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DIGITAL DATA PROCESSING TECHNIQUES FOR RADAR MAPPING

by

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DIGITAL DATA PROCESSING TECHNIQUES FOR RADAR MAPPING

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DIGITAL DATA PROCESSING TECHNIQUES FOR RADAR MAPPING

by

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Submitted to the Department of Electrical Engineering on 17 January 1968 in partial fulfillment of the requirements for the Degree of Doctor of Science.

ABSTRACT

Alternatives for the organization of the data processing problem for high quality radar mapping are examined under the constraint of a "digital" implementation. The comparison is made on the basis of the functional requirements that each formulation, of the organization problem, places on the implementing apparatus. Quantization effects on the quality of the radar maps produced, are studied by generalizing the "spread function" quality indication of linear radar theory to a statistical measure, termed the "M" function. This is essentially the covariance between the radar map function and the scatterer density function which gives rise to the radar signals. It is shown that the "M" function is computable in useful form for a radar in which the data is quantized. The results show that a quantizer step size equal to the rms quantizer input will be adequate in most cases of practical interest.

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LIST OF SYMBOLS*

$a(\mu)$	pulse amplitude modulation function
A_a	aperture area
$A_d(\overline{u}_m - \overline{u}_d)$	overlap area of point-scatterer-signal-function and processor-weighting-function apertures in signal space.
A_L	area under main lobe of $\underline{L}_m(\overline{u}_\alpha; \overline{u}_\beta)$; also area of resolution cell in certain cases.
A_N	normalizing constant in statistical model of $\underline{\Psi}(\overline{u}_m)$
A_s	sampling area in signal space
c	radiation propagation velocity
$C(\overline{u}_d; \overline{u}_m)$	part of $M(\overline{u}_d; \overline{u}_m)$ due to quantization
C_R	range compression ratio
C_S	azimuth compression ratio
$\underline{e}_c(t)$	transmitted signal (in complex analytic form)
$\underline{e}_r(t; \overline{u}_m)$	received signal due to unit point scatter located at $\overline{u}_m$

*The general scheme of notation is as follows: overscores indicate a set of coordinates; an underscore indicates a complex quantity; lack of an underscore indicate a real quantity; the real and imaginary parts of a complex quantity are indicated by subscripts 1 and 2 respectively attached to the same symbol as the complex quantity but without underscores, i.e. $\underline{e}_c(t) = e_{c1}(t) + j\, e_{c2}(t)$. The subscript q indicates a quantity that has been sampled and quantized; the subscript s indicates a quantity that has been sampled but not quantized.

$\underline{E}_r(t)$	received signal
$\mathcal{E}$	$L_m(\bar{u}_a;\bar{u}_a)$
$\underline{f}(t;\bar{u}_d)$	temporal weighting function for map point $\bar{u}_d$.
$g_m(t)$	antenna gain at time t relative to unity for scatterer at $\bar{u}_m$.
$G_m(S)$	geometrical attenuation factor relative to unity for scatterer at $\bar{u}_m$ and radar at S.
$\underline{h}(t, t'; \bar{u}_m)$	response at time t from a unit point scatterer located at $\bar{u}_m$, to an impulse transmitted at time t'
$\underline{i}(\bar{u}_a, \bar{u}_a'; \bar{u}_m)$	response at $\bar{u}_a$ from a unit point scatterer located at $\bar{u}_m$, to an impulse transmitted from $\bar{u}_a'$.
$\underline{I}(\bar{u}_d)$	complex map function
$\mathcal{I}m\{\cdot\}$	imaginary part of
j	$\sqrt{-1}$
$\underline{K}(\bar{u}_a;\bar{u}_d)$	spatial weighting function for map point $\bar{u}_d$
$\underline{L}_m(\bar{u}_a;\bar{u}_\beta)$	Inverse matched smoothing function
$M(\bar{u}_d;\bar{u}_m)$	correlation coefficient of $\lvert \underline{\Psi}(\bar{u}_m) \rvert^2$ and $\lvert \underline{I}(\bar{u}_d) \rvert^2$ defined as the radar map quality function.
$\underline{n}(t)$	noise component of $\underline{E}_r(t)$
$\underline{N}(\bar{u}_d)$	component of $\underline{I}(\bar{u}_d)$ due to noise

N_m	width of strip map in number of resolution cells.
$Q\{\cdot\}$	quantization operator
q	quantizer step size
$\mathcal{R}e\{\cdot\}$	Real part of
$R_{mx}(t)$	distance from antenna to $\bar{u}_m$ at time t
$R_{mx}(S)$	distance from antenna to $\bar{u}_m$ when radar is at S
R	propagation distance coordinate of signal space
R_a	$= \frac{c}{2} T_p$ = propagation distance aperture
R_d	"range" coordinate of map space
R_m, R_x, R_y	"range" coordinate of scatterer space
S	trajectory distance coordinate of signal space
S_a	trajectory distance aperture
S_d	"azimuth" coordinate of map space
S_m	"azimuth" coordinate of scatterer space
S_n	radar position along trajectory at time of transmission of n^{th} pulse
t	time
$T(\bar{u}_x - \bar{u}_y)$	scatterer density correlation function
T_n	reference time or time of transmission of n^{th} pulse
T_p	transmitted pulse length

$\bar{u}_m, \bar{u}_x, \bar{u}_y$	scatterer space coordinates
$\bar{u}_\alpha, \bar{u}_\beta$	signal space coordinates
$\bar{u}_d$	map space coordinate
$\bar{u}_\alpha(t)$	mapping of t onto a one-dimensional subset of signal space.
U_o	speed along the trajectory
$\underline{v}_r(\bar{u}_\alpha : \bar{u}_m)$	spatial model of received signal from unit point scatterer at u_m
$\underline{V}_r(\bar{u}_\alpha)$	spatial model of received signal
V	$2\pi/q$ = quantizing "frequency"
(VR)	ratio of the volume under the incoherent part of $M(\bar{u}_d; \bar{u}_m)$ to the volume under the side-lobes of the dominant part
$w(\bar{u}_\alpha ; \bar{u}_d)$	weighting function amplitude or "aperture" function
$w_1 w_2, w_3, w_4$	amplitude weightings due to quantization
$w_a(\bar{u}_\alpha ; \bar{u}_d)$	effective amplitude weighting, for dominant part of $M(\bar{u}_d; \bar{u}_m)$, due to quantization "after-the-multiplier".
W	spatial bandwidth in radian measure
β	ratio of bit storage capacity required for immediate formulation of processing to that required for the delayed formulation

$\underline{\Gamma}(\bar{u}_d;\bar{u}_m)$	the smoothing function
$\underline{\Gamma}_m(\bar{u}_d;\bar{u}_m)$	smoothing function for a "matched" processor
$\underline{\Gamma}_q(\bar{u}_d;\bar{u}_m)$	smoothing function for a quantized weighting function (but without data quantization)
$\delta(\bar{u}_x-\bar{u}_m)$	Dirac delta function in one or more dimensions
$\underline{\eta}(\bar{u}_a)$	spatial noise component of $\underline{V}_r(\bar{u}_a)$
λ	wavelength
μ	dummy time variable
$\underline{\pi}(\bar{u}_a;\bar{u}_d)$	product of $\underline{V}_r(\bar{u}_a)$ and $\underline{K}^*(\bar{u}_a;\bar{u}_d)$
$\Phi(\mu)$	pulse phase modulation function
$\Phi_e(\bar{u}_a)$	equivalent phase error due to quantization
$\underline{\Psi}_m$	complex scattering coefficient for a point scatterer located at $\bar{u}_m$
$\underline{\Psi}(\bar{u}_m)$	scatterer density function
ω	radian frequency

I. INTRODUCTION

A. Objective

The purpose of the research reported in this thesis has been to develop analysis techniques which would be useful in the design and evaluation of fine-resolution radar mapping systems in which the extraction of map data from signal data is accomplished "digitally". By this we mean systems in which the data manipulation or "processing" is accomplished, at least in part, on digitized number representations of the data. Specifically, we have sought and attained useful results in

(1)Modeling radar signals, particularly the conversion from "temporal" to "spatial" models of the signals encountered in airborne mapping radars.

(2)Organizing the data-processing operations and identifying two competing formulations of data processing for detailed study.

(3)Determining the effects of quantization of radar data on radar mapping performance by generalizing the conventional "spread function" (or spatial impulse response) criterion of radar performance to a statistical performance function which is calculable for "quantized" data processing.

(4)Specifying the functional requirements which various radar performance objectives will impose on the implementing digital apparatus.

B. Background and Motivation

The generic type of radar system under study is the so-called "pulse-compression, synthetic-aperture" type, intended for use in making "fine-resolution" maps of the earth's surface from a moving vehicle, in particular an aircraft. As is well known, the "synthetic-aperture" radar obtains an improvement in azimuth resolution over a conventional radar by performing a weighted, coherent summation of the received signals over an interval of time during which the carrier vehicle moves an appreciable distance. This summation is comparable to that accomplished by a real antenna on the field vectors in its physical aperture; hence the term "synthetic aperture". The first-order theory of such radars is well understood and operating models have been constructed.[2,5]

Existing implementations of combined synthetic aperture and pulse compression radars are almost exclusively in the domain of analog apparatus, with the optical processor the most successful to date.[5] In operation, the radar data processor must accomplish a separate summation for each resolvable element in the map finally obtained. The rate at which resolvable elements can be scanned by synthetic-aperture radars, carried on modern aircraft, is very high, being upwards of 10^5 resolvable elements per second. The time duration of the radar signals which enter into the summation pertinent to a single resolvable element is relatively long by electronic standards, being on the order of seconds. These facts have combined to make high-density analog signal storage media, such as photographic film, a prerequisite for successful implementation of analog processors. The first-order theory of synthetic aperture radars shows that the required summation is very similar to the computation of a Fourier transform, an operation which optical systems can perform quite naturally and accurately on signals recorded as photographic transparencies. Hence, the present pre-eminence of optical processors.

There are two areas of application of fine-resolution mapping radars where optical processors have some serious disadvantages. One is in the area of "real-time viewing" of processed radar data, i.e. on the aircraft carrying the radar. In fact, optical processing of radar data is customarily done at a ground-based facility, due to the difficulties involved in the mechanical and chemical manipulation of photographic film. Furthermore, the optical processor requires precise adjustments which are difficult to maintain in an aircraft environment. This difficulty is compounded when the radar is required to operate over a wide range of geometrical parameters (i.e. range intervals, "squint" angles, etc.), since precise mechanical adjustments of processor components (e.g. lens positions and orientations) are necessary in order to change from one mode of operation to another.

There is, of course, a tremendous incentive to develop an all-electronic alternative to the optical processor for this real-time application. Electronic analog implementations are, however, caught between the choice of providing a large array of summers each stable over intervals on the order of seconds, or providing temporary analog-electronic data storage to compress the signal time duration. The former choice is impractical from an equipment size and weight standpoint, while a satisfactory technology does not exist for the latter. Various hybrid schemes (e.g. electro-optical) can be proposed to fill the operational need, each sharing to some extent the problems of the optical processor in the vehicle environment.

The second area where the optical processor is less than totally satisfactory is that of extremely fine resolution, approaching to within an order of magnitude of the theoretical resolution limit of half the transmitted wavelength. In this regime the first-order theory of synthetic aperture radars is no longer adequate, and the "natural" fit of optical systems to the desired processing operation no longer exists. More exotic optical components than simple lenses and masks are required. An even more troublesome problem arises when the random motions of the carrier vehicle are considered.

When the motion of the carrier vehicle departs from a defined nominal trajectory, these deviations must be measured and accounted for in the data-processing operation. The "first-order" method of satisfying this requirement is to operate on the received radar signal in such a way that the modified signal is very nearly the same as that which would have resulted if the carrier vehicle had exactly followed the desired trajectory. The extent to which this procedure can be successful depends on the magnitude of the deviations, and on the resolution desired from the processed radar data. For the very fine resolution regime we are considering here, this procedure will, in general, be inadequate and the departures from the nominal trajectory will require a separate modification of the radar data and/or processor weighting function for each and every map "point" (i.e. resolvable element)to be produced. It is difficult to conceive of a practical way of doing this with an optical processor.

In contrast with the "real-time" processing regime, there appears to be very little effort to extend the synthetic-aperture radar technology into this very fine-resolution area. This is possibly because the need is less urgent and possibly because the full potential of synthetic-aperture radars have not yet been realized at the lower levels of performance for which optical processors are eminently suited.

With this qualitative assessment of the "state-of-the-art" in synthetic-aperture radars, we can now motivate our concern with "digital" implementations of radar data processing for fine-resolution mapping radars. In performing computations involving weighted sums, or integrals, digital apparatus has three potentially advantageous characteristics:

(1)stability of integration over arbitrary time intervals,

(2) accurary of computation,

(3) flexibility in modification of weighting functions.

A concomitant disadvantage is a relatively low density of data storage, although the trend of technological advance is steadily providing increased storage density. The exploitation of the advantages and the minimization of the disadvantages may lead to radar mapping systems which compare favorably with alternative analog systems in capability per unit cost. Our objective is to make some valid judgements among various ways of organizing a "digital" system to accomplish a given set of radar performance specifications.

In comparing digital implementations, we must distinguish between categories of radar performance. Our discussions will be primarily concerned with categories of radar performance that lie in the "real-time viewing" and the very fine-resolution categories discussed previously, i.e. in the operating regimes where optical processors are least efficient. While these categories are the logical starting points for motivating the investigation of digital techniques, they need not be regarded as their limits of applicability. The results in the present research will be applicable to the design of a digital processor at any level of performance. Where numerical results are given we will confine ourselves to these two extremes of radar-performance requirements.

The advantageous characteristics of digital instrumentation are just the ones required to circumvent the difficulties which electronic-analog implementations encounter in the "real-time" viewing category of radar operation. The stability of digital integration over arbitrary time intervals enables the designer to consider many alternate formulations of the data-processing problem which are denied to the analog implementation. In particular, a formulation where the processing operations are performed directly on the received signal samples can be considered. No mechanical tolerances are involved in digital processing and the "processor weighting functions" can be programmed to any desired accuracy* for any mode of operation in a straightforward manner. Processing delay is at a minimum since only electronic operations are involved. If the area coverage and desired resolution are modest, (i.e. if we require a map in which not over 10^5 resolvable elements per second are to be processed for immediate display) the digital data storage requirements will not be found to be a-priori unreasonable in the light of present-day technology.

Under the "non-real-time", wide-area-coverage, fine-resolution category of radar operation, the digital computer provides the necessary flexibility to allow the construction of arbitrarily accurate weighting

*We distinguish accuracy from precision - the former refers to functions being evaluated, the latter to the number of figures retained in the computation.

functions. In fact the digital computer may inherit this operational regime by default, just as has already occurred in the largest and most highly-developed ground-based search radars. Since we are not placing any a-priori time or equipment constraints (either directly or implied) on the implementation of a processor for this category of performance, we can conceive of this processing being accomplished on a large general purpose digital computer with perhaps some specialized peripheral equipment. Implementation thus becomes more of a programming than an equipment problem, a situation which is very likely to lead to real economic advantages for digital radar-data processing.

C. Summary of Results

The problems that will normally occur in the design of a digital radar-data processor may be classified, perhaps somewhat arbitrarily, as formulation, function, and implementation. The designer will first investigate the methods by which the desired results may be obtained and from these select a formulation for organizing the data-processing operation. Each formulation will specify a sequence of operations which must be performed on the raw data to obtain the final result. The functional requirements that each operation places on the apparatus which will accomplish that operation must be determined and will also influence the selection of a formulation. Finally, hardware must be designed (or selected) and assembled to implement the processing operation.

In this research program we have been primarily concerned with the closely related areas of formulation and function, under the constraint of a digital implementation. We have explored the most promising formulation possibilities and deduced the functional requirements implied for each choice.

As in most engineering problems, the first task has been to develop a model of the physical situation, in this case, a cause and effect relationship between "scatterers" and radar signals, and a definition of the end result sought from the "processing" operations to be performed on the received signals. Our starting point has been "dense scatterer" theory[1] modified to fit the geometry of synthetic aperture radar. This gives what may be termed a "temporal" model of radar signals and processing since the signals and weighting functions are defined as a one-dimensional time variable, and the processing integration is over this variable. Inherent in most expositions of synthetic aperture radar[2, 5] is a transformation to a spatial domain of signal and weighting function definition, followed by a processing integration over this spatial domain. The method of transformation has always seemed to the author to be excessively hueristic and intuitive, and inherently difficult to apply except for very idealized geometries.

In this thesis, in Section II, we accomplish the transformation from a temporal to a spatial model by <u>defining</u> a spatial model of radar signals, based on a redefinition of the temporal "unit-point-scatterer" impulse response of dense-scatterer theory as a sampling of a spatial function over a defined spatial domain. The temporal radar model can then be obtained as a particular "sampling" of this spatial model and conditions of equivalance are easily established. The definition of the spatial domain is not unique, but must be justified, in any particular case, by showing that any valid sampling over this spatial domain leads to a possible radar signal in the temporal domain. The method is applied to the usual simplified geometry case in Section II, and to a more realistic situation in Appendix A. The justification for a spatial model is that the resulting weighting functions are simplier in form and the spatial integrations are easier to implement.

Based on this model we have identified two methods of organizing the data processing operation which appear to be advantageous and which, in some form or combination, encompass a whole range of possibilities, although we consider them only in their "pure" form. These are defined in Section III as the "delayed" formulation and the "immediate" formulation. In the delayed formulation, each map point representation, of the scatterer distribution viewed by the radar, is computed sequentially from all the data pertinent to that map point. Hence, the data must be held in storage until all that is pertinent to that point is available, and the storage medium "interrogated" repeatedly for the computation of many map point representations. In the immediate formulation each data value is multiplied by all the weightings appropriate to all the map points which will be affected by that data, and the integrations appropriate to many map points are carried forward simultaneously. Hence, the data storage takes the form of an array of partial sums. Most present day processors use the "delayed" formulation, the optical processor[5] being the premier example.

Before we can compare these competing formulations in the context of a digital implementation, we must be able to say something about the effects of data quantization and obtain some estimate of the allowed coarseness of quantization. (This analysis is given in Section IV supplemented by Appendices B and C.) We accomplish this by developing a generalization of the conventional "spread function" measure of radar performance, which applies to "linear" radars. The "spread function" is essentially the response of the radar over the map coordinate space, to a scatterer configuration consisting of only a single unit point-scatterer at a given location. We show that the meaning of the spread function in terms of map quality can be interpreted as being a special case of the correlation between the radar map produced and the "scatterer-density function" which gave rise to the radar signal, when the scatterer density function is modeled as an ensemble member of an infinite (spatial) bandwidth random process, and the radar is linear. We term this correlation an

"M" function. It is defined for every mapping radar, linear or not, and we show that it is computable in a useful way for radars in which the data is quantized. The computations depend on a generalization of a result by Widrow[3] which is developed in Appendix B. An "exact" computation of the "M" function is given in Appendix C together with an approximate solution which has a simple interpretation.

The approximate solution for the "M" function is composed of the sum of a "dominant" part, which is closely related to the spread function of the corresponding "linear" (not quantized) radar system, and an "incoherent" part which is proportional to a weighted sum of signal amplitudes. The effect of the incoherent part is to raise the "side-lobe" level of the "M" function relative to that obtained with infinitely fine quantization of data. We take as a criterion of the allowed coarseness of quantization, the the "volume" under the incoherent part should be less than the volume under the "sidelobes" of the dominant part of the "M" function. For the case of "uniform" aperture functions (i.e. complex, unit-point-scatterer signal functions and processor weighting functions with constant magnitude over a finite region of signal space and zero outside this region) a quantizer step size equal to the rms quantizer input makes the volume under the incoherent part negligible with respect to the volume under the sidelobes of the dominant part. From the form of the result it is evident that the same criterion will be adequate for "tapered" aperture functions even though these can have "M" functions with dominant parts with much lower side-lobe levels than the uniform aperture functions.

We also distinguish between systems in which quantization of data is accomplished "after multiplication" by the weighting function, and those systems in which the received signals are quantized "before multiplication" by the weighting function. For uniform aperture weighting functions, the "M" function is identical for both systems. For tapered aperture weighting functions, the dominant part of the "M" function for "after-the-multiplier" quantization of data, is proportional to a linear spread function in which the tapering of the processor weighting function has been distorted. In Section IV we show that the effective tapering can be restored to a predetermined desired form by a modification of the actual tapering.

With the results on quantization available, we can make some direct comparisons between the delayed and immediate formulations of data processing. This is the subject of Section V. Here we are making comparisons of a very practical nature in the context of very general properties of the competing formulations. In the actual development of a radar system, ingenuities of design in the implemating hardware may make one formulation preferable to the other for reasons we cannot consider here.

Among the clearly established facts are the following:

(1)In certain cases (in particular for azimuth only processing and moderate resolution applications and where uniform aperture weighting functions are adequate) the generation of weighting functions is simplified by the immediate formulation of data processing.

(2)The immediate formulation will always require more data storage capacity than the delayed formulation by a factor ranging between 3 and 10, the higher ratios accompanying the finest resolution requirements.

(3)The delayed formulation will always require faster summing circuits, the exact ratio depending on the number of parallel summers provided in the delayed formulation. However, large numbers of parallel summers, become, in effect, additional data storage capacity which then reduce the relative advantage of the delayed formulation in this regard.

Because of the large quantities of data that will be required for the very fine-resolution, wide-area-coverage, non-real-time applications, the advantage of the delayed formulation in data storage capacity will probably be decisive. The real-time processing applications, where resolution requirements are modest and area coverage requirements (in terms of resolution cells to be evaluated per unit of distance traveled) are low, may be better served by the immediate formulation. This will be particularly true if large arrays of analog multipliers can be economically fabricated and the data is quantized "after-the-multiplier".

II. A MODEL OF THE RADAR MAPPING PROBLEM

From the viewpoint of data-processing, the system consisting of a radar and a group of reflecting bodies is simply a stochastic "modulator", a device which modifies a given input signal in a way which is functionally dependent on the particular configuration of remote bodies, in the radar field of view during the conduct of a particular radar "observation" experiment. A sufficient radar signal model is one which specifies this functional dependence in terms of the "size", or reflectivity of their remote bodies, their coordinates relative to the radar, and the variations of these parameters with time over the radar observation interval. No known model of radar signals purports accomplish this task exactly, and it is likely that any model which did would be too complex for practical use.

The discussion in this thesis is concerned only with "mapping" radar observations, in particular with mappings of the earth from aircraft. Such observations are differentiated from other types by the following characteristics.

(1) The remote bodies are linear time-invariant "scatterers" of electromagnetic waves and their positions are fixed relative to each other.*

(2) The distribution of scatterers is "dense" in the scatterer space.

*Extension of the model to include slowly moving isolated scatterers, e.g. terrestrial vehicles, can be accomplished but will not be considered here.

(3) Time variations in the geometry of the radar-scatterer system are completely described by the motion of the radar in a coordinate system in which the scatterers are at rest, and this motion is presumed known or measurable to arbitrary accuracy.

An appropriate model for this situation is the "dense" scatterer model used by Bar-David[1], extended to include certain geometric attenuation factors. In this model an elementary object called a "point scatterer" is defined, and real scatterer configurations are modelled as simple summations of these objects. The functional dependence of the received signal on the scatterer parameters is made independent of the form of the transmitted signal by assuming linearity for the elementary radar-point-scatterer system, i.e. by assuming the existence of an impulse response description.

Let $\bar{u}_x$ (or $\bar{u}_y$) represent coordinates in a "scatterer space" being mapped by our radar, and let $\bar{u}_m$ denote a particular point in this scatterer space. Then a "point-scatterer" is defined by a function*, $\underline{h}(t, t'; \bar{u}_m)$, which is the response (at the radar's "receiving" terminals) at time t to a unit impulse transmitted at time t' when only a "unit" point scatterer is present, at the point $\bar{u}_m$, in the scatterer space. With $\underline{e}_c(t)$ representing the complex analytic form of the real transmitted signal $e_{c1}(t)$, we have

$$\underline{e}_r(t; \bar{u}_m) \equiv \int \underline{h}(t, t'; \bar{u}_m)\, \underline{e}_c(t')dt' \qquad (1)$$

*In general, a complex function representing the received signal in complex analytic form.

for the complex received signal and

$$e_{r_1}(t;\bar{u}_m) = \mathcal{R}_e\left\{\underline{e}_r(t;\bar{u}_m)\right\}$$

for the real received signal* from a unit point scatterer at $\bar{u}_m$. When a collection of scatterers is present in the scatterer space, the complex received signal is taken as

$$\underline{E}_r(t) = \sum_m \underline{\Psi}_m \underline{e}_r(t;\bar{u}_m) + \underline{n}(t) \qquad (2)$$

where

$\underline{\Psi}_m$ = a complex reflection coefficinet for the m^{th} point scatterer

$\underline{n}(t)$ = an arbitrary additive noise process in complex analytic form.

In the limit of a continuous distribution of point scatterers in the scatterer space being observed, we generalize to $\underline{\Psi}(\bar{u}_x)$, a complex "scatterer density function" and let

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*The transmitted signal is "analytic", by assumption, and we will assume that $\underline{h}(t, t'; \bar{u}_m)$ is such that $\underline{e}_r(t; \bar{u}_m)$ retains this analytic character. By "analytic" we mean that $\underline{e}_r(\tau; \bar{u}_m)$ for $\tau = t + j\sigma$ is finite over the whole half-plane where $\sigma \geq 0$. As is well known this implies that the real and imaginary parts of $\underline{e}_r(t; \bar{u}_m)$ are a Hilbert Transform pair and that the Fourier Transform of $\underline{e}_r(t; \bar{u}_m)$ is zero for negative frequencies. Consequently reception of $e_{r_1}(t; \bar{u}_m)$ is equivalent to reception of $\underline{e}_r(t; \bar{u}_m)$.

$$\begin{aligned}\underline{E}_r(t) &= \int d\bar{u}_x\, \Psi(\bar{u}_x)\, \underline{e}_r(t;\bar{u}_x) + \underline{n}(t) \\ &= \int d\bar{u}_x \int dt'\, \underline{\Psi}(\bar{u}_x)\, \underline{e}_c(t')\, \underline{h}(t,t';\bar{u}_x) + \underline{n}(t)\end{aligned} \tag{3}$$

Again the actually received real signal is

$$E_{r_1}(t) = \mathcal{R}e\left\{\underline{E}_r(t)\right\}$$

The central point is that the specific form assumed for $\underline{h}(t,t';\bar{u}_m)$ becomes the definition of a point scatterer. If we associate a particular type of physical body with a "point scatterer", e.g. a perfectly conducting sphere, then $\underline{h}(t,t';\bar{u}_m)$ will be related to the solution of a rather complex boundary value problem. A more practical approach is to assume an $\underline{h}(t,t';\bar{u}_m)$ with a simple form, which obeys the known energy and propagation velocity constraints on electromagnetic radiation, and adopt the convention that for every $\underline{E}_r(t)$ there exists a $\underline{\Psi}(\bar{u}_x)$ which could have generated it according to Eq. 3. The defined purpose of our radar will be to extract a representation of $\left|\underline{\Psi}(\bar{u}_x)\right|^2$ from $\underline{E}_r(t)$; the interpretation of $\left|\underline{\Psi}(\bar{u}_x)\right|^2$ in terms of actual reflecting bodies is then a separate problem which we will not consider. As a practical matter, the interpretation problem is always solved empirically, i.e. by testing.

Let $\bar{u}_d$ represent the coordinates of a point in "map space", i.e. the coordinate system in which the radar data processor will construct a representation of $|\underline{\Psi}(\bar{u}_x)|^2$. Let $\underline{I}(\bar{u}_d)$ be a complex map function and let $|\underline{I}(\bar{u}_d)|^2$ be the desired representation of $|\underline{\Psi}(\bar{u}_x = \bar{u}_d)|^2$. We will consider linear operations on $\underline{E}_r(t)$ of the form

$$\underline{I}(\bar{u}_d) = \int \underline{E}_r(t)\, \underline{f}^*(t;\bar{u}_d)\,dt \tag{4}$$

in which the integral is to be evaluated for every $\bar{u}_d$ point where a value of $|\underline{I}(\bar{u}_d)|^2$ is desired. The function $\underline{f}^*(t;\bar{u}_d)$ is termed the "processor weighting function". Its selection to optimize some aspect of $|\underline{I}(\bar{u}_d)|^2$ is a topic that has received considerable attention.[1,6,7] Bar-David[1] showed that when $\underline{\Psi}(\bar{u}_x)$ and $\underline{n}(t)$ are complex normal random processes, and $\underline{f}(t;\bar{u}_d)$ is correctly chosen, the linear operation given in Eq. 4 yields a maximum a-posteriori probability estimate for $|\underline{\Psi}(\bar{u}_x = \bar{u}_d)|^2$ for all $\bar{u}_d$, and he obtained a solution for this $\underline{f}(t;\bar{u}_d)$. If we put Eq.3 into Eq. 4 we obtain

$$\underline{I}(\bar{u}_d) = \int dt \int d\bar{u}_x\, \underline{\Psi}(\bar{u}_x)\, \underline{e}_r(t;\bar{u}_x)\underline{f}^*(t;\bar{u}_d) + \underline{N}(\bar{u}_d) \tag{5}$$

or

$$\underline{I}(\bar{u}_d) = \int d\bar{u}_x\, \underline{\Psi}(\bar{u}_x)\, \underline{\Gamma}(\bar{u}_d;\bar{u}_x) + \underline{N}(\bar{u}_d)$$

$$\text{where} \quad \underline{N}(\bar{u}_d) = \int dt\, \underline{n}(t)\underline{f}^*(t;\bar{u}_d) = \text{"filtered noise"} \tag{6}$$

and

$$\underline{\Gamma}(\bar{u}_d; \bar{u}_x) = \int dt\, \underline{e}_r(t; \bar{u}_x)\, \underline{f}^*(t; \bar{u}_d) \qquad (7)$$

is termed the radar "smoothing function". Its squared magnitude is called the radar "spread function". We may note the "spread function", $\left| \underline{\Gamma}(\bar{u}_d; \bar{u}_m) \right|^2$, is an "impulse response" in the sense that a single "unit point scatterer" located at $\bar{u}_x = \bar{u}_m$ has a scatterer density function representation of $\underline{\Psi}(\bar{u}_x) = \delta(\bar{u}_x - \bar{u}_m)$. The "map" which would result from the radar "viewing" this "scene" is just $\left| \underline{\Gamma}(\bar{u}_d; \bar{u}_m) \right|^2$.

The interchange of the order of integrations in Eq. 5 which allows us to write Eqs. 6 and 7 is valid because of the linearity of the defined data processing operations. In Section IV of this thesis we will be concerned with the effect that quantizing $\underline{E}_r(t)$ has on $\left| \underline{I}(\bar{u}_d) \right|^2$ and this interchange of order of integration will no longer be valid. However, we will want to compare radar operation in the quantized case to the linear model of Eqs. 6 and 7. This will be done by devising a representation of radar operation in which $\underline{\Psi}(\bar{u}_x)$ is a random process.

As a practical matter, Eq. 7 is the usual basis for selecting the processor weighting function, $\underline{f}(t; \bar{u}_d)$, rather than some theoretical optimization procedure. When the spread function, $\left| \underline{\Gamma}(\bar{u}_d; \bar{u}_m) \right|^2$, has a sharply peaked character near $\bar{u}_d = \bar{u}_m$ and falls off to very small values outside the small region occupied by this peak, we have an intuitive feeling that the radar maps produced will be of high "quality", i.e. will show fine details in the structure of $\left|\underline{\Psi}(\bar{u}_x)\right|^2$ with good fidelity. The more sharply peaked the spread function, and the lower the skirts, or "sidelobes", the better the expected "quality" of the radar map. Quantitative measures of radar mapping quality are usually defined in terms of the spread function, and we will have occasion to discuss these more fully in subsequent sections. For the present we need only the well known fact that the weighting functions, $\underline{f}(t;\bar{u}_d)$, which produce the most desirable spread functions have arguments (i.e. phase functions) which are equal to the argument of $\underline{e}_r(t;\bar{u}_d)$ and have magnitude functions which are <u>similar</u> to $\underline{e}_r(t;\bar{u}_d)$ (i.e. $\left|\underline{f}(t;\bar{u}_d)\right|$ will have appreciable value only where $\left|\underline{e}_r(t;\bar{u}_d)\right|$ has appreciable value) but not necessarily exactly the same. The phase matching condition is essential to producing a sharp central peak in the spread function. The magnitude of the processor weighting is often adjusted empirically to produce a desirable shape in the spread function in the region outside the central peak. Of course the result actually obtained depends on both $\underline{f}(t;\bar{u}_d)$ and $\underline{e}_r(t;\bar{u}_m)$ but the freedom to select $\underline{e}_r(t;\bar{u}_m)$ will be limited by practical design problems and the physics of the situation.

Freedom of choice for $\underline{f}(t;\overline{u}_d)$ is limited only by the available computational power.* When $\underline{f}(t;\overline{u}_d)$ is equal to $\underline{e}_r(t;\overline{u}_d)$ over the major portion of its extent in t, the processor is said to be "matched".+

The foregoing radar mapping model may be called a "temporal" model, since the desired processing operations are defined as integrations on time, t, and the received radar signal $\underline{E}_r(t)$ is obtained as a function of the single independent variable, t. There is, however, an alternate model which we shall term "spatial", which will lead us to better ways of synthesizing the processor weighting functions and to simpler processing. In this section we will give a general development of the spatial model and illustrate it by an idealized example. In Appendix A we give an example using a more realistic geometry.

Consider the unit-point-scatterer impulse response function, $\underline{h}(t,t';\overline{u}_m)$. The simplest form that this function can take, consistant with a constant radiation propagation velocity and with conservation of energy, is an impulse delayed by the round-trip propagation time,

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*Once $\underline{e}_r(t;\overline{u}_m)$ is specified, there is, of course, only limited freedom in shaping the spread function through choice of $\underline{f}(t;\overline{u}_d)$.

+It is well known that a matched processor maximizes the expectation of the ratio of $\left|\underline{I}(\overline{u}_d)\right|^2$ to $\left|\underline{N}(\overline{u}_d)\right|^2$ when $\underline{n}(t)$ is "white noise" attenuated by the spherical-wave law and amplified by the directional characteristics of the radar antenna,* i.e.

$$h(t, t'; \overline{u}_m) = \frac{g_m(t) g_m(t')}{\mathcal{R}_{mx}(t) \mathcal{R}_{mx}(t')} \delta \left[t - t' - \frac{1}{c} (\mathcal{R}_{mx}(t) + \mathcal{R}_{mx}(t')) \right] \quad (8)$$

where $\mathcal{R}_{mx}(t)$ = distance from the radar to the point scatterer at point $\overline{u}_m$ at time t

$g_m(t)$ = antenna "gain", relative to unity for the antenna orientation and position at time t, in the direction of the point $\overline{u}_m$.

c = propagation velocity

and $\delta(t)$ = the unit impulse (Dirac delta function) at $t = 0$

If the radar antenna follows a known (or measurable) path in the fixed coordinate system containing the scatterer space, and its velocity along that path is never zero, then a one-to-one correspondence can be made between distance, S, along the path and time, t. We can then write $S = S(t)$ and we can always find a form for $\mathcal{R}_{mx}(t)$ like

$$\mathcal{R}_{mx}(t) = R_{mx}(S(t))$$

Similarly

$$G_m(S(t)) = \frac{g_m(t)}{\mathcal{R}_{mx}(t)}$$

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*This particular model for $\underline{h}(t, t'; \overline{u}_m)$ is real. Hence we shall drop the underscore in the sequel.

Now, if we define a "propagation distance", R, by

$$R(t) = \frac{c}{2}(t - T_n)$$

where $\quad T_n$ = an arbitrary reference time

we can write for $h(t, t'; \overline{u}_m)$

$$h(t, t'; \overline{u}_m) = i(\overline{u}_\alpha(t), \overline{u}_\alpha(t'); \overline{u}_m) \tag{9}$$

where

$$\overline{u}_\alpha(t) = (R(t), S(t))$$

and

$$i(\overline{u}_\alpha, \overline{u}_\alpha'; \overline{u}_m) = G_m(S) G_m(S') \delta\left[\frac{2}{c}\left(R - R' - \frac{R_{mx}(S) + R_{mx}(S')}{2}\right)\right] \tag{9A}$$

with $\quad \overline{u}_\alpha = (R, S)$

$\overline{u}_\alpha' = (R', S')$

We have obtained the spatial function $i(\overline{u}_\alpha, \overline{u}_\alpha'; \overline{u}_m)$ for a special $h(t, t'; \overline{u}_m)$ but the process is by no means limited to this special choice of $h(t, t'; \overline{u}_m)$. Furthermore, the i we have obtained is dependent upon the particular choice made for the spatial variables R and S, and indeed these need not be limited to two. In general, if we are given an $i(\overline{u}_\alpha, \overline{u}_\alpha'; \overline{u}_m)$ function and a one-to-one mapping from "t" onto a one dimensional subset of the $\overline{u}_\alpha$ space, i.e. $\overline{u}_\alpha = \overline{u}_\alpha(t)$ (implies $\overline{u}_\alpha' = \overline{u}_\alpha(t')$), such that $h(t, t'; \overline{u}_m)$ is given by

Eq. 9, then we can define a spatial radar model by+

$$\underline{v}_r(\bar{u}_a; \bar{u}_m) \equiv \int i(\bar{u}_a, \dot{\bar{u}}_a(t'); \bar{u}_m)\underline{e}_c(t')dt' \tag{10}$$

$$\underline{V}_r(\bar{u}_a) \equiv \int d\bar{u}_x \underline{\Psi}(\bar{u}_x)\, \underline{v}_r(\bar{u}_a; \bar{u}_x) + \underline{\eta}(\bar{u}_a) \tag{11}$$

$$\underline{I}'(\bar{u}_d) \equiv \int d\bar{u}_a\, \underline{V}_r(\bar{u}_a)\underline{K}^*(\bar{u}_a; \bar{u}_d) \tag{12}$$

where again we desire to have $\left|\underline{I}'(\bar{u}_d)\right|^2$ be a representation of $\left|\underline{\Psi}(\bar{u}_m = \bar{u}_d)\right|^2$. We now inquire as to the conditions under which $\left|\underline{I}'(\bar{u}_d)\right|^2$ will be equivalent to $\left|\underline{I}(\bar{u}_d)\right|^2$; under what conditions the actually received radar signal, $\underline{E}_r(t)$, can be used to obtain $\underline{V}_r(\bar{u}_a)$; and what are the advantages of processing according to the spatial model rather than the temporal model.

The questions of equivalence between $\left|\underline{I}'(\bar{u}_d)\right|^2$ and $\left|\underline{I}(\bar{u}_d)\right|^2$ and construction of $\underline{V}_r(\bar{u}_a)$ from $\underline{E}_r(t)$ are easier to answer in reverse. That is, simple comparison of Eqs. 9, 10, and 11 with Eqs. 3 and 4 shows that

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+For completeness we introduce a spatial noise process, $\underline{\eta}(\bar{u}_a)$, which should correspond in some sense to $\underline{n}(t)$, the temporal noise process in Eq. 2. The effects due to this spatial noise process will be carried along in the equations but we will temporarily ignore the noise in discussing the relationship between the temporal and spatial models of the radar signals. We will return to noise processes at the end of this section.

$$\underline{E}_r(t) = \underline{V}_r(\bar{u}_\alpha(t))$$

$$\underline{e}_r(t; \bar{u}_m) = \underline{v}_r(\bar{u}_\alpha(t); \bar{u}_m)$$

and

$$\underline{I}(\bar{u}_d) = \int dt\, \underline{V}_r(\bar{u}_\alpha(t))\, \underline{K}^*(\bar{u}_\alpha(t); \bar{u}_d) \tag{13}$$

provided

$$\underline{n}(t) = \underline{\eta}(\bar{u}_\alpha(t))$$

and

$$\underline{f}(t; \bar{u}_d) = \underline{K}(\bar{u}_\alpha(t); \bar{u}_d)$$

(This last condition is certainly easy to meet. It merely requires that $\underline{K}(\bar{u}_\alpha; \bar{u}_d)$ bear the same relationship to $\underline{v}_r(\bar{u}_\alpha; \bar{u}_m = \bar{u}_d)$ as $\underline{f}(t; \bar{u}_d)$ bears to $\underline{e}_r(t; \bar{u}_m = \bar{u}_d)$, i.e. identical argument and "similar" magnitude.) Hence given $\underline{V}_r(\bar{u}_\alpha)$ and the $\bar{u}_\alpha = \bar{u}_\alpha(t)$ curve we could certainly construct $\underline{E}_r(t)$ but the reverse process is not assured, a-priori. But $\underline{E}_r(t)$ is, in fact, a sampling of $\underline{V}_r(\bar{u}_\alpha)$ over the $\bar{u}_\alpha$ space, specifically at the points $\bar{u}_\alpha = \bar{u}_\alpha(t)$. Then a set of conditions under which $\underline{E}_r(t)$ suffices for the construction of $\underline{V}_r(\bar{u}_\alpha)$ is well known, namely that $V_r(\bar{u}_\alpha)$ be strictly "bandlimited" as a function of each $\bar{u}_\alpha$ coordinate, and that the $\bar{u}_\alpha = \bar{u}_\alpha(t)$ points be regularly distributed over the $\bar{u}_\alpha$ space at intervals along each coordinate that are less than the inverse of the bandwidth associated with that coordinate. Similarly it can be shown that $\underline{I}(\bar{u}_d)$ as given by Eq. 13

will be strictly proportional to $\underline{I}'(\bar{u}_d)$ if both $\underline{V}_r(\bar{u}_\alpha(t))$ and $\underline{K}(\bar{u}_\alpha(t);\bar{u}_d)$ are adequate samplings of $\underline{V}_r(\bar{u}_\alpha)$ and $\underline{K}(\bar{u}_\alpha;\bar{u}_d)$. This last statement can be paraphrased in terms of smoothing functions, i.e.

$$\underline{I}'(\bar{u}_d) = \int d\bar{u}_x \underline{\Psi}(\bar{u}_x)\underline{\Gamma}'(\bar{u}_d;\bar{u}_x) + \underline{N}'(\bar{u}_d) \tag{14}$$

where

$$\underline{\Gamma}'(\bar{u}_d;\bar{u}_m) = \int d\bar{u}_\alpha \underline{v}_r(\bar{u}_\alpha;\bar{u}_m)\underline{K}^*(\bar{u}_\alpha;\bar{u}_d) \tag{14A}$$

and

$$\underline{N}'(\bar{u}_d) = \int d\bar{u}_\alpha \underline{\eta}(\bar{u}_\alpha)\underline{K}^*(\bar{u}_\alpha;\bar{u}_d)$$

while repeating Eq. 6

$$\underline{I}(\bar{u}_d) = \int d\bar{u}_x \underline{\Psi}(\bar{u}_x)\underline{\Gamma}(\bar{u}_d;\bar{u}_x) + \underline{N}(\bar{u}_d) \tag{6}$$

and comparing Eqs. 7 and 13

$$\underline{\Gamma}(\bar{u}_d;\bar{u}_m) = \int dt\, \underline{v}_r(\bar{u}_\alpha(t);\bar{u}_m)\underline{K}^*(\bar{u}_\alpha(t);\bar{u}_d)$$

Hence $\underline{I}'(\bar{u}_d)$ and $\underline{I}(\bar{u}_d)$ are proportional if $\underline{\Gamma}(\bar{u}_d;\bar{u}_m)$ and $\underline{\Gamma}'(\bar{u}_d;\bar{u}_m)$ are proportional for every $\bar{u}_m$ point where $\underline{\Psi}(\bar{u}_m) \neq 0$; and the latter will be the case if both $\underline{v}_r(\bar{u}_\alpha(t);\bar{u}_m)$ and $\underline{K}(\bar{u}_\alpha(t);\bar{u}_d)$ are adequate samplings of $\underline{v}_r(\bar{u}_\alpha;\bar{u}_m)$and $\underline{K}(\bar{u}_\alpha;\bar{u}_d)$ respectively, for every $\bar{u}_m$ point which is "illuminated" by the radar.

All of the foregoing suggests that the roles of the spatial model and the temporal model should be reversed, i.e. we should regard the spatial model, and in particular $i(\overline{u}_\alpha, \overline{u}_\alpha'; \overline{u}_d)$, as fundamental and the temporal functions, in particular $h(t, t'; \overline{u}_d)$, as derived quantities pertinent to a particular radar observation "experiment". In view of the arbitrariness of $i(u_\alpha, u_\alpha'; \overline{u}_m)$, it will be impossible to justify such a viewpoint in a completely general context. However, for particular $h(t, t'; \overline{u}_m)$ and $i(\overline{u}_\alpha, \overline{u}_\alpha'; \overline{u}_m)$ combinations, a simple test for the validity of i as a continuous function can be given; namely "does Eq. 9 hold for every sampling of the $\overline{u}_\alpha$ space obtained from $\overline{u}_\alpha = \overline{u}_\alpha(t)$ by independent linear displacements of the $\overline{u}_\alpha(t)$ coordinates relative to t, i.e. is the connection between i and h valid for the sampling given by

$$\overline{u}_{\alpha\tau}(t) = \begin{cases} R = R(t+\tau) \\ S = S(t) \end{cases} \tag{14 B}$$

for every τ?" As a minimum, this will require that $\overline{u}_\alpha(t)$ be a one-to-one mapping from t onto a one-dimensional subset of the $\overline{u}_\alpha$ space.

We will henceforth refer to the $\overline{u}_\alpha$ space (sometimes denoted $\overline{u}_\beta$) as the "signal space". The functions $\underline{v}_r(\overline{u}_\alpha; \overline{u}_m)$ and $\underline{K}(\overline{u}_\alpha; \overline{u}_d)$ are then mappings from scatterer space into signal space, and from signal space into map space respectively. For most of the rest of this thesis we can regard these as "given" functions. Their joint product and summation over the signal space by Eq. 14 defines the "smoothing function," $\underline{\Gamma}$, of the radar, which in turn provides a connection between the "scatterer density function" $\underline{\Psi}(\bar{u}_x)$, and its complex-map representation $\underline{I}(\bar{u}_d)$, via Eq. 14. The mapping $\bar{u}_a = \bar{u}_a(t)$, with a suitable interpolation rule, provides the means of converting the actually received radar signal $\underline{E}_r(t)$ to the spatial signal $\underline{V}_r(\bar{u}_a)$. (The interpolation can be actual, as is provided by the CRT spot size in a "signal film" recorder, or implied as in a computer integration algorithm.) We assume that the $\bar{u}_a = \bar{u}_a(t)$ mapping meets the minimum sampling requirements on $\underline{v}_r(\bar{u}_a; \bar{u}_m)$ (and hence on $\underline{V}_r(\bar{u}_a)$ and $\underline{K}(\bar{u}_a; \bar{u}_d)$) and hence we can drop the primes which distinguish the smoothing ($\underline{\Gamma}$) and ($\underline{I}$) functions in the temporal and spatial models.

The foregoing general development can now be applied to a specific radar mapping problem. Here we will consider only an idealized geometry, which does, however, preserves the essence of the problem. Appendix A gives the extension to a realistic geometry.

First let us stipulate a scatterer space confined to a plane and a radar trajectory which is a straight line at a constant height, H, above this plane. Let S be distance along this trajectory and the radar (to be precise the "phase-center" of the radar antenna) travels along the trajectory at the constant speed U_o. Take the scatterer space coordinates, i.e, the components of $\bar{u}_m$ at the scatterer point "m", to be R_m and S_m where R_m is the length of the perpendicular from "m" to the radar trajectory, and S_m is the value of S where this perpendicular meets the radar trajectory. Then

$$R_{mx}(S) = \sqrt{R_m^2 + (S-S_m)^2} \tag{15}$$

and the geometry is shown in Fig. 1. This will be recognized as an idealization of the usual synthetic aperture geometry for "broadside" operation.

For the transmitted signal, take a complex pulse train of the form

$$\underline{e}_c(t) = \sum_n a(t-T_n)\, e^{j\Phi(t-T_n)}\, e^{j\omega_o(t-T_n)} \tag{16}$$

where

$a(\mu)$ = a pulse amplitude function, non-zero over an interval $0 \leq \mu \leq T_p << T_{n+1} - T_n$

$T_n, T_{n+1}, \ldots$ = a set of time instants, equally spaced on t.

$\Phi(\mu)$ = a non-linear phase modulation function.

ω_o = the carrier "frequency" (in radian measure) of the transmitted radiation

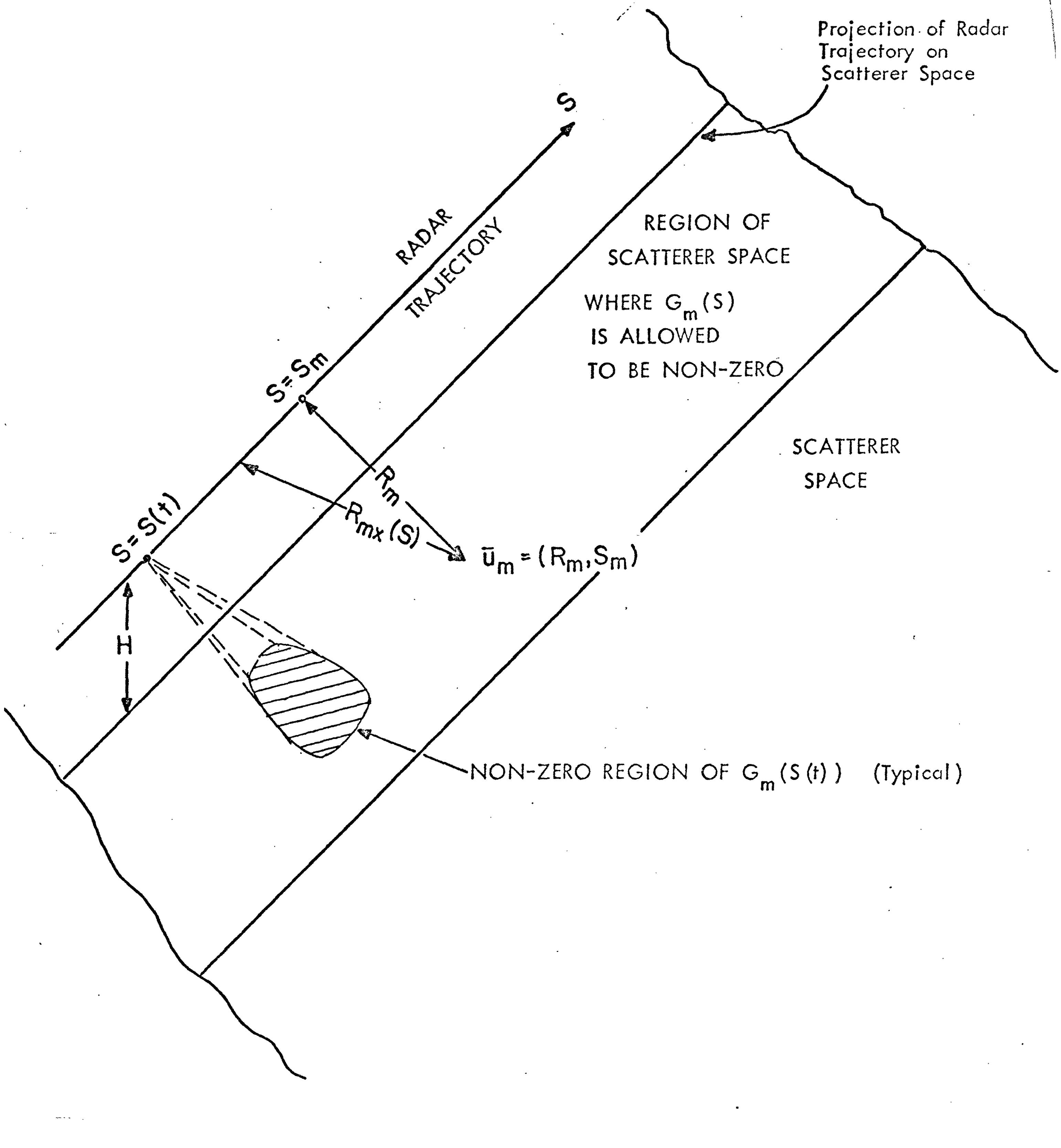


Fig. 1 Radar Mapping Geometry

Starting each pulse at a given phase can be regarded as equivalent to "coherent detection" of the received signal. By limiting the extent of the signal space in the R dimension, we can specify $\bar{u}_a(t)$ as the following one-to-one mapping from t onto the one-dimensional subset of signal space shown in Fig. 2:

$$\bar{u}_a(t) = \begin{cases} R(t) = \frac{c}{2}(t-T_n): & T_n \le t \le T_{n+1} \\ S(t) = U_o t: & \text{all } t \end{cases} \tag{17}$$

Hence $0 < R < \frac{c}{2}(T_{n+1}-T_n)$, $-\infty < S < \infty$ defines a "strip" in signal space over which $\underline{v}_r(\bar{u}_a;\bar{u}_m)$ is sampled.

Combining $\underline{e}_c(t)$ from Eq. 16, the $\bar{u}_a(t)$ mapping of Eq. 17 and the $i(\bar{u}_a\ \bar{u}_a':\bar{u}_m)$ function of Eq. 9A gives for $\underline{v}_r(\bar{u}_a;\bar{u}_m)$ by substitution in Eq. 10

$$\underline{v}_r(\bar{u}_a;\bar{u}_m) = \int_{-\infty}^{\infty} \frac{dS'}{U_o}\, \underline{e}_c\left(\frac{S'}{U_o}\right)\, \delta\left[\frac{2}{c}\left(R - R'(S') - \frac{R_m(S)+R_m(S')}{2}\right)\right] \tag{17A}$$

$$\cdot\, G_m(S)\, G_m(S')$$

where

$$R'(S') = \frac{c}{2U_o}(S' - S_n) \qquad S_n < S' < S_{n+1}$$

and

$$S_n = U_o T_n.$$

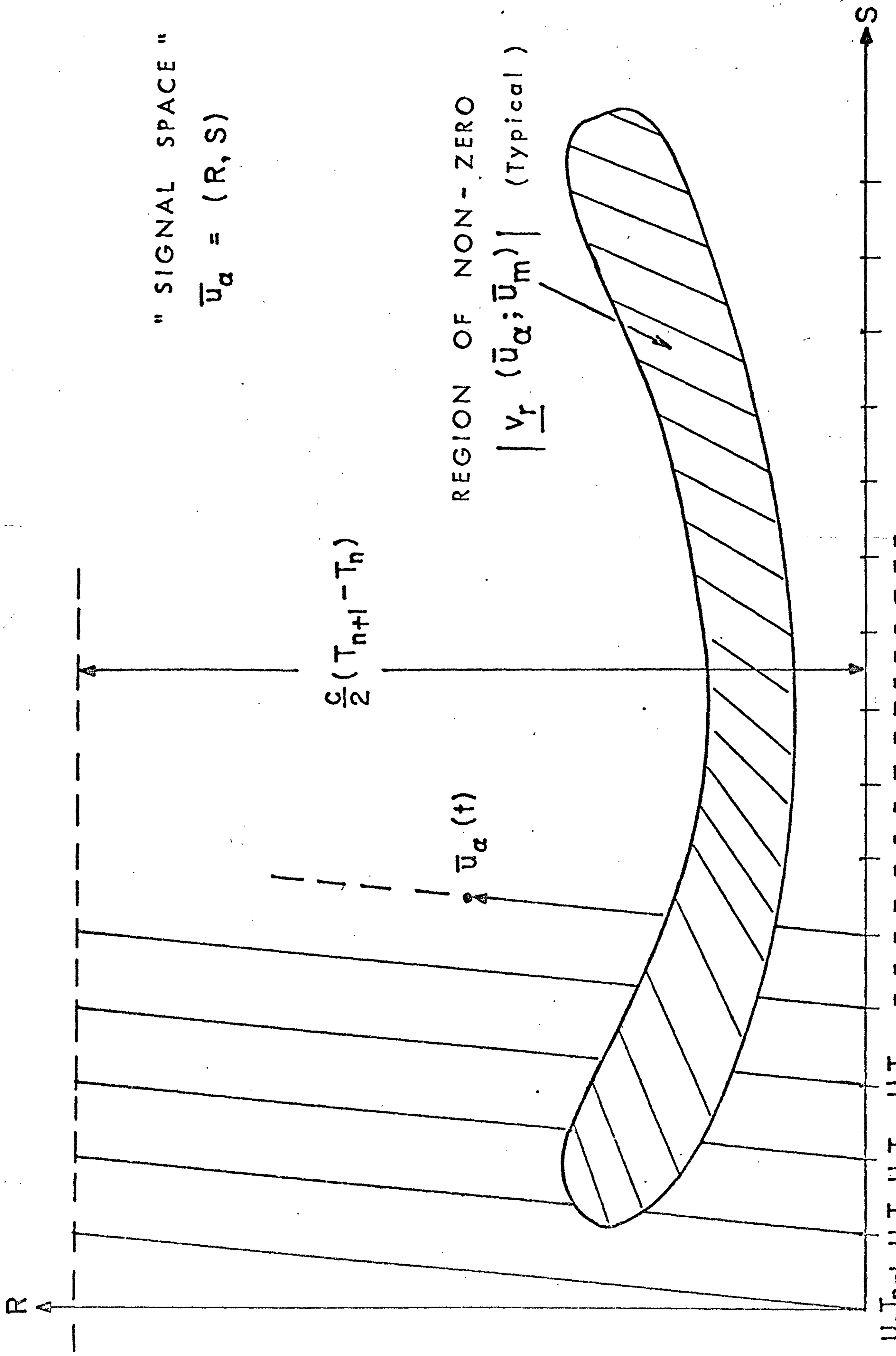


Fig.2 $\overline{u}_\alpha(t)$ Mapping

This integral is of the form

$$\int dx\ f(x)\ \delta(g(x)) = \frac{f(x_o)}{\left|\frac{dg(x)}{dx}\right|_{x = x_o}}$$

where

$$x_o = \left\{ x \,\middle|\, g(x) = o \right\}$$

Finding the values of S' which cause the argument of the delta function in Eq. 17A to go to zero is straightforward but algebraically complex. Sufficiently accurate solutions for most practical applications are

$$S_{on}' = S_n + \frac{2U_o}{c}\left[R - R_m(S) \right]$$

where

$$S_n \leq S_{on}' < S_{n+1}$$

$$0 \leq R < \frac{c}{2U_o}\ (S_{n+1} - S_n)$$

Using $G_m(S') \approx G_m(S)$ since this function varies very little over the intervals $(S_{n+1} - S_n)$, we obtain

$$\underline{v}_r(\overline{u}_a ; \overline{u}_m) = G_m^{\,2}(S)\ a\left\{ \frac{2}{c}\left(R - R_m(S)\right)\right\} e^{j\Phi\left\{\frac{2}{c}\left(R - R_m(S)\right)\right\}} e^{j\frac{4\pi}{\lambda}\left(R - R_m(S)\right)} \tag{18}$$

where $\lambda = \frac{2\pi c}{\omega_o}$ = "wavelength" of the carrier and the sum on n has disappeared because each term of the sum defines $\underline{v}_r(\overline{u}_a;\overline{u}_m)$ over a single rectangle in signal space (the rectangle $(S_{n+1}-S_n) \times \frac{c}{2}(T_{n+1}-T_n)$) and the $\underline{v}_r(\overline{u}_a;\overline{u}_m)$ so obtained is continuous across the rectangle boundaries.

Having found $\underline{v}_r(\overline{u}_a;\overline{u}_m)$, our model is essentially complete. There remains only the task of specifying the T_n instants (or the S_n positions) so that $\overline{u}_a(t)$ is an adequate sampling of signal space, for every $\overline{u}_m$ point which is illuminated by the radar. But $\underline{v}_r(\overline{u}_a;\overline{u}_m)$ will be confined to the allowed strip of signal space only if

$$G_m(S) = 0$$

for $\quad R_m > \frac{c}{2}(T_{n+1}-T_n) - R_a - \left(R_{mx}(S_m + \frac{S_a}{2}) - R_{mx}(S_m)\right)$

where S_a = the "azimuth aperture", i.e. the length of the interval for which $G_m(S)$ is non-zero.

$R_a = \frac{c}{2} T_p$ = the "range aperture".

The "bandwidth" of $\underline{v}_r(\overline{u}_a;\overline{u}_m)$ along S, the sampled dimension, is determined by the $G_m^{\,2}(S)\, e^{j\frac{4\pi}{\lambda}R_{mx}(S)}$ factor. The non-zero width of $G_m(S)$, i.e. S_a, must be finite if we are going to map a non-zero map width (by the foregoing limits on $G_m(S)$ as a function of R_m) so $\underline{v}_r(\overline{u}_a;\overline{u}_m)$ cannot be strictly "bandlimited" along S. This is the usual dilema of sampling; so we will not belabor the point. The "effective" bandwidth. W_S, of $\underline{v}_r(\bar{u}_a ; \bar{u}_m)$ along S can easily be shown to be $W_S = \frac{4\pi}{\lambda} \frac{S_a}{R_m}$ rad. /unit dist. The usual criterion for sampling of complex waveforms is that $(S_{n+1} - S_n) < \frac{2\pi}{W_S}$. The sampling requirements thus yield a set of constraints on $G_m(S)$ as a function of S and R_m, and on the spacing of transmitted pulses.

We can now consider the question of the utility of the spatial model vs. the temporal model, at least for the example we have just given. A weighting function based on Eq. 19 for $\underline{v}_r(\bar{u}_a ; \bar{u}_m)$ would have the form

$$\begin{aligned} \underline{K}(\bar{u}_a ; \bar{u}_d) = w(\bar{u}_a ; \bar{u}_d) e^{j\Phi\left\{\frac{2}{c}(R - R_{dx}(S))\right\}} e^{j\frac{4\pi}{\lambda i}\left(R - \frac{\lambda i}{\lambda} R_{dx}(S)\right)} \\ \cdot e^{-j 4\pi R\left(\frac{1}{\lambda i} - \frac{1}{\lambda}\right)} e^{j\frac{4\pi}{\lambda} R_d} \end{aligned} \quad (20)$$

where

$w(\bar{u}_a ; \bar{u}_d)$ = an aperture weighting function

$\bar{u}_d = (R_d, S_d)$

$R_{dx}(S) = \sqrt{R_d^{\,2} + (S - S_d)^2}$

Here the factor $e^{-j 4\pi R\left(\frac{1}{\lambda i} - \frac{1}{\lambda}\right)}$ has been segregated to adjust the "spatial carrier" frequency of the remainder of $\underline{K}$ along the R coordinate from $4\pi/\lambda$ to $4\pi/\lambda i$ radians per unit distance. Loosely speaking, λi can be regarded as the "intermediate frequency wavelength". When the proper precautions are observed+, the $e^{-j4\pi R\left(\frac{1}{\lambda i} - \frac{1}{\lambda}\right)}$ part of $\underline{K}$ can be introduced into the signal processing before the conversion of $\underline{E}_r(t)$, the received temporal signal, to $\underline{V}_r(\overline{u}_a)$, the "received" spatial signal. The other strange factor, $e^{j\frac{4\pi}{\lambda}R_d}$, is introduced to bring the average spatial frequency of $\underline{K}$ as a function of R_d to zero. This is permissable since the only effect is to shift the phase of $\underline{I}(\overline{u}_d)$ as a function R_d, while only $\left|\underline{I}(\overline{u}_d)\right|^2$ is of interest in the final result.

+ The factor $e^{-j4\pi R\left(\frac{1}{\lambda i} - \frac{1}{\lambda}\right)} = e^{+j(t-T_n)(\omega_o-\omega_i)}$ (where $\omega_i = \frac{2\pi c}{\lambda i}$) represents a frequency "translation" when applied to the received signal. Hence, the "proper precautions" consist of (1)insuring that the full complex form of $\underline{E}_r(t)e^{j(t-T_n)(\omega_o-\omega_i)}$ is available after the translation if ω_i is less than the bandwidth of $\underline{E}_r(t)$ and (2)insuring that the "local-oscillator" phase is reset to zero at each $t = T_n$ instant or some equivalent operation is performed to retain pulse-to-pulse phase continuity in the received signal. Requirement (1) above may be avoided if $G_m(S)$ is such that the $\underline{V}_r(\overline{u}_a)$ constructed from $\underline{E}_r(t)e^{j(t-T_n)(\omega_o-\omega_i)}$ is analytic along the S coordinate in signal, $\overline{u}_a$, space. Or the "local-oscillator" may be phase modulated as a function of S to accomplish this same purpose. As long as $\underline{V}_r(\overline{u}_a)$ is analytic along one coordinate in $\overline{u}_a$, only its real part is required for the evaluation of $\underline{I}(\overline{u}_d)$.

The temporal weighting function $\underline{f}(t; \bar{u}_d)$ which corresponds to the $\underline{K}(\bar{u}_\alpha ; \bar{u}_d)$ in Eq. 20 would be obtained by the substitution $\bar{u}_\alpha = \bar{u}_\alpha(t)$ where $\bar{u}_\alpha(t)$ is given by Eq. 17. It is fairly evident that the two dimensional function, $\underline{K}$, will be easier to work with in most types of physical apparatus than the one dimensional function, $\underline{f}$. At the very least it is easier to visualize.

The method we have used to derive $\underline{v}_r(\bar{u}_\alpha ; \bar{u}_m)$, the complex spatial return signal from a point scatterer, is not unique. The more usual procedure is to visualize the construction of a "signal film" by recording successive "range traces", which are intensity modulated by the return signal, side by side. For the simple geometry of our example either method is equally effective. However, when the geometry is not so simple, in particular when the radar trajectory is somewhat unpredicatable (although still measureable), the method developed here provides a sound and easily applied analytical connection between a spatial signal model and its temporal counterpart. The present method also provides a model which is independent in conception from any particular implementation of processing, but is equally applicable to all implementations.

There remains only the question of the correspondence of the noise processes $\underline{n}(t)$ and $\underline{\eta}(\bar{u}_\alpha)$. If $\underline{n}(t)$ is assumed to be broadband "white-noise", as is usually done, then $\underline{\eta}(\bar{u}_\alpha)$ should be taken as a multidimensional "white-noise" process, i.e. having a flat broadband spectral density along every coordinate in signal space. Then every sampling of signal space, by Eq. 14.B, will yield a statistically equivalent white noise $\underline{n}(t) = \underline{\eta}(\overline{u}_a(t))$. However, it is also clear that no sampling of the signal space is "adequate" from the standpoint of the noise processes. This means that no construction of $\underline{V}_r(\overline{u}_a)$ from $\underline{E}_r(t)$ gives a broadband $\underline{\eta}(\overline{u}_a)$ component but instead yields a $\underline{V}_r(\overline{u}_a)$ with an $\underline{\eta}(\overline{u}_a)$ component which has "bandlimited" spectral distribution, of bandwidth approximately equal to 2π divided by the sampling distance (in radians per unit distance) along each coordinate.

The distinction is largely academic, however. For this bandwidth is greater than the bandwidth of $\underline{K}(\overline{u}_a; \overline{u}_d)$ along each coordinate in signal space. This must be so because $\underline{K}(\overline{u}_a(t); \overline{u}_d)$ is an adequate sampling of $\underline{K}$ by assumption. Since the only use to be made of $\underline{V}_r(\overline{u}_a)$ is to "filter" it by $\underline{K}(\overline{u}_a; \overline{u}_d)$, no significant penalty is incurred by considering the $\underline{V}_r(\overline{u}_a)$ obtained from $\underline{E}_r(t)$ to have a broadband "white-noise" component along every coordinate in signal space.

III. ALTERNATIVES FOR ORGANIZATION OF DATA PROCESSING

In this section we will discuss the functional requirements implied by the processing equation given in Section II(Eq. 12)

$$\underline{I}(\bar{u}_d) = \int d\bar{u}_a \, \underline{V}_r(\bar{u}_a) \underline{K}^*(\bar{u}_a ; \bar{u}_d) \qquad (12)$$

By this we mean sequences of operations which must be performed on the received signal, $\underline{E}_r(t)$ to obtain $|\underline{I}(\bar{u}_d)|^2$ for every $\bar{u}_d$ point of interest. There are, in fact, several alternative sequences, of which two will be discussed in detail here. We refer to these alternatives as "formulations" of the data-processing problem.

In this and succeeding sections we will assume signal space coordinate, $\bar{u}_a$, of propagation distance, R, and trajectory distance, S, as used in Section II and in the more detailed example of Appendix A. (In Appendix A, S is a "nominal-trajectory" distance.) Although other signal space coordinates are possible, these will suffice for our purposes. We also assume a two-dimensional scatterer space $\bar{u}_x = (R_x, S_x)$ (or $\bar{u}_y$, or, when a particular point is to be designated, $\bar{u}_m = (R_m, S_m)$) and a two-dimensional map space of coordinates $\bar{u}_d = (R_d, S_d)$.

The operations for obtaining $|\underline{I}(\bar{u}_d)|^2$ defined by the processing equation (Eq. 12), given $\underline{v}_r(\bar{u}_a ; \bar{u}_m)$ (and hence by implication $\underline{K}(\bar{u}_a ; \bar{u}_d)$) present a significant realization problem only because the number of map points to be processed is enormous. It is generally desired t: evaluate $|\underline{I}(\bar{u}_d)|^2$ for values of $\bar{u}_d$ which correspond to an interior point of each contiguous resolvable element (or "box") in the scatterer spaced scanned by the radar. A rather ordinary radar carried by a modern aircraft is capable of scanning through upwards of 10^5 resolvable elements per second. In addition, better resolution requirements invariably lead to more stringent accuracy requirements in the realization of $\underline{K}(\bar{u}_a ; \bar{u}_d)$, particularly the phase factor. The net result is that a practical data-processor for radar mapping applications must be based on a simultaneous evaluation of $|\underline{I}(\bar{u}_d)|^2$ for many $\bar{u}_d$ points. The required processing must be subdivided into simple operations, each to be simultaneously accomplished in a large array of simple elements, and the whole array to operate sequentially on input data to produce sequences of output map points.

Evaluation of $\underline{I}(\bar{u}_d)$ for a particular $\bar{u}_d$ point requires that $\underline{V}_r(\bar{u}_a)$ be available over the whole range of $\bar{u}_a - \bar{u}_d$ for which $|\underline{K}(\bar{u}_a ; \bar{u}_d)|$ is non-zero. The input $\underline{V}_r(\bar{u}_a)$ becomes available very rapidly along the propagation distance coordinate, R, and relatively slowly along the trajectory distance coordinate, S. We can conceive of the evaluation of $\underline{I}(\bar{u}_d)$ by storing $\underline{V}_r(\bar{u}_a)$ until all the required values are available, then performing weighting by $\underline{K}^*(\bar{u}_a ; \bar{u}_d)$ and integration over the signal space. We call this the "delayed" processing formulation. A "block" diagram of this formulation is shown in Fig. 3. The basic functional requirements

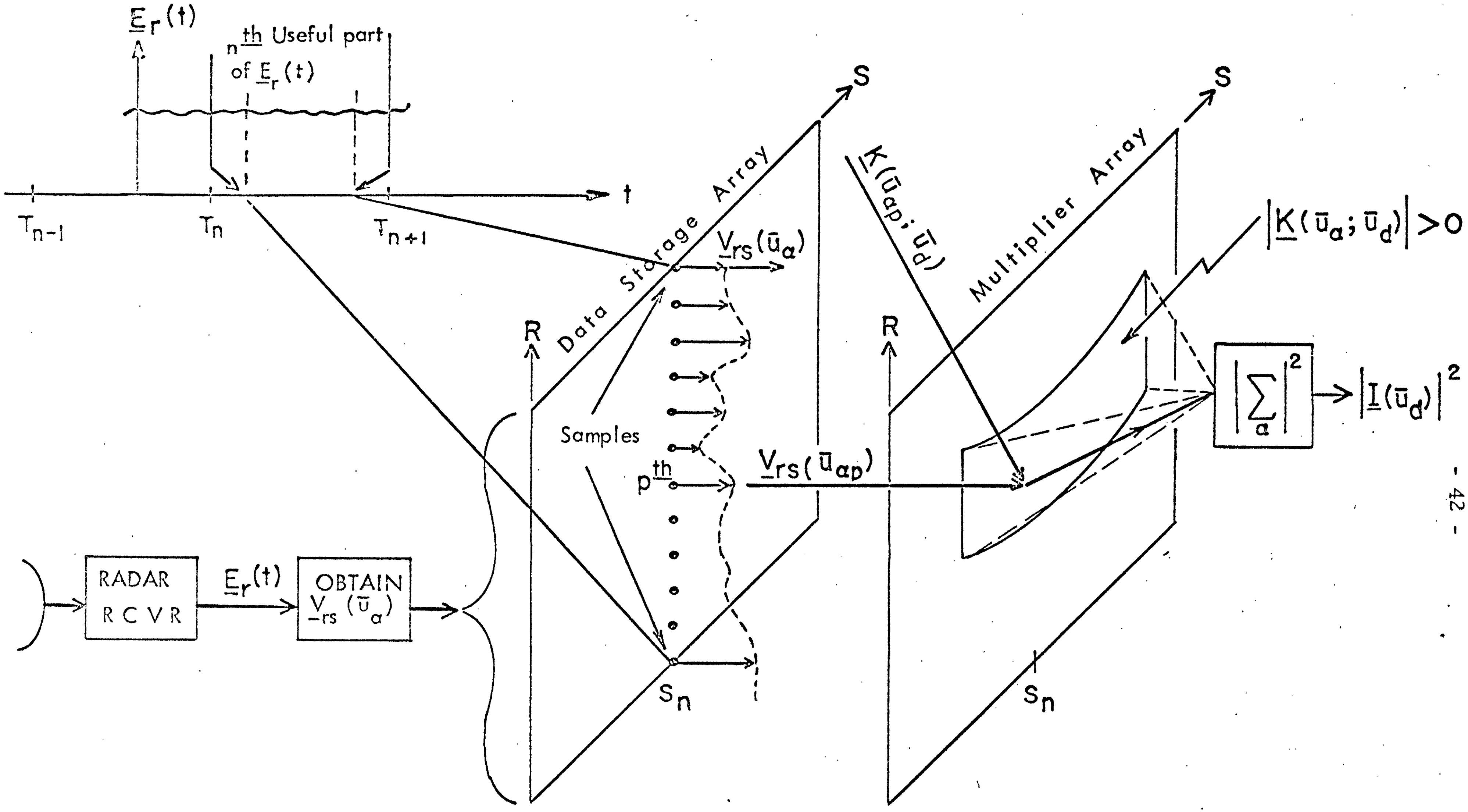


Fig. 3 Delayed Processing

include data storage, simultaneous (or at least very rapid) readout of all the data pertinent to a particular map point, multiplication by the appropriate $\underline{K}^*(\bar{u}_a ; \bar{u}_d)$ values, summation of all the products, detection of the magnitude of $\underline{I}(\bar{u}_d)$, and finally transfer to a display or other post-processing operation. These operations must be repeated for each map point of interest and the same data points, $\bar{u}_a$, will enter into the evaluation of $\underline{I}(\bar{u}_d)$ for many map points.

An alternative formulation, called "immediate" processing is block diagramed in Fig. 4. Here each data point, $\bar{u}_a$ is weighted by the appropriate values from every $\underline{K}^*(\bar{u}_a ; \bar{u}_d)$ (for which $\underline{K}$ is non-zero) as soon as it is obtained. Each product is then added to the contents of a storage "bin" assigned to a particular map point. When all the data points pertaining to a particular map point have been received, the "sum" of the increments in the "bin" is proportional to $\underline{I}(\bar{u}_d)$. In this formulation the functional requirements may be listed as simultaneous (or again very rapid) multiplication of each data point by the appropriate $\underline{K}^*(\bar{u}_a ; \bar{u}_d)$ values, storage of the products as increments in the appropriate bins, summation of all the increments in each bin, detection of the magnitude of $\underline{I}$ and transfer to a display.

In comparing these two formulations, it is evident that the roles of the variables $\bar{u}_a$ and $\bar{u}_d$ in $\underline{K}$ are interchanged. In "delayed"

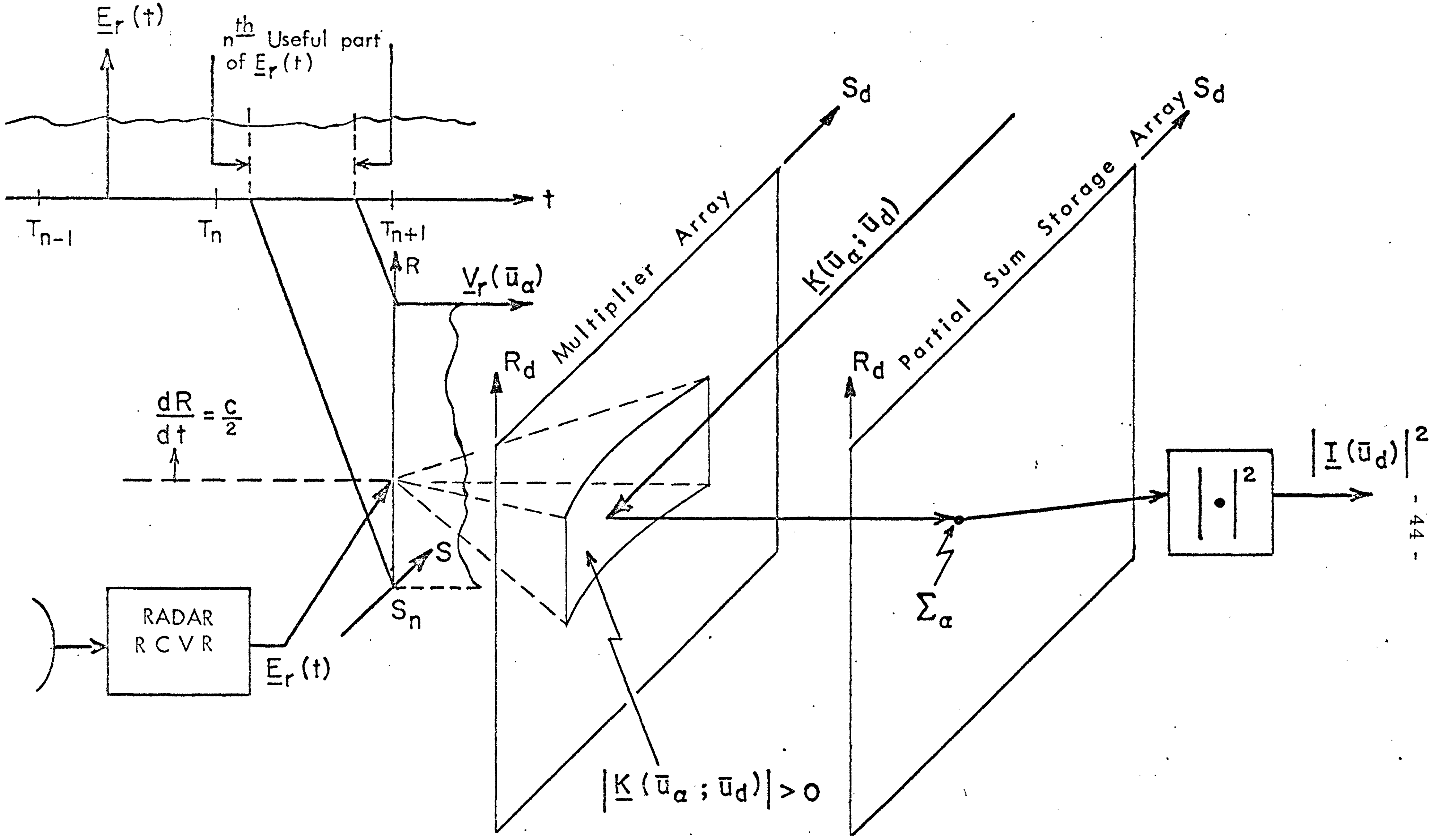


Fig. 4 Immediate Processing

processing $\underline{K}^*$ is developed as a function of $\bar{u}_a$ and signal space is "viewed" through $\underline{K}^*$ from the vantage point of particular output map points, $\bar{u}_d$, in sequence. In "immediate" processing, $\underline{K}^*$ is developed as a function of $\bar{u}_d$ and the data at each $\bar{u}_a$ point is "projected" in sequence through this $\underline{K}^*$ onto the output plane. Delayed processing may be likened to a "correlator", in which the data is shifted and weighted relative to a fixed function, each shift corresponding to a particular output point. Immediate processing may be likened to a "filter bank" where each output is a summation of responses to a sequence of input impulses. The equivalence of these devices is well known.

The distinction between "immediate" and "delayed" processing does not lie, necessarily, in the length of time that elapses between availability of data and availability of output; nor does the preservation of raw data in a recording medium exclude "immediate" processing. Thd distinction lies in the manner of handling data. Hence a processor which operates on permanently recorded data in the manner of an immediate processor, will still be calssed as an immediate processor.

It is clear that the number of multiplications of input data by values of $\underline{K}^*$ will be the same in each formulation and that the best maps obtainable from each formulation, for given signals and $\underline{K}$ functions, will be of essentially equal quality. However, a significant difference arises in the data storage operations. In "delayed" processing the storage medium needs to accept data only once but hold it for multiple subsequent interrogations. In "immediate" processing the storage medium must accept its data in increments at a sequence of times and hold it for only a single interrogation. Clearly, it will be most convenient, in the latter case, if the data storage is in the form of partial sums, i.e., the storage medium also serves as an integrator, and, in fact, it makes very little sense to talk of immediate processing in any other context. In either formulation the final operation of transfer of the $|\underline{I}(\bar{u}_d)|^2$ values to the "display", can be, and often is, the physical removal of the summing or magnitude detecting device to the display machine.

A second significant difference between the two formulations arises in the area of integration (or summation), in particular in the time scale of the integration. For immediate processing this summation is carried on over a period of time which is dependent on speed along the trajectory, and the resolution sought, and is typically on the order of seconds. In delayed processing, all the data pertinent to a particular output point is available simultaneously; hence, the integration can proceed as rapidly as products with the $\underline{K}^*$ function can be obtained. In the conventional optical processor, which would be classified as a "delayed" processor by our definition, the products are formed simultaneously, and the integration is instanteous, being provided by the linearity of addition of electromagnetic field vectors. Only the magnitude detection device (i.e. photographic film) requires a measurable time interval for operation. In analog electronic processors, the delayed formulation is a practical necessity since it is impractical to build electronic integrators which are stable over the time spans required for immediate processing, in arrays of the required size.

It is not hard to recognize other formulations of the processing problem which have elements of both immediate and delayed processing. The summations over propagation distance, R, and trajectory distance, S, can be separated and each accomplished by a different formulation. A common example is a "chirp" radar where the received signal is "dechirped" "immediately" on a pulse-to-pulse basis while the azimuth (S-dimension) processing is accomplished via delayed processing in an optical processor. It is possible to go even further and decompose the weighted summations on R and S into a sequence of such weighted summations, in the manner of a cascade of filters, and each member of the sequence can be accomplished by either a delayed or immediate formulation. We will not pursue any of these alternatives because no advantage is presently seen for any of them. The one common case of a "dechirp" on receive radar with subsequent azimuth (S-dimension) processing can be handled as if it were a simple narrow-pulse radar. During the rest of this proposal we will deal only with the two defined alternatives of "immediate" and "delayed" processing.

In the processing equation (Eq. 12) and Figs. 3 and 4, a literal interpretation of the operation of complex multiplication implies four real multiplications for each data point and map point combination. Similarly complex integration implies, literally, two real integrations. By exploiting the known equivalences between the real and imaginary parts of "analytic" complex functions, it is possible to reduce these literal requirements to two real multiplications and two real integrations, and in certain cases to one real multiplication and one real integration. The requirements for accomplishing these reductions are well known. They are principally concerned with the various frequency-translation operations (i.e., "r-f" to "i-f" to "video") which are necessary in practical systems but are not shown in Figs. 3 and 4. Whether one wishes to utilize any of these reductions in a particular case is dependent on the type of physical apparatus that will be used in implementing the processing operation. In the particular case o. digital implementation, minimum bit rate requirements at any step of the processing are invariant to the choices available, i.e. if we arrange the computation so that only two real multiplications are required per data point and map point combination, we find that the minimum number of data points required per processed map point is doubled.

When a digital implementation of the processing equation is being designed, it is, of course, necessary to rewrite the equation in fully sampled form, i.e.

$$\underline{I}(\bar{u}_d) = \sum_{\alpha} \underline{V}_{rs}(\bar{u}_\alpha)\underline{K}_s^*(\bar{u}_\alpha ; \bar{u}_d) \qquad (21)$$

where the s subscripts signify that $\underline{V}_r$ and $\underline{K}$ have been sampled along all the coordinates in signal, $\bar{u}_\alpha$, space. Minimum sampling requirements remain unchanged, i.e. $w_\alpha/2\pi$ complex samples per unit distance along each coordinate where w_α is the effective bandwidth along that coordinate in radians per unit distance. $\underline{V}_r(\bar{u}_\alpha)$ is obtained from the sampling $\underline{E}_r(t) = \underline{V}_r[\bar{u}_\alpha(t)]$ which implies that $\underline{V}_r[\bar{u}_\alpha(t)]$ contains a finite number of samples per unit distance along one coordinate of signal space, i.e. the trajectory distance coordinate. Hence $\underline{V}_{rs}(\bar{u}_\alpha)$ merely implies completing the process by sampling along the other (propagation distance) coordinate.

Suppose, however, that the sampling implied by $\bar{u}_a = \bar{u}_a(t)$ is much denser along the trajectory distance coordinate than the minimum required by bandwidth considerations. Then it is possible to construct $\underline{V}_r(\bar{u}_a)$ (by an implicit interpolation of these dense samples) and "resample" $\underline{V}_r(\bar{u}_a)$ to obtain a $\underline{V}_{rs}(\bar{u}_a)$ where the sampling rate is closer to the minimum requirements dictated by the bandwidths involved. An advantage accrues to such a procedure only if the radar transmitter is peak-power limited (as opposed to average power), for it is then theoretically possible to raise the signal energy to noise power ratio per sample of $\underline{V}_{rs}(\bar{u}_a)$ by a factor equal to the ratio of original to "resample" sampling rates per unit distance. The cost extracted is that the maximum amount of map area that can be produced per unit "flight" time is reduced by this same ratio, as can be seen from the limits on $G_m(S)$ as a function of R_m given in Section II, Eq. 19 and Appendix A. This "resampling procedure sometimes goes under the pseudonym of "pre-summing", i.e. forming a single sample from a group of dense samples by a weighted summation.

Since sampling is a linear operation we can extend the "smoothing function" formalism to Eq. 21, i.e.

$$\underline{I}(\bar{u}_d) = \int d\bar{u}_x \, \underline{\Psi}(\bar{u}_x) \, \underline{\Gamma}_s(\bar{u}_d; \bar{u}_x) + \underline{N}(\bar{u}_d) \tag{22}$$

where

$$\underline{\Gamma}_s(\bar{u}_d; \bar{u}_m) = \sum_a \underline{v}_{rs}(\bar{u}_a; \bar{u}_m) \, \underline{K}_s^{*}(\bar{u}_a; \bar{u}_d) \tag{23}$$

$$\underline{N}(\bar{u}_d) = \sum_a \underline{\eta}_s(\bar{u}_a) \underline{K}_s(\bar{u}_a; \bar{u}_d)$$

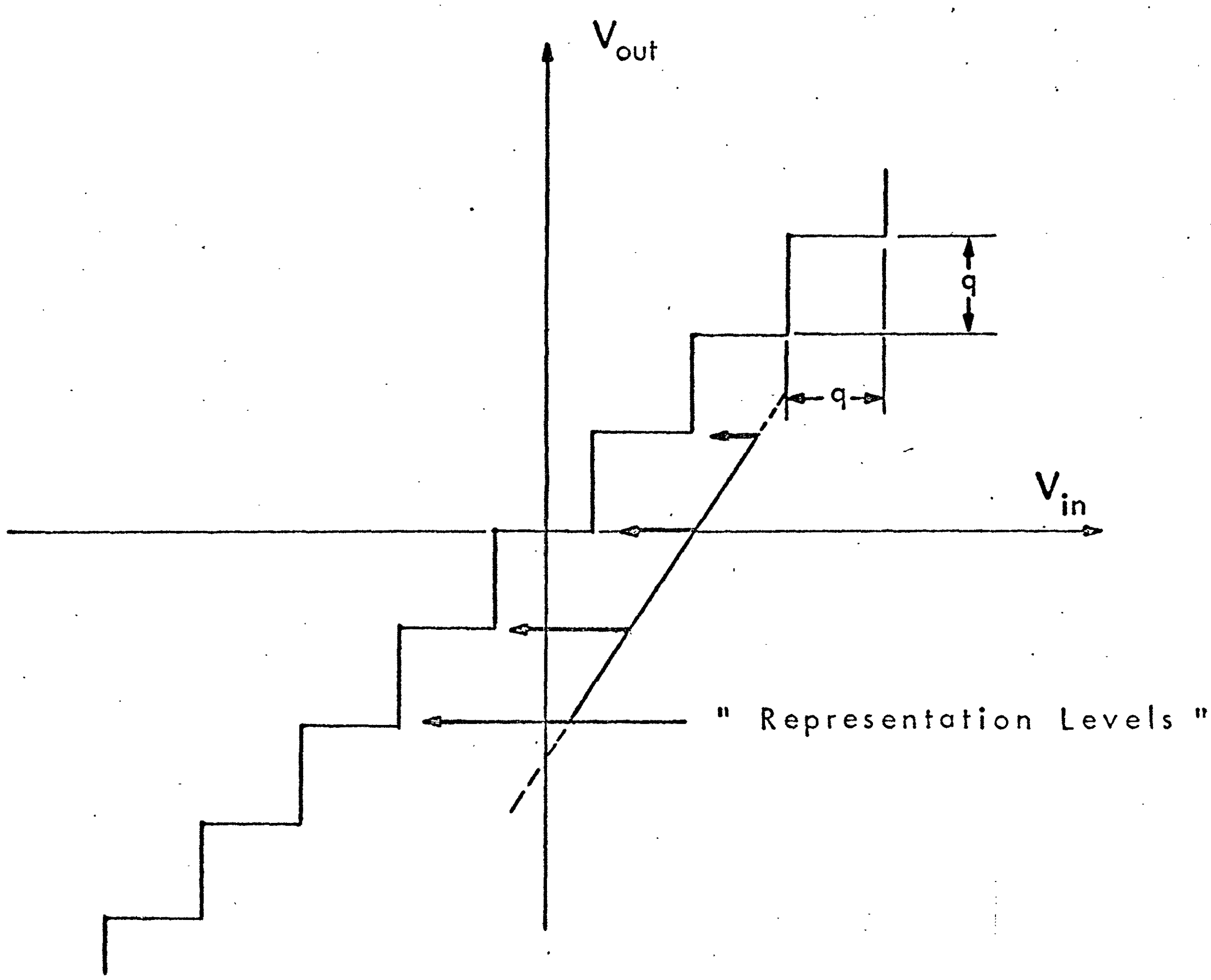
IV. QUANTIZATION EFFECTS

The fully sampled version of the processing equation (Eq. 21 of Section III) appropriate to a discussion of a processor to be implemented by digital techniques is repeated below.

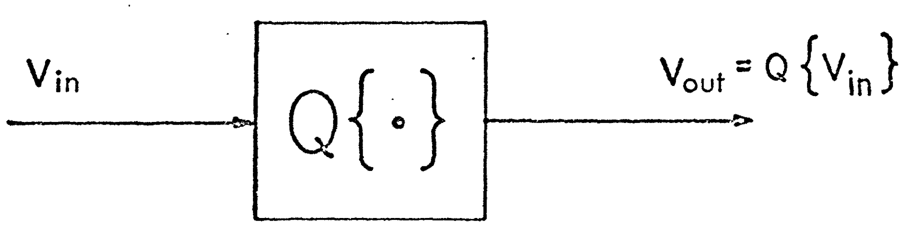
$$\underline{I}(\bar{u}_d) = \sum_{\alpha} \underline{V}_{rs}(\bar{u}_\alpha)\, \underline{K}_s^*(\bar{u}_\alpha ; \bar{u}_d) \qquad (21)$$

By "digital techniques" we mean that the operationsin Eq. 21 are to be accomplished, at least in part, by using digitized number representations of the functions involved. This in turn implies a "quantization" of these functions which we define by the nonlinear gain function shown in Fig. 5 a and by the symbology shown in Fig. 5 b. The purpose of this section is to present a tractable and useful way of computing the effects of quantization on radar operations.

By quantization of complex signals we mean independent quantization of the real and imaginary part. It is conceivable that the quantizer could operate on the magnitude and phase of the complex input. However, a magnitude and phase quantizer will be relatively inefficient unless the angle quantizer is coupled to the magnitude quantizer in a way which tends to keep the vector-quantization uncertainty constant over the complex-signal plane. This is a more complicated device than a quantizer which operates on the real and imaginary parts independently. Having signals in the form of magnitude



a) Input-Output Relationship



b) Symbology

Fig. 5 Quantizer

and phase simplifies multiplication (if it is done digitally as in Fig. 6 c) but complicates addition so no obvious advantage accrues to the magnitude and phase quantizer.

Most studies of quantization have been concerned with the quantizer itself, and in the study of "quantization noise", which may be defined as the difference between the quantizer output and input.[3, 4] Here our objective is different, namely to compute the effects of quantization in terms of the "quality" of the radar maps produced. This is a more particular problem than past investigations but the result will be applicable to any digital implementation of an "imaging" process of the form of Eq. 21.

To begin, we first examine the ways in which quantization can enter into the processing operation. Figure 6 depicts three potentially useful ways in which this can occur. These are all essentially different because the operation of quantization does not commute with the operations of multiplication or addition. We should note, however, that the operations of sampling and quantization do commute.

In Fig. 6 a the quantizer is shown following the formation of the $\underline{V}_{rs}(\overline{u}_\alpha)\underline{K}_s^*(\overline{u}_\alpha ; \overline{u}_d)$ products, the products themselves being formed in an "analog" device. The summer (operating over the signal space

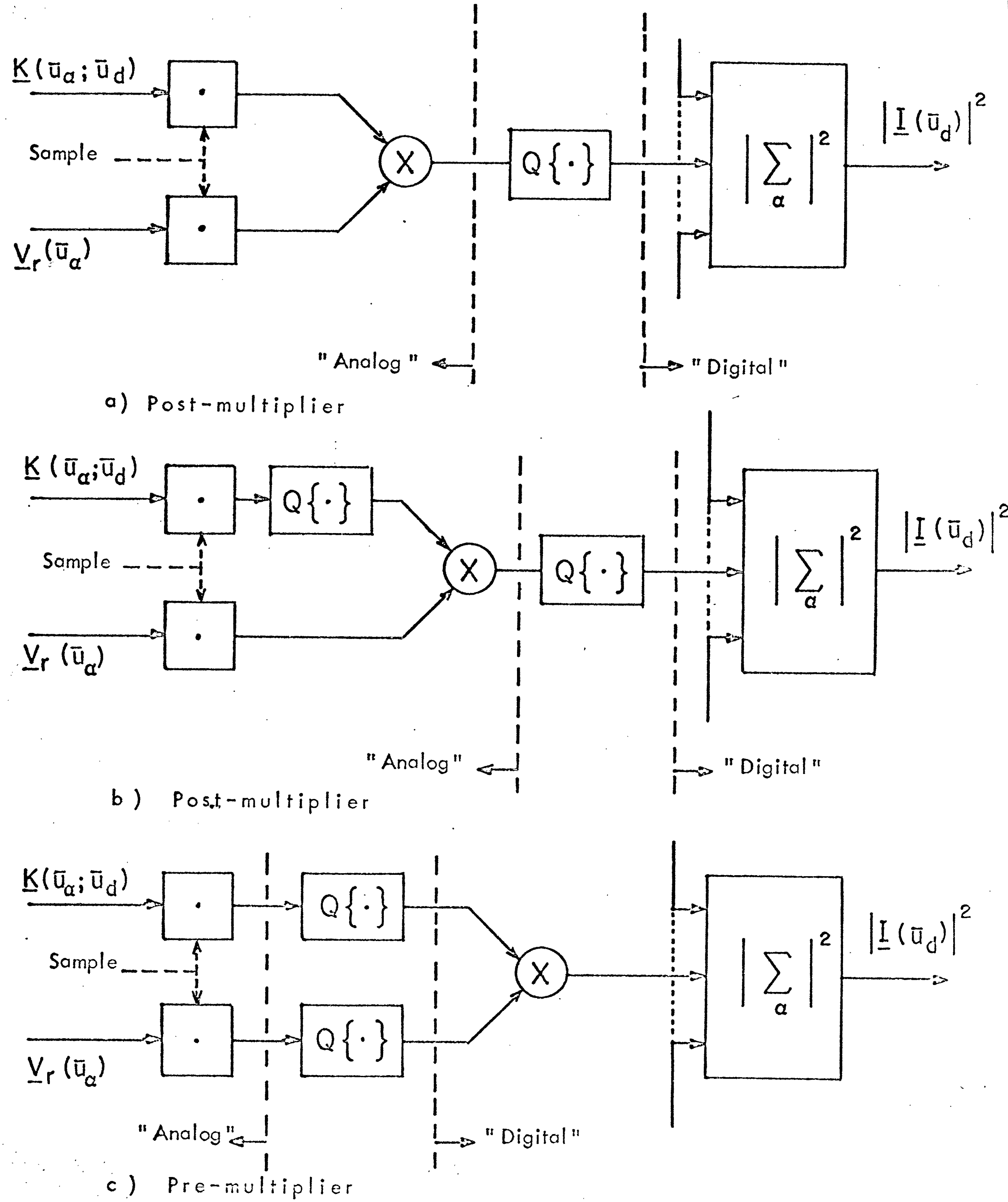


Fig. 6 Variations in data quantization

as symbolized by the multiple inputs to the summing box)and squared magnitude detector are the only "digital" devices in the system. As noted in Section I, the stability and ease of implementation of large arrays of summers (i.e. integrators) is one potentially useful characteristic of a digital implementation. The system of Fig. 6a then attempts to combine the favorable characteristics of digital integrators and analog multipliers in a case where $\underline{K}$ is not required to take on many different forms. In Fig. 6b a quantizer has been added to the $\underline{K}^*$ (weighting function) branch leading to the multiplier which forms $\underline{K}_s^*\underline{V}_{rs}$ products, although the multiplier is still assumed to be an analog device, since the data, $\underline{V}_{rs}(\bar{u}_a)$ has not been quantized. This is a refinement of the system of Fig. 6a, applicable when it is desirable to select from a large number of $\underline{K}$ functions, depending on the mode of operation desired, and these $\underline{K}$ functions are generated by computer or are stored in a digital computer memory. In Fig. 6c the quantizers appear in both the $\underline{V}_{rs}$ and $\underline{K}_s^*$ branches leading to the multiplier which is now assumed to be a digital device leading to what may be termed an "all-digital" processor.

For the following analysis, it is actually irrelevant whether $\underline{K}_s^*$ is quantized or not. For any given map point, $\bar{u}_d$, for which $\left|\underline{I}(\bar{u}_d)\right|^2$ is to be evaluated, $\underline{K}_s^*(\bar{u}_a;\bar{u}_d)$ is a fixed function of $\bar{u}_a$

whether it is quantized or not. The variable part of the problem is the data $\underline{V}_r(\bar{u}_a)$ and the underlying variable functions (or processes) $\underline{\Psi}(\bar{u}_x)$ and $\underline{\eta}(\bar{u}_a)$ which give rise to $\underline{V}_r(\bar{u}_a)$ by Eq. 11 of Section I, i.e.

$$\underline{V}_r(\bar{u}_a) \equiv \int d\bar{u}_x \underline{\Psi}(\bar{u}_x) \underline{v}_r(\bar{u}_a ; \bar{u}_x) + \underline{\eta}(\bar{u}_a) \tag{11}$$

Hence quantization of $\underline{K}_s(\bar{u}_a ; \bar{u}_d)$ does not in itself prevent a smoothing function description of the relationship between radar output and input. Let

$$\underline{\Gamma}_q(\bar{u}_d; \bar{u}_m) = \sum_{\alpha} \underline{v}_{rs}(\bar{u}_a ; \bar{u}_m) \underline{K}_q^*(\bar{u}_a ; \bar{u}_d)$$

where (24)

$$\underline{K}_q(\bar{u}_a ; \bar{u}_d) = Q\left\{\underline{K}_s(\bar{u}_a ; \bar{u}_d)\right\}$$

Then if, in the system shown in Fig. 6b the quantizer which follows the multiplier is removed (or its step size q is made infinitely small), we can write for the output of the summer

$$\underline{I}(\bar{u}_d) = \int d\bar{u}_x \underline{\Psi}(\bar{u}_x) \underline{\Gamma}_q(\bar{u}_d; \bar{u}_x) + \underline{N}(\bar{u}_d) \tag{25}$$

showing a "linear" system despite the fact that $\underline{K}_s$ is quantized. To be sure, $\underline{\Gamma}_q(\bar{u}_d; \bar{u}_m)$ is a <u>different</u> smoothing function than the $\underline{\Gamma}_s(\bar{u}_d; \bar{u}_m)$ given by Eq. 23 of Section III, i.e.

$$\underline{\Gamma}_s(\bar{u}_d; \bar{u}_m) = \sum_{\alpha} \underline{v}_{rs}(\bar{u}_\alpha; \bar{u}_m)\underline{K}_s{}^*(\bar{u}_\alpha; \bar{u}_d) \tag{23}$$

However, evaluation of $\underline{\Gamma}_q$ for given q is a straightforward problem and the design problem of selecting the step size, q, for the quantizer which operates on $\underline{K}_s{}^*$ is essentially "solved". One has merely to note that the major effect of the quantization of $\underline{K}_s{}^*$ is to produce a weighting function which has a "phase-error". Then to a close approximation

$$\underline{\Gamma}_q(\bar{u}_d; \bar{u}_m) \approx \sum_{\alpha} \underline{v}_{rs}(\bar{u}_\alpha; \bar{u}_m)\underline{K}_s{}^*(\bar{u}_\alpha; \bar{u}_d)e^{j\Phi_e(\bar{u}_\alpha)} \tag{26}$$

where $\Phi_e(\bar{u}_\alpha)$ is a deterministic phase error which depends on the quantizer step size. The degrading effects of deterministic phase errors have been widely studied and there is little that needs to be added to this subject.+

+A conservative but easily applied estimate on the allowed coarseness of quantization for $\underline{K}_q$ may be obtained in the following way. The phase error, $\Phi_e(\bar{u}_d)$, in Eq. 26 is defined by the equality

$$\underline{K}_q = Q\{K_{s1}\} + jQ\{K_{s2}\} = |\underline{K}_q|[\cos(\theta+\Phi_e) + j\sin(\theta+\Phi_e)]$$

where

$$\theta(\bar{u}_\alpha; \bar{u}_d) = \arg\{\underline{K}_s(\bar{u}_\alpha; \bar{u}_d)\}$$

Then ignoring the effects of quantization on the magnitude of $\underline{K}_q$ i.e. assuming $|\underline{K}_q| = |\underline{K}_s|$ we obtain

$$\cos(\theta+\Phi_e) \approx \frac{Q\{K_{s1}\}}{|\underline{K}_s|}$$

Footnote continued

This approximation to $\Phi_e(\bar{u}_a)$ will have a peak at each point where $Q\left\{K_{s1}(\bar{u}_a;\bar{u}_d)\right\}$ jumps from one representation level to the next such that

$$\cos(\theta+\Phi_{ep}) \approx \cos\theta + \frac{q}{2\left|\underline{K}_s\right|}$$

where Φ_{ep} = the value of $\Phi_e(\bar{u}_a)$ at these peaks
The largest values of $\left|\Phi_{ep}\right|$ will occur when θ is near a multiple of π. If $\left|\Phi_{ep}\right|_{max}$ is the largest phase error we will tolerate, we have an approximate bound on q.

$$q < 2\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|_{min} (1 - \cos\left|\Phi_{ep}\right|_{max})$$

The rms phase error, σ_e, will be no greater than a third of $\left|\Phi_{ep}\right|_{max}$ so this is equivalent to

$$q < 2\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|_{min} (1 - \cos 3\sigma_e)$$

Although $\Phi_e(\bar{u}_a)$ is a deterministic phase error we can treat it as a psuedo-random process and apply a tolerance on σ_e derived for random phase errors. The best results for such phase errors appear in the classified literature and hence cannot be disclosed here.

For the remainder of this section we will concern ourselves only with the effects of quantizers which appear in the direct input-output channel, either before or after the multiplier arrays which appear in Figs. 3 and 4.

We will consider first the arrangement shown in Figs. 6b and 6a where only the quantier which follows the multiplier is of immediate concern. The output of the summer in this system is given by

$$\begin{aligned} \underline{I}_a(\bar{u}_d) &= \sum_{\alpha} Q\left\{\underline{V}_{rs}(\bar{u}_\alpha)\underline{K}_s^*(\bar{u}_\alpha;\bar{u}_d)\right\} \\ &= \sum_{\alpha} Q\left\{\underline{K}_s^*(\bar{u}_\alpha;\bar{u}_d)\int d\bar{u}_x \underline{v}_{rs}(\bar{u}_\alpha;\bar{u}_x)\Psi(\bar{u}_x) + \underline{K}_s^*\underline{\eta}_s(\bar{u}_\alpha)\right\} \end{aligned} \tag{27}$$

(The subscript "a" identifies an "after the multiplier" quantizer) Since the quantizer introduces an essential non-linearity into the processing, the order of summation and integration cannot be interchanged in Eq. 27) and a smoothing function formalism cannot be applied. As discussed in Section II, the smoothing function and particularly its squared magnitude, the "spread" function $\left|\underline{\Gamma}(\bar{u}_d;\bar{u}_m)\right|^2$, is the basis for conventional assessments of the "quality" of radar systems. For quantized systems we require an alternate way to assess quality and devising a method to do this has been a major part of this research.

Before presenting our method of assessing the "quality" of the radar maps produced by a "quantized radar", it will be worthwhile to examine the spread function as a measure of quality for a "linear radar". We should first note that it is a function of two sets of coordinates, the map space coordinates, $\bar{u}_d$, and the scatterer space coordinates, $\bar{u}_m$. If we fix the point $\bar{u}_m$, then as a function of $\overline{u}_d$ the spread function is an indication of the manner in which a single "point scatterer" at $\overline{u}_m$ contributes to the radar response at each map point. In this sense the term "spread function" is very descriptive. Conversely, if we fix the point $\overline{u}_d$, then as a function of $\overline{u}_m$ the spread function is an indication of the contribution that point scatterers at various points in the scatterer space will make to the output at the point $\overline{u}_d$ in map space.

The word "indication" in the foregoing was carefully chosen. The contributions of individual scatterers to $\left| \underline{I}(\overline{u}_d) \right|^2$ are not additive under weighting by the spread function but require the complex weighting given in Eq. 28 where we assume for the moment that the noise, $\underline{N}(\overline{u}_d)$, is zero.

$$\left| \underline{I}(\overline{u}_d) \right|^2 = \iint d\overline{u}_x d\overline{u}_y \underline{\Psi}(\overline{u}_x) \underline{\Psi}^*(\overline{u}_y) \underline{\Gamma}(\overline{u}_d; \overline{u}_x) \underline{\Gamma}(\overline{u}_d; \overline{u}_y) \tag{28}$$

This certainly does not give any direct justification for our intuitive notion that $\left| \underline{\Gamma}(\overline{u}_d; \overline{u}_m) \right|^2$ is a function which is representative of map "quality". There must be a deeper reason for using the spread function since we accept as empirical fact the theory that nicely shaped spread functions lead to high quality maps. To find the rationale we must view Eq. 28 statistically, that is we must regard $\left| \underline{I}(\overline{u}_d) \right|^2$ as a member of an ensemble of functions which are generated from an ensemble of $\underline{\Psi}(\overline{u}_x)$ functions, statistically described. It can then be shown that the spread function is equal to an ensemble average of a particular function of $\left|\underline{I}(\bar{u}_d)\right|^2$ and $\left|\underline{\Psi}(\bar{u}_x)\right|^2$, for a particular type of $\underline{\Psi}(\bar{u}_x)$ ensemble.

Define a map quality function, $M(\bar{u}_d;\bar{u}_m)$, by the following covariance;

$$\begin{aligned} M(\bar{u}_d;\bar{u}_m) &= E\left\{\left(\left|\underline{I}(\bar{u}_d)\right|^2 - E\left\{\left|\underline{I}(\bar{u}_d)\right|^2\right\}\right)\left(\left|\underline{\Psi}(\bar{u}_m)\right|^2 - E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\}\right)\right\} \\ &= E\left\{\left|\underline{I}(\bar{u}_d)\right|^2\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\} - E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\}E\left\{\left|\underline{I}(\bar{u}_d)\right|^2\right\} \end{aligned} \tag{29}$$

We will refer to this as an "M" function. Like the spread function, it depends on both the map space coordinate, $\bar{u}_d$, and the scatterer space coordinate, $\bar{u}_m$. Now model $\underline{\Psi}(\bar{u}_m)$ as a complex "normal" process, i.e.

$$\underline{\Psi}(\bar{u}_m) = \Psi_1(\bar{u}_m) + j\,\Psi_2(\bar{u}_m)$$

where $\Psi_1(\bar{u}_m)$ and $\Psi_2(\bar{u}_m)$ are stationary, normally distributed, have zero mean, and are uncorrelated (and hence are statistically independent). Define a correlation function for this ensemble by

$$T(\Delta\bar{u}) = E\left\{\underline{\Psi}^*(\bar{u}_m + \Delta\bar{u})\,\underline{\Psi}(\bar{u}_m)\right\}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (30)

$$T(\bar{u}_x - \bar{u}_y) = E\left\{\underline{\Psi}^*(\bar{u}_x)\underline{\Psi}(\bar{u}_y)\right\}$$

Note that T is real because Ψ_1 and Ψ_2 are uncorrelated. With this $\underline{\Psi}(\bar{u}_m)$ model, $M(\bar{u}_d;\bar{u}_m)$ is easily computed from Eq. 29 for a linear radar characterized by a smoothing function, $\underline{\Gamma}$. The result is

$$M(\bar{u}_d; \bar{u}_m) = \left| \int d\bar{u}_x \underline{\Gamma}(\bar{u}_d; \bar{u}_x) T(\bar{u}_x - \bar{u}_m) \right|^2 \tag{31}$$

Finally, if the $\underline{\Psi}(\bar{u}_x)$ ensemble is generated by an infinite bandwidth ("white noise") process, then

$$T(\Delta\bar{u}) = 2A_n \delta(\Delta\bar{u}) \tag{32}$$

where

A_n = a normalizing constant with the units of area

$\delta(\Delta\bar{u})$ = a two-dimensional Dirac delta function with units of area $^{-1}$.

giving

$$M(\bar{u}_d; \bar{u}_m) = 4A_n^2 \left| \underline{\Gamma}(\bar{u}_d; \bar{u}_m) \right|^2 \tag{33}$$

Hence, in this special case the "M" function and the spread function are identical in form.

Now the "M" function, being a covariance between $\left|\underline{I}(\bar{u}_d)\right|^2$ and $\left|\underline{\Psi}(\bar{u}_m)\right|^2$ can be interpreted as giving the average contribution that a point scatterer at $\bar{u}_m$ makes to the radar output $\left|\underline{I}(\bar{u}_d)\right|^2$ at various map space points $\bar{u}_d$, the average being taken over an ensemble of scatterer density functions. In view of the equivalence between the "M" function and the spread function of a linear radar given in Eq. 33, we can make this a precise definition of what we mean by the term "indication" in our previous discussion of the significance of the spread function. In other words, we can give a justification of the use of the spread function for assessment of map quality in terms of its relationship to the "M" function. Historically, however, the spread function (or its equivalent) has been the accepted measure of radar quality while the "M" function is unknown in radar applications.[+] We submit here that the "M" function is the logical generalization of the spread function. In fact the use of the spread function as a measure of radar performance must be justified by its relationship to the "M" function. The spread function is equivalent to the "M" function for a certain subclass of radar systems, namely those for which a smoothing function formalism can be used to describe the system input-output relationship, and a "white noise" type of scatterer density function is an appropriate "input". While the spread function exists only for linear radars, the "M" function is defined for <u>every</u> radar.

The justification for our assumed model of the $\underline{\Psi}(\bar{u}_m)$ ensemble can be approached from several different directions. On the one hand we can go back to our original definition of the scatterer density

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[+] Some analysts working in optics[9] have considered input-output correlations of intensity functions which resemble the "M" function. However, the averages have been spatial rather than ensemble. Input-output correlations have long been used to characterize communication systems, but not usually in terms of the correlation of magnitude of complex inputs and outputs.

function as the continuous limit of a discrete distribution of point scatterers, as the number of such scatters per unit area (in scatterer space) increased without limit while the average sum of scatterer "sizes", $|\underline{\Psi}_m|$, per unit area remains constant. If $|\underline{\Psi}_m|$ is a random variable with a Rayleigh distribution, while $\arg(\underline{\Psi}_m)$ is uniformly distributed over 0 to 2π radians, each $\underline{\Psi}_m$ being independently selected and positioned, and continuous interpolation functions are used to construct a scatterer density function by $\underline{\Psi}_m(\overline{u}_x) = \sum_{m=1}^{M} \underline{\Psi}_m \Phi_m(\overline{u}_x)$, then when $M \to \infty$, $\Psi_m(\overline{u}_x) \to$ the $\underline{\Psi}(\overline{u}_x)$ process with correlation function given by Eq. 32, i.e. the "infinite bandwidth" or "white noise" process.

There is, of course, no way to prove that nature provides scatterer density functions by the above described statistical rule or any single rule we might hypothesize. The radar designer is interested in having a $\underline{\Psi}(\overline{u}_x)$ model which can be interpreted as being broadly representative of all the situations which are likely to be encountered in the operational use of the radar. A second approach to modeling $\underline{\Psi}(\overline{u}_x)$ is thus to regard it as a "test" ensemble against which the relative performance of radar systems can be measured. From this standpoint it is possible to argue that the infinite bandwidth $\underline{\Psi}(\overline{u}_x)$ ensemble provides the most general test available because it can simulate any continuous $\underline{\Psi}(\overline{u}_x)$ function. This argument was first developed by Weiner.[8] We can paraphrase Weiner's argument by the statement that given any finite sized piece of any arbitrary but continuous $\underline{\Psi}_a(\bar{u}_x)$ function, and a suitably defined tolerance, there is a non-zero probability that a similarly sized piece of any member of the infinite bandwidth $\underline{\Psi}(\bar{u}_x)$ ensemble will duplicate $\underline{\Psi}_a(\bar{u}_x)$ to within the given tolerance.

Having justified the "M" function as a generalization of the spread function and selected the infinite bandwidth $\underline{\Psi}(\bar{u}_m)$ ensemble as the proper "test" input, we must now shown that "M" is computable in a useful form for "quantized" radar systems. Considering first the systems with an "after the multiplier" quantizer, as in Figs. 6 a and 6 b, and designating the "M" function for this system by $M_a(\bar{u}_d;\bar{u}_m)$ we obtain using Eqs. 27 and 29

$$M_a(\bar{u}_d;\bar{u}_m) = \sum_{\alpha,\beta} E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2 \underline{\pi}_q(\bar{u}_\alpha;\bar{u}_d)\underline{\pi}_q^*(\bar{u}_\beta;\bar{u}_d)\right\} - E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\} \sum_{\alpha,\beta} E\left\{\underline{\pi}_q(\bar{u}_\alpha;\bar{u}_d)\underline{\pi}_q^*(\bar{u}_\beta;\bar{u}_d)\right\} \tag{34}$$

where

$$\underline{\pi}_q(\bar{u}_\alpha;\bar{u}_d) = Q\left\{\underline{K}_s^*(\bar{u}_\alpha;\bar{u}_d)\,\underline{V}_{rs}(\bar{u}_\alpha)\right\} \tag{35}$$

Since M_a is thus given by sums of joint moments of quantized and analog functions, it is computable, in principle, from the corresponding joint characteristic functions. These characteristic functions can be obtained from a generalization of Widrow's results[3], which are basically methods for obtaining the characteristic function of the output of a quantizer when the characteristic function of the input is known. The required generalization and the aspects of Widrow's results which bear on the present investigation are derived and discussed in Appendix B.

The $\underline{V}_{rs}(\overline{u}_a)$ which appears in Eqs. 27 and 35 is composed of a "signal" part, which is related to $\underline{\Psi}(\overline{u}_m)$, and a "noise" part, equal to $\underline{\eta}(\overline{u}_a)$. These parts are statistically independent by definition (or assumption) so the characteristic function of $\underline{\pi}(\overline{u}_a;\overline{u}_d)$, the input to the quantizer, will consist of two factors related to the characteristic functions of the signal part and noise part respectively. Computations of M_a retaining this full generality (implying that the expectations are taken over joint $\underline{\Psi}(\overline{u}_m)$ and $\underline{\eta}(\overline{u}_a)$ ensembles) are excessively cumbersome and probably not worth the effort, at least for an initial evaluation of quantization effects.

In a well designed airborne mapping radar, thermal noise (produced by either noise fields in the antenna aperture or by internally generated noise in the first mixer or amplifier stage of the radar receiver) is usually very small compared to the signal levels received from the terrain being mapped. Neglecting the thermal noise contribution to "M" amounts to an assumption that the net contribution to $\left| \underline{I}(\overline{u}_d) \right|^2$ from all points in scatterer space which are more than a few resolution elements removed from the point $\overline{u}_m = \overline{u}_d$ will be significantly greater than the thermal noise contribution. Satisfaction of this condition is desirable and depends on the transmitted power, on the average "reflectivity" of the area being mapped, and on the noise filtering properties of $\underline{K}(\overline{u}_a ; \overline{u}_d)$. The minimum transmitted power should be selected to satisfy this condition for areas where there are concentrations of "cultural" targets. Such areas are generally the most interesting to map with fine resolution systems. Other areas where the $\left| \underline{\Psi}(\overline{u}_m) \right|^2$ function arises from "natural" objects will then be mapped with relatively more "background noise" but such areas are usually intrinsically less interesting from a fine resolution standpoint. Hence the background noise can be reduced by decreasing the "bandwidth" of $\underline{K}_s$, i.e. by sacrificing resolution. We will limit our computation of M_a (and any other "M" functions of interest) to the noise-free case.

The computation of $M_a(\overline{u}_d; \overline{u}_m)$ starting with the statistical model of $\underline{\Psi}(\overline{u}_m)$ and using Eqs. 34 and 35 is quite complex in detail although straightforward in conception. The various steps are given in Appendix C, where the computation is first carried out for an arbitrary $\underline{\Psi}(\overline{u}_m)$ process correlation function, $T(\Delta\overline{u})$, and then specialized to the infinite bandwidth process defined by the correlation function of Eq. 32. This result is given in Eq. C-20 of Appendix C. There is little point in repeating it here since very little can be deduced from it directly.

It is of the general form

$$M_a(\overline{u}_d;\overline{u}_m) = 4A_n^2\left|\underline{\Gamma}_s(\overline{u}_d;\overline{u}_m)\right|^2 + A_n^2 C_a(\overline{u}_d;\overline{u}_m;q) \qquad (36)$$

where C_a is a correction which tends to zero (as q becomes small) in proportion to $e^{-\frac{4\pi^2}{q^2}}$, and $\left|\underline{\Gamma}_s(\overline{u}_d;\overline{u}_m)\right|^2$ is the spread function of the radar with the quantizer removed.

An interesting function which plays an important part in C_a is the inverse matched smoothing function defined by

$$\underline{L}_m(\overline{u}_\alpha;\overline{u}_\beta) = \int d\overline{u}_x \underline{v}_r^*(\overline{u}_\alpha;\overline{u}_x)\underline{v}_r(\overline{u}_\beta;\overline{u}_x) \qquad (37)$$

which may be compared to the conventional matched (i.e. for $\underline{K}(\overline{u}_\alpha;\overline{u}_d) = \underline{v}_r(\overline{u}_\alpha;\overline{u}_m = \overline{u}_d)$) smoothing function of a linear radar.

$$\underline{\Gamma}_m(\overline{u}_d;\overline{u}_m) = \int d\overline{u}_\alpha\, \underline{v}_r^*(\overline{u}_\alpha;\overline{u}_d)\,\underline{v}_r(\overline{u}_\alpha;\overline{u}_m) \qquad (38)$$

If $\underline{v}_r(\bar{u}_\alpha ; \bar{u}_m)$ depends only on coordinate differences, then it is easy to show that

$$\underline{L}_m(\bar{u}_\alpha ; \bar{u}_\beta) = \underline{\Gamma}_m(\bar{u}_d = -\bar{u}_\alpha ; \bar{u}_m = -\bar{u}_\beta)$$

Even if $\underline{v}_r(\bar{u}_\alpha ; \bar{u}_m)$ is a more complex function of $\bar{u}_\alpha$ and $\bar{u}_m$, it is clear that $|\underline{\Gamma}_m|$ and $|\underline{L}_m|$ will have the same general character, i.e. a narrow main lobe near the point where $\bar{u}_d = \bar{u}_m$ or $\bar{u}_\alpha = \bar{u}_\beta$ respectively and relatively small values elsewhere. Using this property of $\underline{L}_m$ an approximation to $M_a(\bar{u}_d ; \bar{u}_m)$ is derived in Appendix C which can be interpreted very simply and can be used for initial estimates of quantization effects. This result for $M_a(\bar{u}_d ; \bar{u}_m)$ is

$$\begin{aligned} M_a(\bar{u}_d ; \bar{u}_m) \approx\ & 4A_n^{\,2} \left| \underline{\Gamma}_a(\bar{u}_d ; \bar{u}_m) \right|^2 \\ & + 4A_n^{\,2} \frac{A_L}{A_s} \sum_\alpha \left| \underline{v}_r(\bar{u}_\alpha ; \bar{u}_m) \right|^2 \left| \underline{K}_s(\bar{u}_\alpha ; \bar{u}_d) \right|^2 w_2(\bar{u}_\alpha ; \bar{u}_d) \end{aligned} \tag{39}$$

where $V = \frac{2\pi}{q} =$ "quantizing frequency"

$\underline{\Gamma}_a(\bar{u}_d ; \bar{u}_m)$ = a modified smoothing function

$$= \sum_\alpha \underline{v}_r(\bar{u}_\alpha ; \bar{u}_m) \underline{K}_s{}^*(\bar{u}_\alpha ; \bar{u}_d)\, w_1(\bar{u}_\alpha ; \bar{u}_d) \tag{40}$$

$$w_1(\bar{u}_\alpha;\bar{u}_d) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-\frac{k^2V^2A_n}{2}\left|\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\right|^2 \mathcal{E}} \tag{41}$$

$$\mathcal{E} \equiv \underline{L}_m(\bar{u}_\alpha;\bar{u}_\alpha) = \int d\bar{u}_x \left|\underline{v}_r(\bar{u}_\alpha;\bar{u}_x)\right|^2 \tag{42}$$

$$\begin{aligned} w_2(\bar{u}_\alpha;\bar{u}_d) = & \sum_{k,\,l=1}^{\infty} (-1)^{k+1} \frac{(k+1)^2}{kl} e^{-\frac{(k+1)^2V^2}{2}\mathcal{E} A_n\left|\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\right|^2} \\ & - \sum_{k,\,l=1}^{\infty} (-1)^{k+1} \frac{(k-1)^2}{kl} e^{-\frac{(k-1)^2V^2}{2}\mathcal{E} A_n\left|\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\right|^2} \\ & - 2V^2A_n \left|\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\right|^2 \sum_{k=1}^{\infty} (-1)^k k^2 e^{-\frac{k^2V^2}{2}\mathcal{E} A_n\left|\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\right|^2} \end{aligned} \tag{43}$$

A_L = the area under the "main-lobe" of $\underline{L}_m(\bar{u}_\alpha;\bar{u}_\beta)$

A_s = the sampling area in signal space (area per sample)

Note that $\mathcal{E}$ is real and may be interpreted as being proportional to the average "energy" associated with the signal sample at $\bar{u}_\alpha$ due to the contributions from all of scatterer space. For any well designed radar system it is safe to assume that $\mathcal{E}$ will be essentially independent of $\bar{u}_\alpha$ for all points in signal space which contribute to the desired final map, i.e. those $\bar{u}_\alpha$ points where $\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)$ is non-zero for the $\bar{u}_d$ points of interest.

The mean square input to the quantizer (or any quantizer if many operate in parallel) for any particular combination of $\bar{u}_a$ and $\bar{u}_d$ is equal to $2\mathcal{E} A_n \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2$. The easiest way to normalize the problem is to set $A_n = \frac{1}{\mathcal{E}}$ (A_n is quite arbitrary in any case) and adjust $\left|\underline{K}_s\right|$ so that it has a maximum value of unity. This has the effect of giving q, the step size of each individual (real and imaginary) quantizer the units of root mean square input as measured at the $\bar{u}_a$ and $\bar{u}_d$ combinations where $\left|\underline{K}_s\right|$ has its maximum value. For a system with a "fixed" quantizer this can be interpreted as a specification on the "gain" of the radar receiver.

The interpretation of Eq. 39 is simple. Quantization causes $M_a(\bar{u}_d;\bar{u}_m)$ to be the sum of two functions which can be individually interpreted as spread functions. The first, which dominates the solution for any reasonable quantizer step size (i.e. $q < 3$) is a modification of the spread function that applies to the corresponding linear radar. The modification consists only of a change in the magnitude of the weighting function, $\underline{K}_s(\bar{u}_a;\bar{u}_d)$. The second function is proportional to a summation over signal space which involves only magnitudes of $\underline{v}_r$ and $\underline{K}_s$ and hence may be termed the "incoherent" part of $M_a(\bar{u}_d;\bar{u}_m)$. We would expect this incoherent part to be broadly shaped, having its maximum value at $\bar{u}_d = \bar{u}_m$ but decreasing very slowly as we move away from that point. Selection of an accaptable value of quantizer step size, q, should then be based on the allowable departure of the shape of $\left| \underline{\Gamma}_a(\bar{u}_d; \bar{u}_m) \right|^2$ from its non-quantized counterpart $\left| \underline{\Gamma}_s(\bar{u}_d; \bar{u}_m) \right|^2$, and on the allowable contribution of the incoherent part of $M_a(\bar{u}_d; \bar{u}_m)$ to the background noise level of the map.

In order to set a quality criterion on the "M" function, we first define a "resolution cell" as the area, in scatterer space, enclosed by the curve connecting points where $M(\bar{u}_d; \bar{u}_m)$ for fixed, u_d, has decreased to half the value it has at it maximum. All of "M" outside the resolution cell is defined to be "sidelobes". Our intuitive desire is to have $\left| \underline{I}(\bar{u}_d) \right|^2$ depend primarily on the portion of the scatterer density function, $\underline{\Psi}(\bar{u}_x)$, enclosed within the "resolution cell", and very little on the portion of $\underline{\Psi}(\bar{u}_x)$ outside the resolution cell. In addition, we desire to obtain a resolution cell size adequate to the operational needs of the radar user. For M_a as given by Eq. 39, the dominant part (the first term on the right) will be the primary determinant of the resolution cell size, while both terms will contribute to the side-lobes of M_a. Then by our previous interpretation of the "M" function as the average contribution to $\left| \underline{I}(\bar{u}_d) \right|^2$ due to a scatterer at $\bar{u}_m$, we are led to consider the ratio of the volume under M_a inside the resolution cell, to the volume under the "sidelobes" of M_a, i.e., everything outside the resolution cell. As large a value of this ratio as is possible, consistant with a small resolution cell size, will lead to high quality maps. As is well known, for a linear radar with fixed aperture size and signal bandwidth, there is a definite trade-off between resolution cell size and the volume ratio defined above.

In the following we will show that for reasonable value of quantizes step size, q, the domiant part of M_a in Eq. 39 can be made to correspond very closely in form to the spread function of any realizable linear radar, by adjustment of the magnitude of the weighting function, $\underline{K}_s(\overline{u}_a ; \overline{u}_d)$. Then so far as quantization effects are concerned we have only to consider the volume under the incoherent part of M_a, i.e. the second term on the right of Eq. 39. Assuming that the dominant part of M_a has been shaped to our liking (this implies that the proper phase matching conditions have been satisfied as well as the proper tapering of the magnitude of the weighting function), it will be sufficient to choose the quantizer step size such that the volume under the incoherent part of M_a is less than the volume under the side-lobes of the dominant part. For then both the side-lobe to main lobe volume ratio and the size of the resolution cell will be determined by the dominant part of M_a.

The combination of $\underline{K}_s{}^*$ and w_1 in Eq. 40 can be viewed as a new "effective" weighting function, whose phase part is unchanged from that of $\underline{K}_s^*$ but whose magnitude variation over signal space has been distorted. This distorted magnitude can be written as

$$
\begin{aligned}
w_a(\bar{u}_a;\bar{u}_d) &\equiv \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right| \, w_1(\bar{u}_a;\bar{u}_d) \\
&\approx \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right| \left(1-2e^{-\frac{2\pi^2}{q^2}\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2}\right)
\end{aligned}
\tag{44}
$$

where only the $k=1$ term in the sum in Eq. 42 for w_1 has been retained and the previously discussed normalization has been used. Note that if $\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|$ is "uniform" over the processing aperture in signal space, so is $w_a(\bar{u}_a;\bar{u}_d)$ and the only effect is to reduce the magnitude of the dominant part of $M_a(\bar{u}_d;\bar{u}_m)$ relative to the incoherent part. If, however, $\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|$ is "tapered", the effect of the second factor on the right side of Eq. 43 is to increase this taper, in the manner,

$$
1 \geq \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right| \geq \frac{w_a(\bar{u}_a;\bar{u}_d)}{1-2e^{-\frac{2\pi^2}{q^2}}}
$$

and equality holds only if $\left|\underline{K}_s\right| = 1$. Then to constrain $\left|\underline{\Gamma}_a(\bar{u}_d;\bar{u}_m)\right|^2$ to have the form of some desired $\left|\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m)\right|^2$, we can specify $\dfrac{w_a(\bar{u}_a;\bar{u}_d)}{1-2e^{-\frac{2\pi^2}{q^2}}}$ and, for a given q, solve Eq. 44 for $\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|$.

In this way a reasonably accurate approximation to the desired tapering will be attainable provided q is not too large and $\left|\underline{\Gamma}_a(\overline{u}_d;\overline{u}_m)\right|^2$ will have the form corresponding to this desired tapering.

In Fig. 7 we show a sketch of $\dfrac{w_a(\overline{u}_\alpha;\overline{u}_d)}{1-2e^{-2\pi^2/q^2}}$ vs. $\left|\underline{K}_s(\overline{u}_\alpha;\overline{u}_d)\right|$ from Eq. 44. Note that for every value of $\dfrac{w_a(\overline{u}_\alpha;\overline{u}_d)}{1-2e^{-2\pi^2/q^2}}$ between zero and unity, we can find a corresponding value of $\left|\underline{K}_s(\overline{u}_\alpha;\overline{u}_d)\right|$ and this value of $\left|\underline{K}_s\right|$ will lie between $0.188\,q$ and unity. This implies that we must have $q< 5$. (Recall that q is given in units of rms quantizer input for those $\overline{u}_\alpha$ and $\overline{u}_d$ combinations where $\left|\underline{K}_s(\overline{u}_\alpha;\overline{u}_d)\right| = 1$.) In fact, $q< 3$ is the only practical range that need be considered. In every practical case of interest we will be able to find an actual aperture weighting $\left|\underline{K}_s(\overline{u}_\alpha;\overline{u}_d)\right| \leq 1$ such that the effective aperture weighting, $w_a(\overline{u}_\alpha;\overline{u}_d)$ has the desired form, to a close approximation.

We turn now to the system of Fig. 6c where the quantizer in the signal channel appears before the multiplier. Using the subscript "b" to identify the functions related to this system we have

$$\begin{aligned}\underline{I}_b(\overline{u}_d) &= \sum_\alpha \underline{K}_s^*(\overline{u}_\alpha;\overline{u}_d)\, Q\left\{\underline{V}_{rs}(\overline{u}_\alpha)\right\} \\ &= \sum_\alpha \underline{K}_s^*(\overline{u}_\alpha;\overline{u}_d)\, Q\left\{\int d\overline{u}_x \underline{v}_{rs}(\overline{u}_\alpha;\overline{u}_x)\,\underline{\psi}(\overline{u}_x)+\underline{\eta}(\overline{u}_\alpha)\right\}\end{aligned} \qquad (45)$$

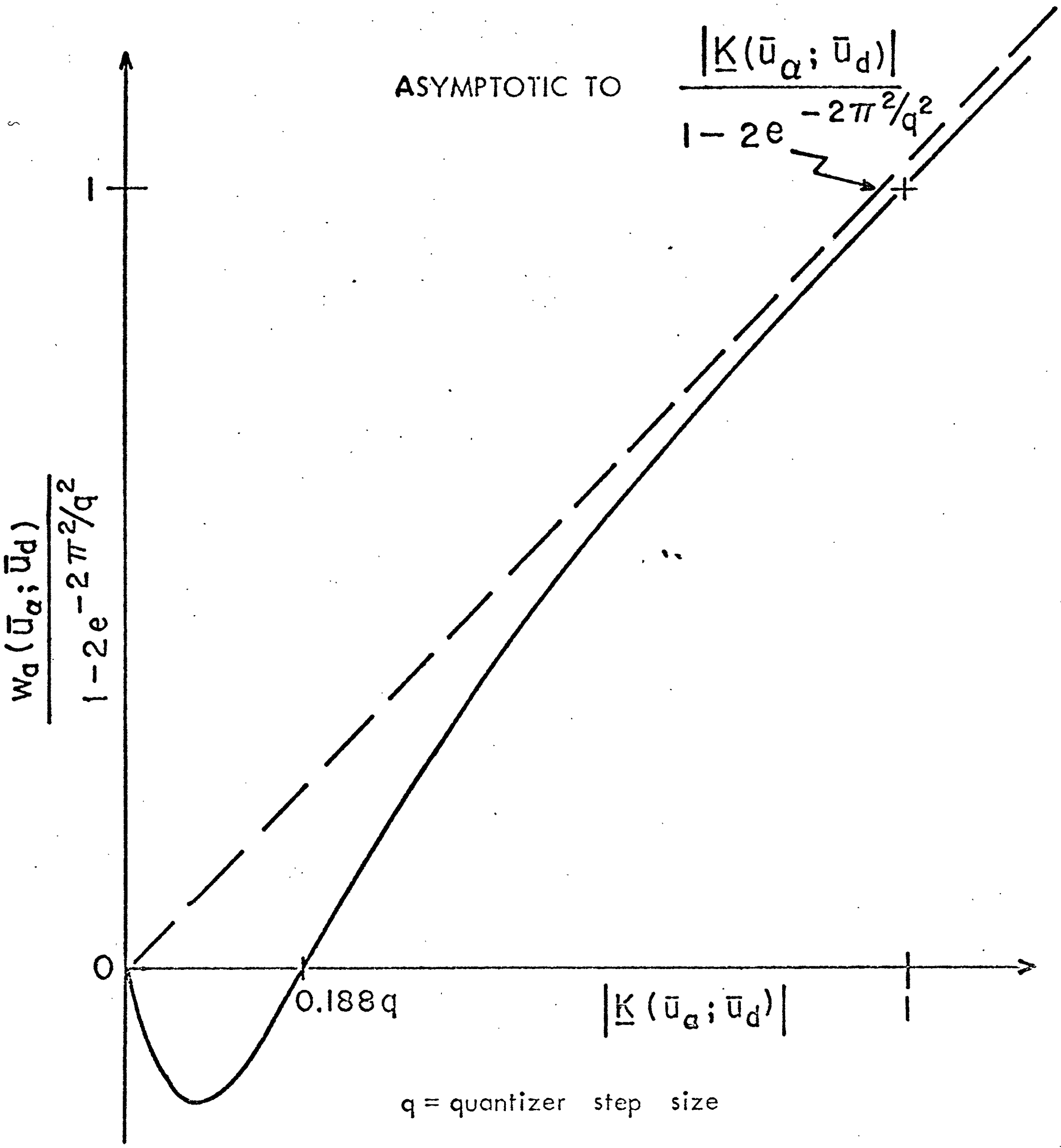

Fig. 7 Sketch of Effective Tapering vs. Actual Tapering

$$M_b(\bar{u}_d;\bar{u}_m) = \sum_{\alpha,\beta} \underline{K}_s^*(\bar{u}_\alpha;\bar{u}_d)\underline{K}_s(\bar{u}_\beta;\bar{u}_d)\, E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2 \underline{V}_{rq}^*(\bar{u}_\alpha)\underline{V}_{rq}^*(\bar{u}_\beta)\right\} \tag{46}$$

where

$$\underline{V}_{rq}(\bar{u}_\alpha) = Q\left\{\underline{V}_{rs}(\bar{u}_\alpha)\right\} \tag{47}$$

Comparing Eqs. 46 and 47 to Eqs. 34 and 35 we see that the solution for M_b can be easily obtained from that for M_a by making a few obvious associations in the final result. The details are given in Appendix C where the solution for M_b for the infinite bandwidth scatterer density function ensemble appears as Eq. C-37. The same type of approximation as was applied to M_a can be applied to M_b giving the result

$$\begin{aligned} M_b(\bar{u}_d;\bar{u}_m) &\approx 4A_n^2\left|\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m)\right|^2 w_3^2 \\ &= 4A_n^2 \frac{A_I}{A_s} w_4 \sum_\alpha \left|\underline{v}_r(\bar{u}_\alpha;\bar{u}_m)\right|^2 \left|\underline{K}(\bar{u}_\alpha;\bar{u}_d)\right|^2 \end{aligned} \tag{48}$$

where

$$w_3 = 1 + 2\sum_{k=1}^{\infty} (-1)^k e^{-\frac{k^2V^2}{2}} A_n \mathcal{E} \tag{49}$$

$$w_4 = \sum_{k,\,l=1}^{\infty} (-1)^{k+1} \frac{(k+1)^2}{kl} e^{-\frac{(k+1)^2 V^2}{2} A_n \mathcal{E}}$$

$$- \sum_{k,\,l=1}^{\infty} (-1)^{k+1} \frac{(k+1)^2}{kl} e^{-\frac{(k-1)^2 V^2}{2} A_n \mathcal{E}} \tag{50}$$

$$- 2V^2 A_n \mathcal{E} \sum_{k=1}^{\infty} (-1)^k k^2 e^{-\frac{k^2 V^2}{2} A_n \mathcal{E}}$$

and V, A_L, and A_s are as defined under Eqs. 39 through 42. It should be noted that $\underline{\Gamma}_s(\overline{u}_d; \overline{u}_m)$ is the smoothing function of the corresponding linear radar (Eq. 23) rather than a modified version, as in Eq. 39 for M_a. Also note that w_3 and w_4 are equal to the values of w_1 and w_2 (Eqs. 41 and 42) obtained by setting $\left|\underline{K}(\overline{u}_a; \overline{u}_d)\right|^2 = 1$ for all $\overline{u}_a$. Hence M_a and M_b are identical (to within the approximations inherent in Eq. 39 and 48) for "uniform" aperture functions $\left|\underline{K}_s(\overline{u}_a; \overline{u}_d)\right|$. For the system of Fig. 6c, the mean square input to the quantizer is $2\mathcal{E}A_n$ so the same normalization is appropriate for M_b as for M_a, i.e. set $A_n = \frac{1}{\mathcal{E}}$ giving q in units of root mean square quantizer input.

The result for M_b given in Eq. 48 is considerably simpler than the result for M_a given in Eq. 39. The quantization effects in Eq. 42 appears only as simple numerical factors w_3 and w_4

in contrast to the appearance of $w_1(\bar{u}_\alpha ; \bar{u}_d)$ and $w_2(\bar{u}_\alpha ; \bar{u}_d)$ inside the α summations in Eqs. 39 and 40. The interpretation of the result is, however, unchanged. The major criterion for selection of q is again that the total volume under the "incoherent" part of M_b (i.e. the second term in Eq. 48) should be less than the total volume under the "sidelobes" of the dominant part of M_b (i.e. the first term).

Although it may not be desirable to use uniform $\left| \underline{K}_s(\bar{u}_\alpha ; \bar{u}_d) \right|$ functions[+], because of their poor sidelobe behavior, it will be instructive to compute the quantization requirements for this special case. We will assume that $\left| \underline{v}_r(\bar{u}_\alpha ; \bar{u}_m) \right|$ is also "uniform" over a finite region of signal space, although this case is never encountered in practice. The computations are, however, simple and will give a reasonable "ball-park" estimate of quantization requirements. To keep the calculations simple we will assume the phase variations of $\underline{v}_r(\bar{u}_\alpha ; \bar{u}_m)$ and $\underline{K}(\bar{u}_\alpha ; \bar{u}_d)$ are quadratic in $\left| \bar{u}_\alpha - \bar{u}_m \right|$ and $\left| \bar{u}_\alpha - \bar{u}_d \right|$ respectively. This will be a reasonable approximation to the forms of these functions for the example given at the end of Section II, provided the apertures of $\left| \underline{v}_r(\bar{u}_\alpha ; \bar{u}_m) \right|$ and $\left| \underline{K}(\bar{u}_\alpha ; \bar{u}_d) \right|$ (i.e. the areas where they are non-zero) in signal space for fixed $\bar{u}_m$ and $\bar{u}_d$ are not too large.

[+] This is a matter of controversy and depends to a great extent on subjective evaluations of map "interpretability".

Specifically let us take

$$\underline{v}_r(\bar{u}_\alpha ; \bar{u}_x) = \begin{cases} \sqrt{\dfrac{\mathcal{E}}{A_a}}\; e^{jp|\bar{u}_\alpha - \bar{u}_x|^2} & \bar{u}_\alpha \in A_a(\bar{u}_x) \\ 0 & \bar{u}_\alpha \notin A_a(\bar{u}_x) \end{cases} \tag{51}$$

$$\underline{K}(\bar{u}_\alpha ; \bar{u}_d) = \sqrt{\frac{A_a}{\mathcal{E}}}\; \underline{v}_r(\bar{u}_\alpha ; \bar{u}_x = \bar{u}_d) \tag{52}$$

where

$A_a(\bar{u}_x)$ = the aperture area of $\underline{v}_r(\bar{u}_\alpha ; \bar{u}_x)$ in signal space

= a square of dimensions $R_a \times S_a = A_a$ centered on the point $\bar{u}_\alpha = \bar{u}_x$

and

p = a constant with units of radians/(unit distance)^2

Also let $A_d(\bar{u}_m - \bar{u}_d)$ be the "overlap" of the aperture areas $A_a(\bar{u}_m)$ and $A_a(\bar{u}_d)$ as shown in Fig. 8.

Assuming that the sampling on signal space is adequate for the bandwidth involved we can approximate $|\underline{\Gamma}_s(\bar{u}_d ; \bar{u}_m)|^2$ by its continuous equivalent $\frac{1}{A_s^2}|\underline{\Gamma}(\bar{u}_d ; \bar{u}_m)|^2$ (given by Eq. 14). Then a rather straightforward computation gives

$$\begin{aligned} M_a(\bar{u}_d ; \bar{u}_m) &= M_b(\bar{u}_d ; \bar{u}_m) \\ &\approx 4\left(\frac{A_n}{A_s}\right)^2 |\underline{\Gamma}(\bar{u}_d ; \bar{u}_m)|^2 w_3^{\,2} \\ &\quad + 4A_n^{\,2}\left(\frac{A_I}{A_s}\right)\left(\frac{\mathcal{E}}{A_a}\right)\left(\frac{A_d(\bar{u}_m - \bar{u}_d)}{A_s}\right) w_4 \end{aligned} \tag{53}$$

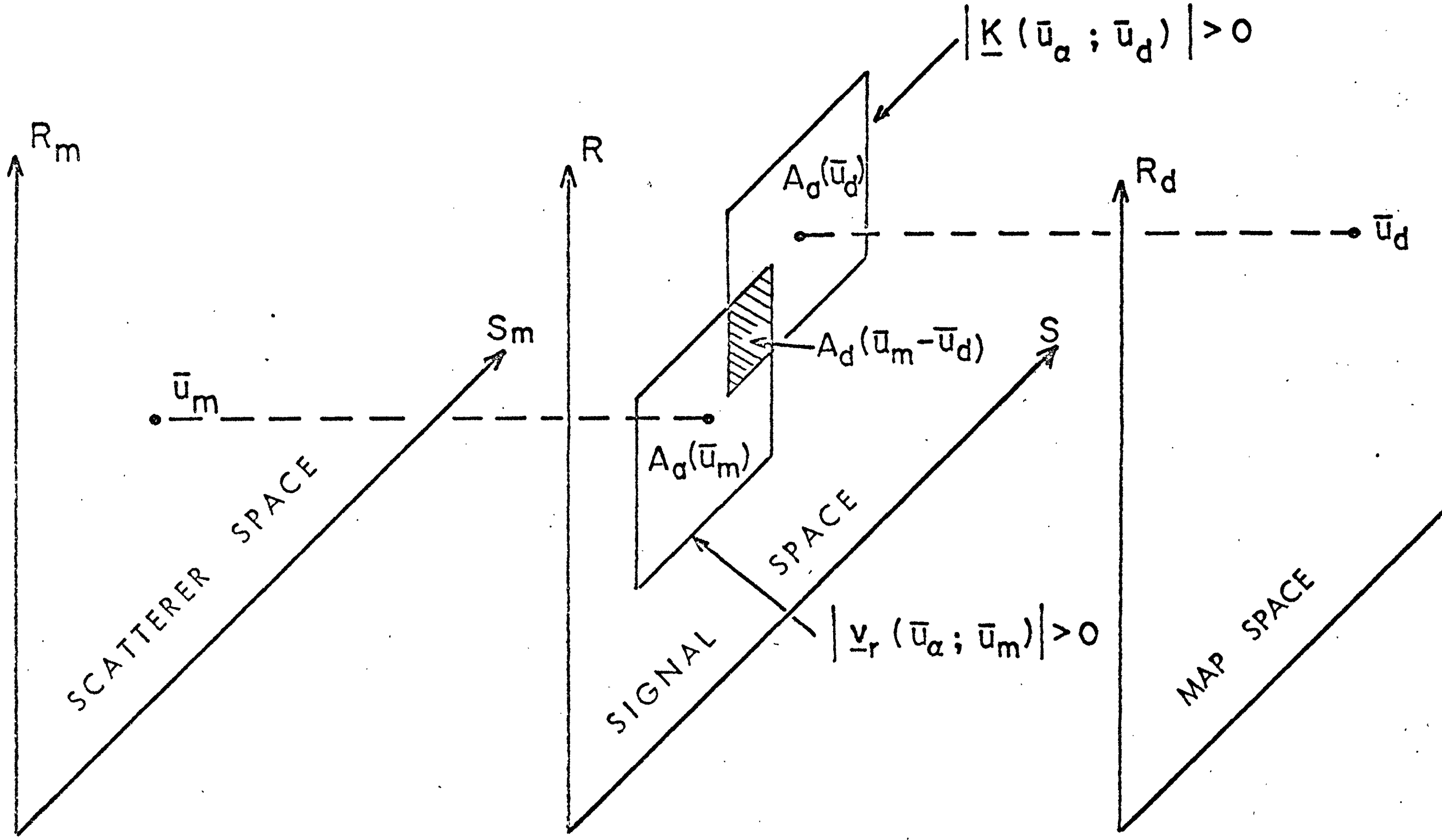

Fig. 8 Processing Apertures

and

$$\left|\underline{\Gamma}(\bar{u}_d;\bar{u}_m)\right|^2 = \frac{\mathcal{E}}{A_a}\left|\int_{A_d(\bar{u}_m-\bar{u}_d)} d\bar{u}_a \; e^{-2jp<\bar{u}_a,\bar{u}_m-\bar{u}_d>}\right|^2 \quad (54)$$

where $<\bar{a},\bar{b}>$ denotes the usual inner product of vectors. Using the coordinates of the example given in Section II, i.e.

$$\bar{u}_a = R,S \; ; \; \bar{u}_d = R_d, S_d \; ; \; \bar{u}_m = R_m, S_m$$

the evaluation of Eq. 54 gives

$$\left|\underline{\Gamma}(\bar{u}_d;\bar{u}_m)\right|^2 = \mathcal{E} A_a \left[\frac{\sin\left[p(R_m-R_d)\,(R_a-|R_m-R_d|)\right]}{p\,R_a(R_m-R_d)}\right]^2 \cdot \left[\frac{\sin\left[p(S_m-S_d)\,(S_a-|S_m-S_d|)\right]}{p\,S_a(S_m-S_d)}\right]^2 \quad (55)$$

for $R_m - R_d \leq R_a$ and $S_m - S_d \leq S_a$ and zero elsewhere. Near $\bar{u}_d = \bar{u}_m$ this spread function is given approximately by

$$\left|\underline{\Gamma}(\bar{u}_d \approx \bar{u}_m;\bar{u}_m)\right|^2 \approx \mathcal{E} A_a \left[\frac{\sin pR_a(R_m-R_d)}{p\,R_a(R_m-R_d)}\right]^2 \left[\frac{\sin p\,S_a(S_m-S_d)}{p\,S_a(S_m-S_d)}\right]^2 \quad (56)$$

The "resolution cell" is thus approximately an ellipse with semi-axes $\frac{1.4}{p R_a}$ and $\frac{1.4}{p S_a}$ and area $\frac{2\pi}{p^2 A_a}$. This is also the area A_L in this case. We are primarily interested in the volume under $\left|\underline{\Gamma}(\bar{u}_d;\bar{u}_m)\right|^2$

outside this resolution cell. An analytic expression for this volume is not obtainable but it can be approximated by the volume under a pyramid of height equal to the height of the first sidelobe of $\left| \underline{\Gamma}(\overline{u}_d; \overline{u}_m) \right|^2$ (equal to 0.047 times its maximum) and base $4A_a$. Using this approximation we obtain for (VR), the ratio of the volume under the incoherent part of M_a (and M_b), to the volume under the sidelobes of the dominant part

$$VR \approx 16 \left(\frac{A_a}{A_L} \right)^{-1} \frac{w_4}{w_3^2} \tag{57}$$

The ratio A_a/A_L is the "area compression ratio", i.e. the ratio of the total processing aperture area to the area in a resolution cell. Typcially this will range from 10^2 to 10^6. In this case it is given by

$$\frac{A_a}{A_L} = \frac{p^2 A_a}{2\pi}$$

It is interesting to note that VR is not a function of the sampling density, $(A_s)^{-1}$, as this appears in the same way in both the dominant part and the incoherent part of M_a (and M_b). Hence "oversampling", i.e. sampling more often than the minimum required, does not give any advantage as far as the effects of quantization are concerned.

Using the normalization $A_n = \frac{1}{\mathcal{E}'}$, which gives q in units of rms, quantizer input, and limiting cur consideration to reasonably small values of q (i.e. $q < 3$ or $V > 2$) we obtain the following very close approximations for w_3 and w_4.

$$w_3 \approx 1-2e^{-\frac{V^2}{2}} = 1-2e^{-\frac{2\pi^2}{q^2}}$$

$$w_4 \approx 2(V^2+1)\,e^{-\frac{V^2}{2}} = 2\left(\frac{4\pi^2}{q^2}+1\right)e^{-\frac{2\pi^2}{q^2}}$$

Then

$$(VR) \approx 32\left(\frac{A_L}{A_a}\right)\left(\frac{4\pi^2}{q^2}+1\right)\frac{e^{-\frac{2\pi^2}{q^2}}}{\left[1-2e^{-\frac{2\pi^2}{q^2}}\right]^2} \tag{58}$$

$$\approx 32\left(\frac{A_L}{A_a}\right)\frac{\left(\frac{4\pi^2}{q^2}+1\right)}{\left(e^{\frac{2\pi^2}{q^2}}-4\right)}$$

For $q = 1$ we have $(VR) \approx 2.5\left(\frac{A_L}{A_a}\right) \times 10^{-6}$ which is certainly negligibly small. Because of the dominance of the exponential factor when $q < 3$ we expect that $q = 1$ will be adequate even for tapered aperture functions, $\left|\underline{K}(\overline{u}_a ; \overline{u}_d)\right|$, which lead to "M" functions with much lower sidelobe volumes.

This selection of the quantizer step size in relation to the RMS quantizer input, assumes that the quantizer has infinite range, i.e. has no "saturation" level or largest (positive or negative) representation level. This is, of course, a practical impossibility but even if such a quantizer were available, some other part of the processor would limit the useable range of representation levels. In the "before-the-multiplier" quantization of Fig. 6 c this limitation would probably occur in the finite length of the data register which receives the quantizer output. In the "after-the-multiplier" quantization of Figs. 6 a and 6b the limitation might be in the linear range of the multiplier or in the length of the summing registers which form the real and imaginary parts of $\underline{I}(\overline{u}_d)$. In the context of our computation of the "M" function, the input to the quantizer (or other limiting device if it occurs before the quantizer) is a zero mean normally distributed random variable. Hence quantizer representation values (or linear ranges of devices which preceed the quantizer) which cover plus or minue three standard deviations of quantizer (or other limiting device) input will avoid saturation better than 99 percent of the time. Choosing the quantizer step sizes equal to one standard deviation of quantizer input then requires 3 "bit" words

(7 representation levels) for the data registers in the "before-the-multiplier" quantization scheme and $\log_2 7\left(\frac{A_a}{A_s}\right)$ bit words for the summing register for "after-the-multiplier" quantization.

We have selected the quantizer step size and the number of representation levels based on the "M" function calculation. The point of departure for this calculation was the "spread function" of linear radar imaging theory. Of course no radar is "linear" for an infinite range of inputs but always saturates at some level. Hence we now have to make contact with another conventional measure of radar system performance, namely "dynamic range". This is conventionally defined as the ratio of the size of the smallest "scatterer" which is discernable above the "noise" background of the radar map to the size of the largest scatterer which the processor can handle without saturating. The "noise" of concern here is usually not thermal noise in the signal input but so-called "high-level" processing noise, such as film graininess in optical processors.

The concept of "dynamic range" is a difficult thing to apply equitably to all radar mapping implementations. On the one hand it depends on some type of "noise" injection mechanism, which will differ for various implementations, and on the other hand by some poorly defined, in general, "limiting" non-linearity. Since a deterministic $\underline{\Psi}(\overline{u}_x)$ function serves as the "test" function for the conventional concept of dynamic range, it is virtually impossible to give a rigorous analytic treatment for a digitally implemented system. What is required is a computation of the "M" function for a radar in which the data is quantized and some form of limiting is placed in the data channel. This will complicate the calculation enormously, since it will be necessary to compute the joint moments of the input and output of a limiter and quantizer in series. However, the task does not appear to be impossible and we suggest it as a possible subject for further research. To make a clearer connection between the conventional concept of dynamic range and the "M" function it may be desirable to use a modified "test" ensemble of $\underline{\Psi}(\bar{u}_x)$ functions, e.g. something resembling a uniform background reflectivity and widely separated discrete "point-scatterers" with random complex reflection coefficients.

Because the allowed quantizer step size which we have deduced from our "M" function calculation for the uniform aperture case, may appear surprisingly coarse, those readers who are familiar with optical processing systems will want some way to relate the performance of such a digital system to the conventional concept of dynamic range. For this purpose we offer the following non-rigorous and very approximate estimate of the dynamic range of a digital processor with N_q representation levels in the quantizer. Using the quantizer model of Fig. 5a, we restrict our attention to those cases where N_q is odd and the same number of representation levels occur above and below zero. (This implies all input magnitude greater than $\frac{N_q-2}{2}q$ are assigned to one of the representation values $\pm\frac{N_q-1}{2}q$). We restrict our attention to the "after-the-multiplier" case of Fig. 6a and uniform aperture "matched" weighting and signal functions. The quantizers which operate on the real and imaginary parts of the complex multiplier outputs are, of course, identical.

We first need a way of characterizing quantizing as "high-level" noise injection. To do this we can use Widrow's "quantizing theorem"[3] which states* that if the joint characteristic function of the inputs to a quantizer is zero outside intervals of width $2\pi/q$ (along each dimension of characteristic function space), then second order j o i n t moments of the quantizer outputs are equal to those of the quantizer inputs. The joint characteristic function of a sequence of independent normally distributed random variables, of standard deviation equal to the quantizer step size, q, satisfies the requirements of this theorem remarkably well (although not exactly). This result means that if we insert such a noise source ahead of the quantizer in Fig. 6a, we can compute the second order joint moments of the quantizer output as if the quantizer were replaced by a simple unity gain. We will interpret this noise source as the equivalent of quantization noise.

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*This is one special form of Widrow's more general theorem.

Now assume that the scatterer "scene" viewed by the radar consists of a single point scatterer at the point $\bar{u}_m$ with scattering coefficient $\underline{\Psi}_m$, i.e. $\underline{\Psi}(\bar{u}_x) = \underline{\Psi}_m\, \delta(\bar{u}_x - \bar{u}_m)$. Then with the additive noise source described above inserted ahead of the quantizer in Fig. 6 a, we have

$$E\left\{\left|\underline{I}_a(\bar{u}_d; \bar{u}_m)\right|^2\right\} = \left|\underline{\Psi}_m\right|^2 + 2\frac{q^2}{A_a/A_s} \qquad (58\,A)$$

where we have assumed the normalization

$$\left|\underline{\Gamma}(\bar{u}_d = \bar{u}_m; \bar{u}_m)\right| = 1$$

If the phase of $\underline{\Psi}_m$ is a random variable uniformly distributed over a 2π range, the mean square input to the quantizer in Fig. 6 a during the evaluation of $\left|\underline{I}(\bar{u}_d = \bar{u}_m)\right|^2$ is found to be

$$E\left\{\mathcal{R}e^2\left\{\underline{\pi}(\bar{u}_a; \bar{u}_d = u_m)\right\}\right\} = E\left\{\mathcal{I}m^2\left\{\underline{\pi}(\bar{u}_a; \bar{u}_d = \bar{u}_m)\right\}\right\} = \frac{1}{2}\left|\underline{\Psi}_m\right|^2 + q^2 \qquad (58\,B)$$

where $\underline{\pi}(\bar{u}_a; \bar{u}_d)$ was previously defined as the products of signal and weighting function. From Eq. 58 A we define a minimum "visible" scatterer, $\left|\underline{\Psi}_m\right|_{min}$ by

$$\left|\underline{\Psi}_m\right|_{min} = 3\sqrt{2}\,\frac{q}{\sqrt{(A_a/A_s)}}$$

i.e. a scatterer "size" which results in a peak response three times the average noise background. From Eq. 58 B we define a "saturating" target $\left|\underline{\Psi}_m\right|_{max}$ by

$$\sqrt{\frac{1}{2}\left|\underline{\Psi}_m\right|^2_{max} + q^2} = \frac{N_q - 1}{2}\, q$$

i.e. a rms quantizer input equal to the representation levels of greatest magnitude. Defining dynamic range, D.R., by

$$\text{D.R.} = \frac{\left|\underline{\Psi}_m\right|_{max}}{\left|\underline{\Psi}_m\right|_{min}}$$

we obtain easily

$$\text{D.R.} = \sqrt{\frac{A_a}{A_s}}\,\sqrt{\frac{(N_q - 1)^2 - 4}{6}}$$

The ratio (A_a / A_s) is just the number of complex samples that occur in the sum defining $\underline{\Gamma}_m(\bar{u}_d; \bar{u}_m)$. The minimum of this ratio occurs when $A_s = A_L$ in which case it is equal to the area compression ratio. Hence, by the above calculation, "oversampling" seems to improve dynamic range even though it does not effect the behavior of the "M" function evaluation. Putting $N_q = 7$ and $A_a / A_s = 10^2$ into Eq. 58 C gives a dynamic range of slightly less than 20 db which is certainly compatible with the sidelobe behavior of uniform aperture weighting functions. The result is quite sensitive to how one defines $\left|\underline{\Psi}_m\right|_{max}$, so we do not put this result forth as an absolute measure of radar performance.

Summarizing our results on quantization we have:

(1) Put forth a new functional measure, $M(\overline{u}_d; \overline{u}_m)$, of radar mapping quality and related it to the more conventional "spread function", $\left| \underline{\Gamma}(\overline{u}_d; \overline{u}_m) \right|^2$, measure.

(2) Computed this measure, "M", for two types of quantized radars, i.e. where the quantizer is "before-the-multiplier" and where the quantizer is "after-the-multiplier". This solution is in the form of the sum of the spread function for the corresponding "linear" radar plus correction terms which depend on the quantizer step size.

(3) Obtained approximations to these solutions in simple forms which are readily interpreted in terms of a "dominant" part, which retains the characteristics of the spread function of the corresponding linear radar, and an "incoherent" part which increases the side-lobe level of the result.

(4) Shown that in the case of "after-the-multiplier" quantization of the radar data, and a tapered processing aperture function $\left| \underline{K}(\overline{u}_a; \overline{u}_d) \right|$, a modified tapering can be specified which results in an effective tapering approximately equal to any desired tapering.

(5) Shown that for uniform aperture functions, a quantizer step size equal to the rms quantizer input will make the incoherent part of the "M" function negligible with respect to the inherent side lobe level of the dominant part. Because of the form of the dependence on quantizer step size we anticipate that this will also be true for tapered aperture functions.

Finally we may note that in the design of an actual radar, the approximate solutions for the "M" function will give a good estimate of quantization effects, but should not be relied upon entirely. Before expending several million dollars on the development, it would be prudent to compute M from the "exact" equations given in Appendix C. While these appear rather formidable, they possess a good deal of symmetry, and the sums on k and l should converge rapidly for reasonable values of q (i.e. $q \approx$ rms quantizer input).

V. COMPARISON OF THE IMMEDIATE AND DELAYED FORMULATIONS FOR DIGITAL IMPLEMENTATION

In Section III we defined two alternatives for organizing the computation of the output map function $\left| \underline{I}(\overline{u}_d) \right|^2$ from the received signal $\underline{V}_r(\overline{u}_\alpha)$ and the given weighting function $\underline{K}(\overline{u}_\alpha ; \overline{u}_d)$. These were designated as the "delayed" and "immediate" formulations and were diagrammed in Figs. 3 and 4 respectively. We now wish to examine the relative merit of these formulations in the context of a digital implementation of the data processing. As in our treatment of quantization effects in Section IV, we can further differentiate digital processors by the location of the quantizer, either before or after the multiplier arrays in Figs. 3 and 4. Of the six possible comparisons that these variations can lead to, only two are of practical importance. These are:

1) Comparison of the delayed and immediate formulations in combination with "before-the-multiplier" quantization of data.

2) Comparison of the delayed formulation and "before-the-multiplier" quantization with the immediate formulation and "after-the-multiplier" quantization.

In Section I we pointed out the most fruitful areas for application of digital techniques appear to be at the extremes of performance requirements, i.e. "real-time" processing at moderate resolution levels and relatively low area coverage, vs. "non-real-time" processing at very fine resolution levels and large area coverage. The first comparison above is pertinent to "all-digital" systems for both extremes of performance. The second comparison above is of concern only for "real-time" processing applications where there may be some advantage to the "hybrid" computation implied by "after-the-multiplier" quantization.

The significant differences between the delayed and immediate formulations from a computational standpoint are:

A) In the delayed formulation of processing, the weighting function, $\underline{K}_s(\overline{u}_\alpha ; \overline{u}_d)$, is developed as a function of the signal space, $\overline{u}_\alpha$, coordinates and all the addends for the sum over α (Eq. 21) pertinent to a particular map point, $\overline{u}_d$, are generated essentially simultaneously. In the immediate formulation, $\underline{K}_s(\overline{u}_\alpha ; \overline{u}_d)$ is developed as a function of map space coordinates, $\overline{u}_d$, and all the addends for a particular signal space coordinate $\overline{u}_\alpha$, which enter into many different sums on α, are generated essentially simultaneously.

B) The type of data storage that is required for each formulation is different. In the delayed formulation, the data-storage-array elements must be interrogated repeatedly for read out of $\underline{V}_{rs}(\overline{u}_\alpha)$ samples, and these elements are simple single-function devices. In the immediate formulation, the data-storage-array elements accept their data in increments, form these increments into partial sums and hold their data for a single interrogation. The amount of data storage required is also different, the degree of difference depending on the performance specifications of the radar.

C) In the delayed formulation, the data flow converges on a single (complex) summer which produces sequences of $\underline{I}(\bar{u}_d)$ values. In immediate processing the data flow diverges from a single (complex) register which holds a sequence of data samples. This will imply some differences in the circuit operating speeds involved which, to some extent, can be traded off against differences in the amount of data storage required.

The first of the above listed differences is concerned with the relative ease with which $\underline{K}_s(\bar{u}_a;\bar{u}_d)$ can be generated, i.e. as a function of $\bar{u}_a$ for fixed $\bar{u}_d$ or vice-versa. Inspection of Eq. 20 in Section II, which gives $\underline{K}(\bar{u}_a;\bar{u}_d)$ for a simple idealized case, reveals that there is certainly some difference between $\underline{K}_s$ as a function of $\bar{u}_d$ or of $\bar{u}_d$. However, there is only one special circumstance where we have found a significant difference in the complexity of $\underline{K}_s(\bar{u}_a;\bar{u}_d)$. This case demands (1) that the actual radar trajectory and the required radar performance parameters are such that the data samples, $\underline{V}_{rs}(\bar{u}_a)$, are (or can be corrected to) a set equivalent to those that would be obtained from the idealized example given in Section II+, (2) that the data processing be done with full complex samples of $\underline{V}_{rs}(\bar{u}_a)$ and $\underline{K}_s(\bar{u}_a;\bar{u}_d)$ (so that the average spatial frequency of $\underline{K}_s(\bar{u}_a;\bar{u}_d)$ on the $\bar{u}_a$ coordinates can be reduced to zero), (3) that a simple narrow pulse (or its equivalent in a "dechirp or receive" variation) be used to obtain range coordinate, R_d, resolution, and, (4) that the amplitude weighting (aperture function $w(\bar{u}_a;\bar{u}_d)$) be uniform over the processing aperture. When these conditions are satisfied the appropriate weighting function is obtained from Eq. 20 by setting $\Phi(\mu) = 0$, letting $\lambda_i \to \infty$, and deleting the factor $e^{j\frac{4\pi}{\lambda}R}$ (which in view of (2) above is presumed to be already incorporated in $\underline{V}_{rs}(\bar{u}_a)$ by prior operations on the received signal), This gives

$$\underline{K}_s(\bar{u}_a;\bar{u}_d) = \begin{cases} e^{-j\frac{4\pi}{\lambda}(R_{dx}(S) - R_d)} & \bar{u}_a \in A_a(\bar{u}_d) \\ 0 & \bar{u}_a \notin A_a(\bar{u}_d) \end{cases} \tag{59}$$

+Non-zero "squint" angles, as defined in Appendix A, can be allowed but departures from the idealized trajectory must be correctable by operations on the received signal.

where

$$A_a(\overline{u}_d) = \text{the aperture area of } \underline{K}_s(\overline{u}_\alpha ; \overline{u}_d)$$

$$\overline{u}_\alpha = R, S$$

$$\overline{u}_d = R_d, S_d$$

$$R_{dx}(S) = \sqrt{R_d^{\,2} + (S-S_d)^2}$$

and λ = the transmitted wavelength

From Eq. 59 we see that the generation of $\underline{K}_s(\overline{u}_\alpha ; \overline{u}_d)$ will be much simpler in the immediate formulation (as a function of $\overline{u}_d$ for various $\overline{u}_\alpha$) than in the delayed formulation (as a function of $\overline{u}_\alpha$ for various $\overline{u}_d$). Referring to Fig. 4, the block diagram of the immediate formulation, we see that the generation of $\underline{K}_s(\overline{u}_\alpha ; \overline{u}_d)$ can take the form of a fixed (complex) function over a plane of coordinates R_d and $(S-S_d)$ which is "masked" (i.e. modified by simple "on-off" transmission functions) as a function of R. The summing array in Fig. 4 is then translated with respect to the multiplier array for each successive S value. In contrast the delayed formulation (Fig. 3) requires that $\underline{K}_s(\overline{u}_\alpha ; \overline{u}_d)$ be regenerated as a non-linear function of R_d for each successive map point to be produced. Stated another way, the weighting function through which the data plane is to be "viewed" from each successive map point (for the delayed formulation) changes significantly as the R_d coordinate of the map point changes.

Because of the limitations under which Eq. 59 for $\underline{K}_s(\overline{u}_a ; \overline{u}_d)$ is appropriate, its applicability is limited to those real-time processing applications where the resolution requirements are modest and uniform aperture functions give adequate side-lobe performance. For the particular variant of "after-the-multiplier" quantization of data, the restriction to uniform aperture functions can perhaps be lifted without undue complication, depending on how the (analog) multipliers are implemented. For instance, if the multiplier array of Fig. 4 can be implemented by intercepting a uniform "flux" (of say optical energy or electron density) made proportional to $V_{rs1}(\overline{u}_a)$ and $V_{rs2}(\overline{u}_a)$, by flux transmission "masks" of variable transmissibility (over spatial coordinates) proportional to $K_{s1}(\overline{u}_a ; \overline{u}_d)$ and $K_{s2}(\overline{u}_a ; \overline{u}_d)$, it would not be too difficult in principle to include an amplitude taper over the aperture area $A_a(\overline{u}_d)$.+

We turn now to the question of the type and quantity of data storage implied by each formulation. As pointed out in (B) of the enumeration of differences between the delayed and immediate formulations, the data-storage elements for the immediate formulation must also serve as summing devices. This must be counted as a disadvantage of the immediate formulation vis-a-vis delayed formulation. However, technology marches on and advent of large-scale

+This is not a recommended design. It is merely a suggestion of a means for circumventing the uniform aperture requirement in this case.

integrated circuitry has generated considerable interest in the possibilities of distributing simple logic functions throughout data-storage arrays made up of "flip-flops" or shift registers. The partial-sum storage array of Fig. 4 is, of course, just such a mixed logic and memory device. It would be futile to try to make a definitive judgement on the relative cost (or size, or weight, etc.) of a data-storage array for the delayed formulation vs. one for the immediate formulation on the basis of these type differences, because the hardward availability situation is changing much too rapidly.

We can, however, say something quantitative about the amount of data storage that will be required in each case. To set up a reasonable basis of comparison, we postulate a radar the purpose of which is to produce a "strip" map of the earth's surface N_m resolution cells wide and indefinitely long, this strip being roughly parallel to the radar trajectory. Let A_L be the area of a resolution cell and A_a be the processing aperture area, and assume that these are independent of the map coordinates, $\bar{u}_d = (R_d, S_d)$. Then A_a/A_L is the area "compression ratio" of the radar and we define C_R and C_S to be the linear range and azimuth compression ratios, so that

$$A_a/A_L \approx C_R C_S$$

Let A_s be the sampling area in signal space so that

A_a/A_s = the number of (complex) data samples that enter into the evaluation of $|\underline{I}(\bar{u}_d)|^2$ for each $\bar{u}_d$ point of interest.

We further postulate that the map function, $|\underline{I}(\bar{u}_d)|^2$, is to be evaluated for one $\bar{u}_d$ point centered within each of a set of contiguous and essentially non-overlapping resolution cells which cover the desired strip map. With these preliminaries, our previous discussion of the relationship between bandwidth, sampling, and resolution, show that $A_a/A_L < A_a/A_s$ and we further assume that the sampling density over signal space is as close to the minimum as possible. Hence for all practical purposes

$$A_a/A_s \approx A_a/A_L \approx C_R C_S$$

Consider first the delayed formulation combined with "before-the-multiplier" quantization of data. The total number of complex data "words" that will have to be in storage at any one time is given by $(N_m + C_R)\, C_S$, i.e. the number of R coordinate samples required to produce N_m contiguous resolution cells along the R_d coordinate multiplied by the number of S dimension samples required to produce a single resolution cell. The number of representation levels required per real data word is equal to the quantization step size divided into twice* the expected maximum value of the real or imaginary part of a data sample. In Section IV we concluded that a quantization step size equal to the rms quantizer input would be adequate for uniform aperture functions and would quite likely be adequate in most other cases of practical interest. Based on this we concluded that seven representation levels, or three "bit" word sizes, should be adequate, giving a total required bit capacity of $6\,C_S(N_m + C_R)$.

Consider next the immediate formulation combined with "after-the-multipler" quantization of data. The total number of partial complex sums at any one time will be $N_m C_S$. Each complex storage element will be required to accept A_a/A_L increments in the course of computing a particular $\underline{I}(\overline{u}_d)$ value. Again using the results of Section IV, each increment will require seven representation levels, requiring a word length of $\log_2 7\frac{A_a}{A_L}$ bits. The total required bit capacity of the partial sum storage array is thus $2N_m C_S \log_2 7\,\frac{A_a}{A_L}$.

Finally we have to consider the immediate formulation combined with "before-the-multiplier" quantization of data. While we can, in this case, compute a required quantizer step size by the methods of Section IV, and estimate the quantization requirements on $\underline{K}$, we

*Twice, to allow for the bipolar nature of the real and imaginary parts of the data samples.

cannot give a neatly derived estimate of the size of the partial sum storage array. This unfortunate situation arises because the output of a complex multiplier, whose inputs are quantized complex numbers with n_1 and n_2 admissable representation levels, will have approximately $n_1 n_2$ unevenly spaced admissable representation levels for its real and imaginary parts. In order to efficiently store the increments produced by the multiplier array, the multiplier outputs will have to be "requantized" into a set of evenly spaced representation levels. Computation of a "requantization" step size by methods analogous to those used in Section IV does not appear to be possible. However, it is inconceivable that the result could be substantially different from that obtained for "after-the-multiplier" quantization of data. Hence we will take the total required bit capacity of the partial sum storage array to be $2N_m C_S \log_2 7\left(\frac{A_a}{A_L}\right)$ for the immediate formulation with either "before-" or "after-the-multiplier" quantization of data.

The ratio, β, of storage-array bit capacity required for the immediate formulation to that required for the delayed formulation is thus

$$\beta \approx \frac{N_m \log_2 7\left(\frac{A_a}{A_L}\right)}{3(N_m + C_R)}$$

This ratio will almost always be greater than unity, but almost never greater than ten. The largest values will occur for the non-real-time, fine-resolution, wide-area-coverage applications where we can assume $N_m >> C_R$, and $\frac{A_a}{A_L} \geq 10^4$, in which case $\beta \geq 5$. Even with $A_a/A_L = 10^8$, we have $\beta \approx 10$. However, these are precisely the cases where this ratio is most significant because of the tremendous quantities of data involved. For instance N_m can conceivably be as large as 10^5 while C_R and C_S of 10^4 is not out of the question, giving 6×10^9 bits of data to be interrogated from storage per resolution-cell length of strip map produced for the delayed formulation. For the immediate formulation the corresponding number is 6×10^{10} bits of data to be incremented into temporary storage. The inescapable conclusion is that the non-real time applications will best be served by the delayed formulation.

The situation is quite different for the real-time, limited-area-coverage, moderate-resolution applications. Hence strip maps on the order of 10^3 resolution cells wide or less are of interest and area compression ratios of 10^2 may be quite interesting. For the narrow pulse radar (or equivalent), where $C_R = 1$, we have $\beta \approx 3$ and a total of 6×10^5 bits of data to be incremented into temporary storage per resolution cell length of strip map produced by the immediate formulation. This is also the required total capacity of the partial sum storage array of Fig. 4 for this set of parameters. While this is a large number, it is not out of the question, and the factor 3 advantage to the delayed formulation is not necessarily of overriding importance.

Our final comparison involves circuit operating speeds, particularly as these are affected by point (C) in our previous enumeration of differences between the immediate and delayed formulations. Recall that this point of comparison involved the divergence of data flow from a complex register in the immediate formulation, vs. the convergence of data flow into a complex summer for the delayed formulation. The complex summers, whether used in the delayed or immediate formulation must accept A_a/A_L addends per map point produced. In real-time applications using the delayed formulation, the summer addend acceptance rate is then $N_m U_o (A_a/A_L)$ where U_o is the velocity of the radar along its trajectory in units of resolution cell widths (S_d coordinate) per unit time. Typically interesting numbers might be $N_m = 10^3, U_o = 10^2, A_a/A_L = 10^2$ making the use of a single complex summer impractical. It is certainly possible to provide a number of summers which operate in parallel, say N_m of them, although this will also require parallel multiplier arrays. Each real summer will have to accommodate sums of bit length equal to $\log_2 7 \dfrac{A_a}{A_L}$ and, in effect, become additional temporary storage decreasing the relative advantage previously computed for the delayed formulation in regard to data storage requirements. By contrast, each storage element in the partial summer storage array for the immediate formulation (Fig. 4) must accommodate addends at a rate given by $\dfrac{U_o (A_a/A_L)}{C_S}$. For a narrow pulse (or equivalent) radar $C_S \approx A_a/A_L$ and this rate becomes simply U_o.

It would be imprudent at this point to give any hard and fast rules for system design on the basis of the foregoing considerations. However, such results really should not be expected from a general study of practical alternatives; too much depends on the ingenuity of the designer in mitigating the affects of apparent obstacles to an economic implementation. Some general trends and quantitative facts, however, are clear. These are summarized as follows:

(1). Weighting-function generation with uniform aperture (i.e. amplitude) functions at moderate-resolution levels is simplified by the immediate formulation.

(2) The immediate formulation will always require more data storage than the delayed formulation by a factor ranging from 3 to 10.

(3) The delayed formulation will always require faster summers, the ratio of speeds being proportional to $N_m C_S$ divided by the number of parallel summers provided for the delayed formulation. However, the relative advantage of the delayed formulation in data-storage requirements is reduced in proportion to the number of such parallel summers.

(4) Non-real-time, wide-area-coverage, fine-resolution applications appear to be best served by the delayed formulation.

(5) Real-time, limited-area-coverage, moderate-resolution applications are an approximate "toss up", with perhaps a slight edge to the immediate formulation. If clever, compact ways of implementing arrays of analog multipliers can be devised, then the advantage will definitely shift toward the immediate formulation combined with "after-the-multiplier" quantization of data.

APPENDIX A

A Spatial Model of Radar Signals For Realistic Geometries

Our purpose here is to show how to apply the formalism developed in Section II to the specification of a "spatial" model of the radar signals that result from a more practical trajectory than that used in the idealized example given there. We will also extend the development to include "squinted" modes of operations.

Apart from the limitation to the "broadside" mode of operation, the example given in Section II does not portray a practical situation in three respects:

(1) The earth is not flat

(2) It is generally impossible to keep an aircraft precisely on a predetermined trajectory

(3) Aircraft speed along any given trajectory will not be constant.

Of these the first arises because the earth is grossly spherical and "rough" in surface contour. For the present purposes we will include only the gross sphericity. The other two points will be handled by assuming a predetermined "nominal" trajectory along which the radar is "navigated", i.e. departures from the nominal trajectory are continuously measured and recorded and the actual trajectory of the radar is continuously corrected back toward the nominal trajectory. The recorded deviations then become data to be used in constructing the radar signal model and, hence, the processor weighting function.* The notation we use here follows that defined in Section II.

First we idealize the earth to a perfect sphere and restrict the domain of non-zero $\underline{\Psi}(\bar{u}_m)$ to the surface of this sphere. Secondly, we will choose nominal trajectories which lie at a constant altitude above the earth's surface, and for which the radius of curvature (at any point) is <u>large</u> compared to $\mathcal{R}_m(t)$ the distance from the radar to the point $\bar{u}_m$ in scatterer space at time t, for any $\bar{u}_m$ point being mapped. The actual trajectory of the radar is constriained to remain "close" to the chosen nominal trajectory, i.e. departure distances are <u>small</u> compared to $\mathcal{R}_m(t)$ for all $\bar{u}_m$ points of interest. With these conventions we can define our coordinate systems in a relatively simple way.

Let $\bar{u}_{vo}(S)$ be an <u>a-priori</u> chosen nominal trajectory with S designating distance along it from an arbitrary zero reference point. Let $\bar{x}(S)$ be the coordinate vector in an orthogonal coordinate

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*We are not at all concerned here with the accuracy of these measurements. We assume they are "perfect". The effects of non-perfect measurements are treated in the classified literature.

system with origin at $\overline{u}_{vo}(S)$. Then measured S and $\overline{x}(S)$ coordinates define the radar position. Similarly scatterer positions can be defined in terms of a position, S_m, along the $\overline{u}_{vo}(S)$ trajectory, and a "range" coordinate R_m to be defined below.

The detailed definition of the parameters and coordinate systems is as follows, and pictured in Fig. A-1. For a set of unit vectors defining the $\overline{x}(S)$ coordinate axes take one unit vector i_{x1} along the tangent to the specified $\overline{u}_{vo}(S)$ trajectory at S, directed toward increasing S, the second i_{x2} downward along the local vertical and the third, i_{x3}, perpendicular to i_{x1} and i_{x2} forming a right-hand coordinate system. A "reference line of sight" direction is defined within the $\overline{x}(S)$ coordinate system by the angles α_o and Ω where $\frac{\pi}{2} - \Omega$ is the angle from the x_1 coordinate axis and α_o is the "depression" angle from the horizontal (i.e. the x_1x_3 plane) to the plane containing the x_1 axis and the reference line of sight direction. We now have four coordinates, S, x_1, x_2, and x_3, for specifying the actual radar position when clearly three are sufficient. The redundancy can be removed by defining a transformation from points on the actual trajectory to points on the nominal trajectory. We do this by considering the

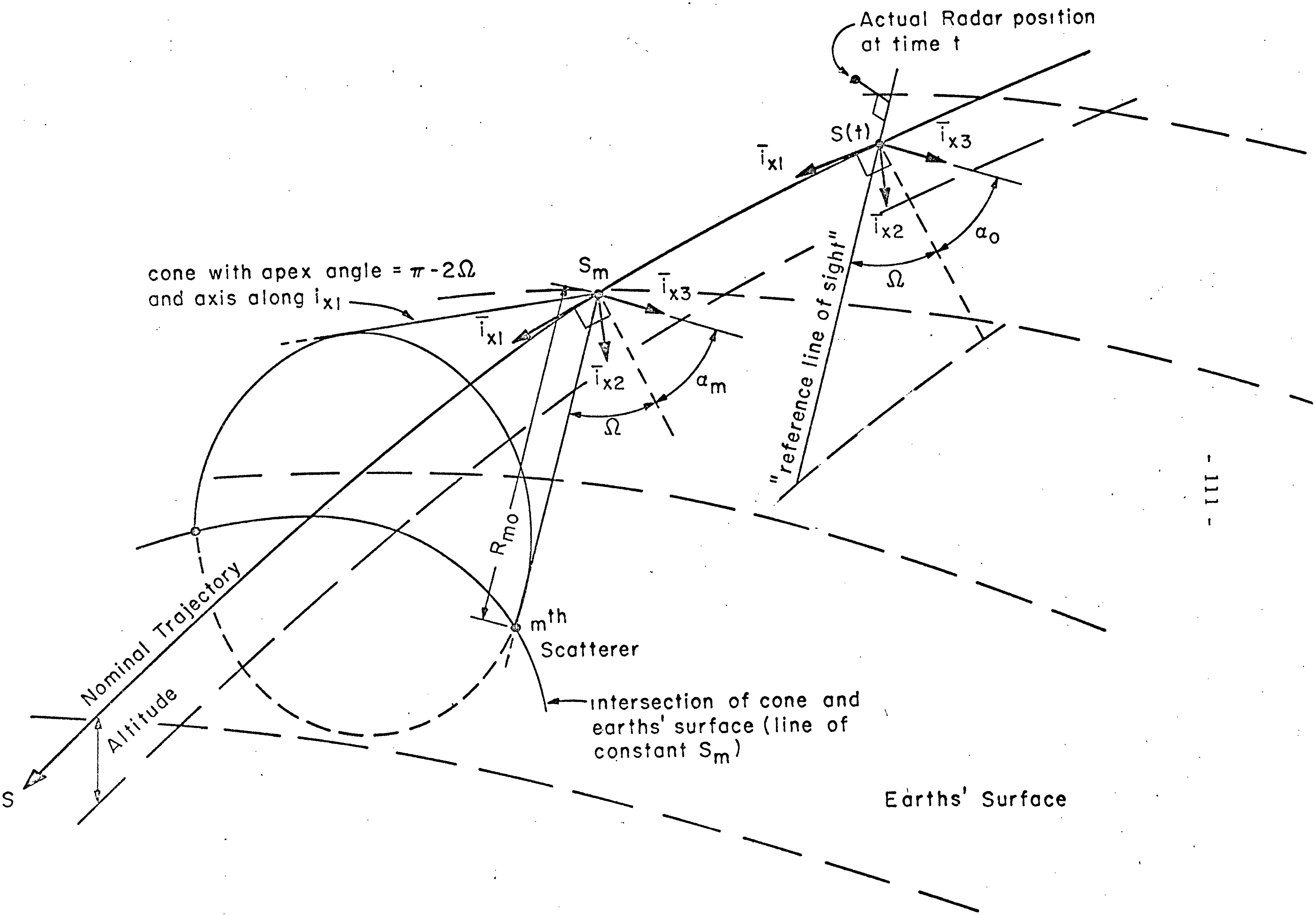

Fig. A-1 Radar and Scatterer Coordinates surface formed by all the "reference lines-of-sight" emanating from the nominal trajectory, and projecting the actual radar position onto this surface. This projection lies on a particular "reference line-of-sight" and the measured S coordinate of the radar is taken as the point on the nominal trajectory from which this particular "reference line-of-sight" emanates.* A little trigonometry will show that this is equivalent to imposing the following constraint on the $\overline{x}(S)$ coordinates

$$x_1 = \tan\Omega(x_3 \cos \alpha_o + x_2 \sin \alpha_o) \qquad \text{(A-1)}$$

To obtain the scatterer coordinates we construct a cone of apex angle $(\pi-2\Omega)$ around the tangent to the specified $\overline{u}_{vo}(S)$ trajectory and define S_m as the point on the trajectory such that the $\overline{u}_m$ point in the scatterer space lies on the surface of this cone when the apex of the cone is at $S = S_m$. The scatterer coordinates can be taken as S_m, the range R_m from the point $\overline{u}_{vo}(S_m)$ to $\overline{u}_m$, and a depression angle α_m measured in a manner similar to α_o. The angle Ω is the "squint" angle of the radar.

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*This procedure is guaranteed to work only because of the restrictions we have placed on the allowable trajectories and only if $\Omega < \pi/2$ rad. In practical cases we usually have $\Omega < 80°$.

As stated earlier, all scatterers are presumed to be located on the surface of the earth. In effect this introduces a fixed relationship between the scatterer coordinates S_m, R_m, and α_m reducing the number of free coordinates to two, which we will choose as S_m and R_m and reducing the scatterer density per unit volume, $\underline{\Psi}(\overline{u}_m)$, to a scatterer density per unit area on the $\overline{u}_m = \{S_m, R_m\}$ coordinate plane.

We can now write for the coordinates, $\overline{u}_v(S)$, of the actual position of the radar

$$\overline{u}_v(S) = (S, x_2(S), x_3(S))$$

and for the coordinates of the point in the scatterer space

$$\overline{u}_m = R_m, S_m$$

although these are meaningless without a defined $\overline{u}_{vo}(S)$ trajectory. If $\frac{dS}{dt} > 0$ for all t we will have a one-to-one correspondence from t onto S for all t. Then we can write $S = S(t)$ and

$$\mathcal{R}_{mx}(t) = \left| \overline{u}_v(S) - \overline{u}_m \right| = R_{mx}(S(t)) \qquad \text{(A-2)}$$

providing we interpret the magnitude signs properly, i.e. as meaning "compute the geometric distance between the points $\overline{u}_v(S)$ and $\overline{u}_m$

located in normal Euclidean 3-space for $S = S(t)$." With these definitions $R_{mx}(S_m)\Big|_{x_2 = x_3 = 0} = R_m$.

For our definition of the "unit-point-scatterer" we will take the same impulse response function $h(t, t'; \bar{u}_m)$ used in Section II, Eq. 8, i.e.

$$h(t, t'; \bar{u}_m) = \frac{g_m(t) g_m(t')}{R_{mx}(t) R_{mx}(t')} \quad \delta\left[t - t' - \frac{1}{c}(R_{mx}(t) + R_m(t'))\right] \tag{8}$$

where c, $\delta(t)$, $g_m(t)$, and $R_{mx}(t)$ are as defined in Section II. Following the procedure used in Section II, we now need to define a "signal space" coordinate system, $\bar{u}_a$, a function on signal space and scatterer space $i(\bar{u}_a, \bar{u}_a', \bar{u}_m)$, and a mapping, $\bar{u}_a(t)$, from time, t, onto a one dimensional subset of $\bar{u}_a$, such that h is given by the composition of functions

$$h(t, t'; \bar{u}_m) = i(\bar{u}_a, \bar{u}_a'; \bar{u}_m) \; 0 \; (\bar{u}_a(t), \bar{u}_a(t')) \tag{A-3}$$

In order to attach some general significance to i, we must be sure that relative displacements among the components of the $\bar{u}_a(t)$ mapping lead to mappings from t onto one dimensional subsets of signal space, such that the composition of Eq. A-3 leads to a possible $h(t, t'; \bar{u}_m)$ function. We can then use $i(\bar{u}_a, \bar{u}_a'; \bar{u}_m)$ as the basis for a spatial signal model in the manner shown in Section II.

We choose for signal space coordinates a set similar to those used in Section II, namely R = "propagation distance, and, S = nominal trajectory distance. Then the i function given by Eq. 9A of Section II, defined on the signal space coordinate $\bar{u}_a = (R, S)$, and the mapping

$$\bar{u}_a(t) = \begin{cases} R = \frac{c}{2}(t - T_n) \\ S = S(t), \end{cases} \qquad \text{(A-4)}$$

where T_n is an arbitrary reference time, and S(t) is now the radar position along the nominal trajectory, as defined above, will meet our requirements.

The development of the spatial signal model can now proceed exactly as in the idealized example given in Section II with the following exceptions:

(1) The solution for the roots of the delta function arguments in Eq. 17A of Section II cannot be known a-priori because of the dependence of $R_{mx}(S)$ on the measured coordinate $x_2(S)$ and $x_3(S)$. However, they can always be computed a-posterior, implying that the weighting function based on $\underline{v}_r(\bar{u}_a; \bar{u}_m)$ will not be known until after the radar signals are received. Series expansions of $R_{mx}(S)$ about the reference positions $S_n = S(T_n)$ can be used to simplify the computations.

(2) The sampling of signal space implied by the mapping $\bar{u}_a(t)$ will not be "regular" over the signal space coordinates, i.e., the lines defining $\bar{u}_a(t)$ in Fig. 2 will be randomly curved (but deterministically related to S(t)) and unevenly spaced. Hence the usual sampling interpolation for a "uniform" sampling (sin x/x) will not be applicable, and there is no known way to assure an exact correspondence between the $\underline{I}'(\bar{u}_d)$ computed from the spatial model (Eq. 12) and the $\underline{I}(\bar{u}_d)$ computed from the temporal model (Eq. 4). However, if the maximum spacing between samples is no greater than the maximum allowed for a uniform sampling, the problem will be more academic than real. Any reasonable interpolation function will yield a $\underline{V}_r(\bar{u}_a)$ from $\underline{E}_r(t)$ which will be adequate to the purposes of the model. In addition it is possible to specify the T_n instants such that the $S(T_n)$ positions are uniformly spaced on S (although non-uniformly spaced on t). However, the samples along any R= constant > 0 section of signal space will still, in general, be non-uniformly spaced.

APPENDIX B

Joint Characteristic Functions of Quantized and Continuous Random Variables

In order to compute the "M" function defined in Section IV for a radar in which the data has been "quantized" we have to be able to compute certain joint moments of random variables, some of which have been quantized and some not. Specifically we need a solution to the following problem:

<u>Given</u> the random variables $\overline{x} = \{x_1, \ldots, x_r\}$ and $\overline{y} = \{y_1, \ldots, y_s\}$ of known joint characteristic function $C_{\overline{x}\,\overline{y}}(\overline{v})$, $\overline{v} = \{v_1, \ldots, v_{r+s}\}$

<u>Find</u> the joint characteristic function $C_{\overline{x}\overline{y}_q}(\overline{v})$ of the random variables $\overline{x} = \{x_1, \ldots, x_r\}$ and $\overline{y}_q = \{(y_1)_q, \ldots, (y_s)_q\}$

where $(y_\alpha)_q = Q\{y_\alpha\}$; $\alpha = 1, \ldots s.$

The quantizing operation $Q\{\cdot\}$ is defined by the non-linear gain function shown in Fig. 5 of Section IV.

Widrow[3] solved this problem for the case j= 0 and k = 1 or 2 specifically. His solution is based on certain analogies with the processing of signals in linear sampled data systems. For a single quantized random variable, the solution proceeds as follows:

Let $\quad p_y(Y) \;=\;$ the probability density function of y

and $\quad p_{yq}(Y) \;=\;$ the probability density function of y_q

Then
$$p_{yq}(Y) = \sum_{n=-\infty}^{\infty} \left[\int_{(n-\frac{1}{2})q}^{(n+\frac{1}{2})q} p_y(Z)\,dZ \right] \delta(Y - nq)$$

$$= \left[\int_{-\infty}^{Y+\frac{q}{2}} p_y(Z)\,dZ - \int_{-\infty}^{Y-\frac{q}{2}} p_y(Z)\,dZ \right] \sum_{n=-\infty}^{\infty} \delta(Y-nq) \tag{B-1}$$

The characteristic function $C_y(v)$ is defined as the Fourier transform of the probability density function, i.e. the expectation of e^{jvy}. Then

$$C_y(v) = \int_{-\infty}^{\infty} p_y(Y)\, e^{jvY}\, dY \tag{B-2}$$

Widrow computed the characteristic function of y_q by noting that the operations in brackets correspond to certain linear system operations on "signals", where the counterpart of the signal is $p_y(Y)$, while multiplication by the sum of Delta functions corresponds to "sampling" the "signal" which results from these linear operations. In the transform (characteristic function) domain Eq. B-1 becomes

$$C_{yq}(v) = \left[\frac{e^{jv\frac{q}{2}} - e^{-jv\frac{q}{2}}}{jv} C_y(v) \right] * \frac{2\pi}{q} \sum_{k=-\infty}^{\infty} \delta\left(v + \frac{2\pi k}{q}\right) \quad \text{(B-3)}$$

where $*$ represents convolution and the term to the right of $*$ is the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta\,(Y - nq)$. Carrying out the convolution and simplifying the result gives

$$C_{y_q}(v) = \sum_{k=-\infty}^{\infty} \frac{\sin \pi\left(\frac{v}{V} + k\right)}{\pi\left(\frac{v}{V} + k\right)} \; C_y(v + kV) \quad \text{(B-4)}$$

where $V = \frac{2\pi}{q}$ = "quantizing frequency".

To extend this result to the problem we posed at the beginning of this appendix, we have only to replace $p_y(Y)$ in Eq. B-1 by the partially transformed function

$$C_{\overline{x}\overline{y}_q{}'p_{y_a}}(\overline{v}', Y_a) \; ; \; 1 \leq a \leq s$$

where
$$\overline{v}' = \left\{ v_1, \ldots, v_{r+a-1}, v_{r+a+1} \ldots v_{r+s} \right\}$$
$$\overline{y}_q{}' = \left\{ (y_1)_q, \ldots (y_{a-1})_q, (y_{a+1})_q, \ldots (y_s)_q \right\}$$

which is defined to be the Fourier transform of the joint probability density of $\overline{x}, \overline{y}_q{}'$, and y_a along every coordinate in $\overline{X} \cdot \overline{Y}$ space except Y_a. Then Eq. B-4 gives us the Fourier transform of $C_{\overline{x}\overline{y}_q{}'p_{y_a}}(v', T_a)$ along Y_a in terms of the Fourier transform of

$C_{\overline{xy}_q, p_{y_a}}(\overline{v}', Y_a)$. But the former is just the desired joint characteristic function and the latter is the joint characteristic function of the random variables $\overline{x}, \overline{y}_q'$, and y_a, i.e., where all the y's have been quantized but the $a^{\underline{th}}$. But if this is true for the $a^{\underline{th}}$ quantized variable, it holds for all the quantized variables. Hence the general result

$$C_{\overline{x}\overline{y}_q}(\overline{v}) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_s=-\infty}^{\infty} \left[\prod_{m=1}^{s} \operatorname{sinc}\left(\frac{v_{r+m}}{V} + k_m\right)\right] \cdot C_{\overline{x}\overline{y}}(\overline{v}_x, v_{r+1} + k_1 V, \ldots, v_{r+s} + k_s V) \tag{B-5}$$

where $\quad \overline{v}_x = \left\{ v_1, \ldots, v_r \right\}$

and $\quad \operatorname{sinc} x = \dfrac{\sin \pi x}{\pi x}$

From Eq. B-5 we see that if $C_{\overline{x}\overline{y}}(\overline{v})$ is non-zero only over single simple intervals of length less than V along every $\overline{v}_y = \left\{ v_{r+1}, \ldots v_{r+s} \right\}$ coordinate, then only terms for which $k_1 = \ldots = k_s = 0$ will contribute to any joint moments computed from $C_{\overline{xy}_q}(\overline{v})$ (by differentiation and letting $\overline{v} \to 0$). Hence, under this condition, s e c o n d o r d e r joint moments of $\overline{x}$ and $\overline{y}_q$ are equal to the corresponding moments and joint moments of $\overline{x}$ and $\overline{y}$. This is Widrow's "quantizing theorem" in simplified form.

APPENDIX C

Computation of the "M" Functions for Radars With Quantized Data Channels

Repeating Eqs. 34 and 35 from Section IV

$$M_a(\bar{u}_d;\bar{u}_m) = \sum_{\alpha,\beta} E\left\{ \left|\underline{\Psi}(\bar{u}_m)\right|^2 \underline{\pi}_q(\bar{u}_\alpha;\bar{u}_d)\underline{\pi}_q^*(\bar{u}_\beta;\bar{u}_d)\right\}$$

$$-E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\} \sum_{\alpha,\beta} E\left\{\underline{\pi}_q(\bar{u}_\alpha;\bar{u}_d)\underline{\pi}_q^*(\bar{u}_\beta;\bar{u}_d)\right\} \tag{34}$$

where

$$\underline{\pi}_q(\bar{u}_\alpha;\bar{u}_d) = Q\left\{\underline{K}_s^*(\bar{u}_\alpha;\bar{u}_d)\,\underline{V}_{rs}(\bar{u}_\alpha)\right\} \tag{35}$$

and restricting an attention to "noise-free" signals

$$\underline{V}_{rs}(\bar{u}_\alpha) = \int d\bar{u}_x \underline{\Psi}(\bar{u}_x)\underline{v}_{rs}(\bar{u}_\alpha;\bar{u}_x) \tag{C-1}$$

To compute M_a using the results of Appendix B, rewrite M_a in terms of real random variables as follows:

$$M_a(\bar{u}_d;\bar{u}_m) = \sum_\alpha \sum_\beta E\left\{\Psi_1^2\pi_{1q}(\alpha)\pi_{1q}(\beta)+\Psi_1^2\pi_{2q}(\alpha)\pi_{2q}(\beta)\right\}$$

$$+\sum_\alpha \sum_\beta E\left\{\Psi_2^2\pi_{1q}(\alpha)\pi_{1q}(\beta)+\Psi_2^2\pi_{2q}(\alpha)\pi_{2q}(\beta)\right\} \tag{C-2}$$

$$-T_o\sum_\alpha \sum_\beta E\left\{\pi_{1q}(\alpha)\pi_{1q}(\beta)+\pi_{2q}(\alpha)\pi_{2q}(\beta)\right\}$$

where $$T_o = T(\Delta\bar{u} = 0) = E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\} \qquad \text{(C-3)}$$

and some notation simplifications have been used, i.e.

$$\begin{aligned}
\Psi_1 &= \Psi_1(\bar{u}_m) = \mathcal{R}e\left\{\underline{\Psi}(\bar{u}_m)\right\} \\
\Psi_2 &= \Psi_2(\bar{u}_m) = \mathcal{I}m\left\{\underline{\Psi}(\bar{u}_m)\right\} \\
\pi_{1q}(\alpha) &= Q\left\{\pi_1(\alpha)\right\} = \mathcal{R}e\left\{\underline{\pi}_q(\bar{u}_\alpha\,;\bar{u}_d)\right\} \\
\pi_{2q}(\alpha) &= Q\left\{\pi_2(\alpha)\right\} = \mathcal{I}m\left\{\underline{\pi}_q(\bar{u}_\alpha\,;\bar{u}_d)\right\}
\end{aligned} \qquad \text{(C-4)}$$

Now define a notation for joint characteristic functions (j.c.f.'s) as follows with $\bar{v} = v_1, v_2, v_3$;

$$\begin{aligned}
C_{111}(\bar{v}) &= \text{the j.c.f. of } \Psi_1, \pi_1(\alpha)\pi_1(\beta) \\
C_{122}(\bar{v}) &= \text{the j.c.f. of } \Psi_1\pi_2(\alpha), \pi_2(\beta) \\
C_{211q}(\bar{v}) &= \text{the j.c.f. of } \Psi_2, \pi_{1q}(\alpha), \pi_{2q}(\beta)
\end{aligned} \qquad \text{(C-5)}$$

etc.

Then we can write for M_a

$$M_a(\bar{u}_d;\bar{u}_m) = \sum_\alpha \sum_\beta \frac{\partial^4}{\partial v_1{}^2 \partial v_2 \partial v_3}\left[C_{111q}(\bar{v}) + C_{122q}(\bar{v}) + C_{211q}(\bar{v}) + C_{222q}(\bar{v})\right]_{\bar{v}=0}$$
$$+ \; T_o \sum_\alpha \sum_\beta \frac{\partial^2}{\partial v_2 \partial v_3}\left[C_{111q}(\bar{v}) + C_{222q}(\bar{v})\right]_{\bar{v}=0} \qquad \text{(C-6)}$$

Applying Eq. B-5 of Appendix B we have for the form of the j.c.f.'s in Eq. C-6

$$C_{111q}(\bar{v}) = \sum_{kl} \operatorname{sinc}\left(\frac{v_2}{V} + k\right) \operatorname{sinc}\left(\frac{v_3}{V} + l\right) C_{111}(v_1, v_2 + kV, v_3 + lV) \tag{C-7}$$

where $\operatorname{sinc} x = \dfrac{\sin \pi x}{\pi x}$

and $V = \dfrac{2\pi}{q}$

Then M_a can be written

$$M_a(\bar{u}_d; \bar{u}_m) = \sum_{\alpha} \sum_{\beta} \frac{\partial^2}{\partial v_2 \partial v_3} \Bigg[\sum_{kl} \operatorname{sinc}\left(\frac{v_2}{V} + k\right) \operatorname{sinc}\left(\frac{v_3}{V} + l\right) \left\{ \left(\frac{\partial^2}{\partial v_1^2} \left\{ C_{111}(\bar{v}) + C_{122}(\bar{v}) + C_{211}(\bar{v}) + C_{222}(\bar{v}) \right\} \right) + T_o \; C_{111}(\bar{v}) + C_{222}(\bar{v}) \right\} \Bigg] \tag{C-8}$$

The random variables $\Psi_1, \Psi_2, \pi_1(\alpha)$, and $\pi_2(\alpha)$ are all jointly normally distributed so their j.c.f.'s are given by the form

$$C_{111}(\bar{v}) = e^{-\frac{1}{2} \bar{v}^T \Lambda_{111} \bar{v}} \tag{C-9}$$

where $\bar{v}$ is considered as a column matrix and $\bar{v}^T$ as its transpose (row matrix) and

$$\Lambda_{111} = \begin{bmatrix} \lambda_{100} & \lambda_{110} & \lambda_{101} \\ \lambda_{110} & \lambda_{010} & \lambda_{011} \\ \lambda_{101} & \lambda_{011} & \lambda_{001} \end{bmatrix} \tag{C-10}$$

is a matrix of covariances.

The notational extension to other combinations of subscripts is fairly obvious. The λ's are covariances which are easily computed, i.e.,

$$\lambda_{100} = E\left\{\Psi_1{}^2\right\} = \frac{1}{2}\, T_o$$

$$\lambda_{110} = E\left\{\Psi_1 \pi_1(\alpha)\right\}$$

$$= \frac{1}{2}\, \mathrm{Re}\left\{\underline{K}_s{}^*(\bar{u}_\alpha ; \bar{u}_d) \int d\bar{u}_x T(\bar{u}_x - \bar{u}_m) \underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x)\right\}$$

$$\lambda_{010} = E\left\{\pi_1{}^2(\alpha)\right\}$$

$$= \frac{1}{2}\, \mathrm{Re}\left\{\left|\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)\right|^2 \int d\bar{u}_x d\bar{u}_y T(\bar{u}_x - \bar{u}_y) \underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x) \underline{v}_{rs}{}^*(\bar{u}_\alpha ; \bar{u}_y)\right\}$$

$$\lambda_{011} = E\left\{\pi_1(\alpha)\pi_1(\beta)\right\} \tag{C-11}$$

$$= \frac{1}{2}\, \mathrm{Re}\left\{\underline{K}_s{}^*(\bar{u}_\alpha ; \bar{u}_d) \underline{K}_s(\bar{u}_\beta ; \bar{u}_d) \int d\bar{u}_x d\bar{u}_y T(\bar{u}_x - \bar{u}_y) \underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x) \underline{v}_{rs}{}^*(\bar{u}_\beta ; \bar{u}_y)\right\}$$

$$\lambda_{120} = E\left\{\Psi_1 \pi_2(\alpha)\right\}$$

$$= \frac{1}{2}\, \mathrm{Im}\left\{\underline{K}_s{}^*(\bar{u}_\alpha ; \bar{u}_d) \int d\bar{u}_x T(\bar{u}_x - \bar{u}_m) \underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x)\right\}$$

Certain equivalences among the convariances are also easily established.

$$
\begin{aligned}
\lambda_{200} &= E\left\{\Psi_2^{\,2}\right\} = \lambda_{100} \\
\lambda_{022} &= E\left\{\pi_2(\alpha)\pi_2(\beta)\right\} = \lambda_{011} \\
\lambda_{020} &= E\left\{\pi_2^{\,2}(\alpha)\right\} = \lambda_{010} \\
\lambda_{210} &= E\left\{\Psi_2\pi_1(\alpha)\right\} = -\lambda_{120} \\
\lambda_{220} &= E\left\{\Psi_2\pi_2(\alpha)\right\} = \lambda_{110}
\end{aligned}
\tag{C-12}
$$

The remaining six covariances of interest $\lambda_{001}, \lambda_{002}, \lambda_{201}, \lambda_{202}, \lambda_{101}, \lambda_{102}$, are obtained by replacing α by β in certain of the other covariances, e.g. λ_{201} is equal to λ_{210} or $-\lambda_{120}$ with α replaced by β.

The process of carrying out the differentiations in Eq. C-8 is tedious but straightforward. The result, after setting $\overline{v} = 0$ is

$$
\begin{aligned}
M_a(\overline{u}_d;\overline{u}_m) &= M_s(\overline{u}_d;\overline{u}_m) + 2\sum_{\alpha,\beta}\sum_{k\neq 0}(-1)^k\Big[2(\lambda_{110}\lambda_{101}+\lambda_{120}\lambda_{102}) \\
&\quad -k^2V^2\lambda_{011}(\lambda_{110}^{\,2}+\lambda_{120}^{\,2})\Big]\, e^{-\frac{k^2V^2}{2}\lambda_{010}} + 2\sum_{\alpha,\beta}\sum_{l\neq 0}(-1)^l \\
&\quad \cdot\Big[2(\lambda_{110}\lambda_{101}+\lambda_{120}\lambda_{102}) - l^2V^2\lambda_{011}(\lambda_{101}^{\,2}+\lambda_{102}^{\,2})\Big]\, e^{-\frac{l^2V^2}{2}\lambda_{001}} \\
&\quad + 2\sum_{\alpha,\beta}\sum_{\substack{k\neq 0\\ l\neq 0}}\frac{(-1)^{k+l}}{kl}\Big[(k\lambda_{110}+l\lambda_{101})^2+(k\lambda_{120}+l\lambda_{102})^2\Big] \\
&\quad \cdot e^{-\frac{V^2}{2}(k^2\lambda_{010}+l^2\lambda_{001}+2kl\lambda_{011})}
\end{aligned}
\tag{C-13}
$$

where

$$M_s(\bar{u}_d;\bar{u}_m) = 4\sum_{\alpha\beta}(\lambda_{110}\lambda_{101}+\lambda_{120}\lambda_{102})$$

$$=\left|\sum_{\alpha}\underline{K}_s^*(\bar{u}_\alpha;\bar{u}_d)\int d\bar{u}_x T(\bar{u}_x-\bar{u}_m)\underline{v}_{rs}(\bar{u}_\alpha;\bar{u}_x)\right|^2 \qquad \text{(C-14)}$$

$$=\left|\int d\bar{u}_x\, T(\bar{u}_x-\bar{u}_m)\,\underline{\Gamma}_s(\bar{u}_d;\bar{u}_x)\right|^2$$

is the "M" function of the corresponding linear (sampled data) radar.

Equation C-13 can be simplified by defining the following complex covariances

$$\underline{\lambda}_{\psi\alpha} = \lambda_{110} - j\lambda_{120} = \frac{1}{2}E\left\{\underline{\psi}(\bar{u}_m)\,\underline{\pi}^*(\bar{u}_\alpha;\bar{u}_d)\right\}$$

$$\underline{\lambda}_{\alpha\beta} = \frac{1}{2}E\left\{\underline{\pi}(\bar{u}_\alpha;\bar{u}_d)\,\underline{\pi}^*(\bar{u}_\beta;\bar{u}_d)\right\} \qquad \text{(C-15)}$$

Note that $\lambda_{011} = \lambda_{022} = \mathcal{R}e\left\{\underline{\lambda}_{\alpha\beta}\right\}$ and that $\underline{\lambda}_{\alpha\alpha} = \lambda_{\alpha\alpha}$ is real and $\lambda_{010} = \lambda_{\alpha\alpha}$. Using these definitions and rearranging some of the sums in Eq. (C-13) gives

$$M_a(\bar{u}_d;\bar{u}_m) = M_s(\bar{u}_d;\bar{u}_m) \qquad \text{(C-16)}$$

$$+\,8\sum_{\alpha,\beta}\mathcal{R}e\left\{\underline{\lambda}_{\psi\alpha}\,\underline{\lambda}_{\psi\beta}^*\right\}\sum_{k=1}^{\infty}(-1)^k\left[e^{-\frac{k^2V^2}{2}\lambda_{\alpha\alpha}}+e^{-\frac{k^2V^2}{2}\lambda_{\beta\beta}}\right]$$

$$-\,4V^2\sum_{\alpha,\beta}\mathcal{R}e\left\{\underline{\lambda}_{\alpha\beta}\right\}\sum_{k=1}^{\infty}(-1)^k k^2\left[\left|\underline{\lambda}_{\psi\alpha}\right|^2 e^{-\frac{k^2V^2}{2}\lambda_{\alpha\alpha}}+\left|\underline{\lambda}_{\psi\beta}\right|^2 e^{-\frac{k^2V^2}{2}\lambda_{\beta\beta}}\right]$$

(Eq. C-16 continued on next page)

$$+ 4 \sum_{\alpha, \beta} \sum_{k=l=1}^{\infty} \frac{(-1)^{k+l}}{kl} \Big[\left| k\underline{\lambda}_{\Psi\alpha} + l\underline{\lambda}_{\Psi\beta} \right|^2 e^{-\frac{V^2}{2}\left(k^2\lambda_{\alpha\alpha} + l^2\lambda_{\beta\beta} + kl\,\mathrm{Re}\{\underline{\lambda}_{\alpha\beta}\}\right)}$$

$$- \left| k\underline{\lambda}_{\Psi\alpha} - l\underline{\lambda}_{\Psi\beta} \right|^2 e^{-\frac{V^2}{2}\left(k^2\lambda_{\alpha\alpha} + l^2\lambda_{\beta\beta} - 2kl\,\mathrm{Re}\{\underline{\lambda}_{\Psi\alpha}\}\right)} \Big] \qquad \text{(C-16) cont'd.}$$

This is the general result, applicable for any "test" ensemble of $\underline{\Psi}(\bar{u}_x)$ functions. The complex covariance $\underline{\lambda}_{\Psi\alpha}$ and $\underline{\lambda}_{\Psi\beta}$ are easily computed to be

$$\underline{\lambda}_{\Psi\alpha} = \frac{1}{2} \underline{K}_s(\bar{u}_\alpha ; \bar{u}_d) \int d\bar{u}_x \, T(\bar{u}_x - \bar{u}_m) \underline{v}_{rs}^*(\bar{u}_\alpha ; \bar{u}_x)$$

$$\underline{\lambda}_{\alpha\beta} = \frac{1}{2} \underline{K}_s^*(\bar{u}_\alpha ; \bar{u}_d) \underline{K}_s(\bar{u}_\beta ; \bar{u}_d) \int d\bar{u}_x d\bar{u}_y \, T(\bar{u}_x - \bar{u}_y) \qquad \text{(C-17)}$$

$$\cdot \, \underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x) \underline{v}_{rs}^*(\bar{u}_\alpha ; \bar{u}_y)$$

Now specializing to the infinite bandwidth ensemble of $\underline{\Psi}(\bar{u}_x)$ functions, we have

$$T(\bar{u}_x - \bar{u}_y) = 2 A_n \, \delta(\bar{u}_x - \bar{u}_y) \qquad \text{(C-18)}$$

and several simplifications immediately result.

$$\underline{\lambda}_{\Psi\alpha} = A_n \underline{K}_s(\bar{u}_\alpha ; \bar{u}_d) \underline{v}_{rs}^*(\bar{u}_\alpha ; \bar{u}_m)$$

$$\underline{\lambda}_{\alpha\beta} = \frac{1}{A_n} \int d\bar{u}_m \, \underline{\lambda}^*_{\Psi\alpha} \, \underline{\lambda}_{\Psi\beta} \qquad \text{(C-19)}$$

$$k^2\lambda_{\alpha\alpha} + l^2\lambda_{\beta\beta} \pm 2kl\,\mathrm{Re}\{\underline{\lambda}_{\alpha\beta}\} = \frac{1}{A_n} \int \left| k\underline{\lambda}_{\Psi\alpha} \pm l\underline{\lambda}_{\alpha\beta} \right|^2 d\bar{u}_m$$

Then the result for M_a can be written as

$$
\begin{aligned}
M_a(\bar{u}_d;\bar{u}_m) = {} & 4A_n^2\left|\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m)\right|^2 \\
& + 16\sum_{\alpha,\beta} \mathcal{R}e\left\{\underline{\lambda}_{\psi\alpha}\underline{\lambda}_{\psi\beta}^*\right\}\sum_{k=1}^{\infty}(-1)^k\, e^{-\frac{k^2V^2}{2A_n}\int\left|\underline{\lambda}_{\psi\alpha}\right|^2 d\bar{u}_m} \\
& - 8\,\frac{V^2}{A_n}\sum_{\alpha,\beta}\mathcal{R}e\left\{\int\underline{\lambda}_{\psi\alpha}^*\,\underline{\lambda}_{\psi\beta}\,d\bar{u}_m\right\}\sum_{k=1}^{\infty}(-1)^k k^2\left|\underline{\lambda}_{\psi\alpha}\right|^2 e^{-\frac{k^2V^2}{2A_n}\int\left|\lambda_{\psi\alpha}\right|^2 d\bar{u}_m} \\
& + 4\sum_{\alpha,\beta}\sum_{\substack{k=1\\l=1}}^{\infty}\frac{(-1)^{k+l}}{kl}\left|k\underline{\lambda}_{\psi\alpha}+l\underline{\lambda}_{\psi\beta}\right|^2 e^{-\frac{V^2}{2A_n}\int\left|k\underline{\lambda}_{\psi\alpha}+l\underline{\lambda}_{\psi\beta}\right|^2 d\bar{u}_m} \\
& - 4\sum_{\alpha,\beta}\sum_{\substack{k=1\\l=1}}^{\infty}\frac{(-1)^{k+l}}{kl}\left|k\underline{\lambda}_{\psi\alpha}-l\underline{\lambda}_{\psi\beta}\right|^2 e^{-\frac{V^2}{2A_n}\int\left|k\underline{\lambda}_{\psi\alpha}-l\underline{\lambda}_{\psi\beta}\right|^2 d\bar{u}_m}
\end{aligned}
\tag{C-20}
$$

which is of the form of Eq. 36 of Section IV. The "correction" terms to the linear-system-spread-function part of M_a clearly tend to zero as $q(V = 2\pi/q)$ becomes small. This expression for M_a could easily be evaluated on a computer, given $\underline{\lambda}_{\psi\alpha}$ (Eq. C-19). Because of the strong exponential dependence on $(k/q)^2$, the sums on k (and l) should converge very rapidly for reasonably small values of q (i.e. $q < 3$ standard deviations of quantizer input). However, it si not possible to deduce any analytical bounds or qualitative insights on quantization effects from Eq. C-20. For this purpose we have to introduce an approximate treatment of M_a which will be applicable in most cases of practical interest.

We first define the "inverse matched smoothing function"

$$\underline{L}_m(\bar{u}_\alpha;\bar{u}_\beta) = \int d\bar{u}_x \underline{v}_r^*(\bar{u}_\alpha;\bar{u}_x)\underline{v}_r(\bar{u}_\beta;\bar{u}_x) \tag{C-21}$$

which may be compared to the conventional matched smoothing function

$$\underline{\Gamma}_m(\bar{u}_d;\bar{u}_m) = \int d\bar{u}_\alpha \underline{v}_r^*(\bar{u}_\alpha;\bar{u}_d)\underline{v}_r(\bar{u}_\alpha;\bar{u}_m) \tag{C-22}$$

If $\underline{v}_r(\bar{u}_\alpha;\bar{u}_m)$ depends only on coordinate differences (between $\bar{u}_\alpha$ and $\bar{u}_m$) then it is possible to show that

$$\underline{L}_m(\bar{u}_\alpha;\bar{u}_\beta) = \underline{\Gamma}_m(\bar{u}_d = -\bar{u}_\alpha;\bar{u}_m = -\bar{u}_\beta) \tag{C-23}$$

Even if $\underline{v}_r(\bar{u}_\alpha;\bar{u}_m)$ is a more complicated function of $\bar{u}_\alpha$ and $\bar{u}_m$, it is clear that $\underline{\Gamma}_m$ and $\underline{L}_m$ will have the same general character, i.e. a narrow peak near $\bar{u}_d = \bar{u}_m$ or $\bar{u}_\beta = \bar{u}_\alpha$ respectively and much smaller values elsewhere. The value of $\underline{L}_m$ at $\bar{u}_\alpha = \bar{u}_\beta$ is given a special symbol, $\mathcal{E}$,

$$\mathcal{E} = \underline{L}_m(\bar{u}_\alpha;\bar{u}_\alpha) = \int d\bar{u}_x \left| \underline{v}_r(\bar{u}_\alpha;\bar{u}_x)\right|^2 \tag{C-24}$$

which is real and proportional to the total energy associated with the signal sample at $\bar{u}_\alpha$ due to contributions from all of scatterer space. In a well designed radar, $\mathcal{E}$ will be essentially independent of $\bar{u}_\alpha$ for all $\bar{u}_\alpha$ points of interest, i.e, those $\bar{u}_\alpha$ points where $\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)$ is non-zero for some $\bar{u}_d$ point of interest.

Define A_L as the area inside the closed curve in $\bar{u}_\beta$ where $\left|\underline{L}_m(\bar{u}_\alpha ; \bar{u}_\beta)\right|^2$ has fallen to one half its value at it maximum (at the point $\bar{u}_\beta = \bar{u}_\alpha$). We assume a reasonably large area compression ratio so that A_L is a small fraction of the total aperture area, A_a, in signal space, (i.e. A_a is the area where $\left|\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)\right|$ is non-zero), say $A_a/A_L > 10$. Then it is reasonable to approximate $\left|\underline{L}_m(\bar{u}_\alpha ; \bar{u}_\beta)\right|$ in the correction terms in Eq. C-20 by a cylinder of height $\mathcal{E}$ and base A_L as a function of $\bar{u}_\beta$ for fixed $\bar{u}_\alpha$, i.e.

$$\left|\underline{L}_m(\bar{u}_\alpha ; \bar{u}_\beta)\right| = \begin{cases} \mathcal{E} & ; \bar{u}_\beta \in A_L(\bar{u}_\alpha) \\ 0 & ; \bar{u}_\beta \notin A_L(\bar{u}_\alpha) \end{cases} \tag{C-25}$$

when $A_L(\bar{u}_\alpha)$ is the area A_L defined above centered on a particular $\bar{u}_\alpha$. Also we can assume with relatively small error

$$\left|\underline{K}_s(\bar{u}_\beta ; \bar{u}_d)\right| \approx \left|\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)\right| \; ; \; \bar{u}_\beta \in A_L(\bar{u}_\alpha) \tag{C-26}$$

since the magnitude of the weighting function will vary relatively slowly over signal space.

The approximation

$$\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)\underline{K}_s^*(\bar{u}_\beta ; \bar{u}_d)\ \underline{L}_m(\bar{u}_\alpha ; \bar{u}_\beta) \approx |\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)|^2 ; \bar{u}_\beta \in A_L(\bar{u}_\alpha) \quad \text{(C-27)}$$

can be made because the first order phase variations, if any, in the $\bar{u}_\alpha ; \bar{u}_\beta$, and $\bar{u}_d$ coordinates will cancel. (Recall that the phase of $\underline{K}_s$ is presumed matched to the phase of $\underline{v}_{rs}$.) Using these approximations we can simplify the integral appearing in the last two terms of Eq. C-20 as follows:

$$\frac{1}{A_n}\int |k\underline{\lambda}_{\psi\alpha} \pm l\underline{\lambda}_{\psi\beta}|^2 \, d\bar{u}_m \approx \begin{cases} (k \pm l)^2 \ |\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)|^2 \mathcal{E}\ A_n ; \bar{u}_\beta \in A_L(\bar{u}_d) \\ k^2 \,|\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)|^2 \mathcal{E}\ A_n + l^2 \,|\underline{K}_s(\bar{u}_\beta ; \bar{u}_d)|^2 \\ \cdot\, \mathcal{E}\ A_n ; \bar{u}_\beta \notin A_L(\bar{u}_\alpha) \end{cases} \quad \text{(C-28)}$$

The integrals appearing in the second term become

$$\frac{1}{A_n}\int \underline{\lambda}_{\psi\alpha}{}^* \underline{\lambda}_{\psi\beta}\, d\bar{u}_m \approx \begin{cases} A_n \quad |\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)|^2 ; \bar{u}_\beta \in A_L(\bar{u}_d) \\ 0 \qquad\qquad ; \bar{u}_\beta \notin A_L(\bar{u}_d) \end{cases} \quad \text{(C-29)}$$

and

$$\frac{1}{A_n}\int |\underline{\lambda}_{\psi\alpha}|^2 \, du_m = A_n \mathcal{E}\, |\underline{K}_s(\bar{u}_\alpha ; \bar{u}_d)|^2 \quad \text{(C-30)}$$

The double sums on α, and β in the last two terms of Eq. C-20 can be now split into two parts corresponding to $\bar{u}_\beta \in A_L(\bar{u}_\alpha)$ and $\bar{u}_\beta \notin A_L(\bar{u}_\alpha)$. Those for $\bar{u}_\beta \notin A_L(\bar{u}_\alpha)$ recombine with some manipulation to give

$$16\,\mathcal{R}e\left\{ \sum_{\alpha} \underline{\lambda}_{\psi\alpha} w_1'(\bar{u}_\alpha;\bar{u}_d) \sum_{\substack{\beta | \\ \bar{u}_\beta \notin A_L(\bar{u}_\alpha)}} \underline{\lambda}_{\psi\beta}^* w_1'(\bar{u}_\beta;\bar{u}_d) \right\}$$

where

$$w_1'(\bar{u}_\alpha;\bar{u}_d) = \sum_{k=1}^{\infty} (-1)^k e^{-\frac{k^2V^2A_n\epsilon}{2}\left|\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\right|^2} \tag{C-31}$$

Since A_L is small compared to A_a, the result will not be seriously affected if we include the terms for $\bar{u}_\beta \in A_L(\bar{u}_\alpha)$ in the second summation above, making the result

$$16\,A_n^2 \left| \sum_{\alpha} v_{rs}(\bar{u}_\alpha;\bar{u}_m)\underline{K}_s(\bar{u}_\alpha;\bar{u}_d) w_1'(\bar{u}_\alpha;\bar{u}_d)\right|^2$$

This will combine with the second term of Eq. C-20, which, with the help of Eq. C-30, can be written as

$$16\,A_n^2\,\mathcal{R}e\left\{\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m) \sum_{\alpha} \underline{v}_{rs}(\bar{u}_d;\bar{u}_m)\underline{K}_s(\bar{u}_\alpha;\bar{u}_d) w_1'(\bar{u}_\alpha;\bar{u}_d)\right\}$$

and the first term to give

$$4A_n^{\,2}\left|\underline{\Gamma}_a(\bar{u}_d;\bar{u}_m)\right|^2$$

where

$$\underline{\Gamma}_a(\bar{u}_d;\bar{u}_m) = \sum_{\alpha} \underline{v}_{rs}(\bar{u}_\alpha;\bar{u}_m)\underline{K}_m(\bar{u}_\alpha;\bar{u}_d)w_1(\bar{u}_\alpha;\bar{u}_d) \tag{C-32}$$

and

$$w_1(\bar{u}_\alpha;\bar{u}_d) = 1 + 2w_1'(\bar{u}_\alpha;\bar{u}_d) \tag{C-33}$$

There remains the third term in Eq. C-20 and the parts of the last two terms in which the sums over β are restricted to $\bar{u}_\beta \in A_L(\bar{u}_\alpha)$. Using Eq. C-29 the second term in Eq. C-20 simplifies very easily but for the rest it is necessary to make the approximation

$$\underline{\lambda}_{\gamma\beta} = A_n\underline{K}_s(\bar{u}_\beta;\bar{u}_d)\underline{v}_{rs}{}^*(\bar{u}_\beta;\bar{u}_m) \approx A_n\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\underline{v}_{rs}{}^*(\bar{u}_\alpha;\bar{u}_m) = \underline{\lambda}_{\gamma\alpha} \tag{C-34}$$

$$;\ \bar{u}_\beta \in A_L(\bar{u}_\alpha)$$

Again first order phase variations in $\bar{u}_\beta$ and $\bar{u}_\alpha$ cancel, but since $\left|\bar{u}_d - \bar{u}_m\right|$ can become fairly large the approximation is really only valid near $\bar{u}_d = \bar{u}_m$. It is conservative, however, since it maximizes the contributions of the last two terms in Eq. C-20, and in any event these last two terms are less important in the final result than the contribution of the third term in Eq. C-20. The final result of all these approximations is the following equation for M_a.

$$M_a(\bar{u}_d;\bar{u}_m) \approx 4A_n{}^2 \left|\underline{\Gamma}_a(\bar{u}_d;\bar{u}_m)\right|^2$$

$$+ 4A_n{}^2 \frac{A_L}{A_s} \sum_a \left|v_{rs}(\bar{u}_a;\bar{u}_m)\right|^2 \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2 \quad \text{(C-35)}$$

$$\cdot\, w_2(\bar{u}_a;\bar{u}_d)$$

where

$$w_2(\bar{u}_a;\bar{u}_d) = \sum_{k,\,l=1}^{\infty} (-1)^{k+1} \frac{(k+1)^2}{kl}\, e^{-\frac{(k+1)^2 V^2}{2}\, \mathcal{E}\, A_n \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2}$$

$$- \sum_{k,\,l=1}^{\infty} (-1)^{k+1} \frac{(k-1)^2}{kl}\, e^{-\frac{(k-1)^2 V^2}{2}\, \mathcal{E}\, A_n \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2} \quad \text{(C-36)}$$

$$- 2V^2 A_n \mathcal{E} \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2 \sum_{k=1}^{\infty} (-1)^k k^2\, e^{\frac{k^2 V^2}{2}\, \mathcal{E}\, A_n \left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2}$$

The factor (A_L/A_s) is just the number of terms in the sums over β which are restricted to $\bar{u}_\beta \in A_L(\bar{u}_a)$. (Recall that A_s is the sampling area in signal space.) Equation C-35, with Eqs. C-32, 33, 36, to define the terms, is the result sought and is discussed in Section IV (Eq. 39).

We have to repeat the process for M_b given by Eq. 46 of Section IV which is repeated as follows:

$$M_b(\bar{u}_d;\bar{u}_m) = \sum_{\alpha,\beta} \underline{K}_s{}^*(\bar{u}_\alpha;\bar{u}_d)\underline{K}_s(\bar{u}_\beta;\bar{u}_d)E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\underline{V}_{rq}(\bar{u}_\alpha)\underline{V}_{rq}{}^*(\bar{u}_\beta)\right\}$$
$$-E\left\{\left|\underline{\Psi}(\bar{u}_m)\right|^2\right\}\sum_{\alpha,\beta}\underline{K}_s{}^*(\bar{u}_\alpha;\bar{u}_d)\underline{K}_s(\bar{u}_\beta;\bar{u}_d)E\left\{\underline{V}_{rq}(\bar{u}_\alpha)\underline{V}_{rq}{}^*(\bar{u}_\beta)\right\} \tag{C-46}$$

Comparing Eq. 46 and Eq. 34, shows that the results for M_b can be obtained from that for M_a by substituting $\underline{v}_{rs}$ for the product $\underline{K}_s{}^*\underline{v}_{rs}$ and introducing the factor $\underline{K}_s{}^*$ after each summation on α (or β). The principle result of interest is that for the infinite bandwidth $\Psi(\bar{u}_x)$ ensemble which is (using Eq. C-20)

$$M_b(\bar{u}_d;\bar{u}_m) = 4A_n{}^2\left|\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m)\right|^2 + 16A_n{}^2\mathrm{Re}\Bigg\{$$
$$\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m)\sum_{\alpha}\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\underline{v}_{rs}{}^*(\bar{u}_\alpha;\bar{u}_m)\sum_{k=1}^{\infty}(-1)^k e^{-\frac{k^2V^2}{2}}A_n\underline{L}_m(\bar{u}_\alpha;\bar{u}_\alpha)\Bigg\}$$
$$-4A_n{}^2\mathrm{Re}\Bigg\{\sum_{\alpha,\beta}\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\underline{K}_s{}^*(\bar{u}_\beta;\bar{u}_d)\underline{L}_m(\bar{u}_\alpha;\bar{u}_\beta)$$
$$\sum_{k=1}^{\infty}(-1)^k k^2 A_n\left|\underline{v}_{rs}(\bar{u}_\alpha;\bar{u}_m)\right|^2 e^{-\frac{k^2V^2}{2}}A_n\underline{L}_m(\bar{u}_\alpha;\bar{u}_\alpha)\Bigg\} \tag{C-37}$$
$$+4A_n{}^2\sum_{\alpha,\beta}\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\underline{K}_s{}^*(\bar{u}_\beta;\bar{u}_d)\,Z_{kl}^{+}(\bar{u}_\alpha;\bar{u}_\beta;\bar{u}_m)$$
$$-4A_n{}^2\sum_{\alpha,\beta}\underline{K}_s(\bar{u}_\alpha;\bar{u}_d)\underline{K}_s{}^*(\bar{u}_\beta;\bar{u}_d)\,Z_{kl}^{-}(\bar{u}_\alpha;\bar{u}_\beta;\bar{u}_m)$$

where

$$Z^{\pm}_{kl}(\bar{u}_\alpha ; \bar{u}_\beta ; \bar{u}_m) = \sum_{k=l=1}^{\infty} \frac{(-1)^{k+l}}{kl} \left| k\underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_m) \pm l\underline{v}_{rs}(\bar{u}_\beta ; \bar{u}_m)\right|^2 \cdot e^{-\frac{V^2}{2} A_n \int \left| k\underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x) + l\underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_x)\right|^2 d\bar{u}_x} \tag{C-38}$$

Essentially the same approximation procedure is applicable to M_b as was used for M_a. However, it is now necessary to make the approximation

$$\underline{K}_s(\bar{u}_\beta ; \bar{u}_d) \approx \underline{K}_s(\bar{u}_\alpha ; \bar{u}_d) \; ; \quad \bar{u}_\beta \in A_L(\bar{u}_\alpha) \tag{C-39}$$

and

$$\underline{v}_{rs}(\bar{u}_\beta ; \bar{u}_m) \approx \underline{v}_{rs}(\bar{u}_\alpha ; \bar{u}_m) \; ; \; \bar{u}_\beta \in A_L(\bar{u}_\alpha) \tag{C-40}$$

in place of that given by Eq. C-34. This is a more stringent approximation because it essentially requires that $\underline{v}_{rs}$ and $\underline{K}_s$ individually have no first order phase variations with $\bar{u}_\alpha$. This in turn limits the validity of the approximation to the case where the signals and weighting functions are translated to zero or near zero average frequency over every coordinate of signal space. However, this is also the most probable design for a digitally implemented system. The approximation given in Eq. C-27 remains valid in any case and the additional approximations of

Eqs. C-39 and C-40 are only required in evaluating the contributions of the last two terms of Eq. C-37. Again the approximations lead to conservative results and affect only the least important point of the final result, with is

$$M_b(\bar{u}_d;\bar{u}_m) \approx 4A_n^{\,2}\left|\underline{\Gamma}_s(\bar{u}_d;\bar{u}_m)\right|^2 w_3^{\,2} + 4A_n^{\,2}\,\frac{A_L}{A_s}\,w_4\sum_a \left|\underline{v}_{rs}(\bar{u}_a;\bar{u}_d)\right|^2\left|\underline{K}_s(\bar{u}_a;\bar{u}_d)\right|^2 \tag{C-41}$$

where

$$w_3 = 1 + 2\sum_{k=1}^{\infty} e^{-\frac{k^2V^2}{2}A_n} .$$

and (C-42)

$$w_4 = \sum_{k=l=1}^{\infty}(-1)^{k+l}\,\frac{(k+l)^2}{kl}\,e^{-\frac{(k+l)^2V^2}{2}A_n\varepsilon} - \sum_{k=l=1}^{\infty}(-1)^{k+l}\,\frac{(k-l)^2}{kl}\,e^{-\frac{(k-l)^2V^2}{2}A_n\varepsilon} - 2V^2A_n\varepsilon\sum_{k=1}^{\infty}(-1)^k k^2\,e^{-\frac{k^2V^2}{2}A_n\varepsilon} \tag{C-43}$$

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BIOGRAPHICAL NOTE

The author, Robert W. Roig, was born on 22 February 1929 in Brooklyn, New York. He received the S. B. Degree in 1950, the S. M. Degree in 1959 and the E. E. Degree in 1964 all from the Massachusetts Institute of Technology. From 1951 to 1962 he was on active duty with the U. S. Air Force. From 1962 until 1966 he was associated with the M. I. T. Electronic Systems Laboratory. During the 1966 to 1967 Academic year, when the majority of the research for this thesis was accomplished, he held a National Science Foundation Fellowship. He is a member of Sigma Xi, IEEE, and the American Association for the Advancement of Science.