A FINITELY CONVERGING CUTTING PLANE TECHNIQUE

by

James B. Orlin

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by: James B. Orlin
Sloan School of Management
MIT

Abstract.

We consider a new finitely convergent cutting plane algorithm for mixed integer linear programs in which the optimal objective value is assumed to be integral. The primary "theoretical" contribution is the simplicity of the proof of convergence.

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Gomory [1958] developed the first cutting plane technique guaranteed to obtain the optimum solution to a bounded integer program in a finite number of steps. Since that time, Gomory [1960], Bowman and Nemhauser [1970], Chvatal [1973], and several others have developed finitely convergent cutting plane techniques. The purpose of this note is to provide a very simple cutting plane rule guaranteed to give finite convergence. The technique is a modification of a procedure provided by Bell [1973]. Bell's procedure recursively changes the objective function, and each cut is derived from the current objective via "rounding."

Our procedure is developed for the following mixed integer linear program (1)

$$\text{Minimize} \quad z_0 \tag{1a}$$

$$\text{Subject to} \quad z_0 - cx - dy \geq 0 \tag{1b}$$

$$Ax + By \geq b \tag{1c}$$

$$0 \leq y \leq u \tag{1d}$$

$$x_j \in \{0, 1\} \text{ for } j \in [1..n] \tag{1e}$$

$$z_0 \quad \text{integer,} \tag{1f}$$

where all data is rational and u is finite. The notation $[i..j]$ is an alternate notation for the set $\{i, i+1, ..., j\}$ of integers.

In order to apply our cutting plane procedure, we modify our problem by changing the objective to a lexico objective and by modifying the variables as follows:

$$\text{Lexico Minimize} \quad z_0, z_1, ..., z_n \tag{1a}$$

$$z_i = 2z_{i-1} + x_i \quad \text{for} \quad i \in [1..n], \tag{1g}$$

$$z_i \geq \ell_i \text{, integer} \quad \text{for} \quad i \in [1..n], \tag{1h}$$

where initially the coefficients ℓ_i are sufficiently small negative integers.

For readers unfamiliar with lexiography, one can alternately exchange the objective in (1a) for "minimize ($\Sigma\, \varepsilon^i z_i$: $i \in [0..n]$)" where ε is a sufficiently small positive number. As is well known (see for example Dantzig (1963)), lexico minimization is a mathematical way of introducing tie breaking rules and is equivalent to perturbation.

Henceforth, we refer to the problem with lexico objective and constraints (1b - 1h) as the MIP, and we refer to the problem derived by relaxing the integrality constraints on x and z as the continuous relaxation of the MIP.

LEMMA 1. Let (x^*, y^*, z^*) be optimal for the continuous relaxation of the MIP, and let j be the minimum index for which z_j^* is non integral. Then the constraints "$z_i \geq z_i^*$ for $i \in [0..j-1]$ and the constraint "$z_j \geq z_j^*$ are valid inequalities for the MIP.

PROOF. Suppose that $\hat{x}, \hat{y}, \hat{z}$ is feasible for the MIP. By assumption $\hat{z}$ is lexico greater than z^*. Let i be the first index for which $\hat{z} > z_i^*$. Then $\hat{z}_k = z_k^*$ for $k \in [1..i-1]$. We first observe that since $\hat{z}_j \neq z_j^*$ it follows that $i \leq j$. If $i = j$, $|\hat{z}_j| \geq |z_j^*|$ and since $\hat{z}$ is integral it follows that $\hat{z}_j \geq z\,|_j^*$and thus the above constraints are satisfied. We now consider the remaining case in which $i > j$ and $\hat{z}_i > z_i^*$. Since both $\hat{z}_i$ and z_i^* are integral, it follows that $\hat{z}_i \geq \overset{*}{z}_i + 1$. Moreover, because $0 \leq x_i \leq 1$, if we assume inductively that for $k \geq i$, $\hat{z}_{k-1} \geq z_{k-1}^* + 1$, then

$$\hat{z}_k \geq 2\hat{z}_{k-1} \geq 2z_{k-1}^* + 2 \geq z_k^* + 1 ,$$

completing the proof. □

THEOREM. If we successively add cuts of the type described in Lemma 1, then the total number of iterations is $O(2^n\hat{z}_0)$, where

$$\hat{z}_0 = \sum_{j=1}^{n} |c_j| + \sum_{j=1}^{p} u_j |d_j|$$

PROOF. Let $\ell'_0, \ldots, \ell'_n$ be the initial bounds on the z variables. It is clear from (1b), (1d) and (1e) that we may take $\ell'_0 = -\hat{z}_0$, and thus by (1g) we may take $\ell'_1 = 2^i \ell'_0$. At each iteration at least one of the lower bounds increases by one. Moreover, we also know that any feasible vector z is bounded above by the bounds u' defined as follows: $u'_0 = \hat{z}_0$ and $u'_j = 2^j(\ell'_0 + 1) - 1$. It follows that the total number of increases in the lower bounds is bounded by $2^{n+2}\ \hat{z}_0$. □

We illustrate the above cutting plane technique on a small 0-1 integer program. The reader may recognize the example as a special case of the simplest class of IP's for which branch and bound takes an exponential number of steps. This class of problems also requires an exponential number of steps if solved by the above cutting plane technique.

$$
\begin{array}{lll}
\text{Minimize} & z_0 & \\
\text{Subject to} & z_0 - 2x_1 - 2x_2 - 2x_3 & \geq 0 \\
 & \quad 2x_1 + 2x_2 + 2x_3 & \geq 3 \\
 & x_j \in \{0, 1\} \ \text{ for } \ j = 1,2,3. &
\end{array}
$$

The resulting lexico problem is:

Lexico minimize z_0, z_1, z_2, z_3

Subject to

$$
\begin{aligned}
z_0 \quad & - 2x_1 - 2x_2 - 2x_3 \geq 0 \\
& 2x_1 + 2x_2 + 2x_3 \geq 3 \\
-2z_0 + z_1 \quad & - x_1 = 0 \\
-2z_1 + z_2 \quad & - x_2 = 0 \\
-2z_2 + z_3 \quad & - x_3 = 0
\end{aligned}
$$

together with the constraints $z_i \geq \ell_i$ integer for $i = 0, 1, 2, 3$, and $0 \leq x_j \leq 1$ for $i = 1, 2, 3$.

(If z is integer, it follows that x must also be integer).

The sequence of solutions obtained are given in Table 1.

ITERATION	**LOWER BOUNDS**				**X**			**Z**			
	ℓ_0	ℓ_1	ℓ_2	ℓ_3	x_1	x_2	x_3	z_0	z_1	z_2	z_3
1	0	0	0	0	0	1/2	1	3	6	25/2	26
2	3	6	13	26	0	1	1/2	3	6	13	53/2
3	3	6	13	27	1/6	1	1/3	3	37/6	40/3	27
4	3	7	13	27	1	0	1/2	3	7	14	57/2
5	3	7	14	29	1	1/2	0	3	7	29/2	29
6	3	7	15	29	1	5/9	0	28/9	65/9	15	30
7	4	7	15	29	0	1	1	4	8	17	35

Conclusion

The cutting plane technique devised above is in a sense an enumerative technique. As such it suffers from the disadvantage that it does not exploit special properties of the feasible region. Another disadvantage is that it may suffer from severe numerical problems if n is large. (The "cutting plane" technique of Bell shares the former disadvantage, but it does not share the numerical problems.)

One potential advantage of this cutting plane technique is the simplicity of implementing cuts. In particular, the constraint matrix of the LP is never changed. Only the right hand side coefficients are changed.

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