BREAKING THE LAW:

Competing Perspectives in Children's Thinking about the Balance Scale

by

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ABSTRACT

As elementary school age children interact with and try to understand the functioning of a balance scale, they develop different notions of how the scale works. They may develop some incorrect notions, they may develop some partially correct notions, and they may even develop "the law for balancing": the scale balances *if and only if* the sum of the products of weight and distance is the same on each arm.

The two major bodies of research on children's development and use of "the law" embody perspectives compatible with a monolithic kind of "law." To Inhelder and Piaget and to Siegler, "the law" reflects a certain level of understanding. Once this level is reached, children use "the law" consistently.

In this thesis, I focus on *inconsistencies* in children's use of "the law." I explore *when* and *how* fourth and fifth grade children use "the law" after first developing it. Do children always use "the law" once they develop it? Do they see "the law" as something that always governs the balance scale? Do competing approaches ever emerge? Are there particular sorts of situations which seem to trigger doubts about or inability to use "the law"? In exploring these questions I consider the nature of "the law" that children develop, the contexts in which children find "the law" relevant, and children's conceptions of "the law."

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Introduction

A fifth grade boy and I sit side by side in front of a table with a balance scale and a pile of metal rings on it. I reach for one end of the scale and hold it steady so the scale cannot tip. I place three rings on an arm and ask, "Can you use one ring to balance the scale?" The boy looks at the scale intently for several seconds and then, smiling and animated, turns to me and says, "It goes on the seventh peg. You put two on the third and one on the first, that's two times three, six, plus one, seven!" He picks up a metal ring from the pile on the table and places it on the seventh peg of the empty arm. He turns to me again, laughs, and says "That was easy! I figured it out the same way as last time. I *know* it will balance when you let go!" He turns back to look at the balance scale again. Suddenly his smile vanishes. "It's not going to work," he says. "Why not?" I ask. "More rings on one side." "But you've balanced the scale lots of times with more rings on one side." "It's impossible ... I don't understand."

At first the boy appears to be using "the law for balancing": the scale balances *if and only if* the sum of the products of weight and distance is the same on each arm (in this case, $2*3 + 1*1 = 1*7$). But, if he is using "the law for balancing" why does he suddenly say the scale will tip to the arm with more rings? Does using "the law" imply understanding that it holds no matter how many rings are on an arm? Or, can one use "the law" without a thorough understanding of when and how it applies?

This anecdote suggests that there are different kinds of "knowing the law for balancing." The boy initially seems to know "the law," but then he comes to question it. Why does he change his mind? Nothing in the *external* context has changed: the same configuration remains on the scale, I continue to hold the scale steady, and I do not make any further comments. Something *internal* seems to have changed. Looking at the balance scale again seems to lead him to doubt that the scale can balance when more rings are on one arm.

In this thesis I look at the contextualization of children's "knowing the law for balancing." I base my discussion on a study of fourth and fifth grade children who, when interacting with and trying to understand the functioning of the balance scale, appear to begin using "the law." In exploring when and how these children use "the law," I consider the nature of "the law" that they develop, the contexts in which they find "the law" relevant, and their conceptions of "the law."

In Chapter 1 I consider the background of the present study. In Chapter 2 I cover my research methods. In Chapter 3 I review the various approaches to the balance scale that children took over the course of the experimental session, and in Chapter 4 I detail the approaches children took after first using "the law." I conclude by comparing my perspectives on children's use of "the law" to perspectives inherent in previous studies. I include a section of a protocol of a typical experimental session in Appendix D.

Chapter One

Background of the present study

1.1 Overview

As elementary school age children interact with and try to understand the functioning of a balance scale, they develop different notions of how the scale works. They may develop some incorrect notions, they may develop some partially correct notions, and they may even develop "the law for balancing":[1] the scale balances *if and only if* the sum of the products of weight and distance is the same on each arm.

The present study explores *when* and *how* fourth and fifth grade children use "the law" after first developing it. Do children always use it once they develop it? Do they see "the law" as something that always governs the balance scale? Do competing approaches ever emerge? Are there particular sorts of situations which seem to trigger doubts about or inability to use "the law"?

Although the two major bodies of research on children's development and use of "the law" do not focus on these questions, they embody perspectives compatible with a monolithic kind of "law." To both Inhelder and Piaget [Inhelder and Piaget 58] and Siegler [Siegler 78, Siegler 81, Siegler and Klahr 82, Klahr and Siegler 78], development of "the law" reflects a certain level of understanding. Once this level is reached, children use "the law" consistently.

Interest in broad developmental consistencies led Inhelder and Piaget to use the

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[1] Throughout this thesis I put *the law for balancing* (or *the law*) in quotations to emphasize the fact that I am questioning the nature of children's understanding of this law.

balance scale to study the development of the concept of proportionality. Clinical interviews led them to postulate several stages in children's understanding of the balance scale (see Table 1 for a summary of these stages). They relate these stages to broader developmental stage. Understanding of "the law" develops with the onset of formal operations, at which point children are able to think of the balance scale in terms of a system of logical operations, dissociated from content area.

A view of the development of cognition as the use of increasingly complex rules led Siegler to a series of balance scale studies. His studies focus on the kinds of rules children use in making predictions about the balance scale and the factors that might influence progression from one rule to the next. He determined that the knowledge children have about the balance scale could be represented as one of four developmentally sequenced rules (see Table 2 for a summary of these rules). To assess rule use, he presented children with series of configurations of weights on a balance scale held steady, and asked them to predict which arm, if either, would fall. Children using the most advanced rule can determine the scale's stability by computing the sum of the products of weight and distance for each arm.

In this thesis, I focus neither on consistencies in children's use of "the law," nor on mechanisms of progression from one broad level of balance scale understanding to another. Instead, I focus on *when* children *do and do not* use the law once they have developed it. I explore the nature of "the law" that children develop, the contexts in which children find "the law" relevant, and children's conceptions of "the law."

1.2 Approach to the knowledge structure

Fundamental to my study is a particular "modular" conception of the knowledge structure and developmental process (e.g. [Minsky 86, diSessa 83, Papert 80, Lawler 85]). Essentially, the knowledge structure consists of interacting bodies of knowledge, and development proceeds by the reorganization of knowledge as well as the acquisition of new knowledge.

This modular approach offers a perspective on the kinds of local processes that may underlie broader developmental change. For example, Minsky [Minsky 86] and Papert [Papert 80] postulate a "society of mind" in which knowledge takes the form of interacting mental agents. In the course of development, "administrative agents" may emerge to organize other agents in new ways. Different "administrations" of a group of agents may result in very different patterns of behavior, perhaps even patterns suggestive of different Piagetian stages ([Papert 80], pp. 167-169, [Minsky 86], pp. 101-102). At a more domain-specific level, Lawler [Lawler 85] shows both the acquisition of new knowledge and the hierarchical reorganization of knowledge in the genesis of arithmetical knowledge. He details the kinds of experiences that led to his six year old daughter's various arithmetical *microviews* (cognitive structures built through particular classes of experience). Each of her microviews was based on what *she experienced* as an independent microcosm of arithmetic, for example, counting, money, decadal calculations, and paper sums. Finding connections among these microviews involved the emergence of new *executive* microviews, which served to subordinate the others.

This modular approach suggests a way in which "the law for balancing" might emerge independently of a leap to the next stage or rule in a strict developmental sequence. Certain sorts of experience may lead to the acquisition of "the law" as a new piece of knowledge. Eventually, this new piece of knowledge may grow to control or subsume other knowledge about the balance scale.

diSessa [diSessa 83, diSessa 85, diSessa 82] offers a way of thinking about how this more gradual kind of transition may take place. His studies of physics students lead him to postulate components of intuitive knowledge about physics and mechanisms for the reorganization of these components. *Phenomenological primitives*, or *p-prims*, are "phenomena in terms of which [physics-naive students] see the world and sometimes explain it ([diSessa 83], p. 16)." For example, a student might think of the transfer of energy when a bouncing ball collides with the floor in terms of the "squishiness" of the ball. P-prims originate with the naive physicist's interpretations of her interactions with the physical world. Two qualities of p-prims determine when they will be invoked to make sense of a given situation: *cuing priority*, the rapidity with which a given p-prim will be called up in a particular context, and *reliability priority*, the degree to which a given p-prim resists abandonment once called up in a particular context.

Over the course of the development of physics understanding, these priorities change. In particular, p-prims that are compatible with the laws of physics may gain status and others may lose status. Two kinds of changes accompany gains in status: p-prims may become more general, such that they apply to a wider set of contexts, and p-prims may cease to be self-contained explanations.

Thus, to diSessa, knowledge need not assume an "all or none" character. As cuing and reliability priorities of a p-prim increase in a wider range of contexts, the p-prim itself may become more general. However, even when the p-prim has become quite general and widely applicable, there still may exist certain contexts in which it is not seen as particularly reliable, or it is not even cued.

1.3 The pilot study

Before beginning the present study, I conducted a study of ten children, aged six to twelve, using and trying to understand the functioning of the balance scale [Kliman 87]. This pilot study provided a sense of the patterns that characterize children's interactions with the balance scale, in particular, the kinds of approaches children take in exploring the balance scale, and the context-specific, tenuous, and sometimes contradictory nature of these approaches. The pilot study also provided a sense of the kinds of situations that lead to the development of more successful and more widely applicable approaches to the balance scale.

My purpose in conducting this study was to explore from a microgenetic perspective a domain bearing on mathematical thinking and education. I chose the balance scale because of its potential for establishing an autonomous problem-solving environment: elementary school children can use it without special prerequisite skills or knowledge, it gives feedback, and it bears an intrinsic problem. In experimental sessions, children explored the balance scale on their own as I occasionally probed them with specific questions and problems. Children used the balance scale for an average of two sessions, each approximately one hour long.

My findings include a collection of strategies, discoveries, misconceptions, conflicts, and special-case pieces of knowledge that emerged over the course of experimental sessions. Most of these fall into one of three groups according to the theme that seems best to characterize them: symmetry, inverse correspondence of weight and distance, and arithmetic. A symmetric approach, for example, might involve placing the same number of weights at corresponding distances on each arm, an approach based on inverse correspondence of weight and distance might involve placing a small number of weights near the end of one arm and several weights close to the fulcrum on the other, and an arithmetic approach might involve using special-case arithmetic relationships that hold when the scale balances, such as "the scale balances when one weight is on one arm and two are at half that distance on the other arm." Only one child appeared to develop "the law."

Over the course of individual sessions and with increasing age, children tended to use arithmetic more frequently. However, they nearly always retained several fairly distinct and sometimes contradictory approaches to the balance scale. They did not seem to seek or expect to find one pervasive arithmetical law governing balance scale stability. They appeared, rather, to see the balance scale as an ever-changing collection of different contexts with different ways of balancing it in each. In the present study, I explore the extent to which these patterns persist as children develop "the law."

Chapter Two

Method

2.1 Components of context

In exploring the contextualization of children's "knowing the law for balancing," I take a broad view of the notion of *context*. I consider as context anything that might bear on children's use and understanding of "the law." One kind of contribution to context is objective: factors that are readily observable. For example, at any given point in a session, context might be created in part by the particular configuration on the balance scale, by a balance scale problem I posed that the child is trying to solve, and by the "history" of balance scale problems, questions, and configurations that the has child encountered so far. In addition, factors less readily observable (and perhaps less readily characterized) contribute to an "internal" mental context that can change independently of the objective "external context." For example, the child in the anecdote in the Introduction first appeared to be using "the law," but after taking another look at a configuration of weights on the balance scale doubted his "law." Nothing external changed, but *something* must have led him to change his mind. His taking another look seemed to trigger some notion that put "the law" into question.

In this thesis I focus on external context. Although I make no attempt to identify all the factors that might contribute to establishing context, I try to be sensitive to the multifaceted nature of context and the fact that some situations -- like the one described in the Introduction -- cannot be explained by changes in external context.

In this chapter I discuss the major external contexts children encountered in the balance scale session. My discussion occasionally touches on internal context, for example when I consider possible consequences of children's not wanting to talk about what they are doing.

2.2 Experimental environment

2.2.1 Apparatus

I use a wooden balance scale and metal rings of uniform size, shape, and weight, which can be hung from these pegs (Figure 2.1). The balance scale has ten pegs on each arm on each side, and one peg in the center on each side. Pegs are 1.5 inches apart. The beam is 31 inches long and 1.5 inches high, and the base on which it rests is 5 inches high.

On one side the pegs on the balance scale are numbered. The first peg out from the center of the scale is marked "1" on each arm, the second is marked "2", and so forth. The other side of the balance scale is blank. Children in the present study used the numbered side almost exclusively. On several occasions, the balance scale was very briefly turned around so that the blank side was facing.

In illustrations in this thesis, I use a schematic representation of the balance scale and weights (Figure 2.2). To represent a number of rings **N** on a peg **P**, I use a stack of **N** small squares over the number **P**. For example, I represent three rings on the second peg on each arm by a stack of three small squares above each "2" (Figure 2.3).

The scale balances if the sum of the products of weight and distance on each arm, or *torque* on each arm.[2] Otherwise, the scale tips to the arm with greater torque. For example, if one arm has a ring on the first peg and two on the fourth and the other arm has a ring on the ninth peg, the scale balances (Figure 2.4). If one arm has a ring on the first peg and two on the third and the other arm has a ring on the ninth peg, the scale tips to the arm with the ring on the ninth (Figure 2.5).

2.2.2 Setting

Balance scale sessions took place at the Hennigan School, an inner-city, multi-racial Boston public school, during school hours. I worked with each child individually. At the start of the session, I walked with the child from her classroom or from an open computer area to a nearby room where the balance scale and an audiotape recorder were set up on a table. As we walked I did not discuss what we were about to do, but rather, led the child into conversation about her interests and activities. When we entered the room I asked if she minded my audiotaping her (no one minded). I then asked several introductory questions, including age and birth date, if she had ever "seen one of these" (the balance scale) before, and if she knew "what it was for."[3] If she did not at this point begin to use the balance scale spontaneously, I encouraged her to "try it." Sessions lasted an average of forty-five minutes.

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[2] I use the term *torque* for convenience, not to suggest that children necessarily thought of this quantity as *torque*.

[3] Two of the nine children on which I focus in this thesis said they had used a balance scale when they were very young. Another said he had used a pan balance the previous year, and another said he had used a balance scale the previous year. Only the child who had used the balance scale the previous year, Dar, seemed to try to use his previous balance scale experience later in the session. His comments suggest, however, that if he indeed *had* used the same sort of balance scale, he had forgotten how it worked.

Two factors suggest that my subjects may have been less inhibited and more comfortable during the session than children normally are in such situations. Firstly, subjects had all seen me before, and in some cases had interacted with me. As part of an MIT-sponsored project, I was in the building approximately twice a week during the school year (sessions took place at least three months into the year). Although I had not worked closely with any of the subjects previously, they all appeared to recognize me as someone who participated in the MIT project and who had in some cases, observed their math classes or discussed computing activities with them. Secondly, throughout the school year, they were given extensive series of tests in circumstances similar to the balance scale session. I believe the children saw the balance scale session as "just another test."

2.3 Structure of the balance scale session

2.3.1 Leading children to explore the balance scale

A primary concern in designing the balance scale session was a need to lead children to explore the balance scale without imposing an environment that would prevent the development and expression of individual approaches and ideas.

In the pilot study I found that some children readily explored and experimented with the balance scale on their own, but that others did not do so without some prodding. For example, at the start of the session, many children balanced the scale by putting rings at corresponding positions on each arm. Some children then began spontaneously to find non-symmetric configurations that would balance the scale. Others stopped at this point, and a few even said something like, "OK, I've balanced the scale. Now what?" Some of these children may have thought that no other ways of balancing the scale existed, others may have thought that the situation did not require considering the possibility. In order to lead children to further exploration, I would intervene, for example by asking them to find different ways of balancing the scale or by asking them to balance the scale with an odd number of weights.

My intervention in the pilot study was not systematic. Since I usually only presented problems or questions when children seemed unable or unwilling to explore on their own (or unaware that I wanted them to explore on their own) the kinds of balance scale situations children encountered varied considerably. Since children seemed to develop context-specific strategies for balancing the scale, *which* strategies they developed may have to some extent depended on *what* they encountered. In particular, certain kinds of questions and problems may have triggered thinking about the balance scale in new ways, finding relationships among different strategies for balancing the scale, and even coming up with a strategy that seemed to be "the law."

I began the present study by proposing a sequence of several such *Situations*, essentially classes of balance scale questions and problems around which I would loosely structure balance scale sessions. These Situations are based on some of the actual problems posed, questions asked, and spontaneous discoveries made in the pilot study, and on my intuitions about the kinds of questions and problems that would prove thought-provoking to children somewhat older (and perhaps more likely to develop "the law"). I expected that the Situations would present children with balance scale phenomena they could not readily explain and would require them to face conflicts or contradictions in their approaches to the balance scale.

The Situations provide a sort of macrostructure for the balance scale session. They constitute what *I see* as relatively distinct classes of balance scale problems carving the session into relatively distinct chapters. I expect, however, that *children may have seen* the session as more of a continuous series of interactions with the balance scale and with me. Frequently, Situations emerged quite naturally from my interactions with children or from their autonomous explorations. As they worked on the balance scale, I formed "working hypotheses" of what their approaches to and notions of the balance scale -- strategies, conflicts, misconceptions, etc. -- were *at that moment.* I posed questions and problems to try to explore those hypotheses. In doing so I sometimes led them to a new Situation. Likewise, children often experimented with the balance scale and posed questions and problems on their own. In doing so, they occasionally began working on something which I considered characteristic of a new Situation.

Although I tried whenever possible to introduce new Situations in a smooth and natural manner, I occasionally simply initiated a new kind of question or problem. I did this primarily when a child was at an impasse or had stopped exploring, and I had no further questions about the ways in which she solved and explained what I saw as a certain class of problems.

Because of the flexibility of the balance scale session, not every child encountered Situations in the same way or in the same sequence. Thus, different children may have approached the same Situation at different points in the session, and with different sorts of balance scale experience behind them. Occasionally a child did not encounter a particular Situation at all. In addition, as children explored on their own, they sometimes encountered balance scale phenomena that other children did not encounter. Despite these variations in the course of the session, I believe that any more rigid a presentation of problems and questions would preclude development and expression of the very processes I seek to observe.

2.3.2 Situations

For each Situation I give: what the child does in that Situation (**What**), a brief summary of what I hoped to find out by including the Situation in balance scale sessions (**Rationale**),[4] and a typical way I might introduce the Situation (**Example**) when it did not arise spontaneously. See Table 3 for a summary of Situations.

I list the Situations in the order in which children most commonly encountered them.

- **SITUATION1:** Balance the scale any way you want

  What: Children attempt to balance the scale any way they want at the very start of the session.

  Rationale: What are children's spontaneous first approaches to the balance scale?

  Example: I ask "Can you put some of these [rings] on so that this [the scale] balances?"[5]

- **SITUATION2a:** One ring per peg, variable number of pegs

  What: Children attempt to explain the stability of, make predictions about, and/or construct non-symmetric configurations with no more than one ring on a peg, any number of pegs used on an arm (Figure 2.6).

  Rationale: The pilot study led me to expect children to construct symmetric configurations in SITUATION1. How readily will they construct non-symmetric configurations? Will they use arithmetic at all?

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[4]Although I focus here on what *I* hoped to find out, I am aware that this depends on what children do and say in the Situation.

[5]Whenever possible, I avoided introducing terms for the balance scale and rings, as I believe my language can influence children's thinking.

Example: I ask "Can you use three [rings] to make this balance?"[6]

- **SITUATION2b:** One peg per arm, variable number of rings

  What: Children attempt to explain the stability of, make predictions about, and/or construct non-symmetric configurations with one peg used per arm (Figure 2.7).[7]

  Rationale: The pilot study led me to expect children to find SITUATION2b easier than SITUATION2a. If children have difficulty with SITUATION2a I can introduce SITUATION2b and later return to SITUATION2a. One of the major functions of this Situation is to keep children's interest in the balance scale, allow them to experience some success with non-symmetric configurations, and to serve as a bridge to SITUATION2a.

  Example: I put a ring on one arm and ask "Can you put two [rings] on the same place so that this balances?".

- **SITUATION3:** Multiple rings on multiple pegs

  What: Children attempt to explain the stability of, make predictions about, and/or construct non-symmetric configurations with multiple pegs used on at least one arm, multiple rings on at least one of the pegs (Figure 2.8).

  Rationale: Any arithmetical approaches to the balance scale that work *only* in SITUATION2a or SITUATION2b will not work in SITUATION3. The best approach to SITUATION3 is "the law." Will children develop it here?

  Example: I put three rings on the sixth peg, one on the fifth, and two on the second on one arm and ask "Can you use three [rings] to make this balance?".

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[6] In several cases, children responded to this request by placing a ring on each of two corresponding pegs and one in the center -- thus constructing a symmetric configuration. In these instances I then asked if they could balance the scale with three rings *without* using the center peg.

[7] See the discussion of ARITHMETIC2 in Chapter 3 for the one exception to this.

- **SITUATION4:** Removing an asymmetric balance from a symmetric one

  What: I put two rings on the third peg of each arm and a ring on the sixth peg of each arm (Figure 2.9). Children attempt to remove three rings so that the scale remains balanced.

  Rationale: Solving this problem involves removing an asymmetric balanced configuration from a symmetric one. Will the symmetry of the configuration on the balance scale interfere with problem solution, in particular, will it interfere with use of "the law"?

  Example: I construct the configuration and ask "Can you remove three [rings] so that it stays balanced?".

- **SITUATION 5:** Role of the numbers on the balance scale

  What: Children talk about the role of the numbers on the balance scale.

  Rationale: Do children understand that distance plays a part in the functioning of the balance scale, but that the *actual numerals* on the balance scale do not? If children use "the law" or other arithmetical approaches to the balance scale, do they relate the numbers they use in their calculations to the numbers on the balance scale or to distance?

  Example: I place a ring on the third peg on each arm. Then I "interchange" the 3 and the 8 on one arm (by putting my finger over the left half of the 8 so it looks like a 3 and changing the 3 into an 8 with a magic marker), hold the scale steady, and ask if the scale will still balance when released and why (Figure 2.10).

- **SITUATION6:** Balanced, with more rings on one arm

  What: Children attempt to explain how the scale can balance when more rings are on one arm (Figure 2.11).

  Rationale: In the pilot study, children sometimes said they did not understand how the scale could balance with more rings on one arm, even though they knew that this was possible. In expressing their confusion they focused on the number of rings on each arm, seemingly ignoring the distance at which those rings were placed.

Will children using "the law" ever doubt the scale can balance with more rings on one arm? What kinds of doubts will they express?

Example: As I hold the scale steady, I construct a configuration (with more rings on one arm) that will balance the scale. I ask the child if the scale will balance when it is released. If the child says it will, I ask "Are you sure it will balance *even though* there are more on one arm?".

2.4 Perspectives on intervention

My interest in the processes a child goes through in coming to solve particular sorts of problems influences the ways I interact with the child in the balance scale session. I view my intervention -- posing questions and problems -- not only as probing the child's knowledge, but also as influencing it.

My perspectives on the role of intervention differ from those of Piaget. The processes I seek to observe over the course of the session, however, are processes with which Piaget was not primarily concerned. Since he focused on broad developmental consistencies underlying performance, he used the clinical interview as a way of taking snapshots of children's thinking at different stages. He did not look for changes that might have taken place as a result of the questions and problems he posed over the course of the clinical interview. Instead, he looked for *what remains invariant.* Unlike Piaget, I look for *what changes*, and I consider my intervention an integral part of the environment I am studying.

In contrast to the Piagetian clinical interview, the Vygotskiian "experimental-developmental" method ([Vygotsky 78], p. 61) considers the experimenter an important part of the experimental environment. In the experimental-developmental method, the experimenter presents the subject with a problem that is just a little too difficult, in that the subject cannot readily solve it without outside assistance. The experimenter provides this assistance and then observes the ways in which the subject makes use of it in coming to solve the problem. For example, the experimenter might provide auxiliary tools for problem-solving or give out particular suggestions, and then observe ways in which the subject uses and builds on these. In Vygotsky's view, the experimental-developmental method provides a reasonable simulation of real-life development.

In Vygotskiian *scaffolding* [Greenfield 84], a system of supports for task performance, the experimenter (or teacher) also plays an important role. At first, performance is a joint effort of teacher and learner. As the learner becomes increasingly adept, supports are gradually removed until performance is fluent and proceeds without outside assistance. Thus, as the learner progresses along a series of graded partial tasks, each involving greater responsibility in accomplishment of the goal, the role of the experimenter/teacher mitigates correspondingly.

My experimental method parallels aspects of the experimental-developmental method and aspects of scaffolding. The assistance I provided appears as a blend of the auxiliary tools for problem-solving of the experimental-developmental method and scaffolding. Throughout the session, I presented a progression of Situations as well as questions and problems concerning particular contexts that arose within and in addition to Situations. Navigating these Situations and other questions and problems appeared to lead children to learn more about the balance scale. Thus, although the form of my assistance was not exactly the same for every child, I did nonetheless provide children with some standard sorts of assistance if they did not reach certain levels of performance on their own.

My method differs from the Vygotskiian paradigm in that I did not simply give out this assistance and then stand back to watch what happened. One reason for this is that the balance scale session was much more of an interaction between the child and myself. Indeed, I was rarely silent for more than a minute: as children worked on the balance scale, I asked them about what they were doing and posed questions and problems. Their responses led *them* to explore further and *me* to pose additional questions and problems. Another reason that I did not provide assistance and then stand back to see if the child could accomplish "the task" on her own is that there was no such "task." I did not pose a pervasive task for the session (such as "find an arithmetical rule by which you can predict the stability of the scale"), and children did not seem to generate such tasks. Even the various problems and questions I posed over the course of the session were generally such that children could solve them fairly readily, although usually not without some difficulty.

Although the absence of a pervasive task for the session may have limited the extent to which I could explore what children could do on their own or with standard sorts of assistance, it allowed me to investigate their *expectations* about the relationship of arithmetic to the balance scale.

2.5 Probing for competing Strategies

In exploring children's navigation of the balance scale session, I focus on their *Strategies.* Strategies are approaches to the balance scale manifested as ways of trying to balance the scale, predicting the scale's stability, and explaining the scale's state of balance. Strategies are not "units" of knowledge, nor do they necessarily each correspond to a distinct body of knowledge. For example, knowing that "distance matters" could contribute to the development of several different Strategies.

I inferred children's Strategies from their actions and verbalizations.[8] Although many children were not readily able to explain what they were doing and why, sometimes the more they talked the more they seemed to focus, clarify and better express their ideas. Not all children, though, seemed to *want to talk*. When they appeared eager to talk or when talking seemed to make them feel more comfortable, I tended to ask more questions. When children seemed reluctant to talk or uncomfortable talking, I limited my questioning. Thus, my perceptions of how eager children were to talk may have affected both *their* progress and *my* ability to infer Strategies.

Over the course of the session, I was not only concerned with *what* Strategies children developed, but also *when* they used these Strategies. In particular, I was interested in when Strategies emerged as competitors. Below I discuss the two primary ways I probed for the presence of competing Strategies.

2.5.1 Confidence

One way I probed for the presence of competing Strategies was by exploring the child's confidence in her responses or solutions. Did her confidence ever waver before she found out whether or not she was correct?

I frequently investigated confidence by holding the scale steady as the child constructed a configuration she expected would balance and, before letting go, asked her what would happen when I released the scale. As she studied what she constructed, did she begin to doubt that the scale would balance? Did she explain why she believed the scale would balance? Did she begin to consider other possible outcomes (thus, suggesting "second choice" Strategies)?

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[8] In Chapter 3 I discuss the types of Strategies children developed and how I inferred them.

Likewise, when a child described (rather than constructed) the solution, I sometimes asked if she wanted to construct it to be sure. If she wanted to, I held scale steady while she constructed it and then, before releasing the scale, asked her if she thought it would balance. If actually *seeing* the configuration appeared to raise doubts, I led her to talk about why she *now* thought the scale would not balance and why she had *previously* thought it would.

Other methods of investigation included noting whether the child's confidence wavered as the scale tipped back and forth in coming to rest, and asking about or showing possible alternative solutions.

An interesting consideration is the extent to which my investigating children's confidence might have actually *reduced* their confidence. For example, by asking a child if she was sure the scale would balance when released might I have inadvertently led her to think she was wrong and that I was asking in order to give her a second chance? I believe this is unlikely. If children are confident in their responses, they will insist they are correct, and perhaps even take the opportunity to explain why. If, on the other hand, they are not confident, my questions may resonate with their doubts, which they may then proceed to express.[9]

2.5.2 Apparently balance scale-related changes in emotion

Sudden displays of affect, such as surprise, excitement, or distress can reflect recognition of conflicts and contradictions among Strategies. When children displayed such affect, I tried to investigate its cause. For example, surprise suggests encountering something unexpected. This can serve as a springboard

[9] Of course, children might have felt compelled to alter their answers if I had bombarded them with an "unreasonable" amount of questioning, but I don't believe this was ever the case.

for investigating what *was* expected and how the child comes to reconcile what was expected with what really happened.

2.6 Subjects

The present analysis is based on the balance scale sessions of three girls and six boys from the Hennigan School, an inner-city, multi-racial public school in Boston. All nine children came from advanced classes. Seven were fifth graders, two were fourth graders. They ranged in age from 9;3 to 12;4. See Table 4 for a listing of subjects by age, grade, and sex.

I chose these children from a group of twenty-nine Hennigan fourth and fifth graders with whom I conducted balance scale sessions during the 1985/6 school year. My data tentatively suggests some trends across age and academic ability groups. Perhaps the most prominent was in development and use of arithmetical Strategies. In general, "advanced" fifth graders developed what appeared to be more sophisticated arithmetical Strategies than did "advanced" fourth graders, who developed what appeared to be more sophisticated arithmetical Strategies than did fourth graders from regular classes. (I have insufficient data to place fifth graders from regular classes in this ordering.)

For the present analysis, I selected the children who developed "the law" over the course of the session. I did not include the four children (all "advanced" fifth graders) who appeared to use "the law" at or very close to the outset of the session, as their sessions did not show any process of coming to use this Strategy in different contexts.[10]

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[10]Although developmental differences across age groups is not a focus of this thesis, it is interesting to note that these four fifth graders in many respects outperformed several college graduates with whom I conducted informal balance scale sessions. Many other factors, including willingness to use arithmetic and expectations of "the law" may relate to the superior performance of these four fifth graders.

Chapter Three

Strategies and their use

3.1 Strategies

In this chapter I discuss the Strategies children used and when they used them.[11] I begin by delineating the Strategies. For each I give a description (**What**), an explanation of how I inferred it (**How inferred**), and an example of a manifestation of the Strategy (**Example**). See Table 5 for a summary of Strategies.

I begin each description with a statement of the approach suggested by the various manifestations of the Strategy. I do not intend this statement to reflect a general heuristic, procedure, or piece of knowledge underlying use of the Strategy. Indeed, *my* characterization of the Strategy may not parallel *the child's* perspective. Although I consider particular kinds of actions and verbalizations as instances of a type of approach to the balance scale, children did not necessarily appear to see their actions and verbalizations in this way.

The explanations of how I inferred each Strategy represent the final stage of a complex process. I did not begin the present study with precise expectations of which Strategies children would use. Instead, I determined Strategies by analyzing what children did and said over the course of the session. My pilot study had led me to expect at least some Strategies based on symmetry, inverse correspondence of weight and distance, and arithmetic. In the first level of protocol analysis, I identified Strategies belonging to one of these three groups and Strategies not clearly belonging to any one of them. In the second level of

[11] See Chapter 2, Section 5 for a discussion of my use of the term "Strategy."

protocol analysis, I looked for major differences among the Strategies within each group. Next, I proposed a precise description of each Strategy and a set of criteria for inferring instances of the Strategy from the protocols. I then returned to the protocols and checked that the proposed criteria would indeed lead to inferring each instance of the Strategy and that the proposed definitions were accurate. For some Strategies, the process was now complete. For others, the process required many more iterations: the criteria or definitions had to be refined to account for particular cases, new Strategies were identified, or instances of Strategies were reclassified.[12]

3.1.1 Delineation of Strategies

- **COMPENSATION1:** Active compensation

  What: If the scale tips, it can be balanced by moving rings toward the fulcrum on the tipping arm, by moving rings away from the fulcrum on the raised arm, by removing rings from the tipping arm, or by adding rings to the raised arm.

  COMPENSATION1 is a way of balancing the scale, apparently based on a qualitative understanding of the inverse correspondence of weight and distance. It does not provide a way of predicting or explaining.

  How inferred: Children made at least one attempt to balance the scale, as described above. Most consecutive attempts of the same sort (e.g. moving rings, but not changing their number) resulted in a series of successive approximations to balance.

  Example: In trying to balance the scale, Rac put a ring on the tenth peg on one arm and one on the fifth on the other (Figure 3.1). When the scale tipped, she moved the ring on the fifth to the ninth

[12]In many instances I could have continued to subdivide Strategies by grouping similar manifestations together. I chose the present level of description in order to facilitate consideration of general trends across subjects.

(Figure 3.2). The scale remained tipped. She then moved the ring on the ninth to the tenth (Figure 3.3) and the scale balanced.

- **COMPENSATION2:** Static compensation

  What: The scale balances if it "looks like" it will balance.

  COMPENSATION2, a prediction or explanation, is a qualitative estimation of balance scale stability usually based either on inverse correspondence of weight and distance or on the visual similarity of one configuration to another known to balance. I also include as COMPENSATION2 an instance in which a child said she thought the scale might not balance because of the "big gap" (several empty pegs) between two rings on one arm. Her comment may have been based on inverse correspondence of weight and distance or on some other notion of what a balanced scale "looks like."

  How inferred: Children gave an explanation or prediction based on inverse correspondence of weight and distance or said they had constructed the configuration or one like it before.

  Example: I put two rings on the fourth peg on one arm and one on the ninth on the other, held the scale steady, and asked Rin if he thought the scale would balance when released (Figure 3.4). He said it might because "this one's closer in [the group of two rings], this one's further out [the single ring]."

- **SYMMETRY1:** Qualitative symmetry

  What: The scale balances if a ring is placed at approximately the same position on each arm.

  SYMMETRY1 is a way of balancing the scale based on a qualitative estimation of distance. Distance, here, is not represented by a particular numbered peg, but rather, by a *section of the balance scale* (e.g. in the middle or toward the end of an arm).

  How inferred: A child, either asked to balance the scale or having stated an intention to balance the scale, put a ring in approximately the same position on each arm. In order to distinguish this Strategy from SYMMETRY2 (directly below), in which rings are placed on corresponding pegs, I looked at what children did immediately after the scale tipped. If they had intended to use SYMMETRY2 but had misplaced a ring, they would readily move one ring so that it corresponding with the other, sometimes noting the mistake. Sometimes children using SYMMETRY1 responded to the scale's tipping by moving one ring so it corresponding with the other, but they did so tentatively or seemed surprised or greatly relieved when the scale balanced.

Example: Tot put a ring on the fifth peg on one arm, then one on the fourth on the other (Figure 3.5). As the scale started to tip, she removed the ring from the fourth, wondered out loud "On the same number, maybe?" and placed the ring on the fifth peg.

- **SYMMETRY2:** Quantitative symmetry

What: The scale balances if the identical configuration is placed on each arm.

Here, corresponding pegs are determined by the number on the peg or by counting pegs from the center of the scale. In some cases children using SYMMETRY2 seemed to see the numbers on the scale as a quantification of distance, in some cases children did not seem aware of a relationship between the numbers and distance, and in some cases I was not readily able to determine the extent to which children related numbers to distance. Thus, SYMMETRY2 appeared to be a surface manifestation of at least two different kinds of understanding.

How Inferred: Children constructed symmetric configurations and predicted that symmetric configurations would balance. Their explanations and other comments suggested that they were not using compensation-based or arithmetic-based Strategies to construct or predict in these instances. If they had used such Strategies already, they appeared to see SYMMETRY2 as a distinct approach.

Example: Krif began the session (SITUATION1) attempting to balance the scale with SYMMETRY1, used COMPENSATION1 to readjust the rings so that the scale balanced, and then used SYMMETRY2 to balance the scale again (Figure 3.6). After that, he had difficulty constructing a non-symmetric balance with three rings (SITUATION2). He made two attempts before he was successful. His first attempt involved constructing a symmetric configuration with three rings, and his second involved constructing an almost-symmetric configuration which did not balance.

- **ARITHMETIC1:** Summing pegs

  What: The scale balances if no more than one ring is on a peg, and the sum of the pegs used is the same on each arm.

  ARITHMETIC1 provided a way of constructing, predicting, and explaining. Using this Strategy did not necessarily reflect relating the numbers on the pegs to distance.

  How inferred: Children balanced the scale and made correct predictions when no more than one ring was on a peg. Their comments when they were constructing and their explanations referred to arithmetical equalities involving the sum of the pegs used on each arm (and in one case to arithmetic relationships between pairs of pegs used on opposite arms). Children did not use arithmetic to construct, predict, or explain when more than one ring was on a peg.

  Example: Nib balanced the scale by placing a ring on the tenth peg on one arm and one on the third and one on the seventh on the other (Figure 3.7). He said that the scale balanced because "seven and three is ten, so it should be even."

- **ARITHMETIC2:** Multiplying rings by pegs

  What: I group together three similar Strategies here in order to facilitate discussion. Each Strategy involves computing and comparing torque of one stack of rings on each arm.

  ARITHMETIC2a: The scale balances if one ring is on an even-numbered peg on one arm and two are half as far on the other arm.[13]

  ARITHMETIC2b: The scale balances if only one peg is used per arm, and the product of the number of rings on the peg and the number of the peg (not necessarily seen as reflecting distance) is the same on each arm.

[13] Children may have focused either on distance or on the numbers of the pegs.

ARITHMETIC2c: The scale balances if superpositions of balances using only one peg per arm are placed on it.

How inferred: Children computed and compared torques of rings on one peg per arm in order to construct, predict, and explain. Children using ARITHMETIC2a and ARITHMETIC2b did not yet use arithmetic in SITUATION3. The child who used ARITHMETIC2c could only use arithmetic when rings were on multiple pegs if he saw the configuration as the superposition of balances using only one ring per arm.[14]

Example: I give an example for each version.

ARITHMETIC2a: Krif constructed several configurations with a single ring on an even-numbered peg on one arm and two half as far on the other arm (e.g. Figure 3.8). He then said "all the halves work."

ARITHMETIC2b: After using ARITHMETIC2a several times, Shas put a ring on the tenth peg on one arm and two on the fifth on the other arm. He then replaced the two on the fifth with five on the second on the same arm (Figure 3.9). Soon after, he replaced the five on the second with two on the fifth. His comments suggest he understood that both configurations balanced one on the tenth because the product of the number of rings and the number of the peg was, in all instances, ten. At this point Shas was not yet able to use arithmetic to construct a non-symmetric balance of three rings.

ARITHMETIC2c: After using ARITHMETIC2a, Sam spontaneously balanced the scale with six rings on the tenth peg on one arm and three on the tenth and six on the fifth on the other arm (Figure 3.10). At this point he was not yet able to construct a non-symmetric balance of three rings, each on a different peg.

- **ARITHMETIC3:** "The law"

What: The scale balances if the sum of the products of the number of rings on a peg and the number of that peg (not necessarily seen as reflecting distance) are the same on each arm.

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[14] I do not consider ARITHMETIC2c a manifestation of "the law" (see ARITHMETIC3 below), as it does not involve *summing* products of weight and distance.

I break ARITHMETIC3 into two sub-Strategies, differing in whether or not arithmetic is seen as a quantification of qualitative compensation of weight and distance. Children using ARITHMETIC3a saw (at least to some extent) ARITHMETIC3 as a quantification of this compensation. Children using ARITHMETIC3b did not appear to ARITHMETIC3 in this way. Thus, ARITHMETIC3 can be thought of as a surface manifestation of these two different kinds of understanding.[15]

I also include as ARITHMETIC3a instances in SITUATION4 in which children who had been using ARITHMETIC3 said that distance, not the numbers on the balance scale, affect the scale's functioning.

Although use of ARITHMETIC3a suggests a fuller understanding of the balance scale, it does not necessarily suggest understanding of relationships between arithmetic and compensation in all circumstances. There appears to exist a multidimensional continuum of ways and contexts in which children related ARITHMETIC3 to compensation. Only when children appeared to see no relationships at all between ARITHMETIC3 and compensation I do consider them to have used ARITHMETIC3b.

How inferred: Children computed torques to construct, predict, and explain when multiple pegs were on at least one arm, and multiple rings were on at least one peg. I used children's comments just before and after they first developed ARITHMETIC3, and their navigation of SITUATION4 and SITUATION6 to differentiate between the two versions.

Example: I give examples of each version.

ARITHMETIC3a: After Krif had begun using ARITHMETIC3, I asked him if interchanging two numbers on the same arm of the balance scale would alter its operation (SITUATION4). He said that this wouldn't make any difference, pointed to the 8 and the 2 on one arm and said: "because this is an eight and this is a two, and it would just *say* something different." I used Krif's response to

[15]From this point on, I use ARITHMETIC3 when I wish to emphasize use of the arithmetical form of the "law" -- whatever the underlying understanding. I use ARITHMETIC3a or ARITHMETIC3b when I wish to emphasize the kind of understanding that appears to underlie this use.

SITUATION4 and some of his other comments to infer that he saw the numbers on the balance scale as a quantification of distance, and that he saw ARITHMETIC3 as arising, in part, from this quantification.

ARITHMETIC3b: When asked the same question after she had begun using ARITHMETIC3, Rac said she would use the new numbers in her calculations. Although she would still be using ARITHMETIC3, she would be basing her calculations on the numbers on the scale, not on distance.

- **ARITHMETIC4:** "Incorrect" arithmetical Strategies

  What: The scale balances if the same number can be calculated to represent each arm on the balance scale.

  I include here several different Strategies, all of which appeared to be based on a misapplication or misremembering of an arithmetical Strategy that had worked in the past.

  The Strategies Rac and Nib came up with are typical.[16]

  Rac: The scale balances if each arm has the same "weight" as determined by a calculation. One possible calculation is as follows: If there are two groups of rings on one arm, one with R1 rings on peg P1 and one with R2 rings on peg P2, the "weight" on that arm is R1*P2 + R2*P1 (Figure 3.11).

  Nib: The scale balances (even when there are multiple rings on a peg) when the sum of the pegs used on each arm is the same (Figure 3.12).

  How inferred: Children used ARITHMETIC4 to construct, predict, and explain. In each instance, I inferred the particular calculation children used by their verbalizations or a combination of verbalizations and actions.

  Rac described the calculation she had done when one arm of the scale had a ring on the sixth peg and three on the fourth: "six times

[16]Since the manifestations of ARITHMETIC4 varied widely, I present examples first in order to facilitate discussion.

three is eighteen and four is twenty-two." Rac had previously used ARITHMETIC3, but she appeared to have somehow "forgotten" what she had multiplied together to balance the scale in the past, perhaps partly because she did not relate the numbers on the scale to distance.

Nib used ARITHMETIC4 when he was attempting to balance a ring on the fifth peg and three on the fourth on one arm with two rings on the other arm. He put the two rings on the ninth peg, and explained that this might work because "four and five is nine, so maybe it's nine." Nib had previously used ARITHMETIC1, and appeared to be trying to apply it in the current context.

- **NUMBERS**: Numbers supersede distance

  What: The numbers on the balance scale affect its functioning.

  In the most common manifestation of this Strategy, children said that if the numbers on the scale were altered, the scale would work in a different way. Any previously-developed arithmetical Strategies could still be used for constructing, predicting, and explaining, but the *new number* would be used in the calculation, not the original number, which reflected distance from the fulcrum.

  In some cases, the Strategies included in ARITHMETIC4 seem related to those included in NUMBERS. For example, Rac's Strategy, described above, may have resulted partly from failure to relate the numbers on the balance scale to distance.

  How inferred: Children specifically discussed the numbers on the scale as influencing the scale's operation.

  Example: I put a ring on each of the third pegs, held the scale steady, and asked Jeo if the scale would still balance if the 3 and 8 were interchanged on one arm (SITUATION4). She said the scale would no longer balance because "it matters where the number is."

- **RINGS**: The number of rings on each arm supersedes all else

  What: The scale balances if the same number of rings is on each arm.

  How inferred: Children used RINGS for constructing and predicting. Their comments and explanations suggested they thought the scale balances whenever the same number of rings is on each arm. In many instances, children also said that the scale doesn't balance when different numbers of rings are on each arm.

Example: Early in the session, Shas put a ring on each odd-numbered peg on one arm and a ring on each even-numbered peg on the other arm (Figure 3.13). When the scale tipped, he expressed surprise: "How come, I've got this on two, four, six, eight, and ten, I've got this on one, three, and five, seven, and nine, and they're not ... but there's five on each side."[17]

3.1.2 Discussion

The eleven Strategies (considering ARITHMETIC3a and ARITHMETIC3b as distinct) could be classified along a variety of dimensions, such as number of factors (e.g. number of rings used, distance, numbers on the scale, etc.) taken into account, accuracy, or reliance on quantification. In order to facilitate later discussion, I have presented the Strategies according to the approximate extent to which they provide a way of predicting, explaining, and/or constructing via precise quantification of weight and distance. COMPENSATION1 and COMPENSATION2 do not require any precise quantification. Both rely on approximate quantity of weight and distance. SYMMETRY1 requires precise quantification of weight but not of distance. SYMMETRY2, ARITHMETIC1, and ARITHMETIC2 require precise quantification of both weight and distance, but can only be invoked in a narrow range of contexts. ARITHMETIC3 in theory always provides a way of predicting, explaining, and constructing via precise quantification of weight and distance. Although children did not always seem to see it or use it in this way, they did use ARITHMETIC3a more widely than ARITHMETIC3b. The various manifestations of ARITHMETIC4 usually involved

[17] I use three dots (...) to indicate a pause of several seconds.

precise quantification of both weight and distance, but they actually worked infrequently. NUMBERS and some of the manifestations of ARITHMETIC4 reflect a precise but incorrect quantification which contradicts the role of distance. RINGS requires precise quantification of weight, but ignores distance altogether.

ARITHMETIC4 and NUMBERS reflect a sort of "overquantification" that emphasizes arithmetic and number over the reality of the balance scale. All the other Strategies can be thought of as manifestations of "the law." Even if Strategies could only be invoked in a narrow range of situations (e.g. ARITHMETIC1) or were not always correct (e.g. RINGS), they could at least in theory have been derived from (perhaps quite limited) experience with the balance scale. ARITHMETIC4, by contrast, seemed to reflect misapplication or misremembering of a previously successful manifestation. NUMBERS bears even less relationship to "the law." Unlike the other Strategies, it attributes a causal role to something that is neither weight nor distance. At times, use of NUMBERS seemed to reflect an attempt to "see" the balance scale only in terms of numbers and arithmetic, and correspondingly, a tendency to ignore compensation.

3.2 Use of Strategies

3.2.1 Summary of Strategies used in different Situations

Table 6 summarizes the Strategies each child used in the different Situations. Situations are listed on the left, and children's names are listed across the bottom. Within each Situation, Strategies appear in order of use, such that those used earlier are closer to the bottom of the page. Multiple occurences of a Strategy within a child's session were not necessarily each manifested in the same way. The first use of ARITHMETIC3 is shown shaded, and the instances in which children used SYMMETRY2 very briefly in SITUATION5 are shown in parentheses. When a child used two Strategies simultaneously, the Strategies are shown connected with vertical dashes.

I break each Situation into *Episodes* (shown separated by horizontal lines in Table 6). An Episode is an attempt to solve a particular problem or answer a particular question. Episodes usually involved either a series of explanations or predictions about a given configuration or an attempt to construct (and perhaps also make predictions about or explain) a configuration according to the constraints of the current Situation. Like Situations, Episodes were triggered either by a question or problem I posed or by a question or problem resulting from the child's explorations.

The total count of Strategies a child used in an Episode reflects the number of times she *changed* Strategies, but not necessarily the time she spent in that Episode. A child may have used a Strategy continuously within the same Episode for several seconds or several minutes. The number of Strategies a child used in an Episode, however, may reflect her ability to navigate the particular question or problem posed in the Episode and her interest in exploring multiple approaches. For example, children trying to balance the scale might vacillate among competing Strategies, unable to use any successfully, might try a range of approaches until they found one that worked, might readily balance the scale and then stop, or might seek multiple ways of balancing the scale.

Like the number of Strategies in an Episode, the number of Episodes in a Situation was not necessarily related to time spent in that Situation. Number of Episodes did, however, seem related to ability to use a given Strategy within the Situation. For example, if a child consistently used one Strategy for consecutive Episodes in a situation, I would not continue to pose problems and questions of the same sort (thus, I would not trigger more Episodes in the same Situation). Likewise, children did not tend to continue posing similar problems and solving them all with *the same* Strategy. On the other hand, children sometimes continued posing similar problems and questions and approaching them with *different* Strategies, perhaps exploring contexts in which a Strategy might apply or seeking multiple solutions.

In most instances, breaking Situations into Episodes was straightforward. In several instances the division was less clear, for example, when children temporarily interrupted a series of attempts to balance the scale by beginning to ask a question or initiate a new kind of exploration. I decide each such ambiguous case individually. I believe that the number of such instances is sufficiently small so that it does not significantly alter the patterns of Episodes shown in Table 6.

As discussed in Chapter 2, not all children encountered Situations in the same order. In Table 6, I indicate a child's temporarily leaving a Situation with a bar across the list of Strategies. The bar shows the point at which she left the Situation and what she did during the interruption. At the bottom of each column in Table 6, I list the order in which each child encountered the Situations. I mark Situations that arose spontaneously with an asterisk.[18] I consider all instances of SITUATION1 to be non-spontaneous, as this Situation was at least to a large extent directly triggered by my initial presentation of the balance scale.

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[18]Children varied in the extent to which they initiated explorations and posed questions and problems on their own, and consequently, in the extent to which they came up with Situations spontaneously. Initiative appeared to be related to several issues, including comfort in the balance scale session, interest in the balance scale, satisfaction with a single solution to a given problem, and gender. Additional research might address the extent to which these factors bear on children's initiative in the balance scale session and in similar learning situations.

3.2.2 Interrelationships of Strategies

Up to this point I have not emphasized interrelationships among Strategies: which Strategies were integrated and which Strategies appeared to evolve from others. Given the nature of the balance scale sessions, I feel less confident making inferences about these relationships than inferences about the nature and usage of the Strategies. Strategies appeared as children's overt responses to the demands of particular events. In the case of interrelationships, however, children may have had less need or opportunity to discuss or demonstrate. For example, after encountering SITUATION5 a child may have come to realize that SYMMETRY2 is a special case of ARITHMETIC3, but may have found no need to mention this or to experiment with the notion on the balance scale.

Integration is an *awareness* of relationships among Strategies. It is not manifested as use of a new Strategy, but as a more powerful understanding of relationships between two or more Strategies. I inferred integration only from verbalizations, although I suspect that children may have integrated more strategies than their verbalizations suggest. Children's predictions, explanations, and comments sometimes suggest that they had found relationships between compensation-based and arithmetic-based Strategies. For example, Nib's explanation of why a ring on the tenth peg on one arm balances one on the third and one on the seventh on the other suggests that he related ARITHMETIC1 to compensation: "Ten would weigh it down a lot, so seven would push it up some, but not as much as three [and seven], so three would push it up the rest of the way." No children explicitly referred to relationships between symmetry and compensation or symmetry and arithmetic.[19]

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[19]It is not clear when, over the course of the balance scale session, children might have found it natural or necessary to make comments suggesting that they had integrated symmetry and compensation or symmetry and arithmetic. I suspect that had children explicitly considered such relationships, they might have seen them as trivial.

I inferred genetic relationships when one Strategy appeared to lead directly to the first use of another. For example, I consider ARITHMETIC1 to have led to ARITHMETIC3 when Dar, who had been using ARITHMETIC1 in SITUATION2, came up with ARITHMETIC3 by superposing two configurations. Each configuration used only one ring per peg, but when they were superposed, one peg had two rings on it (Figure 3.14). He used ARITHMETIC3 for the remaining several Episodes in SITUATION3. I also inferred genetic relationships when a later Strategy resembled a previous one. For example, I consider ARITHMETIC1 to have led to ARITHMETIC4 when Nib appeared to try to use ARITHMETIC1 in an instance in which multiple rings were on a peg (see discussion of ARITHMETIC4 above).

Occasionally, children's verbalizations aided my inferences. I consider a genetic relationship of which a child seemed aware to be an integration. Most of the genetic relationships I inferred were within one of the three major groups (compensation, symmetry, and arithmetic), rather than across groups, and in most cases, I assumed only one Strategy to be a parent of another. However, I suspect that there exists a web of genetic relationships much richer than that uncovered by the present analysis.

Although over the course of the session children tended to use arithmetical Strategies more frequently, the entire range of Strategies used did not appear to fall into a neat hierarchy. In most cases, when children developed a new Strategy, they did not necessarily stop using others. Exceptions occur within the arithmetic-based and the symmetry-based Strategies. Children no longer used ARITHMETIC1 and ARITHMETIC2 once they developed ARITHMETIC3. They no longer used SYMMETRY1 once they developed SYMMETRY2.

3.2.3 Individual balance scale essions

The series of charts in Appendix C shows the course of each child's balance scale session. Charts include the information in Table 6, along with inferred integration of and genetic relationships among Strategies.

3.2.4 Protocol

Situations and Strategies represent the basic units of analysis I use in considering what children did after first using "the law." To provide a more concrete sense of how Situations and Strategies were manifested, I present in Appendix D the section of Tot's protocol surrounding her first use of ARITHMETIC3. The protocol includes all of SITUATION3 and most of SITUATION2a and SITUATION4. I chose Tot primarily because I felt that the course of her session would be relatively easy to follow: she was articulate, she went through the Situations in a linear sequence, and she explored relatively little.

In the protocol, I identify Episodes, Situations, and Strategies.

Chapter Four

Balance scale sessions after "the law"

4.1 Arithmetical trends in balance scale sessions

Over the course of the sessions, children tended to use arithmetical Strategies (excluding ARITHMETIC4) in an increasingly wide range of contexts. This development did not always proceed smoothly. The first successful use of a particular arithmetical Strategy was not necessarily followed immediately by a second. For example, after first using ARITHMETIC1 to construct a balanced configuration with three rings, children could not always readily use ARITHMETIC1 to balance the scale with three rings in a different way. Sometimes children did not seem to know *how* to use a particular Strategy in a new context, and sometimes they did not seem to know *if* they could a particular Strategy. Indeed, what I saw as a single Strategy children may have experienced as a series of remotely related or perhaps even unrelated approaches to the balance scale.

Sometimes inability to use a particular arithmetical Strategy led children to develop a different one. Although *I* distinguish developing a new Strategy from developing a different or more general manifestation of a previous Strategy, I suspect that *children* did not always make this distinction. Nonetheless, in the case of ARITHMETIC3, this distinction is an important one. ARITHMETIC1 and ARITHMETIC2 each provide a way of coping with limited classes of contexts. ARITHMETIC3, however, in theory provides a way of coping with *all* contexts. There is no further arithmetical Strategy to develop.

After first using ARITHMETIC3, eight of the nine children used other Strategies. In this chapter, I discuss their use of these Strategies. For each Strategy used after ARITHMETIC3, I include how and when children used it, why they appeared to use it (e.g. because they weren't sure if they could use ARITHMETIC3), and if/how they appeared to reconcile it with ARITHMETIC3. If relevant, I also discuss ways in which use of the strategy *after* ARITHMETIC3 appeared to relate to use *before*.

Table 7 summarizes the Strategies children used after first using ARITHMETIC3.

4.2 Strategies used after ARITHMETIC3

4.2.1 COMPENSATION2

Two children used COMPENSATION2 after ARITHMETIC3. Both used it soon after using ARITHMETIC3 for the first time.

Nib's use of COMPENSATION2 appears to reflect uncertainty about whether ARITHMETIC3 always works. He used COMPENSATION2 just after he used ARITHMETIC3 for the first time. The first time he used ARITHMETIC3, I had put a ring on the first peg and three on the third on one arm. As he put a ring on the tenth peg of the other arm to balance the scale, he said "That would be ten, so I should put, um, ... one on the ten"[20] (Figure 4.1). Next, I removed the rings from the scale, put three on the fourth peg and one on the fifth of one arm, and held the scale steady. The following interchange occurred as Nib put a ring on the tenth peg and one on the seventh on the other arm (Figure 4.2) while I held the scale steady.

N: Um, let's see, oh, I get it, OK, four times three is twelve plus five is

[20] I use three consecutive dots (...) in protocols to indicate a pause in speech of several seconds.

seventeen, so if I have one on the ten and the seven.

M: Do you think that would work?

N: No.

M: Why not?

N: It's probably too much weight.

The circumstances surrounding Nib's use of the word "weight" suggest he meant that the scale would not balance because one arm (I am not sure which) looked heavier, rather than because more weights (rings) were on one arm. In other words, he didn't think that the scale *looked like it would balance*, even though he had calculated the same torque for each arm.

When I released the scale, Nib seemed surprised and excited to see that it balanced.

N: It works!

M: It did. Why do you think so?

N: Um, because, um, on the left side I had three fours and a five, and on the right side, which equals seventeen [the "three fours and a five"], and on the right side I had a seven and a ten, so I think it would probably work, um, any, anything that equals, that would equal seventeen would probably work.

Although Nib now seemed *more sure* that the scale balances as long as torques are the same on each arm, his use of the word "probably" suggests he still wasn't *entirely* certain. In the two remaining Episodes in his balance scale session, he used ARITHMETIC3.

Rac's first use of COMPENSATION2 was manifested in a similar manner. Soon (although not immediately) after she used ARITHMETIC3 for the first time, she encountered a configuration she thought might not balance because of "too much weight on [one arm]" even though she knew that torque was the same on each arm. When she saw that the scale balanced, she explained "it's still the same number."

After another use of COMPENSATION2, in the next Episode, Rac seemed to come to think that as long as her calculation yielded the same number for each arm, the scale would balance. However, she was sometimes unsure exactly what calculation to perform. In addition, she based her calculations on the numbers on the scale, not on distance. Thus, although she never again used COMPENSATION2 she did use ARITHMETIC4 and NUMBERS.

4.2.2 SYMMETRY2

Seven of the nine children used SYMMETRY2 after ARITHMETIC3. Six used it in SITUATION5, and one used it in both SITUATION5 and SITUATION2a. All seven had used symmetry-based Strategies earlier in the session -- six had used SYMMETRY2 and one had used SYMMETRY1.

4.2.2.1 SYMMETRY2 in SITUATION5

Eight children encountered SITUATION5 after using ARITHMETIC3. Seven of them used SYMMETRY2 before using ARITHMETIC3 in SITUATION5, the other used ARITHMETIC3 immediately. Three of the seven children used SYMMETRY2 only briefly. Before using ARITHMETIC3, one child suggested removing two corresponding rings, another reached for two corresponding rings, and another removed two corresponding rings and then quickly replaced them. The other four children had more difficulty. Before solving the problem with ARITHMETIC3, they spent several minutes removing and replacing corresponding rings, suggesting pairs of corresponding rings that might be removed, and/or explaining that the problem was impossible.[21]

The children who used SYMMETRY2 only briefly never seemed to doubt that ARITHMETIC3 would provide a solution, rather, they seemed to have initial difficulty applying it. The other children seemed to "forget" about arithmetic as an approach to the balance scale. Jeo was typical. When I presented SITUATION5, she first claimed the problem was impossible, next tried to solve it by removing corresponding pairs of rings, and then claimed it was impossible again. She did not at this point seem aware that she could use have used arithmetic.

M: Can you take three off so that this balances?

J: Three? No.

M: No?

J: 'Cause this one is ... (after about seven seconds of looking at the balance scale) Oh yeah, you could.

M: How?

J: (removing a ring on the third peg from each arm) Taking these two off.

M: Uh huh ... that's not ...

J: Oh, before you said three, right?

M: Yes.

J: Oh ... there ain't no way.

---

[21]Presenting SITUATION5 verbally might provide a way of exploring the extent to which *seeing the symmetric configuration* on the balance scale interfered with children's thinking of and applying arithmetical Strategies. If, however, children solved the verbally-presented problem by forming mental images of the balance scale, symmetry would continue to interfere.

Soon after, Jeo explained why the problem was impossible. In doing so, she came to use arithmetic to solve the problem.

J: [It is impossible] 'cause if I got three off anywhere it won't make it balance 'cause it will be an odd number.

M: Why can't it be an odd number?

J: 'Cause if I take these three off, 'cause if I take, like, two from here [one arm] and one from here [the other arm] it won't make it balance ... Oh! (removes two from the third peg on one arm and one from the sixth peg on the other arm, so that the scale balances)

M: What happened?

J: These two [on the third peg] make six!

M: Yes?

J: And this one [on the sixth peg] is a six!

M: So how many did you take off this time?

J: Three.

M: And it worked.

J: I forgot about that [using arithmetic].

Jeo seemed to have come to see the configuration in a new way. She no longer focused on (or *only* on) corresponding pairs of pegs, she was now able to use arithmetic to partition the configuration.

4.2.2.2 SYMMETRY2 in SITUATION2a

Rin, the child who used SYMMETRY2 in SITUATION2a (after having used ARITHMETIC3) also said he had "forgotten" he could use arithmetic. He had never before used arithmetical Strategies in SITUATION2a. Near the beginning of the session, after he had used SYMMETRY2 to balance the scale, I introduced SITUATION2a by asking him if he could balance the scale with three rings. He tried to do this by putting each ring on a different peg. He used SYMMETRY2 (two rings at corresponding positions and one in the center) and both compensation-based Strategies, but was unable to balance the scale. He was, however, able to use arithmetic in SITUATION2b -- in this case, three rings, two on the same peg. The questions and problems I posed in SITUATION2b led to SITUATION3, where he developed ARITHMETIC3.

After Rin used ARITHMETIC3 twice, I introduced SITUATION2a again. I removed all the rings from the scale and asked him if he could balance the scale with three rings, all in different positions. He suggested a symmetric solution, and when I told him he could not use this solution he expressed concern.

R: I can't use the middle one?

M: No, you can't.

R: Uh, oh. ... Oh, I've got it! Six (puts a ring on a sixth peg), four (puts a ring on the fourth on the same arm), and ten (puts one on the tenth peg on the other arm)!

M: When you said "I've got it" what did you get?

R: Like, six and four is the same amount as ten going down.

M: Uh huh ... How come when I first said to you that you can't use the middle one you said "uh, oh"? Why did you think there was going to be a problem?

R: Because I forgot about that.

M: You forgot about what?

R: You could ... I could do that [arithmetic].

M: You forgot you could do that, huh? How did you remember?

R: I don't know ... I don't know (shakes his head).

Although in this case, there was no symmetric configuration on the balance scale to interfere with the use of arithmetic, Rin nonetheless was not readily able to use arithmetic to balance the scale. Perhaps his previous experience with SITUATION2a and his sense of symmetry as an approach to balancing the scale (he was one of two children whose first strategy of the session was SYMMETRY2) contributed to his use of SYMMETRY2 in this instance.

In the next Episode, I presented SITUATION5, where he once again "forgot" he could use arithmetic.

4.2.3 ARITHMETIC4

Only Rac used ARITHMETIC4 after ARITHMETIC3. She seemed to think that the scale balanced when "the same number" could be calculated on each arm, but she did not think that there was necessarily one single way of calculating this "number." ARITHMETIC3 was one possibility, but other methods might also work. Rac appeared to think that choice of calculation was somewhat arbitrary, and she did not seem to tie her use of arithmetic to compensation. She had shown no uncertainty about which calculations to perform in her previous use of arithmetical Strategies -- three consecutive uses of ARITHMETIC1 in SITUATION2a. Since no multiplication was required in these instances, the numbers on the scale may have led her directly to the necessary calculation.

After first using ARITHMETIC3, Rac vacillated between ARITHMETIC3 and other Strategies for several Episodes. She first used ARITHMETIC4 after she had used ARITHMETIC3 twice. I put a ring on the sixth peg and three on the fourth on one arm, held the scale steady, and asked her if she could put five rings on the other arm so that the scale would balance when released (Figure 4.3). Rac put two rings on the third peg, one on the second, and two on the first on the other arm (Figure 4.4). Although I was unable to determine what calculation she had performed to arrive at this solution, her comments as she placed the rings on the scale suggest she *had calculated something.* She seemed aware of ARITHMETIC3, but not sure if she should use it.

> **R:** And ... uh ... that would be ... twelve [the three rings on the fourth peg]. Oh, this isn't going to work. How did I figure that out? (mumbles) ... Um ... two and that's four, so one, two here [on the first peg] and one here [on the second peg] makes four, four and six.
>
> **M**: Do you think that's going to work?
>
> **R:** Yeah, I think so.

When the scale tipped, Rac used ARITHMETIC3 to readjust the rings so that it balanced.

In the next Episode, I removed the rings Rac had placed on the scale, held the scale steady, and asked if she could balance it with two rings on the other arm (Figure 4.5). Although she had just correctly calculated torque for the *same configuration*, this time she calculated a sort of "cross product" of weight and distance. For each of the two pegs used, she multiplied the number of the peg by the number of rings on the other peg. She then added the two products. Since one ring was on the sixth peg and three were on the fourth, she calculated $(6 * 3) + (4 * 1)$ and came up with a "torque" of 22.

After she calculated this, I reminded her of what she had previously calculated.

> **M:** Remember last time you said this was twelve [the three rings on the fourth peg], ... remember, when you put some on two and some on one [to equal twelve on the other arm]?
>
> **R:** Twelve and six would be eighteen.

M: But is this [the three rings on the fourth peg] twelve or is it eighteen?

R: Well, it's twelve in a way.

M: Twelve in a way?

R: It's three, three times four would give you twelve.

M: Uh huh.

R: And six more would give you the same as six times three.

M: How do you figure out what number it is?

R: I don't know, you just ... experiment with it.

M: Just experiment ... well, does it matter if you call it [the torque on one arm] twenty-two or if you call it eighteen?

R: It might.

Rac proceeded to use ARITHMETIC3 to construct a configuration with a torque of twenty-two on the other arm. The scale and it tipped (Figure 4.6).

In the next Episode, I removed everything from the scale and set up a new configuration on one arm. I put two rings on the seventh peg and three on the fifth, held the scale steady, and asked Rac if she could put three on the other arm to balance the scale (Figure 4.7). Rac readily calculated the correct solution with ARITHMETIC3. Before she put any rings on the scale I asked her why she didn't use the Strategy she had used before.

M: (as she places rings) If I put three on the fifth and two on the seventh, can you use three on the other side to make it balance?

R: Three? ... I'd say five times three would be fifteen

M: Uh huh

R: Then seven times two would be fourteen, add it together, it would be twenty-nine.

M: Why wouldn't you say seven times three, that's what you were doing before, remember?

R: Oh ... I don't know ... just different ways of figuring it ...

R placed two rings on the tenth peg and one on the ninth. The dialogue continued as M held the scale steady.

M: How do you get your answer?

R: Well ... it depends on the numbers, I guess. Um ... I don't know why, but sometimes I times from here and here [the number of the peg and the number of rings on that peg], sometimes I times there and there [the "cross products"] and then add them.

After I released the scale and Rac saw that her solution worked, she still did not seem sure that there is only one way to calculate torque. Her uncertainty seemed to continue throughout the rest of the session, although from this point on she always used ARITHMETIC3 whenever she used an arithmetical Strategy.

4.2.4 NUMBERS

Four children used NUMBERS after ARITHMETIC3. Three used it in SITUATION4. The other child, who did not encounter SITUATION4, used it in SITUATION6.

4.2.4.1 NUMBERS in SITUATION4

Jeo, Rac, and Tot, the three children who used NUMBERS in SITUATION4 all appeared to have been using ARITHMETIC3b (rather than ARITHMETIC3a) earlier in the session. They constructed, explained, and/or made predictions about configurations with multiple rings on multiple pegs by calculating torque, but they did not appear to relate these calculations to compensation.

For both Jeo and Rac, this use of ARITHMETIC3b seemed to reflect a failure to relate arithmetic and compensation that had persisted throughout much of the session. Indeed, for Jeo (and perhaps also to some extent for Rac), this failure seemed to reflect an attempt to "see" the balance scale *only* in terms of numbers and arithmetic.[22] For Tot, the issue of relationship between arithmetic and compensation did not arise until SITUATION4. Her initial use of NUMBERS in SITUATION4 led me to infer that she did not see arithmetic as a quantification of compensation (and thus, had been using ARITHMETIC3b). Her experience in SITUATION4, however, seemed to lead her to believe that distance, not numbers, affected the operation of the balance scale. Thus, I inferred that during SITUATION4 she switched to ARITHMETIC3a.

Both Tot and Rac navigated SITUATION4 by oscillating between the notion that distance matters (and not numbers) and the notion that numbers matter (and not distance).

When I first presented SITUATION4 to Rac, the scale was balanced with three rings on the third peg and one on the fifth on one arm, and one on the tenth and one on the fourth on the other (Figure 4.8). Rac thought that altering the numbers would affect the stability of the scale.

> **M:** What if we made this [the 3 marking the third peg with three rings on it] into an eight?
>
> **R:** Then I'd just say eight times three plus five.
>
> **M:** Do you think it would be eight times three plus five
>
> **R:** (interrupting) Yes.
>
> **M:** Instead of three times three plus five?

[22] See [Kliman 86] for an analysis of Jeo's use of numbers in the balance scale session.

R: ... It could be ... but this way, if I just left that [the configuration on the arm with unchanged numbers] like that, it wouldn't work any more [because the other side would now total twenty-nine, instead of fourteen].

I held the scale steady, changed the 3 (marking the third peg with three rings on it) into an 8 with a magic marker, and covered the left half of the 8 on the same arm, so that it looked like a 3 (Figure 4.9). Rac temporarily changed her mind about the stability of the scale.

M: Do you think this is going to work [if I release the scale]? ... What do you think will happen now?

R: That ... it will balance.

M: Why?

R: 'Cause ... even if it's an eight or not, it's still in the same place.

Before releasing the scale I wanted to summarize what I thought was Rac's position -- changing the numbers won't alter the stability of the scale as long as rings remain "in the same place" -- and see if she agreed. But before I got very far, she interrupted me with essentially the opposite position.

M: So even if we fool around with the numbers it wouldn't

R: (interrupting) It wouldn't have to change your answer *too* much, it would just change your answer over here [on the arm on which numbers were not changed].

M: It would change your answer?

R: Yuh, like you'd maybe do eight times three, which is twenty-four, plus five would equal, like, ... twenty-nine.

As Rac said this, she added rings to the "unchanged" arm of the scale so that the torque was twenty-nine. Several of her comments over the course of the session suggest that at least some of the time she thought of the balance scale as an arithmetic equation. The torque on each arm forms half of the equation, and if one half of the equation is changed the other half must be changed accordingly. In the present instance, however, Rac began to doubt that the "equation" would hold. As I started to release the scale she said "but I don't think that would balance." When the scale came to rest tipped, she said "it's not enough." In this last comment she appeared to be referring to the torque on the arm with changed numbers. As SITUATION4 drew to a close, Rac remained puzzled about why the scale had not balanced.

Although throughout SITUATION4 Tot was also unsure of the role of numbers on the balance scale, she seemed to emerge from SITUATION4 with a better understanding.

When I first asked Tot if she thought interchanging the 3 and the 8 on one arm would make a difference, she said that it might because "maybe they're [the numbers are] the most important thing." However, she began to change her mind when I actually "interchanged" the numbers on the balance scale, held the scale steady, and put a ring on each of the "eighth" pegs -- one on the "new" eighth peg and one on the "real" one on the other arm (Figure 4.10).

M: Will it balance now?

T: ... This is confusing now ... no, I don't think so.

M: Why not?

T: Well ...

M: This is an eight and that's an eight, they're both on the same number.

T: ... This isn't as easy as I thought it was.

M: Why

T: (interrupting) I don't know, because the thing doesn't have brains.

Tot seemed uncertain. On one hand, if the scale doesn't have brains, it can't tell what numbers are on it. On the other hand, "it might just think that's an eight [the 'new' 8] because that's an eight [the 'real' 8], so it's in the same shape."

> **M:** How does it know what shape I drew on?
>
> **T:** ... That's what I don't ... it knows this is an eight because it knows that's an eight and they both have the same shape.
>
> **M:** So it can sense what shape I put on?
>
> **T:** ... No ... it ... I don't know, let's try, I'm not sure.

Just before I released the scale, Tot said that the scale would not balance because "that's a three, it's a fake eight." Her comments from this point on suggest that she understood that the numbers on the scale reflect distance, and that altering the numbers will not affect the scale's operation.

Unlike both Tot and Rac, Jeo consistently maintained that the numbers on the scale affect its operation. Like Rac, Jeo seemed to emerge from SITUATION4 *without* a better understanding of the relationship between the numbers on the scale and distance. When I suggested interchanging the 3 and the 8 on one arm and then putting a ring on each of "eighth" pegs (the "real" and the "fake" eighth pegs), Jeo said that the scale would balance. When I tried this (by putting a paper 8 over the "real" 3 and a paper 3 over the "real" 8) and the scale tipped, she said "it didn't work 'cause this is a paper, and this [the 'real' 8] is, like, a sticker." She went on to insist that if the actual *stickers* (the "3" sticker and the "8" sticker) were interchanged, the sale would balance.[23]

4.2.4.2 NUMBERS in SITUATION6

Dar's use of NUMBERS took quite a different form. He used NUMBERS twice, once near the beginning of the session and once near the end. The two instances seemed related.

Early in the session, Dar claimed that a year ago he used to compare quantities of money with a balance scale. He would represent a multidigit number on each arm of the scale by placing a weight on each of the relevant digits on that arm. The scale would tip to the arm with the larger number represented. To demonstrate, he put a ring on the second and first pegs on one arm to represent 21, and a ring on the second and sixth pegs on the other arm to represent 26. The scale tipped to the "26" (Figure 4.11).

I presented a counter example. I held the scale steady and moved a ring on the arm with "21" from the second to the third peg. The scale now had "31" on one arm and "26" on the other (Figure 4.12). When I asked Dar what would happen when I released the scale, he said the scale would tip to the arm with "31." He was surprised and confused when he saw that it tipped in the other direction.

At this point, NUMBERS seemed to be just another addition to Dar's repertoire of sometimes contradictory Strategies. Before NUMBERS, he had used RINGS, COMPENSATION1, COMPENSATION2, and ARITHMETIC1. After NUMBERS, he

[23] I did not actually interchange the stickers on the scale. If I had done so, I doubt Jeo would have readily understood the relationship between numbers on the scale and distance. In addition, I suspect that she might have come up with another reason why the scale did not balance in this particular case.

continued to use a range of Strategies until he developed ARITHMETIC3. He then used only ARITHMETIC3 for six consecutive Episodes, and in the next Episode (SITUATION5) he used SYMMETRY2 very briefly and then returned to ARITHMETIC3.

After SITUATION5, Dar said that he was still confused about something that had bothered him earlier in the session: the fact that the scale can balance when more rings are on one arm.[24] He likened his confusion to his earlier use of NUMBERS: "Like, when I was doing the money thing and ... that still doesn't make sense ... I had more money over here, less money over here, and it went down over here [on the arm with 'less money']." In both cases -- the "money thing" and a balanced scale with more rings on one arm -- he recognized that the scale balances if and only if the torque is the same on each arm. Yet, he seemed to see this as counterintuitive. The scale *shouldn't* balance simply because the torque is the same on each arm. It should balance because the same number of rings is on each arm, whatever the torque. Or, it should tip because a larger number is "represented" on one arm, whether or not torque is greater.[25]

Dar and I set up an example in which the scale tipped to the arm on which the smaller number was "represented." I put "42" on one arm, and he put "25" on the other (Figure 4.13). After studying the configuration for several seconds,

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[24]Dar's use of RINGS here is discussed below.

[25]These two notions are, of course, contradictory. According to the first notion, *any* two-digit numbers "represented" should balance the scale, since two rings are used on each arm. According to the second, *no* two-digit numbers "represented" should balance, since it is impossible to "represent" the same number on each arm: Dar read the numbers he "represented" from left to right, so numbers on the left arm would always have a larger ten's digit and numbers on the right arm would always have a smaller ten's digit. Dar did not appear to have thought through these notions sufficiently to have noticed this internal inconsistency.

Dar related his "money thing" to arithmetic.

> **D**: I know why [it tips] now.
>
> **M**: Why?
>
> **D**: 'Cause over here it's [the torque is] six and over here it's seven. And if I put one on (adds a ring to the first peg on the same arm as "42") it will be even.

Dar's confusion about the "money thing" appeared to be resolved. Neither he nor I brought it up again.

4.2.5 RINGS

Only Dar used RINGS after ARITHMETIC3. He had used it once before, as his very first Strategy in SITUATION1. He used a range of Strategies until he developed ARITHMETIC3, and from that point on, he used ARITHMETIC3 consistently for several consecutive episodes. Near the end of the session, he again expressed confusion about how the scale can balance with more rings on one arm. This confusion surfaced as he explained why the scale balanced with a ring on the sixth and tenth pegs of one arm, and a ring on the seventh and ninth pegs of the other arm. First he gave an arithmetical explanation, then he noted, seemingly as an explanation, "you have two on each side." At this point, Dar seemed to realize that these two kinds of explanation -- one based on arithmetic and one based on number of rings -- might sometimes contradict each other. Furthermore, if the scale always balances when the same number of rings is on each arm, it shouldn't be necessary to calculate torques to determine the scale's stability.

> **D**: So, I was wondering, if you put, like forty (puts three more rings on the tenth peg), six, forty-six. And you put thirty (puts two on the tenth peg on the other arm, another on the ninth, moves the ring on the seventh to the third, and puts one on the fourth and one on the first, so

that the total is forty-six on each arm -- see Figure 4.14) ... forty-six in different ways, what would happen? (The scale comes to rest balanced.) ... But they're both forty-six. But ... but ...

M: What?

D: I don't know ... Before, there's, uh, five over here, and here there's, uh, seven. So I wonder, how come there's seven over here and five over here and it's still even [balanced]?

M: How did you decide where to put ... you said you added ... and you put forty-six on each side.

D: Yuh.

M: So ...

D: It would, it should have come out, it *is* even. But there's more of ... what do you call (motions to rings) ... over there than there is here.

M: Do they add to the same number?

D: ... uh, yes.

M: Isn't that why you put them there?

D: Yuh, but (mumbles).

In trying to resolve his confusion, he continued constructing configurations in which torque was the same on each arm but the number of rings was not. He also constructed configurations in which the number of rings was the same on each arm but torque was not (e.g. Figure 4.15). He continued this experimenting for several minutes,[26] interrupting it only briefly to relate the present series of events to an earlier confusion involving NUMBERS (discussed

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[26] I did not break Dar's series of experiments into Episodes. Because of the way he proceeded -- constructing and altering configurations and oscillating among different explanations and ideas about them -- I felt that for the most part, any such divisions I made would have been arbitrary.

above).

Dar ultimately seemed to have accepted the notion that stability is determined by torque, not number of rings. Although he had "known" earlier that this was the case, he did not until the very end of the session seem comfortable with this notion. His experiments ceased, and when I asked him what he found most interesting about the balance scale session, he replied that he had learned "you don't have to have the same number [of rings] to make it equivalent [balanced, as determined by equal torques]. That *was* funny, though! That was hard, too."

4.3 Summary

Most children neither used ARITHMETIC3 consistently nor seemed sure it could be used consistently. Sometimes they didn't seem to *think to use* it. Other times they weren't sure it would work. Even when they seemed sure it would work, they didn't always use it correctly. Frequently children seemed to treat ARITHMETIC3 as a powerful, but not necessarily pervasive approach to the balance scale. They did not seem to see it as "the law." Indeed, a child's use of other Strategies often seemed to reflect inability to integrate or reconcile her various notions of the balance scale with ARITHMETIC3.

Experience with the balance scale seemed to lead children to become more confident in and more able to use ARITHMETIC3 in a wider range of contexts. Nonetheless, conflicts, misconceptions, and use of other Strategies often persisted throughout the session.

Chapter Five

Discussion and conclusions

5.1 The context of knowledge

A fundamental issue the present study raises is the *relevance* a kind of knowledge takes on in a particular context: it may be seen as *definitely* (or *definitely not*) relevant, it may be seen as *possibly* relevant, or it may *not even come to mind.* Furthermore, seeing knowledge as relevant may or may not accompany knowing exactly *how* it is relevant. Sometimes children used ARITHMETIC3 consistently and seemingly without considering other Strategies, sometimes they did not know *if* they could use ARITHMETIC3, and sometimes they did not seem even *to think to use* ARITHMETIC3. In addition, sometimes they seemed to know that ARITHMETIC3 was relevant but didn't know exactly *how.*

Children did not always seem sure how their various Strategies related to ARITHMETIC3: If ARITHMETIC3 was right was another Strategy wrong? Could they both be right? Was each only right *some of the time*? Did one somehow encompass the other? Does anything change when numbers on the balance scale are switched? By the end of the session many children had still not resolved relationships between ARITHMETIC3 and other Strategies. However, these children did not necessarily seem to think that there *should be* one arithmetic "law" governing balance scale stability.

It is widely accepted that people seek to generate unified theories to explain observations of certain kinds. For example, Karmiloff-Smith and Inhelder [Karmiloff-Smith and Inhelder 75], in discussing children's construction of "theories-in-action" state that "[T]he tendency to explain phenomena by a unified theory, the most general or simplest one possible, appears to be a natural aspect of the creative process, both for the child and the scientist (p. 209)." My data does not contradict this notion, but rather, questions what constitutes a phenomenon about which one forms a theory. In this case, is it the balance scale *per se*? Is it the particular balance scale used in the study and the Situations? Is it a (perhaps dynamic) subset of these Situations? What kind of knowledge is required to master each? How stable is knowledge across such phenomena?

In the present study, both objective and subjective factors seemed to play a part in establishing the contexts in which children used ARITHMETIC3. Objectively, there are classes of questions and problems for which ARITHMETIC3b should be sufficient. Other classes of questions and problems require, in addition, understanding of relationships between ARITHMETIC3b and compensation -- some versions of ARITHMETIC3a may be sufficient, some may not. For the child, however, these classes were not always distinct or steadfast. Children sometimes *changed* a problem from one that required only calculation of torque to one that required understanding of relationships between this calculation and compensation.

For example, one child's consistent use of ARITHMETIC3 throughout several consecutive Episodes in SITUATION3 did not reflect the conflicts between ARITHMETIC3 and RINGS that he showed later in the session. He readily calculated torques in order to predict and explain balance scale stability and construct balanced configurations. Yet, later, when he began to consider how it was possible for the scale to balance with more rings on one arm, his use and understanding of ARITHMETIC3 appeared in a different light. He seemed uncertain about the status of ARITHMETIC3: Why should the scale balance simply because torque is equal on each arm? Isn't it also important to have the same number of rings on each arm? At least at this point, he did not seem to see ARITHMETIC3 as "the law" at all. At the end of the session he seemed to have come to understand that no matter how many rings are on each arm, torques must always be equal if the scale is to balance. However, I suspect that the introduction of some new consideration might have led him once again to question his use of ARITHMETIC3.

Another child's use and understanding of "the law" was thrown into a different light by a change in context that *I* introduced. After some vacillation (albeit not clearly resolved) between ARITHMETIC3 and ARITHMETIC4, she used ARITHMETIC3 for several consecutive episodes. At this point I "interchanged" two numbers on the same arm of the scale. She continued to use ARITHMETIC3, but she did not know if she should use the *number under the relevant peg* or the *distance from the fulcrum to the peg* in calculating torque.

In these and other instances, a child at first appeared to be using "the law," but, with the introduction of additional considerations, no longer appeared to be doing so. What was an adequate approach to the balance scale in one context was no longer adequate. When, then, can we say a child knows "the law"? The answer depends in part on what set of balance scale phenomena "the law" explains. If knowing "the law" requires ability to predict, explain, and construct in all six Situations, then not all children emerged from experimental sessions with "the law." If knowing "the law" only requires ability to predict, explain, and construct in SITUATION3, then more but still not all children emerged with "the law." Even in SITUATION3, children sometimes used "the law" consistently, and then suddenly began to doubt it. Thus, another aspect of knowing "the law" may be *knowing it always works*, and consequently, knowing that anything that contradicts it *doesn't work.* Although many children appeared eventually to see that the scale always balances *as long as torques are equal*, doubts about the status of ARITHMETIC3 still seemed to linger, perhaps in mitigated form. Even children who *usually* seemed to see that ARITHMETIC3 arises from the functioning of the balance scale, occasionally seemed doubtful in particular contexts.

5.2 "The law" in previous balance scale studies

My focus on when and how children use "the law" after first developing it contrasts with the focus on underlying consistencies in both the Inhelder and Piaget and the Siegler balance scale research. (See Chapter 1 for a brief discussion of their balance scale work.) To Inhelder and Piaget and to Siegler "the law for balancing" seems to take on a kind of monolithic quality. Inhelder and Piaget focused on underlying understanding of proportionality reflected in children's use and explanation of "the law," and Siegler focused on "the law" as a rule that children use. Neither explored relationships between knowing "the law" and the contexts in which that knowledge emerges as relevant. Nonetheless, each has a specific view of "the law" and considers what use of this "law" in a certain context implies about balance scale understanding. In particular, Inhelder and Piaget used an experimental environment conducive to the emergence and observation of proportional relationships and Siegler used an experimental environment conducive to the emergence and observation of rules.

Inhelder and Piaget used the balance scale as a way of investigating how the "proportionality schema develops as it is linked with the equilibrium schema ([Inhelder and Piaget 58], p. 164)." Accordingly, they set up their experiment "in a way that would force the question of proportionality (p. 164)." They used a balance scale from which only one group of weights could be hung on each arm. Children at Stage IIIb (the highest level of balance scale understanding) are able to explain the functioning of the balance scale in terms of proportional relationships among weight, distance, and *work* required to raise a given weight at a given distance to a height such that the scale is level. Thus, "the law" that they possess includes ARITHMETIC2, an understanding of the relationship between ARITHMETIC2 and *work*, and some ability to express this understanding.[27]

Siegler's view that "children's problem-solving strategies are rule-governed, with the rules progressing from less sophisticated to more sophisticated ([Siegler 78], p. 111)" led him to "create problem sets that yield sharply differing patterns of correct answers and errors depending on what rule is being used (p. 111/112)." His experimental environment included a balance scale with four pegs on each arm and uniform weights, and sets of prediction problems intended to distinguish use of different rules. Children at the highest level of understanding (Rule IV) can make predictions about the stability of a balance scale by computing torques. However, whenever possible these children use simpler tests, essentially the "correct" aspects of Rules I, II, or III. "The law," then, is ARITHMETIC3b as well as knowledge about how to make predictions by comparing weight and distance on each arm.

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[27]Unlike the children in the Inhelder and Piaget study, the children in my study did not seem to focus on height, and no one mentioned *work*. This difference may relate to specific aspects of the two studies. Inhelder and Piaget were particularly concerned with how children relate the balance scale to *work*. The questions and problems they posed throughout the session may have led children to pay more attention to height, and perhaps consequently, to relate the balance scale to *work*. By contrast, I was more concerned with how children relate arithmetic to the functioning of the balance scale. None of my questions and problems specifically addressed changes in height. The experimental materials may also have influenced the salience of height. A schematic diagram Inhelder and Piaget include suggests that the base of their balance scale is approximately as high as an arm is long. The base of my balance scale is only about 1/3 as high as an arm is long. Thus, with my balance scale, height is less salient. Finally, children's age and education may bear on the extent to which they relate the balance scale to *work*. In particular, the Stage IIIb children Inhelder and Piaget interviewed were two or three years older than the children in my study, and may have already been introduced to *work* in school science classes.

Neither Siegler nor Inhelder and Piaget report that children questioned the status of "the law" once they first began using it. Children's perspectives on "the law" in these studies may have related in part to the sorts of balance scale phenomena they encountered over the course of the experimental session. In addition, children in these studies may not have had the opportunity or felt the need to question the status of "the law" that they used.[28] For example, suppose a child using Rule IV in Siegler's prediction tasks had occasional doubts about whether the scale *always balances* if more rings are on one arm. The prediction tasks would neither reflect any such doubts nor specifically invite expressing or exploring them.[29] Although the experimental environment that Inhelder and Piaget used may have been more conducive to expressing and exploring such doubts, their protocols do not suggest that experimental sessions with Stage IIIb children continued once these children discovered and explained "the law." If the experimenter did not continue to introduce questions and problems (and perhaps if the experimenter appeared satisfied that the child had fully explained "the law" and even appeared ready to terminate the experimental session) the child may have been less likely to prolong the session.

[Hardiman *et al.* 86] in some respects present a perspective intermediate between my focus on the contextualization of balance scale knowledge and the focus on a more monolithic kind of knowledge in the Inhelder and Piaget and the Siegler studies. Hardiman *et al.* seem to consider "the law"

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[28]In some of Siegler's training conditions, children were told "that there were rules underlying the balance scale's behavior that they could discover if they 'watched [the stability of the balance scale with different configurations on it] carefully and thought about it' ([Klahr and Siegler 78], p. 82)." Thus, in these instances children were looking for rules, and therefore might have been more likely to see what they discovered as "rules" and less likely to question any of these "rules."

[29]Another possibility is that children did not actually use one of the four Rules in Siegler's prediction tasks. See [Hardiman *et al.* 86] and [Wilkening and Anderson 82] for discussion of alternative Strategies that might have resulted in the same patterns of response as the Rules.

(ARITHMETIC3b) as something that is either *known* or *not known*. However, they also show that when college students are asked to induce "the law" from prediction problems with feedback, they develop more limited "rules" which they use and appear to view as my subjects used and appeared to view ARITHMETIC3.

One of these limited rules was "the ratio rule", which is essentially ARITHMETIC2, or "the law" of the Inhelder and Piaget study (although not necessarily accompanied by understanding of relationship to *work*). It is not one of Siegler's four Rules. Subjects' first uses of the ratio rule usually involved configurations with weight and distance in small ratio (e.g. one weight at a distance of two units on one arm, two weights at a distance of one unit on the other arm). Only gradually did subjects begin to generalize from these simple special cases.[30] Although at one point subjects may have *appeared to be* using a rule that could provide a way of predicting whenever only one stack of weights was on an arm, the subjects themselves were not always able to use this "rule" so widely, nor did they necessarily think the "rule" should apply so widely.

> When subjects first verbalized a form of the ratio rule, they rarely made comments suggesting that the rule should hold for all ratios, and in some cases expressed doubts that the relationship observed in a few instances was generalizable. For example, one subject said in considering whether the ratio rule verbalized earlier for a smaller ratio would hold for 4:1, "I know we kept the proportion the same, but I thought that there was a point at which you went one too many down here [referring to distance] and the ratio didn't stay the same just because it was so far out on the end. You want to put one more block on there and more that one more? [requesting that the interviewer modify the problem from 1000/4000 to

[30] See [Kliman 87] for discussion of a similar phenomenon.

> 10000/5000.][31] Oh, you mean it's a constant rule, it doesn't change?" [after observing that the beam still balanced] (p. 82).

In addition, use of the ratio rule was inconsistent.

> The ratio rule may not be applied at all or may be applied to make a correct prediction on one problem and not be applied when a similar problem is presented a few trials later. The Strategy employed depends on the particular problem and on problems that were encountered earlier (p. 81).

Hardiman *et al.* terminated experimental sessions when subjects used "the law" five times consecutively, in other words, when subjects made five consecutive correct predictions for configurations with multiple stacks of weights on an arm. They do not say if they found any inconsistencies in subjects' use and apparent views of the "the law," as they found with the ratio rule.

Differences in the apparent status of Strategies both between the Hardiman *et al.* study and my study, and within the Hardiman *et al.* study may relate to subjects' expectations. Hardiman *et al.* asked subjects to find a rule governing all predictions, thus, subjects may have been less likely to question any such rules they developed. In my study the issue of a rule governing the functioning of the balance scale did not arise. However, in neither study were subjects led to expect they would find a rule, such as the ratio rule, governing configurations with only one stack of weights per arm. And in neither study did they readily see ARITHMETIC2 (or its equivalent, the ratio rule) as a reliable approach to the balance scale whenever only one stack of weights was on an arm.

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[31] In the notation convention adopted by Hardiman *et al.*, this means changing a configuration from one weight at a distance of four units from the fulcrum on one arm and four weights at one unit on the other arm to one weight at five units on one arm and five weights at one unit on the other arm. Subjects used a balance scale with units and half units marked along the arms and uniform weights.

5.3 Implications for the structure of balance scale understanding

I believe that neither a rule-based nor a stage-based progression can adequately characterize the series of approaches to the balance scale subjects developed in the Hardiman *et al.* study and in my study. Hardiman *et al.* stress that "It is unlikely that any simple stage analysis can characterize the changes in knowledge states [that they observed] in more than a superficial manner. Our analysis of the protocols does not depict the subject as relentlessly progressing through a well-defined sequence of levels until the product-moment rule [ARITHMETIC3b] is reached (p. 80-81)." Likewise, in my study, children's use of Strategies did not appear to conform to a strict developmental progression. Although children progressed toward increased use of arithmetical Strategies, they continued to use and to develop other sorts of Strategies as well.

Children's use of Strategies is better characterized as reflecting the competition and interaction of ever-changing and sometimes inconsistent bodies of knowledge.[32] In particular, I believe that diSessa's *cuing priority* and *reliability priority* (see Chapter 1) provide a way of thinking about children's "regression" to less advanced Strategies once they began to use "the law." Over the course of the session, arithmetical Strategies (with the exception of ARITHMETIC4) frequently led to successful predictions, explanations, and constructions. As children found more and more contexts in which arithmetical Strategies led to success, these Strategies increased in reliability and cuing priorities. However, after ARITHMETIC3 was first used its reliability priority was not always high enough to insure that it would work *all the time*. In addition, when cuing priority of other Strategies remained high, doubts about which Strategy to use continued to emerge. In some contexts the disparity between cuing priority of

[32] Although, as is discussed in Chapter 2, I do not assume that each Strategy necessarily corresponds directly to one body of knowledge.

ARITHMETIC3 and that of other Strategies was great enough that children didn't even *think to use* ARITHMETIC3.

Changes in priorities can also provide a way of thinking about changes in the child's view of "the law" and the contexts in which she uses it. As the priorities of a Strategy increase in a range of contexts, the Strategy evolves: an approach used with some uncertainty in a narrow set of contexts may grow into an approach used with confidence in a wide range of contexts. Although a Strategy may be *manifested* in the same way (for example, as the computation and comparison of torques) at different points in the session, each manifestation may be accompanied by a different kind of understanding. Thus, what I classify as manifestations of a single Strategy, the child may see as very different approaches to the balance scale. Likewise, as a child comes to use a Strategy more widely, what she sees as distinct contexts may change. For example, if she is able to use a particular Strategy in one context but not another, she may see the two contexts as very different. If she is able to use the same Strategy in both contexts, she may see some relationships between the two contexts, perhaps even see them as "the same" sort of context.

5.4 Conclusions

Although "the law for balancing" may ultimately grow to control or supersede *at least some* other balance scale knowledge, it need not reign sovereign on first emerging. After the children in the present study first developed what appeared to be "the law" they did not always know *how* or *if* it could be used, understand its relationship to other approaches to the balance scale, or see it as "the law." An experimental session structured around a more narrow view of context might have prevented the emergence and observation of this kind of use and understanding of "the law." The child's behavior in *one context* may seem to reflect a certain kind of understanding -- perhaps use of a particular rule or adherence to a particular fundamental principle -- but her behavior in *a slightly altered context* can put this understanding in a very different light.

We must not lose sight of the fact that the child's understanding can only be studied through particular contexts. Sensitivity to these contexts is critical if we are ever to approach the child's perspective.

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Appendix A

Figures

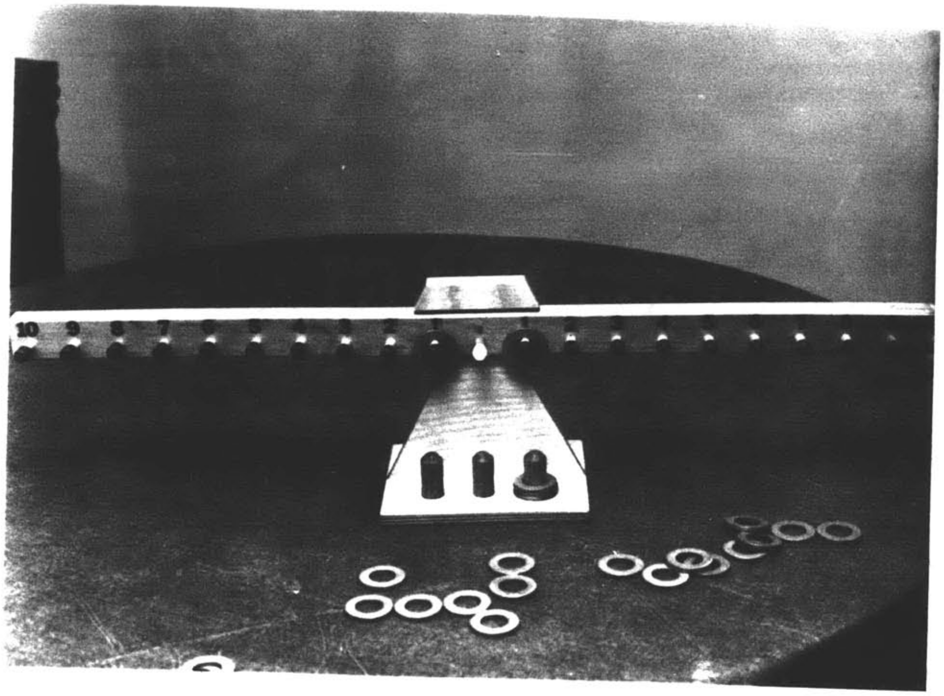

Figure 2.1. The balance scale and weights

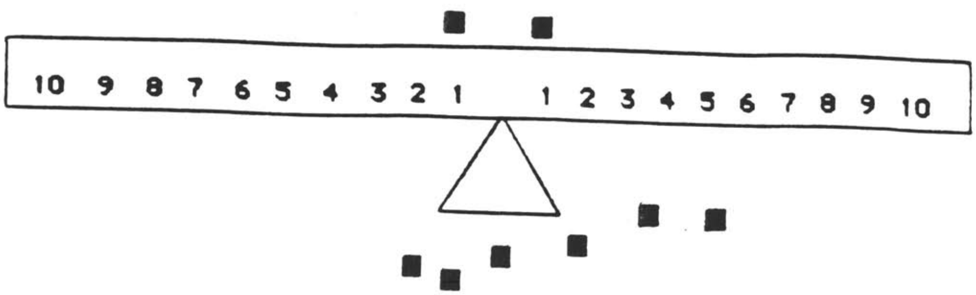

Figure 2.2. Schematic representation of the balance scale and weights

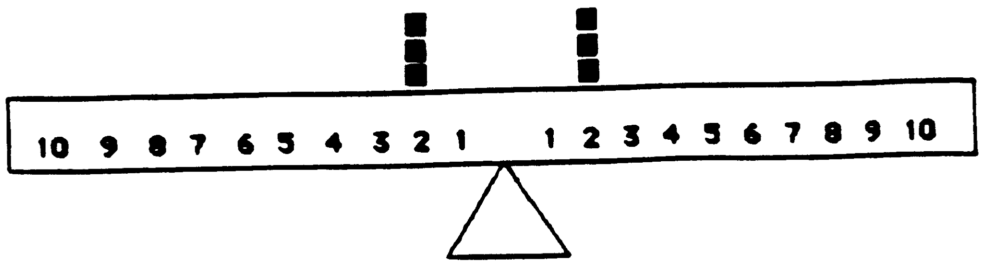

Figure 2.3. Schematic representation of three rings on the second peg of each arm

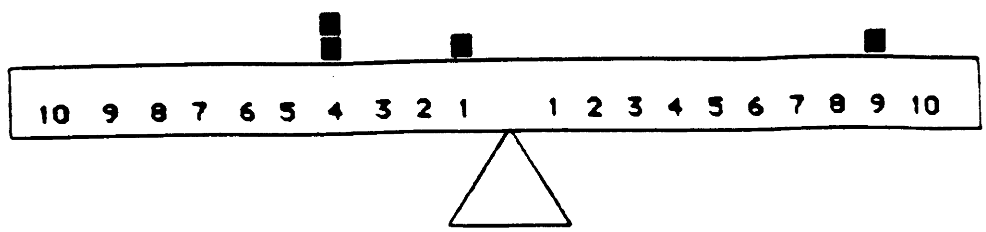

Figure 2.4. Balanced, with a torque of nine on each arm

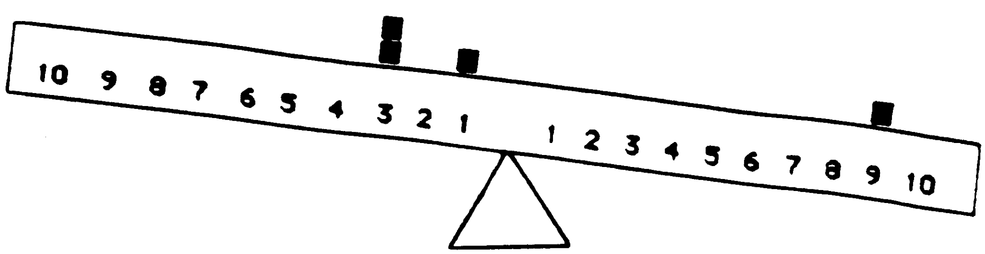

Figure 2.5. Tipped, with a torque of seven on one arm and nine on the other

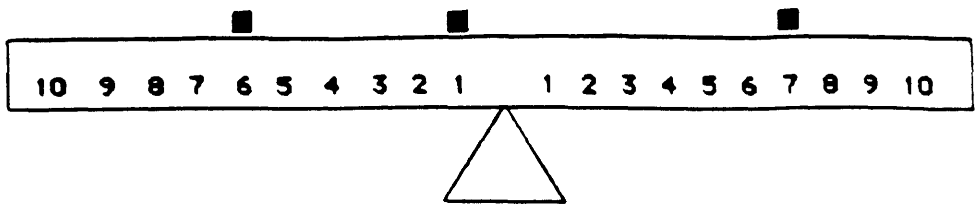

Figure 2.6. A typical SITUATION2a configuration

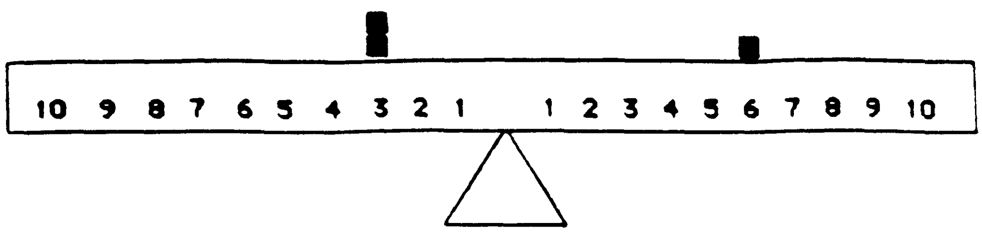

Figure 2.7. A typical SITUATION2b configuration

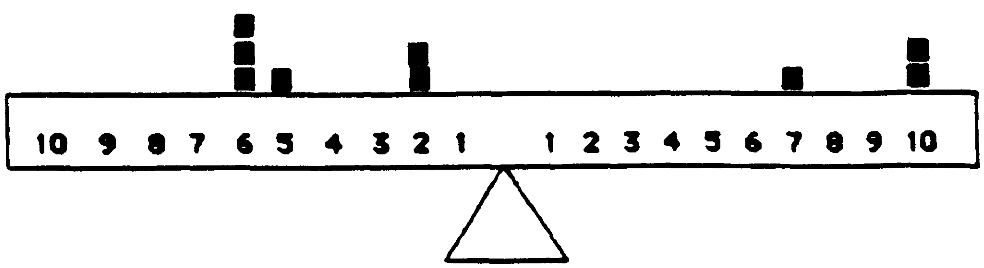

Figure 2.8. A typical SITUATION3 configuration

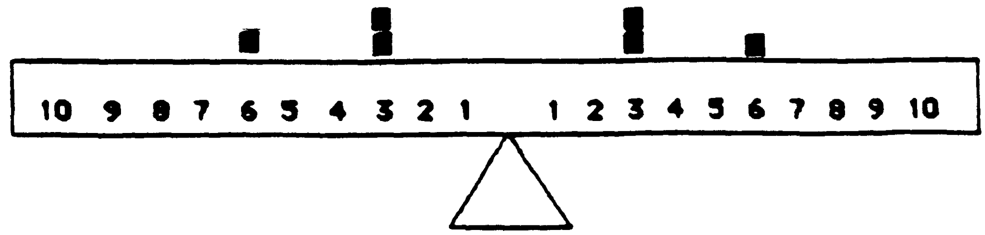

Figure 2.9. The SITUATION4 problem: Can you remove three rings so that the scale stays balanced?

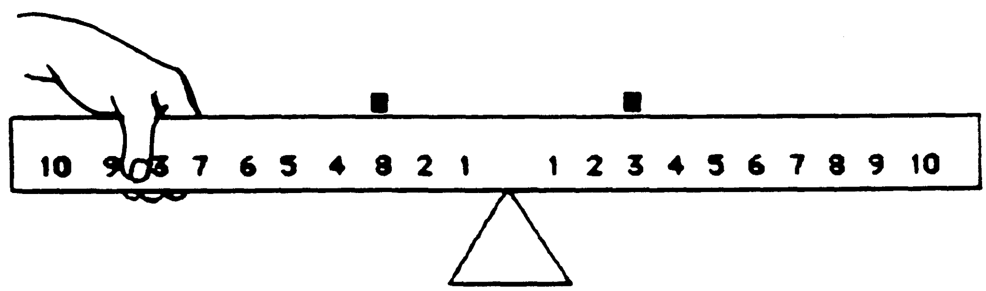

Figure 2.10. 3 and 8 "interchanged" on the left arm

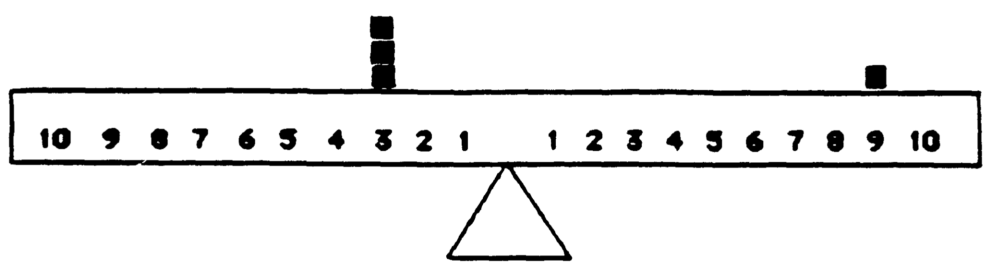

Figure 2.11. The scale balanced with more rings on one arm

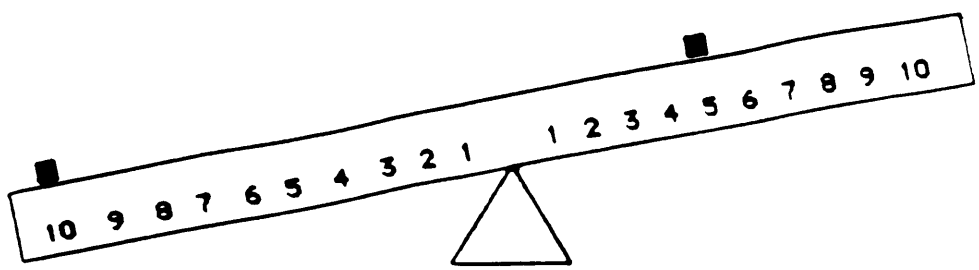

Figure 3.1. An attempt to balance the scale

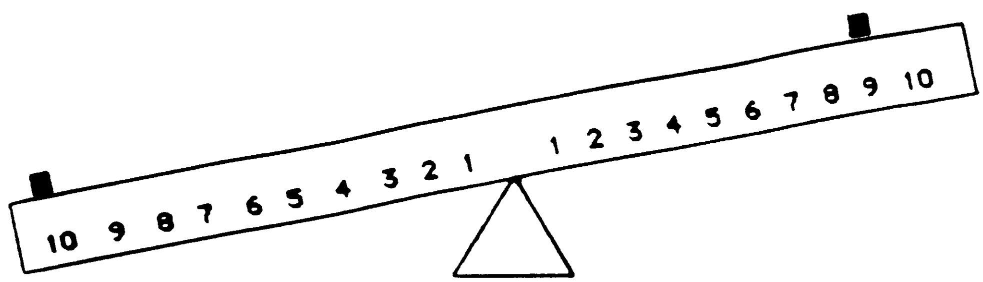

Figure 3.2. An attempt to compensate for tipping by moving the ring on the raised arm away from the fulcrum

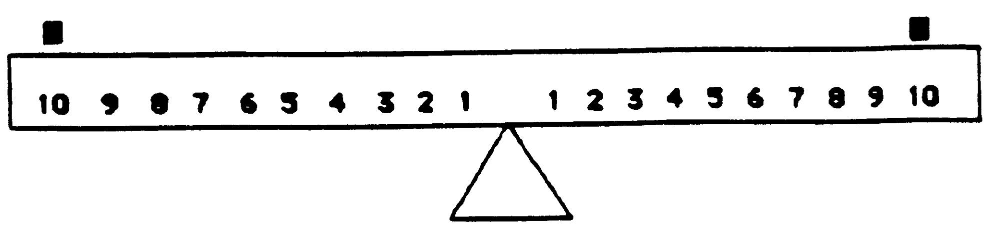

Figure 3.3. A second attempt to compensate for tipping by moving the ring on the raised arm even further away from the fulcrum

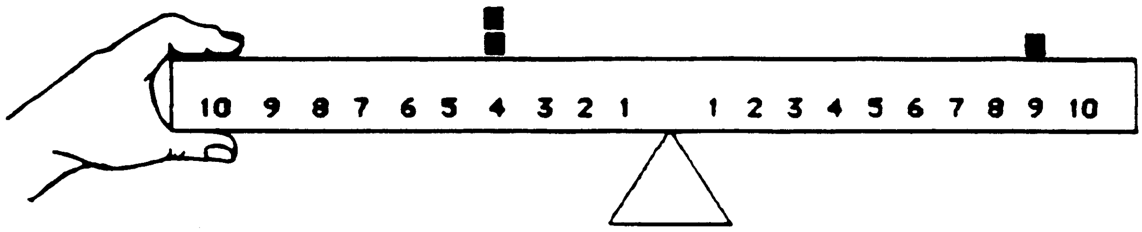

Figure 3.4. The scale will balance because "this one's closer in [the group of two rings] and this one's further out [the single ring]"

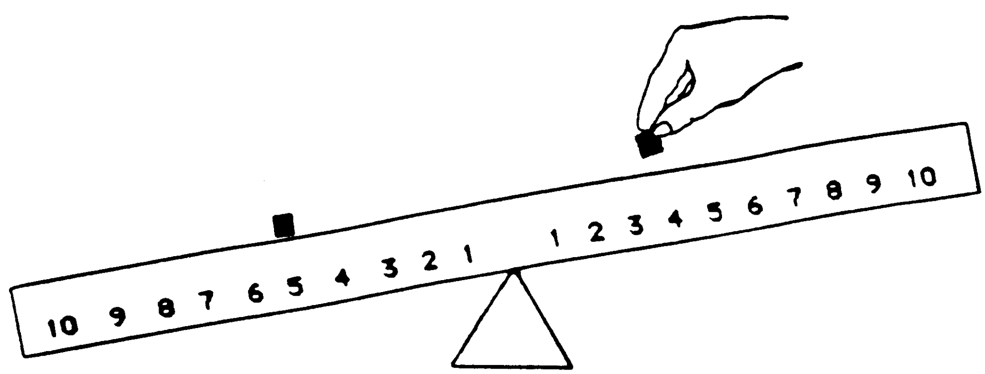

Figure 3.5. An attempt to balance the scale by placing a ring at approximately the same position on each arm

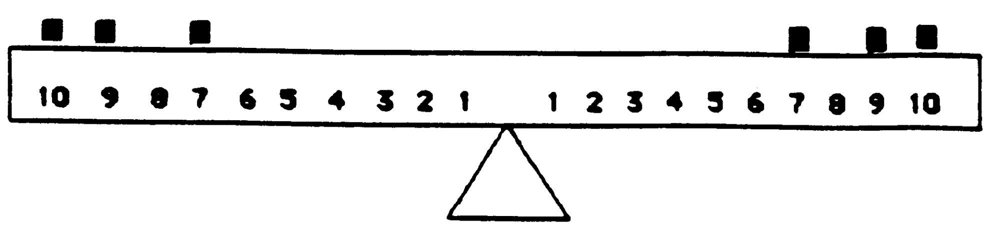

Figure 3.6. A symmetric configuration

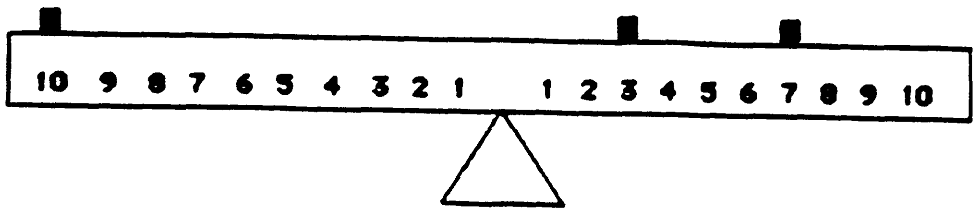

Figure 3.7. An example of Nib's use of ARITHMETIC1

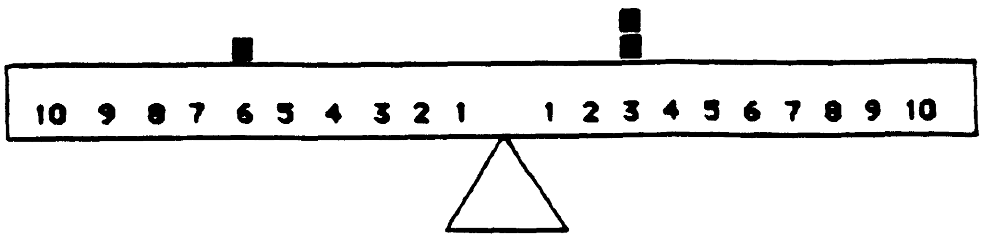

Figure 3.8. An example of Krif's use of ARITHMETIC2a

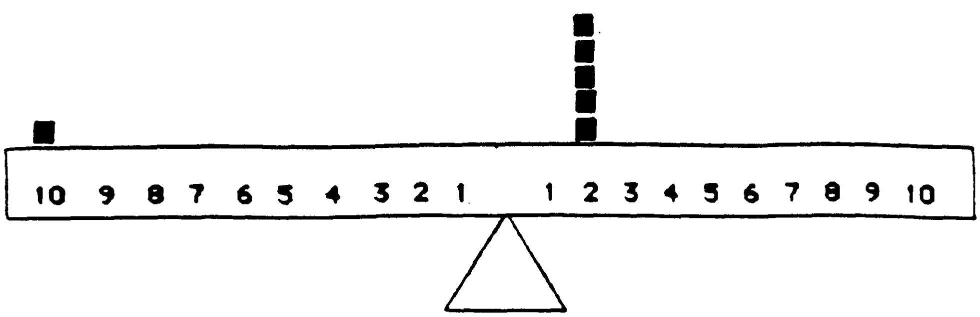

Figure 3.9. An example of Shas 's use of ARITHMETIC2b

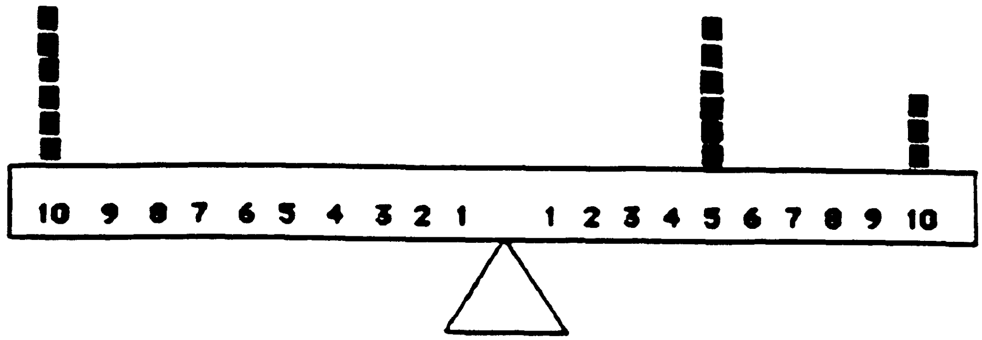

Figure 3.10. An example of Sam's use of ARITHMETIC2c

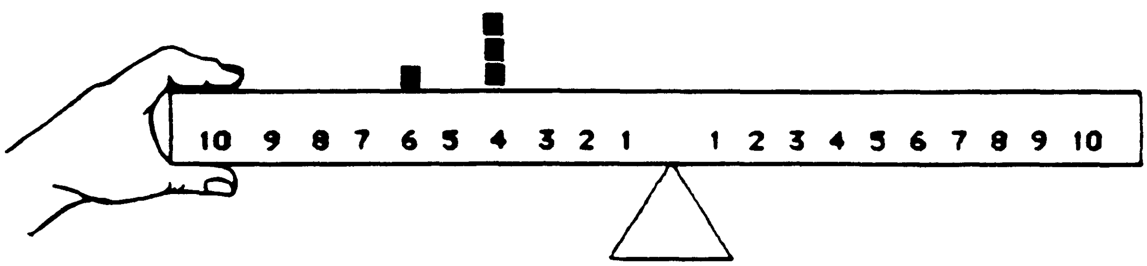

Figure 3.11. The total "torque" on the left arm is twenty-two, since "six times three is eighteen and four is twenty-two"

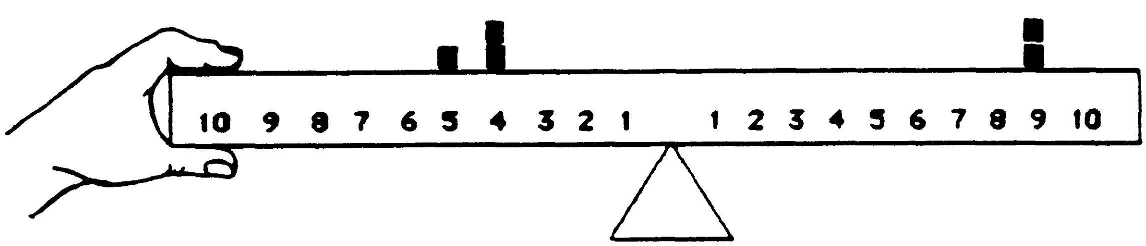

Figure 3.12. The scale will balance because the sum of the pegs used on each arm is the same

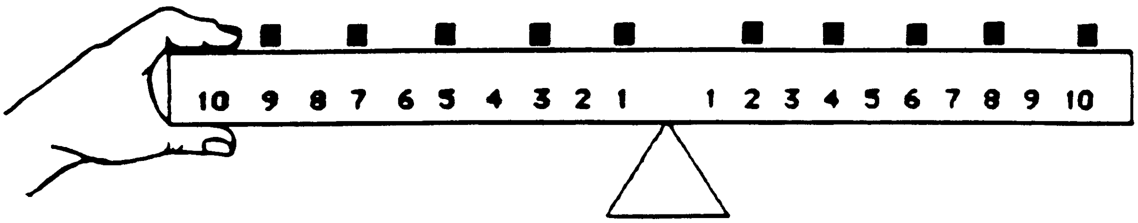

Figure 3.13. The scale will balance because the same number of rings is on each arm

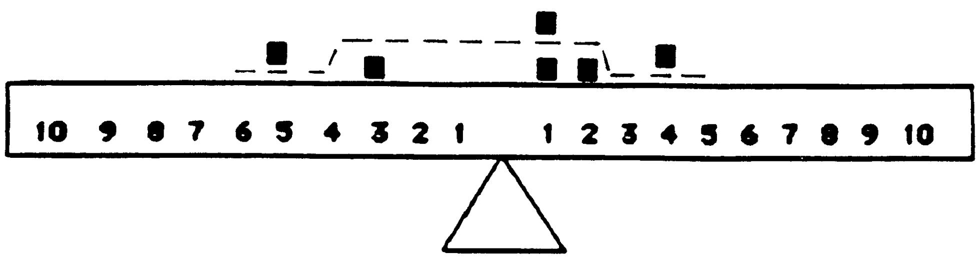

Figure 3.14. Superposition of two configurations using no more than one ring per peg

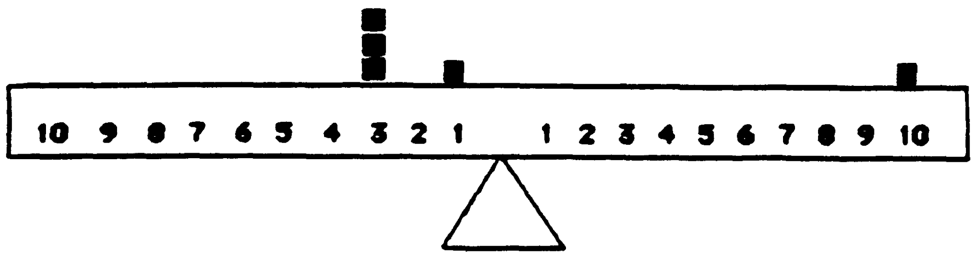

Figure 4.1. Nib: The scale balances because torque is the same on each arm

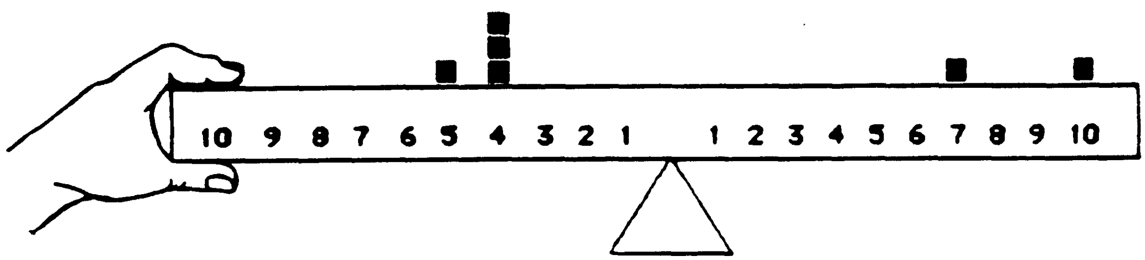

Figure 4.2. Nib: The scale won't balance because "it's probably too much weight"

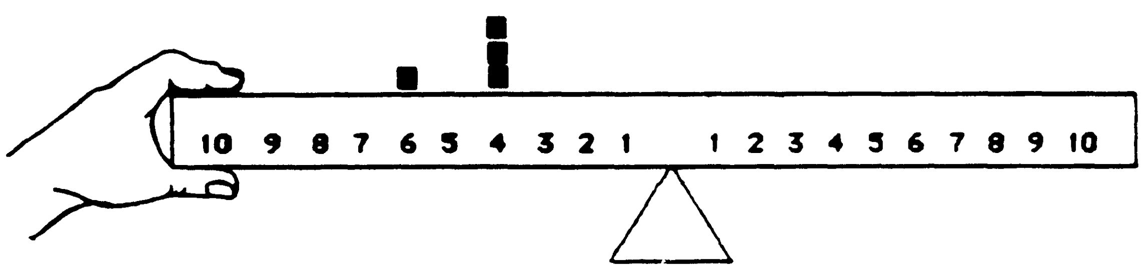

Figure 4.3. Can you put five on the other arm to balance the scale?

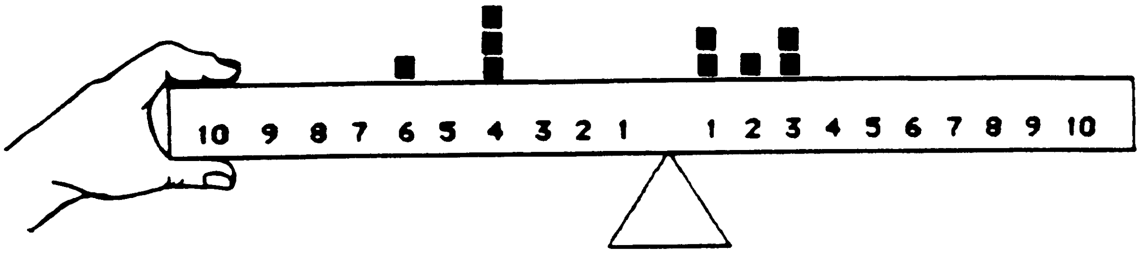

Figure 4.4. Rac's attempt to balance the configuration on the left arm

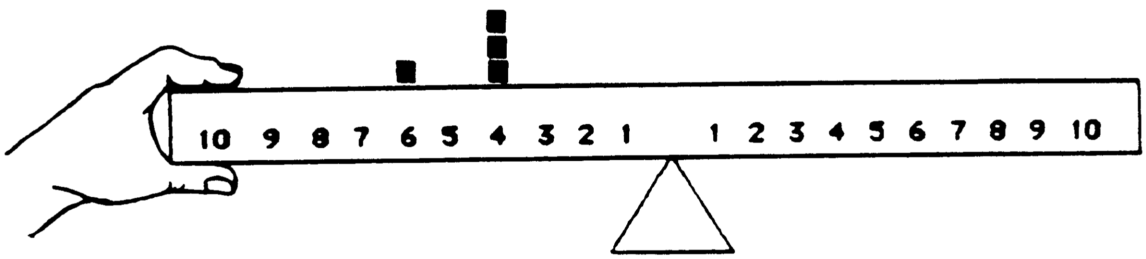

Figure 4.5. Can you put two on the other arm to balance the scale?

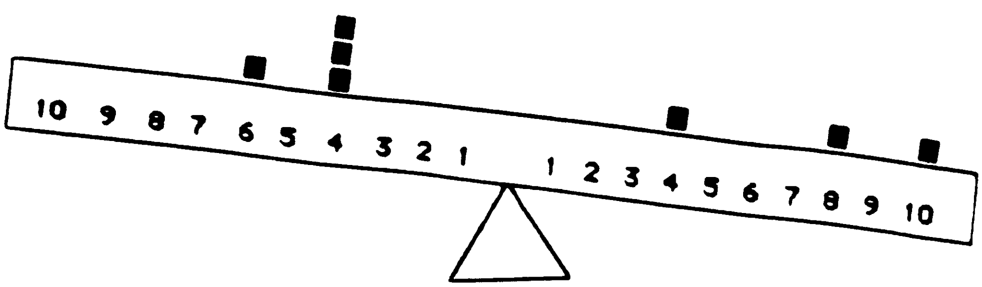

Figure 4.6. Rac's attempt to balance the configuration on the left arm: She used ARITHMETIC4 to calculate a "torque" of twenty-two on the left arm and ARITHMETIC3 to calculate a torque of twenty-two on the right arm

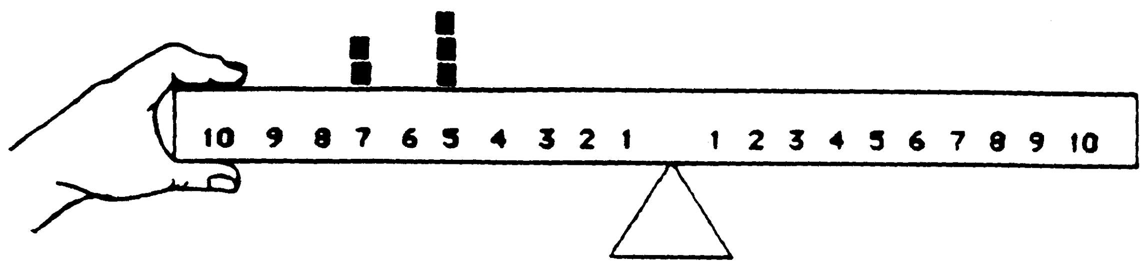

Figure 4.7. Can you put three on the other arm to balance the scale?

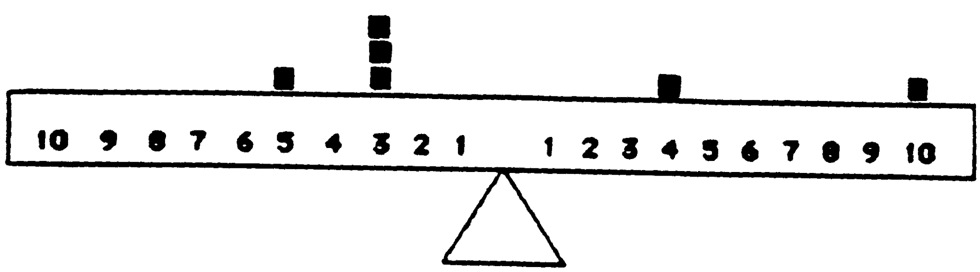

Figure 4.8. The configuration on the scale when Rac entered SITUATION4

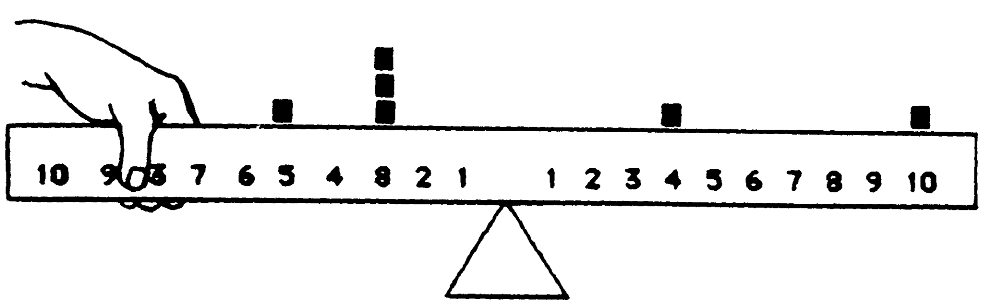

Figure 4.9. The same configuration with 3 and 8 "interchanged"

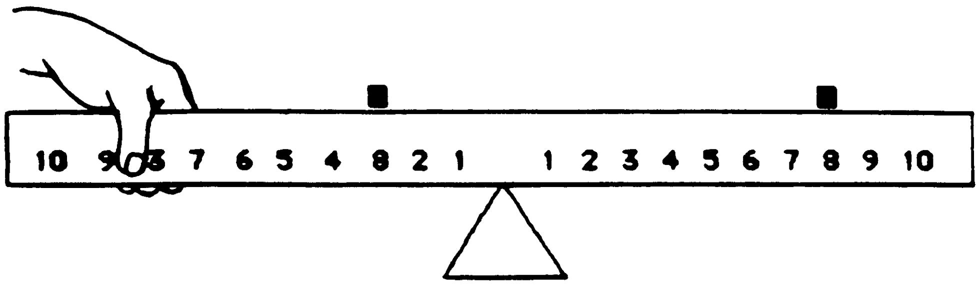

Figure 4.10 A ring on each of the "eighth" pegs

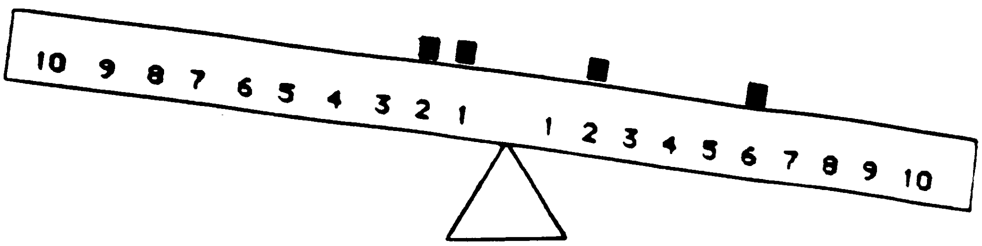

Figure 4.11. Dar's representation of "21" (on the left arm) and "26" (on the right arm)

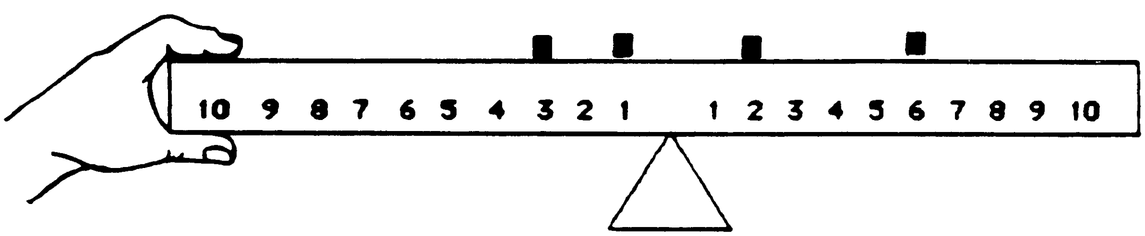

Figure 4.12. "31" and "26"

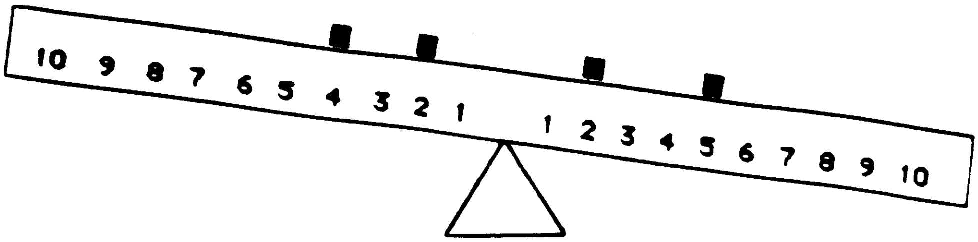

Figure 4.13."42" and "25"

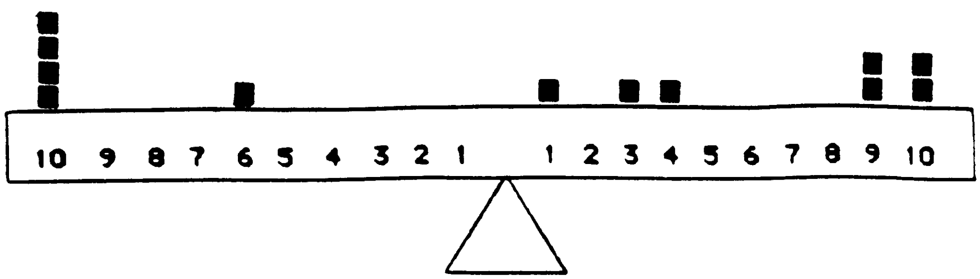

Figure 4.14 A configuration with the same torque but a different number of rings on each arm

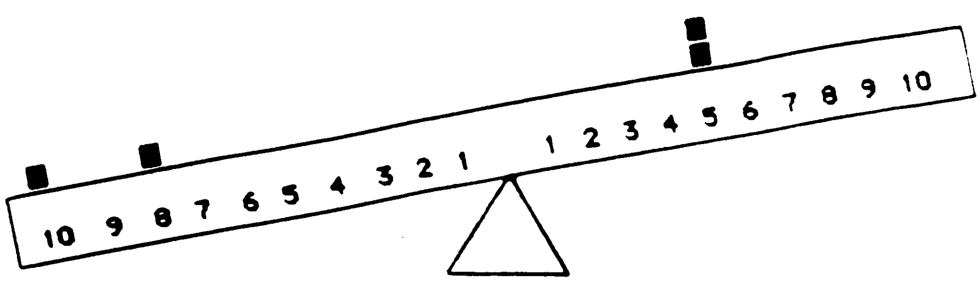

Figure 4.15 A configuration with the same number of rings but a different torque on each arm

Appendix B

Tables

Table 1: Balance scale stages -- Inhelder and Piaget

Stage Ia (3 to 5 years)[1]: inability to distinguish between one's own actions and the actions of objects one is trying to control

Stage Ib (5 to 7 or 8 years): compensation of weight, some unsystematic discoveries about the role of distance

Stage IIa (7 or 8 to 10): concrete operations on weight and distance, but no systematic coordination of weight and distance, in other words, coordination of equal weights and equal distances, equal weights at unequal distances, and unequal weights at equal distances, but only unsystematic discoveries about unequal weights at unequal distances

Stage IIb (10 to 12): understanding of inverse correspondence of weight and distance, relationships between unequal weights at unequal distances resolved qualitatively

Stage IIIa (appears at 12 or later): discovery of the law for balancing but inability to explain it

Stage IIIb (appears at 12 or later): explanation of the law in terms of proportional relationships among weight, distance, and *work* required to raise a given weight at a given distance to a height such that the scale is level

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[1] All ages are approximate.

Table 2: Balance scale rules -- Siegler

Rule I (4 to 8 years)[1]: *if* the number of weights is the same on each arm the scale will balance, otherwise the arm with more weights will drop

Rule II (8 to 12 years): *if* the number of weights is different on each arm use Rule I, otherwise *if* the distance (from the fulcrum) of each stack of weights is the same on each arm the scale will balance, and *if* distances are not the same, the arm with weights at the greater distance will drop

Rule III (appears at 12 or later): *if* the number of weights is the same on each arm use Rule II, otherwise *if* distances are the same the arm with greater weight drops, and *if* distances are not the same *if* the greater number of weights is on the same arm as the greater distance that arm will drop, otherwise the child will "muddle through"[2]

Rule IV (13 years or later, although many high school students still use Rule III): Rule III unless the greater weight is not on the same arm as the greater distance, in which case the child calculates torque

Note: Siegler used sets of configurations specifically designed to distinguish rule use. Not all sets of configurations would have so readily accomplished this. I believe that Siegler's rules may to some extent reflect aspects of the configurations he used, in particular, configurations with only one or two stacks of weights on an arm. For example, Rule II predicts that if the same number of

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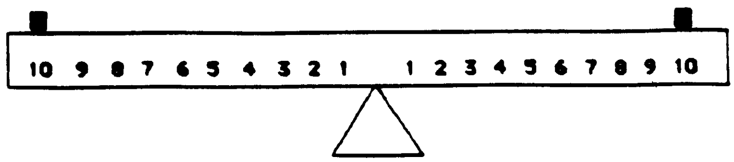
[1] All ages are approximate.

[2] [Klahr and Siegler 78] show that more specific rules may underlie use of Rule III. In addition, [Siegler and Taraban 86] show that any available special-case knowledge about the stability of particular configurations may be used instead of these rules.

weights is on each arm but the distances of these weights from the fulcrum are not equal, the arm with weights at a greater distance will drop. Rule II does not specify how to determine "greater distance" when multiple stacks of weights are placed at different distances on each arm. Would a Rule II child predict that the arm with the stack at the greatest distance would drop -- even if that stack only had one weight in it, but on the other arm a stack just one unit closer had several weights on it? What if the outer stacks of weights were identical on each arm, but the inner stacks varied?

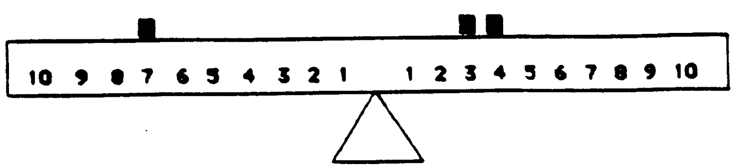
Table 3: Summary of Situations

SITUATION1: Balance the scale any way you want



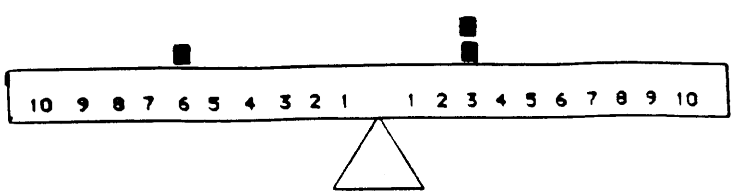
A typical SITUATION1 configuration

SITUATION2a: One ring per peg, variable number of pegs



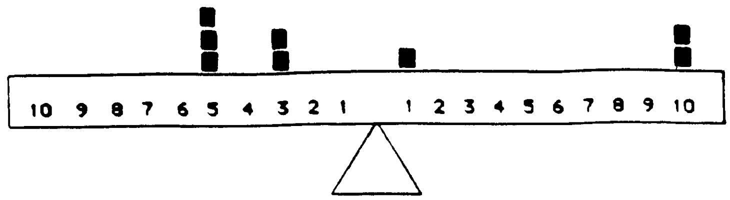
A typical SITUATION2a configuration

SITUATION2b: One peg per arm, variable number of rings



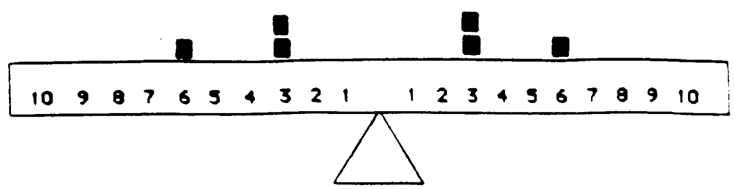
A typical SITUATION2b configuration

SITUATION3: Multiple rings on multiple pegs



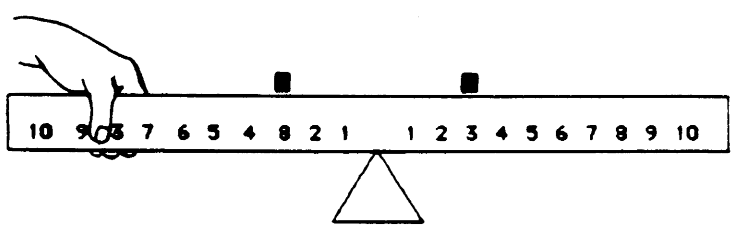
A typical SITUATION3 configuration

SITUATION4: Removing an asymmetric balance from a symmetric one



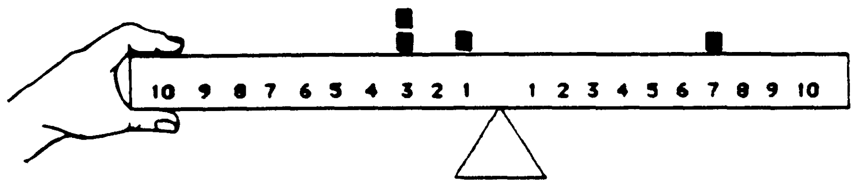
Problem posed: Can you remove three rings so that the scale remains balanced?

SITUATION5: Role of the numbers on the balance scale



Typical question: Will the scale remain balanced now that the 3 and 8 have been interchanged?

SITUATION6: Balanced, with more rings on one arm



Typical question: Will the scale balance even though more rings are on one arm?

Table 4: Subjects

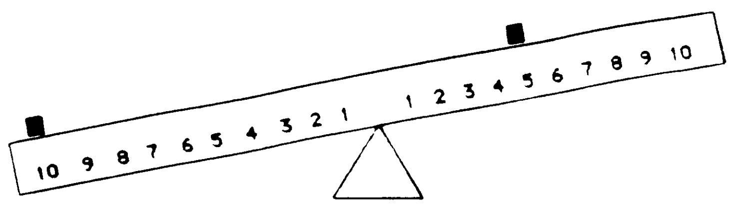
Name	**Age**	**Grade**	**Sex**
Dar	11;8	5	Male
Jeo	9;3	4	Female
Krif	11;7	5	Male
Nib	10;1	5	Male
Rac	11;6	5	Female
Rin	10;7	4	Male
Sam	11;1	5	Male
Shas	12;4	5	Male
Tot	11;4	5	Female

Table 5: Summary of Strategies

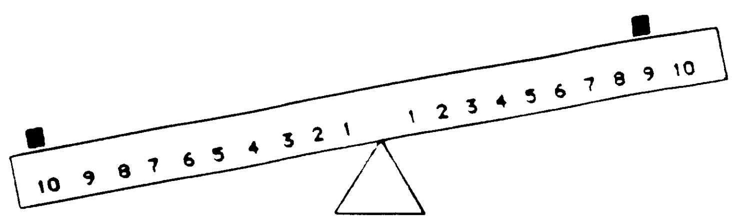
COMPENSATION1: Active compensation

If the scale tips, it can be balanced by moving rings toward the fulcrum on the tipping arm, by moving rings away from the fulcrum on the raised arm, by removing rings from the tipping arm, or by adding rings to the raised arm.

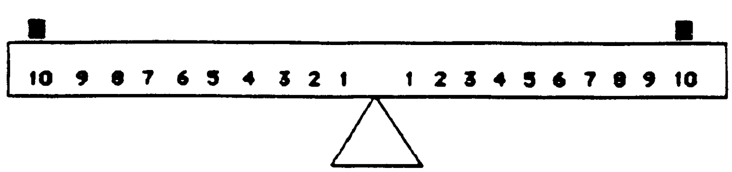
An unsuccessful attempt to balance the scale:



Two consecutive uses of COMPENSATION1:



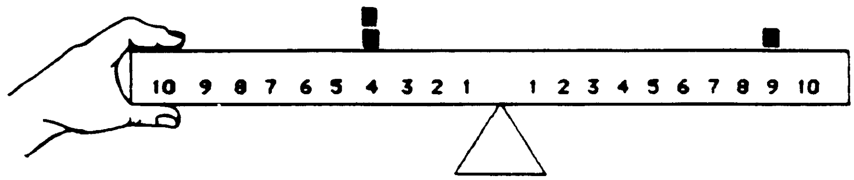
An attempt to compensate for tipping by moving the ring on the raised arm away from the fulcrum



A second attempt to compensate for tipping by moving the ring on the raised arm even further away from the fulcrum

COMPENSATION2: Static compensation

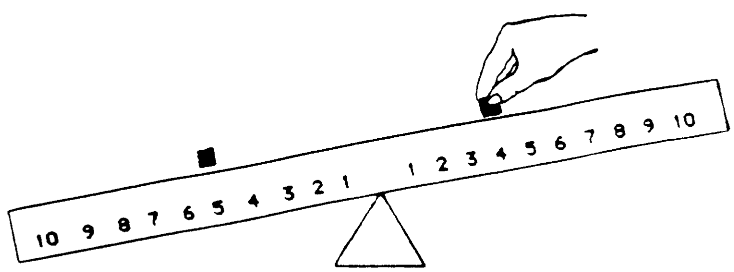
The scale balances if it "looks like" it will balance.



A typical example: The scale will balance because "this one's closer in [the group of two rings] and this one's further out [the single ring]"

SYMMETRY1: Qualitative symmetry

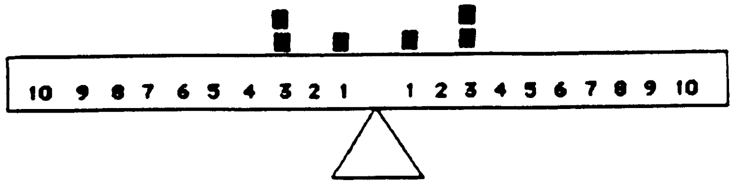
The scale balances if a ring is placed at approximately the same position on each arm.



An attempt to balance the scale by placing a ring at approximately the same position on each arm

SYMMETRY2: Quantitative symmetry

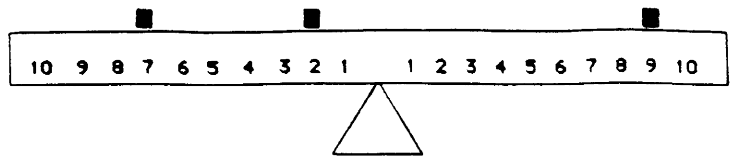
The scale balances if the identical configuration is placed on each arm.



A typical configuration constructed with SYMMETRY2

ARITHMETIC1: Summing pegs

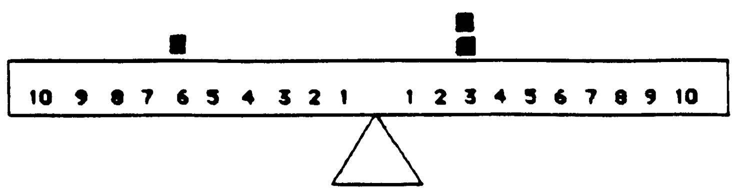
The scale balances if no more than one ring is on a peg, and the sum of the pegs used is the same on each arm.



A typical configuration constructed with ARITHMETIC1

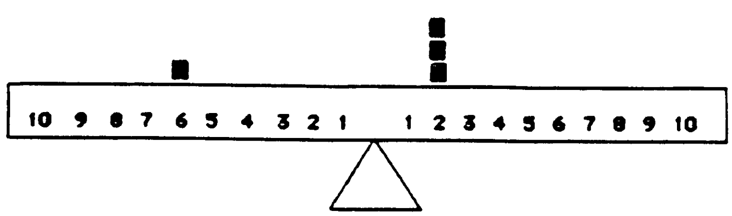
ARITHMETIC2: Multiplying rings by pegs

ARITHMETIC2a: The scale balances if one ring is on an even-numbered peg on one arm and two are half as far on the other arm.



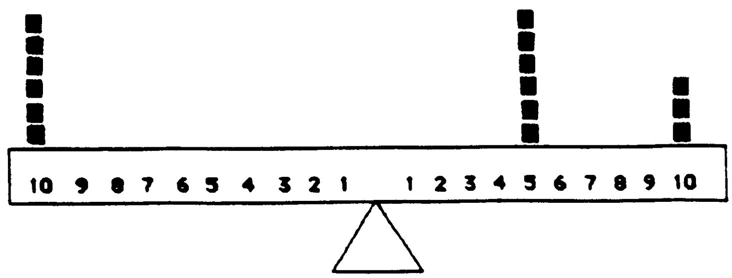
A typical configuration constructed with ARITHMETIC2a

ARITHMETIC2b: The scale balances if only one peg is used per arm, and the product of the number of rings on the peg and the number of the peg is the same on each arm.



A typical configuration constructed with ARITHMETIC2b

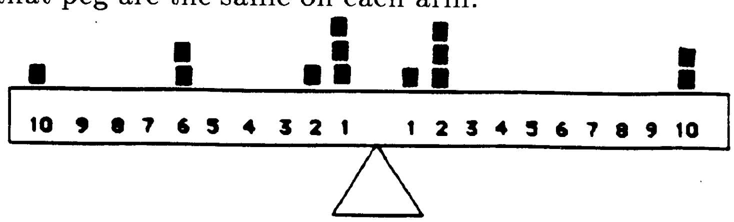
ARITHMETIC2c: The scale balances if superpositions of balances using only one peg per arm are placed on it.



A configuration constructed with ARITHMETIC2c: The child seemed to see this configuration as the superposition of three rings on the tenth peg on one arm and six on the fifth on the other, and three on the tenth on one arm and three on the tenth on the other

ARITHMETIC3: "The law"

The scale balances if the sum of the products of the number of rings on a peg and the number of that peg are the same on each arm.



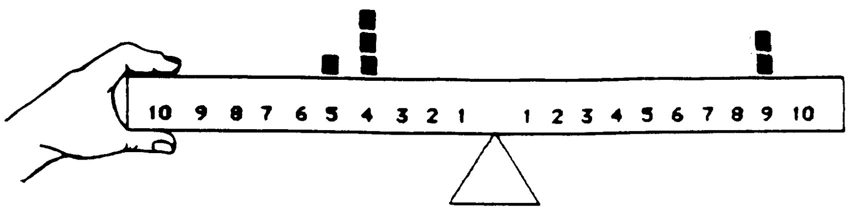
A typical configuration constructed with ARITHMETIC3

Children using ARITHMETIC3a saw (at least to some extent) ARITHMETIC3 as a quantification of qualitative compensation of weight and distance.

Children using ARITHMETIC3b did not appear to see ARITHMETIC3 as a quantification of qualitative compensation of weight and distance.

ARITHMETIC4: "Incorrect" arithmetical Strategies

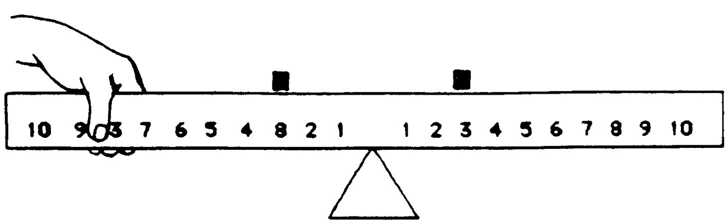
The scale balances if the same number can be calculated to represent each arm on the balance scale.



A typical example: The scale will balance because the sum of the pegs used on each arm is the same

NUMBERS: Numbers supersede distance

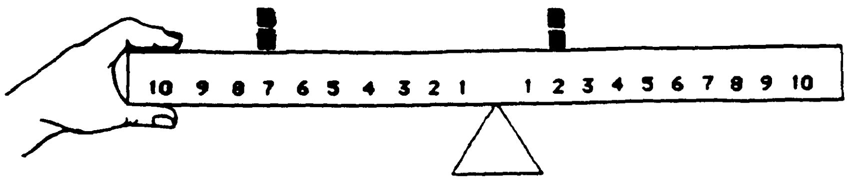
The numbers on the balance scale affect its functioning.



A typical example: The scale will no longer balance because the 3 and 8 have been interchanged and "it matters where the number is"

RINGS: The number of rings on each arm supersedes all else

The scale balances if the same number of rings is on each arm.



A typical example: The scale will balance because the same number of rings is on each arm

Table 6: Use of Strategies

Following is a table showing the Strategies children used in different Situations. Situations are listed on the left, and children's names are listed across the bottom. Situations are divided into Episodes by horizontal lines. Within each Situation, Strategies appear in order of use, such that those used earlier are closer to the bottom of the page. Each child's first use of ARITHMETIC3 is shown shaded, and the instances in which children used SYMMETRY2 very briefly in SITUATION5 are shown in parentheses. When a child used two Strategies simultaneously, the Strategies are shown connected with vertical dashes.

A bar across the list of Strategies indicates a child's temporarily leaving a particular Situation. The bar shows the point at which she left the Situation and what she did during the interruption. When a child did not encounter a Situation, the appropriate section of the table is blank. At the bottom of each column, the order in which the child encountered the Situations is listed. Situations that arose spontaneously are marked with an asterisk.

SIT2b		ARITH2 6 COMP2	ARITH2 ARITH2 ARITH2 ARITH2			ARITH2 ARITH2 COMP2 ARITH2 COMP1 COMP2	COMP1 COMP2 ARITH2 ARITH2	ARITH2 ARITH2 ARITH2 ARITH4 ARITH2 ARITH2 ARITH2 ARITH2	
SIT2a	ARITH1 ARITH1	ARITH1 COMP1 2a, 6, 2a COMP1	ARITH1 ARITH1 ARITH1 COMP1 COMP1 2a COMP1 SYMM2	ARITH1 ARITH1 ARITH1 SYMM2	ARITH1 ARITH1 ARITH1	ARITH3a SYMM2 6, 2a, 3 SYMM2 COMP1 SYMM2	ARITH1 6 ARITH1 ARITH1 COMP1 ARITH2 COMP1 COMP1 SYMM2 2b	ARITH1 ARITH1 COMP2 ARITH1 COMP1 ARITH4	ARITH1 ARITH1 COMP2 ARITH1 ARITH1 COMP1 SYMM2
SIT1	COMP1 COMP2 ARITH1 COMP1 SYMM1 NUMBS COMP1 COMP2 COMP1 COMP2 ARITH1 COMP1 RINGS	SYMM2 COMP1 NUMBS SYMM2 NUMBS SYMM2 COMP1 SYMM1	SYMM2 COMP1 SYMM1	SYMM2 COMP1 SYMM1	SYMM2 COMP1 SYMM1	SYMM2 SYMM2	SYMM2	SYMM2 SYMM2 RINGS COMP1	SYMM2 SYMM1
Order of SITS	6* 5 3* 2* 1	5 3 4 3 2 2a 6 2a 2 1	6 5 4 3* 2 2a 2 1	6 3 5 4 3 2 1	6 5 4 3 2 1	5 2 3 2a 6 2 1	5 3* 2 6* 2 2a* 1	4 6 5 3 2 2a 1	6 5 4 3 2 1
	Dar (11;8)	**Jeo** (9;3)	**Krif** (11;7)	**Nib** (10;1)	**Rac** (11;6)	**Rin** (10;7)	**Sam** (11;1)	**Shas** (12;4)	**Tot** (11;4)

	Dar (11;8)	Jeo (9;3)	Krif (11;7)	Nib (10;1)	Rac (11;6)	Rin (10;7)	Sam (11;1)	Shas (12;4)	Tot (11;4)
SIT6	ARITH3a RINGS ARITH3a RINGS ARITH3a NUMBS ARITH3a RINGS ARITH3a RINGS ARITH3a RINGS ARITH3a	ARITH2	ARITH3a	ARITH3a	ARITH3b	COMP2	RINGS	ARITH3a	ARITH3a
SIT5	ARITH3a (SYMM2)	ARITH3b SYMM2	ARITH3a SYMM2	ARITH2 SYMM2	ARITH3b SYMM2	ARITH3a SYMM2	ARITH3a (SYMM2)	ARITH3a	ARITH3a (SYMM2)
SIT4		NUMBS NUMBS NUMBS	ARITH3a	ARITH1 COMP2	ARITH3a ARITH3b NUMBS ARITH3a ARITH3b NUMBS			ARITH3a	ARITH3a NUMBS ARITH3a NUMBS
SIT3	ARITH3a ARITH3a ARITH3a ARITH3a ARITH3a ARITH3a ARITH3a	ARITH3b 4 ARITH3b	ARITH3a ARITH3a ARITH3a ARITH3a ARITH3a	ARITH3a ARITH3a COMP2 ARITH3a ARITH4 ARITH4 4,5 COMP1 COMP1 ARITH4 COMP1 COMP1 COMP1	ARITH3b ARITH3b ARITH3b ARITH4 ARITH3b ARITH4 ARITH3b COMP2 ARITH4 ARITH3b COMP2 ARITH3b SYMM2 RINGS COMP2	ARITH3a ARITH3a	ARITH3a	ARITH3a ARITH3a ARITH3a	ARITH3b ARITH3b ARITH3b ARITH3b

Table 7: Use of Strategies after first use of ARITHMETIC3

	Dar (11;8)	Jeo (9;3)	Krif (11;7)	Nib (10;1)	Rac (11;6)	Rin (10;7)	Sam (11;1)	Shas (12;4)	Tot (11;4)
COMP1									
COMP2				1	2				
SYMM1									
SYMM2	1/0	1	1		1	2	1/0		1/0
ARITH1									
ARITH2									
ARITH3a	14		7	3	2	3	1	5	4
ARITH3b		2			10[1]				3
ARITH4					3				
NUMBS	1	3			2				2
RINGS	5								
Total	21/20	6	8	4	18	5	2/1	5	10/9
Total *not* ARITH3	7/6	4	1	1	8	2	1/0	0	3/2
% *not* ARITH3	33/30	67	13	25	44	40	50/0	0	30/22

Note: When appropriate, figures including and excluding brief use of SYMMETRY2 in SITUATION5 are shown. In these instances, the two figures are shown divided by a "/". The figure including the brief use of SYMMETRY2 is on the left.

[1] Two of Rac's uses of ARITH3b were simultaneous with NUMBERS.

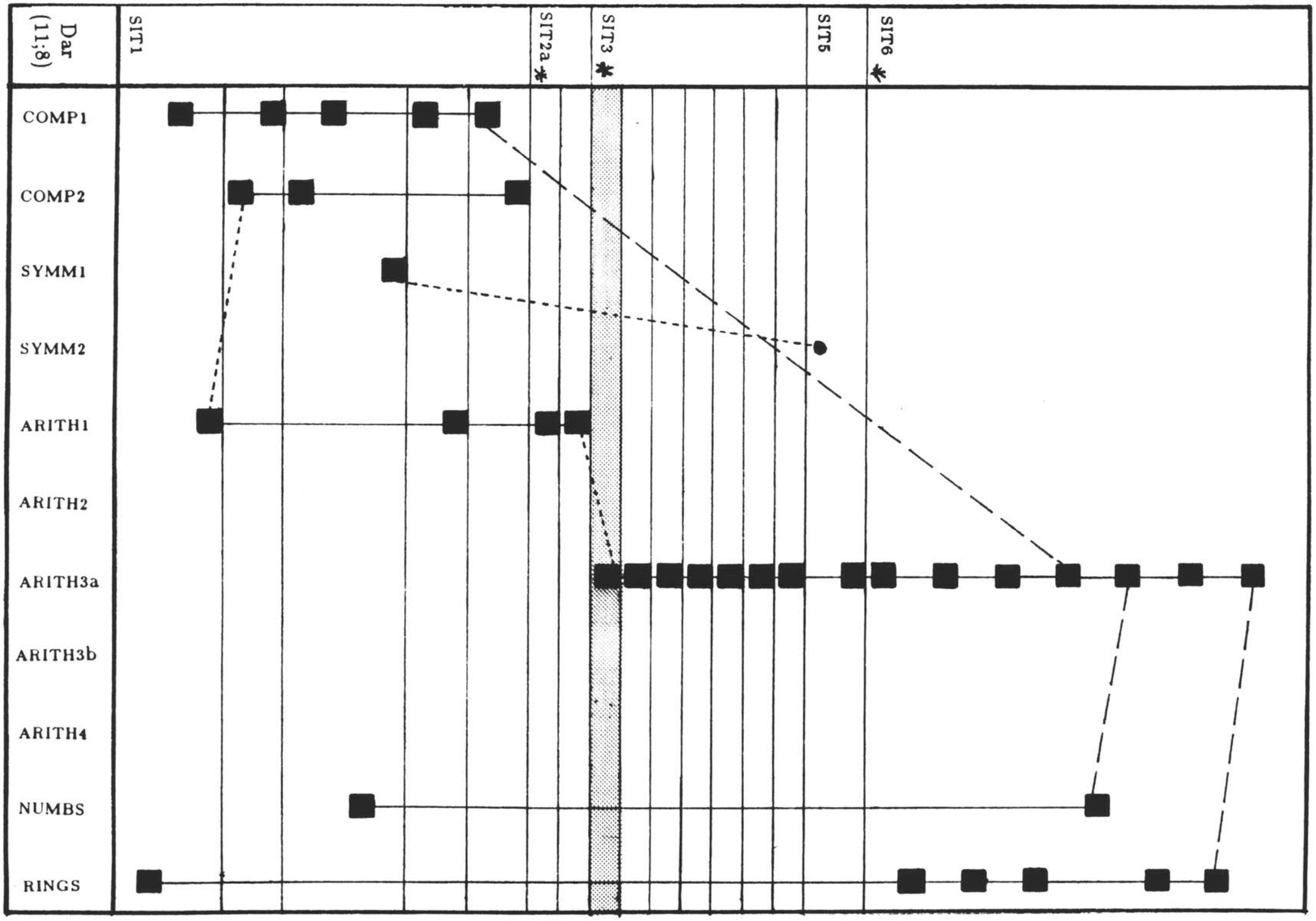
Appendix C

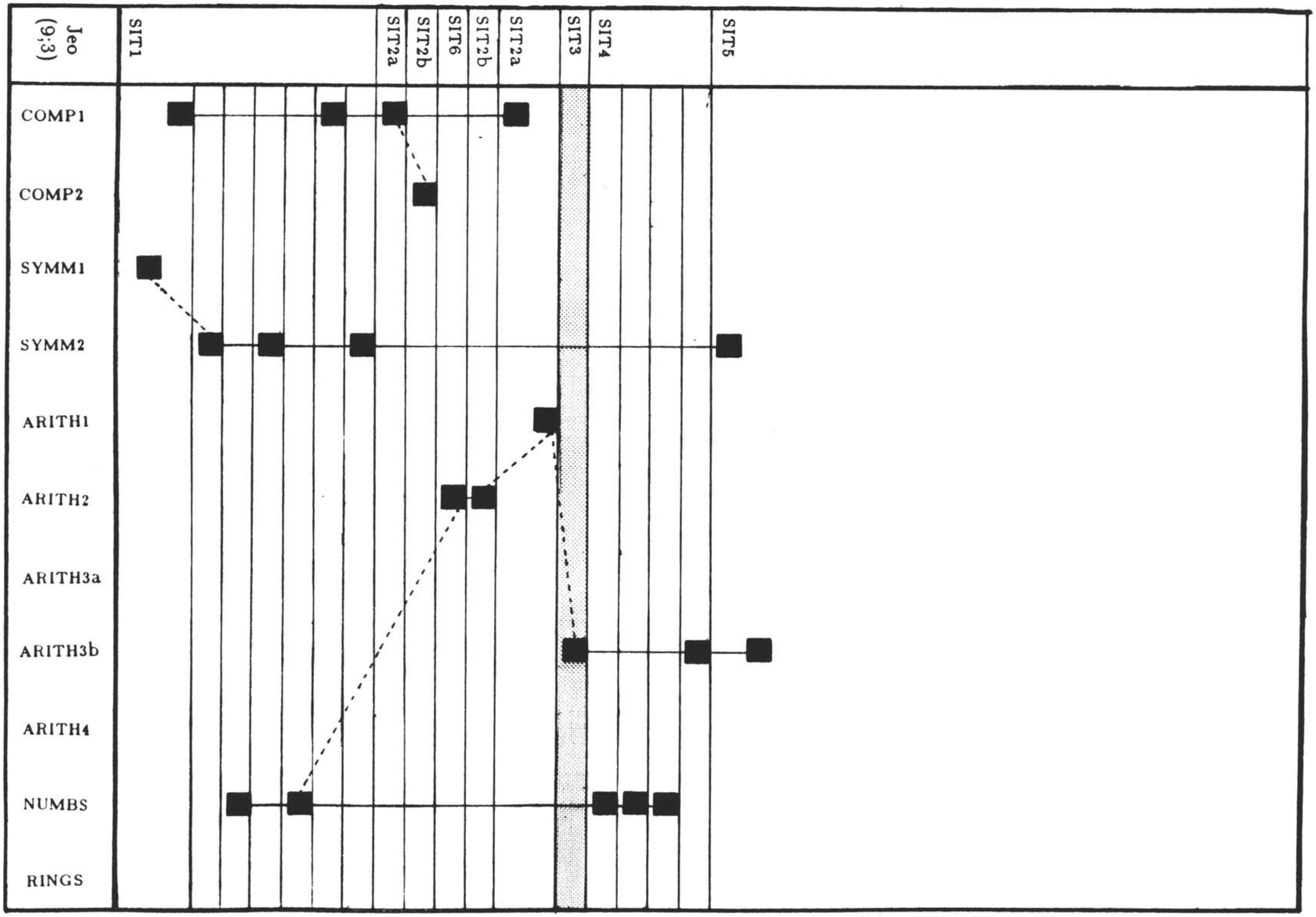
Individual balance scale sessions

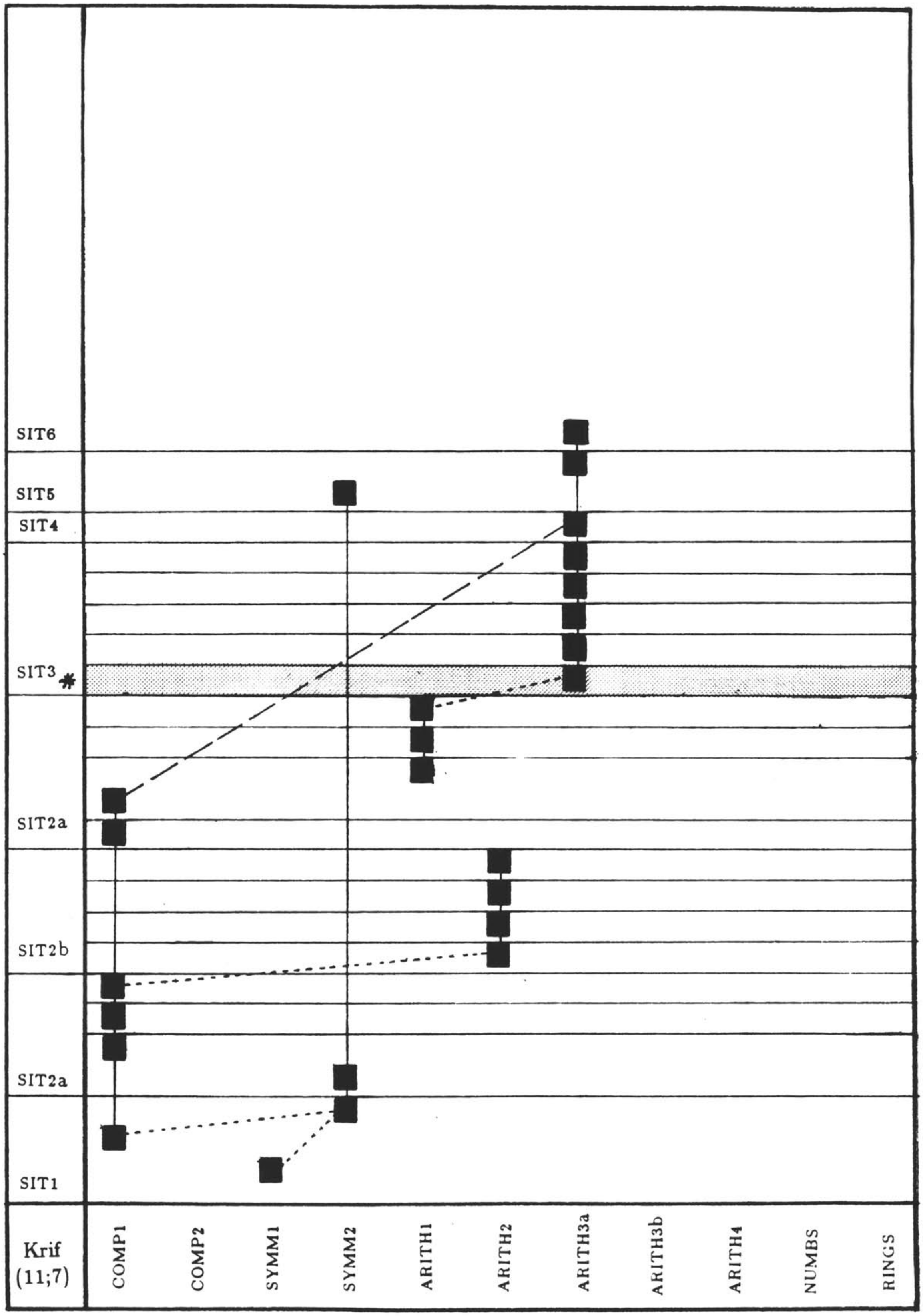
Following are a series of charts showing the course of each child's balance scale session. The child's name is in the lower left corner of each chart. On the left side is a list of Situations the child encountered, in the order in which she encountered them. Situations are divided into Episodes by horizontal lines, and Situations that emerged spontaneously are marked with an asterisk. Along the bottom is a list of the Strategies. A square in row X and column Y indicates the child used the Strategy listed in column Y when working in the Episode to which row X corresponds. The instances in which children used SYMMETRY2 only very briefly in SITUATION5 are indicated with a small circle. The row containing the first use of ARITHMETIC3 is shaded.

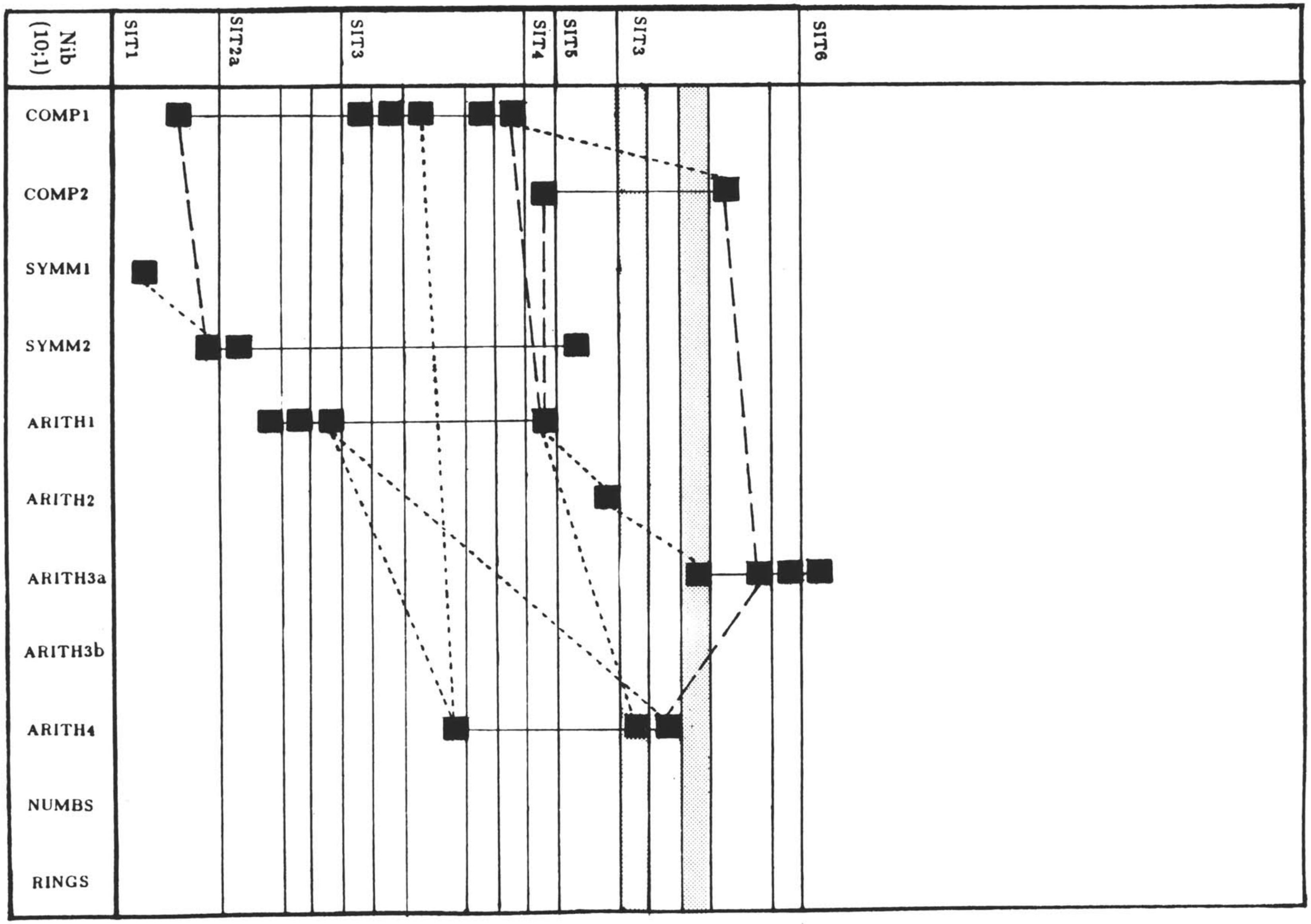
Inferred integration of two Strategies is shown with a large-dashed line connecting the instance of the Strategy in which the integration was manifested to the most recent use of the other Strategy. Inferred genetic relationships are shown with a small-dashed line connecting the parent Strategy to the Strategy that appeared to evolve from it. Any integration manifested in the *first use* of a Strategy is also a genetic relationship. When a child used two Strategies simultaneously, both Strategies are shown in the same row, connected with a large-dashed line.

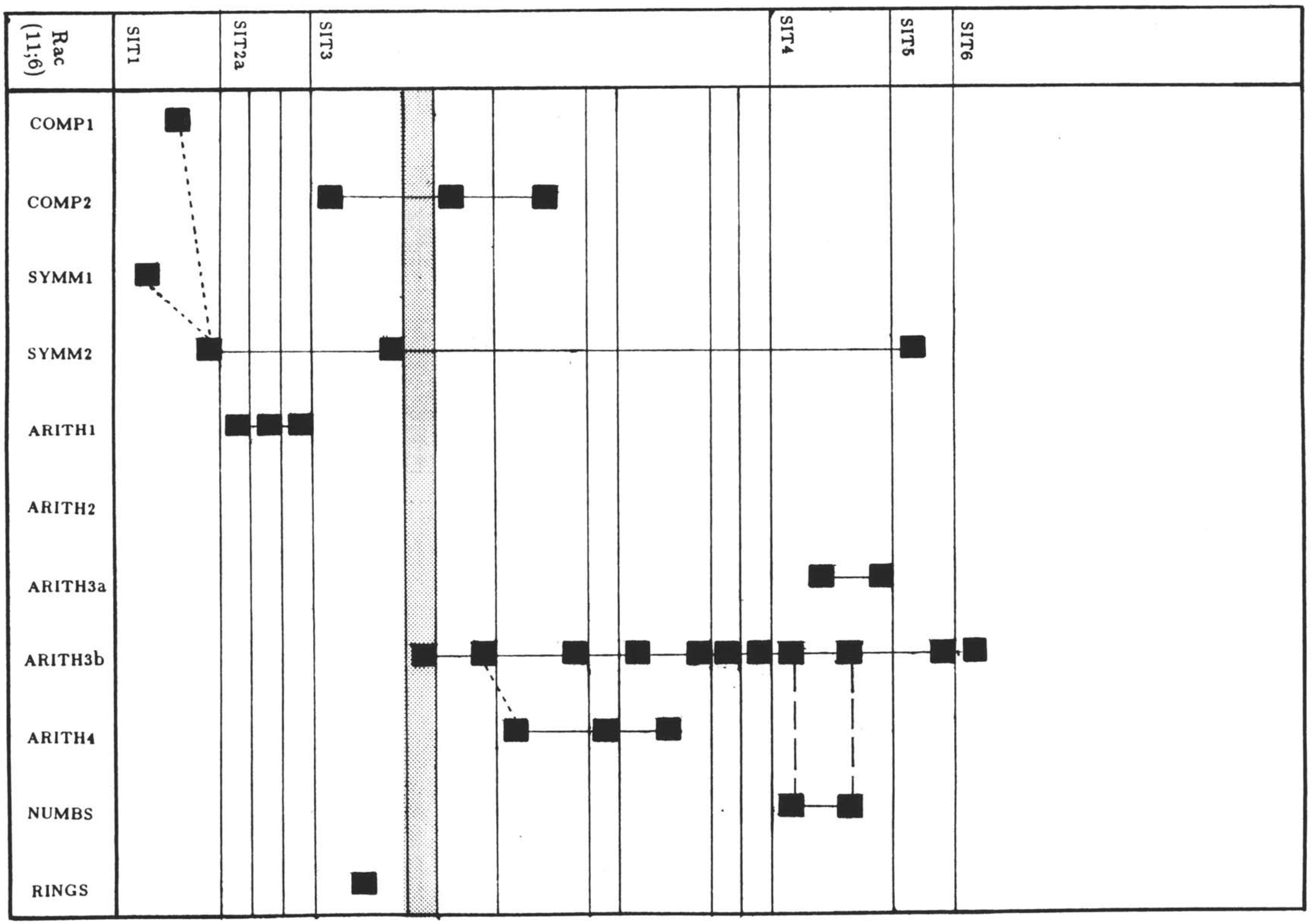
For legibility, instances of the same Strategy are connected with a solid line.

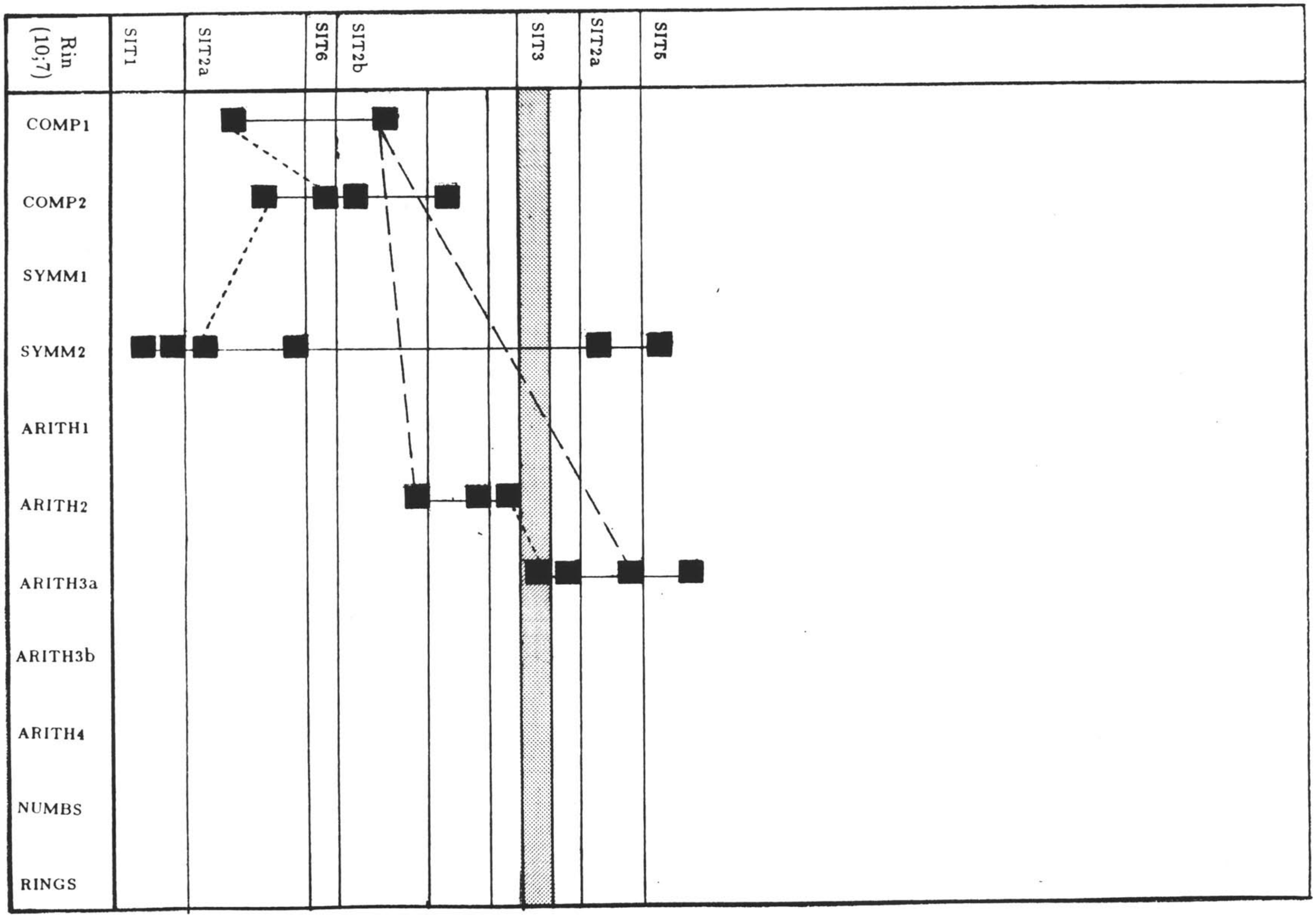

Dar
(11;8)
SIT1
SIT2a
SIT3
SIT5
SIT6
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS

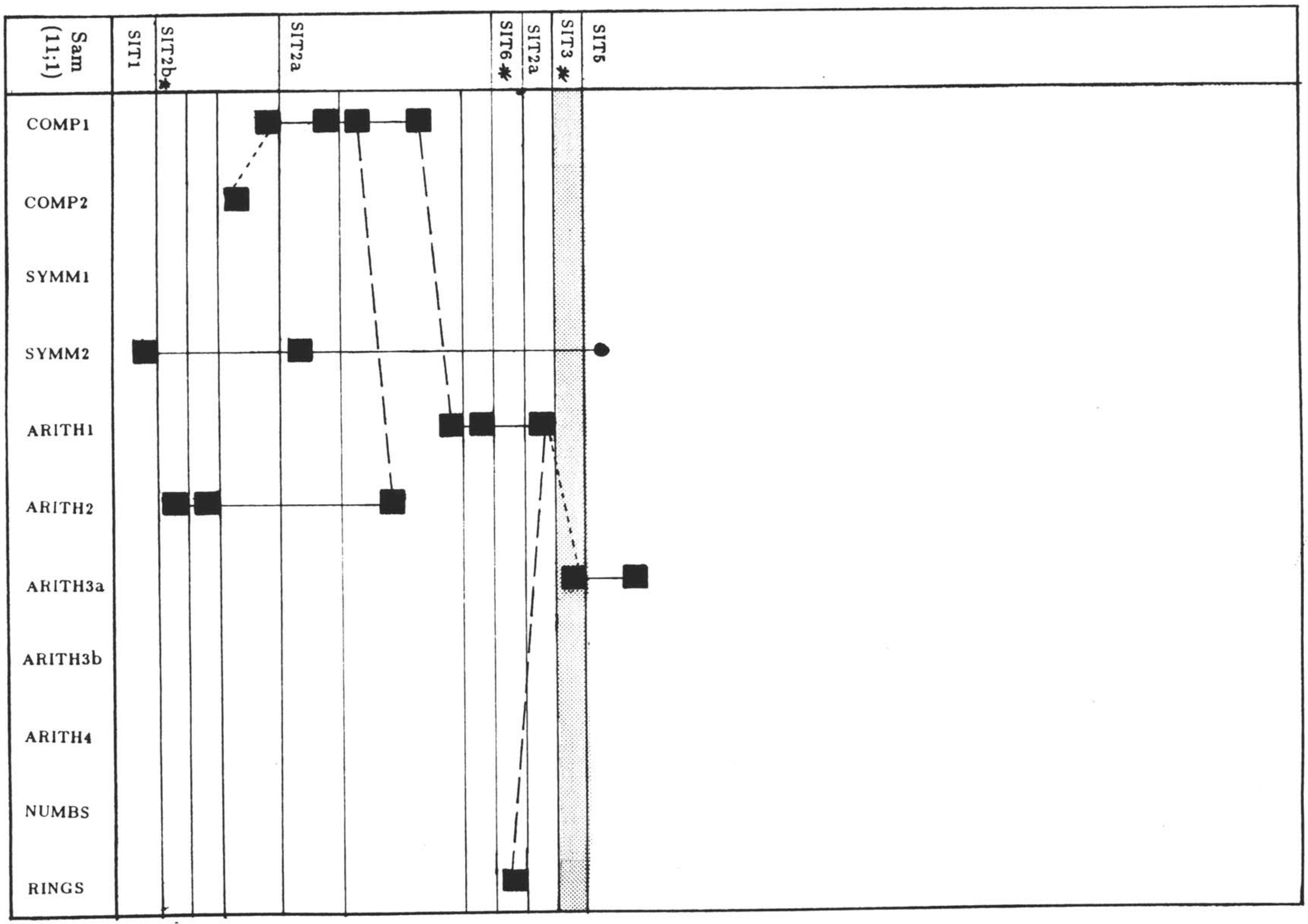

Jeo (9;3)
SIT1
SIT2a
SIT2b
SIT6
SIT2b
SIT2a
SIT3
SIT4
SIT5
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS

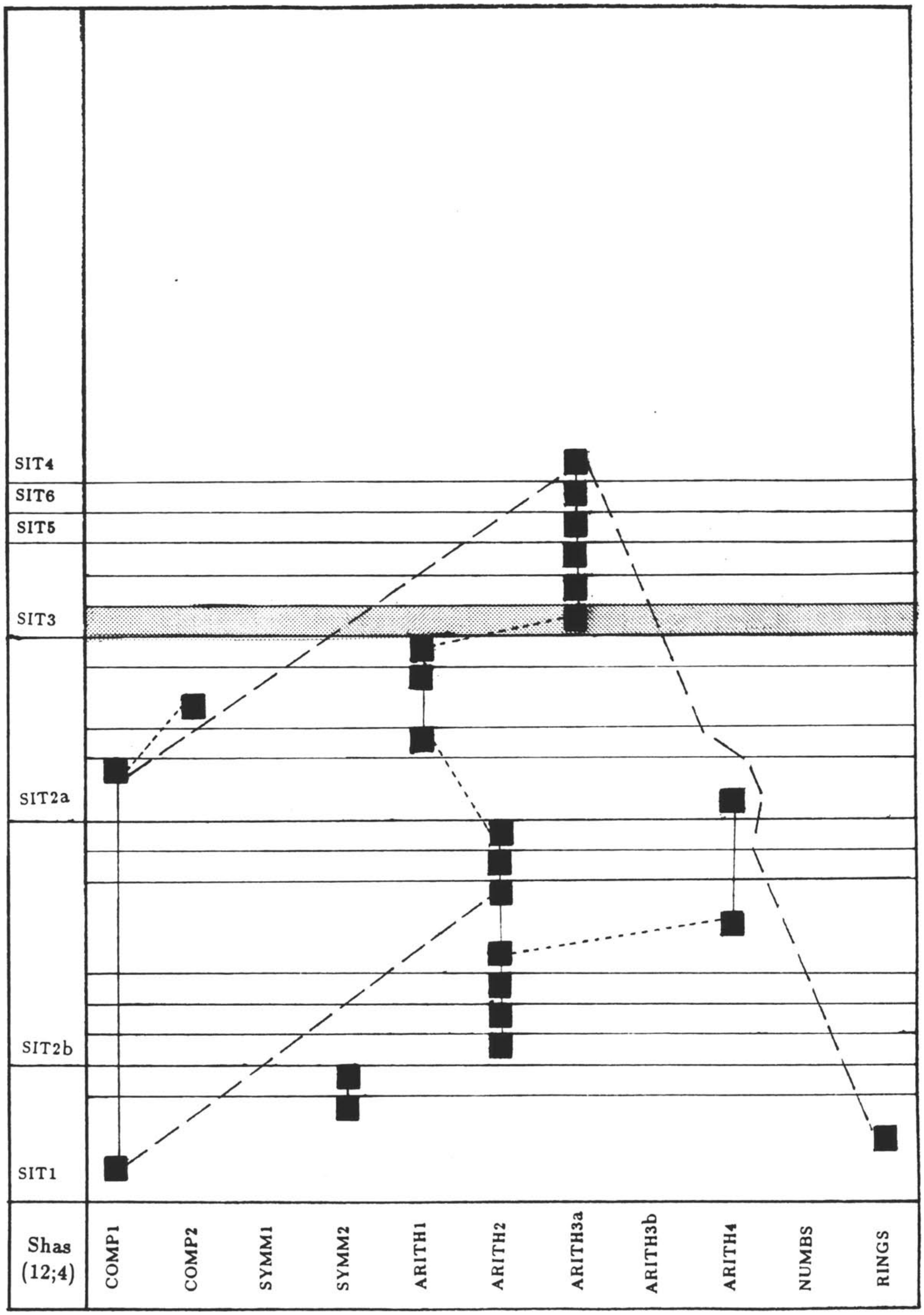

SIT6
SIT5
SIT4
SIT3 *
SIT2a
SIT2b
SIT2a
SIT1
Krif (11;7)
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS

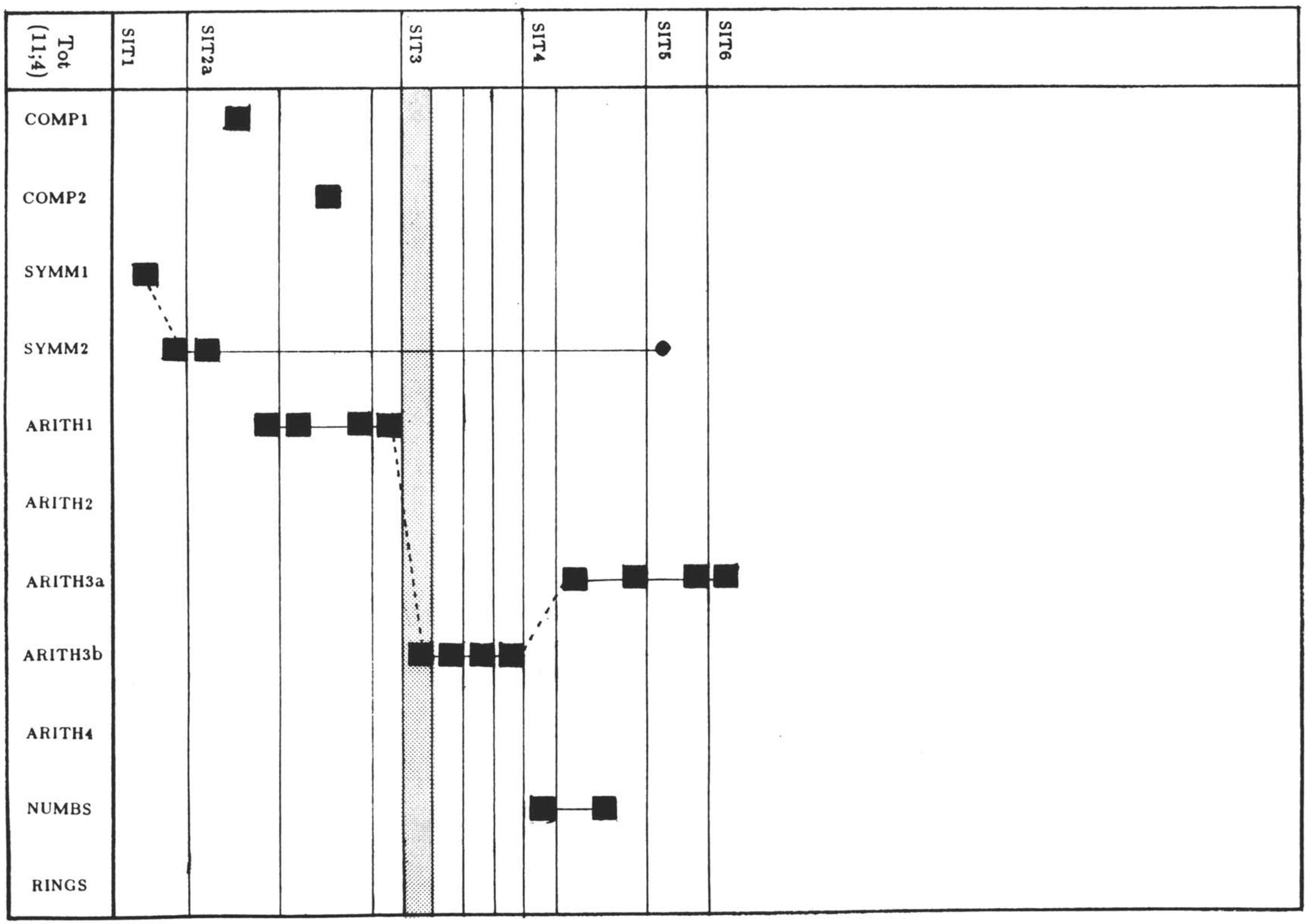

Nib (10;1)
SIT1
SIT2a
SIT3
SIT4
SIT5
SIT3
SIT6
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS


Rac (11;6)
SIT1
SIT2a
SIT3
SIT4
SIT5
SIT6
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS


Rin (10;7)
SIT1
SIT2a
SIT6
SIT2b
SIT3
SIT2a
SIT5
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS


Sam (11;1)
SIT1
SIT2b*
SIT2a
SIT6*
SIT2a
SIT3*
SIT5
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS


SIT4
SIT6
SIT5
SIT3
SIT2a
SIT2b
SIT1
Shas
(12;4)
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS


Tot (11;4)
SIT1
SIT2a
SIT3
SIT4
SIT5
SIT6
COMP1
COMP2
SYMM1
SYMM2
ARITH1
ARITH2
ARITH3a
ARITH3b
ARITH4
NUMBS
RINGS

Appendix D

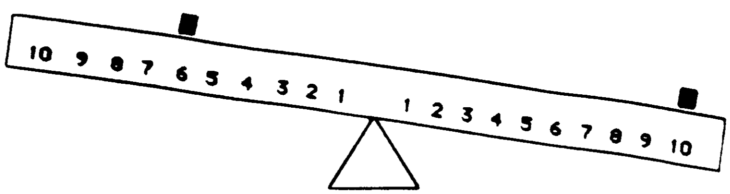
Protocol

Following is the section of Tot's protocol surrounding her initial use of ARITHMETIC3. I begin with her first use of an arithmetical strategy, ARITHMETIC1 in SITUATION2a. In response to my request that she balance the scale with three rings, all on different pegs, she used COMPENSATION1 and then developed ARITHMETIC1.

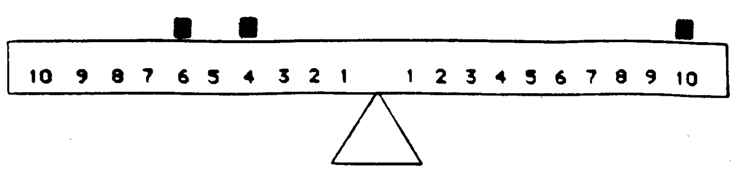
I use three consecutive dots to indicate pauses of several seconds. When the pause is longer than about five seconds, I note the length. I refer to myself as "M" and to Tot as "T."

- SITUATION2a, end of first Episode

  Strategy: ARITHMETIC1



T: Oh ... Oh! I get it now! (puts a ring on the fourth peg of the arm with a ring on the sixth peg) ... You've got to try to equal ten.



M: How did you figure that out?

T: Six and four is ten!

M: ... How did you get that, though?

T: Because, if I put a six there, and I have to put the third one somewhere on this side, and that equals ten [the ring on the tenth peg on the other arm], then I should try to equal it out by the two numbers [the two rings on the same arm] ... and I already put it on the six, so ...

• SITUATION2a, second Episode

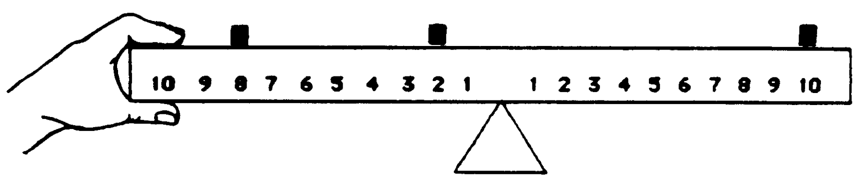
Strategy: ARITHMETIC1

M: Uh huh ... If you had put one [a ring] on the two [on the arm with two rings], where would you put the other one?

T: On the eight.

M: ... Do you think it works, or do you want to try it?

(After a few seconds, T tries it while M holds the scale steady.)



Strategy: COMPENSATION2

M: What do you think?

T: Uh, I don't think it will work. I'm not sure.

M: Why not? ... You just said that you thought it would.

T: In a way I do, in a way I don't, because there's a big gap between here and ... the six and the four are sort of close together.

M: Uh huh.

T: But these two are not.

M: So, maybe it adds up, but maybe there's something else as well?

T: Yuh ...

(M releases the scale, it tips from side to side in coming to rest.)

T: Didn't (softly).

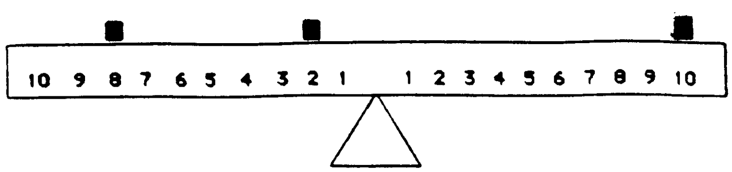
M: It didn't?

T: It didn't add up.

M: It didn't add up?

Strategy: ARITHMETIC1

(The scale comes to rest, balanced.)



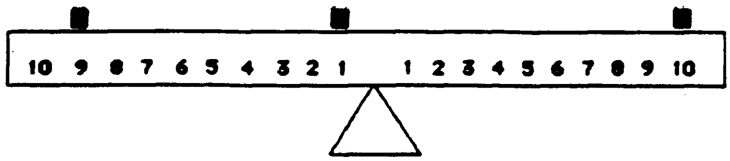
T: Yeah, it did! But maybe it's *just* that it has to add up ...

- SITUATION2a, third Episode

Strategy: ARITHMETIC1

T: I'll try it on the one ... and the nine [another configuration that "adds up" but has a "big gap"]

(T tries this, and after several seconds the scale comes to rest, balanced.)

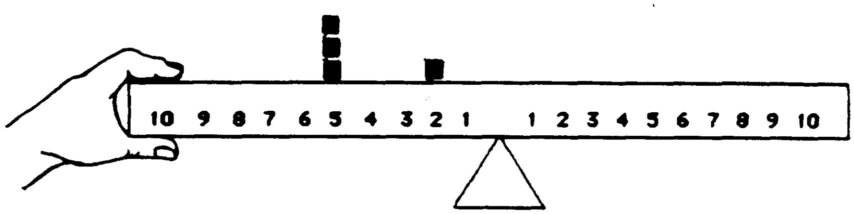


T: Yeah, it *just* needs to add up.

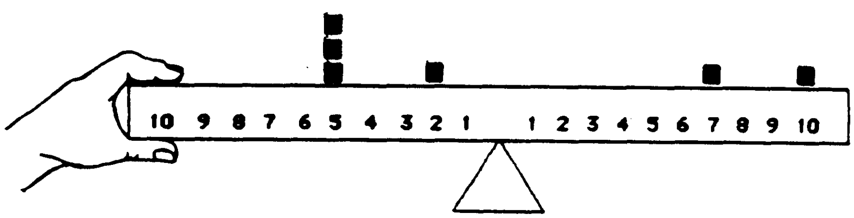
• SITUATION3, first Episode

Strategy: ARITHMETIC3b

M: OK ... If I put three on the fifth here and one on the second, (does this and then holds the scale steady) could you use two to make it balance?



(T puts one on the tenth and one on the seventh.)



M: Why did you try that?

T: Because they equal seventeen, I don't know why, just a funny feeling about that five.

M: What's the funny feeling?

T: There's three, it's like three fives, you've got three rings, so it's like each ring equals five, so that's fifteen, plus the two is seventeen.

M: So you think that will work?

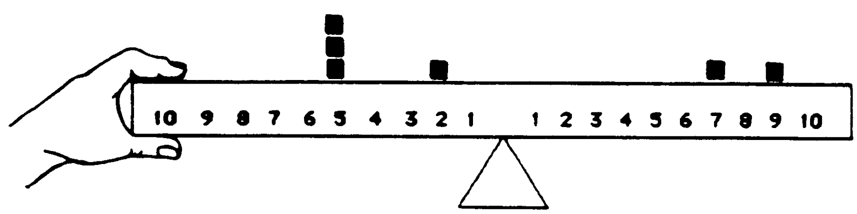
T: I'm not sure. ... um ... let's see what happens ...

(M releases the scale and it balances.)

• SITUATION3, second Episode

Strategy: ARITHMETIC3b

(After several seconds, M holds the scale steady and moves the ring on the tenth peg to the ninth peg on the same arm.)



M: Do you think this would work, too?

T: Nope.

M: Why not?

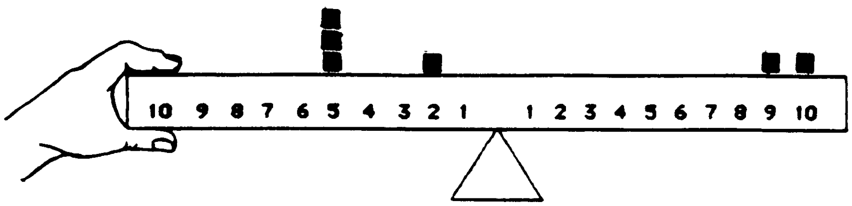
T: Seven and nine don't equal seventeen.

(M releases the scale so that T can see that it tips, and then holds it steady again.)

• SITUATION3, third Episode

Strategy: ARITHMETIC3b

M: ... OK ... and if I put this one here, on the tenth (moves the ring on the seventh to the tenth peg on the same arm), what do you think will happen?



T: This side [the arm with two rings] will go down.

M: Why?

T: Because nine and ten equal nineteen and it's more than seventeen.

(M releases the scale and it tips.)

• SITUATION3, fourth Episode

Strategy: ARITHMETIC3b

M: ... All right ... So now if you were telling someone how to make this balance, what would you tell her?

(M removes the rings from the scale while speaking.)

T: Let's see, it doesn't necessarily have to be on the same number, but, like, you have to add this, like, ten and nineteen equal, no, seven (touches the seventh peg on one arm) and six (touches the sixth peg on the same arm) equal thirteen, then on this scale [arm] over here, then why don't you try to equal thirteen with all different numbers.

• SITUATION4, first Episode

Strategy: NUMBERS

M: OK ... Do you think the numbers make a difference, like, if we changed this three here (the 3 on one arm) into an eight and this eight (the 8 on the same arm) into a three, would it make a difference in the way this works?

T: I think so.

M: Why do you think so?

T: Because the number belongs here, forget that.

M: Forget that?

T: Yuh (giggles).

M: OK, why do you think it would make a difference?

T: ... because ... uh ... maybe they're the most important things ... like, if you're counting by two's and ...

- SITUATION4, second Episode

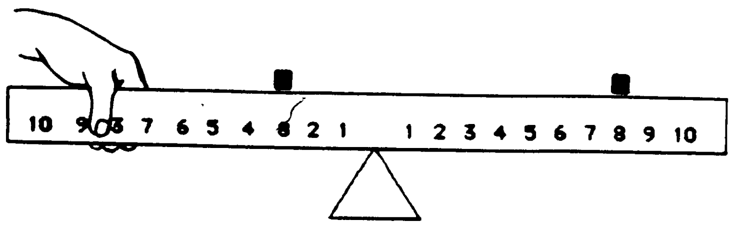
Strategy: ARITHMETIC3a[1]

M: What if you changed this [the 3 on one arm] into an eight, you know, normally if you put one [ring] on the three and one [ring] on the three [on the other arm], it will balance, right?

T: Yuh.

M: OK, now it's an eight (turns the 3 into an 8 with magic marker). If this is now an eight and that's a three (puts her finger over the left part of the 8 on the same arm, also holds the scale steady, and puts a ring on the "new" eighth peg on that arm and a ring on the "real" eighth peg on the the other arm) ... will it balance now?

[1]Although Tot did no calculations at this point, I classify her strategy as ARITHMETIC3a because she had been using ARITHMETIC3 and she now considered distance, rather than the numbers on the scale as affecting the scale's functioning. See the description of ARITHMETIC3 in Chapter 3.



T: ... This is confusing now ... no, I don't think so.

M: Why not?

T: Well ...

M: This is an eight, and that's an eight, they're both on the same number.

T: ... This isn't as easy as I thought it was.

M: Why

T: (interrupting) I don't know, because the thing doesn't have brains.

M: ... It doesn't have brains

T: And I don't think a piece of wood is monogrammed, so ...

M: You don't think a piece of wood is what?

T: Is like, you know, it has, it has senses in it.

M: Uh huh ... What was that word you used?

T: Monogrammed, I don't know why I used it, I make up words ... (laughs) ... I don't know if it works (whines).

M: ... What are the arguments for and against ... You said a piece of wood doesn't have brains, so it doesn't know.

T: Right.

Strategy: NUMBERS

M: But why might it work?

T: Because, it might just think that's an eight, because that's an eight [the 8 on the other arm], so it's in the same shape.

M: How would it know that?

T: Because, it's the shape of an eight.

M: How does it know what shape I drew on?

T: ... That's what I don't ... it knows this is an eight because it knows that's an eight and they both have the same shape.

M: So it can sense what shape I put on?

T: ... No ... It ... I don't know, let's try, I'm not sure.

Strategy: ARITHMETIC3a

M: OK ... should I let go?

T: It won't work.

M: What will happen?

T: This side [the side with the ring on the "real" 8] will go down.

M: Why?

T: Because that's an eight and that's a three, it's a fake eight.