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SAMPLED-DATA CONTROL OF HIGH-SPEED TRAINS

by

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ABSTRACT

This report deals with the control of the positions and velocities of high-speed vehicles in a single guideway. It is assumed that each and every train measures its position and velocity every T seconds. The appropriate accelerations or decelerations to be applied to each vehicle are constrained to be constant during the sampling interval. Through the use of a control cost functional, which penalizes the system for any deviations from the desired headway and velocity, the required control accelerations and decelerations are obtained by deriving the system equations in discrete-time and, through the use of available results in the theory of discrete optimal control, the optimal linear time-invariant sampled-data feedback control system is determined. The general results, as well as the general purpose digital computer programs, are presented and are used to study the effect of changing the sampling time T upon the control-system performance. Since, in general, the cost of the communication system (in terms of required channel capacity, bandwidth, etc.) decreases with increasing values of the sampling time, the system designer has the capability of conducting trade-off studies involving the deterioration of the control system performance vs. the decrease in the cost of communication as the sampling time is increased.

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The computations were performed at the M.I.T. Information Processing Services Center.

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I. INTRODUCTION

This study was motivated by the continuing interest in developing simple but efficient methods for controlling the spacing, velocity and acceleration of each individual vehicle in a tightly packed string of high-speed trains. In work reported previously[1*] the equations of motion of the vehicles were linearized about a set of operating conditions and the optimal control system was determined. The criterion with respect to which the performance of the trains was optimized, was a quadratic functional that penalized the system for deviations from the desired average velocity, for deviations from the specified separation distance between adjacent trains and, finally, for the use of large corrective forces. The optimal design was found to be a linear, time-invariant, state feedback control. Simulation on an analog computer showed that the design was quite satisfactory. Nevertheless, actual implementation in large systems became problematic because of the large number of feedback loops required. This realization motivated research in two directions--both having the same objective: the reduction of the size and complexity of the required communication system, or, alternatively, the reduction of the amount of information that had to be transmitted.

The first approach was based on the practical realization, substantiated by theoretical results, that a vehicle's behavior is affected mostly by the behavior of adjacent vehicles. This permitted the development of suboptimal control designs that required a drastically reduced number of feedback loops for their implementation. Furthermore, results from simulation of typical systems on the analog computer compared well with respect to the optimal system.[1]

The alternative approach, i.e., the reduction of the amount of information to be transmitted, motivated the introduction of sampled-data feedback. In this case, the position and velocity of each vehicle

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* Superscripts refer to numbered items in the References.

in the string are measured once every T seconds, (sampling). On the basis of this information, corrective forces are applied that remain constant until the next sampling of the data.

The basic model and the design philosophy used in this report are similar to those used in the continuous feedback case. In addition, the theoretical development is parallel, but in the sampled-data case the discrete minimum principle is used.[2]

The optimal control was found to be a linear one, i.e., over a sampling interval each corrective force could be expressed as a weighted linear combination of the values of the position and velocity deviations measured at the beginning of the interval. Evaluation of the weighting coefficients required the solution of a general discrete-type matrix Riccati equation. This was implemented successfully on a high-speed digital computer.

In Section II of this report the model of the system is described and the optimization problem is formulated. In Section III, the sampled-data model is transformed to a discrete-time one and an equivalent discrete-time optimization problem is stated. In Section IV, the analytical solution to the problem is presented. As an illustrative example, numerical results for a string of five trains obtained from the digital computer solution of the problem are presented in Section V. In Section VI, the cost of communication systems necessary for the implementation of the optimal control is discussed. Finally, in the Appendix, the digital computer program is listed along with instructions for its use.

Although the research reported here is centered on the control of vehicles moving in a single guideway, the results are also applicable to the sampled-data control of high-speed trains for the merging problem. The reason is that the merging problem can be transformed into a problem involving the control of vehicles in a single guideway. This approach is explained in more detail in the report by Athans and Levine[3] and it is not discussed here. For the purpose of this report, it suffices to state that the same type feedback system presented here can be used to control the vehicles in the merging problem.

II. DEFINITION OF THE OPTIMIZATION PROBLEM

The investigation is restricted, as in the past, to the properties of a string of vehicles moving along a single guideway at high speed. The problem of injection (i.e., merging) of vehicles from the guideway is treated in Ref. 3. The n vehicles that comprise the string are taken to be identical and are moving with an average, "cruising" velocity v_o. This implies that the starting and stopping operations near stations are excluded from the model.

Since the description of the physical model and the derivation of the governing mathematical relations are presented in detail in Ref. 1, only a brief exposition of the salient features of the resulting system of equations will be given here.

Each vehicle is modeled as a second-order dynamical system with nonlinear damping. If, instead of actual vehicle position, the deviation from a desired separation distance between adjacent trains is considered as the position state variable, then the equations of adjacent vehicles become coupled. Furthermore, if instead of actual vehicle velocity, the deviation from the prescribed mean velocity v_o is considered as the velocity state variable, then the equations of motion can be linearized.

The linearized string equations are

$$\frac{d}{dt}\,\delta\omega_k(t) = \delta y_k(t) - \delta y_{k+1}(t) \tag{2.1}$$

$$m\,\frac{d}{dt}\,\delta y_k(t) = -\alpha_o \delta y_k(t) + \delta f_k(t) \tag{2.2}$$

where $\delta\omega_k(t)$ is the deviation from the desired separation distance between the k-th and the (k+1)st vehicles, δy_k is the velocity deviation of the k-th vehicle, and δf_k is the incremental corrective force applied to the k-th vehicle. Since the vehicles were assumed identical, m is the mass of each vehicle and $\alpha_o \delta y_k(t)$ is the linearized drag force. The separation distance does not appear explicitly in the relations. It should be noted that Eq. 2.1 is valid for $k=1, \ldots, n-1$ while Eq. 2.2 is valid for $k=1, \ldots, n$.

The introduction of the following dimensionless variables simplifies the equations and makes the results applicable to many dynamically similar models.

For the k-th vehicle:

dimensionless distance deviation: $\chi_k(t) = \delta\omega_k(t)/(\frac{mv_o}{a_o})$ (2.3)

dimensionless velocity deviation: $\psi_k(t) = \delta y_k(t)/v_o$ (2.4)

dimensionless incremental force: $\phi_k(t) = \delta f_k(t)/a_o v_o$ (2.5)

dimensionless time: $\tau = t/(m/a_o)$ (2.6)

The equations of motion (2.2) and (2.1) then take the normalized form

$$\frac{d}{d\tau}\psi_k(\tau) = -\psi_k(\tau) + \phi_k(\tau) \tag{2.7}$$

$$\frac{d}{d\tau}\chi_k(\tau) = \psi_k(\tau) - \psi_{k+1}(\tau) \tag{2.8}$$

By interlacing the velocity deviations ψ_k with the deviations from the desired position χ_k, the following matrix representation of the dynamical equations of motion for a string of N vehicles is obtained:

$$\frac{d}{d\tau}\underbrace{\begin{bmatrix}\psi_1\\ \chi_1\\ \psi_2\\ \chi_2\\ \vdots\end{bmatrix}}_{\underline{x}(\tau)} = \underbrace{\begin{bmatrix}-1 & 0 & 0 & 0 & \cdots\\ 1 & 0 & -1 & 0 & \\ 0 & 0 & -1 & 0 & \cdots\\ 0 & 0 & 1 & 0 & \\ & \vdots & & \vdots & \vdots\end{bmatrix}}_{\underline{A}}\underbrace{\begin{bmatrix}\psi_1\\ \chi_1\\ \psi_2\\ \chi_2\\ \vdots\end{bmatrix}}_{\underline{x}(\tau)} + \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ \vdots & & & \vdots\end{bmatrix}}_{\underline{B}}\underbrace{\begin{bmatrix}\phi_1\\ \phi_2\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}}_{\underline{u}(\tau)} \tag{2.9}$$

Using the indicated vector-matrix notation, Eq. 2.9 can be written compactly as

$$\frac{d}{d\tau}\underline{x}(\tau) = \underline{A}\,\underline{x}(\tau) + \underline{B}\,\underline{u}(\tau) \tag{2.10}$$

where:

$\underline{x}$ is the (2n-1) state vector of the system

$\underline{u}$ is the n control vector of the system

$\underline{A}$ is the (2n-1) x (2n-1) system matrix

$\underline{B}$ is the (2n-1) x n gain matrix

Next, a crucial assumption on the structure of the time function $\underline{u}(\tau)$ is made. The interval of definition of the system $[\tau_o, \tau_f]$, where τ_o is the initial time and τ_f is the terminal time, is subdivided into a sequence of N intervals of equal length T:

$$NT = \tau_f - \tau_o \tag{2.11}$$

<u>It is then assumed that the control vector $\underline{u}$ takes some constant value over each interval.</u>

$$\underline{u}(\tau) = \underline{u}(\tau_o + kT) \overset{\Delta}{=} \underline{u}_k \; ; \; \tau_o + kT \leq \tau < \tau_o + (k+1)T \tag{2.12}$$

This assumed structure of the control (Fig. 1) will yield a <u>sampled-data</u> feedback control as the optimal one. In order to simplify the expression that will follow, but with no conceptual loss in generality, the initial time τ_o will be taken to coincide with the origin, i.e., $\tau_o = 0$.

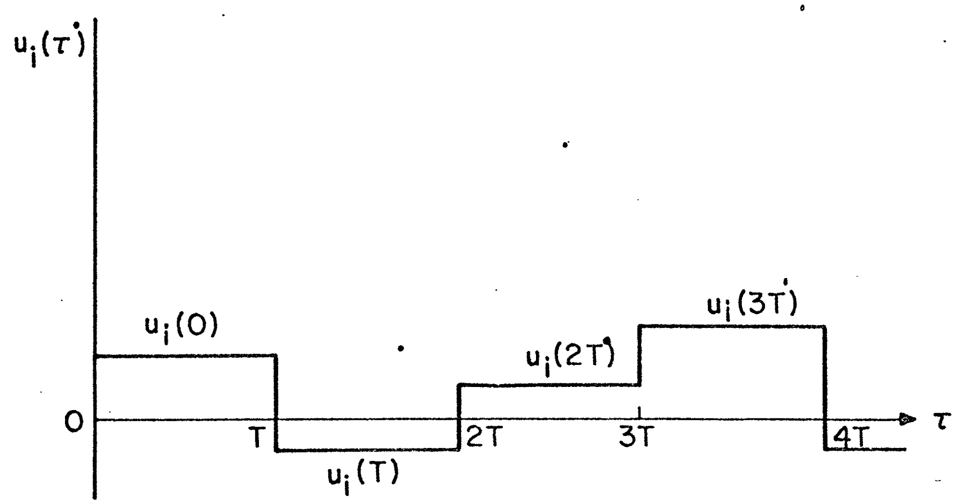

Fig. 1 Typical Piecewise Constant Element of the Control Vector

The cost criterion, with respect to which the performance of the system is optimized, is of the form of a quadratic functional

$$J = \lim_{N\to\infty} \frac{1}{2}\int_0^{NT} \{[q\sum_{k=1}^{n-1} \delta\omega_k^2(\tau)]+[p\sum_{k=1}^{n}\delta y_k^2(\tau)]+[r\sum_{k=1}^{n}\delta f_k^2(\tau)]\}d\tau \qquad (2.13)$$

The first sum of quadratic terms penalizes the system for deviations, positive or negative, from the desired separation distance; the second sum reflects the cost of deviations from the string velocity v_o; the last sum penalizes the system for requiring large corrective forces because that implies sudden accelerations and decelerations. The non-negative coefficients p, q, and r permit flexibility in assigning relative importance to the various errors.

The above criterion can be expressed compactly with vector-matrix notation as follows

$$J = \lim_{N\to\infty} \frac{1}{2}\int_0^{NT} \{\underline{x}'(\tau)\underline{Q}\,\underline{x}(\tau) + \underline{u}'(\tau)\underline{R}\,\underline{u}(\tau)\}d\tau \qquad (2.14)$$

where $'$ denotes transposition and where $\underline{Q}$ is a constant positive semidefinite $(2n-1)\times(2n-1)$ matrix defined by

$$\underline{Q} = \text{diag}\{p, q, p, q, \ldots, q, p\} \qquad (2.15)$$

and $\underline{R}$ is constant, positive definite, $n\times n$, diagonal matrix given by

$$\underline{R} = \text{diag}\{r, r, \ldots, r\} = r\underline{I} \qquad (2.16)$$

The optimization problem that is now under consideration is the following:

<u>Given a dynamical system characterized by a linear differential equation</u>

$$\dot{\underline{x}}(\tau) = \underline{A}\,\underline{x}(\tau) + \underline{B}\,\underline{u}(\tau) \;;\; \underline{x}(0) \triangleq \underline{x}_o \triangleq \underline{\xi} \;;\; \tau \in [0, \tau_f] \qquad (2.17)$$

<u>determine a discrete control sequence</u>

$$\underline{u}(\tau) = \underline{u}(kT) \triangleq \underline{u}_k \;;\; kT \le \tau < (k+1)T \qquad (2.18)$$

where

$$\{k\} = \{0, 1, \ldots, N\} \tag{2.19}$$

that minimizes the quadratic cost functional

$$J(\underline{x}, 0, \infty, \underline{u}) = \lim_{\tau_f \to \infty} \frac{1}{2} \int_0^{\tau_f} [\underline{x}'(\tau)\underline{Q}\,\underline{x}(\tau) + \underline{u}'(\tau)\underline{R}\,\underline{u}(\tau)]\, d\tau \tag{2.20}$$

where T is the sampling period, τ_f is the final time such that

$$NT = \tau_f \tag{2.21}$$

III. THE EQUIVALENT DISCRETE-TIME OPTIMIZATION PROBLEM

In this section the optimization problem that was posed at the end of the previous section will be transformed to an equivalent discrete-time one in a form which permits the direct application of the discrete minimum principle. The type of transformations involved first appeared in this context in Ref. 4.

If $\underline{\Phi}(\tau, \tau_o)$ is the transition matrix of the homogeneous differential equation

$$\dot{\underline{x}}(\tau) = \underline{A}\,\underline{x}(\tau) \tag{3.1}$$

associated with the system described by Eq. 2.17, then the solution to Eq. 2.17 expressing the state at time $(k+1)T$ in terms of the state at kT and the constant control $\underline{u}(kT)$ is given by

$$\underline{x}[(k+1)T] = \underline{\Phi}(T,0)\underline{x}(kT) + \underline{D}(T,0)\underline{u}(kT) \quad ; \quad \underline{x}(0) = \underline{x}_o \stackrel{\Delta}{=} \xi \tag{3.2}$$

It is observed that the matrices $\underline{\Phi}$ and $\underline{D}$ are time-invariant and depend only on the sampling period T. This can be shown as follows

$$\underline{\Phi}[(k+1)T, kT] = e^{\underline{A}[(k+1)T - kT]} = e^{\underline{A}T} = \underline{\Phi}(T,0) \tag{3.3}$$

and

$$\underline{D}[(k+1)T, kT] = \int_{kT}^{(k+1)T} \underline{\Phi}[(k+1)T, t]\,\underline{B}dt = \int_0^T e^{\underline{A}t}\underline{B}dt =$$

$$\int_0^T \underline{\Phi}(t,0)\,\underline{B}dt = \underline{D}(T,0) \tag{3.4}$$

The cost functional J of Eq. 2.20 can be expressed as the sum over k of N integrals

$$J(\underline{x}, 0, \infty, \underline{u}) = \lim_{N \to \infty} \sum_{k=0}^{N-1} \frac{1}{2} \int_{kT}^{(k+1)T} [\underline{x}'(\tau)\underline{Q}\,\underline{x}(\tau) + \underline{u}'(\tau)\underline{R}\,\underline{u}(\tau)]\,d\tau \tag{3.5}$$

If the formal solution of Eq. 2.17 is substituted in each integral and if the fact that $\underline{u}$ is constant over the interval of each integration is taken into account, then the following expression for J can be derived

$$J(\underline{x}, 0, \infty, \underline{u}) = \lim_{N\to\infty} \frac{1}{2} \sum_{k=0}^{N-1} [\underline{x}'(kT)\hat{\underline{Q}}(T,0)\underline{x}(kT) +$$

$$2\underline{x}'(kT)\underline{M}(T,0)\underline{u}(kT) + \underline{u}'(kT)\hat{\underline{R}}(T,0)\underline{u}(kT)] \tag{3.6}$$

Again, the observation is made that the matrices $\hat{\underline{Q}}$, $\underline{M}$ and $\hat{\underline{R}}$ defined below are time invariant and depend only on the sampling period T. This can be seen directly from the definitions

$$\hat{\underline{Q}}[(k+1)T, kT] = \int_{kT}^{(k+1)T} \underline{\Phi}'(t,kT)\underline{Q}\,\underline{\Phi}(t,kT)\,dt = \int_0^T e^{\underline{A}'t}\underline{Q}\,e^{\underline{A}t}dt = \hat{\underline{Q}}(T,0) \tag{3.7}$$

$$\underline{M}[(k+1)T, kT] = \int_{kT}^{(k+1)T} \underline{\Phi}'(t,kT)\underline{Q}\,\underline{D}(t,kT)\,dt = \int_0^T e^{\underline{A}'t}\underline{Q}[\int_0^t e^{\underline{A}s}ds]\,\underline{B}\,dt$$

$$= \underline{M}(T,0) \tag{3.8}$$

and

$$\hat{\underline{R}}[(k+1)T, kT] = \int_{kT}^{(k+1)T} [\underline{R}+\underline{D}'(t,kT)\underline{Q}\,\underline{D}(t,kT)]\,dt$$

$$= \underline{R}T + \underline{B}'\cdot\int_0^T [\int_0^t e^{\underline{A}'s}ds]\underline{Q}[\int_0^t e^{\underline{A}s}ds]\,dt\cdot\underline{B} = \hat{\underline{R}}(T,0) \tag{3.9}$$

Furthermore, it can be shown that if $\underline{R}$ and $\underline{Q}$ are positive definite and semidefinite respectively, so are $\hat{\underline{R}}$ and $\hat{\underline{Q}}$. It was necessary that the above transformations preserve these properties, since they are required for the existence of the solution.

With the above transformations the problem can now be formulated as a discrete optimization one. The following notational correspondence is assumed in order to simplify the appearance of the relations

$$\underline{x}[(k+1)T] \triangleq \underline{x}_{k+1}$$

$$\underline{u}(kT) \triangleq \underline{u}_k$$

and the arguments $(T, 0)$ of the various transformed matrices are suppressed, i.e.,

$$\underline{\Phi}(T,0) \triangleq \underline{\Phi}, \quad \underline{\hat{R}}(T,0) \triangleq \underline{\hat{R}}$$

The Equivalent Discrete-Time Optimization Problem

Given the linear discrete-time system

$$\underline{x}_{k+1} = \underline{\Phi}\,\underline{x}_k + \underline{D}\,\underline{u}_k \quad ; \quad \underline{x}_o = \underline{\xi} \tag{3.10}$$

determine the control sequence

$$\{\underline{u}^*_k, \ k=0, \ 1, \ldots, \ N-1\} \tag{3.11}$$

and corresponding trajectory $\{\underline{x}^*_k\}$, such that the cost functional

$$J(\{\underline{u}_k\}) = \lim_{N\to\infty} \frac{1}{2} \sum_{k=0}^{N-1} [\underline{x}'_k \underline{\hat{Q}}\,\underline{x}_k + 2\underline{x}'_k \underline{M}\,\underline{u}_k + \underline{u}'_k \underline{\hat{R}}\,\underline{u}_k] \tag{3.12}$$

attains its minimum value at

$$\{\underline{u}_k\} = \{\underline{u}^*_k\} \quad ; \quad \{\underline{x}_k\} = \{\underline{x}^*_k\} \tag{3.13}$$

It should be noted that the above problem is a generalized discrete-time analogue to the continuous-time linear regulator problem, as described in Ref. 1.

IV. SOLUTION OF THE OPTIMIZATION PROBLEM

The optimization problem, as formulated in Section III, satisfies the assumptions necessary for the valid application of the discrete minimum principle.[2] In particular, the problem consists of a set of linear difference equations with a quadratic cost functional, where $\hat{\underline{Q}}$ is positive semidefinite and $\hat{\underline{R}}$ is positive definite for any choice of sampling interval. The system matrix $\underline{\Phi}$ is nonsingular for any T since it is a matrix exponential.

The control sequence which minimizes the cost functional J at the limit $N \rightarrow \infty$ for any set of initial conditions $\underline{x}_o$ is given by

$$\underline{u}_k^* = -\{\hat{\underline{R}}^{-1}\underline{M}' + [\hat{\underline{R}}+\underline{D}'\hat{\underline{K}}\underline{D}]^{-1}\underline{D}'\hat{\underline{K}}[\underline{\Phi}-\underline{D}\hat{\underline{R}}^{-1}\underline{M}']\}\underline{x}_k^* \triangleq -\underline{G}\underline{x}_k^* \qquad (4.1)$$

The matrix $\hat{\underline{K}}$ is the steady-state solution, i.e.,

$$\hat{\underline{K}} = \lim_{k \rightarrow -\infty} \underline{K}_k \qquad (4.2)$$

of the nonlinear matrix difference equation

$$\underline{K}_k=[\underline{\Phi}-\underline{D}\hat{\underline{R}}^{-1}\underline{M}']'[\underline{K}_{k+1}-\underline{K}_{k+1}\underline{D}(\hat{\underline{R}}+\underline{D}'\underline{K}_{k+1}\underline{D})^{-1}\underline{D}'\underline{K}_{k+1}][\underline{\Phi}-\underline{D}\hat{\underline{R}}^{-1}\underline{M}']+[\hat{\underline{Q}}-\underline{M}\underline{R}^{-1}\underline{M}'] \qquad (4.3)$$

with the boundary condition

$$\underline{K}_N = \underline{0} \qquad (4.4)$$

The concepts involved in the derivation of Eq. 4.3, the form of the equation itself and the way the solution matrix $\hat{\underline{K}}$ appears in the optimal control law indicate that Eq. 4.3 is the discrete analog to the matrix Riccati differential equation. Furthermore, the analogy carries over to most of the properties of the matrix solution $\hat{\underline{K}}$.

Although a difference equation always has a solution, the nonlinear nature of Eq. 4.3 does not permit the derivation of an analytic expression for it. It is necessary, therefore, to compute the elements of the constant matrix $\hat{\underline{K}}$ by solving the difference equation on a digital computer until steady-state is reached. The existence and uniqueness of the steady-state solution is guaranteed by the complete controllability of the system.[5] Since the equation's coefficient matrices are constant, there are no approximation errors involved, only round-off ones, and a direct iterative algorithm can be used successfully.

This would have been the case, if the constant coefficient matrices did not have to be evaluated by numerical integration. It is clear that if the approximation errors introduced by the numerical integration are not controlled, the resulting errors in $\underline{\hat{K}}$ will exceed by far the round-off errors. In order to limit the effect of such approximations, some error bounds were derived[6] for these matrix computations, which permit evaluation of the coefficient matrices to the desired accuracy.

Once the $\underline{\hat{K}}$ matrix has been computed numerically, the feedback gain matrix $\underline{G}$, defined from Eq. 4.1 by

$$\underline{G} = \underline{\hat{R}}^{-1}\underline{M}' + (\underline{\hat{R}}+\underline{D}'\underline{\hat{K}}\underline{D})^{-1}\underline{D}'\underline{\hat{K}}(\underline{\Phi}-\underline{D}\underline{\hat{R}}^{-1}\underline{M}') \tag{4.5}$$

can be evaluated. The optimal control law can be expressed simply as

$$\underline{u}_k^* = -\underline{G}\underline{x}_k^* \tag{4.6}$$

This is the sampled-data control law. It states that in order to evaluate the constant control for the interval $[kT, (k+1)T)$, only the state at time kT has to be measured. Alternatively, the elements of the matrix $\underline{G}$ specify the optimal weighting of the states, i.e., the position and velocity deviations, measured only at time kT, for the construction of the corrective forces to be applied to each and every vehicle in the string over that interval of time.

Finally, the control law of Eq. 4.6 can be substituted in the original equation of the system to obtain the optimal closed-loop sampled-data system design

$$\dot{\underline{x}}^*(\tau) = \underline{A}\underline{x}^*(\tau) - \underline{B}\underline{G}\underline{x}^*(kT) \; ; \; kT \leq \tau < (k+1)T \tag{4.7}$$

The structure of the optimal sampled-data system is shown graphically in Fig. 2.

The minimum cost associated with the optimal sampled-data feedback control law has been shown to be

$$J^* = \frac{1}{2}\underline{x}'(0)\underline{\hat{K}x}(0) \tag{4.8}$$

which is also analogous to the optimal J for the continuous-time case.

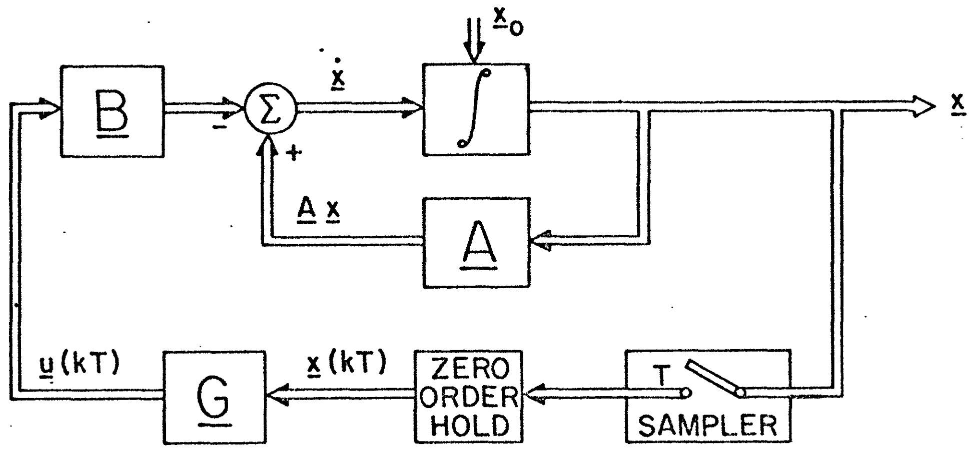

Fig. 2 The Structure of the Optimal Sampled-Data Feedback Control System

V. EXAMPLE: A STRING OF FIVE TRAINS

The application of the theory developed in the previous sections can be demonstrated best by an illustrative example. The physical system described in Ref. 1 has continued to serve as the basic model for computer simulation. A string of five identical vehicles was chosen for the purposes of illustration. This provided invaluable insight in the structure of the solution which, in turn, facilitated the development of more efficient and accurate computer algorithms. The existing algorithm can handle up to eleventh order systems (six trains) and by a simple modification of the dimensioning statements it can be made to handle any number of vehicles, the only limitation being the size of the storage of the digital computer.

The computer programs were written in FORTRAN IV and were tested at the facilities of the M.I.T. Information Processing Services Center.* For a given set of input data, i.e., matrices $\underline{A}$, $\underline{B}$, $\underline{Q}$, and $\underline{R}$ and a sampling interval T, the corresponding matrices $\underline{\Phi}$, $\underline{D}$, $\hat{\underline{Q}}$, $\underline{M}$, and $\hat{\underline{R}}$ of the discrete-time optimization problem are computed. This permits the iterative solution of the Riccati equation until steady state is reached. The steady-state solution matrix $\hat{\underline{K}}$ is then used to evaluate the matrix of the feedback gains and to obtain, finally, the closed-loop optimal discrete system matrix. The mathematical model for a string of five identical trains is summarized below.

The dimensionless state vector has nine components

$$\underline{x}' = [\psi_1, \chi_1, \psi_2, \chi_2, \psi_3, \chi_3, \psi_4, \chi_4, \psi_5] \tag{5.1}$$

and the control vector has five

$$\underline{u}' = [\phi_1, \phi_2, \phi_3, \phi_4, \phi_5] \tag{5.2}$$

The system matrix $\underline{A}$ and the input gain matrix $\underline{B}$ are given by

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* The IBM System 360/65-40 was used.

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} ; \quad \underline{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.3}$$

The value of the elements of the weighting matrices $\underline{R}$ and $\hat{\underline{Q}}$ are taken to be the same as in Ref. 1, i.e.,

$$\underline{R} = \text{diag}\{1, 1, 1, 1, 1\} = I_{5x5} \tag{5.4}$$

and either

$$\underline{Q} = \text{diag}\{0, 10, 0, 10, 0, 10, 0, 10, 0\} \tag{5.5}$$

or

$$\underline{Q} = \text{diag}\{6, 10, 6, 10, 6, 10, 6, 10, 6\} \tag{5.6}$$

where the first $\underline{Q}$ corresponds to the case in which the system is penalized only for deviations from the desired separation distance (case I), while the second $\underline{Q}$ corresponds to the case in which the system is penalized for velocity as well as separation distance deviations (case II).

For each choice of sampling interval T, an optimal closed-loop system can be determined. Therefore, a rational way to study the performance of the system with respect to the choice of sampling time is to display the effect of T on certain characteristic quantities, such as the optimal cost, the eigenvalues of the closed-loop system, etc.

It has been shown[1] that the optimal cost is a very good indicator of the system's performance, even though it depends on the initial conditions. In order to study the parametric dependence of J^* on the sampling interval T, two sets of initial conditions were chosen (Fig. 3). In Case A all separation distances are the minimum

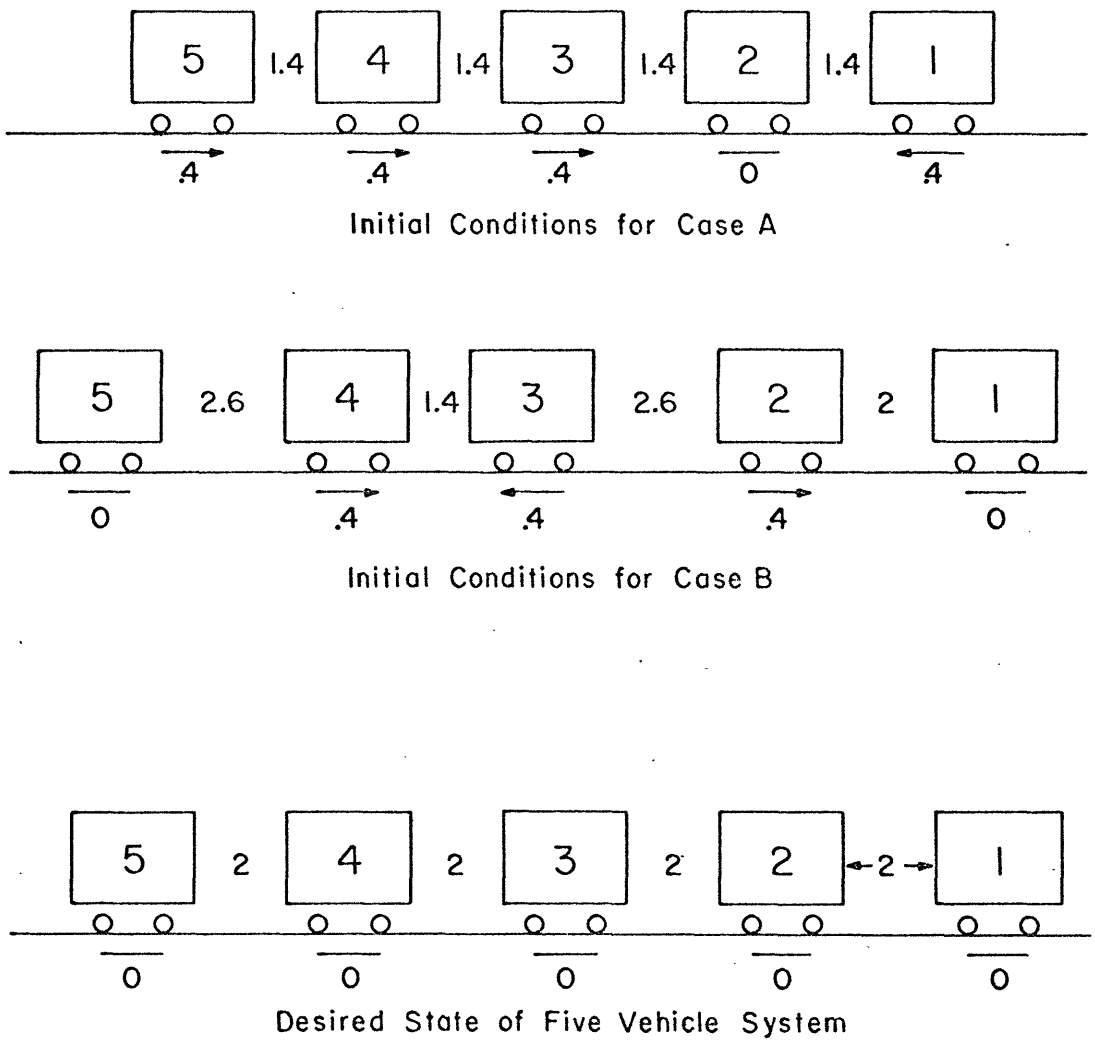

Fig. 3 Initial Conditions of Five Vehicle System

allowable by the linearization procedure. The velocity deviations have been set in such a manner that all trains tend to decrease their separation distances and group about the second train. In Case B, the separation distances are uneven and the velocity deviations tend to accentuate the grouping of the trains in pairs. For a desired separation distance of two dimensionless units, the initial state vectors for the two cases are given by

$$\underline{x}'_{oA} = [.4, -.6, .4, -.6, .4, -.6, 0, -.6, -.4] \tag{5.7}$$

$$\underline{x}'_{oB} = [0, .6, .4, -.6, -.4, .6, .4, 0, 0] \tag{5.8}$$

Ideally, it would be desirable to obtain an analytical expression for the optimal cost J^* as an explicit function of T and then investigate the change of J^* with T, i.e., evaluate $\frac{\partial J^*}{\partial T}$ with respect to T. Since the optimal cost is given by

$$J^*(T) = \frac{1}{2}\underline{x}'_o \hat{\underline{K}}(T)\underline{x}_o \tag{5.9}$$

which is a highly nonlinear function of T, the above approach does not yield a useful expression.[7] Alternatively, if the optimal cost is normalized with respect to the initial conditions, it is then possible to obtain upper and lower bounds that are independent of the initial state.

It can be shown[8] that if $\lambda_{min}\{\hat{\underline{K}}\}$ and $\lambda_{max}\{\hat{\underline{K}}\}$ denote the minimum and maximum eigenvalues of the symmetric positive semi-definite matrix $\hat{\underline{K}}$, then

$$\lambda_{min}\{\hat{\underline{K}}\} \le \frac{\frac{1}{2}\underline{x}'_o \hat{\underline{K}}\underline{x}_o}{\frac{1}{2}\underline{x}'_o \underline{x}_o} \le \lambda_{max}\{\hat{\underline{K}}\} \tag{5.10}$$

For simplicity, the normalized ratio is denoted by

$$\zeta^*(T) \triangleq \frac{J^*(T)}{\frac{1}{2}\langle \underline{x}_o, \underline{x}_o \rangle} \tag{5.11}$$

so that Eq. 5.10 becomes

$$\lambda_{min}\{\hat{\underline{K}}\} \leq \zeta^*(T) \leq \lambda_{max}\{\hat{\underline{K}}\} \tag{5.12}$$

The behavior of ζ^* as a function of the sampling interval is exhibited in Figs. 4 and 5.

In Fig. 4 the results for case I are shown. The minimum eigenvalue is zero since $\hat{\underline{K}}$ is positive semidefinite due to the semidefiniteness of the $\underline{Q}$ matrix. In Fig. 5 the normalized cost curves for case II are shown. As expected the numerical values of the cost are higher since deviations in velocity are included in computations. In all cases the cost increases very slowly for small T, and for large T the curves become straight lines. This implies that there are two basic modes of behavior of the system. The first mode, for small T, is essentially similar to that of the optimal continuous system and it exhibits fast oscillatory response. In the second mode the effect of the feedback is to make all the closed-loop eigenvalues real and negative, i.e., the optimal system is "overdamped." In the transition region between the two modes each successive pair of eigenvalues becomes a pair of negative real ones. It can be concluded, therefore, that a satisfactory upper bound for design values of the sampling interval such that the cost increase remains small is the smallest T for which all eigenvalues of the optimal closed-loop system become real.

Since the actual cost is a function of the initial state, other measures of the system's performance that are independent of the initial state can be considered. Typical examples are the Hölder norms of $\hat{\underline{K}}$, the trace of $\hat{\underline{K}}$ etc. None of these, though, yield substantially more information than the maximum and minimum eigenvalues of $\hat{\underline{K}}$ which can be evaluated easily. Therefore, a design which is to be independent of initial conditions can be based on these two curves that bound between them all possible normalized cost curves. As an example, it can be seen from Figs. 4 and 5 that a sampling interval one and one-half times the dominant time constant of the continuous open-loop system increases the cost by only 15 percent. This is a very small increase in the cost in view of the

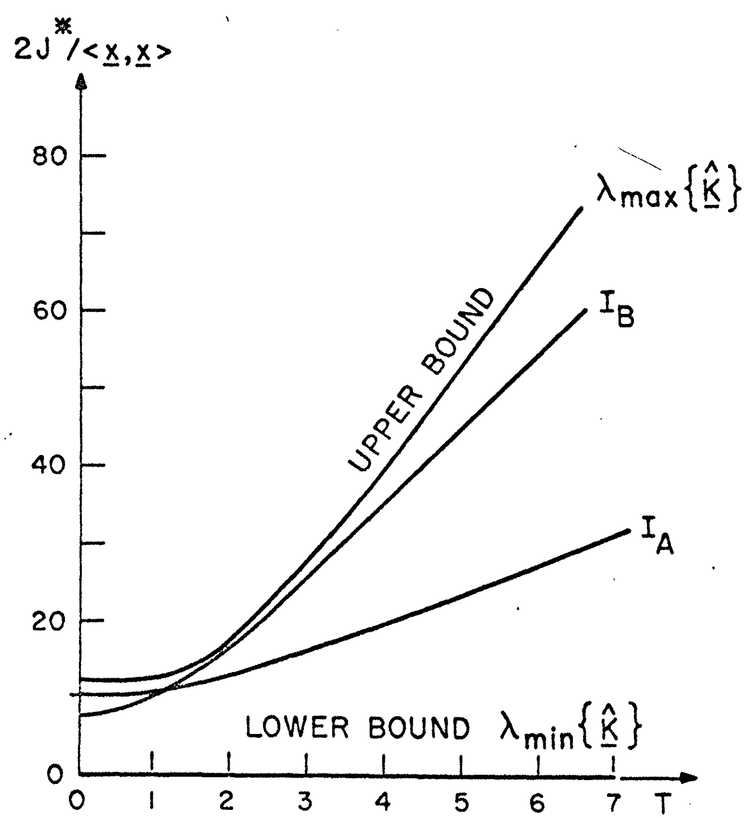

Fig. 4 The Normalized Optimal Cost as a Function of the Sampling Period. Cases I_A, I_B

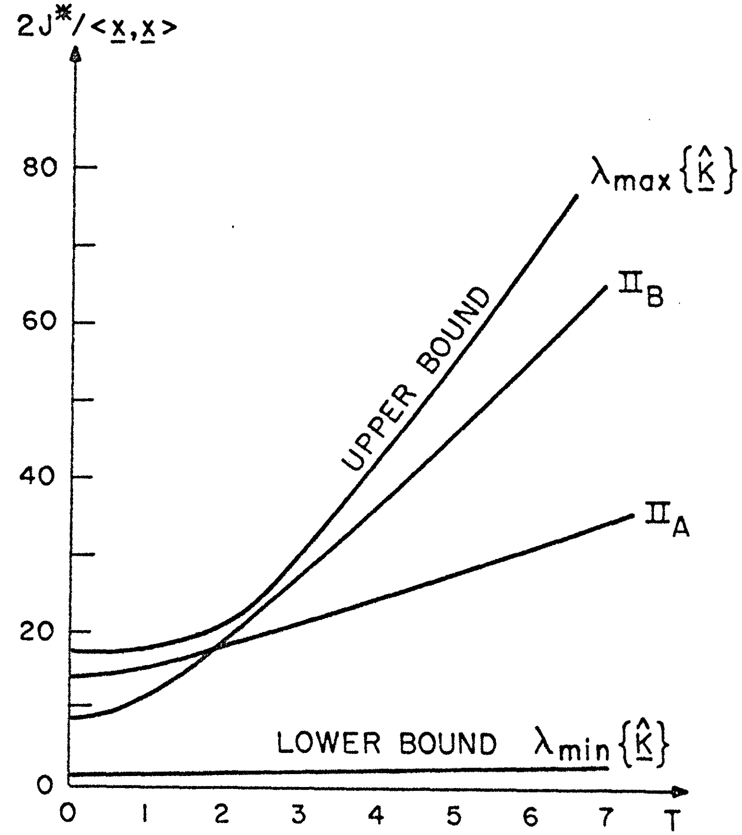

Fig. 5 The Normalized Optimal Cost as a Function of the Sampling Period. Cases II_A, II_B

advantages the sampled-data system offers with regards to the communication system requirements.

VI. DISCUSSION OF RESULTS

In the preceding sections the results of the theoretical investigations were outlined and a specific example was presented. In this section an attempt will be made to place these mathematical results into an engineering perspective.

The first topic of discussion concerns measurements. It has been assumed that every T dimensionless units of time, reasonably accurate measurements of

a. The velocity deviation of each and every vehicle from the desired average string velocity,

b. The deviation of adjacent vehicles from their desired separation distance,

can be made. These error measurements are then used to define the state vector of the system at the discrete instances of time at which sampling occurs. Thus the sequence of state vectors

$$\underline{x}_1, \underline{x}_2, \underline{x}_3, \ldots, \underline{x}_k, \underline{x}_{k+1}, \ldots \tag{6.1}$$

where

$$\underline{x}_k \triangleq \underline{x}(kT) \tag{6.2}$$

represents the sequence of measurements in time available for feedback control.

On the basis of these measurements the optimal sampled-data control can be computed. It should be stressed that this control remains constant during the sampling interval. Thus, for the time t in the interval

$$kT \leq t < (k+1)T \tag{6.3}$$

the control is given by

$$\underline{u}^*(t) = \underline{u}_k = -\underline{G}\underline{x}_k \tag{6.4}$$

where $\underline{G}$ is the $n \times (2n-1)$ matrix of optimal constant feedback gains, and $\underline{x}_k$ is the state measured at $t = kT$.

In order to implement this particular scheme of control it is necessary that each train communicates, every T units of time, its position and velocity to a command-and-control center (CCC). At the center the required computations are performed and the components of the control vector, i.e., the appropriate accelerations and decelerations, are transmitted back to the trains. For a string of n trains a total of $2n$ numbers are received by the CCC and a total of n are transmitted to the trains every sampling interval.

In the method of control described above there is no reason why the optimal solution cannot be used. What remains to be chosen is the sampling rate. This involves a study of the trade-off between system performance and the cost of satisfying the communication requirements. It has been shown that optimal cost increases with increasing T while on the other hand this involves lower rate of data transmission. A procedure is then required that will yield that value of T which will best satisfy the two simultaneous requirements. A possible approach to the problem is the following:

Express the cost of the communication system necessary to implement the optimal control as a function of the sampling interval T, normalize it so that it is commensurate with some normalized form of the optimal cost J^* and plot both on a cost versus sampling interval set of coordinates. The lowest point of intersection defines the best choice of sampling rate.

The above procedure can be demonstrated best by a simple example. An index or measure that reflects the communication costs associated with the transmission of information at a certain rate can be the rate itself or the capacity of a channel. The channel capacity[9] is a number that indicates the maximum rate of information transfer of a channel when the signal power P and the power N added by the noise in the channel are known. It can be defined as

$$C = \frac{1}{T} \log \left(1 + \frac{P}{N}\right)^{1/2} \tag{6.5}$$

where T is the sampling interval and the units of measurement are bits per second. For a given channel then the channel capacity is inversely proportional to the sampling time. Since the above expression is not meaningful for very small or very large T

the following modifications are made. For very small sampling times the fixed cost of the continuous-time optimal feedback control implementation is taken as an upper bound while for large sampling times the fixed cost for any basic low-rate information transmission system is taken as the lower bound. For the intermediate region the channel capacity multiplied by an appropriate constant that scales it with respect to the two bounds is used.

The maximum and minimum eigenvalues of $\hat{\underline{K}}$ scaled by the appropriate constant that makes their values commensurate with the cost of communication are taken as the measure that describes the effect of the sampling on the system's performance. A possible sketch of the two costs is shown in Fig. 6. The possible choice of a

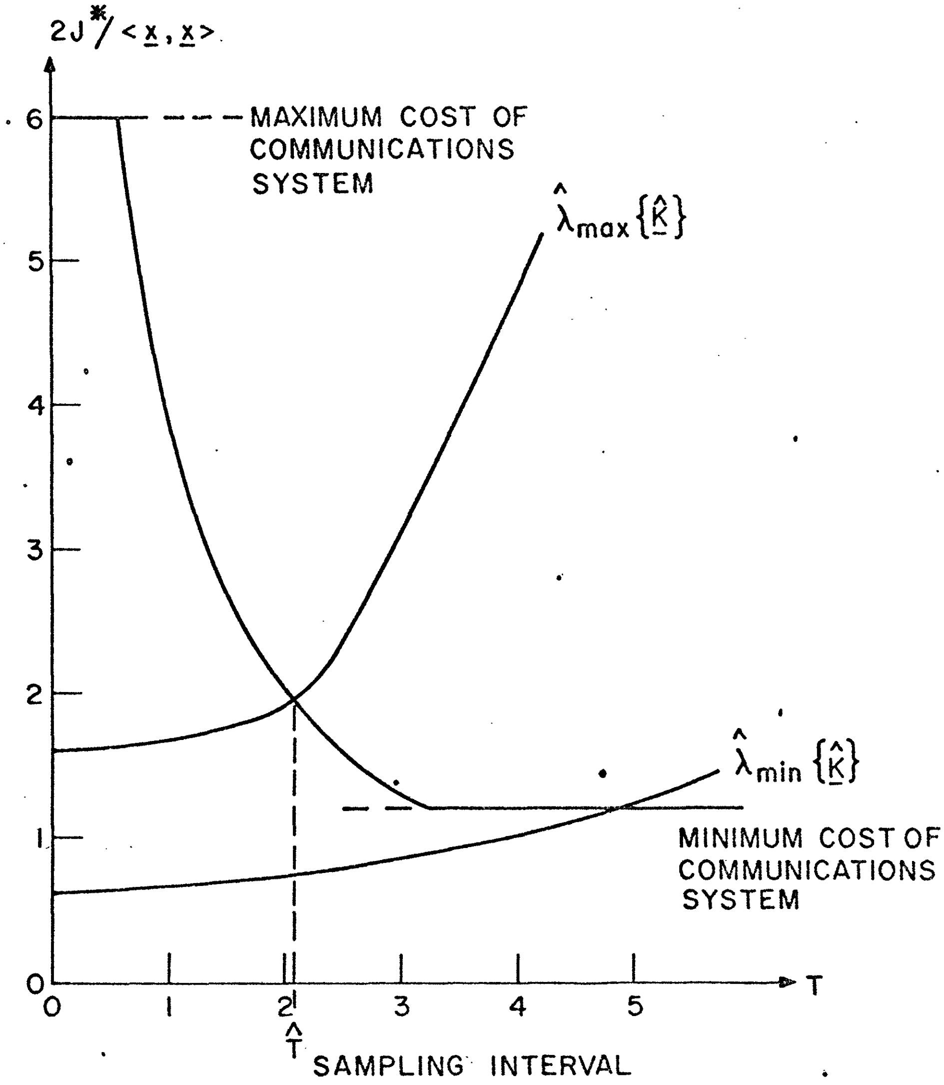

Fig. 6 Trade-off Curves for Normalized Cost vs. Sampling Interval

sampling interval according to these curves is $T = 2+$. Since this is determined by the intersection of an upper bound of the cost curve it is a conservative estimate. With this sampling time as a basis, a detailed analysis that takes into consideration specific data transmission systems that can handle the data flow specified by $\hat{T}$ can be undertaken and a more accurate estimate of the best sampling time can be made.

As a final comment on this control scheme it should be noted that a strong subjective element is introduced when coefficients that multiply the two cost indices to make them comparable are chosen. Although the cost of a communications system can be expressed readily in some units such as dollars, it is very difficult to assign a similar cost to concepts such as safety or comfort limit that are inherent in the structure of the quadratic cost functional.

Another method of control may just involve the transmission of position and velocity information from each and every train in the string to every other train. Then each train computes the position and velocity deviations and then the required acceleration or deceleration to be used for the next T seconds. In such a scheme, it may be desirable to have communication and interchange of information only between trains that are physically near each other.

If only limited communication is desirable, then the string of n vehicles can be divided into substrings, and the general techniques outlined in Ref. 1 can be used. This would involve

a. the subdivision of the string of n vehicles into substrings each containing m vehicles (where $m = 2, 3, \ldots$)

b. optimization of each substring using the appropriate performance criterion

c. determination of the optimal gains to be used for each substring using methods identical to these described in this report

d. superposition of the gains to obtain the overall linear sampled-data feedback control system.

Since both "substring control" and sampling of the data have similar effects on the system performance, it can be concluded that schemes which combine large sampling times with very few vehicles per substring will be unacceptable due to the severe deterioration of the dynamic behavior of the complete system. On the other hand intermediate size substrings with conservative sampling times may provide the best scheme if local control is desired.

VII. CONCLUSIONS

The theory of the optimal sampled-data feedback control of linear systems with respect to quadratic cost functionals has been developed. The theoretical results have been applied to the design of optimal position and velocity control of strings of high-speed trains. The necessary algorithms for the solution of the ensuing difference equations have been developed and the gains necessary for the design of particular systems can be evaluated.

The above design procedure has the advantage that it permits great flexibility, and therefore reduced cost, in the implementation of the communications system implicit in the feedback structure of the control. It is quite clear that periodic measurements of the position and velocity of a moving vehicle are much easier to implement than continuous ones. In addition, it should be noted that the sub-optimal design presented in Ref. 1, by which the number of the feedback loops can be reduced drastically, can be applied with equal validity in the sampled-data case.

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APPENDIX

THE COMPUTER PROGRAM AND ITS USE

1. INTRODUCTION

The organization of the program follows closely the analytical solution of the problem. A given linear, time-invariant, continuous-time system with a quadratic cost functional is transformed into the equivalent discrete-time linear regulator. All the required matrices are evaluated numerically for a specific sampling interval and the Riccati difference equation is solved. The steady-state solution to the Riccati equation yields the matrix of the feedback gains and the optimal closed-loop system matrix.

The program is versatile and can be used for the solution of a variety of regulator problems. For example, the continuous-time regulator problem with finite terminal time can be solved by setting the sampling time sufficiently small (same as in a Runge-Kutta scheme) and by printing the solution to the Riccati equation at every iteration in order to obtain the time-varying gains. A discrete-time regulator can be designed by omitting the subroutines that make the transformations and by starting the computations directly with the Riccati difference equation.

A sequence of tests appears throughout the program, some optional, such as the test for positive semidefiniteness of the modified $\hat{\underline{Q}}$, and some integrated in the computations, such as the test for convergence of the matrix exponential. These tests permit the user to control the error in the numerical computations. The techniques used to determine the necessary parameters are presented in Ref. 5.

The algorithm was programmed in FORTRAN IV, level G, and was tested on the IBM System 360/65-40 of the M.I.T. Information Processing Services Center. The program is virtually self-sufficient in that it contains all the mathematical subroutines required (except the optional use of an EIGENVALUE subroutine). This makes it easily adaptable to other installations.

2. THE INPUT VARIABLES

A list of the input variables in the order that they appear in the program follows.

a. Parameters

(Card No. 1)

MORE	= 1	A set of data follows.
	≠ 1	End of data.
OPT 1	= 1	Nonzero boundary condition for the Riccati equation to be read.
	≠ 1	Boundary condition set equal to zero in the program.
OPT 2	= 1	Test eigenvalues of $\hat{\underline{Q}}$.
	≠ 1	Omit test.
OPT 3	= 1	Evaluation of eigenvalues of $\hat{\underline{K}}$.
	≠ 1	Omit above operation.
OPT 4	= 1	Evaluation of $\underline{G}$ and optimal closed-loop matrix.
	≠ 1	Omit above operations.
OPT 5	= 1	$\hat{K}$ and optimal closed-loop system matrix are punched on cards.
	≠ 1	No punched card output.

(Card No. 2)

N	=	Dimension of system state vector $\underline{x}$.
M	=	Dimension of control vector $\underline{u}$.
T	=	Length of sampling interval.
NEVEN	=	Even number of points for parabolic rule of numerical integration.

(Card No. 3)

GEST	=	Iteration at which test of convergence starts in subroutines MATX and LAMDA. Set GEST $\geq$ 4.
LEND	=	Maximum allowable number of iterations in subroutine RICCAT. Set LEND = t_{final}/T for time-varying feedback case.

		Set LEND > $t_{\text{steady state}}/T$ for $\hat{K}$.
JAY	=	Modulus for printing the Riccati matrix (every JAY iterations).
EPSLO	=	ϵ, small number used for convergence and error tests.

(Card No. 4)

This card contains the variable FORMAT statements for reading and printing matrices.

RN	=	Format for reading row vector of dimension N.
RM	=	Format for reading row vector of dimension M.
PN	=	Format for printing row vector of dimension N.
PM	=	Format for printing row vector of dimension M.

b. <u>Input Arrays</u>

A	=	Array containing the system matrix.
B	=	Array containing the input gain matrix.
Q	=	Array containing the matrix weighting the state.
R	=	Array containing the matrix weighting the input.

These four matrices are read sequentially, row by row, according to the previously specified FORMAT statements. If the first option is taken (OPT 1 = 1) then the array containing the boundary condition for the Riccati equation is read after array R.

3. THE OUTPUT ARRAYS

ARK	=	Array $\underline{K}$ containing the solution to the Riccati equation.
GMAT	=	Array $\underline{G}$ containing the feedback gain matrix.
AM	=	Array containing the optimal discrete system matrix.

It should be noted that the above array names are often used for temporary storage of intermediate results. Furthermore, all the arrays containing intermediate results are defined in the legend preceding the subprogram in which they appear.

4. EXAMPLE

As an illustrative example, consider the two train case for which no punched output is required. The input data card deck should read as follows, where each line corresponds to one card.

```
    1      0     1     1     1     0
         3       2    0.1500000E 01          6
    4.0         100            25   0.1000E-03
(3F8.4)            (2F8.4)          (3F12.4)        (2F12.4)
   -1.          0.          0.
    1.          0.         -1.          A
    0.          0.         -1.
    1.          0.
    0.          0.                      B
    0.          1.
    0.          0.          0.
    0.         10.          0.          Q
    0.          0.          0.
    1.          0.
    0.          1.                      R
    0
```

5. ORGANIZATION AND LISTING OF PROGRAM

The organization of the main program is shown in the Flow Chart, Fig. 7. The specific function of each subprogram and its usage is described in the legend preceding its listing. The main program and the ten subprograms are listed in the following order

MAIN	
Subroutine	MATX
"	MULT
"	LAMDA
"	MODIFY
"	SMVECT
"	VECT
"	INTEG
"	RICCAT
"	GAIN
"	INVERT

In addition, for some optional operations, the subroutine EIGEN of the System 360 Scientific Subroutine Package is used for the determination of the eigenvalues of real symmetric matrices.

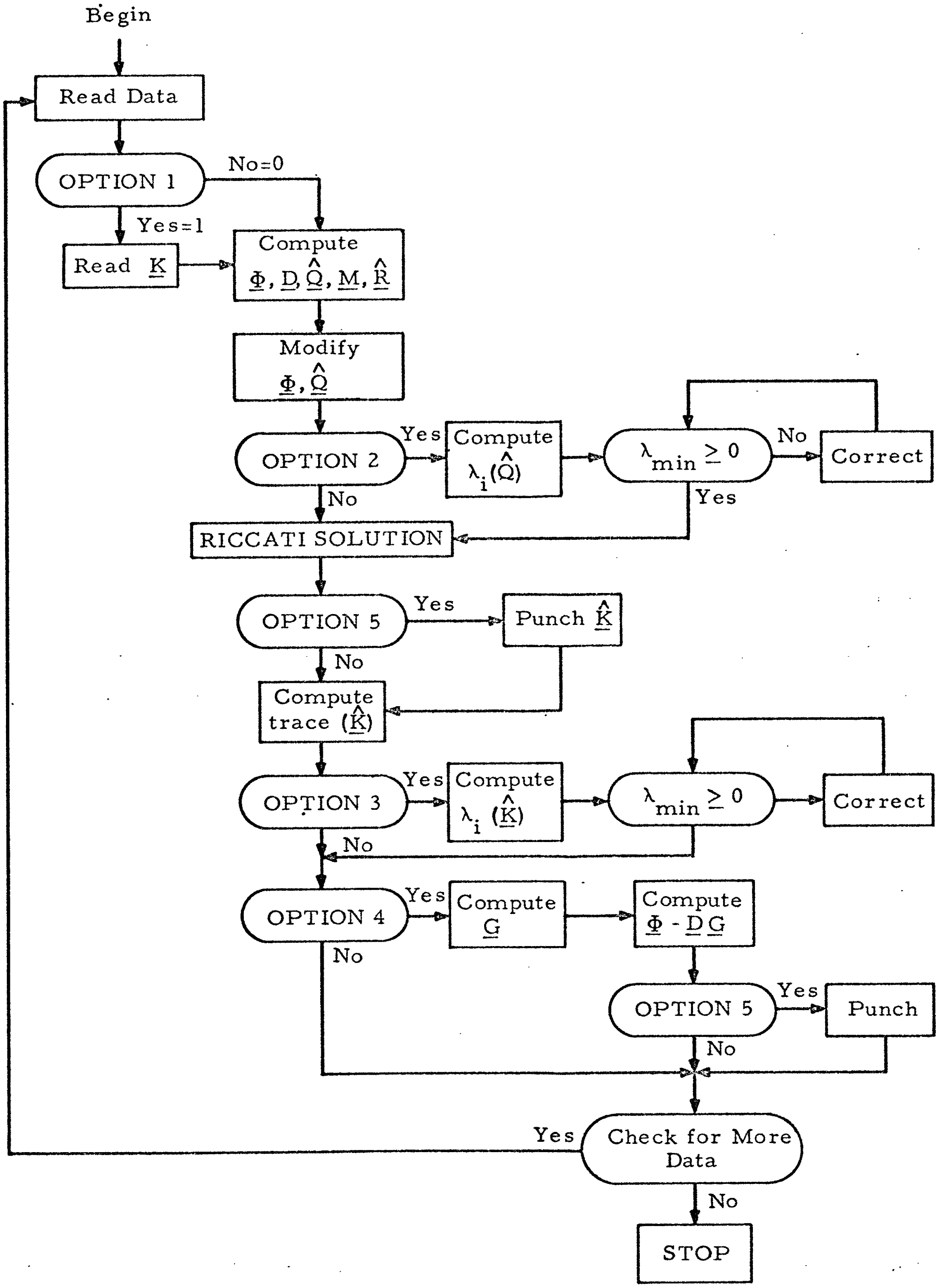

Fig. 7 Flow Chart of MAIN Program

```
C     MAIN PROGRAM FOR THE COMPUTATION  OF THE OPTIMAL SAMPLED-DATA CONTROL
C     OF LINEAR REGULATORS
C     DIMENSION STATEMENTS
      DIMENSION RN(3),RM(3),PN(3),PM(3)
      DIMENSION B(11,11),R(11,11),PHI(11,11),C(11,11),QHAT(11,11),EM(11,
     111),RHAT(11,11),ARK(11,11),GMAT(11,11)
      DIMENSION QINT(11,11),EMINT(11,11),RINT(11,11),AM(11,11)
C     INPUT PARAMETERS STORED IN COMMON.
      COMMON N,EPSLO,GEST,A(11,11),Q(11,11)
      INTEGER OPT1,OPT2,OPT3,OPT4,OPT5
    2 CONTINUE
C     MAGNITUDE PARAMETERS
      READ(5,1000)MORE,OPT1,OPT2,OPT3,OPT4,OPT5
      IF(MORE.EQ.0)GO TO 999
      WRITE(6,2000)
      READ (5,1001)N,M,T,NEVEN
      WRITE(6,2001)N,M,T,NEVEN
      READ (5,1002)GEST,LEND,JAY,EPSLO
      WRITE(6,2002)GEST,LEND,JAY,EPSLO
      READ(5,1003)(RN(I),I=1,3),(RM(J),J=1,3),(PN(K),K=1,3),(PM(L),L=1,3
     1)
C     THE CONTINUOUS SYSTEM'S MATRICES
C     THE A MATRIX
      WRITE(6,2003)
      READ (5,RN)((A(I,J),J=1,N),I=1,N)
      WRITE(6,PN)((A(I,J),J=1,N),I=1,N)
C     THE B MATRIX
      WRITE(6,2004)
      READ (5,RM)((B(I,J),J=1,M),I=1,N)
      WRITE(6,PM)((B(I,J),J=1,M),I=1,N)
C     THE Q MATRIX
      WRITE(6,2005)
      READ(5,RN)((Q(I,J),J=1,N),I=1,N)
      WRITE(6,PN)((Q(I,J),J=1,N),I=1,N)
C     THE R MATRIX
      WRITE(6,2006)
      READ(5,RM)((R(I,J),J=1,M),I=1,M)
      WRITE(6,PM)((R(I,J),J=1,M),I=1,M)
C     THE BOUNDARY CONDITION FOR THE RICCATI EQUATION
      IF(OPT1.NE.1)GO TO 4
C     THE NONZERO K(T-FINAL)
      READ (5,RN)((ARK(I,J),J=1,N),I=1,N)
      WRITE(6,2017)
      WRITE(6,PN)((ARK(I,J),J=1,N),I=1,N)
      GO TO 10
    4 CONTINUE
C     THE ZERO K(T-FINAL)
      DO 6 I=1,N
      DO 6 J=1,N
      ARK(I,J)=0.0
    6 CONTINUE
   10 CONTINUE
```

```
C     THE DISCRETE SYSTEM'S MATRICES
C     COMPUTATION OF PHI
      WRITE(6,2007)
      CALL MATX(PHI,T)
      WRITE(6,PN)((PHI(I,J),J=1,N),I=1,N)
C     COMPUTATION OF D
      CALL LAMDA(AM,T)
      WRITE(6,2008)
      CALL MULT(AM,B,D,N,M,N)
      WRITE(6,PM)((D(I,J),J=1,M),I=1,N)
C     COMPUTATION OF Q-HAT,AM,R-HAT
      CALL INTEG(QINT,EMINT,RINT,0.)
      CALL INTEG(QHAT,EM,RHAT,T)
      DT=T/NEVEN
      DO 50 I=1,N
      DO 50 J=1,N
      QHAT(I,J)=(DT/3.)*(QINT(I,J)+QHAT(I,J))
      EM(I,J)=(DT/3.)*(EMINT(I,J)+EM(I,J))
      RHAT(I,J)=(DT/3.)*(RINT(I,J)+RHAT(I,J))
   50 CONTINUE
      MAX=NEVEN-1
      DO 60 IND=1,MAX,2
      BETA=IND*DT
      CALL INTEG(QINT,EMINT,RINT,BETA)
      DO 60 I=1,N
      DO 60 J=1,N
      QHAT(I,J)=QHAT(I,J)+(4.*DT/3.)*QINT(I,J)
      EM(I,J)=EM(I,J)+(4.*DT/3.)*EMINT(I,J)
      RHAT(I,J)=RHAT(I,J)+(4.*DT/3.)*RINT(I,J)
   60 CONTINUE
      NEWMAX=NEVEN-2
      DO 70 IND=2,NEWMAX,2
      BETA=IND*DT
      CALL INTEG(QINT,EMINT,RINT,BETA)
      DO 70 I=1,N
      DO 70 J=1,N
      QHAT(I,J)=QHAT(I,J)+(2.*DT/3.)*QINT(I,J)
      EM(I,J)=EM(I,J)+(2.*DT/3.)*EMINT(I,J)
      RHAT(I,J)=RHAT(I,J)+(2.*DT/3.)*RINT(I,J)
   70 CONTINUE
C     THE Q-HAT MATRIX
      WRITE(6,2009)
      WRITE(6,PN)((QHAT(I,J),J=1,N),I=1,N)
C     THE AM MATRIX
      WRITE(6,2010)
      CALL MULT(EM,B,AM,N,M,N)
      WRITE(6,PM)((AM(I,J),J=1,M),I=1,N)
C     THE R-HAT MATRIX
      WRITE(6,2011)
      CALL MULT(RHAT,B,GMAT,N,M,N)
      DO 80 I=1,N
      DO 80 J=1,M
      EM(J,I)=B(I,J)
   80 CONTINUE
```

```
      CALL MULT(EM,GMAT,RHAT,M,M,N)
      DO 90 I=1,M
      DO 90 J=1,M
      RHAT(I,J)=T*R(I,J)+RHAT(I,J)
   90 CONTINUE
      WRITE(6,PM)((RHAT(I,J),J=1,M),I=1,M)
C     MODIFICATION OF PHI AND Q-HAT.
      CALL MODIFY(PHI,D,QHAT,AM,RHAT,N,M)
C     PRINT MODIFIED PHI AND Q-HAT
      WRITE(6,2012)
      WRITE(6,PN)((PHI(I,J),J=1,N),I=1,N)
      WRITE(6,2013)
      WRITE(6,PN)((QHAT(I,J),J=1,N),I=1,N)
C     OPTIONAL TEST FOR EIGENVALUES OF Q-HAT
      IF(OPT2.NE.1)GO TO 92
      WRITE(6,2018)
      CALL SMVECT(QHAT,N)
   92 CONTINUE
C     SOLUTION OF THE RICCATI EQUATION
      WRITE(6,2014)
      CALL RICCAT(PHI,D,QHAT,RHAT,ARK,LEND,N,M,PN,EPSLO,JAY)
C     OPTIONAL PUNCH CARD OUTPUT OF THE SOLUTION
      IF(OPT5.NE.1)GO TO 91
      DO 85 I=1,N
      PUNCH RN,(ARK(I,J),J=1,N)
   85 CONTINUE
   91 CONTINUE
C     COMPUTATION OF THE TRACE OF K-HAT
      TRACE=0.0
      DO 93 I=1,N
      TRACE = TRACE + ARK(I,I)
   93 CONTINUE
      WRITE(6,2015)TRACE
C     OPTIONAL EVALUATION OF THE EIGENVALUES OF K-HAT
      IF(OPT3.NE.1)GO TO 94
      WRITE(6,2019)
      CALL SMVECT(ARK,N)
   94 CONTINUE
C     OPTIONAL EVALUATION OF GAIN AND CLOSED LOOP MATRICES
      IF(OPT4.NE.1)GO TO 96
      WRITE(6,2016)
      CALL GAIN(GMAT,PHI,D,AM,RHAT,ARK,N,M)
      WRITE(6,PN)((GMAT(I,J),J=1,N),I=1,M)
      CALL MULT(D,GMAT,EMINT,N,N,M)
      CALL MATX(PHI,T)
      DO 97 I=1,N
      DO 97 J=1,N
      AM(I,J)=PHI(I,J)-EMINT(I,J)
   97 CONTINUE
      WRITE(6,2020)
      WRITE(6,PN)((AM(I,J),J=1,N),I=1,N)
```

```
C     OPTIONAL PUNCH CARD OUTPUT OF THE OPTIMAL SYSTEM MATRIX
      IF(OPT5.NE.1)GO TO 96
      DO 98 I=1,N
      PUNCH RN,(AM(I,J),J=1,N)
   98 CONTINUE
   96 CONTINUE
      GO TO 2
  999 CONTINUE
C     FORMAT STATEMENTS
C        READING FORMATS
 1000 FORMAT(6I3)
 1001 FORMAT(2I5,E14.7,I5)
 1002 FORMAT(F5.1,2I8,E12.4)
 1003 FORMAT(3A4,3A4,3A4,3A4)
C        WRITING FORMATS
 2000 FORMAT('1   *// THE INPUT DATA //*')
 2001 FORMAT('0  N=',I3,'     M=',I3,3X/,'0 SAMPLING INTERVAL=',F10.6,3X/
     1,'0 NEVEN=',I4/)
 2002 FORMAT('0 GEST=',F5.1,3X/,'0 LEND=',I5,3X/,'0  JAY=',I5,3X/,'0EPSL
     1O=',E12.4//)
 2003 FORMAT('0 THE A MATRIX')
 2004 FORMAT('0 THE B MATRIX')
 2005 FORMAT('0 THE C MATRIX')
 2006 FORMAT('0 THE R MATRIX')
 2007 FORMAT('0   *// THE TRANSFORMED MATRICES //*',3X//,'0 THE PHI MATR
     1IX')
 2008 FORMAT('0 THE D MATRIX')
 2009 FORMAT('0 THE Q-HAT MATRIX')
 2010 FORMAT('0 THE M MATRIX')
 2011 FORMAT('0 THE R-HAT MATRIX')
 2012 FORMAT('0   *// THE MODIFIED MATRICES //*',3X//,'0 THE MODIFIED PH
     1I MATRIX')
 2013 FORMAT('0 THE MODIFIED Q-HAT MATRIX')
 2014 FORMAT('0   *// RESULTS //*')
 2015 FORMAT('0 THE TRACE OF K-HAT =',F12.5//)
 2016 FORMAT('0 THE FEEDBACK GAIN MATRIX  G')
 2017 FORMAT('0 THE BOUNDARY CONDITION FOR K')
 2018 FORMAT('0 THE EIGENVALUES OF Q-HAT')
 2019 FORMAT('0 THE EIGENVALUES OF K-HAT')
 2020 FORMAT('0   THE OPTIMAL CLOSED LOOP SYSTEM')
      END
```

```
C     -----------------------------------------------------------------
C
C     SUBROUTINE MATX
C
C     PURPOSE
C        TO COMPUTE THE MATRIX EXPONENTIAL
C
C     USAGE
C        CALL MATX(EMAT,CT)
C
C     DESCRIPTION OF PARAMETERS
C        A     - AN  N X N  REAL CONSTANT MATRIX IN COMMON
C        CT    - TIME VARIABLE
C        EMAT  - THE  N X N  MATRIX EXP(A*CT)
C        N     - DIMENSION OF MATRIX A
C        EPSLO- AN EPSILON FOR THE CONVERGENCE CRITERION
C        GEST  - AN INDEX FOR STARTING TEST FOR CONVERGENCE
C
C     METHOD
C        EVALUATION ACCOMPLISHED BY TRUNCATION OF POWER SERIES.
C        SERIES TRUNCATEC WHEN EVERY RATIO OF CORRESPONDING ELEMENTS
C        OF LAST TERM TO SUM OF PREVIOUS TERMS IS LESS THAN EPSILON.
C
C     REMARKS
C        A,N,GEST,EPSLO ARE ALL IN COMMON.
C
C     -----------------------------------------------------------------
C
      SUBROUTINE MATX(EMAT,DT)
      COMMON N,EPSLO,GEST,A(11,11),Q(11,11)
      DIMENSION EMAT(11,11),EVEC(11),DVEC(11),AT(11,11)
C     COMPUTE AT MATRIX AND ENTER IDENTITY MATRIX
      DO 200 I=1,N
      DO 200 J=1,N
      AT(I,J)=A(I,J)*CT
      IF(I.NE.J)GC TO 151
      EMAT(I,J)=1.0
      GO TO 200
  151 EMAT(I,J)=C.0
  200 CONTINUE
C     COMPUTATION OF MATRIX EXPONENTIAL (TO STATEMENT 300)
      DO 300 I=1,N
      DO 224 J=1,N
      EVEC(J)=EMAT(I,J)
  224 CONTINUE
      G=1.0
  225 CONTINUE
      DO 270 J=1,N
      C=0.0
      DO 250 K=1,N
  250 C=(EVEC(K)*(AT(K,J)/G))+C
      DVEC(J)=C
  270 CONTINUE
```

```
      G=G+1.0
      DO 276 J=1,N
      EMAT(I,J)=EMAT(I,J)+DVEC(J)
      EVEC(J)=DVEC(J)
  276 CONTINUE
C     CONVERGENCE TEST
      IF(G.LT.GEST)GO TO 225
      DO 275 J=1,N
      IF(EMAT(I,J))273,275,273
  273 RATIO=ABS(DVEC(J)/EMAT(I,J))
      RNOT=RATIO-EPSLO
      IF(RNOT.GT.0.0)GO TO 225
  275 CONTINUE
  300 CONTINUE
      RETURN
      END
C
C     ..................................................................
C
C     ..................................................................
C
C     SUBROUTINE MULT
C
C     PURPOSE
C        TO COMPUTE THE PRODUCT OF TWO MATRICES.
C        GAMMA(N X M) = ALPHA(N X L) * BETA(L X M)
C
C     USAGE
C        CALL MULT(ALPHA,BETA,GAMMA,N,M,L)
C
C     DESCRIPTION OF PARAMETERS
C        ALPHA- N X L  REAL MATRIX
C        BETA - L X M  REAL MATRIX
C        GAMMA-  N X M  REAL MATRIX
C        N     - NUMBER OF ROWS IN ALPHA
C        M     - NUMBER OF COLUMNS IN BETA
C        L     - NUMBER OF COLUMNS(ROWS) IN ALPHA(BETA)
C
C     ..................................................................
C
      SUBROUTINE MULT(ALPHA,BETA,GAMMA,N,M,L)
      DIMENSION ALPHA(11,11),BETA(11,11),GAMMA(11,11)
      DO 10 I=1,N
      DO 10 J=1,M
      GAMMA(I,J)=0.0
      DO 10 K=1,L
      GAMMA(I,J)=GAMMA(I,J)+ALPHA(I,K)*BETA(K,J)
   10 CONTINUE
      RETURN
      END
C
C     ..................................................................
```

```
C      ..................................................................
C
C      SUBROUTINE LAMDA
C
C      PURPOSE
C         TO EVALUATE THE INTEGRAL OF EXP(A*T) FROM 0 TO SIGMA
C
C      USAGE
C         CALL LAMDA(LAMSIG,SIG)
C
C      DESCRIPTION OF PARAMETERS
C         A     - AN  N X N  REAL CONSTANT MATRIX IN COMMON
C         SIG   - THE UPPER LIMIT OF THE INTEGRAL
C         LAMSIG-THE INTEGRAL OF THE MATRIX EXPONENTIAL FROM  0 TO SIG
C         N      - DIMENSION OF MATRIX A
C         EPSLO- AN EPSILON FOR THE CONVERGENCE CRITERION
C         GEST - AN INDEX FOR STARTING TEST FOR CONVERGENCE
C
C      METHOD
C         EVALUATION ACCOMPLISHED BY TRUNCATION OF POWER SERIES.
C         SERIES TRUNCATED WHEN EVERY RATIO OF CORRESPONDING ELEMENTS
C         OF LAST TERM TO SUM OF PREVIOUS TERMS IS LESS THAN EPSILON.
C
C      REMARKS
C         A,N,GEST,EPSLO ARE ALL IN COMMON.
C
C      ..................................................................
C
       SUBROUTINE LAMDA(LAMSIG,SIG)
       DIMENSION LAMSIG(11,11),ASIG(11,11),AVEC(11),BVEC(11)
       COMMON N,EPSLO,GEST,A(11,11),Q(11,11)
       REAL LAMSIG
       DO 30 I=1,N
       DO 30 J=1,N
       ASIG(I,J)=A(I,J)*SIG
       IF(I.EQ.J)GO TO 20
       LAMSIG(I,J)=0.0
       GO TO 30
    20 LAMSIG(I,J)=1.0
    30 CONTINUE
       DO 300 I=1,N
       DO 100 J=1,N
       AVEC(J)=LAMSIG(I,J)
   100 CONTINUE
       G=1.0
   110 CONTINUE
       DO 130 J=1,N
       C=0.0
       DO 120 K=1,N
       C=((AVEC(K)*ASIG(K,J))/G) + C
   120 CONTINUE
       BVEC(J)=C
   130 CONTINUE
```

```
      G=G+1.0
      DO 140 J=1,N
      LAMSIG(I,J)=LAMSIG(I,J) + BVEC(J)/G
      AVEC(J)=BVEC(J)
  140 CONTINUE
C     TEST FOR CONVERGENCE
      IF(G.LT.GEST)GO TO 110
      DO 200 J=1,N
C     TEST FOR ZERO DIVISOR
      IF(LAMSIG(I,J).EQ.0.0)GO TO 200
      RATIO=ABS(BVEC(J)/LAMSIG(I,J))
      IF(RATIO.GT.EPSLO)GO TO 110
  200 CONTINUE
      DO 210 J=1,N
      LAMSIG(I,J)=LAMSIG(I,J)*SIG
  210 CONTINUE
  300 CONTINUE
      RETURN
      END
C
C     .................................................................




C     .................................................................
C
C     SUBROUTINE MODIFY
C
C     PURPOSE
C        TO COMPUTE THE MODIFIED MATRICES
C        PHI = PHI - D*(R INVERSE)*M'
C        Q-HAT = Q-HAT - M*(R INVERSE)*M'
C        AND THUS CONVERT THE RICCATI EQUATION TO STANDARD FORM
C
C     USAGE
C        CALL MODIFY(PHI,D,Q,AM,R,N,M)
C
C     DESCRIPTION OF PARAMETERS
C        PHI   - DISCRETE SYSTEM MATRIX OF DIMENSION  N X N
C        D     - CONTROL GAIN MATRIX OF DIMENSION  N X M
C        Q     - STATE WEIGHTING MATRIX OF DIMENSION  N X N
C        AM    - N X M MATRIX WEIGHTING STATE AND CONTROL
C        R     - CONTROL WEIGHTING MATRIX OF DIMENSION  M X M
C        N     - DIMENSION OF STATE VECTOR
C        M     - DIMENSION OF CONTROL VECTOR
C
C     SUBROUTINES REQUIRED
C        MULT
C        VECT (INVERT)
```

```
C     ..................................................................
C
      SUBROUTINE MODIFY(PHI,D,Q,AM,R,N,M)
      DIMENSION PHI(11,11),D(11,11),AM(11,11),R(11,11),Q(11,11)
      DIMENSION RMAT(11,11),AMTR(11,11),S1(11,11),S2(11,11)
C     INVERSION OF R
      DO 10 I=1,M
      DO 10 J=1,M
      RMAT(I,J)=R(I,J)
   10 CONTINUE
      CALL VECT(RMAT,M)
C     CONSTRUCT M' AND COMPUTE D*(R INVERSE)*M'
      DO 12 I=1,N
      DO 12 J=1,M
      AMTR(J,I)=AM(I,J)
   12 CONTINUE
      CALL MULT(RMAT,AMTR,S1,M,N,M)
      CALL MULT(D,S1,S2,N,N,M)
C     SUBTRACT RESULT FROM PHI AND RENAME PHI
      DO 14 I=1,N
      DO 14 J=1,N
      PHI(I,J)=PHI(I,J)-S2(I,J)
   14 CONTINUE
C     COMPUTE M*(R INVERSE)*M'
      CALL MULT(AM,S1,S2,N,N,M)
C     SUBTRACT S2 FROM Q AND RENAME Q
      DO 16 I=1,N
      DO 16 J=1,N
      Q(I,J)=Q(I,J)-S2(I,J)
   16 CONTINUE
      RETURN
      END
C
C     ..................................................................




C     ..................................................................
C
C     SUBROUTINE SMVECT
C
C     PURPOSE
C        TO  CONVERT A SYMMETRIC REAL MATRIX TO VECTOR STORAGE MODE=1
C        TO EVALUATE ITS EIGENVALUES AND TO TEST AND CORRECT FOR
C        NEGATIVE VALUE OF MINIMUM EIGENVALUE.
C
C     USAGE
C        CALL SMVECT(RMAT,M)
```

```
C      DESCRIPTION OF PARAMETERS
C         RMAT - POSITIVE SEMIDEFINITE SYMMETRIC MATRIX
C         M    - DIMENSION OF SQUARE MATRIX
C
C      REMARKS
C         MAXIMUM DIMENSION IS  11 X 11. USE OF SUBROUTINE OPTIONAL
C
C      SUBROUTINES REQUIRED
C         EIGEN (SYSTEM/360 SCIENTIFIC SUBROUTINE PACKAGE.VERSION II)
C
C      ..................................................................
C
       SUBROUTINE SMVECT(RMAT,M)
       DIMENSION RMAT(11,11),AMAT(66)
       INDEX=1
     6 CONTINUE
       IF(INDEX-M)8,20,20
C      MATRIX TO VECTOR CONVERSION
     8 DO 10 J=1,M
       KONE=(J*(J-1)+2)/2
       KEND= KONE+J-1
       DO 12 K=KONE,KEND
       I=K-KONE+1
       AMAT(K) = RMAT(I,J)
    12 CONTINUE
    10 CONTINUE
C      EVALUATION AND PRINTING OF EIGENVALUES
       WRITE(6,300)
       CALL EIGEN(AMAT,REIGEN,M,1)
       DO 15 K=1,KEND
       IF((K.NE.1).AND.(K.NE.KEND))GO TO 15
       WRITE(6,400)K,AMAT(K)
    15 CONTINUE
C      ERROR CORRECTIONS
       IF(AMAT(KEND))50,20,20
    50 COR=-0.5*M*AMAT(KEND)
       DO 52 I=1,M
       RMAT(I,I)=RMAT(I,I)+COR
    52 CONTINUE
       INDEX=INDEX+1
       GO TO 6
   300 FORMAT('0 THE MAXIMUM AND MINIMUM EIGENVALUES')
   400 FORMAT(7X,'K=',I3,5X,F12.5)
    20 RETURN
       END
C
C      ..................................................................
```

```
C      ..........................................................
C
C      SUBROUTINE VECT
C
C      PURPOSE
C         TO CONVERT A SQUARE MATRIX TO VECTOR MODE=0,
C         TO CALL THE MATRIX INVERSION SUBROUTINE AND
C         TO RECONVERT THE INVERTED VECTOR TO MATRIX FORM.
C
C      USAGE
C         CALL VECT(RMAT,M)
C
C      DESCRIPTION OF PARAMETERS
C         M    - THE DIMENSION OF THE SQUARE MATRIX
C         RMAT - THE MATRIX TO BE INVERTED AND ITS INVERSE
C
C      REMARKS
C         THE INVERSE IS STORED IN THE LOCATIONS OF THE INPUT MATRIX.
C
C      SUBROUTINES REQUIRED
C         INVERT
C
C      ..........................................................
C
       SUBROUTINE VECT(RMAT,M)
       DIMENSION RMAT(11,11),AMAT(121)
C      MATRIX TO VECTOR CONVERSION
       JNOT=0
   150 JNOT=JNOT+1
       IF(M.LT.JNOT)GO TO 180
       KONE=1+M*(JNOT-1)
       KEND=M*JNOT
       DO 170 K=KONE,KEND
       I=K-M*(JNOT-1)
       AMAT(K)=RMAT(I,JNOT)
   170 CONTINUE
       GO TO 150
   180 CONTINUE
C      MATRIX INVERSION
       CALL INVERT(AMAT,M,M)
C      VECTOR TO MATRIX CONVERSION
       KNOT=M*M
       DO 190 K=1,KNOT
       J=(K-1)/M+1
       I=K-M*(J-1)
       RMAT(I,J)=AMAT(K)
   190 CONTINUE
       RETURN
       END
C
C      ..........................................................
```

```
C     ..................................................................
C
C     SUBROUTINE INTEG
C
C     PURPOSE
C        TO COMPUTE THE INTEGRANDS FOR THE EVALUATION OF THE DISCRETE
C        WEIGHTING MATRICES.
C
C     USAGE
C     CALL INTEG(QINT,EMINT,RINT,BETA)
C
C     DESCRIPTION OF PARAMETERS
C        QINT - INTEGRAND FOR Q-HAT, NXN
C        EMINT- INTEGRAND FOR AM, NXN
C        RINT - INTEGRAND FOR R-HAT, NXN
C        BETA - TIME INSTANT AT WHICH INTEGRANDS ARE EVALUATED
C        N    - DIMENSION OF STATE VECTOR
C        A    - THE NXN CONTINUOUS SYSTEM MATRIX
C        Q    - THE STATE WEIGHTING MATRIX, NXN
C
C     REMARKS
C        N,A,Q ARE IN COMMON
C
C     SUBROUTINES REQUIRED
C        MATX
C        LAMDA
C
C     ..................................................................
C
      SUBROUTINE INTEG(QINT,EMINT,RINT,BETA)
      COMMON N,EPSLO,GEST,A(11,11),Q(11,11)
      DIMENSION QINT(11,11),EMINT(11,11),RINT(11,11),EMAT(11,11),EMATR(1
     11,11),ZMAT(11,11),ZMATR(11,11),SM(11,11)
      CALL MATX(EMAT,BETA)
      CALL LAMDA(ZMAT,BETA)
      DO 10 I=1,N
      DO 10 J=1,N
      EMATR(I,J)=EMAT(J,I)
      ZMATR(I,J)=ZMAT(J,I)
   10 CONTINUE
      CALL MULT (EMATR,Q,SM,N,N,N)
      CALL MULT (SM,EMAT,QINT,N,N,N)
      CALL MULT (SM,ZMAT,EMINT,N,N,N)
      CALL MULT (ZMATR,Q,SM,N,N,N)
      CALL MULT (SM,ZMAT,RINT,N,N,N)
      RETURN
      END
C
C     ..................................................................
```

```
C     ..................................................................
C
C     SUBROUTINE RICCAT
C
C     PURPOSE
C        TO SOLVE THE RICCATI DIFFERENCE EQUATION
C
C     USAGE
C        CALL RICCAT(PHI,D,Q,R,ARK,LEND,N,M,PN,E,JAY)
C
C     DESCRIPTION OF PARAMETERS
C        PHI  - DISCRETE SYSTEM MATRIX OF DIMENSION  N X N
C        D    - CONTROL GAIN MATRIX OF DIMENSION  N X M
C        Q    - STATE WEIGHTING MATRIX OF DIMENSION  N X N
C        R    - CONTROL WEIGHTING MATRIX OF DIMENSION  M X M
C        ARK  - RICCATI MATRIX.INITIAL VALUE MUST BE ASSIGNED IN MAIN
C        LEND - MAXIMUM NUMBER OF ITERATIONS
C        N    - DIMENSION OF STATE VECTOR
C        M    - DIMENSION OF CONTROL VECTOR
C        PN   - VARIABLE PRINTING FORMAT SPECIFIED IN MAIN
C        E    - CONVERGENCE CRITERION
C        JAY  - MODULUS FOR PRINTING RICCATI MATRIX
C
C     REMARKS
C        E SHOULD BE SPECIFIED ONE ORDER LESS THAN DESIRED ACCURACY OF
C        CONVERGENCE. MAXIMUM DIMENSION IS  11 X 11.
C
C     SUBROUTINES REQUIRED
C        MULT
C        VECT ( INVERT )
C
C     METHOD
C        DIRECT ITERATIVE SOLUTION OF DIFFERENCE EQUATION
C
C     ..................................................................
C
      SUBROUTINE RICCAT(PHI,D,Q,R,ARK,LEND,N,M,PN,E,JAY)
      DIMENSION PN(3)
      DIMENSION S1(11,11),S2(11,11),S3(11,11),DTRA(11,11)
      DIMENSION PHI(11,11),D(11,11),Q(11,11),R(11,11),ARK(11,11)
C     INITIALIZATION
      L=0
      LNOT=JAY
      E=10.*E
   80 IF(LEND.EQ.L)GO TO 800
      L=L+1
C     MULTIPLY  K*D
      CALL MULT(ARK,D,S1,N,M,N)
C     MULTIPLY D TRANSPOSE BY S1
      DO 10 I=1,N
      DO 10 J=1,M
      DTRA(J,I)=D(I,J)
   10 CONTINUE
```

```
      CALL MULT(DTRA,S1,S2,M,M,N)
C     ADD R TO S2
      DO 15 I=1,M
      DO 15 J=1,M
      S2(I,J)=R(I,J)+S2(I,J)
   15 CONTINUE
C     INVERT S2
      CALL VECT(S2,M)
C     MULTIPLY S1 BY S2
      CALL MULT(S1,S2,S3,N,M,M)
C     TRANSPOSE S1
      DO 20 I=1,N
      DO 20 J=1,M
      DTRA(J,I)=S1(I,J)
   20 CONTINUE
C     MULTIPLY S3*DTRA
      CALL MULT(S3,DTRA,S2,N,N,M)
C     SUBTRACT S2 FROM K
      DO 25 I=1,N
      DO 25 J=1,N
      S2(I,J)=ARK(I,J)-S2(I,J)
   25 CONTINUE
C     MULTIPLY S2*PHI
      CALL MULT(S2,PHI,S3,N,N,N)
C     TRANSPOSE PHI
      DO 30 I=1,N
      DO 30 J=1,N
      S1(J,I)=PHI(I,J)
   30 CONTINUE
C     MULTIPLY S1*S3
      CALL MULT(S1,S3,S2,N,N,N)
C     ADD Q
      DO 35 I=1,N
      DO 35 J=1,N
      S3(I,J)=S2(I,J)+Q(I,J)
   35 CONTINUE
C     CHECK FOR STEADY STATE
      DO 40 I=1,N
      DO 40 J=1,N
      S1(I,J)=ABS(S3(I,J)-ARK(I,J))
      IF(S1(I,J)-E)40,160,160
   40 CONTINUE
      H=0.
      GO TO 180
C     CHECK FOR PRINTING K AT EVERY JAY ITERATIONS
  160 H=1.
      IF(LNOT.NE.L)GO TO 180
  170 LNOT=LNOT+JAY
      WRITE(6,3000)L
      WRITE(6,3001)
      WRITE(6,PN)((S3(I,J),J=1,N),I=1,N)
```

```
C      RESETTING FOR THE NEXT ITERATION
  180 DO 185 I=1,N
      DO 185 J=1,N
      ARK(I,J)=S3(I,J)
  185 CONTINUE
      IF(H.NE.0.)GO TO 80
C      PRINTING FINAL RESULT
  800 WRITE(6,3000)L
      WRITE(6,3001)
      WRITE(6,PN)((S3(I,J),J=1,N),I=1,N)
 3000 FORMAT (20HOTHE VALUE OF L IS =I6)
 3001 FORMAT('0 THE SOLUTION TO THE RICCATI EQUATION')
      RETURN
      END
C
C      ...............................................................
```

```
C      ...............................................................
C
C      SUBROUTINE GAIN
C
C      PURPOSE
C         TO COMPUTE THE MATRIX OF THE FEEDBACK GAINS
C
C      USAGE
C         CALL GAIN(GMAT,PHI,D,AM,RHAT,ARK,N,M)
C
C      DESCRIPTION OF PARAMETERS
C         GMAT - THE  M X N  MATRIX OF FEEDBACK GAINS
C         PHI  - THE MODIFIED DISCRETE SYSTEM MATRIX,NXN
C         D    - THE DISCRETE SYSTEM'S CONTROL GAIN MATRIX , NXM
C         AM   - THE STATE AND CONTROL WEIGHTING MATRIX, NXM
C         RHAT - THE CONTROL WEIGHTING MATRIX, MXM
C         ARK  - THE SOLUTION TO THE RICCATI DIFFERENCE EQUATION, NXN
C         N    - DIMENSION OF THE STATE VECTOR
C         M    - DIMENSION OF THE CONTROL VECTOR
C
C      SUBROUTINES REQUIRED
C         MULT
C         VECT
```

```
C       ..................................................................
C
        SUBROUTINE GAIN (GMAT,PHI,D,AM,RHAT,ARK,N,M)
        DIMENSION GMAT(11,11),PHI(11,11),D(11,11),AM(11,11),RHAT(11,11)
        DIMENSION ARK(11,11),AMTR(11,11),RMAT(11,11),STMAT(11,11)
C       COMPUTATION OF RHAT INVERSE
        DO 10 I=1,M
        DO 10 J=1,M
        RMAT(I,J)=RHAT(I,J)
   10   CONTINUE
        CALL VECT(RMAT,M)
C       TRANSPOSE AM
        DO 12 I=1,M
        DO 12 J=1,N
        AMTR(I,J)=AM(J,I)
   12   CONTINUE
        CALL MULT(RMAT,AMTR,GMAT,M,N,M)
C       TRANSPOSE D
        DO 14 I=1,M
        DO 14 J=1,N
        AMTR(I,J)=D(J,I)
   14   CONTINUE
        CALL MULT(AMTR,ARK,STMAT,M,N,N)
        CALL MULT(STMAT,D,AMTR,M,M,N)
        DO 16 I=1,M
        DO 16 J=1,M
        RMAT(I,J)=RHAT(I,J)+AMTR(I,J)
   16   CONTINUE
        CALL VECT(RMAT,M)
        CALL MULT(RMAT,STMAT,AMTR,M,N,M)
        CALL MULT(AMTR,PHI,STMAT,M,N,N)
        DO 20 I=1,M
        DO 20 J=1,N
        GMAT(I,J)=GMAT(I,J)+STMAT(I,J)
   20   CONTINUE
        RETURN
        END
C
C       ..................................................................
```

```
C      ..................................................................
C
C      SUBROUTINE INVERT
C
C      PURPOSE
C         TO INVERT A REAL SQUARE MATRIX
C
C      USAGE
C         CALL INVERT(A,NN,N)
C
C      DESCRIPTION OF PARAMETERS
C         A     - REAL SQUARE MATRIX TO BE INVERTED
C         NN    - ORDER OF MATRIX A
C         N     - MAXIMUM ORDER OF A. SET EQUAL TO NN.
C
C      METHOD
C         THE INVERSE OF A IS COMPUTED AND STORED IN A.
C
C      REMARKS
C         THIS SUBROUTINE IS A SLIGHTLY MODIFIED VERION OF THE
C         IBM SHARE NO. 1533 MATRIX INVERSION SUBROUTINE.
C
C      ..................................................................
C
       SUBROUTINE INVERT(A,NN,N)
       DIMENSION A(121),M(11),C(11)
       IF(NN.NE.1)GO TO 80
       A(1)=1./A(1)
       GO TO 300
    80 DO 90 I=1,NN
       M(I)=-I
    90 CONTINUE
       DO 140 I=1,NN
C      LOCATE LARGEST ELEMENT
       D=0.0
       DO 112 L=1,NN
       IF(M(L).GT.0)GO TO 112
       J=L
       DO 110 K=1,NN
       IF(M(K).GT.0)GO TO 108
       IF(ABS(D)-ABS(A(J)))105,105,108
   105 LD=L
       KD=K
       D=A(J)
   108 J=J+N
   110 CONTINUE
   112 CONTINUE
C      INTERCHANGE ROWS
       TEMP=-M(LD)
       M(LD)=M(KD)
       M(KD)=TEMP
```

```
      L=LD
      K=KD
      DO 114 J=1,NN
      C(J)=A(L)
      A(L)=A(K)
      A(K)=C(J)
      L=L+N
  114 K=K+N
C     DIVIDE COLUMN BY LARGEST ELEMENT
      NR=(KD-1)*N+1
      NH=NR+N-1
      DO 115 K=NR,NH
  115 A(K)=A(K)/D
C     REDUCE REMAINING ROWS AND COLUMNS
      L=1
      DO 135 J=1,NN
      IF(J.NE.KD)GO TO 130
      L=L+N
      GO TO 135
  130 DO 134 K=NR,NH
      A(L)=A(L)-C(J)*A(K)
  134 L=L+1
  135 CONTINUE
C     REDUCE ROW
      C(KD)=-1.0
      J=KD
      DO 140 K=1,NN
      A(J)=-C(K)/D
      J=J+N
  140 CONTINUE
C     INTERCHANGE COLUMNS
      DO 200 I=1,NN
      L=0
  150 L=L+1
      IF(M(L).NE.I)GO TO 150
      K=(L-1)*N+1
      J=(I-1)*N+1
      M(L)=M(I)
      M(I)=I
      DO 200 L=1,NN
      TEMP=A(K)
      A(K)=A(J)
      A(J)=TEMP
      J=J+1
  200 K=K+1
  300 CONTINUE
      RETURN
      END
C
C     ...................................................................
```

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Unclassified report

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