differential functions \mathcal{V} .

3.4. Evolution PDE and Hamiltonian PDE over an algebra of differential functions. An *evolution PDE* over the algebra of differential functions \mathcal{V} has the form

$$\frac{du_i}{dt} = P_i \in \mathcal{V}, \quad i \in I = \{1, \dots, \ell\}. \quad (3.28)$$

Assuming that time derivative commutes with space derivative, we have $\frac{du_i^{(n)}}{dt} = \partial^n P_i$, and, by the chain rule, a function $f \in \mathcal{V}$ evolves according to

$$\frac{df}{dt} = \sum_{(i,n) \in I \times \mathbb{Z}_+} m \left((\partial^n P_i) \otimes_1 \frac{\partial f}{\partial u_i^{(n)}} \right) = X_P(f). \quad (3.29)$$

An *integral of motion* is a local functional $\int f \in \mathcal{V}/\partial\mathcal{V} + [\mathcal{V}, \mathcal{V}]$ constant in time:

$$\frac{d\int f}{dt} = \int \sum_{(i,n) \in I \times \mathbb{Z}_+} m \left((\partial^n P_i) \otimes_1 \frac{\partial f}{\partial u_i^{(n)}} \right) = 0. \quad (3.30)$$

A *vector field* on \mathcal{V} is a derivation $X : \mathcal{V} \rightarrow \mathcal{V}$ of the form (cf. (2.44))

$$X(f) = \sum_{(i,n) \in I \times \mathbb{Z}_+} m \left(P_{i,n} \otimes_1 \frac{\partial f}{\partial u_i^{(n)}} \right), \quad (3.31)$$

where $P_{i,n} \in \mathcal{V}$ for all i, n . Note that the RHS of (3.31) is a finite sum since, by assumptions on \mathcal{V} , $\frac{\partial f}{\partial u_i^{(n)}} = 0$ for all but finitely many choices of indices (i, n) .

An *evolutionary vector field* is a vector field commuting with ∂ . By (3.22), it has the form (cf. (3.29))

$$X_P(f) = \sum_{(i,n) \in I \times \mathbb{Z}_+} \mathfrak{m} \left((\partial^n P_i) \otimes_1 \frac{\partial f}{\partial u_i^{(n)}} \right), \quad (3.32)$$

for $P = (P_i)_{i=1}^\ell \in \mathcal{V}^\ell$, called the *characteristics* of the evolutionary vector field X_P .

Vector fields form a Lie algebra, denoted by $\text{Vect}(\mathcal{V})$, and evolutionary vector fields form a Lie subalgebra, denoted $\text{Vect}^\partial(\mathcal{V})$. The Lie bracket is given by the same formulas as in the finite case, equation (2.45).

Equation (3.28) is called *compatible* with another evolution PDE $\frac{du_i}{d\tau} = Q_i$, $i \in I$, if the corresponding evolutionary vector fields commute.

The *Hamiltonian equation* on a double Poisson vertex algebra \mathcal{V} , associated to the *Hamiltonian functional* $\int h \in \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$ is

$$\frac{du}{dt} = \{ \int h, u \}, \quad (3.33)$$

for any $u \in \mathcal{V}$. An *integral of motion* for the Hamiltonian equation (2.46) is an element $\int f \in \mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$ such that $\{ \int h, \int f \} = 0$. In this case, by Theorem 3.6(g), the derivations $X^h = \{ \int h, - \}$ and $X^f = \{ \int f, - \}$ (called Hamiltonian vector fields) commute, and equations $\frac{du}{dt} = \{ \text{tr } f, u \}$ and (3.33) are *compatible*.

In the special case when \mathcal{V} is an algebra of differential functions and the 2-fold λ -bracket $\{\{-\lambda-\}\}$ is given by a Poisson structure $H(\partial)$ on \mathcal{V} via (3.24), where $\{\{u_i \lambda u_j\}\} = H_{ji}(\lambda)$, the Hamiltonian equation (3.33) becomes the following evolution equation

$$\frac{du_i}{dt} = \mathfrak{m} \sum_{j=1}^\ell H_{ij}(\partial) \bullet \left(\frac{\delta h}{\delta u_j} \right)^\sigma, \quad (3.34)$$

where $\frac{\delta h}{\delta u_j} \in \mathcal{V} \otimes \mathcal{V}$ denotes the *variational derivative* of h :

$$\frac{\delta h}{\delta u_j} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial h}{\partial u_j^{(n)}}. \quad (3.35)$$

In equation (3.34) ∂ is moved to the right of the \bullet product, acting on $\frac{\delta h}{\delta u_j}$. Furthermore the Lie algebra bracket $\{-, -\}$ on $\mathcal{V}/([\mathcal{V}, \mathcal{V}] + \partial\mathcal{V})$ defined by (3.20), becomes $(f, g \in \mathcal{V})$:

$$\{ \int f, \int g \} = \int \sum_{i,j \in I} \mathfrak{m} \left(\frac{\delta g}{\delta u_j} \right)^\sigma \mathfrak{m} \left(H_{ji}(\partial) \star_1 \mathfrak{m} \left(\frac{\delta f}{\delta u_i} \right)^\sigma \right). \quad (3.36)$$

In this case the notions of compatibility and of integrals of motion, are consistent with those for general evolution equations, due to Theorem 3.10.

Remark 3.13. For $H(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\partial]$ and $F \in \mathcal{V}^{\oplus \ell}$ let $H(\partial)F \in \mathcal{V}^\ell$ be defined by (where we write $H_{ij}(\partial) = \sum_{n \in \mathbb{Z}_+} (H'_{ij,n} \otimes H''_{ij,n}) \partial^n$)

$$(H(\partial)F)_i = \sum_{j \in I} \mathfrak{m}(H_{ij}(\partial) \star_1 F_j) = \sum_{j \in I, n \in \mathbb{Z}_+} H'_{ij,n} (\partial^n F_j) H''_{ij,n}. \quad (3.37)$$

Then, formula (3.36) can be written in the more traditional form

$$\{ \int f, \int g \} = \int \delta g \cdot (H(\partial) \delta f), \quad (3.38)$$

where \cdot denotes the usual dot product of vectors and $(\delta f)_i = m \left(\frac{\delta f}{\delta u_i} \right)^\sigma$. The latter notation is compatible with the theory of the variational complex (see equation (3.46)).

Remark 3.14. The theory of the Lenard-Magri scheme discussed in Section 2.5 for double Poisson algebras V holds verbatim for double Poisson vertex algebras \mathcal{V} if we replace the matrix H by the matrix differential operator $H(\partial)$ and the map tr by the map \int .

3.5. de Rham complex over an algebra of differential functions. Let \mathcal{V} be an algebra differential functions. We define the de Rham complex $\tilde{\Omega}(\mathcal{V})$ as the free product of the algebra \mathcal{V} and the algebra $\mathbb{F}\langle \delta u_i^{(n)} \mid i \in I = \{1, \dots, \ell\}, n \in \mathbb{Z}_+ \rangle$ of non-commutative polynomials in the variables $\delta u_i^{(n)}$. The action of ∂ is extended from \mathcal{V} to a derivation of $\tilde{\Omega}(\mathcal{V})$ by letting $\partial(\delta u_i^{(n)}) = \delta u_i^{(n+1)}$ for all $(i, n) \in I \times \mathbb{Z}_+$. This makes $\tilde{\Omega}(\mathcal{V})$ a differential algebra. It has a \mathbb{Z}_+ -grading, such that $f \in \mathcal{V}$ has degree 0 and the $\delta u_i^{(n)}$'s have degree 1. We consider it as a superalgebra, with superstructure compatible with the \mathbb{Z}_+ -grading. The subspace $\tilde{\Omega}^k(\mathcal{V})$ of degree n consists of linear combinations of elements of the form

$$\tilde{\omega} = f_1 \delta u_{i_1}^{(m_1)} f_2 \delta u_{i_2}^{(m_2)} \dots f_k \delta u_{i_k}^{(m_k)} f_{k+1}, \text{ where } f_1, \dots, f_{k+1} \in \mathcal{V}. \quad (3.39)$$

In particular, $\tilde{\Omega}^0(\mathcal{V}) = \mathcal{V}$ and $\tilde{\Omega}^1(\mathcal{V}) \simeq \bigoplus_{(i,n) \in I \times \mathbb{Z}_+} \mathcal{V} \delta u_i^{(n)}$.

Define the de Rham differential δ on $\tilde{\Omega}(\mathcal{V})$ as the odd derivation of degree 1 on the superalgebra $\tilde{\Omega}(\mathcal{V})$ by letting

$$\delta f = \sum_{(i,n) \in I \times \mathbb{Z}_+} \left(\frac{\partial f}{\partial u_i^{(n)}} \right)' \delta u_i^{(n)} \left(\frac{\partial f}{\partial u_i^{(n)}} \right)'' \in \tilde{\Omega}^1(\mathcal{V}) \text{ for } f \in \mathcal{V}, \text{ and } \delta(\delta u_i^{(n)}) = 0. \quad (3.40)$$

The proof that $\delta^2 = 0$ is the same as for the algebra of ordinary differential functions (cf. Section 2.7), so we can consider the corresponding cohomology complex $(\tilde{\Omega}(\mathcal{V}), \delta)$.

Given a vector field $X_P = \sum_{(i,n) \in I \times \mathbb{Z}_+} m \circ \left(P_{i,n} \otimes_1 \frac{\partial}{\partial u_i^{(n)}} \right) \in \text{Vect}(\mathcal{V})$ (cf. (3.31)), we define the corresponding *Lie derivative* $L_P : \tilde{\Omega}(\mathcal{V}) \rightarrow \tilde{\Omega}(\mathcal{V})$ as the even derivation of degree 0 extending X_P from \mathcal{V} , and such that $L_P(\delta u_i^{(n)}) = \delta P_{i,n}$, $i \in I$, $n \in \mathbb{Z}_+$. Next, we define the corresponding *contraction operator* $\iota_P : \tilde{\Omega}(\mathcal{V}) \rightarrow \tilde{\Omega}(\mathcal{V})$ as the odd derivation of degree -1 defined on generators by $\iota_P(f) = 0$, for $f \in \mathcal{V}$, and $\iota_P(\delta u_i^{(n)}) = P_{i,n}$. It is easy to show, in analogy to Proposition 2.17 and Theorem 2.18, that $\tilde{\Omega}(\mathcal{V})$ is a $\text{Vect}(\mathcal{V})$ -complex and the complex $(\tilde{\Omega}(\mathcal{R}_\ell), \delta)$ is acyclic: $H^n(\tilde{\Omega}(\mathcal{R}_\ell), \delta) = \delta_{n,0} \mathbb{F}$.

3.6. Variational complex over an algebra of differential functions.

Proposition 3.15. *In the de Rham complex $(\tilde{\Omega}(\mathcal{V}), \delta)$ we have:*

- (a) *The commutator subspace $[\tilde{\Omega}(\mathcal{V}), \tilde{\Omega}(\mathcal{V})]$ is compatible with the \mathbb{Z}_+ -grading and is preserved by δ .*
- (b) *δ and ∂ commute: $\partial \circ \delta = \delta \circ \partial$; hence, $\partial \tilde{\Omega}(\mathcal{V})$ is compatible with the \mathbb{Z}_+ -grading and is preserved by δ .*
- (c) *The Lie derivative L_P and the contraction operator ι_P , associated to an evolutionary vector field X_P of characteristics $P = (P_i)_{i=1}^\ell$ (cf. (3.32)), commute with the action of ∂ on $\tilde{\Omega}(\mathcal{V})$.*

Proof. Part (a) is immediate, since δ is a derivation of the associative product in $\tilde{\Omega}(\mathcal{V})$. For part (b) we just need to prove that $\delta(\partial\tilde{\omega}) = \partial(\delta\tilde{\omega})$ for every $\tilde{\omega} \in \tilde{\Omega}(\mathcal{V})$. Since $[\delta, \partial]$ is a derivation of $\tilde{\Omega}(\mathcal{V})$, and it is obviously zero on $\delta u_i^{(n)}$, it suffices to check that it is zero on $f \in \mathcal{V}$. We have, by (3.40),

$$\begin{aligned} \partial(\delta f) &= \sum_{(i,n) \in I \times \mathbb{Z}_+} \left(\frac{\partial f}{\partial u_i^{(n)}} \right)' \delta u_i^{(n)} \left(\frac{\partial f}{\partial u_i^{(n)}} \right)'' \\ &+ \sum_{(i,n) \in I \times \mathbb{Z}_+} \left(\frac{\partial f}{\partial u_i^{(n)}} \right)' \delta u_i^{(n+1)} \left(\frac{\partial f}{\partial u_i^{(n)}} \right)'' , \end{aligned}$$

and

$$\delta(\partial f) = \sum_{(i,n) \in I \times \mathbb{Z}_+} \left(\frac{\partial(\partial f)}{\partial u_i^{(n)}} \right)' \delta u_i^{(n)} \left(\frac{\partial(\partial f)}{\partial u_i^{(n)}} \right)'' .$$

Hence, $\partial(\delta f) = \delta(\partial f)$ by (3.22).

As for part (c), since L_P , ι_P and ∂ are derivations of the associative product of $\tilde{\Omega}(\mathcal{V})$, it suffices to check that $[L_P, \partial]$ and $[\iota_P, \partial]$ act as zero on generators, i.e. on $f \in \mathcal{V}$ and on $\delta u_i^{(n)}$. This is straightforward. \square

Thanks to Proposition 3.15(a-b), we can consider the \mathbb{Z}_+ -graded *variational complex*

$$\Omega(\mathcal{V}) = \tilde{\Omega}(\mathcal{V}) / (\partial\tilde{\Omega}(\mathcal{V}) + [\tilde{\Omega}(\mathcal{V}), \tilde{\Omega}(\mathcal{V})]) = \bigoplus_{n \in \mathbb{Z}_+} \Omega^n(\mathcal{V}) , \quad (3.41)$$

with the induced action of δ . Furthermore, by Proposition 3.15(c) the Lie derivatives L_P and contraction operators ι_P , associated to the evolutionary vector field X_P of characteristics $P \in \mathcal{V}^\ell$, induce well defined maps on the variational complex $\Omega(\mathcal{V})$, and they define a structure of $\text{Vect}^\partial(\mathcal{V})$ -complex on $\tilde{\Omega}(\mathcal{V})$.

Since the total degree vector field X_Δ , with characteristics $\Delta = (u_i)_{i=1}^\ell$, is an evolutionary vector field on \mathcal{R}_ℓ , the same argument in the proof of Theorem 2.18 shows that the complex $(\Omega(\mathcal{R}_\ell), \delta)$ is acyclic, i.e.

$$H^k(\Omega(\mathcal{R}_\ell), \delta) = \delta_{k,0} \mathbb{F} . \quad (3.42)$$

As we did in the finite case, Section 2.8, we give an explicit description of the complex $(\Omega(\mathcal{V}), \delta)$. We obviously have $\Omega^0(\mathcal{V}) = \mathcal{V}/(\partial\mathcal{V} + [\mathcal{V}, \mathcal{V}])$. For $k \geq 1$, we identify $\Omega^k(\mathcal{V})$ with the space $\Sigma^k(\mathcal{V})$ of arrays $(A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}))_{i_1, \dots, i_k=1}^\ell$ with entries $A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}) \in \mathcal{V}^{\otimes k}[\lambda_1, \dots, \lambda_{k-1}]$, satisfying the following skewadjointness condition ($i_1, \dots, i_k \in I$):

$$A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}) = -(-1)^k (A_{i_2 \dots i_k i_1}(\lambda_2, \dots, \lambda_{k-1}, -\lambda_1 - \dots - \lambda_{k-1} - \overleftarrow{\partial}))^\sigma , \quad (3.43)$$

where σ denotes the action of the cyclic permutation on $\mathcal{V}^{\otimes k}$ as in (1.25), and the arrow means that ∂ is acting on the coefficients of the polynomial. To prove the isomorphism $\Omega^k(\mathcal{V}) \simeq \Sigma^k(\mathcal{V})$ we write explicitly the maps in both directions. Given the coset $\omega = [\tilde{\omega}] \in \Omega^k(\mathcal{V})$, where $\tilde{\omega}$ is as in (3.39), we map it to the array $A = (A_{j_1 \dots j_k}(\lambda_1, \dots, \lambda_{k-1}))_{i_1, \dots, i_k=1}^\ell \in \Sigma^k(\mathcal{V})$, with entries $A_{j_1 \dots j_k}(\lambda_1, \dots, \lambda_{k-1}) = 0$ unless (j_1, \dots, j_k) is a cyclic permutation of (i_1, \dots, i_k) , and

$$\begin{aligned} A_{j_1 \dots j_k}(\lambda_1, \dots, \lambda_{k-1}) &= \frac{1}{k} (-1)^{s(k-s)} \lambda_1^{n_{s+1}} \dots \lambda_{k-s}^{n_k} \lambda_{k-s+1}^{n_1} \dots \lambda_{k-1}^{n_{s-1}} \\ &(-\lambda_1 - \dots - \lambda_{k-1} - \partial)^{n_s} (f_{s+1} \otimes \dots \otimes f_k \otimes f_{k+1} f_1 \otimes f_2 \otimes \dots \otimes f_s) , \end{aligned} \quad (3.44)$$

for $(j_1, \dots, j_k) = (i_{\sigma^s(1)}, \dots, i_{\sigma^s(k)})$. The inverse map $\Sigma^k(V) \rightarrow \Omega^k(V)$ is given by (in Sweedler's notation):

$$\begin{aligned} & \left(\sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}_+} A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}} \lambda_1^{n_1} \dots \lambda_{k-1}^{n_{k-1}} \right)_{i_1, \dots, i_k=1}^\ell \mapsto \\ & \sum_{\substack{i_1, \dots, i_k \in I \\ n_1, \dots, n_{k-1} \in \mathbb{Z}_+}} [(A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}})' \delta u_{i_1}^{(n_1)} \dots (A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}})' \dots \delta u_{i_{k-1}}^{(n_{k-1})} (A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}})' \dots \delta u_{i_k}^{(n_k)}]. \end{aligned} \quad (3.45)$$

It is easy to check that the maps (3.44) and (3.45) are well defined, and they are inverse to each other, thus proving that the space $\Omega^k(\mathcal{V})$ and the space of arrays $\Sigma^k(\mathcal{V})$ can be identified using these maps.

It is also not hard to find the formula for the differential $\delta : \Sigma^k(\mathcal{V}) \rightarrow \Sigma^{k+1}(\mathcal{V})$ corresponding to the differential δ of the reduced complex $\Omega(\mathcal{V})$ under this identification. For $k = 0$, we have

$$\delta(ff) = \left(\sum_{n \in \mathbb{Z}_+} (-\partial)^n m \left(\frac{\partial f}{\partial u_i^{(n)}} \right)^\sigma \right)_{i=1}^\ell \left(= \left(m \left(\frac{\delta f}{\delta u_i} \right)^\sigma \right)_{i=1}^\ell \right). \quad (3.46)$$

For $A = (A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}))_{i_1, \dots, i_k=1}^\ell \in \Sigma^k(\mathcal{V})$, where $k \geq 1$, we have, using notation (1.22) and recalling (1.26) and (1.27),

$$\begin{aligned} (\delta A)_{i_1 \dots i_{k+1}}(\lambda_1, \dots, \lambda_k) &= \frac{1}{k+1} \sum_{s=1}^{k+1} \sum_{t=1}^k (-1)^{sk+t-1} \times \\ & \times \sum_{n \in \mathbb{Z}_+} \left(\left(\frac{\partial}{\partial u_{i_{\sigma^s(t)}}^{(n)}} \right)_{(t)} A_{i_{\sigma^s(1)} \dots i_{\sigma^s(k+1)}} \overset{t}{\dots} (\lambda_{\sigma^s(1)} \dots \lambda_{\sigma^s(k)}) \lambda_{\sigma^s(t)}^n \right)_{\sigma^s}^{\sigma^s}, \end{aligned} \quad (3.47)$$

where $\overset{t}{\dots}$ means that we skip the index $i_{\sigma^s(t)}$ and, as before, we need to replace λ_{k+1} , when it occurs, by $-\lambda_1 - \dots - \lambda_k - \partial$, with ∂ acting on the coefficients (here σ denotes the action of the cyclic element $(1, 2, \dots, k+1)$). We can use Lemma 1.4 to rewrite formula (3.47) as follows

$$\begin{aligned} & (\delta A)_{i_1 \dots i_{k+1}}(\lambda_1, \dots, \lambda_k) \\ &= \frac{k}{k+1} \sum_{n \in \mathbb{Z}_+} \left(\sum_{s=1}^k (-1)^{s+1} \left(\frac{\partial}{\partial u_{i_s}^{(n)}} \right)_{(s)} A_{i_1 \dots i_{k+1}} \overset{s}{\dots} (\lambda_1, \dots, \lambda_k) \lambda_s^n \right. \\ & \left. + (-1)^k (-\lambda_1 - \dots - \lambda_k - \partial)^n \left(\left(\frac{\partial}{\partial u_{i_{k+1}}^{(n)}} \right)_{(1)} A_{i_1, \dots, i_k}(\lambda_1, \dots, \lambda_{k-1}) \right)_{\sigma^k}^{\sigma^k} \right). \end{aligned} \quad (3.48)$$

In particular, for $k = 1$, we have, for $F = (F_j)_{j=1}^\ell \in \mathcal{V}^\ell = \Sigma^1(\mathcal{V})$,

$$(\delta F)_{ij}(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}_+} \left(\frac{\partial F_j}{\partial u_i^{(n)}} \lambda^n - (-\lambda - \partial)^n \left(\frac{\partial F_i}{\partial u_j^{(n)}} \right)^\sigma \right). \quad (3.49)$$

For $k = 2$, let $A = (A_{ij}(\lambda))_{i,j=1}^\ell \in \Sigma^2(\mathcal{V})$, namely $A_{ij}(\lambda) \in \mathcal{V}^{\otimes 2}[\lambda]$ and $(A_{ji}(-\lambda - \partial))^\sigma = -A_{ij}(\lambda)$. We have

$$\begin{aligned} (\delta A)_{ijk}(\lambda, \mu) &= \frac{2}{3} \sum_{n \in \mathbb{Z}_+} \left(\left(\frac{\partial}{\partial u_i^{(n)}} \right)_L A_{jk}(\mu) \lambda^n \right. \\ & \left. - \left(\frac{\partial}{\partial u_j^{(n)}} \right)_R A_{ik}(\lambda) \mu^n + (-\lambda - \mu - \partial)^n \left(\left(\frac{\partial}{\partial u_k^{(n)}} \right)_L A_{ij}(\lambda) \right)_{\sigma^2}^{\sigma^2} \right). \end{aligned} \quad (3.50)$$

Remark 3.16. In analogy with [BDSK09], we introduce the *Beltrami 2-fold λ -bracket* $\{\{-\lambda-\}\}^B$ using equation (3.24) where we let $\{\{u_i \lambda u_j\}\}^B = \delta_{ij}(1 \otimes 1)$, for $i, j \in I$ (note that this 2-fold λ -bracket is commutative and does not define a double Poisson vertex algebra structure on \mathcal{V}). Then equation (3.50) can be rewritten as follows

$$(\delta A)_{ijk}(\lambda, \mu) = \frac{2}{3} \left(\{\{u_i \lambda A_{jk}(\mu)\}\}_L^B - \{\{u_j \mu A_{ik}(\lambda)\}\}_R^B + \{\{A_{ij}(\lambda)_{\lambda+\mu} u_k\}\}_L^B \right).$$

More generally, generalizing notation (3.7), we set ($s = 1, \dots, k$ and $a, b_i \in \mathcal{V}$)

$$\{\{a_\lambda b_1 \otimes \dots \otimes b_k\}\}_{(s)}^B = b_1 \otimes \dots \otimes b_{s-1} \otimes \{\{a_\lambda b_s\}\}^B \otimes b_{s+1} \otimes \dots \otimes b_k,$$

and

$$\{\{b_1 \otimes \dots \otimes b_k \lambda a\}\}_{(1)}^B = \{\{b_{1\lambda+\partial} a\}\}_{\rightarrow}^B \otimes_1 (b_2 \otimes \dots \otimes b_k).$$

Then we can rewrite equation (3.48) as follows:

$$\begin{aligned} (\delta A)_{i_1 \dots i_{k+1}}(\lambda_1, \dots, \lambda_k) &= \frac{k}{k+1} \left(\sum_{s=1}^k (-1)^{s+1} \{\{u_{i_s} \lambda_s A_{i_1 \dots i_{k+1}}^s(\lambda_1, \dots, \lambda_k)\}\}_{(s)}^B \right. \\ &\quad \left. + (-1)^k \{\{A_{i_1, \dots, i_k}(\lambda_1, \dots, \lambda_{k-1})_{\lambda_1 + \dots + \lambda_k} u_{i_{k+1}}\}\}_{(1)}^B \right). \end{aligned}$$

As an application of (3.42), we get the following

- Corollary 3.17.** (a) A 0-form $\int f \in \Omega^0(\mathcal{R}_\ell)$ is closed if and only if $f \in \mathbb{F} + [\mathcal{R}_\ell, \mathcal{R}_\ell] + \partial \mathcal{R}_\ell$.
(b) A 1-form $F = (F_i)_{i=1}^\ell \in \mathcal{R}_\ell^{\oplus \ell} = \Sigma^1(\mathcal{R}_\ell)$ is closed if and only if there exists $\int f \in \mathcal{R}_\ell / ([\mathcal{R}_\ell, \mathcal{R}_\ell] + \partial \mathcal{R}_\ell)$ such that $F_i = m \left(\frac{\delta f}{\delta u_i} \right)^\sigma$ for every $i = 1, \dots, \ell$.
(c) A 2-form $\alpha = (A_{ij}(\lambda))_{i,j=1}^\ell \in \Sigma^2(\mathcal{R}_\ell)$ is closed if and only if there exists $F = (F_i)_{i=1}^\ell \in \mathcal{R}_\ell^{\oplus \ell}$ such that

$$A_{ij}(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}_+} \left(\frac{\partial F_j}{\partial u_i^{(n)}} \lambda^n - (-\lambda - \partial)^n \left(\frac{\partial F_i}{\partial u_j^{(n)}} \right)^\sigma \right),$$

for every $i, j = 1, \dots, \ell$.

Remark 3.18. For $F \in \mathcal{V}^{\oplus \ell} = \Sigma^1(\mathcal{V})$, define the *Frechet derivative*

$$D_F(\lambda) = \left(\sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \lambda^n \right)_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\lambda].$$

Therefore, recalling Remark 3.11, we see that $\delta F = 0$ if and only if $D_F(\partial)$ is selfadjoint.

Note also that, using the Beltrami 2-fold λ -bracket introduced in Remark 3.16 we have

$$D_F(\lambda)_{ij} = \{\{u_j \lambda F_i\}\}^B.$$

Furthermore, for $f \in \mathcal{V}$, we have ($i \in I$)

$$\left(\frac{\delta f}{\delta u_i} \right)^\sigma = \{\{f \lambda u_i\}\}^B \Big|_{\lambda=0}.$$

3.7. Poisson vertex algebra structure on \mathcal{V}_m . In this subsection for each double PVA \mathcal{V} and a positive integer m we construct a Poisson vertex algebra \mathcal{V}_m . This construction is similar to the construction of a Poisson algebra V_m , associated to a double Poisson algebra V in [VdB08].

The construction of V_m is motivated by the following considerations. Let V be a unital finitely generated associative algebra, and let Y_m be the affine algebraic variety of all homomorphisms from V to $\text{Mat}_{m \times m} \mathbb{F}$. Let $V_m := \mathbb{F}[Y_m]$ be the algebra of polynomial functions on Y_m . We have a natural map $V \rightarrow \text{Mat}_{m \times m} V_m$, which induces the map

$$\varphi_m : V/[V, V] \rightarrow V_m.$$

It was shown in [VdB08] that if V is a double Poisson algebra with a 2-fold bracket $\{\{-, -\}\}$, then V_m is a Poisson algebra with the bracket

$$\{a_{ij}, b_{hk}\} = \{\{a, b\}'_{hj}\}\{\{a, b\}''_{ik}\}.$$

It is easy to check that φ_m is then a Lie algebra homomorphism.

Note that by the above interpretation of the algebra V_m , it follows that for $V = R_N = \mathbb{F}\langle x_1, \dots, x_N \rangle$ the algebra V_m is a polynomial algebra as well (on generators $(x_s)_{ij}$, $s = 1, \dots, N$, $i, j = 1, \dots, m$).

Given a differential algebra \mathcal{V} we define for each positive integer m a commutative differential algebra \mathcal{V}_m , as the commutative algebra generated by the symbols a_{ij} , for $a \in \mathcal{V}$ and $i, j = 1, \dots, m$, subject to the relations ($k \in \mathbb{F}$, $a, b \in \mathcal{V}$):

$$(ka)_{ij} = ka_{ij}, \quad (a+b)_{ij} = a_{ij} + b_{ij}, \quad (ab)_{ij} = \sum_{k=1}^m a_{ik}b_{kj}. \quad (3.51)$$

The derivation (which we still denote by ∂) on \mathcal{V}_m is defined by

$$\partial(a_{ij}) = (\partial a)_{ij}.$$

Example 3.19. Let $\mathcal{V} = \mathcal{R}_I = \mathbb{F}\langle u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+ \rangle$ be the algebra of non-commutative differential polynomials in the variables u_i , $i \in I$. Then, by the above discussion, for every $m \geq 1$,

$$\mathcal{V}_m = \mathbb{F}[u_{i,ab}^{(n)} \mid i \in I, a, b \in \{1, \dots, m\}, n \in \mathbb{Z}_+],$$

where $u_{i,ab}^{(n)} = (u_i^{(n)})_{ab}$, is the (commutative) algebra of differential polynomials in the variables $u_{i,ab}$.

Proposition 3.20. *Let \mathcal{V} be a differential algebra endowed with a 2-fold λ -bracket $\{\{-\lambda-\}\}$, written as $\{\{a_\lambda b\}\} = \sum_{n \in \mathbb{Z}_+} ((a_n b)' \otimes (a_n b)'') \lambda^n$, for all $a, b \in \mathcal{V}$. Then, for every $m \geq 1$, we have a well-defined λ -bracket on \mathcal{V}_m , given by $(a_{ij}, b_{hk} \in \mathcal{V}_m)$*

$$\{a_{ij\lambda} b_{hk}\} = \sum_{n \in \mathbb{Z}_+} (a_n b)'_{hj} (a_n b)''_{ik} \lambda^n, \quad (3.52)$$

and extended to a bilinear map $\{\cdot_\lambda \cdot\} : \mathcal{V}_m \otimes \mathcal{V}_m \rightarrow \mathcal{V}_m[\lambda]$ by sesquilinearity (0.1) and the left and right Leibniz rules (0.2) and (0.3). This λ -bracket is skewsymmetric (i.e., it satisfies (0.4)) if the 2-fold λ -bracket is.

Proof. First we need to verify that the map $\{\cdot_\lambda \cdot\} : \mathcal{V}_m \otimes \mathcal{V}_m \rightarrow \mathcal{V}_m[\lambda]$ is well defined, that is, it is compatible with the defining relations (3.51) of \mathcal{V}_m . Clearly, the RHS of (3.52) is linear in a and b since $\{\{-\lambda-\}\}$ is a linear map. Hence, we are left to show that

$$\{a_{ij\lambda}(bc)_{hk}\} = \sum_{l=1}^N \{a_{ij\lambda} b_{hl} c_{lk}\} \quad (3.53)$$

and

$$\{(bc)_{hk\lambda}a_{ij}\} = \sum_{l=1}^N \{b_{hl}c_{lk\lambda}a_{ij}\}. \quad (3.54)$$

Using the definition of the λ -bracket (3.52) and (0.2), we have

$$RHS(3.53) = \sum_{l=1}^N \sum_{n \in \mathbb{Z}_+} ((a_nb)'_{hj}(a_nb)''_{il}c_{lk} + (a_nc)'_{lj}(a_nc)''_{ik}b_{hl}) \lambda^n.$$

On the other hand, by the first formula in (3.6),

$$\{\{a_\lambda bc\}\} = b\{\{a_\lambda c\}\} + \{\{a_\lambda b\}\}c = \sum_{n \in \mathbb{Z}_+} (b(a_nc)' \otimes (a_nc)'' + (a_nb)' \otimes (a_nb)''c) \lambda^n.$$

Hence, using (3.52), we have

$$\begin{aligned} LHS(3.53) &= \sum_{n \in \mathbb{Z}_+} ((b(a_nc)'_{hj}(a_nc)''_{ik} + (a_nb)'_{hj}((a_nb)''c)_{ik}) \lambda^n \\ &= \sum_{l=1}^N \sum_{n \in \mathbb{Z}_+} (b_{hl}(a_nc)'_{lj}(a_nc)''_{ik} + (a_nb)'_{hj}(a_nb)''_{il}c_{lk}) \lambda^n, \end{aligned}$$

where in the second identity we used the third relation in (3.51). Hence, the identity (3.53) follows by the commutativity of \mathcal{V}_m . For (3.54), using the right Leibniz rule (0.3) and the definition of the λ -bracket (3.52) we have

$$RHS(3.54) = \sum_{l=1}^N \sum_{n \in \mathbb{Z}_+} ((b_na)'_{il}(b_na)''_{hj}(\lambda + \partial)^n c_{lk} + (c_na)'_{ik}(c_na)''_{lj}(\lambda + \partial)^n b_{hl}).$$

On the other hand, by the second formula in (3.6),

$$\begin{aligned} \{bc_\lambda a\} &= \{b_{\lambda+\partial}a\} \rightarrow \star_1 c + (e^{\partial \frac{d}{dx}} b) \star_1 \{c_\lambda a\} \\ &= \sum_{n \in \mathbb{Z}_+} ((b_na)'(\lambda + \partial)^n c \otimes (b_na)'' + (c_na)' \otimes ((\lambda + \partial)^n b)(c_na)''). \end{aligned}$$

Hence, using (3.52), we have

$$\begin{aligned} LHS(3.54) &= \sum_{n \in \mathbb{Z}_+} \left(((b_na)'(\lambda + \partial)^n c)_{ik} (b_na)''_{hj} + (c_na)'_{ik} (((\lambda + \partial)^n b)(c_na)''_{hj}) \right) \\ &= \sum_{n \in \mathbb{Z}_+} \left((b_na)'_{il}(\lambda + \partial)^n c_{lk} (b_na)''_{hj} + (c_na)'_{ik} ((\lambda + \partial)^n b_{hl})(c_na)''_{lj} \right), \end{aligned}$$

where in the second identity we used the third relation in (3.51). Again, the identity (3.54) follows by the commutativity of \mathcal{V}_m .

Let us prove skewsymmetry of the λ -bracket given by equation (3.52) provided that the skewsymmetry (3.8) of the 2-fold λ -bracket holds. We have to show that

$$\{a_{ij\lambda}b_{hk}\} = -\{b_{hk-\lambda-\partial}a_{ij}\}, \quad (3.55)$$

for all $a_{ij}, b_{hk} \in \mathcal{V}_m$. We can rewrite (3.8) as the following identity

$$\sum_{n \in \mathbb{Z}_+} ((a_nb)' \otimes (a_nb)'') \lambda^n = - \sum_{n \in \mathbb{Z}_+} (-\lambda - \partial)^n ((b_na)'' \otimes (b_na)').$$

Hence, we have

$$\sum_{n \in \mathbb{Z}_+} (a_nb)'_{hj}(a_nb)''_{ik} \lambda^n = - \sum_{n \in \mathbb{Z}_+} (-\lambda - \partial)^n (b_na)''_{hj}(b_na)'_{ik}, \quad (3.56)$$

for all $i, j, h, k = 1, \dots, m$. Using (3.52) and the commutativity of \mathcal{V}_m , it follows that (3.56) is equivalent to (3.55), thus concluding the proof. \square

Proposition 3.21. *Let \mathcal{V} be a differential algebra with a 2-fold λ -bracket $\{\{-\lambda-\}$. Let*

$$\{\{a_\lambda b_\mu c\}\} = \sum_{p,q \in \mathbb{Z}_+} (h'_{pq} \otimes h''_{pq} \otimes h'''_{pq}) \lambda^p \mu^q,$$

and

$$\{\{b_\mu a_\lambda c\}\} = \sum_{p,q \in \mathbb{Z}_+} (g'_{pq} \otimes g''_{pq} \otimes g'''_{pq}) \lambda^p \mu^q,$$

be the associated, by (3.10), 3-fold λ -bracket. Then, for any $m \geq 1$, we have $(a_{ij}, b_{hk}, c_{ln} \in \mathcal{V}_m)$:

$$\begin{aligned} & \{a_{ij} \lambda \{b_{hk} \mu c_{ln}\}\} - \{b_{hk} \mu \{a_{ij} \lambda c_{ln}\}\} - \{\{a_{ij} \lambda b_{hk}\}_{\lambda+\mu} c_{ln}\} \\ = & \sum_{p,q \in \mathbb{Z}_+} \left((h'_{pq})_{lj} (h''_{pq})_{ik} (h'''_{pq})_{hn} - (g'_{pq})_{lk} (g''_{pq})_{hj} (g'''_{pq})_{in} \right) \lambda^p \mu^q. \end{aligned} \quad (3.57)$$

Proof. Expanding in powers of λ and μ equation (3.10) using (3.7), we get

$$\begin{aligned} \{\{a_\lambda b_\mu c\}\} = & \sum_{p,q \in \mathbb{Z}_+} \left((a_p(b_q c)')' \otimes (a_p(b_q c)')'' \otimes (b_q c)'' \right. \\ & \left. - (a_p c)' \otimes (b_q(a_p c)''')' \otimes (b_q(a_p c)''')'' \right) \lambda^p \mu^q \\ & - \sum_{p,q \in \mathbb{Z}_+} \left(((a_p b)'_q c)' \otimes (\lambda + \mu + \partial)^q (a_p b)'' \otimes ((a_p b)'_q c)'' \right) \lambda^p. \end{aligned}$$

Exchanging a with b and λ with μ , we get

$$\begin{aligned} \{\{b_\mu a_\lambda c\}\} = & \sum_{p,q \in \mathbb{Z}_+} \left((b_q(a_p c)')' \otimes (b_q(a_p c)')'' \otimes (a_p c)'' \right. \\ & \left. - (b_q c)' \otimes (a_p(b_q c)''')' \otimes (a_p(b_q c)''')'' \right) \lambda^p \mu^q \\ & + \sum_{p,q \in \mathbb{Z}_+} \left(((a_p b)''_q c)' \otimes (\lambda + \mu + \partial)^q (a_p b)' \otimes ((a_p b)''_q c)'' \right) \lambda^p, \end{aligned}$$

where in the last term we used skewsymmetry (3.8) and Lemma 3.3(a). From the above equations it follows that

$$\begin{aligned} RHS(3.57) = & \sum_{p,q \in \mathbb{Z}_+} \left((a_p(b_q c)')'_{lj} (a_p(b_q c)')''_{ik} (b_q c)''_{hn} \right. \\ & \left. - (a_p c)'_{lj} (b_q(a_p c)''')'_{ik} (b_q(a_p c)''')''_{hn} \right) \lambda^p \mu^q \\ & - \sum_{p,q \in \mathbb{Z}_+} \left(((a_p b)'_q c)'_{lj} ((\lambda + \mu + \partial)^q (a_p b)''_{ik}) ((a_p b)'_q c)''_{hn} \right) \lambda^p \\ & - \sum_{p,q \in \mathbb{Z}_+} \left((b_q(a_p c)')'_{lk} (b_q(a_p c)')''_{hj} (a_p c)''_{in} \right. \\ & \left. - (b_q c)'_{lk} (a_p(b_q c)''')'_{hj} (a_p(b_q c)''')''_{in} \right) \lambda^p \mu^q \\ & - \sum_{p,q \in \mathbb{Z}_+} \left(((a_p b)''_q c)'_{lk} ((\lambda + \mu + \partial)^q (a_p b)'_{hj}) ((a_p b)''_q c)''_{in} \right) \lambda^p. \end{aligned} \quad (3.58)$$

On the other hand, using (3.52) and left and right Leibniz rules (0.4) and (0.5), we get

$$\begin{aligned} \{a_{ij\lambda}\{b_{hk\mu}c_{ln}\}\} &= \sum_{p,q \in \mathbb{Z}_+} \{a_{ij\lambda}(b_q c)'_{lk}(b_q c)''_{hn}\}\mu^q \\ &= \sum_{p,q \in \mathbb{Z}_+} \left((a_p(b_q c)')'_{lj}(a_p(b_q c)''_{ik}(b_q c)''_{hn} \right. \end{aligned} \quad (3.59)$$

$$\begin{aligned} &\quad \left. - (b_q c)'_{lk}(a_p(b_q c)''_{hj}(a_p(b_q c)''_{in})) \right) \lambda^p \mu^q; \\ - \{b_{hk\mu}\{a_{ij\lambda}c_{ln}\}\} &= - \sum_{p,q \in \mathbb{Z}_+} \{b_{hk\mu}(a_p c)'_{lj}(a_p c)''_{in}\} \lambda^p \\ &= - \sum_{p,q \in \mathbb{Z}_+} \left((b_q(a_p c)')'_{lk}(b_q(a_p c)''_{hj}(a_p c)''_{in} \right. \end{aligned} \quad (3.60)$$

$$\begin{aligned} &\quad \left. + (a_p c)'_{lj}(b_q(a_p c)''_{ik}(b_q(a_p c)''_{hn})) \right) \lambda^p \mu^q; \\ - \{a_{ij\lambda}b_{hk}\}_{\lambda+\mu}c_{ln} &= - \sum_{p,q \in \mathbb{Z}_+} \{(a_p b)'_{hj}(a_p b)''_{ik}\}_{\lambda+\mu}c_{ln} \lambda^p \\ &= - \sum_{p,q \in \mathbb{Z}_+} \left(((a_p b)'_q c)'_{lj}((a_p b)'_q c)''_{hn}(\lambda + \mu + \partial)^q (a_p b)''_{ik} \right. \end{aligned} \quad (3.61)$$

$$\left. + ((a_p b)''_q c)'_{lk}((a_p b)''_q c)''_{in}(\lambda + \mu + \partial)^q (a_p b)'_{hj} \right) \lambda^p.$$

Adding up (3.59), (3.60) and (3.61) and using the commutativity of \mathcal{V}_m we see that the RHS of (3.56) is equal to (3.58), thus concluding the proof. \square

Theorem 3.22. *If \mathcal{V} is a double Poisson vertex algebra, then, for any positive integer m , \mathcal{V}_m is a Poisson vertex algebra with λ -bracket defined by (3.52).*

Proof. By Proposition 3.20, formula (3.52) gives a well-defined skewsymmetric λ -bracket on \mathcal{V}_m . Since \mathcal{V} is a double Poisson vertex algebra, by Definition 3.2, $\{\{a_\lambda b_\mu c\}\} = 0$, for all $a, b, c \in \mathcal{V}$. Hence, by Proposition 3.21, the RHS of (3.57) is equal to zero, for all $a, b, c \in \mathcal{V}$, proving that the λ -bracket defined by (3.52) satisfies the Jacobi identity (0.5). \square

3.8. Examples: affine and AGD double PVA structures.

3.8.1. *Affine Poisson vertex algebra for $\mathfrak{g} = \mathfrak{gl}_m$.* Let us consider the differential algebra $\mathcal{V} = \mathcal{R}_1 = \mathbb{F}\langle u, u', u'', \dots \rangle$ and let

$$\{\{u_\lambda u\}\} = 1 \otimes u - u \otimes 1 + c(1 \otimes 1)\lambda \in \mathcal{V}^{\otimes 2}[\lambda], \quad (3.62)$$

where $c \in \mathbb{F}$. It is obvious that the skewsymmetry (3.8) holds for the pair u, u , and it is easy to check that the Jacobi identity (3.9) holds for the triple u, u, u . Hence, by Proposition 3.10, we have a family of compatible double Poisson vertex algebra structures on \mathcal{V} , uniquely extending (3.62). By Theorem 3.22, we get Poisson vertex algebra structures on the commutative differential algebra $\mathcal{V}_m = \mathbb{F}[u_{ij}^{(n)} \mid i, j = 1, \dots, m, n \in \mathbb{Z}_+]$ (see Example 3.19). Using (3.62) and (3.52), we get the explicit formula for the λ -bracket among the generators of \mathcal{V}_m :

$$\{u_{ij\lambda}u_{hk}\} = \delta_{jh}u_{ik} - \delta_{ki}u_{hj} + \delta_{jh}\delta_{ik}c\lambda,$$

for all $i, j, h, k = 1, \dots, m$. This is the λ -bracket among the generators of the affine Poisson vertex algebra for \mathfrak{gl}_m (where we fixed the nondegenerate invariant bilinear form to be a scalar multiple of the trace form).

Example 3.23. Consider the differential algebra $\mathcal{R}_2 = \mathbb{F}\langle u, v, u', v', \dots \rangle$ with the double lambda bracket $\{\{u_\lambda u\}\}$, given by (3.62) with v central. Then a similar computation as the one in Section 2.6 shows that $h_0 = 1$, $h_n = \frac{1}{n}(u+v)^n$ for $n > 0$, are in involution, and the corresponding integrable hierarchy of Hamiltonian equations is:

$$\frac{du}{dt_n} = v(u+v)^n u - u(u+v)^n v + c\partial(u+v)^{n+1}, \quad dv/dt_n = 0, \quad n \in \mathbb{Z}_+.$$

3.8.2. Adler type non-commutative pseudodifferential operators. Let us recall from [Kac98] that the δ -function is, by definition, the \mathbb{F} -valued formal distribution

$$\delta(z-w) = \sum_{n \in \mathbb{Z}_+} z^n w^{-n-1} \in \mathbb{F}[[z, z^{-1}, w, w^{-1}]].$$

Let $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$ be a formal Laurent series in z^{-1} with values in some vector space \mathcal{V} . Then, we have

$$a(z)\delta(z-w) = a(w)\delta(z-w). \quad (3.63)$$

We denote by i_z the power series expansion for large $|z|$. For example,

$$i_z(z-w)^{-1} = \sum_{k \in \mathbb{Z}_+} z^{-k-1} w^k, \quad i_z(z-w-\lambda-\partial)^{-1} = \sum_{k \in \mathbb{Z}_+} z^{-k-1} (w+\lambda+\partial)^k.$$

Using this notation, the δ -function can be rewritten as follows:

$$\delta(z-w) = i_z(z-w)^{-1} - i_w(z-w)^{-1}. \quad (3.64)$$

In the sequel, we use the following notation: if $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ and $b(z)$ are two formal Laurent series in z^{-1} , then

$$a(z+\lambda+\partial) \otimes b(z) = \sum_{n \in \mathbb{Z}} a_n \otimes i_z(z+\lambda+\partial)^n b(z),$$

namely, for any $n \in \mathbb{Z}$, we expand $(z+\lambda+\partial)^n$ in non-negative powers of $\lambda+\partial$ and we let powers of ∂ act to the right, on the coefficients of $b(z)$. In order to emphasize that ∂ does not act on some factors, we enclose in parenthesis the terms on which ∂ does act.

We denote by Res_z the coefficient of z^{-1} . The following identity holds for every formal power series $a(z) = \sum_n \in \mathbb{Z} a_n z^n$,

$$\text{Res}_z a(z) i_z(z-w)^{-1} = a(w)_+, \quad (3.65)$$

where $a(w)_+ = \sum_{n=0}^{\infty} a_n w^n$ is the positive part of $a(z)$. Furthermore, the following identity is a consequence of integration by parts:

$$\text{Res}_z (\iota_z a(z+t) \otimes b(z)) = \text{Res}_z (a(z) \otimes \iota_z b(z-t)). \quad (3.66)$$

Definition 3.24. Let \mathcal{V} be a differential algebra endowed with a 2-fold λ -bracket $\{\{-\lambda-\}\}$. We call a pseudodifferential operator $L(\partial) \in \mathcal{V}((\partial^{-1}))$ of *Adler type* for $\{\{-\lambda-\}\}$ if the following identity holds in $\mathcal{V}^{\otimes 2}[\lambda, \mu]((z^{-1}, w^{-1}))$ (cf. [DSKV14a]):

$$\begin{aligned} \{\{L(z)_\lambda L(w)\}\} &= L(z) \otimes i_z(z-w-\lambda-\partial)^{-1} L(w) \\ &- L(w+\lambda+\partial) \otimes i_z(z-w-\lambda-\partial)^{-1} L^*(-z+\lambda), \end{aligned} \quad (3.67)$$

where $L^*(\partial)$ is the formal adjoint of the pseudodifferential operator $L(\partial)$, and $L^*(z)$ is its symbol.

Proposition 3.25. Let \mathcal{V} be a differential algebra, let $\{\{-\lambda-\}\}$ be a 2-fold λ -bracket on \mathcal{V} , and let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be an Adler type pseudodifferential operator. Then:

(a) The following identity holds in $\mathcal{V}^{\otimes 2}[\lambda]((z^{-1}, w^{-1}))$:

$$\{\{L(z)_\lambda L(w)\}\} = -\{\{L(w)_{-\lambda-\partial} L(z)\}\}^\sigma.$$

(b) The following identity holds in $\mathcal{V}^{\otimes 3}[\lambda, \mu]((z_1^{-1}, z_2^{-1}, z_3^{-1}))$:

$$\begin{aligned} & \{\{L(z_1)_\lambda \{\{L(z_2)_\mu L(z_3)\}\}\}_L - \{\{L(z_2)_\mu \{\{L(z_1)_\lambda L(z_3)\}\}\}_R \\ & = \{\{\{L(z_1)_\lambda L(z_2)\}\}_{\lambda+\mu} L(z_3)\}_L. \end{aligned}$$

Proof. By (3.67) we get

$$\begin{aligned} \{\{L(w)_{-\lambda-\partial} L(z)\}\}^\sigma &= L(w + \lambda + \partial) \otimes i_w(z - w - \lambda - \partial)^{-1} L^*(-z + \lambda) \\ & - L(z) \otimes i_w(z - w - \lambda - \partial)^{-1} L(w). \end{aligned}$$

Hence, in order to prove (a) we are left to show that

$$\begin{aligned} & L(z) \otimes i_z(z - w - \lambda - \partial)^{-1} L(w) \\ & - L(w + \lambda + \partial) \otimes i_z(z - w - \lambda - \partial)^{-1} L^*(-z + \lambda) \\ & = L(z) \otimes i_w(z - w - \lambda - \partial)^{-1} L(w) \\ & - L(w + \lambda + \partial) \otimes i_w(z - w - \lambda - \partial)^{-1} L^*(-z + \lambda), \end{aligned}$$

which can be rewritten, using equation (3.64), as

$$L(z) \otimes \delta(z - w - \lambda - \partial) L(w) = L(w + \lambda + \partial) \otimes \delta(z - w - \lambda - \partial) L^*(-z + \lambda). \quad (3.68)$$

Identity (3.68) holds by applying the property of the δ -function given by equation (3.63).

In order to prove part (b) let us compute the three terms separately using (3.7) and (3.67). We have

$$\begin{aligned} & \{\{L(z_1)_\lambda \{\{L(z_2)_\mu L(z_3)\}\}\}\}_L \\ & = L(z_1) \otimes (i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L(z_2)) \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L(z_3) \quad (3.69) \end{aligned}$$

$$\begin{aligned} & - (L(z_2 + \lambda + \partial) \otimes i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda)) \\ & \quad \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L(z_3) \quad (3.70) \end{aligned}$$

$$\begin{aligned} & - L(z_1) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} L(z_3 + \mu + \partial) \\ & \quad \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L^*(-z_2 + \mu) \quad (3.71) \end{aligned}$$

$$\begin{aligned} & + L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} L^*(-z_1 + \lambda) \\ & \quad \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L^*(-z_2 + \mu); \quad (3.72) \end{aligned}$$

$$\begin{aligned} & - \{\{L(z_2)_\mu \{\{L(z_1)_\lambda L(z_3)\}\}\}\}_R \\ & = -L(z_1) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} L(z_2) \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L(z_3) \quad (3.73) \end{aligned}$$

$$\begin{aligned} & + L(z_1) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} L(z_3 + \mu + \partial) \\ & \quad \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L^*(-z_2 + \mu) \quad (3.74) \end{aligned}$$

$$\begin{aligned} & - L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(\lambda + \mu + \partial - z_1) \otimes i_{z_2}(z_1 - z_2 - \lambda - \partial)^{-1} L(z_2))^\sigma \quad (3.75) \end{aligned}$$

$$\begin{aligned} & + L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(-z_2 + \mu) \otimes i_{z_2}(z_1 - z_2 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda))^\sigma; \quad (3.76) \end{aligned}$$

$$\begin{aligned} & \{\{L(z_1)_\lambda L(z_2)\}_{\lambda+\mu} L(z_3)\}_L \\ &= L(z_1) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} (L(z_3) \otimes i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L(z_2))^\sigma \end{aligned} \quad (3.77)$$

$$\begin{aligned} & - L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(\lambda + \mu + \partial - z_1) \otimes i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L(z_2))^\sigma \end{aligned} \quad (3.78)$$

$$\begin{aligned} & - (L(z_2 + \lambda + \partial) \otimes i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda)) \\ & \quad \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L(z_3) \end{aligned} \quad (3.79)$$

$$\begin{aligned} & + L(z_3 + \lambda + \mu + \partial) \otimes (i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda)) \\ & \quad \otimes i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L^*(-z_2 + \mu). \end{aligned} \quad (3.80)$$

Note that

$$(3.70) = (3.79) \quad \text{and} \quad (3.71) + (3.74) = 0.$$

Next, we claim that

$$(3.69) + (3.73) = (3.77).$$

Indeed, we can rewrite the above identity as

$$\begin{aligned} & i_{z_1}(z_1 - z_2 - x)^{-1} i_{z_2}(z_2 - z_3 - y)^{-1} L(z_1) \otimes (\big|_{x=\lambda+\partial} L(z_2)) \otimes (\big|_{y=\mu+\partial} L(z_3)) \\ &= i_{z_1}(z_1 - z_3 - x - y)^{-1} (i_{z_1}(z_1 - z_2 - x)^{-1} \\ & \quad + i_{z_2}(z_2 - z_3 - y)^{-1}) L(z_1) \otimes (\big|_{x=\lambda+\partial} L(z_2)) \otimes (\big|_{y=\mu+\partial} L(z_3)). \end{aligned} \quad (3.81)$$

(Here and further, for a Laurent series $P(z) = \sum_{n=-\infty}^N c_n z^n \in \mathcal{V}((z^{-1}))$ and elements $a, b \in \mathcal{V}$, we let

$$a \left(\big|_{x=\nu+\partial} P(z+x)b \right) = \sum_{n=-\infty}^N a i_z(z + \nu + \partial)^n (c_n b).$$

The identity (3.81) follows from the obvious identity

$$a^{-1} b^{-1} = i_a(a+b)^{-1} b^{-1} + i_a(a+b)^{-1} a^{-1}.$$

In order to prove part b) we are left to show that

$$(3.72) + (3.75) + (3.76) = (3.78) + (3.80). \quad (3.82)$$

Note that, using the definition and the properties of the δ -function we have

$$\begin{aligned} & (3.75) - (3.78) \\ &= L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(\lambda + \mu + \partial - z_1) \otimes \delta(z_1 - z_2 - \lambda - \partial) L(z_2))^\sigma \end{aligned} \quad (3.83)$$

$$\begin{aligned} &= L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(-z_2 + \mu) \otimes \delta(z_1 - z_2 - \lambda - \partial) L^*(-z_1 + \lambda))^\sigma; \end{aligned}$$

$$\begin{aligned} & (3.76) \\ &= L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(-z_2 + \mu) \otimes i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda))^\sigma \end{aligned} \quad (3.84)$$

$$\begin{aligned} & - L(z_3 + \lambda + \mu + \partial) \otimes i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \\ & \quad \times (L^*(-z_2 + \mu) \otimes \delta(z_1 - z_2 - \lambda - \partial) L^*(-z_1 + \lambda))^\sigma. \end{aligned}$$

Using (3.83) and (3.83), identity (3.82) is proved once we show that

$$\begin{aligned} & i_{z_1}(z_1 - z_2 - x)^{-1} i_{z_2}(z_2 - z_3 - y)^{-1} \\ & \quad \times L(z_3 + x + y) \otimes (\big|_{x=\lambda+\partial} L^*(-z_1 + \lambda)) \otimes (\big|_{y=\mu+\partial} L^*(-z_2 + \mu)) \\ & = i_{z_1}(z_1 - z_3 - x - y)^{-1} (i_{z_1}(z_1 - z_2 - x)^{-1} + i_{z_2}(z_2 - z_3 - y)^{-1}) \times \\ & \quad \times L(z_3 + x + y) \otimes (\big|_{x=\lambda+\partial} L^*(-z_1 + \lambda)) \otimes (\big|_{y=\mu+\partial} L^*(-z_2 + \mu)), \end{aligned}$$

which can be proved as we did in the previous claim. \square

Theorem 3.26. *Let \mathcal{V} be a differential algebra, endowed with a 2-fold λ -bracket $\{\{-\lambda-\}$. Let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be a pseudodifferential operator, and let $\mathcal{U} \subset \mathcal{V}$ be the differential subalgebra generated by the coefficients of $L(\partial)$. If $L(\partial)$ is of Adler type, then $\{\{-\lambda-\}$ restricts to a double PVA λ -bracket on \mathcal{U} .*

Proof. By (3.67) $\{\{-\lambda-\}$ restricts to a 2-fold λ -bracket of \mathcal{U} , and by Proposition 3.25, together with Proposition 3.10, we have that \mathcal{U} is a double PVA. \square

3.8.3. *AGD Poisson vertex algebra structures for generic matrix (pseudo)differential operators of order N .* Let \mathcal{V} be the algebra of non-commutative differential polynomials in the variables u_i , $i \in I$, namely

$$\mathcal{V} = \mathcal{R}_I = \mathbb{F}\langle u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+ \rangle.$$

The index set I will be either $I = \{i \in \mathbb{Z} \mid i \geq -N\}$ or $I_- = \{-N, -N+1, \dots, -1\}$. Let us collect the differential generators of \mathcal{V} into the generating series

$$L(z) = z^N + \sum_{i \in I} u_i z^{-i-1} \in \mathcal{V}((z^{-1})).$$

By Example 3.19 we have that $\mathcal{V}_m = \mathbb{F}\langle u_{i,ab} \mid i \in I, a, b \in \{1, \dots, m\}, n \in \mathbb{Z}_+ \rangle$, for every $m \geq 1$. Let us denote

$$L_{ab}(z) = z^N + \sum_{i \in I} u_{i,ab} z^{-i-1} \in \mathcal{V}_m((z^{-1})),$$

for every $a, b = 1, \dots, m$.

Corollary 3.27. *\mathcal{V} has two compatible double Poisson vertex algebra structures with 2-fold λ -brackets defined on generators by the following generating series:*

$$\begin{aligned} \{L(z)_\lambda L(w)\}_H &= L(z) \otimes i_z(z - w - \lambda - \partial)^{-1} L(w) \\ & - L(w + \lambda + \partial) \otimes i_z(z - w - \lambda - \partial)^{-1} L^*(-z + \lambda) \end{aligned} \quad (3.85)$$

and

$$\begin{aligned} \{L(z)_\lambda L(w)\}_K &= i_z(z - w - \lambda)^{-1} (L(z) - L(w + \lambda)) \otimes 1 \\ & + 1 \otimes i_z(z - w - \lambda - \partial)^{-1} (L(w) - L^*(-z + \lambda)). \end{aligned} \quad (3.86)$$

Furthermore, for all $m \geq 1$, \mathcal{V}_m has two compatible Poisson vertex algebra structures, with λ -brackets on generators given by the generating series

$$\begin{aligned} \{L_{ab}(z)_\lambda L_{cd}(w)\} &= L_{cb}(z) i_z(z - w - \lambda - \partial)^{-1} L_{ad}(w) \\ & - L_{cb}(w + \lambda + \partial) i_z(z - w - \lambda - \partial)^{-1} L_{ad}^*(\lambda - z) \end{aligned}$$

and

$$\begin{aligned} \{L_{ab}(z)_\lambda L_{cd}(w)\}_K &= \delta_{ad} i_z(z - w - \lambda)^{-1} (L_{cb}(z) - L_{cb}(w + \lambda)) \\ & + \delta_{cb} i_z(z - w - \lambda - \partial)^{-1} (L_{ad}(w) - L_{ad}^*(\lambda - z)), \end{aligned}$$

for all $a, b, c, d = 1, \dots, m$.

Proof. The fact that \mathcal{V} has two compatible double Poisson vertex algebra structures follows by Theorem 3.26 applied to the pseudodifferential operator $L - c$, $c \in \mathbb{F}$. The second claim follows by Theorem 3.22. \square

Remark 3.28. The compatible PVA structures on \mathcal{V}_m are the same as the one appearing in [DSKV14a, Eq. (4.7)].

The following results will be used in Section 4.1.

Lemma 3.29. *In the double Poisson vertex algebra \mathcal{V} , with λ -bracket defined by (3.85), we have:*

- (a) $\{\{u_{-N\lambda}L(w)\}\}_H = 1 \otimes L(w) - L(w + \lambda) \otimes 1;$
- (b) $\{\{L(z)_\lambda u_{-N}\}\}_H = 1 \otimes L^*(-z + \lambda) - L(z) \otimes 1;$
- (c) $\{\{u_{-N\lambda}u_{-N}\}\}_H = 1 \otimes u_{-N} - u_{-N} \otimes 1 - (1 \otimes 1)N\lambda.$

Proof. Same proof as Lemma 2.17 in [DSKV14a]. \square

The following results follow immediately from the previous lemma and equation (3.52).

Corollary 3.30. *For $m \geq 1$, in the Poisson vertex algebra \mathcal{V}_m we have $(a, b, c, d = 1, \dots, m)$:*

- (a) $\{u_{-N,ab\lambda}L_{cd}(w)\} = \delta_{cb}L_{ad}(w) - \delta_{ad}L_{cb}(w + \lambda);$
- (b) $\{L_{ab}(z)_\lambda u_{-N,cd}\} = \delta_{cb}L_{ad}^*(-z + \lambda) - \delta_{ad}L_{cb}(z);$
- (c) $\{u_{-N,ab\lambda}u_{-N,cd}\} = \delta_{cb}u_{-N,ad} - \delta_{ad}u_{-N,cb} - \delta_{ad}\delta_{cb}N\lambda.$

4. NON-LOCAL DOUBLE POISSON VERTEX ALGEBRAS AND THEIR DIRAC REDUCTION

4.1. Non-local double Poisson vertex algebras. The notion of a non-local double PVA is similar to that of a non-local PVA, introduced in [DSK13]. As in that paper, given a vector space V , we denote

$$V_{\lambda,\mu} := V[[\lambda^{-1}, \mu^{-1}, (\lambda + \mu)^{-1}][\lambda, \mu],$$

namely, the quotient of the $\mathbb{F}[\lambda, \mu, \nu]$ -module $V[[\lambda^{-1}, \mu^{-1}, \nu^{-1}][\lambda, \mu, \nu]$ by the submodule $(\nu - \lambda - \mu)V[[\lambda^{-1}, \mu^{-1}, \nu^{-1}][\lambda, \mu, \nu]$. Recall that we have the natural embedding $i_\mu : V_{\lambda,\mu} \hookrightarrow V((\lambda^{-1}))((\mu^{-1}))$ (see the beginning of Section 3.8.2).

Let \mathcal{V} be a differential algebra with a derivation ∂ . A (non-local) 2-fold λ -bracket on \mathcal{V} is a linear map $\{\{-\lambda-\}\} : \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}((\lambda^{-1}))$ satisfying the *sesquilinearity* conditions (3.5) and the left and right *Leibniz rules* (3.6). Here and further an expression $\{\{a_{\lambda+\partial}b\}\} \rightarrow \star_1 c$ is interpreted as follows: if $\{\{a_\lambda b\}\} = \sum_{n=-\infty}^N (h'_n \otimes h''_n)\lambda^n$, then $\{\{a_{\lambda+\partial}b\}\} \rightarrow \star_1 c = \sum_{n=-\infty}^N h'_n ((\lambda + \partial)^n c) \otimes h''_n$, where we expand $(\lambda + \partial)^n$ in non-negative powers of ∂ acting on c . The (non-local) 2-fold λ -bracket $\{\{\cdot_\lambda \cdot\}\}$ is called *skewsymmetric* if equation (3.8) holds. The RHS of the skewsymmetry condition should be interpreted as follows: we move $-\lambda - \partial$ to the left and we expand its powers in non-negative powers of ∂ , acting on the coefficients on the 2-fold λ -bracket. We say that the (non-local) 2-fold λ -bracket $\{\{-\lambda-\}\}$ is *admissible* if

$$\{\{a_\lambda \{\{b_\mu c\}\}\}\}_L \in \mathcal{V}_{\lambda,\mu}^{\otimes 3} \quad \text{for all } a, b, c \in \mathcal{V}. \quad (4.1)$$

Here we are identifying the space $\mathcal{V}_{\lambda,\mu}^{\otimes 3}$ with its image in $\mathcal{V}^{\otimes 3}((\lambda^{-1}))((\mu^{-1}))$ via the embedding i_μ .

Remark 4.1. If $\{\{-\lambda-\}\}$ is a skewsymmetric admissible (non-local) 2-fold λ -bracket on \mathcal{V} , by equation (3.13) we have also $\{\{b_\mu \{\{a_\lambda c\}\}\}\}_R \in \mathcal{V}_{\lambda,\mu}^{\otimes 3}$, since $\mathcal{V}_{\lambda,\mu}^{\otimes 3} = \mathcal{V}_{\mu,\lambda}^{\otimes 3}$, and similarly $\{\{\{a_\lambda b\}\}_{\lambda+\mu} c\}\}_L \in \mathcal{V}_{\lambda,\mu}^{\otimes 3}$, for all $a, b, c \in \mathcal{V}$.

Definition 4.2. A *non-local double Poisson vertex algebra* is a differential algebra \mathcal{V} endowed with a non-local 2-fold λ -bracket, $\{\{-\lambda-\}\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}((\lambda^{-1}))$ satisfying skewsymmetry (3.8), admissibility (4.1), and Jacobi identity (3.9), where the latter is understood as an identity in the space $\mathcal{V}_{\lambda,\mu}^{\otimes 3}$.

4.2. Dirac reduction for non-local double Poisson vertex algebras. Dirac reduction for a double PVA is similar to that of a PVA, constructed in [DSKV14a]. Let \mathcal{V} be a non-local double Poisson vertex algebra with 2-fold λ -bracket $\{\{-\lambda-\}\}$. Let $\theta_1, \dots, \theta_m$ be elements of \mathcal{V} and let us consider the matrix pseudodifferential operator

$$C(\partial) = (C_{\alpha\beta}(\partial))_{\alpha,\beta=1}^m \in \text{Mat}_{m \times m} \mathcal{V}^{\otimes 2}((\partial^{-1})),$$

where the symbol of the pseudodifferential operator $C_{\alpha\beta}(\partial)$ is

$$C_{\alpha\beta}(\lambda) = \{\{\theta_{\beta\lambda}\theta_{\alpha}\}\}. \quad (4.2)$$

Given a pseudodifferential operator $A(\partial) = \sum_{n \leq N} A_n \partial^n \in \text{Mat}_{m \times m} \mathcal{V}^{\otimes 2}((\partial^{-1}))$, recall that we defined its adjoint by

$$A(\partial)^\dagger = \sum_{n \leq N} (-\partial)^n \circ (A_n^t)^\sigma,$$

where A_n^t denotes the transpose matrix of A_n . By the skewsymmetry condition (3.8), the pseudodifferential operator $C(\partial)$ is skewadjoint. We shall assume that the matrix pseudodifferential operator $C(\partial)$ is invertible with respect to the \bullet -product (1.1), and we denote its inverse by $C^{-1}(\partial) = ((C^{-1})_{\alpha\beta}(\partial))_{\alpha,\beta=1}^m \in \text{Mat}_{m \times m} \mathcal{V}^{\otimes 2}((\partial^{-1}))$.

Definition 4.3. The *Dirac modification* of the 2-fold λ -bracket $\{\{-\lambda-\}\}$, associated to the elements $\theta_1, \dots, \theta_m$, is the map $\{\{-\lambda-\}\}^D : \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}((\lambda^{-1}))$ given by $(a, b \in \mathcal{V})$:

$$\{\{a_\lambda b\}\}^D = \{\{a_\lambda b\}\} - \sum_{\alpha,\beta=1}^m \{\{\theta_{\beta\lambda+\partial} b\}\} \rightarrow \bullet (C^{-1})_{\beta\alpha}(\lambda + \partial) \bullet \{\{a_\lambda \theta_\alpha\}\}. \quad (4.3)$$

Theorem 4.4. Let \mathcal{V} be a non-local double Poisson vertex algebra with 2-fold λ -bracket $\{\{-\lambda-\}\}$. Let $\theta_1, \dots, \theta_m \in \mathcal{V}$ be elements such that the corresponding matrix pseudodifferential operator $C(\partial) = (C_{\alpha\beta}(\partial))_{\alpha,\beta=1}^m \in \text{Mat}_{m \times m} \mathcal{V}^{\otimes 2}((\partial^{-1}))$ given by (4.2) is invertible.

- (a) The Dirac modification $\{\{-\lambda-\}\}^D$ given by equation (4.3) is a 2-fold λ -bracket on \mathcal{V} , giving \mathcal{V} a structure of a non-local double PVA.
- (b) All the elements θ_i , $i = 1, \dots, m$, are central with respect to the Dirac modified λ -bracket: $\{\{a_\lambda \theta_i\}\}^D = \{\{\theta_i \lambda a\}\}^D = 0$ for all $i = 1, \dots, m$ and $a \in \mathcal{V}$.

Before proving the theorem we state some lemmas that we will use in the proof.

Lemma 4.5. Let $A(\lambda, \mu) \in \mathcal{V}_{\lambda,\mu}^{\otimes 3}$ and $B(\lambda, \mu) \in \mathcal{V}_{\lambda,\mu}^{\otimes 2}$, and let $S, T : \mathcal{V}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}$ be endomorphisms of $\mathcal{V}^{\otimes 3}$ (viewed as a vector space). Then

$$\begin{aligned} A(\lambda + S, \mu + T) \bullet_1 B(\lambda, \mu) &\in \mathcal{V}_{\lambda,\mu}^{\otimes 3}, \\ A(\lambda + S, \mu + T) \bullet_2 B(\lambda, \mu) &\in \mathcal{V}_{\lambda,\mu}^{\otimes 3}, \\ A(\lambda + S, \mu + T) \bullet_3 B(\lambda, \mu) &\in \mathcal{V}_{\lambda,\mu}^{\otimes 3}, \end{aligned}$$

where we expand the negative powers of $\lambda + S$ and $\mu + T$ in non-negative powers of S and T , acting on the coefficients of B , and

$$\begin{aligned} B(\lambda + S, \mu + T) \bullet_1 A(\lambda, \mu) &\in \mathcal{V}_{\lambda,\mu}^{\otimes 3}, \\ B(\lambda + S, \mu + T) \bullet_2 A(\lambda, \mu) &\in \mathcal{V}_{\lambda,\mu}^{\otimes 3}, \\ B(\lambda + S, \mu + T) \bullet_3 A(\lambda, \mu) &\in \mathcal{V}_{\lambda,\mu}^{\otimes 3}, \end{aligned}$$

where now S and T act on the coefficients of A .

Proof. The proof follows the same lines as the proof of Lemma 2.3 in [DSKV13] using the definition of the \bullet -products given by equation (1.18). \square

Lemma 4.6. Let $\{\{-\lambda-\}\} : \mathcal{V}^{\otimes 2} \times \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}((\lambda^{-1}))$ be a 2-fold λ -bracket on the differential algebra \mathcal{V} . Suppose that $C(\partial) = (C_{ij}(\partial))_{i,j=1}^{\ell} \in \text{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}((\partial^{-1}))$ is an invertible $\ell \times \ell$ matrix pseudodifferential operator with coefficients in $\mathcal{V}^{\otimes 2}$, and let $C^{-1}(\partial) = ((C^{-1})_{ij}(\partial))_{i,j=1}^{\ell} \in \text{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}((\partial^{-1}))$ be its inverse. The following identities hold for every $a \in \mathcal{V}$ and $i, j = 1, \dots, \ell$:

$$\begin{aligned} \{\{a_{\lambda}(C^{-1})_{ij}(\mu)\}\}_L &= - \sum_{r,t=1}^{\ell} i_{\mu}(C^{-1})_{ir}(\lambda + \mu + \partial) \\ &\bullet_2 \{\{a_{\lambda} C_{rt}(y)\}\}_L \bullet_1 \left(\left|_{y=\mu+\partial} (C^{-1})_{tj}(\mu) \right. \right) \in \mathcal{V}^{\otimes 3}((\lambda^{-1}))((\mu^{-1})), \end{aligned} \quad (4.4)$$

$$\begin{aligned} \{\{a_{\lambda}(C^{-1})_{ij}(\mu)\}\}_R &= - \sum_{r,t=1}^{\ell} i_{\mu}(C^{-1})_{ir}(\lambda + \mu + \partial) \\ &\bullet_2 \{\{a_{\lambda} C_{rt}(y)\}\}_R \bullet_3 \left(\left|_{y=\mu+\partial} (C^{-1})_{tj}(\mu) \right. \right) \in \mathcal{V}^{\otimes 3}((\lambda^{-1}))((\mu^{-1})), \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \{\{(C^{-1})_{ij}(\lambda)_{\lambda+\mu} a\}\}_L^{\sigma^2} &= - \sum_{r,t=1}^{\ell} \{\{C_{rt}(x)_{x+y} a\}\}_L^{\sigma^2} \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{tj}(\lambda) \right. \right) \\ &\bullet_1 \left(\left|_{y=\mu+\partial} i_{\lambda}(C^{*-1})_{ri}(\mu) \right. \right) \in \mathcal{V}^{\otimes 3}(((\lambda + \mu)^{-1}))((\lambda^{-1})), \end{aligned} \quad (4.6)$$

where $i_{\mu} : \mathcal{V}_{\lambda,\mu}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}((\lambda^{-1}))((\mu^{-1}))$ and $i_{\lambda} : \mathcal{V}_{\lambda,\mu}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}(((\lambda + \mu)^{-1}))((\lambda^{-1}))$ are the natural embeddings defined above. In equations (4.4) and (4.6), $C(\lambda) \in \text{Mat}_{\ell \times \ell} \mathcal{V}((\lambda^{-1}))$ denotes the symbol of the matrix pseudodifferential operator C and C^* denotes its adjoint (its inverse being $(C^{-1})^*$).

Proof. The identity $C \circ C^{-1} = 1 \otimes 1$ becomes, in terms of symbols,

$$\sum_{t=1}^{\ell} C_{rt}(\mu + \partial) \bullet (C^{-1})_{tj}(\mu) = \delta_{rj}(1 \otimes 1).$$

For all $a \in \mathcal{V}$, we have, by sesquilinearity and the left Leibniz rule (the first equation in (3.23)):

$$\begin{aligned} 0 &= \sum_{t=1}^{\ell} \{\{a_{\lambda} C_{rt}(\mu + \partial) \bullet (C^{-1})_{tj}(\mu)\}\}_L \\ &= \sum_{t=1}^{\ell} \{\{a_{\lambda} C_{rt}(y)\}\}_L \bullet_1 \left(\left|_{y=\mu+\partial} (C^{-1})_{tj}(\mu) \right. \right) \\ &\quad + \sum_{t=1}^{\ell} i_{\mu} C_{rt}(\lambda + \mu + \partial) \bullet_2 \{\{a_{\lambda}(C^{-1})_{tj}(\mu)\}\}. \end{aligned}$$

Note that $i_{\mu} C(\lambda + \mu + \partial)$ is invertible in $\text{Mat}_{\ell \times \ell}(\mathcal{V}^{\otimes 2}[\partial]((\lambda^{-1}))((\mu^{-1})))$, its inverse being $i_{\mu} C^{-1}(\lambda + \mu + \partial)$. We then apply $i_{\mu}(C^{-1})_{ir}(\lambda + \mu + \partial) \bullet_2$ on the left to both sides of the above equation, and using Lemma 1.1(a) and summing over $r = 1, \dots, \ell$, we get

$$\begin{aligned} \sum_{t=1}^{\ell} \delta_{it} \{\{a_{\lambda}(C^{-1})_{tj}(\mu)\}\}_L &= - \sum_{r,t=1}^{\ell} i_{\mu}(C^{-1})_{ir}(\lambda + \mu + \partial) \bullet_2 \\ &\quad \{\{a_{\lambda} C_{rt}(y)\}\}_L \bullet_1 \left(\left|_{y=\mu+\partial} (C^{-1})_{tj}(\mu) \right. \right), \end{aligned}$$

proving equation (4.4). The proof of the equation (4.5) is done in a similar way using the second equation in (3.23).

For the third equation we have, by the right Leibniz rule (the third equation in (3.23)):

$$\begin{aligned}
0 &= \sum_{t=1}^{\ell} \{ \{ C_{rt}(\lambda + \partial) \bullet (C^{-1})_{tj}(\lambda)_{\lambda+\mu} a \} \}_L \\
&= \sum_{t=1}^{\ell} \{ \{ C_{rt}(x)_{\lambda+\mu+\partial} a \} \}_L \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{tj}(\lambda) \right. \right) \\
&\quad + \sum_{t=1}^{\ell} \{ \{ (C^{-1})_{tj}(\lambda)_{\lambda+\mu+\partial} a \} \}_L \bullet_1 (i_{\lambda} C_{tr}^{\dagger}(\mu)).
\end{aligned}$$

We next replace in the above equation μ (placed at the right) by $\mu + \partial$, and we apply the resulting differential operator to $i_{\lambda}(C^{\dagger-1})_{ri}(\mu)$. As a result we get, after summing over $r = 1, \dots, \ell$:

$$\begin{aligned}
\sum_{t=1}^{\ell} \{ \{ (C^{-1})_{tj}(\lambda)_{\lambda+\mu+\partial} a \} \}_L \delta_{ti} &= - \sum_{r,t=1}^{\ell} \{ \{ C_{rt}(x)_{\lambda+\mu+\partial} a \} \}_L \\
&\quad \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{tj}(\lambda) \right. \right) \bullet_1 (i_{\lambda}(C^{\dagger-1})_{ri}(\mu)),
\end{aligned}$$

proving equation (4.6). \square

Corollary 4.7. *Let $\{ \{ -_{\lambda} - \} \} : \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}((\lambda^{-1}))$ be a 2-fold λ -bracket on the differential algebra \mathcal{V} . Let $C(\partial) = (C_{ij}(\partial))_{i,j=1}^{\ell} \in \text{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}((\partial^{-1}))$ be an invertible $\ell \times \ell$ matrix pseudodifferential operator with coefficients in $\mathcal{V}^{\otimes 2}$, and let $C^{-1}(\partial) = ((C^{-1})_{ij}(\partial))_{i,j=1}^{\ell} \in \text{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}((\partial^{-1}))$ be its inverse. Let $a \in \mathcal{V}$, and assume that*

$$\{ \{ a_{\lambda} C_{ij}(\mu) \} \}_L \in \mathcal{V}_{\lambda, \mu}^{\otimes 3} \text{ for all } i, j = 1, \dots, \ell. \quad (4.7)$$

(As before, we identify $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ with its image $i_{\mu}(\mathcal{V}_{\lambda, \mu}^{\otimes 3}) \subset \mathcal{V}^{\otimes 3}((\lambda^{-1}))((\mu^{-1}))$.) Then, we have $\{ \{ a_{\lambda}(C^{-1})_{ij}(\mu) \} \}_L, \{ \{ (C^{-1})_{ij}(\lambda)_{\lambda+\mu} a \} \}_L \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}$. In fact, the following identities hold in the space $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$:

$$\begin{aligned}
&\{ \{ a_{\lambda}(C^{-1})_{ij}(\mu) \} \}_L = \\
&\quad - \sum_{r,t=1}^{\ell} (C^{-1})_{ir}(\lambda + \mu + \partial) \bullet_2 \{ \{ a_{\lambda} C_{rt}(y) \} \}_L \bullet_1 \left(\left|_{y=\mu+\partial} (C^{-1})_{tj}(\mu) \right. \right), \quad (4.8)
\end{aligned}$$

$$\begin{aligned}
&\{ \{ a_{\lambda}(C^{-1})_{ij}(\mu) \} \}_R = \\
&\quad - \sum_{r,t=1}^{\ell} (C^{-1})_{ir}(\lambda + \mu + \partial) \bullet_2 \{ \{ a_{\lambda} C_{rt}(y) \} \}_R \bullet_3 \left(\left|_{y=\mu+\partial} (C^{-1})_{tj}(\mu) \right. \right), \quad (4.9)
\end{aligned}$$

$$\begin{aligned}
&\{ \{ (C^{-1})_{ij}(\lambda)_{\lambda+\mu} a \} \}_L = \\
&\quad - \sum_{r,t=1}^{\ell} \{ \{ C_{rt}(x)_{\lambda+\mu+\partial} a \} \}_L \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{tj}(\lambda) \right. \right) \bullet_1 ((C^{\dagger-1})_{ri}(\mu)). \quad (4.10)
\end{aligned}$$

Proof. It is an immediate corollary of Lemmas 4.5 and 4.6. \square

Remark 4.8. Lemmas 4.5 and 4.6 are the double PVA analogues of (respectively) [DSK13, Lemmas 2.3 and 3.9]. Corollary 4.7 is the double PVA analogue of [DSKV14b, Corollary 1.4]. Naturally, if we put $\lambda = 0$ in (4.3), we obtain Dirac modification of the corresponding double Poisson algebra.

Proof of Theorem 4.4. Both sesquilinearity conditions (3.5) for the Dirac modified 2-fold λ -bracket (4.3) are immediate to check. The skewsymmetry condition (3.8) for the Dirac modified 2-fold λ -bracket (4.3) can also be easily proved: it follows by the skewsymmetry of the 2-fold λ -bracket $\{ \{ -_{\lambda} - \} \}$, by the fact that the matrix

$C(\partial)$ (hence $C^{-1}(\partial)$) is skewadjoint, and by equation (1.3). Moreover, by equation (1.16), it follows that the Dirac modified 2-fold λ -bracket $\{\{-\lambda-\}\}^D$ satisfies the Leibniz rules (3.6).

Next, we prove that the Dirac modified 2-fold λ -bracket $\{\{-\lambda-\}\}^D$ is admissible, in the sense of equation (4.1). For this, we compute $\{\{a_\lambda\{b_\mu c\}\}^D\}_L^D$ using the definition (4.3), the sesquilinearity conditions (3.5), the Leibniz rule (3.6) and equation (3.23). We get

$$\{\{a_\lambda\{b_\mu c\}\}^D\}_L^D = \{\{a_\lambda\{b_\mu c\}\}\}_L \quad (4.11)$$

$$- \sum_{\gamma, \delta=1}^m \{\{a_\lambda\{\theta_{\delta y} c\}\}\}_L \bullet_1 \left(\Big|_{y=\mu+\partial} (C^{-1})_{\delta\gamma}(\mu+\partial) \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.12)$$

$$- \sum_{\gamma, \delta=1}^m \{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} \bullet_2 \{\{a_\lambda (C^{-1})_{\delta\gamma}(y)\}\}_L \bullet_1 \left(\Big|_{y=\mu+\partial} \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.13)$$

$$- \sum_{\gamma, \delta=1}^m (\{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial)) \bullet_2 \{\{a_\lambda\{b_\mu \theta_\gamma\}\}\}_L \quad (4.14)$$

$$- \sum_{\alpha, \beta=1}^m \{\{\theta_{\beta\lambda+\partial}\{b_\mu c\}\}\}_{L \rightarrow} \bullet_3 ((C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\}) \quad (4.15)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta=1}^m \{\{\theta_{\beta x}\{\theta_{\delta y} c\}\}\}_L \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \bullet_1 \left(\Big|_{y=\mu+\partial} (C^{-1})_{\delta\gamma}(\mu+\partial) \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.16)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta=1}^m \{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} \bullet_2 \{\{\theta_{\beta x} (C^{-1})_{\delta\gamma}(y)\}\}_L \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \bullet_1 \left(\Big|_{y=\mu+\partial} \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.17)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta=1}^m (\{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial)) \bullet_2 \{\{\theta_{\beta\lambda+\partial}\{b_\mu \theta_\gamma\}\}\}_{L \rightarrow} \bullet_3 ((C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\}). \quad (4.18)$$

All the terms (4.11), (4.12), (4.14), (4.15), (4.16), and (4.18), lie in $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ by the admissibility assumption on $\{\{-\lambda-\}\}$ and Lemma 4.5. Moreover, by the admissibility of $\{\{-\lambda-\}\}$ and the definition (4.2) of the matrix $C(\partial)$, condition (4.7) holds. Hence, we can use Corollary 4.7 and Lemma 4.5 to deduce that the terms (4.13) and (4.17) lie in $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ as well. Therefore, $\{\{a_\lambda\{b_\mu c\}\}^D\}_L^D$ lies in $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ for every $a, b, c \in \mathcal{V}$, i.e. the Dirac modification $\{\{-\lambda-\}\}^D$ is admissible.

In order to complete the proof of part (a) we are left to check the Jacobi identity (3.9) for the Dirac modified 2-fold λ -bracket. We can use equation (4.8) in Corollary

4.7 and Lemma 1.1(a,c) to rewrite the terms (4.13) and (4.17). As a result, we get

$$\{\{a_\lambda \{\{b_\mu c\}^D\}\}_L^D = \{\{a_\lambda \{\{b_\mu c\}\}\}_L \quad (4.19)$$

$$- \sum_{\gamma, \delta=1}^m \{\{a_\lambda \{\{\theta_{\delta y} c\}\}\}_L \bullet_1 \left(\Big|_{y=\mu+\partial} (C^{-1})_{\delta\gamma}(\mu+\partial) \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.20)$$

$$+ \sum_{\gamma, \delta, \eta, \zeta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\zeta}(\lambda+\mu+\partial) \right) \bullet_2 \{\{a_\lambda \{\{\theta_{\eta y} \theta_\zeta\}\}\}_L \bullet_1 \left(\Big|_{y=\mu+\partial} (C^{-1})_{\eta\gamma}(\mu+\partial) \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.21)$$

$$- \sum_{\gamma, \delta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial) \right) \bullet_2 \{\{a_\lambda \{\{b_\mu \theta_\gamma\}\}\}_L \quad (4.22)$$

$$- \sum_{\alpha, \beta=1}^m \{\{\theta_{\beta\lambda+\partial} \{\{b_\mu c\}\}\}_L \rightarrow \bullet_3 \left((C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \quad (4.23)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta=1}^m \{\{\theta_{\beta x} \{\{\theta_{\delta y} c\}\}\}_L \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \bullet_1 \left(\Big|_{y=\mu+\partial} (C^{-1})_{\delta\gamma}(\mu+\partial) \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.24)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta, \eta, \zeta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\zeta}(\lambda+\mu+\partial) \right) \bullet_2 \{\{\theta_{\beta x} \{\{\theta_{\eta y} \theta_\zeta\}\}\}_L \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \bullet_3 \left(\Big|_{y=\mu+\partial} (C^{-1})_{\eta\gamma}(\mu+\partial) \{\{b_\mu \theta_\gamma\}\} \right) \quad (4.25)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial) \right) \bullet_2 \{\{\theta_{\beta\lambda+\partial} \{\{b_\mu \theta_\gamma\}\}\}_L \rightarrow \bullet_3 \left((C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right). \quad (4.26)$$

Next, we compute the second term in the Jacobi identity using the definition (4.3), the sesquilinearity conditions (3.5), the Leibniz rules (3.6), equations (3.23) and

(4.9) and Lemma 1.1(a,c). We get

$$\{\{b_\mu \{\{a_\lambda c\}^D\}\}_R^D = \{\{b_\mu \{\{a_\lambda c\}\}\}_R \quad (4.27)$$

$$- \sum_{\alpha, \beta=1}^m \{\{b_\mu \{\{\theta_{\beta_x} c\}\}\}_R \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda + \partial) \{\{a_\lambda \theta_\alpha\}\} \right. \right) \quad (4.28)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta}^m (\{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} (C^{-1})_{\delta\gamma}(\lambda + \mu + \partial)) \bullet_2 \{\{b_\mu \{\{\theta_{\beta_x} \theta_\gamma\}\}\}_R \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda + \partial) \{\{a_\lambda \theta_\alpha\}\} \right. \right) \quad (4.29)$$

$$- \sum_{\gamma, \delta=1}^m (\{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} (C^{-1})_{\delta\gamma}(\lambda + \mu + \partial)) \bullet_2 \{\{b_\mu \{\{a_\lambda \theta_\gamma\}\}\}_R \quad (4.30)$$

$$- \sum_{\gamma, \delta=1}^m \{\{\theta_{\delta\mu+\partial} \{\{a_\lambda c\}\}\}_R \bullet_1 ((C^{-1})_{\delta\gamma}(\mu + \partial) \{\{b_\mu \theta_\gamma\}\}) \quad (4.31)$$

$$+ \sum_{\alpha, \beta, \gamma, \delta=1}^m \{\{\theta_{\delta_y} \{\{\theta_{\beta_x} c\}\}\}\}_R \bullet_1 \left(\left|_{y=\mu+\partial} (C^{-1})_{\delta\gamma}(\mu + \partial) \{\{b_\mu \theta_\gamma\}\} \right. \right) \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda + \partial) \{\{a_\lambda \theta_\alpha\}\} \right. \right) \quad (4.32)$$

$$- \sum_{\alpha, \beta, \gamma, \delta, \eta, \zeta=1}^m (\{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} (C^{-1})_{\delta\zeta}(\lambda + \mu + \partial)) \bullet_2 \{\{\theta_{\eta_y} \{\{\theta_{\beta_x} \theta_\zeta\}\}\}\}_R \bullet_1 \left(\left|_{y=\mu+\partial} (C^{-1})_{\eta\gamma}(\mu + \partial) \{\{b_\mu \theta_\gamma\}\} \right. \right) \bullet_3 \left(\left|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda + \partial) \{\{a_\lambda \theta_\alpha\}\} \right. \right) \quad (4.33)$$

$$+ \sum_{\gamma, \delta, \eta, \zeta=1}^m (\{\{\theta_{\delta\lambda+\mu+\partial} c\}\}_{\rightarrow} (C^{-1})_{\delta\zeta}(\lambda + \mu + \partial)) \bullet_2 \{\{\theta_{\eta_{\mu+\partial}} \{\{a_\lambda \theta_\zeta\}\}\}\}_R \bullet_1 ((C^{-1})_{\eta\gamma}(\mu + \partial) \{\{b_\mu \theta_\gamma\}\}). \quad (4.34)$$

In a similar way we compute the RHS of the Jacobi identity for the Dirac modified 2-fold λ -bracket, using the definition (4.3), the sesquilinearity (3.5), the right Leibniz rule (3.6), equation (4.10) (recall that the matrix C is skewadjoint), and Lemma

1.1(a,c). We get

$$\{\{a_\lambda b\}\}_{\lambda+\mu}^D \Big|_L^D = \{\{a_\lambda b\}\}_{\lambda+\mu}^c \Big|_L \quad (4.35)$$

$$- \sum_{\alpha,\beta=1}^m \{\{\theta_{\beta_x} b\}\}_{\lambda+\mu+\partial}^c \Big|_{L \rightarrow} \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \quad (4.36)$$

$$- \sum_{\alpha,\beta,\gamma,\delta=1}^m \{\{\theta_{\beta_x} \theta_\gamma\}\}_{\lambda+\mu+\partial}^c \Big|_{L \rightarrow} \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \\ \bullet_1 \left((C^{-1})_{\delta\gamma}(\mu+\partial) \{\{\theta_{\gamma-\mu-\partial} b\}\}^\sigma \right) \quad (4.37)$$

$$- \sum_{\gamma,\delta=1}^m \{\{a_\lambda \theta_\delta\}\}_{\lambda+\mu+\partial}^c \Big|_{L \rightarrow} \bullet_1 \left((C^{\dagger-1})_{\delta\gamma}(\mu+\partial) \{\{\theta_{\gamma-\mu-\partial} b\}\}^\sigma \right) \quad (4.38)$$

$$- \sum_{\gamma,\delta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial) \right) \bullet_2 \{\{a_\lambda b\}\}_{\lambda+\mu} \theta_\gamma \Big|_L \quad (4.39)$$

$$+ \sum_{\alpha,\beta,\gamma,\delta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial) \right) \\ \bullet_2 \{\{\theta_{\beta_x} b\}\}_{\lambda+\mu+\partial} \theta_\gamma \Big|_{L \rightarrow} \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \quad (4.40)$$

$$+ \sum_{\alpha,\beta,\gamma,\delta,\eta,\zeta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\gamma}(\lambda+\mu+\partial) \right) \\ \bullet_2 \{\{\theta_{\beta_x} \theta_\eta\}\}_{\lambda+\mu+\partial} \theta_\zeta \Big|_{L \rightarrow} \bullet_3 \left(\Big|_{x=\lambda+\partial} (C^{-1})_{\beta\alpha}(\lambda+\partial) \{\{a_\lambda \theta_\alpha\}\} \right) \\ \bullet_1 \left((C^{-1})_{\eta\gamma}(\mu+\partial) \{\{\theta_{\gamma-\mu-\partial} b\}\}^\sigma \right) \quad (4.41)$$

$$+ \sum_{\gamma,\delta,\eta,\zeta=1}^m \left(\{\{\theta_{\delta\lambda+\mu+\partial} c\}\} \rightarrow (C^{-1})_{\delta\zeta}(\lambda+\mu+\partial) \right) \\ \bullet_2 \{\{a_\lambda \zeta_\alpha\}\}_{\lambda+\mu+\partial} \theta_\eta \Big|_{L \rightarrow} \bullet_1 \left((C^{\dagger-1})_{\eta\gamma}(\mu+\partial) \{\{\theta_{\gamma-\mu-\partial} b\}\}^\sigma \right). \quad (4.42)$$

The following equations hold due to the skewsymmetry (3.8), the Jacobi identity (3.9), and the fact that the matrix C is skewadjoint:

$$\text{RHS(4.19)} - \text{RHS(4.27)} = \text{RHS(4.35)}, \quad (4.20) - (4.31) = (4.38), \\ (4.23) - (4.28) = (4.36), \quad (4.22) - (4.30) = (4.39) \\ (4.21) - (4.34) = (4.42), \quad (4.26) - (4.29) = (4.40).$$

Moreover, using Lemma 1.1(c), the skewsymmetry (3.8) and the Jacobi identity (3.9), we also get

$$(4.24) - (4.32) = (4.37), \quad (4.25) - (4.33) = (4.41).$$

This concludes the proof of the Jacobi identity for the Dirac modified 2-fold λ -bracket, and of part (a).

Note that the identities $C(\partial)C^{-1}(\partial) = C^{-1}(\partial)C(\partial) = 1 \otimes 1$ read, in terms of the symbols of the pseudodifferential operators $C(\partial)$ and $C^{-1}(\partial)$, as

$$\sum_{\beta=1}^m \{\{\theta_{\beta\lambda+\partial} \theta_\alpha\}\} \rightarrow \bullet (C^{-1})_{\beta\gamma}(\lambda) = \delta_{\alpha\gamma}(1 \otimes 1), \quad \sum_{\beta=1}^m (C^{-1})_{\alpha\beta}(\lambda+\partial) \bullet \{\{\theta_{\gamma\lambda} \theta_\beta\}\} = \delta_{\alpha\gamma}(1 \otimes 1).$$

Part (b) is an immediate consequence of these identities and the definition (4.3) of the Dirac modified 2-fold λ -bracket. \square

Definition 4.9. Let $\mathcal{I} \subset \mathcal{V}$ be a two-sided differential ideal. We say that \mathcal{I} is a *double Poisson vertex algebra ideal* if

$$\{\{\mathcal{V}_\lambda \mathcal{I}\}, \{\{\mathcal{I}_\lambda \mathcal{V}\}\} \in (\mathcal{V} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{V})[\lambda].$$

If $\mathcal{I} \subset \mathcal{V}$ is a double PVA ideal then we have an induced double PVA structure on the quotient differential algebra \mathcal{V}/\mathcal{I} .

Corollary 4.10. *The two-sided differential ideal $\mathcal{I} = \langle \theta_1, \dots, \theta_m \rangle_{\mathcal{V}} \subset \mathcal{V}$, generated by $\theta_1, \dots, \theta_m$, is a double PVA ideal with respect to the Dirac modified double λ -bracket $\{\{\cdot, \cdot\}_\lambda\}^D$. Hence, the quotient space \mathcal{V}/\mathcal{I} is a (non-local) double PVA, with 2-fold λ -bracket induced by $\{\{-\lambda-\}\}^D$, which we call the double Dirac reduction of \mathcal{V} by the constraints $\theta_1, \dots, \theta_m$.*

Proof. The statement follows by the sesquilinearity conditions and the left and right Leibniz rules for the Dirac modified 2-fold λ -bracket. \square

Example 4.11. Let us consider the non-commutative algebra of differential polynomials $\mathcal{V} = \mathcal{R}_I = \mathbb{F} \langle u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+ \rangle$ (the index set may be either $I = \{i \in \mathbb{Z} \mid i \geq -N\}$ or $I_- = \{-N, -N+1, \dots, -1\}$) with the double Poisson vertex algebra structure defined by equation (3.85). Let us denote $C(\lambda) = \{\{u_{-N\lambda} u_{-N}\}\}_H$. By Lemma 3.29 we have that

$$C(\lambda) = 1 \otimes u_{-N} - u_{-N} \otimes 1 - (1 \otimes 1)N\lambda,$$

thus $C(\partial) \in \mathcal{V}^{\otimes 2}((\partial^{-1}))$ is an invertible pseudodifferential operator. Denote by \mathcal{I} the two-sided differential ideal of \mathcal{V} generated by u_{-N} . Note that $\mathcal{V}/\mathcal{I} \simeq \mathbb{F} \langle u_i^{(n)} \mid i \in I \setminus \{-N\}, n \in \mathbb{Z}_+ \rangle$. Using the explicit expression for $C(\lambda)$, the formula for the Dirac reduction given by (4.3), and the expression for the 2-fold λ -bracket $\{\{-\lambda-\}\}_H$ given by (3.85) and Lemma 3.29, the Dirac reduced 2-fold λ -bracket on the quotient space $\mathbb{F} \langle u_i^{(n)} \mid i \in I \setminus \{-N\}, n \in \mathbb{Z}_+ \rangle$, which we denote by $\{\{-\lambda-\}\}_{H^D}$, becomes

$$\begin{aligned} \{\{L(z)_\lambda L(w)\}\}_{H^D} &= L(z) \otimes i_z(z-w-\lambda-\partial)^{-1}L(w) \\ &\quad - L(w+\lambda+\partial) \otimes i_z(z-w-\lambda-\partial)^{-1}L^*(-z+\lambda) \\ &\quad - \frac{1}{N}L(w+\lambda+\partial) \otimes (\lambda+\partial)^{-1}L^*(-z+\lambda) - \frac{1}{N}((\lambda+\partial)^{-1}L(z)) \otimes L(w) \\ &\quad + \frac{1}{N}L(w+\lambda+\partial)(\lambda+\partial)^{-1}L(z) \otimes 1 + \frac{1}{N} \otimes ((\lambda+\partial)^{-1}L^*(-z+\lambda))L(w). \end{aligned} \tag{4.43}$$

Using equations (4.43) and (3.52), the corresponding λ -bracket on $(\mathcal{V}/\mathcal{I})_m = \mathbb{F} \langle u_{ab,i}^n \mid i \in I \setminus \{-N\}, a, b = 1, \dots, m, n \in \mathbb{Z}_+ \rangle$ is given by the following generating series

$$\begin{aligned} \{\{L_{ab}(z)_\lambda L_{cd}(w)\}\}_{H^D} &= L_{cb}(z)i_z(z-w-\lambda-\partial)^{-1}L_{ad}(w) \\ &\quad - L_{cb}(w+\lambda+\partial)i_z(z-w-\lambda-\partial)^{-1}L_{ad}^*(-z+\lambda) \\ &\quad - \frac{1}{N}L_{cb}(w+\lambda+\partial)(\lambda+\partial)^{-1}L_{ad}^*(-z+\lambda) - \frac{1}{N}L_{ad}(w)(\lambda+\partial)^{-1}L_{cb}(z) \\ &\quad + \frac{1}{N} \sum_{k=1}^m \delta_{ad}L_{ck}(w+\lambda+\partial)(\lambda+\partial)^{-1}L_{kb}(z) \\ &\quad + \frac{1}{N} \sum_{k=1}^m \delta_{cb}L_{kd}(w)(\lambda+\partial)^{-1}L_{ak}^*(-z+\lambda). \end{aligned} \tag{4.44}$$

This is the same as equation (4.11) in [DSKV14a].

Example 4.12. For $N = 2$, we have $\mathcal{R}_2/\mathcal{I} = \mathbb{F} \langle u^{(n)} \mid n \in \mathbb{Z}_+ \rangle$ (where u is the image of u_{-1}). The two compatible double PVA structures (H^D, K) given by the

equations (4.44) and (3.86) (using that $L(z) = z^2 + u$) are

$$\begin{aligned}\{\{u_\lambda u\}\}_{H^D} &= (1 \otimes 1) \frac{\lambda^3}{2} + \frac{1}{2}(2\lambda + \partial)(u \otimes 1 + 1 \otimes u) - \frac{1}{2}u \otimes (\lambda + \partial)^{-1}u \\ &\quad - \frac{1}{2}((\lambda + \partial)^{-1}u) \otimes u + \frac{1}{2}u(\lambda + \partial)^{-1}u \otimes 1 + \frac{1}{2} \otimes ((\lambda + \partial)^{-1}u) u, \\ \{\{u_\lambda u\}\}_K &= 2(1 \otimes 1)\lambda.\end{aligned}$$

The above double Poisson structures are (up to a scalar factor) the same as that appearing in formula (6.15) in [OS98]. Note that the Olver and Sokolov formula is given in terms of operators on $\mathcal{R}_2/\mathcal{I}$, rather than as an element of $(\mathcal{R}_2/\mathcal{I} \otimes \mathcal{R}_2/\mathcal{I})[\lambda]$. The rule to recover their formula is the following: we should replace an element $p \otimes q \in \mathcal{R}_2/\mathcal{I} \otimes \mathcal{R}_2/\mathcal{I}$ by the operator $\mathcal{L}_p \circ \mathcal{R}_q$, where \mathcal{L}_p (respectively \mathcal{R}_q) denotes the left (respectively right) multiplication by p (respectively q).

Example 4.13. For $N = 3$, we have $\mathcal{R}_3/\mathcal{I} = \mathbb{F}\langle u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_+ \rangle$ (where u is the image of u_{-2} and v is the image of u_{-1}). The two compatible double PVA structures (H^D, K) given by the equations (4.44) and (3.86) (using that $L(z) = z^3 + uz + v$) are

$$\begin{aligned}\{\{u_\lambda u\}\}_{H^D} &= \frac{1}{3}(u(\lambda + \partial)^{-1}u \otimes 1 + 1 \otimes ((\lambda + \partial)^{-1}u) u) \\ &\quad - \frac{1}{3}(u \otimes (\lambda + \partial)^{-1}u + ((\lambda + \partial)^{-1}u) \otimes u) + 1 \otimes v - v \otimes 1 \\ &\quad + (1 \otimes u)\lambda + (\lambda + \partial)u \otimes 1 + 2(1 \otimes 1)\lambda^3, \\ \{\{u_\lambda v\}\}_{H^D} &= \frac{1}{3}(v(\lambda + \partial)^{-1}u \otimes 1 + 1 \otimes ((\lambda + \partial)^{-1}u) v) \\ &\quad - \frac{1}{3}(v \otimes (\lambda + \partial)^{-1}u + ((\lambda + \partial)^{-1}u) \otimes v) + \frac{1}{3}(u^2 \otimes 1 - u \otimes u) \\ &\quad + 1 \otimes (2\lambda + \partial)v + (v \otimes 1)\lambda + \frac{1}{3}(\lambda + \partial)^2(u \otimes 1 - 1 \otimes u) + (u \otimes 1)\lambda^2 \\ &\quad + (1 \otimes 1)\lambda^4, \\ \{\{v_\lambda v\}\}_{H^D} &= \frac{1}{3}(v(\lambda + \partial)^{-1}v \otimes 1 + 1 \otimes ((\lambda + \partial)^{-1}v) v) + \frac{2}{3}(u \otimes v - v \otimes u) \\ &\quad + \frac{1}{3}(uv \otimes 1 - 1 \otimes uv) - \frac{2}{3}(u \otimes (\lambda + \partial)u) + \frac{2}{3}((\lambda + \partial)^2(1 \otimes v) - (v \otimes 1)\lambda^2) \\ &\quad + \frac{1}{3}((\lambda + \partial)^2(v \otimes 1) - (1 \otimes v)\lambda^2) - \frac{2}{3}((\lambda + \partial)^3(1 \otimes u) - (u \otimes 1)\lambda^3) \\ &\quad - \frac{2}{5}(1 \otimes 1)\lambda^5, \\ \{\{u_\lambda u\}\}_K &= 0, \quad \{\{u_\lambda v\}\}_K = 3(1 \otimes 1)\lambda, \quad \{\{v_\lambda v\}\}_K = u \otimes 1 - 1 \otimes u.\end{aligned}$$

5. ADLER-GELFAND-DICKEY NON-COMMUTATIVE INTEGRABLE HIERARCHIES

In this section we want to show how to apply the Lenard-Magri scheme of integrability (see Section 2.5) in order to obtain integrable hierarchies for the compatible pair of PVAs we constructed in Section 3.8.2.

First we state a technical lemma.

Lemma 5.1. *Let \mathcal{V} be an arbitrary differential algebra endowed with a 2-fold λ -bracket $\{\{-\lambda-\}\}$. Let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be a monic pseudodifferential operator of order $N > 0$. Then, for all $k \geq 1$, the following identity holds in $\mathcal{V}((w^{-1}))$:*

$$\begin{aligned}\text{Res}_z \text{m} \left(\left\{ \left\{ L^{\frac{k}{N}}(z)_\lambda L(w) \right\} \right\} \Big|_{\lambda=0} \right) \\ = \frac{k}{N} \text{Res}_z \text{m} \left(\left\{ \left\{ L(z+x)_x L(w) \right\} \right\} \star_1 \left(\Big|_{x=\partial} L^{\frac{k}{N}-1}(z) \right) \right).\end{aligned}\tag{5.1}$$

Proof. Since, $L^{\frac{k}{N}}(z) = L^{\frac{1}{N}}(z + \partial)L^{\frac{1}{N}}(z + \partial)\dots L^{\frac{1}{N}}(z)$ (k times), we have, by sesquilinearity and the right Leibniz rule,

$$\begin{aligned} \{\{L^{\frac{k}{N}}(z)\}_\lambda L(w)\} &= \sum_{l=1}^k \left(\left|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z + \lambda) \right. \right. \\ &\left. \left. \star_1 \{\{L^{\frac{1}{N}}(z+x)\}_{\lambda+x+y} L(w)\} \star_1 \left(\left|_{x=\partial} L^{\frac{k-l}{N}}(z) \right. \right) \right). \end{aligned} \quad (5.2)$$

Since $m(a \star_1 X \star_1 b) = m(X \star_1 ba)$, for all $a, b \in \mathcal{V}$ and $X \in \mathcal{V}^{\otimes 2}$, we have

$$\begin{aligned} m\{\{L^{\frac{k}{N}}(z)\}_\lambda L(w)\} &= \sum_{l=1}^k m\left(\{\{L^{\frac{1}{N}}(z+x)\}_{\lambda+x+y} L(w)\} \right. \\ &\left. \star_1 \left(\left|_{x=\partial} L^{\frac{k-l}{N}}(z) \right. \right) \left(\left|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z + \lambda) \right. \right) \right). \end{aligned} \quad (5.3)$$

Taking the residue of both sides of equation (5.3) and using (3.66) with $t = \lambda + y$, we get

$$\begin{aligned} \text{Res}_z m\{\{L^{\frac{k}{N}}(z)\}_\lambda L(w)\} &= \text{Res}_z \sum_{l=1}^k m\left(\{\{L^{\frac{1}{N}}(z + \lambda + x + y)\}_{\lambda+x+y} L(w)\} \right. \\ &\left. \star_1 \left(\left|_{x=\partial} L^{\frac{k-l}{N}}(z + \lambda + y) \right. \right) \left(\left|_{y=\partial} L^{\frac{l-1}{N}}(z) \right. \right) \right), \end{aligned}$$

and setting $\lambda = 0$ we get

$$\begin{aligned} \text{Res}_z m\left(\{\{L^{\frac{k}{N}}(z)\}_\lambda L(w)\} \Big|_{\lambda=0}\right) &= \\ k \text{Res}_z m\left(\{\{L^{\frac{1}{N}}(z+x)\}_x L(w)\} \star_1 \left(\left|_{x=\partial} L^{\frac{k-1}{N}}(z) \right. \right) \right). \end{aligned} \quad (5.4)$$

On the other hand, letting $k = N$ in (5.2), we have

$$\begin{aligned} \{\{L(z)\}_\lambda L(w)\} &= \sum_{l=1}^N \left(\left|_{x=\partial} L^{\frac{N-l}{N}}(z) \right. \right) \\ &\star_1 \{\{L^{\frac{1}{N}}(z+x)\}_{\lambda+x+y} L(w)\} \star_1 \left(\left|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z + \lambda) \right. \right). \end{aligned} \quad (5.5)$$

If we replace in equation (5.5), z by $z + \partial$ and λ by $\lambda + \partial$ acting on $L^{\frac{k}{N}-1}(z)$, and we apply the multiplication map m we get

$$\begin{aligned} m\left(\{\{L(z+x)\}_{\lambda+x} L(w)\} \star_1 \left(\left|_{x=\partial} L^{\frac{k}{N}-1}(z) \right. \right) \right) \\ = \sum_{l=1}^N m\left(\{\{L^{\frac{1}{N}}(z+x)\}_{\lambda+x+y} L(w)\} \star_1 \left(\left|_{x=\partial} L^{\frac{k-l}{N}}(z) \right. \right) \left(\left|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z + \lambda) \right. \right) \right). \end{aligned} \quad (5.6)$$

Taking, as before, residues of both sides of equation (5.6) and using (3.66) with $t = \lambda + y$, we get, after setting $\lambda = 0$,

$$\begin{aligned} \text{Res}_z m\left(\{\{L(z+x)\}_x L(w)\} \star_1 \left(\left|_{x=\partial} L^{\frac{k}{N}-1}(z) \right. \right) \right) \\ = N \text{Res}_z m\left(\{\{L^{\frac{1}{N}}(z+x)\}_x L(w)\} \left(\left|_{x=\partial} L^{\frac{k-1}{N}}(z) \right. \right) \right). \end{aligned} \quad (5.7)$$

Equation (5.1) follows from equations (5.4) and (5.7). \square

As in Section 3.8.2, let \mathcal{V} be the non-commutative algebra of differential polynomials in the variables u_i , $i \in I$, namely

$$\mathcal{V} = \mathcal{R}_I = \mathbb{F}\langle u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+ \rangle.$$

(The index set I may be either $I = \{i \in \mathbb{Z} \mid i \geq -N\}$ or $I_- = \{-N, -N + 1, \dots, -1\}$). Let us collect the differential generators of \mathcal{V} into the generating series

$$L(z) = z^N + \sum_{i \in I} u_i z^{-i-1} \in \mathcal{V}((z^{-1})),$$

and let us consider the compatible double PVA structures (H, K) on \mathcal{V} defined by Corollary 3.27. Let, for $k \geq 1$,

$$h_k = \frac{N}{k} \operatorname{Res}_z L^{\frac{k}{N}}(z) \in \mathcal{V}, \quad (5.8)$$

where $L^{\frac{1}{N}}(\partial) \in \mathcal{V}((\partial^{-1}))$ is uniquely defined (see Proposition 1.1 in [DSKV14a]).

Lemma 5.2. *For the compatible pair of double PVA structures (H, K) on \mathcal{V} , we have for $k \geq 1$:*

$$(a) \operatorname{m} \{ \{ h_{k\lambda} L(w) \} \}_H \Big|_{\lambda=0} = L^{\frac{k}{N}}(w + \partial)_+ L(w) - L(w + \partial) L^{\frac{k}{N}}(w)_+;$$

$$(b) \operatorname{m} \{ \{ h_{k\lambda} L(w) \} \}_K \Big|_{\lambda=0} = L^{\frac{k}{N}-1}(w + \partial)_+ L(w) - L(w + \partial) L^{\frac{k}{N}-1}(w)_+.$$

Proof. By Lemma 5.1 and equation (3.85) we have

$$\operatorname{m} \{ \{ h_{k\lambda} L(w) \} \}_H \Big|_{\lambda=0} = \operatorname{Res}_z L^{\frac{k}{N}}(z) i_z (z - w - \partial)^{-1} L(w) - L(w + \partial) \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z (z - w - \partial)^{-1} L^*(-z). \quad (5.9)$$

Using (3.65), we have

$$\operatorname{Res}_z L^{\frac{k}{N}}(z) i_z (z - w - \partial)^{-1} = L^{\frac{k}{N}}(w + \partial)_+, \quad (5.10)$$

while, using (3.66) and equation (3.65), we have

$$\begin{aligned} \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z (z - w - \partial)^{-1} L^*(-z) &= \operatorname{Res}_z L^{\frac{k}{N}-1}(z + \partial) i_z (z - w)^{-1} L(z) \\ &= \operatorname{Res}_z L^{\frac{k}{N}}(z) i_z (z - w)^{-1} = L^{\frac{k}{N}}(w)_+. \end{aligned} \quad (5.11)$$

Combining equations (5.9), (5.10) and (5.11), we get part (a). Similarly, for part (b), we use Lemma 5.1 and equation (3.86) to get

$$\begin{aligned} \operatorname{m} \{ \{ h_{k\lambda} L(w) \} \}_K \Big|_{\lambda=0} &= \operatorname{Res}_z i_z (z - w)^{-1} (L(z + \partial) - L(w + \partial)) L^{\frac{k}{N}-1}(z) \\ &+ \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z (z - w - \partial)^{-1} (L(w) - L^*(-z)). \end{aligned} \quad (5.12)$$

By equations (3.65) and (3.66), we have

$$\begin{aligned} \operatorname{Res}_z i_z (z - w)^{-1} L(z + \partial) L^{\frac{k}{N}-1}(z) &= L^{\frac{k}{N}}(w)_+ \\ &= \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z (z - w - \partial)^{-1} L^*(-z). \end{aligned} \quad (5.13)$$

Moreover, by equation (3.65) we also have

$$\operatorname{Res}_z i_z (z - w)^{-1} L(w + \partial) L^{\frac{k}{N}-1}(z) = L(w + \partial) L^{\frac{k}{N}-1}(w)_+, \quad (5.14)$$

and

$$\operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z (z - w - \partial)^{-1} L(w) = L^{\frac{k}{N}-1}(w + \partial)_+ L(w). \quad (5.15)$$

Combining equations (5.12), (5.13), (5.14), and (5.15), we get the claim. \square

The following Theorem says that the Lenard-Magri scheme of integrability works for the bi-Poisson structure (H, K) , see Remark 3.14:

Theorem 5.3. (a) *For every $\varepsilon \in \{1, \dots, N\}$, we have*

$$\{ \int h_\varepsilon, u \}_K = 0, \quad \text{for all } u \in \mathcal{V}. \quad (5.16)$$

(b) *For every $k \geq 1$, we have the Lenard-Magri recursion*

$$\{ \int h_k, u \}_H = \{ \int h_{k+N}, u \}_K, \quad \text{for all } u \in \mathcal{V}. \quad (5.17)$$

Proof. For $1 \leq \varepsilon < N$, we have $L^{\frac{\varepsilon}{N}-1}(w)_+ = 0$, and therefore, recalling (3.19), equation (5.16) holds by Lemma 5.2(b). Moreover, $\operatorname{m} \{ \{ h_{N\lambda} L(w) \} \}_K \Big|_{\lambda=0} = L(w) - L(w + \partial) \cdot 1 = 0$. This proves part (a). For part (b), by Lemma 5.2, the recursion (5.17) holds for $u = L(w)$, the generating series of the generators of \mathcal{V} . Hence, (5.17) holds for all $u \in \mathcal{V}$ by the left Leibniz rule. \square

Remark 5.4. It follows from Lemma 5.2 that the Hamiltonian equation corresponding to the Hamiltonian functional $\int h_k$, $k \geq 1$, can be written as (in terms of generating series)

$$\frac{dL(w)}{dt_k} = [(L^{\frac{k}{N}})_+, L](w), \quad (5.18)$$

where on the RHS we have to take the symbol of the usual commutator of pseudodifferential operators. This equation is the symbol of the usual *Lax pair* representation of the AGD hierarchies of Hamiltonian equations.

Example 5.5. On $\mathbb{F}\langle u_i^{(n)} \mid i, n \in \mathbb{Z}_+ \rangle$, we have $L(\partial) = \partial + \sum_{i \in \mathbb{Z}_+} u_i \partial^{-i-1}$. By an explicit computation we get

$$\begin{aligned} L^2(\partial) &= \partial^2 + 2u_0 + (2u_1 - u'_0)\partial^{-1} + (2u_2 + u'_1 + u_0^2)\partial^{-2} + \dots, \\ L^3(\partial) &= \partial^3 + 3u_0\partial + 3(u_1 + u'_0) + (3u_2 + 3u'_1 + 3u_0^2 + u''_0)\partial^{-2} + \dots. \end{aligned}$$

Hence, the first few integrals of motion are

$$\int h_1 = \int \text{tr}(u_0), \quad \int h_2 = \int \text{tr}(u_1), \quad \int h_3 = \int \text{tr}(u_2 + u_0^2), \dots$$

To find the corresponding bi-Hamiltonian equations, we use Lemma 5.2. We have $L(w)_+ = w$, $L^2(w)_+ = w^2 + 2u_0$, $L^3(w)_+ = w^3 + 3u_0w + 3(u'_0 + u_1)$. Hence,

$$\begin{aligned} \frac{dL(w)}{dt_1} &= \partial L(w), \quad \frac{dL(w)}{dt_2} = \partial^2 L(w) + 2w\partial L(w) + 2(u_0L(w) - L(w + \partial)u_0), \\ \frac{dL(w)}{dt_3} &= \partial^3 L(w) + 3w\partial^2 L(w) + 3w^2\partial L(w) + 3u_0\partial L(w), \\ &\quad + 3(((w + \partial)u_0 + u_1)L(w) - L(w + \partial)((w + \partial)u_0 + u_1)) \dots \end{aligned} \quad (5.19)$$

Consider the first two equations in the second system of the hierarchy (5.19), and the first equation in the third system of (5.19). Namely,

$$\begin{cases} \frac{du_0}{dt_2} = u''_0 + 2u'_1, \\ \frac{du_1}{dt_2} = u''_1 + 2u'_2 + 2u_0u'_0 + 2u_0u_1 - 2u_1u_0, \\ \frac{du_0}{dt_3} = u'''_0 + 3u''_1 + 3u'_2 + 3u_0u'_0 + 3u'_0u_0. \end{cases}$$

We can eliminate the variable u_2 from this system. After relabeling $t_2 = y$, $t_3 = t$, $u = 2u_0$ and $w = 4u_1 + 2u'_0$, we get the system

$$\begin{cases} u_y = w', \\ 3w_y = 4u_t - u''' - 3(u^2)' + 3[u, w], \end{cases} \quad (5.20)$$

We call the system (5.20) the non-commutative Kadomtsev-Petviashvili (KP) equation.

Example 5.6. On $\mathbb{F}\langle u^{(n)} \mid n \in \mathbb{Z}_+ \rangle$, we have $L(\partial) = \partial^2 + u$. By an explicit computation we get

$$\begin{aligned} L^{\frac{1}{2}}(\partial) &= \partial + \frac{u}{2}\partial^{-1} - \frac{u'}{4}\partial^{-2} + \frac{1}{8}(u'' - u^2)\partial^{-3} \dots, \\ L^{\frac{3}{2}}(\partial) &= \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u' + \frac{1}{8}(3u^2 + u'')\partial^{-1} + \dots. \end{aligned}$$

Hence, the first two integrals of motion are $\int h_1 = \int \text{tr}(u)$ and $\int h_3 = \int \frac{1}{4}\text{tr}(u^2)$. To find the corresponding bi-Hamiltonian equations, we use Lemma 5.2. We have $L^{\frac{1}{2}}(w)_+ = w$ and $L^{\frac{3}{2}}(w)_+ = w^3 + \frac{3}{2}uw + \frac{3}{4}u'$. Hence, $\frac{du}{dt_1} = u'$ and

$$\frac{du}{dt_3} = \frac{1}{4}(u''' + 3uu' + 3u'u). \quad (5.21)$$

The equation (5.21) is the non-commutative KdV equation.

Example 5.7. On $\mathbb{F}\langle u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_+\rangle$, we have $L(\partial) = \partial^3 + u\partial + v$. By an explicit computation we get

$$\begin{aligned} L^{\frac{1}{3}}(\partial) &= \partial + \frac{u}{3}\partial^{-1} + \frac{1}{3}(v - u')\partial^{-2} + \frac{1}{9}(2u'' - u^2 - 3v')\partial^{-3} \dots, \\ L^{\frac{2}{3}}(\partial) &= \partial^2 + \frac{2}{3}u + \frac{1}{3}(2v - u')\partial^{-1} + \frac{1}{9}(3u'' - u^2 - 3v')\partial^{-2} + \dots \end{aligned}$$

Hence, the first two integrals of motion are $\int h_1 = \int \text{tr}(u)$ and $\int h_2 = \int \text{tr}(v)$. To find the corresponding bi-Hamiltonian equations, we use Lemma 5.2. We have $L^{\frac{1}{3}}(w)_+ = w$ and $L^{\frac{2}{3}}(w)_+ = w^2 + \frac{2}{3}u$. Hence, the corresponding Hamiltonian equations are $\frac{du}{dt_1} = u'$, $\frac{dv}{dt_1} = v'$ and

$$\begin{cases} \frac{du}{dt_2} = 2v' - u'' \\ \frac{dv}{dt_2} = v'' - \frac{2}{3}u''' - \frac{2}{3}uu' + \frac{2}{3}[u, v]. \end{cases} \quad (5.22)$$

We call the equation (5.22) the non-commutative Boussinesq equation (since it reduces to the classical Boussinesq equation if $[u, v] = 0$).

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