A Simplified Mathematical Model for Two-Sided Market Systems With an Intervening Engineered Platform

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A SIMPLIFIED MATHEMATICAL MODEL FOR TWO-SIDED MARKET SYSTEMS WITH AN INTERVENING ENGINEERED PLATFORM

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ABSTRACT

A two-sided market involves two different user groups whose interactions are enabled over a platform that provides a distinct set of values to either side. In such market systems, one side's participation depends on the value created by presence of the other side over the platform. Two-sided market platforms must acquire enough users on both sides in appropriate proportions to generate value to either side of the user market. In this paper, we present a simplified, generic mathematical model for two-sided markets with an intervening platform that enables interaction between the two different sets of users with distinct value propositions. The proposed model captures both the same side as well as cross-side effects (i.e., network externalities) and can capture any behavioral asymmetry between the different sides of the two-sided market system. The cross-side effects are captured using the notion of affinity curves while same side effects are captured using four rate parameters. We demonstrate the methodology on canonical affinity curves and comment on the attainment of stability at the equilibrium points of two-sided market systems. Subsequently a stochastic choice-based model of consumers and developers is described to simulate a two-sided market from grounds-up and the observed affinity curves are documented. Finally we discuss how the two-sided market model links with and impacts the engineering characteristics of the platform.

***Keywords**: two-sided markets, affinity curves, critical mass, equilibrium, stability conditions, choice model, open platforms.*

1. INTRODUCTION

A two-sided market involves two different user groups whose interactions are usually enabled over an open platform [2]. The two sides of the market represent the two primary sets of economic agents while the platform acts as an enabler or catalyst for bringing together the two distinct set of economic agents. In such markets, both categories of economic agents have to be present on the platform in right proportions to create enough value to both sides and thereby accelerate and/or sustain the platform. Hence, in such market systems, one side's participation depends on the value created by presence of the other side over the platform. This is termed as the cross-side network effect or *network externalities*.

For example, video game platforms like Sony PlayStation need to attract users and game developers. Users prefer platforms with many games, while developers prefer platforms with many users. The users get attracted to the video game platform when there is a large number and variety of games available on the platform.

Developers of such two-sided market platforms must acquire enough number of users on both sides in appropriate proportion to generate value to either side of the user market and thereby provide the foundation for sustained future growth. In other words, developers must attain and overcome the *critical mass* on either side of the market for sustainable future growth. This is a difficult market coordination problem and is often described as the chicken-and-egg problem in the literature [6]. This aspect is central to any model that aims to simulate two-sided market systems. Most two-sided markets tend to die early (i.e., infant mortality) because they cannot bring the minimum viable number of members on both sides in appropriate proportion for subsequent self-sustenance and growth [6, 7, 9].

Much of the prior work on two-sided markets have employed game theory, primarily by economists but a growing body of work instead studies competition in such markets as a random process with an embedded diffusion process modeling technology adoption processes [3, 5, 10] with emphasis almost entirely on the pricing structure of two sided markets under steady state. There are relatively few references [1, 7, 9] that have looked at modeling two-sided markets during the initiation phase that involves transient effects. This can be achieved by modeling the system as a time evolving dynamical system [12, 13, 14].

While the market dynamics does not directly concern engineering designers, understanding platform users' needs and use behavior is fundamental for the definition of requirements and product strategy. These aspects have already been amply highlighted in product development literature [2, 7], but in the case of open platforms, they assume specific connotations. First, the presence of network externalities has a strong influence on the adoption rate of a platform [2, 7], thus altering the traditional project financing and production forecasts. Secondly, as different sides of the platform have different needs, the definition of requirements should consider both sides at the same time as interactions between the two sides. This new set of requirements make two-sided markets unique and determine the success and failure of the associated platform [6]. Furthermore, the definition of the initial offerings over the platform is expected to be decisive in determining the critical mass [6, 7]. Finally, the designers of the platform do not have complete control over the platform design space and the platform evolution, as they are partially determined by external developers. It is therefore of vital importance to anticipate potential future needs, as the ability to meet them can be constrained by initial design decisions. This aspect is even more critical for open platforms with an expected long life-cycle.

In this paper, we present a simplified, generic mathematical model for two-sided markets with an intervening platform that enables interaction between the two different sets of users with distinct value propositions. One set is interested in using the platform, while the other provides the means for the usage; and the relationship between the two sides is mediated by the exchange over the enabling platform. The proposed model captures both the same side as well as cross-side effects (i.e., network externalities) and can capture any behavioral asymmetry between the different sides of the two-sided market system. The cross-side effects are captured using the notion of *affinity curves* while same side effects are captured using four rate parameters. This dynamic model is used to find the equilibrium points of the overall market system under consideration and investigate the stability of such equilibrium points [1, 13, 14]. We derive the stability conditions for the system equilibrium points and show that they can be defined in terms of the rate parameters only and are not directly dependent on the class of *affinity curves* chosen.

A simple choice model is employed to model the behavior of users and derive two families of canonical affinity curves.

We apply the methodology to canonical affinity curves. We also comment about the attainment of stable equilibrium for the two-sided market systems with these affinity curves.

Finally, a general perspective on integrative open platform design, linking the two-sided market model to engineering decisions/artifacts is presented.

2. MATHEMATICAL MODEL OF THE TWO-SIDED MARKET

Having described an outline of the mathematical model of the two-sided market system, let us get into the details of the model.

In a two-sided market, there are two primary modes that lead to evolution of the market size on either side of the market: (1) the same-side or direct network effect, and (2) the cross-side network effect or network externalities.

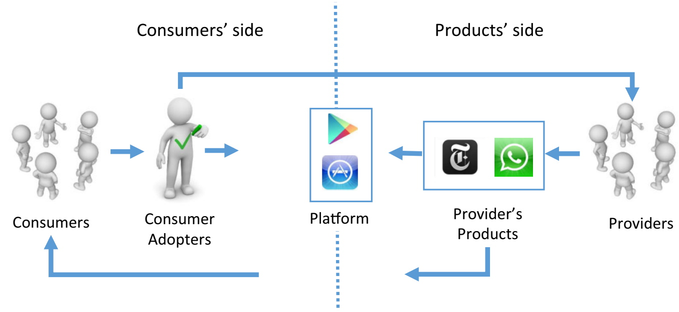

Figure 1: Representation of the two-sided market with consumers on one side and product or modules on the other. The Providers are the developers of the products that are exchanged over the platform.

The same-side effects include factors that originate from the side the participant is in. For example, in Figure 1 above, the same-side effects on the consumer side will not depend explicitly on the characteristics of the product side. Such effects will include quality of the device or services, recurring costs of the products, unmet requirements, economic variables, etc.

The cross-side effect is key to the growth and sustainability of two-sided markets and is the focus of this paper. This network externality effect acts as a “catalyst” that facilitates the interaction between the economic agents on two sides of the “curtain” (see Figure 1). The purely cross-side effects are captured by the *affinity curves*. They indicate the attractiveness of the size of one side of the market to the other side and are said to hold the key for eventual development of the market system [2, 4, 5]. Notice that the two-sided market in this case includes behavioral asymmetry (see Figure 1). In this case, the first side is not interested in the second side directly, as their relation is mediated by a third element (i.e., the product or the module) offered by the second side (i.e., the developers of the engineered products or modules) through the platform. This is an important departure from other cases [3, 11] where there is relative symmetry between the two sides of the market and how they interact. Typical examples include customizable electronic components or devices, app stores for electronic devices and videogame consoles (i.e. Sony PlayStation, Nintendo, GameBox, etc.).

At any time *t+1,* the size of the consumer side of the market can be expressed as the sum of the number of returning consumers from the previous time (t) and the number of new consumers that arrive between the previous and current time.

In the proposed simplified model, we assume distant history does not play a significant role in determining the market size evolution and only market size in the previous time step is required to model the evolution of the market.

On the consumer side, the number of returning consumers is computed using the *death rate* parameter (ε_1), defined as:

$$\varepsilon_1 = \frac{\text{\#consumers left from the previous state}}{\text{\#consumers in previous state}}$$

The number of new consumers is modeled as:

$$G_u(.) = \varepsilon_2 \underbrace{g_u(n_m(t))}_{\text{affinity value}}$$

where $g_u(.)$ stands for the consumer side affinity curve and is a function of the number of modules $n_m(t)$ available over the platform. Here the parameter ε_2 is defined as the *birth rate* and is a function of the same-side effects on the consumer side and depends on the adoption of new products or modules by the consumers [10, 11], given the description of the other side of the market (i.e., visibility over the "curtain"). The birth rate parameter acts as a multiplicative factor on the affinity value. Note that both ε_2 and $g_u(.)$ have to be non-negative to have finite number of new arrivals at any time step.

The factors influencing the *birth rate* parameter include advertisements, endorsements, the bandwagon effect, incentives and others [2, 3, 4, 5, 11].

Please note that the rate parameters $(\varepsilon_1, \varepsilon_2)$ need not be stationary and could be time-varying under a general setting. In such situations, we can use and expected value or a projected value of the rate parameters for use in the analysis.

Similarly we can model the evolution of the product side of the market with the death and birth rate parameters $(\varepsilon_3, \varepsilon_4)$ on the product side.

Combining all the element of the model, we arrive at the final description of the simplified model for the two-sided market:

$$\left.\begin{aligned} n_u(t+1) &= (1-\varepsilon_1)n_u(t) + \varepsilon_2 g_u(n_m(t)) \\ n_m(t+1) &= (1-\varepsilon_3)n_m(t) + \varepsilon_4 g_m(n_u(t)) \end{aligned}\right\} \quad \text{Eq. (1)}$$

Notice that there is an important asymmetry between the two sides in this case. On the product side, there are developers who will engineer the product or modules while the consumers are interested in the products alone and not directly affected by the developers. This introduces asymmetry in their behavior. The number of products and the number of developer teams' is related by:

$$n_m(t) = h(n_d(t)) \qquad \text{Eq. (2)}$$

where $n_d(.)$ stands for the number of developers or development organizations that are developing modules for the platform. Going forward, we will adopt a primarily consumer side view and work with the number of modules $n_m(.)$ in subsequent analysis. Please note that a developer side centric analysis could work with $n_d(.)$ as a primary independent variable.

The evolution of the market size is given as:

$$\left.\begin{aligned} \Delta n_u(t) &\equiv n_u(t+1) - n_u(t) = \varepsilon_2 g_u(n_m(t)) - \varepsilon_1 n_u(t) \\ \Delta n_m(t) &\equiv n_m(t+1) - n_m(t) = \varepsilon_4 g_m(n_u(t)) - \varepsilon_3 n_m(t) \end{aligned}\right\} \quad \textit{Eq. (3)}$$

The evolution of the combined market size is given as:

$$\Delta m(t) = [\varepsilon_2 g_u(n_m(t)) + \varepsilon_4 g_m(n_u(t))] - [\varepsilon_1 n_u(t) + \varepsilon_3 n_m(t)]$$

From Eq. (3) and the above relation, notice that the evolution of any side depends on the number of new arrivals and the number of departing consumers at any time step. For equilibrium of any side of the market, we require the number of departures to be exactly balanced by the number of new consumers. This ensures that there is no eventual growth in the market size as such and the market is in steady state.

For the discussion herein, we have assumed that the affinity curves as well as the four rate parameters are available or estimable. In that sense, the mathematical model introduced here is a *macroscopic* model that embeds *microscopic* information about the market through the affinity curves and the rate parameters. Later in this paper, we will demonstrate an application of stochastic choice-based model using a bottom-up approach to estimation of affinity curves.

3. EQUILIBRIUM OF TWO-SIDED MARKETS

In this section, we discuss about the equilibrium of different sides of the market and also about equilibrium of the overall market system.

At equilibrium, the market system is in *steady state* with the number of arrivals to any side is being balanced by the number of departures of participants from that side. The overall market system is *frozen* in a state and requires external impetus to move out of this *frozen* state. Ability to find the system equilibrium states could provide guidelines for downstream platform engineering and module technology roadmap.

For consumer side equilibrium, we should have:

$$n_u(t+1) = n_u(t) \equiv n_u^*$$

For module or product side equilibrium, we have:

$$n_m(t+1) = n_m(t) \equiv n_m^*$$

Combining them with the growth relations in Eq. (3), we arrive at the following system equilibrium condition:

$$\left.\begin{aligned} \varepsilon_2 g_u(n_m^*(t)) - \varepsilon_1 n_u^*(t) &= 0 \\ \varepsilon_4 g_m(n_u^*(t)) - \varepsilon_3 n_m^*(t) &= 0 \end{aligned}\right\} \quad \textit{Eq. (4)}$$

where (n_u^*, n_m^*) are the fixed-points of the system of nonlinear equations described in Eq. (4). These fixed-points represent the equilibrium of the two-sided market system. The fixed-points can be computed analytically in some cases with canonical affinity curves (as shown in Figure 2), but will have to be solved numerically under more general conditions. In this paper, the above system of equations is solved using MATLAB™ [15] that handles more realistic cases where the affinity curves are defined piece-wise and represented in an incremental fashion over time [14]. As an example, in Figure 2 below, system equilibrium points are shown for resource-constrained affinity curves where Eq. (4) can be represented as: $n_u^* = \dfrac{K_u}{1+e^{-\alpha_u(n_m-\beta_u)}}$ and $n_m^* = \dfrac{K_m}{1+e^{-\alpha_m(n_u-\beta_m)}}$, where $\{K,\alpha,\beta\}$ are the model parameters. They represent the product/module side and the consumer side equilibrium conditions respectively.

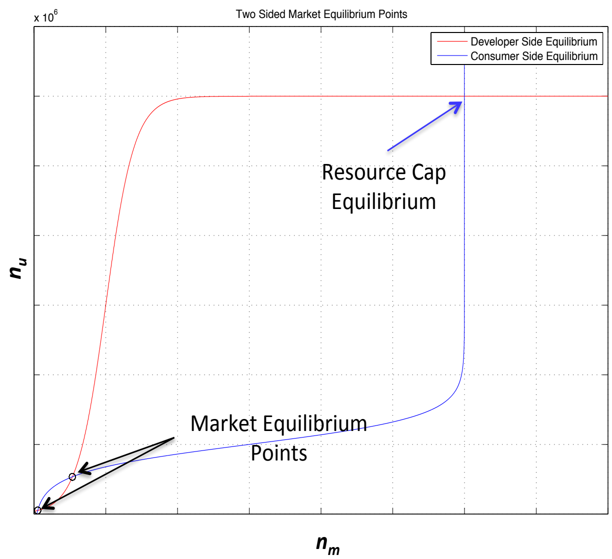

Figure. 2: Equilibrium points for a market system with resource-constrained affinity curves.

Note that there are three equilibrium points with one being the resource cap equilibrium where both sides of the market have been saturated and another being very close to the origin. For attaining critical mass, one needs to reach the second equilibrium point in this case. The number and characterization of the equilibrium points are strongly dependent on the affinity curves.

These equilibrium points can be further classified into stable and unstable equilibrium. In the next section we will investigate the stability of the market system around its equilibrium points.

4. STABILITY CONDITIONS AND AFFINITY CURVES

In case of a stable equilibrium state, small perturbations in the form of external stimuli are insufficient to move the market system away from the equilibrium state. This provides the steady-state condition for the market system. Attainment of a non-zero equilibrium point within a shorter timeframe is important for the two-sided market system since this indicates achievement of critical mass and a plausible path to future sustainability and growth.

Let us look at the stability of the fixed points of the two-sided market system using Eq. (4). The two-sided market system represents a discrete dynamical system [12, 13, 14] of the form:

$$\left.\begin{aligned} x_{t+1} &= f(x_t, y_t) \\ y_{t+1} &= g(x_t, y_t) \end{aligned}\right\}$$

The system (as in Eq. (4)) at fixed points (n_u^*, n_m^*) can be written as:

$$\left.\begin{aligned} n_u^* &= f(n_u^*, n_m^*) \\ n_m^* &= g(n_u^*, n_m^*) \end{aligned}\right\}$$

The corresponding Jacobian [13,14] for the market system is:

$$J \equiv \begin{bmatrix} \left.\dfrac{\partial f}{\partial n_u}\right|_{(n_u^*, n_m^*)} & \left.\dfrac{\partial f}{\partial n_m}\right|_{(n_u^*, n_m^*)} \\ \left.\dfrac{\partial g}{\partial n_u}\right|_{(n_u^*, n_m^*)} & \left.\dfrac{\partial g}{\partial n_m}\right|_{(n_u^*, n_m^*)} \end{bmatrix}$$

Using Eq. (4) with the above definition, we have the following Jacobian matrix for the two-sided market system:

$$J = \begin{bmatrix} (1-\varepsilon_1) & \theta \\ \varphi & (1-\varepsilon_3) \end{bmatrix}$$

where $\theta \equiv \varepsilon_2 \dfrac{\partial g_u(.)}{\partial n_m}; \varphi \equiv \varepsilon_4 \dfrac{\partial g_m(.)}{\partial n_u}$.

For local stability, the absolute eigenvalues $\lambda_{1,2}$ of the Jacobian matrix at the fixed-point (i.e., equilibrium points) should be less than one. Hence for local stability at the equilibrium point, we should have:

$$\left|\lambda_{1,2}(J)_{(n_u^*, n_m^*)}\right| < 1$$

The eigenvalues of the Jacobian matrix, J are given by the solution of the following quadratic equation:

$$\lambda^2 - \lambda(tr(J)) + \det(J) = 0 \qquad \text{Eq. (5)}$$

where, $\det(J) = (1-\varepsilon_1)(1-\varepsilon_3) - \theta\varphi$; $tr(J) = 2 - (\varepsilon_1 + \varepsilon_3)$ and the eigenvalues are related by:

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= \frac{tr(J)}{2} \\ \lambda_1 \lambda_2 &= \det(J) \end{aligned}\right\} \qquad \text{Eq. (6)}$$

Since both roots of Eq.(5) should have their absolute value smaller than 1, the stability conditions can also be expressed as:

$$\frac{1}{4}\left[2[tr(J)]^2 - 4\det(J) + 2tr(J)\sqrt{[tr(J)]^2 - 4\det(J)}\right] < 1$$

$$\frac{1}{4}\left[2[tr(J)]^2 - 4\det(J) - 2tr(J)\sqrt{[tr(J)]^2 - 4\det(J)}\right] < 1$$

The above two inequalities can be combined to derive the stability:

$$[tr(J)]^2 < 2(1+\det(J))$$

This yields the following condition for stability of the equilibrium point:

$$\theta\varphi < (\varepsilon_1+\varepsilon_3) - \frac{1}{2}(\varepsilon_1^2+\varepsilon_3^2)$$

If the eigenvalues were complex, the corresponding stability condition reduces to:

$$\det(J) < 1$$
$$\Rightarrow \theta\varphi > \varepsilon_1\varepsilon_3 - (\varepsilon_1+\varepsilon_3)$$

Combining the above conditions, we can write the overarching stability condition at system equilibrium points as:

$$\varepsilon_1\varepsilon_3 - (\varepsilon_1+\varepsilon_3) < \theta\varphi < (\varepsilon_1+\varepsilon_3) - \frac{1}{2}(\varepsilon_1^2+\varepsilon_3^2)$$

For very small values of the death rate parameters $(\varepsilon_1, \varepsilon_3)$, which is realistic in most real-world applications, we can approximate the stability condition further:

$$-(\varepsilon_1+\varepsilon_3) < \theta\varphi < (\varepsilon_1+\varepsilon_3)$$
$$\Rightarrow |\theta\varphi| < (\varepsilon_1+\varepsilon_3)$$

Note that the stability condition can be expressed in terms of the death rate parameters on either side of the platform only. Also notice that we cannot have a stable equilibrium if the death rates are zero on both sides.

In the following section, we will apply this methodology to canonical affinity curves and identify stability conditions in such cases.

Example: Stability conditions for canonical affinity curves

It is amply clear from the above discussion that we require at least some knowledge about the affinity curves $g_u(.)$ and $g_m(.)$ to find the market equilibrium points and assess their stability.

Let us look at few simple canonical affinity curves and examine the stability of equilibrium points in each case.

Case 1: Simple S-curves -

Let us assume that both sides of the market can be characterized by simple S curves of the form:

$$g_u(.) \equiv g_u^{\max}\left[\frac{n_m}{1+n_m}\right]$$
$$g_m(.) \equiv g_m^{\max}\left[\frac{n_u}{1+n_u}\right]$$

Assuming small death rate parameter values, thereby neglecting the 2nd order effects, the stability condition at equilibrium points can be expressed as:

$$(1+n_u^*)(1+n_m^*) > \sqrt{\left(\frac{\varepsilon_2\varepsilon_4}{\varepsilon_1+\varepsilon_3}\right) g_u^{\max} g_m^{\max}}$$

This stability condition can further be approximated as:

$$n_u^* n_m^* > \sqrt{\left(\frac{\varepsilon_2\varepsilon_4}{\varepsilon_1+\varepsilon_3}\right) g_u^{\max} g_m^{\max}}$$

Hence, for stable equilibrium, the product of market size of both sides should be higher than the square root of the product of limiting values of affinity on both sides. This is the "square root" law for stability.

We can derive similar stability conditions for more complex functional forms of S-curve like affinity curves, but they become increasingly complicated.

Case 2: Power law affinity curves -

Let us assume affinity curves to be of power law form (which includes the linear affinity curves as a special case) where the powers represent the degree of super (sub)-linearity associated with the affinity curves:

$$g_u(.) \equiv a n_m^{\alpha};\ \alpha > 0, a > 0$$
$$g_m(.) \equiv b n_u^{\beta};\ \beta > 0, b > 0$$

where, $\theta\varphi = (\varepsilon_2\varepsilon_4)(ab)(\alpha\beta)(n_m^*)^{\alpha-1}(n_u^*)^{\beta-1} > 0$. If death rate parameters are small, we can neglect the 2nd order effects and arrive at the following stability condition:

$$\frac{\left(n_m^*\right)^{\alpha-1}}{\left(n_u^*\right)^{1-\beta}} < \frac{(\varepsilon_1+\varepsilon_3)}{\left(\varepsilon_2\varepsilon_4\right)(ab)(\alpha\beta)}$$

This condition is quite interesting for different regimes with different values of exponents α and β. Since the numerator on the right hand side of the above inequality is rather small, the market size on the product side has to be much smaller than the consumer side to have stable equilibrium.

In case of linear affinity curves on both sides (i.e., α = β = 1), the stability condition is independent of the market sizes:

$$ab < \frac{(\varepsilon_1+\varepsilon_3)}{\left(\varepsilon_2\varepsilon_4\right)}$$

This is understandable since there could be only one equilibrium point and this point is stable only if the product of the rates of affinity (i.e., coefficients a and b in the affinity curve equations) is limited.

If the product side of the market shows linear affinity, we have β = 1 and for stability, we require:

$$\left(n_m^*\right)^{\alpha-1} < \frac{(\varepsilon_1+\varepsilon_3)}{\left(\varepsilon_2\varepsilon_4\right)(ab)(\alpha)}$$

This implies that for $\alpha > 1$, we are likely to attain stable equilibrium with much smaller number of products. This is a desirable scenario from the platform's perspective.

If the consumer side affinity is super-linear (i.e., $\alpha > 1$) with sub-linear (i.e., $\beta < 1$) affinity on the product/module side, then it is possible to attain stable equilibrium with small number of products (only if there exists an equilibrium point with small number of products). This will likely be the most desirable situation for sustainability of two-sided market systems. Please note that this condition of super-linear affinity on consumer side is not trivial to achieve in a short time.

The next section provides details on construction of affinity curves using a bottom-up, stochastic choice model. This approach can be extended and transitioned into an agent-based approach in the future. From initial results we can see that such bottom-up affinity curves can be well approximated using canonical affinity curves.

5. A BOTTOM-UP APPROACH FOR GENERATION OF AFFINITY CURVES

Correlating the shape and the magnitude of the affinity curves with economic, social and strategic factors is challenging. The questions about modeling the affinity curves and the factors impacting them are still open. In order to answer these questions, we employ simple stochastic choice models to mathematically represent individuals.

In this section, we introduce two simple stochastic choice models and demonstrate how we could leverage such choice models in a bottom-up approach to generate affinity curves.

The stochastic choice model consists of three separate entities related one to the others. In this context, we will call mathematical representation of a user an "agent". Agents that are interested in exploiting the value of the platform are called "Users". Agents that develop products/modules over the platform are called "Developers". Modules do not have any active behavior, but they increase the attractiveness of the platform for Users and determine the investment opportunity for Developers.

The general structure and behavior of agents was derived by direct observation and expert opinion. Consumer behavior is consistent with the Theory of Planned Behavior [16] and the bounded rationality assumption [17], while Developers behave as perfectly rational economic agents, evaluating the perspective profits given a limited amount of information [18].

Products/modules are the key element in the model. They are passive and their attributes affect both sides. These attributes are:

- Type: the class of module a single module represents
- Innovation level: the novelty of the modules in terms of technology, functionality and performance
- Cost of investment K_0: the total cost of investment required to offer the modules on the market
- Mark Up MU: the fraction of the Price of a single module that goes into profits
- Time in the market T: the length of the time during which the module is available in the market
- Costs c_m: cost to sell a single module

Modules are both the reason why a consumer is attracted by the platform and a source of costs and revenues for the developers. Users and Providers are the active parts of the model; they join an industry platform for different reasons, as the two-sided market is asymmetric.

The consumers are attracted by modules that are available over the platform. It is assumed that they join the platform once a certain number of modules are offered. In fact, the more modules/products are present on the platform, the more likely that the consumers are able to satisfy their needs.

Consumers are characterized by two classes of parameters:

- Module preference $\bar{P}_m$: vector quantifying how much a consumer is attracted by a certain class of module
- Threshold T_g: value between 0 and 1 quantifying the propensity to join the platform;

Given a certain number of modules present in the platform, each consumer is subject to the following computation:

$$I_m = \bar{P}_m \cdot \bar{G}^* \qquad (7)$$

Where $\bar{G}^*$ is a vector of i components calculated according to:

$$G_i^* = \min(G_i, G_{\max}) \qquad (8)$$

Here G is a vector that contains the number of modules in the i^{th} class and G_{max} is the maximum number of modules per class that a consumer is able to consider at the same time. The value of G_{max} plays an important role in the shape of the affinity functions. This limit was introduced to model the limits in cognitive bandwidth when a choice is made, in accordance with the bounded rationality assumption. If the interest I_m is greater than the threshold T_g, the agent joins the platform; otherwise, it does not.

Preferences and threshold can be derived from several sources, like surveys, conjoint analysis and focus groups. Later in this paper, we derive the interests from a set of online reviews of smartphones.

Developers are modeled using four types of parameters:

- Module types offered: the type of modules that the developer is able to provide to the platform
- Module variety offered: the number of modules types that any developer is able to provide to the platform
- Variation in expected market share w_h: the minimum and maximum market penetration agents can expect
- Expected uncertainty in market Trend w_u: the minimum and maximum change in forecasted market trend over the investment horizon.

Each developer agent chooses to join the platform given a financial feasibility based on the return of investment or ROI. A developer joins the platform if and only if the ROI is positive, otherwise, does not. The ROI is estimated as follows:

$$ROI = \frac{VA_0 - K_0}{K_0}$$

$$VA_0 = p\left(\frac{w_h n_u}{1+r} + w_h w_u \Delta n_u \sum_{t=2}^{T} \frac{1}{(1+r)^t}\right)$$

$$p = \left(\frac{MU}{1-MU}\right)c_m$$

Here K_0 is the cost of investment, VA_0 is the discounted benefit of the investment, dependent on the price of the module p. Here, w_h is the expected market share, n_u is the number of consumers that have already adopted the platform, and r is the discount rate. The uncertainty in market trend is given by w_u, while T represents the investment horizon. The anticipated average increase in number of consumers during the investment horizon is Δn_u, MU is the module's mark-up and c_m is the cost for selling the module.

It is important to remark that the agents' forecast is approximated with a linear average increase over time; in reality, developers can potentially carry out analyses with a higher degree of fidelity, which are not considered in this paper for the sake of computational simplicity.

Example: Generation of Affinity Curves using stochastic choice models

The choice model is sensitive to the behavioral and economic parameters that are given as an input. The platform considered as a case study is a modular electronic device, like the ones proposed in [19]. The platform provider is in charge of the central bus structure, while independent developers are in charge of designing sensors, batteries and processing units. The two-sided market is therefore composed of phone purchasers (the consumers or users) and module developers. We did not consider secondary market for modules where existing consumers can exchange modules with other consumers.

It is hypothesized that the modules fall into twelve categories, of which six are optional sensors and the others are essential modules like power and storage units (see below Table 1). It is also supposed that the modules can be classified according to three innovation levels, which influence both the consumer interests and the cost of investment required.

According to experts, the cost of investment ranges from 100,000 $ for adapted or optimized modules (i.e., modules that are available in some form or the technology is fairly matured), to 3 million $ for very innovative processor unit or medical sensors. Each baseline investment cost was then varied randomly by 15% around the average value to simulate contingencies in project costs.

Typical values for module mark-up were obtained from online resources [20, 21]. The same sources provided some data regarding the costs of components in traditional smartphones. Since the market considered is very young, a value of 20% was chosen for the mark-up. Finally, it is forecasted that a module will remain available on the market for 12 months.

Optimized	Novel	Radical
Optic Sensors	Optic Sensors	Optic Sensors
Electromagnetic Sensors	Electromagnetic Sensors	Electromagnetic Sensors
Environmental Sensors	Environmental Sensors	Environmental Sensors
Movement Sensors	Movement Sensors	Movement Sensors
Medical Sensors	Medical Sensors	Medical Sensors
Security Sensors	Security Sensors	Security Sensors
Screen	Screen	Screen
Battery	Battery	Battery
Audio	Audio	Audio
Processor	Processor	Processor
Storage	Storage	Storage
Appearance	Appearance	Appearance

Table 1: Modules classes with their variants

Five different user agents were modeled with the help of technical parameters and review data from *www.verizonwireless.com* and *www.gsmarena.com*. The five user agent types were individuals who indicated that their primary mobile phone use was for texting, talking, communicating on social media, surfing the web and interacting with applications. It was assumed that module satisfaction was proportional to module preference and normalized module performance. Data was collected regarding 5 areas of satisfaction and a large number of technical performance variables from 15 popular mobile phones. Our analysis used twelve module classes (*j*) and a subset of 24 performance variables (*k*) that were mapped to the five areas of user satisfaction (*i*). Performance variables *k* were also mapped to module classes *j*. Preferences π_k for individual performance variables *k* were then calculated for each agent type. Preferences for individual modules P_j were then determined via uniformly weighted average of preferences for performance variables *k*.

$$\pi_k = \frac{Satisfaction_i}{performance_k / \max(performance_k)}$$

$$P_j = \sum_{k \in Module(j)} w_k \pi_k$$

The various P_j then form the vector $\bar{P}$ present in Eq. (7). Note that module preferences are normalized by the maximum preference for optimized modules across all 5 agents.

As mentioned before, each of the 12 module types was assumed to have three innovation levels. Notional preference multipliers of 1, 1.5 and 2 were used to denote the increases in preferences for modules with increasing novelty features.

Information about Developers is fuzzier, as the community is still very young and there are no historical data on these platforms. We made the simplifying assumption that each developer can introduce one module at a time. Market penetration between 20% and 80% was randomly set for each developer. The expected uncertainty in future market projection was derived from app market data [20], which shows that predictions about future market can vary in between -25% and +25% (if the initial phase is not considered). For this reason, each developer was assigned a random value of w_u between 0.75 and 1.25.

The consumer side was simulated with number of modules available ranging from 10 to 5000. The variety of Modules was randomly selected for each trial, and 200 trial results were averaged to reduce the noise in the data. In a similar fashion, the developer's side was simulated 200 times; each time, a different developer population was generated. It was assumed that each developer could introduce only one module into the market at any given time step. Developers' affinity function depends not only on the number of consumers, but also on the projected future consumer bases. As mentioned before, this quantity cannot be known a priori, but it must be estimated. Each simulation consists of a full-factorial combination of number of consumers and mean, forecasted growth of the consumer base, the first one increasing linearly from 0 to 30,000, and the second one from 0 to 50,000. For the consumer side, the baseline value of 106 consumers was chosen.

We propose two different regression models for the consumer affinity curve and one for the developer affinity curve. The differences in the consumer model are due to the different saturation effects in their choice model. In the first scenario, it is supposed that the preference P is not bounded by a maximum number of modules G_{max} and therefore the likelihood of purchasing a device increases continuously with the number of modules being offered on the market. This scenario leads to the following family of consumer affinity curve:

$$g_{u1}(n_m) = A_0 + A_1 n_m + A_2 n_m^2 + A_3 n_m^3$$

Where g_{u1} is the affinity curve, n_m is the number of modules and the constants A_i are derived from regression analysis. The values of the constants, as well as other statistical indices are shown in Table 2 below:

Table 2: Coefficient from Users' curve linear regression analysis in absence of saturation

Coefficient	*Value*	*p-value of t-test*
A_0	$7.47\ 10^{2}$	0.67
A_1	2.30	0.45
A_2	$9.68\ 10^{-4}$	0.49
A_3	$4.97\ 10^{-7}$	$0.79\ 10^{-2}$
RMSE	$4.55\ 10^{3}$	---
R^2	0.909	---
p-value F-statistic	$< 10^{-3}$	---

From the R^2 value, as well as from Figure 3, we observe that the regression model capture trends in the data quite well.

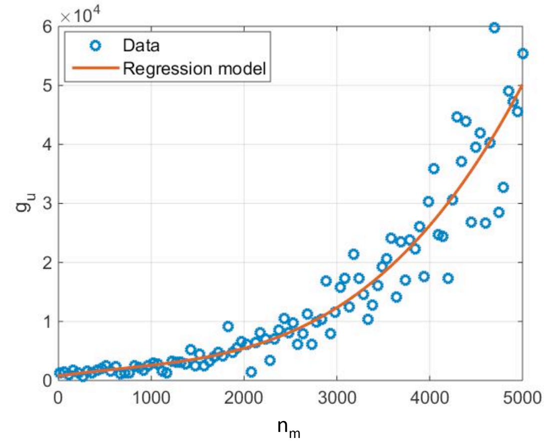

Figure 3: Users' affinity curve when saturation effects are absent

In the second scenario, a limit of 50 Modules is imposed on the consumer choice model, so that if more than 50 modules of a type are offered on the market, the surplus does not affect the overall preferences. This second constraint, expressed in Eq. 8, is introduced in order to account for the limits in cognitive bandwidth when a choice is made.

In this case, consumer interest is limited by a threshold on the maximum number of modules per class that the agent can consider simultaneously in its choice. In this case, the data present a typical S-curve shape (see Figure 4 below):

$$g_{u2}(n_m) = \frac{B_0}{1+\exp(-B_1(n_m - B_2))}$$

Table 3: Coefficient from Users' curve linear regression analysis in presence of saturation

Coefficient	*Value*
B_0	80188
B_1	0.0114
B_2	425
RMSE	$1.23\ 10^{5}$
R^2	0.72

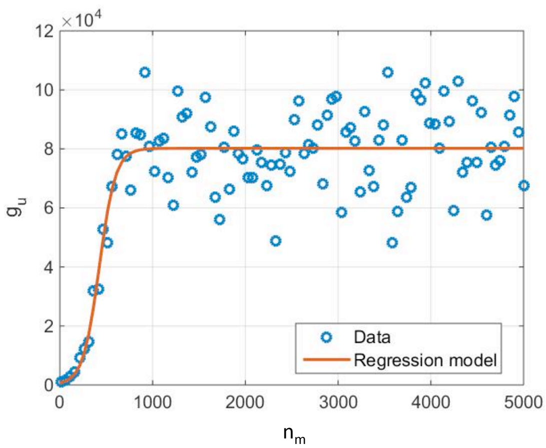

Figure 4: Users' affinity curve when saturation effects are present

The two consumer affinity models are compatible, since it can be said that in the presence of saturation effects, the first curve does not increase forever, but saturates around an asymptote.

Focusing on the other side of the market, the developer affinity curve shape was derived from regression analysis on a subset of the total computational results. The affinity curve is non-linear (see Figure 5). Two areas can be identified: one in which the affinity curve remains equal to zero for various combinations of n_u and Δn_u, and another one in which the affinity curve grows. This logically leads to a function defined in branches:

$$g_m(n_u, \Delta n_u) = f(n_u, \Delta n_u) \quad \text{if } L \geq 0$$
$$g_m(n_u, \Delta n_u) = 0 \quad \text{Otherwise}$$

where g_m is the developer affinity curve and L is a relationship linking n_u and Δn_u. Analysis of the transition zone results in the following definition of L:

$$L = \Delta n_u - l_1 n_u - l_2$$

Where l_1 and l_2 are constant coefficients derived from regression analysis. The Table 4 below provides the coefficients and other relevant statistical indices:

Table 4: Coefficient from linear regression analysis for transient zone

Coefficient	*Value*	*p-value of t-test*
l_1	$3.82\ 10^3$	$< 10^{-3}$
l_2	-0.074	$< 10^{-3}$
RMSE	74.5	---
R^2	0.995	---
p-value F-statistic	$< 10^{-3}$	---

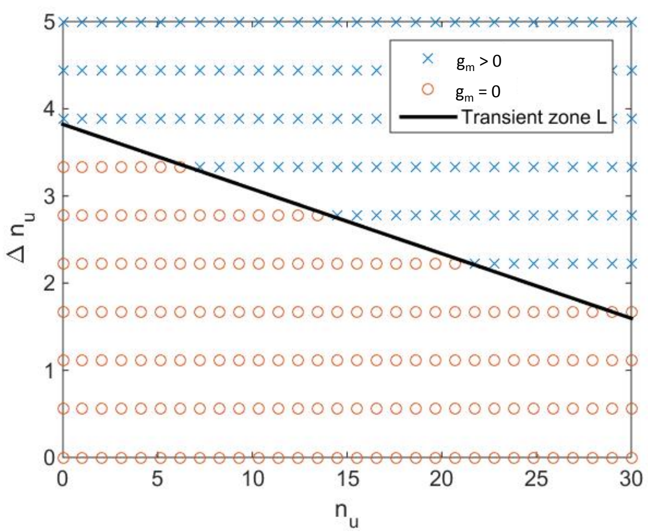

Figure 5. Representation of the transient zone for Developers' affinity curve

The resulting best-fit developer affinity curve (see Figure 6) is a fourth-order polynomial whose coefficients are reported below:

Table 5: Coefficient from developers curve analysis in absence of consumer side saturation

Term in the polynomial	*Estimated Value of the coefficient*	*p-value of t-test*
Δn_u	$6.31\ 10^{-1}$	$< 10^{-3}$
$n_u \Delta n_u$	$-3.33\ 10^{-6}$	$< 10^{-3}$
Δn_u^2	$-3.47\ 10^{-5}$	$< 10^{-3}$
$n_u^2 \Delta n_u$	$4.60\ 10^{-11}$	$< 10^{-3}$
$n_u \Delta n_u^2$	$1.11\ 10^{-9}$	$< 10^{-3}$
Δn_u^3	$6.00\ 10^{-9}$	$< 10^{-3}$
$n_u^2 \Delta n_u^2$	$-6.35\ 10^{-12}$	$< 10^{-3}$
$n_u^3 \Delta n_u$	$-9.94\ 10^{-17}$	$< 10^{-3}$
$n_u \Delta n_u^3$	$-8.03\ 10^{-14}$	$< 10^{-3}$
Δn_u^4	$-3.06\ 10^{-13}$	$< 10^{-3}$
RMSE	0.94	---
R^2	0.999	---
p-value F-statistic	$< 10^{-3}$	---

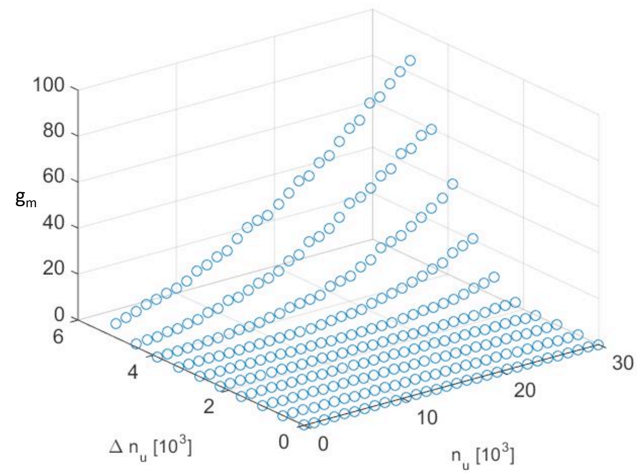

Figure 6: Developers affinity curve

The proposed curves represent an initial attempt at characterizing affinity curves from numerical simulations. The choice model needs to be refined with more features, and the robustness of the results must be tested against real data. Nevertheless, these affinity curve shapes allow us to characterize the stability of the equilibrium points resulting from the intersection of these curves.

Notice that the approximated affinity curves can be represented by well known canonical affinity curves like additive power law curves and S curves.

6. IMPACT OF TWO-SIDED MARKET MODEL ON ENGINEERING CHARACTERISTICS

In contrast to products in isolation, the set of architectures supported via a single two-sided platform grows over time. This growth occurs through the evolution of a two-sided market that connects users and module developers through the platform. The coupling between the possible technical evolutions of the platform and the evolution of the two-sided market necessitates taking into account economic and platform engineering early on via two-sided market simulation.

An evolving architecture over time can be examined by conducting enumeration of different combinations of modules subject to engineering feasibility constraints. Feasibility constraints can be anything from simple experience-based rules pertaining to the existence of individual components/connections (i.e., "node constraints") to more sophisticated constraints that leverage engineering principles and heuristics and apply to groups of connections/components

We generated 3072 feasible module bundles from the library of 12 module classes at 3 different innovation levels (see Table 1). First cost was determined through the summation of the assumed prices of individual modules. Potential satisfaction was determined based on normalized preferences for modules and their performance metrics.

The enumeration was carried out as follows:

1. *Design Space Definition:* All possible connections and components were expressed via a design structure matrix based adjacency matrix that contained all possible pairwise feasible connections using an enhanced Design Structure Matrix, called DS2M. For this analysis the DS2M were generated based primarily on simple connection type compatibility (e.g. outputs can only flow to inputs and vice versa for each connection type).

2. *Component Enumeration:* All possible components in the DS2M matrix were expressed via a binary decision vector. A depth first search was conducted subject to rule-bases to determine feasible module bundles.

3. *Connection Enumeration:* The output of component enumeration was fed into connection enumeration. Every feasible bundle of components had a set of pairwise feasible connections in the DS2M defined previously. This set of potentially feasible connections was converted into a binary decision vector. A depth first search was conducted subject to connection rules until a way to feasibly connecting the modules in bundle was found.

The design of a two-sided platform and the initial set of modules available on the platform strongly influence the evolution of the two-sided market. Note that each point in Figure 7 represents a module bundle as evaluated by one of the 5 user types mentioned earlier.

In Figure 7, the set of initial modules and the platform define the possible architectures as well as initial architectural evolution pathways available to users.

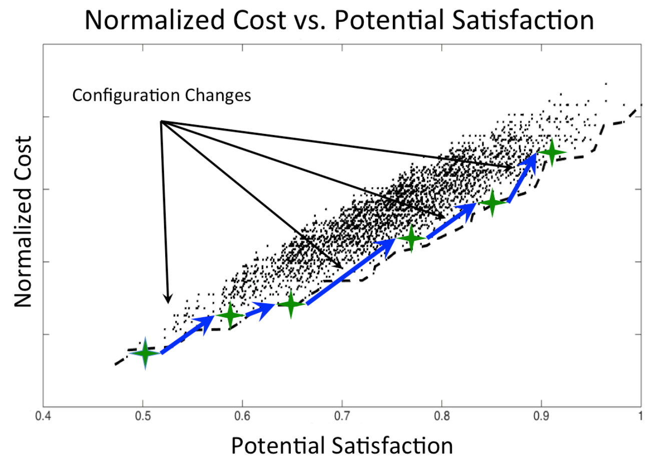

Figure 7. Pareto-front architectures for the modular electronic device

To successfully design the platform and choose an initial set of modules we therefore need to simulate the behavior of the two-sided market enabled by platform between consumers and module developers. For example, the overarching two-sided market analysis could provide a module development roadmap that needs to be exercised for sustainable evolution of the platform.

7. CONCLUSIONS AND FUTURE WORK

This paper proposes s simplified mathematical model of the two-sided markets based on dynamical system theory and captures the cross-side effects using affinity curves and same side effects using the birth and death rate parameters for both sides of the market. In order to have new adopters on either side, we require positive affinity from cross-side interactions and positive birth rate parameter from the same side effects.

We leveraged the dynamical model to compute the fixed-points or the equilibrium points of the two-sided market system and derived conditions for achieving stable equilibrium. Stable equilibrium indicates a steady state of operation of the system where there is no net growth on either side of the market and requires external impetus to get unstuck from this state.

We derived the two-sided markets' affinity curves from a bottom-up perspective using a statistical choice model, consisting of two types of agents. A modular electronic device was introduced as a case study. The agent's behavioral parameters were mainly derived from an online database for the consumers' side and from expert opinion for the product/module developers' side. Analytical affinity curves were derived using nonlinear regression analyses on the choice model's simulation results. The affinity curves were found to be representative of well-known canonical forms for which analytical results on equilibrium and stability of the market system can be arrived at.

We derived the "square root law" for stability if the affinity curves can be approximated using simple S-curve profile. It was shown that the stability condition is independent of market size if we have linear affinity curves for both sides. The most desirable situation from platform's perspective is when the consumer affinity grows super-linearly since one can achieve stability with relatively small number of products or module offerings over the platform. But attaining this "favorable condition" may be difficult in practice for two-sided markets.

When there is no or little "real market", providing external incentive to improve the affinity usually has positive impact.

When there exists a real module or products, it might be useful to focus external incentives to the consumer side to attain $\varepsilon_2 > 1$ in the market initialization phase (i.e., before reaching the critical mass).

Many research directions can be envisioned in order to refine the choice model. Consumers' choice does not consider the effect of price in the simplified example discussed herein. Several model from marketing literature, like the logit model, can improve the agents' behavior on this aspect. Secondly, many parameters in the models are based on experience and opinions, and they need to be checked against other platforms. Furthermore, more case studies need to be considered to prove the generality of the affinity curves' equations.

In future, we can extend the model to multi-sided platforms and develop the equilibrium and stability conditions for such market systems. Further refinement of affinity curves to include distribution of module or product types that are available is another area of improvement. Integration of the two-sided market model with characteristics of the underlying engineering platform creates a framework to study what if scenarios as well as sequential evolution path for optimal evolution of the platform.

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