On the Arakelov Chow group of arithmetic abelian schemes and other spaces with symmetries

by

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Abstract

We construct a Fourier transform for Arakelov Chow groups of arithmetic abelian schemes and prove its basic properties which are analogous to those in the classical Chow group setting. We show that the arithmetic Chow groups do not possess such a transform. As applications we rederive some previous results on canonical heights, obtain a global Bost formula for intersection numbers and also show the existence of a Lefschetz decomposition for Arakelov Chow groups. We compare our transform to the Nahm transform and show that their difference is a degree 4 form which is generally not zero. We establish a numerical criteria for lifting eigenvectors of operators preserving harmonic forms from the classical Chow group to the Arakelov Chow group. We give three examples, the first that of abelian varieties allowes us to recapture some of our previous results, the second that of P^{∞} viewed as a relative infinite symmetric product of P^1 allowes us to establish a Fourier transform theory for the Arakelov Chow groups of that space and the third that of Kuga-Sato varieties produces some eigenvectors for Hecke correspondences which might be usefull for producing interesting height functions.

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1 Introduction

Our main purpose in this thesis is to study the Arakelov Chow groups of arithmetic abelian schemes and other spaces with Symmetries. The main algebraic tool for studying the classical Chow groups of abelian schemes has been the geometric Fourier transform. This transform was first introduced by Mukai [Mu] who gave a transform between the derived category of coherent sheaves on an abelian variety over a field k and the same category for the dual abelian variety. This transform had some very nice properties (to be listed later similar to those possessed by the classical Fourier transform hence the name. In [Be1,2] Beauville introduced a similar construction (compatible by Grothendieck-Riemann-Roch to Mukai's) for the rational Chow groups of an abelian variety and in [Be2] he used the construction to give a spectral filtration on the Chow groups of an abelian variety which conjecturally is an example of the conjectured Beilinson-Bloch filtration. in [D-M] Deninger and Murre extended his results to the context of an abelian scheme over a quasi projective connected scheme over a field. In [K1,2] Küennemann used the above results to provide a Lefshcetz theorem for the spectral decomposition of the Chow groups. Finally in [K-V] Kimura and Vistoli extended the Fourier transform to infinite symmetric products of curves and provided a conjectural spectral decomposition for the infinite Symmetric product of any smooth scheme over a field. It was asked in [G-S2] (question raised by Bloch, Gillet, Soulé, Deninger ,Küennemann) whether one can construct a similar Fourier transform with the properties shared by the above ones for the rational Arakelov Chow groups (or more generally the rational arithmetic Chow groups) of an abelian scheme over the integer ring of a number field. One of our main goals is to give such a construction for the rational Arakelov Chow groups. In addition we will show the existence of a rather trivial obstruction to the construction in the arithmetic Chow group case. By using a lemma which is proved in the contruction we show that all the appllications of the Fourier transform to to the study of classical Chow groups still hold in this context. We also show that a global Bost formula formally follows from the Fourier transform and then give a more precise criterion for the existence of such formulas. In [G-S2] the Nahm transform for certain vector bundles with certain hermitian metrics on abelian surfaces is mentioned. We shall look at the relation (or lack of it) between these two transforms. Finally we look at the problem of constructing canonical elements in Arakelov theory.

We prove a lifting theorem for eigenvectors of nice operators and use it to provide a spectral theory for the infinite projective space viewed as the infinite symmetric product of the projective line. This seems to be the only case where one can generalize the work of Kimura and Vistoli. We also give an example related to Hecke correspondences.
Theorem 1 was obtained at about the same time independently by K.Kuenneman (except the inversion formula which he proved later) . He used it to give a conditional proof of some standard conjectures (see [G-S4] [Ku3]) for arithmetic abelian schemes. Later L.Weng also announced independently a proof of theorem 1.

2 Background

All the material in this section is due to C.Soule and H.Gillet for more details one should look at [G-S1] and also [G-S2].
Arakelov Chow groups: Let S=Spec(A) denote an arithmetic ring (see [G-S1] for a general definition) . For the purposes of this section it means that $A = O_k$, the spectrum of the ring of integers of a number field k , or a localization of such a ring, together with the set σ of embeddings $\sigma : k \hookrightarrow \mathbf{C}$
Definition: An arithmetic variety X is a nonsingular projective scheme X/S such that X_k is smooth.
if X is an arithmetic variety then

$$X(\mathbf{C}) = \coprod_{\sigma:\mathbf{k}\hookrightarrow\mathbf{C}} \mathbf{X}_\sigma(\mathbf{C})$$

We endow $X(\mathbf{C})$ with a Kähler metric $\omega = \{\omega_\sigma\}$ in a way which is compatible with the conjugation action on $X(\mathbf{C})$. Let Z(X) denote the group of cycles on X.
For any cycle $y \in Z(X(\mathbf{C}))$ denote by H(y) the harmonic representative of the cohomology class of δ_y.
For a current $\alpha \in D(X(\mathbf{C}))$ we denote by $H(\alpha)$ the projection of α onto the harmonic forms with respect to the kähler metric ω.
Definition: The Arakelov Chow group of (X,ω) denoted $CH(\bar{X})$ is the group of equivalence classes generated by pairs (y, g) , $y \in Z^p(X)$
$g \in D(X(\mathbf{C}))^{p-1,p-1}_{\mathbf{R}}$ (the space of real (p-1,p-1) currents on X) which satisfy

the condition

$$-\frac{1}{\pi i}\partial\bar{\partial}g = \delta_{y_\infty} - H(\delta_{y_\infty})$$

where y_∞ is the part of y at ∞ (∞ will henceforth be omitted). subject to the relations:
1) $(0, \partial u + \bar{\partial}v)$
2) $(div(f), log|f|)$ for $f \in K^*(y)$ $y \subset X$ of codimension p-1.
Definition: The arithmetic Chow group of X which will be denoted $\hat{CH}(X)$ is defined as above replacing the condition on $\partial\bar{\partial}g$ by

$$-\frac{1}{\pi i}\partial\bar{\partial}g = \delta_{y_\infty} - \beta$$

where β is any smooth form.
We also define $\omega(g) := \beta$.
The arithmetic chow group forms a commutative ring under an intersection product

$$\hat{CH}^p(X) \times \hat{CH}^q(X) \to \hat{CH}^{p+q}(X)$$

which is defined by the following "formal" formula

$$(y_1, g_1) * (y_2, g_2) = (y_1 y_2, \delta_{y_1} g_2 + g_1\omega(g_2))$$

As one can notice from this formula the Arakelov Chow group is generally not a ring under this product since the product of two harmonic forms is generally not harmonic. In all the cases we shall consider, the Arakelov Chow group will be closed under the intersection product, or at least harmonic forms will be sent to harmonic forms by the morphisms under discussion.
In case $p+q = dim(X) = dim(X_k)+1 = n+1$ and $A = O_k$ one can compose the intersection product with the structure degree maps

$$\hat{CH}^{n+1}(X) \to \hat{CH}(S) \to \hat{CH}(spec(\mathbf{Z})) = \mathbf{R}$$

to get an intersection pairing. We shall denote the class of a pair (y, g) by $[(y, g)]$ and by $[y]$ the class of y in $CH(A)$.
Remark :[G-S1] uses $\frac{1}{2\pi i}$ instead of $\frac{1}{\pi i}$ but it does not matter for our purposes.
We need the following 3 facts([G-S 1,2]):
Fact 1: The Arakelov Chow group sits in an exact sequence
$H^{p-1,p-1}(X) \xrightarrow{a} CH^p(\bar{X}) \xrightarrow{\zeta} CH^p(X) \to 0$

where $H^{p-1,p-1}(X)$ denotes (p-1,p-1) harmonic forms $:a(\eta) = [(0,\eta)]$ and $\zeta(y,g) = [y]$.
If $p = 1$ then $ker(a) = im(\tilde{\rho})$ where $\tilde{\rho} : O_k^* \rightarrow H^{0,0}$ sends a unit h to the locally constant function whose value on X_σ is $\int_{X_\sigma} log|h|_\sigma \omega$.
Fact 2: Let A,B be abelian schemes with invariant Kähler metrics at the infinite places and let $f : A \rightarrow B$ be a homomorphism. Then $CH(\bar{A})$ is closed under intersections, $f^*(CH(\bar{B})) \subseteq CH(\bar{A})$ and if f is smooth over k $f_*(CH(\bar{A})) \subseteq CH(\bar{B})$.
Fact 3: If $y \in Z^p(X)$ then there is a unique (up to $Im\partial + Im\bar{\partial}$) current which will be denoted g_y such that $(y, g_y) \in CH(\bar{A})$ and H(g_y)=0.
We want to consider abelian schemes over S, i.e. abelian varieties A_k with good reduction everywhere. In this case, by theorems of Mumford and Raynaud about the representability of the Picard functor, one can attach to A a dual abelian scheme $\hat{A}$ and a normalized Poincaré bundle $\wp$ in a functorial fashion [Ch-Fa,pg 1-5]. Endow $A_\sigma(\mathbf{C})$, $\hat{A}_\sigma(\mathbf{C})$ and their products with invariant Kähler metrics .
By abuse of notation we shall denote the volume form on all these spaces by ω.
Denote by [n] the multiplication by $n \in \mathbf{Z}$ operation on A.
Let e_A:$S \rightarrow A$ denote the zero section.
$m : A \times A \rightarrow A$ denote the multiplication operation.
Define for $x, y \in CH(\bar{A})$ the Pontrjagin product by the formula $x * y := m_*(p^{1*}x \cdot p^{2*}y)$ where p_i i=1,2 denote the projections $p_i : A \times A \rightarrow A$.
Let g:=dim(A/S).
For an integrable top degree current h we shall denote by $\int_A h$ the locally constant function on $A(\mathbf{C})$ whose value on $A_\sigma(\mathbf{C})$ is $\int_{A_\sigma} h_\sigma$.
All Chow groups are with rational coefficients, all products are over S.

3 Arakelov Fourier transform

In this section we will define an Arakelov Fourier transform between $CH(\bar{A})$ and $CH(\bar{\hat{A}})$ and prove its basic properties. In order to define the transform and obtain some of its properties one has to show the existence of a lifting of the Poincaré line bundle to an element $\bar{l} \in CH(\overline{A \times \hat{A}})$ which satisfies $(Id \times n)^*\bar{l} = n\bar{l}$. We shall give three proofs of this fact, two of them in this section. Then in order to prove all other properties one proves an existence and uniqueness lemma for liftings of certain eigenvectors of the multiplication by n operations that allows one to transfer the proofs from the classical case.

Lemma 1 *Let $y \in Z^1(A \times \hat{A})$ be such that $(Id \times [n])^* y = ny$ in $CH^1(A \times \hat{A})$ and let $f_n \in K^*(A)$ be such that $div(f_n) = ny - (Id \times [n])^* y$. Then in $CH^1(A \times \hat{A})$ we have*

$$(Id \times [n])^*[(y, g_y)] = n[(y, g_y)] + a(\int_{A \times \hat{A}} log|f_n| \cdot \omega))$$

Proof: Since the kähler metric is invariant we have for any current $\alpha \in D(A \times \hat{A})$

$$H(\alpha) = \int_{A \times \hat{A}} i_t^*(\alpha) dv(t)$$

where i_t^* denotes the map on currents induced by the shift by $t \in A \times \hat{A}$ operation and dv denotes Haar measure. Harmonic means translation invariant. By fact 2 $[((id \times [n])^* y, (Id \times [n])^* g_y]) \in CH^1(A \times \hat{A})$. We also have

$$\int_{A \times \hat{A}} (Id \times [n])^* g_y \cdot \omega = \int_{A \times \hat{A}} g_y (Id \times [n])_* \cdot \omega = n^{2g} \int_{A \times \hat{A}} g_y \cdot \omega = 0$$

the last equality by the definition of g_y. So $(Id \times [n])^* g_y = g_{(Id \times [n])^* y}$. We also know by the assumptions that

$$ny - (Id \times [n])^* y = div(f_n)$$

and so by the Poincaré Lelong formula (see [G-S1])

$$-\frac{1}{\pi i} \partial \bar{\partial} (g_{Id \times [n])^* y} + log|f_n|) = \delta_{ny} - H(ny)$$

so

$$g_{ny} = g_{(Id \times [n])^* y} + log|f_n| - H(log|f_n)$$

. Since $[(div(f_n), log|f_n|)] = 0$ and obviously $g_{ny} = n g_y$ we have our result.
Remark: The lemma still holds for a smooth endomorphism μ on an abelian scheme A under the analogous assumptions.
Remark: The same proof as above actually proves the following:
Lemma 1': Let $f : A \to B$ be a homomorphism ,smooth over k, of abelian schemes. Then for $y \in Z^p(B)$ $f^*[(y, g_y)] = [(f^* y, g_{f^* y})]$ and similarly for f_*.

Lemma 2 *Let λ be an automorphism of B which commutes with a homomorphism μ and let $y \in Z^1(A \times \hat{A})$ be such that $\lambda^*(y) = -y$ (as cycles) and $\mu^*[y] = n[y]$ let $f \in K^*(B)$ be such that $div(f) = \mu y - ny$. Then*

$$a(\int_{A\times\hat{A}} log|f_n| \cdot \omega) = 0$$

Proof: Since $\lambda\mu = \mu\lambda$ we have

$$\lambda^*(ny - \mu^* y) = -(\mu^* y - ny)$$

hence in the notation of lemma 1

$$\lambda^*(div(f)) = -div(f) = div(f)^{-1}$$

Since λ acts trivially on the top dimensional cohomology $\lambda_*(\omega) = \omega$ and therefore

$$\int_B (\lambda^* log|f|) \cdot \omega = \int_B log|f| \cdot \omega$$

But also

$$\lambda^*(log|f|) = -log|f| + log|h|$$

where h is invertible and hence is defined by a unit in O_k. Therefore

$$2a(\int_B log|f| \cdot \omega) = a(\int_B log|h| \cdot \omega) = a(log|h|) = 0$$

the last equality by fact 2.
Remark: Lemma 2 explicitly identifies eigenvectors for certain operators. For the Poincaré bundle see the corollary below. Other examples are anti symmetric bundles L such that $n^*L = nL$ or elements in higher codimension which are antisymmetric and are divisors of an abelian subvariety of A.
We view the normalized Poincaré bundle as an element l of $CH^1(A \times \hat{A})$. By functoriality $(Id \times [-1])^*\wp = -\wp$ and therefore we can represent $2\wp$ by a cycle $\rho \in Z^1(A \times \hat{A})$ which is antisymmetric with respect to $Id \times [-1]$.

Corollary 1 *Let $\bar{\rho}, \bar{\rho'} \in CH^1(\overline{A \times \hat{A}})$ $\bar{\rho} = [(\rho, g_\rho)], \bar{\rho'} = [(\rho', g_{\rho'})]$ where ρ, ρ' represent $2\wp$ and are antisymmetric as cycles with respect to $\lambda = Id \times [-1]$. Then $\bar{\rho} = \bar{\rho'}$ and $(Id \times n)^*(\bar{\rho}) = n\bar{\rho}$*

Proof: For the first assertion look at f such that $div(f) := \rho' - \rho$. div(f) satisfies the cycle condition of lemma 2 since ρ and ρ' do, and therefore $log|f| = g_{div(f)}$ so

$$[(\rho, g_\rho)] = [(\rho, g_\rho)] + [(div(f), log|f|)] = [(\rho, g_\rho)] + [(div(f), g_{div(f)})] = [(\rho', g_{\rho'})]$$

The second assertion follows from lemmas 1 and 2.
Let ρ be as in corollary 1 set $\bar{l} := 1/2[(\rho, g_\rho)]$

Definition: We define the Fourier transform $\bar{F} : CH(\bar{A}) \to CH(\bar{\hat{A}})$ to be the map given by the correspondence defined by the cycle

$$e^{\bar{l}} = 1 + \bar{l} + \frac{\bar{l}^2}{2!} \ldots \in CH(\overline{A \times \hat{A}})$$

using the standard formula

$$\bar{F}(x) = p_{2*}(p_1^*(x) \cdot e^{\bar{l}})$$

.

Theorem 1 *$\bar{F}$ as defined above satisfies the following properties:*
A) $\bar{F}(\bar{1}_A) = [((-1)^g e_*(1_S), g_{(-1)^g e_*(1_S)})]$ where $\bar{1}_X$ denotes the class corresponding to the total space in $CH^0(\bar{X})$ for a space X and 1_X denotes the total space as a cycle.
B) (inversion formula)

$$\bar{\hat{F}} \circ \bar{F} = (-1)^g [\Gamma_{[-1]}]$$

as correspondences, where $[\Gamma_{[-1]}]$ is the graph of $[-1]$ and $\bar{\hat{F}}$ is the transform for $\hat{A}$.
C) $\bar{F}$ is functorial i.e. if $f : A \longrightarrow B$ is an isogeny ,then we have the following identity:

1) $\bar{F}_A \circ [{}^t\Gamma_f] = [\Gamma_{\hat{f}}] \circ \bar{F}_B$ where ${}^t\Gamma$ denotes the transpose of Γ.
and more generally for a smooth over k homomorphism f we have :
2) $\bar{F}_B \circ [\Gamma_f] = [{}^t\Gamma_{\hat{f}}] \circ \bar{F}_A$

D) Let $x \in CH^p(\bar{A})$ and $\bar{F}(x) = \sum_{q=0}^{g+1} y_q$ where $y_q \in CH^q(\bar{\hat{A}})$. Then $[n]^*(y_q) = n^{g-p+q} y_q$ for all $n \in \mathbf{Z}$.

As a corollary of the above properties we also have:

E) (eigenspace decomposition)

$$CH^p(\bar{A}) = \bigoplus_{s=p''}^{s=p'} CH^p_s(\bar{A})$$

where p'=min(2p,p+1) , p''=max(p-g,2p-2g) and $CH^p_s(\bar{A})$:={ $x \in CH^p(\bar{A}) | [n]^*(x) = n^{2p-s}x, \forall n \in \mathbf{Z}\}$.

F) (Isomorphism of Chow and pontrjagin ring structures)

1) $\bar{F}(x * y) = \bar{F}(x) \cdot \bar{F}(y)$

2) $\bar{F}(x \cdot y) = (-1)^g(\bar{F}(x) * \bar{F}(y))$.

G) (Compatibility with the intersection product) If $x \in CH^p_s(\hat{A})$ *and* $y \in CH^{g-p+1}_r(\hat{A})$ *then* $< x, y >= 0$ *unless* $r + s = 2$*; moreover* $x \cdot y = 0$ *unless* $r + s = 2$.

Proof: Define $F : CH(A) \to CH(\hat{A})$ to be the map given by the correspondence defined by $e^l \in CH(A \times \hat{A})$. Then using the methods of [D-M] and [Kü] one sees that after replacing $\bar{F}$ by F and $CH(\bar{A})$ by $CH(A)$ all the statements in the theorem hold.

Statement D follows from corollary 1 by the argument in [D-M] prop 2.16 (or [Be2]). Denote $0_A := e_{A*}(S)$.

Lemma 3 *(analog of lemma 2.8 in [D-M])* $n\bar{F}(\bar{1}_A) = (-1)^g[(0_{\hat{A}}, g_{0_{\hat{A}}})] + z$ *where* $z \in \bigoplus_{q \neq p} CH^q(\bar{\hat{A}})$

Proof: We know that $F(\bar{1}_A) = (-1)^g 0_{\hat{A}}$ and that the part of $\bar{F}(1_A)$ lying in $CH^g(\bar{\hat{A}})$ is an eigenvector for $[n]_*$. $[(0_{\hat{A}}, g_{0_{\hat{A}}})]$ is an eigenvector by lemma 1' and $[n]_* 0_{\hat{A}} = 0_{\hat{A}}$, therefore the lemma follows from:

Lemma 4 *If* $y \in CH^p_m(A)$ $m \neq 2$ *then there is a unique* $\bar{y} \in CH(\bar{\hat{A}})$ *which maps to y under* ζ *and* $\bar{y} \in CH^p_m(\bar{A})$.

Proof: Since our Fourier transform satisfies $F \circ \zeta = \zeta \circ \bar{F}$ and F is surjective, property D proves the existence. To prove uniqueness we use the exact sequence from fact 1. Suppose $\bar{y}, \bar{y'}$ are 2 elements with the above property. Then

$$\bar{z} := \bar{y} - \bar{y'} \in CH^p_m(\hat{A}) \cap a(H^{(p-1,p-1)})$$

but for any $z \in H^{(p-1,p-1)}$ i.e any harmonic form representing some cohomology class cl(z) , $[n]^* z$ is harmonic and represents $[n]^*(cl(z)) = n^{2p-2} cl(z)$ so $[n]^*(a(z)) = n^{2p-2} a(z)$. Therefore m=2, a contradiction.

Lemma 5 $\bar{\hat{F}} \circ \bar{F}(y) = (-1)^g(-1)^* y + z$ *where* $y \in CH^p(\bar{A})$ *and* $z \in \bigoplus_{q \neq p} CH^q(\bar{A})$ *(the analog of [D-M] lemma 2.9).*

Proof: (As in [Be] or [D-M]) Define $\Sigma : A \times \hat{A} \times A \to A \times \hat{A}$ by $(a, \hat{a}, a') \xrightarrow{\Sigma} (a + a', \hat{a})$ and $s : \hat{A} \times A \to A \times \hat{A}$ by $(a, \hat{a}) \xrightarrow{s} (\hat{a}, a)$. Then we want to show that

$$p_{12}^*(\bar{l}) + P_{23}^*(s^*(\bar{l})) = \Sigma^*(\bar{l})$$

in $CH^1(\overline{A \times \hat{A} \times A})$. This is known on the level of $CH^1(A \times \hat{A} \times A)$ (theorem of the cube) and now we use lemma 1' and a variant of corollary 1 with $Id \times [-1] \times Id$ instead of $\times[-1]$ to get the statement. Then we use a cycle level equality and lemma 1' to get

$$\bar{\hat{F}} \circ \bar{F} = p_{13*}(\epsilon^{\Sigma^*(\bar{l})}) = m^* p_{1*}(\epsilon^{\bar{l}})$$

where $p_1 : A \times \hat{A} \to A$. Now

$$m^* p_{1*}(\epsilon^{\bar{l}}) = m^* \bar{\hat{F}}(\bar{1}_{\hat{A}})$$

by definition, and by lemma 3

$$m^* \bar{\hat{F}}(\bar{1}_{\hat{A}}) = m^*((-1)^g[(0_A, g_{0_A})] + z) = (-1)^g[([\Gamma_{-1}], g_{[\Gamma_{-1}]})] + z'$$

where $z \in \bigoplus_{q \neq g} CH^q(\hat{A})$ and so $z' \in \bigoplus_{q \neq g} CH^q(\overline{A \times \hat{A}})$. To conclude the lemma we have to show that as correspondences on $CH(\bar{A})$ $[(\Gamma_{-1}, g_{\gamma_{-1}})] = [-1]^*$. By [G-S1] we have $[(\Delta, g_\Delta)] = Id$ where Δ is the diagonal, now apply $[-1]^*$ to both sides to get the lemma.

Property C of the theorem follows by combining the argument of [D-M] proposition 2.11 , lemma 3 for $\bar{l}$ and the functoriality properties of lemma 1'. Property F follows from C formally. The only point to note is that $F_{A \times A}(x \times y) = F_A(x) \times F_A(y)$ which followes from lemma 3 by decomposing x,y into eigenvectors with respect to [n] and looking at the resulting eigenvectors with respect to $Id \times [n]$ and $[m] \times Id$.

A, B and E follow from what we have shown so far (without the use of F) in a purely formal fashion involving basically only the grading on $CH(\bar{A})$ (see [Be1,2] [D-M] and [Ku1] for details).

To prove G we note the formula

$$< n^* x, y > = < x, n_* y >$$

(see G-S1). Now the first statement of G follows from a simple numerical calculation. To prove the second statement we note that

$$Z^{g+1}(\hat{A}) = H^{g,g}(A)_{\mathbf{R}} \bigoplus Z^{g+1}(A)$$

. Now $a(H^{g,g}) \in CH_2^{g+1}(\hat{A})$ by a theorem of Bloch [Be2] (also proved by Kuennemann [Ku2]) $CH^*(A_{F_v}) = CH_0^*(A_{F_v})$ for all finite places v. Therefore by compatibility of the Fourier transform for A and A_{F_v} and the shift by one in codimension one has that $CH^{g+1}(\hat{A}) = CH_2^{g+1}(\hat{A})$. As in [G-S4] one can show easily using Kato and Saito's theorem that $CH^{g+1}(A)$ (see [K-S]) is finite that actually $CH^{g+1}(\hat{A}) = im(a)$.
Remark: Even when one considers A_k (no assumptions on reduction) and replaces $<,>$ by $<,>_h$ which is the height pairing on homologically equivalent to zero cycles, the first statement of G is still true with the same proof.
We shall briefly present another proof of the theorem (essentially this proof has been independently found by K.Kuennemann, see [ku3]). To obtain the preferred lifting of $\wp$ one uses a theorem of Faltings which states that there exists a unique hermitian metric with harmonic curvature on a rigidified line bundle L (i.e. a line bundle with an isomorphism $e^*(L) \to O_S$) which turns the isomorphism of the cube (which holds for rigidified bundles)

$$p_{123}^*L \otimes p_{12}^*L^{-1} \otimes p_{13}^*L^{-1} \otimes p_{23}^*L^{-1} \otimes P_1^*L \otimes p_2^*L \otimes p_3^*L = O_A$$

into an isometry (O_A with the trivial metric). It is then easy to show that $\wp$ with this metric is an eigenvector for [n] on $A \times \hat{A}$ and then by another easy argument for $Id \times [n]$. To complete the proof one notices that we can replace lemma 2 by lemma 3 in our arguments (see also section 5.3 later on).
Now let us turn to the nonexistence of the Fourier transform for the arithmetic Chow group of A (see [G-S]1). We also refer the reader to [G-S4] for a definition and discussion of arithmetic correspondences.
Proposition 1: There exists no arithmetic correspondence $AF : \hat{CH}(A) \to \hat{CH}(\hat{A})$ satisfying properties B and F of theorem 1.
Proof: If AF did exist then $\hat{CH}(A)$ would have had a unit z for the Pontrjagin product since the intersection product has a unit and AF is a ring isomorphism by properties B and F. Denote by m_- the operation

$$(a,b) \xrightarrow{m_-} a - b$$

. We want to show that $m_-^*(z)$ acts as the identity correspondence. We have for all $y \in \hat{CH}(A)$

$$y * z = m_*(p_1^* z \cdot p_2^* y)$$
$$= p_{1*}(m_-^* z \cdot p_2^* y)$$

the last equality since $m = p_1 \alpha$ where $\alpha : A \times A \to A \times A$ is given by $(a, b) \to (a + b, b)$ and $m_-^* = \alpha^* p_1^*$. In the case of arithmetic intersection theory the identity correspondence is not represented by a cycle in the product (see [G-S1]) and therefore not of the form $m_-^*(z)$ for any cycle z so it has no unit for the Pontrjagin product.
Remark: one can replace property F in the above proposition with property C and the property
(*) $AF_{A\times A}(x \times y) = AF_A(x) \times AF_A(y)$
This follows from the proof of property F using B and C.
Remark :If we believe motivic theory then on the level of CH(A) we have for horizontal cycles only $CH_0^p(A)$ and $CH_1^p(A)$. $CH_2^p(A)$ consists of finite divisors which are not homologically equivalent to zero.In that case we see that any horizontal $x \in CH_k^p(A)$ has a unique lifting to $\bar{x} \in CH_k^p(\bar{A})$ ($x \in CH_1^p(A)$ if it is homologically equivalent to zero on every fiber) and that the harmonic forms play exactly the role of vertical cycles at infinity.
Remark: The Fourier transform described above is by fact 1 an extension of the Fourier transform on CH(A) by the one on cohomology described in [Bel].

4 Generalizations and variants

The following are examples of other situations where one can apply the Arakelov Fourier transform.
A) Suppose (X, ω_X) is an arithmetic variety. One can define the Arakelov Fourier transform of $X \times A$, viewed as an abelian scheme over X with Kahler metric $\omega_X + \omega_A$, to be the correspondence given by the cycle $p^*(\epsilon^{\bar{l}})$ where p is the projection $X \times A \times \hat{A} \to A \times \hat{A}$. Although $CH(\overline{X \times A})$ is not a ring one can see from the structure of the correspondence and the structure of harmonic forms that the transform sends $CH(\overline{X \times A})$ to $CH(\overline{X \times \hat{A}})$. Also since the homomorphisms used in lemma 2 behave nicely with respect to harmonic forms all the theory still holds.

B) Suppose (A,ω) is an abelian scheme with a Kahler metric such that multiplication by n preserves harmonic forms. One notices that in such a case the proofs of lemmas 1 and 2 still hold. Therefore we get an element $\bar{l}$ which satisfies $n^*\bar{l} = n\bar{l}$. This formally gives property D). Property C) also follows for homomorphisms sending harmonic forms to harmonic forms. The existence part of lemma 4 still holds. An example of such a situation is given by Kuga-Sato varieties which are described in section 7.

C) Another case is obtained by letting $S = spec(\mathbf{C})$. In this case A is any complex abelian variety and the correspondence is uniquely defined by l since $\rho : \mathbf{C} \to \mathbf{H}^{0,0}_{\mathbf{R}}$ is surjective, i.e. l has a unique lifting to $\bar{l} \in CH(\overline{A \times \hat{A}})$.

D) One can define the Fourier transform on the arithmetic K_0 group to be given by the formula

$$F_K((V,h,\beta)) = p_{2*}(p_1^*(V,h,\beta) \odot \hat{\wp})$$

where $\hat{\wp}$ denotes the Poincaré bundle with the unique hermitian metric (up to scalar) whose curvature is harmonic. Since the tangent bundle to A is trivial an application of the arithmetic Riemann Roch formula shows that the transform for Chow groups and the transform for K groups are compatible under $\hat{ch}$ i.e. $F_K\hat{ch} = \hat{ch}F$.

This construction can also be applied to the case of good reduction everywhere (here the metric is scaled so as to give $\bar{l}$ of section 3 by $\hat{c}_1$) and the Riemann-Roch theorem gives one commutativity in this case as well. Since by [G-S3] the Chern character is an isomorphism one sees that F_K is given by a projection onto the inverse image of the Arakelov Chow group which we may call the Arakelov K_0 group.

5 Applications

We shall now present a few applications although the first two do not require the Fourier transform for their proof.

1) Global Bost formula:

Denote by$<,>$: $CH^p(\bar{A}) \times CH^{g-p+1}(\bar{A}) \to \mathbf{R}$ the intersection pairing on $CH(\bar{A})$ (see [G-S1] for definition). The Fourier transform implies the following formula which is a global form of the formula which was proved by Bost [Bo1] for the height pairing at infinity (see also question at the end of [Bo1]):

Theorem 2 *For $x \in CH^p(\bar{A}), y \in CH^{g-p-1}(\bar{A})$ we have*

$$< x, y >=< x * (-1)^* y, [(0_A, g_{0_A})] >$$

Proof: For simplicity we shall assume that A is principally polarized i.e $A = \hat{A}$. In that case, by lemma 2' and the fact that $Id \times n$ and $n \times Id$ commute, $\bar{F}$ is symmetric so $\bar{F}^* = \bar{F}_*$ .We have, using theorem 1,

$$
\begin{aligned}
&< x, y > \\
&=< (-1)^g \bar{F}^* \bar{F}^* (-1)^* x, y > \\
&= (-1)^g < \bar{F}^* (-1)^* x, \bar{F}_* y > \\
&= (-1)^g < \bar{F}^* (-1)^* x, \bar{F}^* y > \\
&= (-1)^g < \bar{F}^* (-1)^* x \cdot \bar{F}^* y, 1_A > \\
&= (-1)^g < \bar{F}^* ((-1)^* x * y), \bar{F}^* [(0_A, g_{0_A})] > \\
&= (-1)^g < \bar{F}^* ((-1)^g x * y), \bar{F}_* [(0_A, g_{0_A})] > \\
&= (-1)^g < \bar{F}^* \bar{F}^* ((-1)^* x * y), [(0_A, g_{0_A})] > \\
&= (-1)^g (-1)^g < x * (-1)^* y, [(0_A, g_{0_A})] >
\end{aligned}
$$

In general we first notice that it is enough to prove the theorem after a change in the ground field since both sides of the equation are invariant under such change. Now one looks at an isogeny $f : A \rightarrow \hat{A}$ (equivalent by Neron model theory to an isogeny $f' : A_k \rightarrow \hat{A}_k$) and using the functoriality of the transform reduces the computation to the previous case.

Remark: Actually in a fashion similar to Bost's original proof one can show that the existence of a unit for the Pontrjagin product implies the formula (basically by the argument in proposition 1 and the projection formula). On the other hand if the intersection pairing is nondegenerate (for a discussion of the case of arithmetic Chow groups see [G-S2]) one can show that a formula of the type $< x, y >=< x * (-1)^* y, z >$ implies that z is a unit for the Pontrjagin product and therefore under that assumption arithmetic intersection theory does not satisfy a formula of that type.

2)Theory of heights:

The construction and proof of properties using lemma 2 of the Fourier transform did not require Faltings' theorem. As a result one obtains from lemma 3 a new proof of the existence of Neron-Tate Heights which are quadratic (or

linear if L is antisymmetric). We may take the self intersection of these elements to produce canonical heights for cycles of any dimension. This produces the heights of Philippon in the case of good reduction (see also [B-G-S]).
Remark: Even without good reduction one should still be able to produce canonical heights by normalizing these eigenvectors and using moving lemmas for divisors.
Now let d be an element in $CH_0^1(\bar{A})$. Such elements come from liftings of symmetric ample line bundles rigidified along the zero section of S (i.e come with an isomorphism $e^*d = O_S$). These line bundles give rise to morphisms $D : A \to \hat{A}$. Define $\nu(D)$ to be the square root of the degree. we still denote by d its canonical lifting to $CH(\bar{A})$
We have the following properties of these canonical elements
A) $d^{g+1} = 0$
This is true since d^{g+1} has eigenvalue $n^{2(g+1)}$ and therefore belongs to $CH_0^{g+1}(\hat{A}) = 0$.
$d^g = \nu(d)g!(0, g_0)$
This is true since it was proved after applying ζ by Beauville [Be1] (see also [Ku1] , [Ku3]) and since both the right and left hand side belong to $CH_0^g(\hat{A})$.
Remark: These properties become characterizations (i.e uniqueness) if we assume the conjecture $H^{p,p} = im(\rho) \bigoplus im(cl)$ of Beilinson. This will follow from 3) below.
3) Lefschetz theory (see also [Ku3])
In [Kú2] Kuennemann uses the Fourier transform to give a Lefschetz decomposition for $CH(A)$. Using Lemma 3 one can lift all his arguments to the context of Arakelov Chow groups. In order not to be overly repetitive I shall only highlight some of the main definitions and formula. We shall see that in all these cases both sides are in $CH_0^p(A)$ (sometimes one has to check the proof to establish this). First let us define $CH_{k,l}^p$ to consist of those elements $x \in CH^p(\overline{A \times A})$ satisfying $(Id \times n)^* x = n^l x$ and $(m \times Id)^* x = m^k x$. From the inversion formula and property C) one sees with the help of lemma 3 that

$$\begin{aligned} Id &\\ &= (-1)^g[-1]^* F_A \circ F_{\hat{A}} \\ &= (-1)^g[-1]^*(\sum \bar{l}_{\hat{A}}) \circ (\sum \bar{l}_A) \\ &= (-1)^g[-1]^* \sum \bar{l}^i_{\hat{A}} \circ \bar{l}^{2g-i}_A \end{aligned}$$

$$= \sum \pi_i$$

where by property C) and the above formula $\pi_i \in CH^g_{i,2g-i}$ and π_i are mutually orthogonal projections onto CH^*_i.
Let d be as in 2) above Define

$$c = \frac{d^{g-1}}{\nu(d)(g-1)!}$$

obviously $c \in CH_0^{g-1}$. We define the Lefschetz operator $L : CH(\hat{A}) \to CH(\hat{A})$ by the formula $L(x) = d \cdot x$ and the operator Λ by $\Lambda(x) = c * x$. Kuenneman shows that after applying ζ these operators satisfy the Kahler-Lefschetz commutation relations i.e.

$$[\Lambda, L] = \sum (g-i)\pi_i$$

By composing both sides with π_i and again using the spectral decomposition via the π_i one sees that both sides are equal in $CH(A \times \hat{A})$ and hence as operators. From these identities one obtains a Lefschetz decomposition and
Corollary: the operator L^{g-s} gives an isomorphism between $CH^p_{2g-s}(A)$ and CH^{p+g-s}_{2g-s}
Remark: in [Ku3] Kuennemann shows (by reduction to the above corollary that assuming some motivic conjectures the Lefschetz conjectures of [G-S4] hold for the whole arithmetic Chow group of the abelian scheme.

6 The Fourier and Nahm transforms

See [D-K] for a detailed account of the Nahm transform.
In [G-S4] The Nahm transform is mentioned as possibly related to the arithmetic Fourier transform. This transform is defined for asd (anti self dual connections) on stable vector bundles over a complex abelian surface A. We shall now Briefly describe this transform. Let p_i denote the projection of $A \times \hat{A}$. Endow $\wp$ with the unique metric (up to multiplication by a scalar) whose induced curvature is harmonic. Let (V,h) be a metrized stable vector bundle which induces an asd connection. We shall say that (V,h) is WFF (without flat factors) if V has no direct summand which is flat with respect to the connection and isomorphic to L_z (the flat line bundle corresponding

to the point $z \in \hat{A}$) . If (V,h) is WFF then $R^i p_2(p_1^* V \times \wp) = 0$ for i=0,2 . For a WFF (V,h) we define $F_N(V,h)$ to be given by the pair $(\hat{V},\hat{h})$ where $\hat{V} = R^1 p_2(p_1^* V \times \wp)$ and $\hat{h}$ is the L_2 metric induced on the pushforward by the theory of harmonic forms.
Remark: A change by scaling the metric on $\wp$ translates to a scaling of the $\hat{h}$ metric which does not affect the connection.
The main theorem regarding the Nahm transform is the following:
Theorem A: F_N is an isomorphism between WFF ASD connections on bundles over A and WFF ASD connections on bundles over $\hat{A}$.
Remark: There is a metric on the moduli space of stable vector bundles which turns F_N into an isometry.
One can try to compare this to the Arakelov Fourier transform by viewing (V,h) as an element in arithmetic K theory, and applying F_K (see section 3) to it. Basically by definition $T = F_K - F_N$ is the analytic torsion of $p_1^*(V,h) \times \wp$ for the map p_2. Let us look at $\hat{ch}(T)$
Claim: $\hat{ch}(T)^0$ i.e. the degree 0 part is zero (i.e. the Quillen metric is constant.
Proof: Both transforms are compatible with Mukai's if we forget the metric so $\zeta(T) = 0$. Now As remarked in section 3 $\omega(\hat{ch}(F_K(V,h))$ is harmonic. On the other hand since $(\hat{V},\hat{h})$ is ASD $ch_1(\hat{V},\hat{h})$ is harmonic and by compatibility of $c_1 = ch_1$ with $\hat{ch}$ we see that $\omega(\hat{ch}(T)^0) = 0$ and therefore is constant. Finally as remarked in section 3 constant functions are zero in $CH(\bar{A})$.
Unfortunately $\hat{ch}(T)^2$ is generally not zero. Indeed as V approaches the boundary of the moduli space of stable bundles by having nearly flat sections, $\hat{h}$ becomes concentrated around some points and so $ch_2(V,h)$ is not harmonic.
Remark: The first claim is complemented by the following claim due to Bost [Bo2]
Claim: If $dim(A) > 1$ then the analytic torsion for $\wp$ is constant.
Sketch of proof: Note that $det(R^* p_2(\wp)) = O_{\hat{A}}$ .One can show that the analytic torsion is a function with a pole only at 0 with zero curvature. But since 0 is of codimension at least two that implies that our function has no poles and hence is constant.

7 Canonical elements

In [C-S] Call and Silverman prove the following generalization of Tate's theorem on canonical heights.

Theorem B: Let $T : X \to X$ be a morphism of a smooth variety X defined over a number field k into itself.suppose L is a line bundle which satisfies $T^*(L) = \lambda L$ in $PIC(X) \otimes \mathbf{R}$ for λ real $|\lambda| > 1$. Then there exists a unique height function $\hat{h}_L$ on the points of X which is a height function for L and satisfies $\hat{h}_L(T(p)) = \lambda \hat{h}_L(p)$.

Suppose we extend X to a regular model over S and suppose that T also extends. If L is a line bundle such that $T^*L = \lambda L$ we can ask for a canonical lifting of L to $\hat{L} \in \hat{CH}(X)$ which still satisfies $T^*(\hat{L}) = \lambda \hat{L}$. such an element will give rise to the canonical height via intersection theory. In general the cannonical heights do not come from such elements because T does not preserve any nice metrics. However we have the following:

Theorem 3: Let (X, ω) be an arithmetic variety and let $T : X \to X$ be a correspondence such that T^* sends harmonic forms to harmonic forms.let $y \in CH^p(A)$ be such that $T^*y = \lambda y$ with λ real and $|\lambda| > 1$. Let B be a ring containing λ. Further denote by λ_q the maximal eigenvalue for the action of T^* on $H^{q,q}_{\mathbf{R}}$, then if $|\lambda| > |\lambda_{p-1}|$ we can lift y uniquely to an element $\bar{y} \in CH(\bar{X})_B$ such that $T^*(\bar{y}) = \lambda \bar{y}$.

Proof: Let $\bar{y_1}, \bar{y_2}$ be two liftings of y, then their difference z is in $H^{p-1,p-1}$. Choose a euclidian norm $|| \ ||$ on that space. By assumption we have

$$||1/\lambda T^*(\bar{y_1}) - 1/\lambda T^*(\bar{y_2})||$$

$$= ||1/\lambda T^* z|| < \lambda_{p-1}/\lambda ||z||$$

Now we see that using the usual telescoping argument the sequence $\bar{y_n} = 1/\lambda^n T^*(\bar{y_0})$ converges to an element $\bar{y}$ which does not depend on y_0. This proves existence and uniqueness.

Corollary: If X and T satisfy the theorem's assumptions and L is a line bundle with $|\lambda| > 1$ L lifts to an $\bar{L}$ satisfying $T^*\bar{L} = \lambda \bar{L}$

Proof: It is enough to show that regardless of T $\lambda_0 = 1$. But H^0 consists of locally constant functions.

Examples:

1) Abelian schemes: As usual any T is a candidate. If we choose $T = Id \times [n]$ on $A \times \hat{A}$ and $\wp$ the Poincaré bundle ($\lambda = n$) we get a third proof of the

existence of the canonical lifting. If we choose T=[n] we get a lifting for any element in$CH^p_s(A)$ where $s < 2$. Again by motivic theory that covers any horizontal element.
In the next example we shall assume for simplicity of notation that $S = spec(\mathbf{Z})$.
2) P^∞: In this example we shall use pushforward maps for injections (see [G-S1] and [Ku3]). These are defined as follows.let i be an injection of X into Y. Let (y, g_y) be an element in the arithmetic Chow group. We send it to $(i_* y, g_{i_* y})$ such that $\omega(g_{i_y})$ is harmonic and $H(g_{i_y}) = H(g)$ For the maps we will consider these pushforwards are functorial and obey the projection formula (see [Ku3]).
Let us look at $P(O_S^{N+1})$ for all natural N. Equip $\mathbf{C^{N+1}}$ with standard hermitian metrics which are naturally compatible. Let $i_N : P^N \hookrightarrow P^{N+1}$ denote the standard embedding, Then we can form the direct limit space of the P^N and call it P^∞. We attach two Arakelov Chow groups to P^∞. The first graded by codimension consists of the inverse limit under the maps i_N of elements in $CH^p(\overline{P^N})$. It will be called the Chow cohomology group. Since i_N^* is a ring homomorphism component wise intersection defines an intersection product on $CH^*(P^\infty)$. The second which will be called the Chow homology group will be graded by dimension and will consist of the direct limit under i_* of $CH^{N-p}(\overline{P^N})$. We also have on P^∞ multiplication by N maps which are given by viewing P^{kN} as the N-th symmetric product of P^k and taking the diagonal map which over the fibers of P^k has the form $x \rightarrow \underbrace{(x, x, ..., x)}_{N}$. These are compatible with the standard inclusions. More generally by viewing P^k and P^l as the k-th and l-th symmetric product of P^1 we obtain an additive structure by sending $(x_1, ..., x_k)$,$(y_1, ..., y_l)$ to $(x_1, ..., x_k, y_1, ..., y_l)$. This induces an action on P^∞ which we shall denote by $m : P^\infty \times P^\infty \rightarrow P^\infty$. We can now define a Pontrjagin product on the Chow homology by the usual formula.
Our choice of the standard embeddings corresponds to the choice of a point $p : S \hookrightarrow P^1$ which in our case is the point at infinity
There is a standard element in $CH^*(P^\infty)$. The metric on $\mathbf{C^{N+1}}$ induces a natural metric on $O(1)_N$ such that its dual on O(-1) is induced by the inclusion into $\pi^* O^{N+1}$ where $\pi : \mathbf{P} \rightarrow \mathbf{S}$. We shall denote $O(1)_N$ with this metric by $\hat{O}(1)$. It can be seen that these $\hat{O}(1)_N$ are compatible under the i_N^*

operations. Since taking Chern classes commutes with pullbacks $\hat{c}_1(\hat{O}(1)_N)$, which will still be denoted $\hat{O}(1)$ forms an element in $CH^1(\overline{P^\infty})$. Also harmonic forms such as ω^k are compatible under i_N^* and so one can see that the general element of $CH^k(P^\infty)$ is of the form $a(\hat{c}_1(\hat{O}(1)))^k + b\omega^k$ where a,b are independent of k.
One can indeed check that these elements are closed under intersection.
Given an element in $g \in U(k+1)$ it induces an action on $\underbrace{P^k \times ... \times P^k}_{N}$ by $(x_1, ..., x_N) \to (g(x_1), ..., g(x_N))$. This action commutes with the action of the permutation group Σ_N and preserves im(N). Therefore N^* sends harmonic forms to harmonic forms. In [K-V] it is shown that $N^*(O(1)) = NO(1)$ Therefore we may apply theorem 3 to this situation and get a unique element $\bar{O}(1) \in CH^1(\overline{P^\infty})$ which satisfies $N^*(\bar{O}(1)) = N\bar{O}(1)$
Remark: Compatibility of $\bar{O}(1)$ with i_N^* follows from the construction in theorem 3. If we did not require compatibility with i_N^* the element would not have been unique.
We also have a cup product

$$CH^p(\overline{P^\infty}) \times CH_q(\overline{P^\infty}) \to CH_{q-p}(\overline{P^\infty})$$

given by componentwise intersection. It is compatible with i_{N*} by the projection formula which holds in this case since the product of harmonic forms is harmonic.
As in the case of abelian schemes we see using the projection formula that the Pontrjagin product has a unit denoted $\bar{p}$ which is the element obtained from iterating i_* starting with the element $[S] \in CH^0(S) = CH_0(\bar{P}^0)$.
Let us define $\bar{L}$ to be the unique element satisfying $\bar{O}(1) \cup \bar{L} = \bar{p}$. It is unique since $a(\omega^{N-2})\bar{O}(1) = a(\omega^{N-1})$.
We define the following element in the inverse system of $CH^*(\overline{P^N \times P^N})$ with respect to $(i_N, i_N)^*$

$$F = EXP(p_1^*(\bar{O}(1))p_2^*(\bar{O}(1)))$$

One can easily show that F caries $CH_*(\overline{P^\infty})$ to $CH^*(\overline{P^\infty})$. We now have the following theorem which generalizes the Fourier transform for P^∞ of [K-V] (for classical Chow groups this is the most trivial case of the Fourier transform of infinite symmetric products of curves):
Theorem 4: The correspondence F satisfies the following properties:

A) F sends $\bar{L}$ to $\bar{O}(1)$
B) F induces a bijection
C) F sends Pontrjagin product to intersection product.
D) F sends N_* to N^* .
E) The image of the k-th part of the expansion of EXP has eigenvalue N^k for N^*.
F) F has an inverse in the direct limit $CH_*(\overline{P^\infty \times P^\infty})$ given by

$$F_- = EXP(\bar{L} \times \bar{L})$$

Sketch of proof: The proof is basically the same as the proof in [K-V] of the same result (theorems 3.12,3.17) although simplified (because we are only dealing with the projective line). One starts by showing that $\cup\bar{O}(1)$ is a derivation. This follows formally from the identity $m^*\bar{O}(1) = p_1^*\bar{O}(1) + p_2^*\bar{O}(1)$ which still holds by theorem 3 since both sides are eigenvectors. The derivation property implies that $\bar{O}(1) \cup \bar{p} = 0$ since $\bar{p}$ is the identity of the Pontrjagin product. This in turn easily implies property A. C, D, E and F use theorem 3 in a similar way to lift the equalities which appear in the proofs of [K-V]. Property B can be shown directly but follows from F.
Remark: Finally let us note that for our choice of p $\bar{O}(1) = \hat{O}(1)$. This is shown by using the fact that we have mentioned that $\bar{O}(1) \cup p = 0$ but this product is exactly the height of p which in our case is indeed 0 with respect to $\hat{O}(1)$. In general the height of p controls the situation giving us some sort of torsor which depends on more then geometric data.
3) Hecke correspondences: Hecke correspondences on complete smooth varieties send harmonic forms to harmonic forms and thus are candidates for an application of theorem 3. We shall give here one example where we can find an eigenvalue satisfying the conditions of the theorem. We hall consider a special type of Kuga-Sato varieties which was considered by Kuga and Shimura. In general Kuga-Sato varieties are complete smooth abelian schemes together with a canonical Kahler metric.They are attached to representations of a real Lie group G with certain integrality conditions with respect to a lattice Γ. In the case we shall consider $G = SL_2(\mathbf{R})$ Γ will be a Fuchsian group isomorphic to the units of an indefinite quaternion algebra whose quotient is compact. More precisely Γ will be the units of a maximal order in $M_2(\mathbf{R})$ such that $\gamma \cdot QId$ is a division algebra (that guarantees cocompactness). More generally for an integer b define Γ_b to be the set of

units ϵ such that $\epsilon - 1$ is divisible by b in the order. We fix b and denote Γ_b by Γ We take k copies of $M_2(\mathbf{R})$ and denote it by $\bar{F}$. We also take the lattice of k copies of the order which will be denoted by L. Now we define $V = G \times \bar{F}/\Gamma \times L$. This is almost our variety. One still has to make some changes to define a good complex structure in this case (see [Kug] for details). V is defined over $\mathbf{Q}$. It is an abelian scheme over a curve and its fibers are k copies of elliptic curves. Its harmonic forms decompose by the Leray-Serre spectral sequence relative to the base curve $H^p = \sum H^{a,p-a}$ a=(0,1,2). If we choose b which is prime to the discriminant of the order (denoted by d) and any prime p which does not divide $b \cdot d$ we can define Hecke correspondences T(p) and T(p,p) which act on V and have eigenvalues $p(p+1)$ and p respectively on $H^{0,2}$. Now V is also an abelian scheme and therefore we can look at $CH^1_0(V)$. By compatibility of the eigenvector decomposition with the Lerray-Serre spectral sequence we see that such elements must map to $H^{0,2}$ under the class map. Also one sees that each such cohomology class has at most one eigenvector maping to it. The rationally defined ample line bundle of V must map nontrivially to $H^{0,2}$ and the Fourier decomposition will now provide us with a rational cycle ρ in $CH^1_0(V)$. Since T(p) and T(p,p) commute with [n] by uniqueness of eigenvectors and the action of T(p) and T(p,p) on $H^{0,2}$ we conclude that ρ is an eigenvector for these Hecke operators in $CH^1(V)$. The eigenvalues are bigger than 1 so the assumptions of theorem 3 hold. Unfortunately V does not have good reduction everywhere so we only obtain cannonical elements over an open set of $spec(\mathbf{Z})$. One can show that harmonic forms on V are closed under $[n]^*$ and therefore V also has a Fourier transform whose properties are currently being studied.
Remark: We should still be able to obtain canonical heights from these canonical elements by avoiding the bad fibers using moving lemma's for divisors and choosing canonical representatives that satisfy the eigen vector condition without the help of multiplication by bad primes which are invertible over the open set.

8 Bibliography

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