

Path Integrals on Ultrametric Spaces

by

Alan Blair

B.Sc., B.A. (Hons.)
University of Sydney
1988, 1989

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

May 1994

© 1994 Alan Blair. All rights reserved.

The author hereby grants to MIT permission to reproduce and
to distribute copies of this thesis document in whole or in part.

Signature of Author.....

Department of Mathematics

April 29, 1994

Certified by.....

Gian-Carlo Rota

Professor of Mathematics

M.I.T. Thesis Advisor

Accepted by.....

David Vogan

Science
Chair, Departmental Committee on Graduate Studies

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

AUG 11 1994

LIBRARIES



PATH INTEGRALS ON ULTRAMETRIC SPACES

by

ALAN BLAIR

Submitted to the Department of Mathematics on April 29, 1994

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

ABSTRACT

A framework for the study of path integrals on adèlic spaces is developed, and it is shown that a family of path space measures on the localizations of an algebraic number field may, under certain conditions, be combined to form a *global* path space measure on its adèle ring. An operator on the field of p -adic numbers analogous to the harmonic oscillator operator is then analyzed, and used to construct an Ornstein-Uhlenbeck type process on the adèle ring of the rationals.

Thesis Advisors: Andrew Lesniewski, Professor of Physics (Harvard)
 Arthur Jaffe, Professor of Mathematics (Harvard)



Acknowledgments

To my father, who first introduced me to the lofty heights of mathematics, and my mother and brother, who kept me in touch with the real world below.

To Andrew Lesniewski, for his wit, patience and generosity. I am grateful to him for providing many valuable insights and suggestions as I was working on this thesis, and for encouraging me to submit it for publication.

To Arthur Jaffe, for his helpful advice and support, especially in the early stages.

To Gian-Carlo Rota, for taking an interest in my work and generating a great many intriguing ideas.

To Jordan Pollack, Paul Smolensky, James Pustejovsky and Uri Wilensky, for their encouragement, vocational guidance and fruitful discussions on a variety of topics.

To the mathematics department and its support staff, whose hobbies, witticisms, musical talents, and daily displays of aerobic fitness were an inspiration to us all.

To Mark and Andrew, with whom I have shared gas, food, lodging and many good times throughout the past three years, and to my previous house-mates - Matthew, Farshid, Rick, David, Janice, Ted, Ceci and Liz - for making me feel at home in a new country.

To the residents of Ashdown House - particularly Richard, Diane and Papa Rao - who sustained me in mind and body with their good cheer and exotic 'Asian' cuisine.

To the many friends near and far who have helped me in so many ways throughout my graduate school career - especially Nozomi, in whose company I was always able to see the light at the end of the tunnel.

This work was partially supported by a University of Sydney Travelling Scholarship.

Dziękuję! Grazie! Tack! Danke! Arigato! & Thanks!

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$



Contents

0. Introduction	9
1. Local Fields	12
1.1 Introduction to p-adic numbers	12
1.2 p-adic Fourier transform	14
2. Path Space Measures	15
2.1 Topology of adèlic space	15
2.2 Kolmogorov theorems	17
3. Heat Kernels	20
4. p-Wiener Measure	26
4.1 Diffusions with independent increments	26
4.2 p-Wiener measure as a Poisson process	29
5. Feynman-Kac Formula	33
6. Oscillator Process on \mathbb{Q}_p	35
7. Orthogonal Component	37
8. Radial Component	45



0. Introduction

An *ultrametric* space is a metric space in which the following ‘strong’ version of the triangle inequality holds:

$$(0.1) \quad \text{dist}(A, C) \leq \max\{\text{dist}(A, B), \text{dist}(B, C)\}.$$

Such spaces have a topology quite different from that of ordinary Euclidean space. In particular balls do not have well-defined centers, all triangles are isosceles and every open ball is also closed. An ultrametric space divides naturally into a hierarchy of clusters and sub-clusters - with the distance between any two points equal to the diameter of the smallest cluster containing them both.

Though the definition and general properties outlined above were first introduced by Marc Krasner in a 1944 address to the French Academy [Kr44], ultrametric spaces really date back to the invention by Kurt Hensel in 1897 of p -adic number fields [Hen97], which have proven invaluable in number theory ever since.

Ultrametricity has recently been discovered in a number of physical systems [RTV86], and much interest has been shown in the idea of extending physical theories to a setting in which a p -adic field replaces the usual field of real or complex numbers [BF93]. There are several motivations for these developments. Some researchers have studied p -adic analogs of physical systems in the hope that they may shed light on classical problems in number theory [Jul90, Sp90]; others have proposed more direct physical applications for such theories, speculating that the topology of space-time itself may become totally disconnected or even p -adic when looked at on the scale of the Planck length [VV89, Zab89]. Vivaldi and others have suggested that local fields provide an ideal setting for dynamical systems [Viv92, VH92], because their non-archimedean nature allows numerical computations to be

carried out without fear of accumulating round-off errors. Connes and Bost have studied models of quantum statistical mechanics on adèlic spaces [CB92], which give rise to phase transitions involving broken symmetry of the Galois group. Much work has also been done on p -adic string theory [FW87], [Zab89] (see also references listed in [BF93]).

Previous approaches to p -adic quantum mechanics [FO88, Me89, RTVW, VV89] have typically been based on the Weyl quantization of a semigroup of operators indexed by a p -adic time parameter. We propose a new approach, modeled on the Heisenberg formulation, in which the time parameter is taken to be real, and the infinitesimal generator H of time translations, acting on L^2 (complex valued) functions of a (p -adic) state space variable may be used to construct a measure on the space of paths from real ‘time’ to p -adic ‘space’. This view brings p -adic quantum mechanics within the realm of diffusion processes on p -adic and ultrametric spaces, which have arisen in the study of spin glasses [BO89] and other physical systems [OS85].

Following the discovery by Mezard, Parisi et al. of an ultrametric structure on the pure spin glass states in the mean field theory of Sherrington and Kirkpatrick [MP84, Par93], dynamical processes on ultrametric spaces have emerged as useful models for studying the behavior of so-called ‘disordered’ systems. Such systems - in a ‘frustrated’ attempt to satisfy as best they can a tangled web of conflicting constraints - typically progress slowly towards equilibrium according to a hierarchy of relaxation regimes. They have arisen in a variety of settings - including neural networks [KT84, PV85], protein folding [St85], and computing structures [HK85].

In trying to gain a better understanding of ultrmetric dynamics, it is natural to study p -adic examples in the first instance, because their algebraic structure brings to hand familiar tools like the Fourier transform, to facilitate a more detailed analysis than is possible in the general case.

From the number theory point of view, a natural question to ask is whether a family of diffusions on ‘local’ p -adic space may be ‘lifted’ to ‘global’ *adèlic* space.

For the reader unfamiliar with analytic number theory, a word of explanation is in order: Consider the field \mathbb{Q} of rational numbers. Every prime p determines a special kind of metric on \mathbb{Q} , according to which two numbers are ‘close’ to each other if their difference is divisible by a high power of p . The completion \mathbb{Q}_p of \mathbb{Q} with respect to this metric is called the field of p -adic numbers. Because of an analogy with algebraic geometry, it is also called a *localization* or *local field* of \mathbb{Q} . Forming the restricted direct product of all these local fields yields a ‘global’ object called the *adèle ring* \mathcal{O} . (these constructions may be generalized to arbitrary algebraic number fields, and are summarized in §2.1). A common strategy for solving an equation on \mathbb{Q} is to first find solutions on all of these local fields, and then check certain conditions which would allow this family of local solutions to be ‘lifted’ to a global solution on \mathcal{O} . This ‘local to global’ principle has been applied in many different areas (see [Gel84] for a general discussion). It is natural, then, to ask whether it may also be applied to diffusion processes, and we show in §3 that a family of path space measures on the localizations of an algebraic number field may indeed, under certain conditions, be lifted to a path space measure on its adèle ring.

In the second half of the thesis a particular example is constructed of a diffusion process on the adèle ring of \mathbb{Q} , which is analogous in some sense to the Ornstein-Uhlenbeck velocity process in Euclidean space. The local infinitesimal generator H_p of this process is introduced in §6 and shown to be trace class, using an analog of the Feynman-Kac formula developed in §5. The spectrum of H_p divides naturally into orthogonal and radial components, which are analyzed in detail in §7 and §8, respectively.

This notion of combining ‘harmonic oscillators’ on each local field \mathbb{Q}_p to form a path space measure on \mathcal{O} may be thought of in analogy with the splitting of the Euclidean free quantum field into subspaces of differing momenta - where the convergence of the integral in this case is achieved not by a steady increase in the size of the ‘mass term’, but by the topology of the local fields becoming more and more ‘grainy’ as p gets bigger, thus providing ever larger potential barriers to confine particles within the ‘unit ball’ of p -adic integers.

1. Local Fields

1.1 Introduction to p-adic numbers

Let p be a prime and consider a rational number c whose denominator is a power of p . Then c may be expressed in p -ary (or *base p*) notation as a finite sequence of digits $c_i \in \{0, \dots, p-1\}$, punctuated with a ‘decimal point’:

$$c_j \dots c_{-1} c_0 . c_1 \dots c_k = c = c_j p^{-j} + \dots + c_{-1} p + c_0 + c_1 p^{-1} + \dots + c_k p^{-k}$$

The same notation may be used to represent an arbitrary rational (yielding an infinite but recurring sequence of digits), or indeed any real number (by approximating it with a sequence of rationals). Note that the p -ary expression for a rational (or real) number is always of finite length to the left of the decimal point, but may extend infinitely to the right of it.

In 1897 Hensel devised an alternative scheme - called p -adic notation - that assigns to each rational number a similar sequence of digits which is of finite length to the **right** of the decimal point, but may extend infinitely to the **left** of it.

For example, while the ternary expression for $\frac{1}{2}$ is $0.11111\dots$, its 3-adic expression is $\dots 11112.0$ - as can be verified by 3-adic ‘long division’, or by multiplying to check that $2 \times (\dots 11112.0) = \dots 00001.0$. Sense can be made of such expressions by introducing to the rational numbers a radically different notion of size and distance - determined by the following *valuation* on \mathbb{Q} :¹

Definition 1.0. For a fixed prime p , we define the p -adic norm $|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{R}^{\geq 0}$ by

$$\left| \frac{a}{b} \cdot p^k \right| = p^{-k}, \quad \text{if } a \text{ and } b \text{ are coprime to } p.$$

For example, in terms of 3-adic distance, $41_{10} = 1112_3$ is a good approximation to $\frac{1}{2}$, because the size of their difference is $|41 - \frac{1}{2}|_3 = |\frac{1}{2} \cdot 3^4|_3 = 3^{-4}$.

¹a *valuation* is a metric defined on a number field that is compatible with its algebraic structure in a sense that will be made precise in §2.1.

Definition 1.1. The field of p -adic numbers is the completion \mathbb{Q}_p of \mathbb{Q} , with respect to the p -adic norm $|\cdot|_p$.

In general the p -adic distance between two numbers is p^k if, when written in p -adic notation, the k^{th} digit is the rightmost digit in which they differ. Thus p -adic numbers may be thought of as comprising an infinite tree, with the branches at height k indexed by the k^{th} digit in the p -adic expansion.

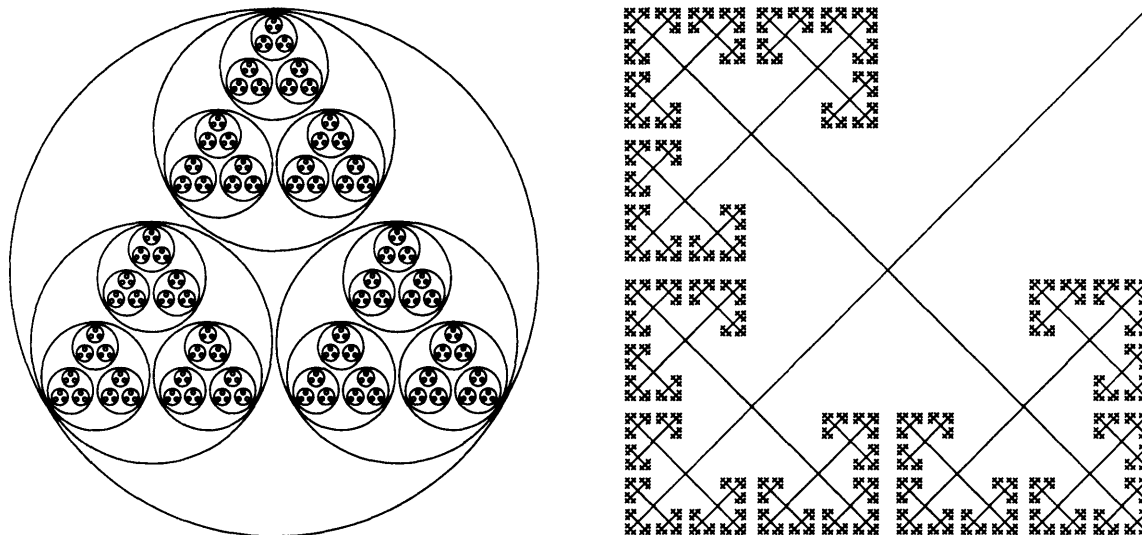


FIGURE 1.1. Two schematic views of 3-adic space.

1.2 p -adic Fourier transform

Definition 1.2. The fundamental character $\chi : \mathbb{Q}_p \rightarrow \mathbb{C}$ is the function $\chi(c) = e^{2\pi i \lambda(c)}$, where $\lambda : \mathbb{Q}_p \rightarrow [0, 1) \subset \mathbb{R}$ is given by

$$\lambda(\dots c_{-1}c_0.c_1c_2\dots c_k) = 0.c_1c_2\dots c_k$$

Definition 1.3. The Fourier transform $\mathcal{F} : L^2(\mathbb{Q}_p) \rightarrow L^2(\mathbb{Q}_p)$ is

$$(\mathcal{F}v)(\xi) = \int_{\mathbb{Q}_p} \chi(x\xi) v(x) dx.$$

Many of the properties of the usual Fourier transform carry over to the p -adic case [Tai75], [V188]. We will mostly be concerned with the special case where v is ‘radially symmetric’ in the sense that $v(x)$ depends only on the norm $|x|_p$ of x . In this case, we write $v(x) = v_k$ if $|x|_p = p^k$, and the Fourier transform \hat{v} of v (which is also radially symmetric) may be written as

$$(1.1) \quad \hat{v}_l = (1 - p^{-1}) \sum_{k \leq -l} p^k v_k - p^{-l} v_{-l+1}.$$

All of the above constructions may be generalized to the case where \mathbb{Q} is replaced by an arbitrary algebraic number field k , and p by a prime ideal \mathfrak{p} of k [We67]. The \mathfrak{p} -adic norm is then $|x|_{\mathfrak{p}} = N\mathfrak{p}^{-j}$, where $N\mathfrak{p}$ is the number of cosets of \mathfrak{p} in the ring of algebraic integers $\mathfrak{o}_{\mathfrak{p}}$ in $k_{\mathfrak{p}}$, and j is the largest number such that $x \in \mathfrak{p}^j$.

$$\text{Here } \mathfrak{p}^0 = \mathfrak{o}_{\mathfrak{p}}, \quad \text{while} \quad \mathfrak{p}^j = \begin{cases} \{x_1 \dots x_j \mid x_1, \dots, x_j \in \mathfrak{p}\}, & \text{if } j > 0, \\ \{x \mid x \cdot y \in \mathfrak{o}_{\mathfrak{p}} \text{ for all } y \in \mathfrak{p}^{-j}\}, & \text{if } j < 0. \end{cases}$$

The Fourier transform may be defined in exactly the same way once a fundamental character is chosen (see [Ta50]). The generalization of (1.1) is as follows:

$$(1.1') \quad \hat{v}_l = (1 - N\mathfrak{p}^{-1}) \sum_{k \leq -l} N\mathfrak{p}^k v_k - N\mathfrak{p}^{-l} v_{-l+1}.$$

2. Path Space Measures

2.1 The topology of adèlic space

Let k be an algebraic number field, of finite degree over \mathbb{Q} .

A *valuation* on k is a function $|\cdot|_{\mathfrak{p}} : k \rightarrow \mathbf{R}^+$ satisfying:

- (i) $|x \cdot y|_{\mathfrak{p}} = |x|_{\mathfrak{p}} |y|_{\mathfrak{p}}$,
- (ii) $|x|_{\mathfrak{p}} = 0 \Leftrightarrow x = 0$,
- (iii) $|x + y|_{\mathfrak{p}} \leq |x|_{\mathfrak{p}} + |y|_{\mathfrak{p}}$.

There are two types of valuations on k :

- (i) Non-Archimedean valuations, indexed by the set \mathfrak{P}_f of prime ideals in k .
- (ii) Archimedean valuations, which are indexed by the (finite) set \mathfrak{P}_{∞} of embeddings of k into \mathbf{C} or \mathbf{R} (the so-called “infinite primes”).

For example, if $k = \mathbb{Q}$, then \mathfrak{P}_f is the collection of p -adic norms defined in §1.1, and \mathfrak{P}_{∞} has as its single element the standard norm on \mathbb{Q} (which may be thought of as the unique embedding of \mathbb{Q} into \mathbf{R} , composed with the absolute value on \mathbf{R}).

We will use \mathfrak{P} to denote any subset of $\mathfrak{P}_{\infty} \cup \mathfrak{P}_f$. For each \mathfrak{p} in \mathfrak{P} , we define $k_{\mathfrak{p}}$ to be the completion of k with respect to $|\cdot|_{\mathfrak{p}}$. Following the usual conventions, we use $\mathfrak{o}_{\mathfrak{p}}$ to denote the unit ball or ring of integers in $k_{\mathfrak{p}}$, $k_{\mathfrak{p}}^{\times}$ the invertible elements of $k_{\mathfrak{p}}$, and $\mathfrak{u}_{\mathfrak{p}}$ the group of units in $\mathfrak{o}_{\mathfrak{p}}$. Non-Archimedean valuations satisfy a stronger version of (iii) known as the *ultrametric* triangle inequality:

$$(2.1) \quad |x + y|_{\mathfrak{p}} \leq \max(|x|_{\mathfrak{p}}, |y|_{\mathfrak{p}}),$$

making the topology of $k_{\mathfrak{p}}$ quite different from that of Euclidean space. $k_{\mathfrak{p}}$, $k_{\mathfrak{p}}^{\times}$, $\mathfrak{o}_{\mathfrak{p}}$ and $\mathfrak{u}_{\mathfrak{p}}$ are complete, separable metric spaces, with topology generated by balls of the form

$$\text{Ball}(z_{\mathfrak{p}}, r_{\mathfrak{p}}) = \{x_{\mathfrak{p}} : |x_{\mathfrak{p}} - z_{\mathfrak{p}}|_{\mathfrak{p}} < r_{\mathfrak{p}}\}.$$

These open balls also generate a σ -algebra of Borel sets, and there is a standard invariant measure on $k_{\mathfrak{p}}$ with respect to which $\mu(\{x : |x|_{\mathfrak{p}} \leq 1\}) = 1$. For $\mathfrak{p} \in \mathfrak{P}_f$ the measure of $\text{Ball}(z_{\mathfrak{p}}, r_{\mathfrak{p}})$ is $N\mathfrak{p}^{-j}$ where $j \in \mathbf{Z}$ is chosen so that $N\mathfrak{p}^{-j} < r_{\mathfrak{p}} \leq N\mathfrak{p}^{-j+1}$ ($N\mathfrak{p}$ = cardinality of $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$). There is also an invariant measure on $k_{\mathfrak{p}}^{\times}$ which for $\mathfrak{p} \in \mathfrak{P}_f$ is normalized so that $\mu(\{x : |x|_{\mathfrak{p}} = 1\}) = 1$.

We formulate the following results in terms of \mathcal{O} , $k_{\mathfrak{p}}$ and $\mathfrak{o}_{\mathfrak{p}}$, but with slight modifications they are also valid for \mathcal{U} , $k_{\mathfrak{p}}^{\times}$ and $\mathfrak{u}_{\mathfrak{p}}$. The *Adèle Group* \mathcal{O} of k is the *restricted direct product* of the $k_{\mathfrak{p}}$'s with respect to the $\mathfrak{o}_{\mathfrak{p}}$'s; this means the group of ordered sequences $x = (x_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{P}}$ such that each $x_{\mathfrak{p}} \in k_{\mathfrak{p}}$ and $x_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$ for all but finitely many \mathfrak{p} . \mathcal{O} carries a topology generated by cylinder sets of the form

$$\mathcal{U}_{\mathcal{S},+}^X = X \times \prod_{\mathfrak{p} \notin \mathcal{S}} \mathfrak{o}_{\mathfrak{p}},$$

where \mathcal{S} denotes a finite subset of \mathfrak{P} , and X a measurable subset of $\prod_{\mathfrak{p} \in \mathcal{S}} k_{\mathfrak{p}}$. These cylinders also generate a σ -algebra of Borel sets, making \mathcal{O} into a measure space. There is a standard measure on \mathcal{O} with respect to which the measure of $\mathcal{U}_{\mathcal{S}}^X$ in \mathcal{O} is the same as the measure of X in $\prod_{\mathfrak{p} \in \mathcal{S}} k_{\mathfrak{p}}$. For $n \geq 1$, $\mathcal{O}^n = \mathcal{O} \times \dots \times \mathcal{O}$ carries the product topology generated by "box" sets of the form

$$\mathcal{U}_{\mathcal{S},+}^{\{X^{(i)}\}} = \prod_{i=1}^n \mathcal{U}_{\mathcal{S},+}^{X^{(i)}},$$

which also generate the σ -algebra $\mathcal{B}_n(\mathcal{O}^n)$ of Borel sets in \mathcal{O}^n .

We next consider the space \mathcal{O}^{∞} of ordered sequences of numbers in \mathcal{O} , and the space \mathcal{O}^T of functions $\omega : T \rightarrow \mathcal{O}$, where T is a subset of \mathbf{R} . $\mathcal{B}_{\infty}(\mathcal{O}^{\infty})$ and $\mathcal{B}_T(\mathcal{O}^T)$ are the σ -algebras of Borel sets generated in each case by the open sets in the product topology. It is convenient to give the following alternate formulation of \mathcal{B}_{∞} and \mathcal{B}_T : The collection Σ of finite subsets of T form a partially ordered set with respect to inclusion. For each $\tau = \{t_1, \dots, t_n\}$ in Σ , there is a projection $\pi_{\tau} : \mathcal{O}^T \rightarrow \mathcal{O}^{\tau}$ given by

$$\pi_{\tau}(\omega) = (\omega(t_1), \dots, \omega(t_n)).$$

And for any $\sigma, \tau \in \Sigma$ with $\tau \subseteq \sigma$, there is a restriction map $\pi_{\sigma\tau} : \mathcal{O}^\sigma \rightarrow \mathcal{O}^\tau$. There are also natural projections $\pi_n : \mathcal{O}^\infty \rightarrow \mathcal{O}^n$ and restrictions $\pi_{nk} : \mathcal{O}^n \rightarrow \mathcal{O}^k$. It is easy to see that $\mathcal{B}_\infty(\mathcal{O}^\infty)$ is the σ -algebra generated by the pull-backs of Borel sets under these projections:

$$\{\pi_n^*(A_n) : A_n \in \mathcal{B}(\mathcal{O}^n), n \geq 1\},$$

while $\mathcal{B}_T(\mathcal{O}^T)$ is the σ -algebra generated by the pull-backs

$$\{\pi_\sigma^*(A_\sigma) : A_\sigma \in \mathcal{B}_\sigma(\mathcal{O}^\sigma), \sigma \in \Sigma\}.$$

In fact every Borel set in $(\mathcal{O}^T, \mathcal{B}_T)$ is of the form $\pi_\rho^*(A_\infty)$, where $\rho = (t_1, t_2, \dots)$ is a countable subset of T , and A_∞ is a Borel set in $(\mathcal{O}^\infty, \mathcal{B}_\infty)$.

2.2 Kolmogorov theorems

In this section we present two well-known theorems which were due originally to Kolmogorov, but have since been adapted to a more general context. The exposition is based closely on ([Sh84], Ch.II §3 - but see also [Tor71]).

In order to establish the existence of path space measures, we shall need the following standard result from measure theory (which we quote without proof - see [Sh84]):

Theorem 2.0. *Carathéodory's Theorem.*

Let Ω be a space, \mathcal{A} an algebra of its subsets and $\mathcal{B} = \sigma(\mathcal{A})$ the σ -algebra generated by \mathcal{A} . Let μ_0 be a σ -finite measure on (Ω, \mathcal{A}) . Then there is a unique measure μ on (Ω, \mathcal{B}) which extends μ_0 .

Theorem 2.1. *Komolgorov Theorem for \mathcal{O}^∞ .*

Let P^1, P^2, \dots be a sequence of regular probability measures on $\mathcal{O}, \mathcal{O}^2, \dots$ which are consistent in the sense that

$$P^{n+1}(A_n \times \mathcal{O}) = P^n(A_n), \quad \text{for } A_n \in \mathcal{B}_n(\mathcal{O}^n).$$

Then there is a unique probability measure P^∞ on $(\mathcal{O}^\infty, \mathcal{B}_\infty)$ such that

$$P^\infty(\pi_n^*(A_n)) = P^n(A_n), \quad \text{for } A_n \in \mathcal{B}_n(\mathcal{O}^n).$$

Proof. P^n is *regular* means that $P^n(A_n) = \sup\{P^n(C_n) : C_n \subset A_n, C_n \text{ compact}\}$ - a condition we shall make use of later. Let $\mathcal{A}(\mathcal{O}^\infty)$ be the algebra of all cylinder sets

$$\mathcal{A}(\mathcal{O}^\infty) = \{\pi_n^*(B_n) : B_n \in \mathcal{B}_n(\mathcal{O}^n), n \geq 1\}.$$

We define a measure P_0 on $(\mathcal{O}^\infty, \mathcal{A})$ by $P_0(\pi_n^*(B_n)) = P^n(B_n)$. It is straightforward to check that this definition is consistent and that P_0 is finitely additive. We check σ -additivity below. Since \mathcal{B} is the σ -algebra generated by \mathcal{A} , it follows from Carathéodory's Theorem that there is a unique measure P^∞ on $(\mathcal{O}^\infty, \mathcal{B}_\infty)$ such that

$$P^\infty(A) = P_0(A), \quad \text{for } A \in \mathcal{A}.$$

P_0 will be σ -additive as long as it is "continuous at zero". This means that

$\lim_{n \rightarrow \infty} P_0(B_n) = 0$ for any sequence of sets $\{B_n\}$ which satisfy $B_n \downarrow \emptyset, n \rightarrow \infty$.

Suppose to the contrary that $P_0(B_n) \rightarrow \delta > 0$. We may assume without loss of generality that $B_n = \pi_n^* \hat{B}_n$, for some $\hat{B}_n \in \mathcal{B}_n(\mathcal{O}^n)$. Since $(\mathcal{O}^n, \mathcal{B}_n, P^n)$ is a *regular* measure, for each $\hat{B}_n \in \mathcal{B}_n(\mathcal{O}^n)$ we can find a compact subset \hat{A}_n of \mathcal{O}^n with $P^n(\hat{B}_n \setminus \hat{A}_n) \leq \delta/2^{n+1}$. Form the set

$$\hat{C}_n = \bigcup_{k=1}^n \pi_{nk}^*(\hat{A}_k) \quad \text{and let} \quad C_n = \pi_n^*(\hat{C}_n).$$

Then since the sets B_n decrease,

$$P_0(B_n \setminus C_n) \leq \sum_{k=1}^n P^n(\hat{B}_n \setminus \hat{A}_n) \leq \delta/2.$$

By assumption $\lim_{n \rightarrow \infty} P_0(B_n) = \delta$; therefore $\lim_{n \rightarrow \infty} P_0(C_n) \geq \delta/2 > 0$.

Let us show that this contradicts the condition $C_n \downarrow \emptyset$.

Choose a point $(x_1^{(n)}, x_2^{(n)}, \dots)$ in C_n . Then $(x_1^{(n)}, \dots, x_n^{(n)}) \in \hat{C}_n$, for $n \geq 1$.

Let (n_1) be a subsequence of (n) such that the sequence $x_1^{(n_1)}$ converges to a point x_1^0 of \hat{C}_1 . Let (n_2) be a subsequence of (n_1) such that $(x_1^{(n_2)}, x_2^{(n_2)}) \rightarrow (x_1^0, x_2^0) \in \hat{C}_2$.

Similarly let $(x_1^{(n_1)}, \dots, x_k^{(n_k)}) \rightarrow (x_1^0, \dots, x_k^0) \in \hat{C}_k$.

Finally form the diagonal sequence (m_k) , where m_k is the k^{th} term of (n_k) . Then $x_i^{(m_k)} \rightarrow x_i^0$ as $m_k \rightarrow \infty$ for $i = 1, 2, \dots$; and $(x_1^0, x_2^0, \dots) \in C_n$ for $n = 1, 2, \dots$, which evidently contradicts the assumption that $C_n \downarrow \emptyset$ as $n \rightarrow \infty$. \square

Theorem 2.2. *Komolgorov Theorem for \mathcal{O}^T .*

Let $\{P^\tau\}_{\tau \in \Sigma}$ be a family of regular measures on $\{(\mathcal{O}^\tau, \mathcal{B}_\tau)\}$ which are consistent in the sense that

$$P^\sigma(\pi_{\sigma\tau}^*(A_\tau)) = P^\tau(A_\tau), \quad \text{for } A_\tau \in \mathcal{B}_\tau(\mathcal{O}^\tau), \quad \text{whenever } \sigma \supseteq \tau.$$

Then there is a unique probability measure P^T on $(\mathcal{O}^T, \mathcal{B}_T)$ such that

$$P^T(\pi_\tau^*(A_\tau)) = P^\tau(A_\tau), \quad \text{for } A_\tau \in \mathcal{B}_\tau(\mathcal{O}^\tau).$$

Proof. Let $B \in \mathcal{B}_T(\mathcal{O}^T)$. Then there is a finite subset S of T such that $B = \pi_S^*(\hat{B}_S)$, for some $\hat{B}_S \in \mathcal{B}_S(\mathcal{O}^S)$. We define the required measure P^T by

$$P^T(B) = P^S(\hat{B}_S),$$

where $P^S(\mathcal{O}^S, \mathcal{B}_S)$ is the measure whose existence was established by the previous theorem.

It is easy to check that this definition is consistent. To show countable additivity, let $\{B_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{B}_T(\mathcal{O}^T)$. Then each $B_n = \pi_{S_n}^*(\hat{B}_{S_n})$ for some countable subset S_n of T and the union S of these countable sets S_n is countable. So

$$P^T\left(\bigcup_{n=1}^{\infty} B_n\right) = P^S\left(\bigcup_{n=1}^{\infty} \hat{B}_{S_n}\right) = \sum_{n=1}^{\infty} P^S(\hat{B}_{S_n}),$$

by the countable additivity of P^S , which concludes the proof of the theorem. \square

3. Heat Kernels

We turn now to the question of how such a consistent family of regular measures may be constructed. We will consider two kinds of measures - *conditional* measures, on spaces of paths which start from a specified basepoint b at an initial time t_0 , and *unconditional* measures, on spaces of paths which may be sampled at any time $t \in \mathbf{R}$. Suppose we have, for each $\mathfrak{p} \in \mathfrak{P}$, an (unbounded) positive, self-adjoint operator $H_{\mathfrak{p}}$ on $L^2(k_{\mathfrak{p}})$, and that the heat kernel $k_t^{\mathfrak{p}}$ of $e^{-tH_{\mathfrak{p}}}$ satisfies the properties:

$$(3.1) \quad k_t^{\mathfrak{p}}(x, y) \geq 0 \quad \text{and} \quad \int_{k_{\mathfrak{p}}} k_t^{\mathfrak{p}}(x, y) dy = 1.$$

Suppose further that we are given an “initial time” $t_0 \in \mathbf{R}$ and a basepoint $b \in \mathcal{O}$ and that there is a finite subset \mathcal{R}_0 of \mathfrak{P} such that

$$(3.2) \quad \prod_{\mathfrak{p} \in \mathfrak{P} \setminus \mathcal{R}_0} \int_{\mathfrak{o}_{\mathfrak{p}}} k_t^{\mathfrak{p}}(x, b_{\mathfrak{p}}) dx > 0, \quad \text{for all } t > 0.$$

Alternatively, assume that each $H_{\mathfrak{p}}$ has a unique ground state $\Omega_{\mathfrak{p}}$, with

$$(3.1') \quad k_t^{\mathfrak{p}}(x, y) \geq 0, \quad \Omega_{\mathfrak{p}}(y) \geq 0, \quad \|\Omega_{\mathfrak{p}}\|_{L^2} = 1 \quad \text{and} \quad H_{\mathfrak{p}} \Omega_{\mathfrak{p}} = 0,$$

and that the following condition is satisfied:

$$(3.2') \quad \sum_{\mathfrak{p} \in \mathfrak{P} \setminus \mathcal{R}_0} \int_{k_{\mathfrak{p}} \setminus \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x)^2 dx < \infty.$$

We will use this family of heat kernels to define the consistent family of regular measures required. It follows from (3.1) (resp. (3.1')) that $H_{\mathfrak{p}}$ defines a *local* path space measure on $k_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathfrak{P}$. We will show that if (3.2) (resp. (3.2')) is also satisfied, these local measures may be combined to form a *global* measure on the path space of \mathcal{O} . We will need the following standard result from measure theory (which we quote without proof - see [Rao87], §2.3):

Theorem 3.0. *Henry's Extension.*

Let Ω be a topological space, \mathcal{A} a subalgebra of the Borel σ -algebra \mathcal{B} generated by the open sets of Ω , and $\mu : \mathcal{A} \rightarrow \mathbf{R}^+$ a function that is (i) finitely additive and (ii) inner regular in the sense that for each $A \in \mathcal{A}$,

$$\mu(A) = \sup \{ \mu(C) : C \subset A, C \in \mathcal{A}, C \text{ compact (and closed)} \}.$$

Then μ can be extended to a Radon (therefore regular) measure on \mathcal{B} . If the σ -algebra generated by \mathcal{A} is \mathcal{B} , then the extension is unique.

Let \mathcal{A}_1 be the set of subsets of \mathcal{O} which are finite unions of sets of the form $U_{\mathcal{S},\sigma}^X$, where \mathcal{S} is a finite subset of \mathfrak{P} , X is a measurable subset of $\prod_{\mathfrak{p} \in \mathcal{S}} k_{\mathfrak{p}}$, $\sigma \in \{0, +, -\}$ and

$$U_{\mathcal{S},0}^X = X \times \mathcal{G}_{\mathcal{S}}^0, \quad U_{\mathcal{S},+}^X = X \times \mathcal{G}_{\mathcal{S}}^+, \quad U_{\mathcal{S},-}^X = X \times (\mathcal{G}_{\mathcal{S}}^0 \setminus \mathcal{G}_{\mathcal{S}}^+),$$

where $\mathcal{G}_{\mathcal{S}}^+ = \prod_{\mathfrak{p} \in \mathfrak{P} \setminus \mathcal{S}} \mathfrak{o}_{\mathfrak{p}}$ and $\mathcal{G}_{\mathcal{S}}^0$ is the restricted direct product of the $k_{\mathfrak{p}}$'s with respect to the $\mathfrak{o}_{\mathfrak{p}}$'s, over $\mathfrak{p} \in \mathfrak{P} \setminus \mathcal{S}$. Note that $\mathcal{O} = U_{\emptyset,0}^{\emptyset} \in \mathcal{A}_1$. We check that \mathcal{A}_1 is closed under finite complements and intersections and is therefore an algebra. For complements, we have

$$(U_{\mathcal{S},0}^X)^c = U_{\mathcal{S},0}^{(X^c)}, \quad (U_{\mathcal{S},+}^X)^c = U_{\mathcal{S},-}^X \cup U_{\mathcal{S},0}^{(X^c)}, \quad (U_{\mathcal{S},-}^X)^c = U_{\mathcal{S},+}^X \cup U_{\mathcal{S},0}^{(X^c)}.$$

We next consider intersections of the form $U_{\mathcal{S}_1,\sigma_1}^X \cap U_{\mathcal{S}_2,\sigma_2}^Y$. In view of the following relations we may assume without loss of generality that $\mathcal{S}_1 = \mathcal{S}_2$: If \mathcal{R} is a finite subset of $\mathfrak{P} \setminus \mathcal{S}$ then, using the notation $k_{\mathcal{R}} = \prod_{\mathfrak{p} \in \mathcal{R}} k_{\mathfrak{p}}$, $\mathfrak{o}_{\mathcal{R}} = \prod_{\mathfrak{p} \in \mathcal{R}} \mathfrak{o}_{\mathfrak{p}}$, we have

$$(3.3) \quad U_{\mathcal{S},0}^X = U_{\text{SUR},+}^{X \times k_{\mathcal{R}}} \cup U_{\text{SUR},-}^{X \times k_{\mathcal{R}}} \quad U_{\mathcal{S},+}^X = U_{\text{SUR},+}^{X \times \mathfrak{o}_{\mathcal{R}}} \quad U_{\mathcal{S},-}^X = U_{\text{SUR},+}^{X \times (k_{\mathcal{R}} \setminus \mathfrak{o}_{\mathcal{R}})} \cup U_{\text{SUR},-}^{X \times k_{\mathcal{R}}}$$

$$\begin{aligned} U_{\mathcal{S},0}^X \cap U_{\mathcal{S},0}^Y &= U_{\mathcal{S},0}^{X \cap Y}, & U_{\mathcal{S},+}^X \cap U_{\mathcal{S},+}^Y &= U_{\mathcal{S},+}^{X \cap Y}, & U_{\mathcal{S},-}^X \cap U_{\mathcal{S},-}^Y &= U_{\mathcal{S},-}^{X \cap Y}, \\ U_{\mathcal{S},+}^X \cap U_{\mathcal{S},0}^Y &= U_{\mathcal{S},+}^{X \cap Y}, & U_{\mathcal{S},-}^X \cap U_{\mathcal{S},0}^Y &= U_{\mathcal{S},-}^{X \cap Y}, & U_{\mathcal{S},+}^X \cap U_{\mathcal{S},-}^Y &= \emptyset. \end{aligned}$$

Let \mathcal{A}_n be the set of subsets of \mathcal{O}^n which are finite (w.l.o.g. disjoint) unions of products of sets in \mathcal{A}_1 . It is easy to check that \mathcal{A}_n is an algebra. Let $\tau = \{t_1, \dots, t_n\}$ be an unordered subset of \mathbf{R} with $t_n > \dots > t_1$ (also $t_1 > t_0$ in the conditional case). We define the function $\mu_\tau : \mathcal{A}_n \rightarrow \mathbf{R}^{\geq 0}$ initially on sets of the form $A_{\mathcal{S}, \sigma}^{\{X_i\}} = \prod_{i=1}^n \mathcal{U}_{\mathcal{S}, \sigma(t_i)}^{X_i}$, where $\sigma : \tau \rightarrow \{0, +, -\}$, and then extend by finite additivity. Let $\mu_\tau(A) = \mathcal{I}_{\mathcal{S}, \tau}^{\{X_i\}} \times \mathcal{J}_{\mathcal{S}, \tau}^\sigma$, where in the conditional case

$$\mathcal{I}_{\mathcal{S}, \tau}^{\{X_i\}} = \int \cdots \int_{\prod_{i=1}^n X_i} k_{t_n - t_{n-1}}^{\mathcal{S}}(x_n, x_{n-1}) \cdots k_{t_2 - t_1}^{\mathcal{S}}(x_2, x_1) k_{t_1 - t_0}^{\mathcal{S}}(x_1, b_{\mathcal{S}}) d^{\mathcal{S}}x_n \cdots d^{\mathcal{S}}x_1,$$

with $k_i^{\mathcal{S}}(x, y) = \prod_{\mathfrak{p} \in \mathcal{S}} k_i^{\mathfrak{p}}(x_{\mathfrak{p}}, y_{\mathfrak{p}})$, $b_{\mathcal{S}} = (b_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{S}}$, and $\mathcal{J}_{\mathcal{S}, \tau}^\sigma$ is defined by induction on the number of $t_i \in \tau$ for which $\sigma(t_i) = '-'$ as follows:

(0) If τ is empty, $\mathcal{J}_{\mathcal{S}, \tau}^\sigma = 1$

(1) If $\sigma(t_i) = '+'$ for all $t_i \in \tau$,

$$\mathcal{J}_{\mathcal{S}, \tau}^\sigma = \prod_{\mathfrak{p} \in \mathfrak{P} \setminus \mathcal{S}} \int \cdots \int_{\prod_{i=1}^n \mathfrak{o}_{\mathfrak{p}}} k_{t_n - t_{n-1}}^{\mathfrak{p}}(x_n, x_{n-1}) \cdots k_{t_2 - t_1}^{\mathfrak{p}}(x_2, x_1) k_{t_1 - t_0}^{\mathfrak{p}}(x_1, b_{\mathfrak{p}}) dx_n \cdots dx_1$$

(2) If $\sigma(t_i) \neq '-'$ for all $t_i \in \tau$, then $\mathcal{J}_{\mathcal{S}, \tau}^\sigma = \mathcal{J}_{\mathcal{S}, \tau'}^{\sigma'}$, where $\tau' = \{t_i \in \tau : \sigma(t_i) = +\}$, and σ' is the restriction of σ to τ'

(3) Otherwise choose a $t_i \in \tau$ for which $\sigma(t_i) = '-'$, and let σ' and σ'' be identical with σ except that $\sigma'(t_i) = '0'$ and $\sigma''(t_i) = '+'$. Then

$$\mathcal{J}_{\mathcal{S}, \tau}^\sigma = \mathcal{J}_{\mathcal{S}, \tau'}^{\sigma'} - \mathcal{J}_{\mathcal{S}, \tau''}^{\sigma''}.$$

In the unconditional case, let

$$\mathcal{I}_{\mathcal{S}, \tau}^{\{X_i\}} = \int \cdots \int_{\prod_{i=1}^n X_i} \Omega_{\mathcal{S}}(x_n) k_{t_n - t_{n-1}}^{\mathcal{S}}(x_n, x_{n-1}) \cdots k_{t_2 - t_1}^{\mathcal{S}}(x_2, x_1) \Omega_{\mathcal{S}}(x_1) d^{\mathcal{S}}x_n \cdots d^{\mathcal{S}}x_1,$$

where $\Omega_{\mathcal{S}}(x) = \prod_{\mathfrak{p} \in \mathcal{S}} \Omega_{\mathfrak{p}}(x_{\mathfrak{p}})$, and in Case (1) above we put

$$\mathcal{J}_{\mathcal{S}, \tau}^{\sigma} = \prod_{\mathfrak{p} \in \mathfrak{P} \setminus \mathcal{S}} \int \cdots \int \Omega_{\mathcal{S}}(x_n) k_{t_n - t_{n-1}}^{\mathfrak{p}}(x_n, x_{n-1}) \cdots k_{t_2 - t_1}^{\mathfrak{p}}(x_2, x_1) \Omega_{\mathcal{S}}(x_1) dx_n \cdots dx_1.$$

Theorem 3.1. *Let (3.1) and (3.2) (resp. (3.1') and (3.2')) be satisfied. Then the above formulas for μ_{τ} define a consistent family of regular measures, and hence there is a global conditional (resp. unconditional) measure μ on $(\mathcal{O}^T, \mathcal{B}_T)$ for $T = [t_0, \infty)$ (resp. \mathbf{R}), such that*

$$\mu(\pi_{\tau}^*(\mathcal{A}_{\tau})) = \mu_{\tau}(\mathcal{A}_{\tau}),$$

for all finite subsets τ of T , $\mathcal{A}_{\tau} \in \mathcal{B}_T(\mathcal{O}^T)$.

Proof. μ_{τ} is finitely additive and we will show in Lemma 3.2 below that it is inner regular. So, by Henry's Extension, μ_{τ} can be extended to a Radon measure P^{τ} on \mathcal{O}^{τ} . This can be done for each finite subset τ of T . It is easy to check that the family of measures $\{P^{\tau}\}$ is consistent in the sense of Theorem 2.2, which therefore establishes the existence of the required measure μ . \square

Lemma 3.2. *μ_{τ} is inner regular.*

proof. Let $A \in \mathcal{A}_n$, $\varepsilon > 0$ be given. We need to find a compact subset C of \mathcal{O}^n such that $\mu_{\tau}(C) > \mu_{\tau}(A) - \varepsilon$. We may assume without loss of generality that A is of the form $A = \prod_{i=1}^n \mathcal{U}_{\mathcal{S}, \sigma(i)}^{X(i)}$, since any $A \in \mathcal{A}_n$ can be written as a finite disjoint union of such.

Note that $\prod_{i=1}^n \mathcal{U}_{\mathcal{S}, \sigma(i)}^{X_i}$ is compact iff each X_i is compact, and $\sigma(t_i) = '+'$ for all i . Our strategy will be first to replace each $\mathcal{U}_{\mathcal{S}, \sigma(i)}^{X_i}$ with $\mathcal{U}_{\mathcal{S} \times \mathcal{R}, +}^{X_i \times Y_i}$, where \mathcal{R} is a finite subset of $\mathfrak{P} \setminus \mathcal{S}$ and each Y_i is a measurable subset of $\prod_{\mathfrak{p} \in \mathcal{R}} k_{\mathfrak{p}}$, and then to approximate X_i and Y_i with compact sets \hat{X}_i and \hat{Y}_i . Equations (3.3) allow us to write

$$\mathcal{J}_{\mathcal{S}, \tau}^{\sigma} = \sum_{\sigma'} \mathcal{I}_{\mathcal{R}, \tau}^{\sigma, \sigma'} \cdot \mathcal{J}_{\mathcal{S} \cup \mathcal{R}, \tau}^{\sigma'}$$

where the sum is over all $\sigma' : \tau \rightarrow \{+, -\}$ such that $\sigma(t_i) = '+' \implies \sigma'(t_i) = '+'$, and

$$\mathcal{I}_{\mathcal{R},\tau}^{\sigma,\sigma'} = \mathcal{I}_{\mathcal{R},\tau}^{\{Y_i\}}, \quad \text{where} \quad Y_i(\sigma,\sigma') = \begin{cases} k_{\mathcal{R}}, & \text{if } \sigma(t_i) = '0' \\ \mathbf{0}_{\mathcal{R}}, & \text{if } \sigma(t_i) = '+' \\ k_{\mathcal{R}}, & \text{if } \sigma(t_i) = \sigma'(i) = '-' \\ k_{\mathcal{R}} \setminus \mathbf{0}_{\mathcal{R}}, & \text{if } \sigma(t_i) = '-', \sigma'(t_i) = '+' \end{cases}$$

We define $\sigma^+ : \tau \rightarrow \{+, -\}$ by $\sigma^+(t_i) = '+'$ for all $t_i \in \tau$. We assign a *degree* to all other σ' by $\deg(\sigma') = k$ if $\sigma'_k = '-'$ and $\sigma'_i = '+'$ for $i < k$. Then

$$\mathcal{J}_{S,\tau}^{\sigma} = \mathcal{I}_{\mathcal{R},\tau}^{\sigma,\sigma^+} \cdot \mathcal{J}_{S \cup \mathcal{R},\tau}^{\sigma^+} + \sum_{k=1}^n \sum_{\deg(\sigma')=k} \mathcal{I}_{\mathcal{R},\tau}^{\sigma,\sigma'} \cdot \mathcal{J}_{S \cup \mathcal{R},\tau}^{\sigma'}$$

From the definition of $\mathcal{J}_{S \cup \mathcal{R},\tau}^{\sigma'}$, it follows (in the conditional case) that

$$\begin{aligned} \sum_{\deg(\sigma')=k} \mathcal{J}_{S \cup \mathcal{R},\tau}^{\sigma'} &\leq \prod_{\mathfrak{p} \in \mathfrak{P} \setminus S \cup \mathcal{R}} \int \cdots \int_{\prod_{i=1}^{k-1} \mathfrak{o}_{\mathfrak{p}}} k^{\mathfrak{p}}(x_{k-1}, x_{k-2}) \cdots k^{\mathfrak{p}}(x_1, b_{\mathfrak{p}}) dx_{k-1} \cdots dx_1 \\ &\quad - \prod_{\mathfrak{p} \in \mathfrak{P} \setminus S \cup \mathcal{R}} \int \cdots \int_{\prod_{i=1}^k \mathfrak{o}_{\mathfrak{p}}} k^{\mathfrak{p}}(x_k, x_{k-1}) \cdots k^{\mathfrak{p}}(x_1, b_{\mathfrak{p}}) dx_k \cdots dx_1 \\ &\leq 1 - \prod_{\mathfrak{p} \in \mathfrak{P} \setminus S \cup \mathcal{R}} \int_{\mathfrak{o}_{\mathfrak{p}}} k^{\mathfrak{p}}(x, b_{\mathfrak{p}}) dx. \end{aligned}$$

Choose \mathcal{R} large enough that

$$\prod_{\mathfrak{p} \in \mathfrak{P} \setminus S \cup \mathcal{R}} \int_{\mathfrak{o}_{\mathfrak{p}}} k^{\mathfrak{p}}(x, b_{\mathfrak{p}}) dx > 1 - \frac{\varepsilon}{3n}, \quad \text{for } 1 \leq k \leq n.$$

In the unconditional case, we similarly obtain

$$\sum_{\deg(\sigma')=1} \mathcal{J}_{S \cup \mathcal{R},\tau}^{\sigma'} \leq 1 - \prod_{\mathfrak{p} \in \mathfrak{P} \setminus S \cup \mathcal{R}} \int_{\mathfrak{o}_{\mathfrak{p}} \times \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x) k^{\mathfrak{p}}(x, y) \Omega_{\mathfrak{p}}(y) dx dy.$$

Now

$$\begin{aligned} \int_{\mathfrak{o}_{\mathfrak{p}} \times \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x) k_{t_k - t_1}^{\mathfrak{p}}(x, y) \Omega_{\mathfrak{p}}(y) dx dy &\geq 1 - 2 \int_{k_{\mathfrak{p}} \setminus \mathfrak{o}_{\mathfrak{p}} \times k_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x) k_{t_k - t_1}^{\mathfrak{p}}(x, y) \Omega_{\mathfrak{p}}(y) dx dy \\ &= 1 - 2 \int_{k_{\mathfrak{p}} \setminus \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x)^2 dx. \end{aligned}$$

So using (3.2') and choosing \mathcal{R} sufficiently large, we may ensure that

$$\prod_{\mathfrak{p} \in \mathfrak{P} \setminus S \cup \mathcal{R}} \int_{\mathfrak{o}_{\mathfrak{p}} \times \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x) k_{t_k - t_1}^{\mathfrak{p}}(x, y) \Omega_{\mathfrak{p}}(y) dx dy > 1 - \frac{\varepsilon}{3n}, \quad \text{for } 1 \leq k \leq n.$$

In either case, we have
$$|\mathcal{J}_{S, \tau}^{\sigma} - \mathcal{I}_{\mathcal{R}, \tau}^{\sigma, \sigma^+} \cdot \mathcal{J}_{S \cup \mathcal{R}, \tau}^{\sigma^+}| \leq \sum_{k=1}^n \sum_{\deg(\sigma')=k} \mathcal{J}_{S \cup \mathcal{R}, \tau}^{\sigma'} < \frac{\varepsilon}{3}.$$

Finally, using the fact that $k_S \times k_{\mathcal{R}}$ is a complete, separable metric space (and therefore any finite measure on it is regular), we can choose compact subsets \hat{X}_i of X_i and \hat{Y}_i of $Y_i(\sigma, \sigma')$ such that

$$\mathcal{I}_{S, \tau}^{\{X_i\}} - \mathcal{I}_{S, \tau}^{\{\hat{X}_i\}} < \frac{\varepsilon}{3} \quad \text{and} \quad \mathcal{I}_{\mathcal{R}, \tau}^{\{Y_i\}} - \mathcal{I}_{\mathcal{R}, \tau}^{\{\hat{Y}_i\}} < \frac{\varepsilon}{3}.$$

So $C = \prod_{i=1}^n \mathcal{U}_{S \cup \mathcal{R}, +}^{\hat{X}_i \times \hat{Y}_i}$ is the required compact subset, and satisfies

$$\begin{aligned} |\mu_{\tau}(A) - \mu_{\tau}(C)| &= |\mathcal{I}_{S, \tau}^{\{X_i\}} \times \mathcal{J}_{S, \tau}^{\sigma} - \mathcal{I}_{S, \tau}^{\{\hat{X}_i\}} \times \mathcal{I}_{\mathcal{R}, \tau}^{\{\hat{Y}_i\}} \times \mathcal{J}_{S \cup \mathcal{R}, \tau}^{\sigma^+}| \\ &\leq |\mathcal{I}_{S, \tau}^{\{X_i\}} - \mathcal{I}_{S, \tau}^{\{\hat{X}_i\}}| + |\mathcal{I}_{\mathcal{R}, \tau}^{\{Y_i\}} - \mathcal{I}_{\mathcal{R}, \tau}^{\{\hat{Y}_i\}}| + |\mathcal{J}_{S, \tau}^{\sigma} - \mathcal{I}_{\mathcal{R}, \tau}^{\sigma, \sigma^+} \cdot \mathcal{J}_{S \cup \mathcal{R}, \tau}^{\sigma^+}| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

completing the proof of Lemma 3.2. □

4. \mathfrak{p} -Wiener Measure

We now give examples of families of Hamiltonian operators $H_{\mathfrak{p}}$ that may be used to construct measures in the manner described in §3.

4.1 Diffusions with independent increments

Interest has been shown in diffusion processes on locally compact groups by mathematicians [Bin71, Hey77, Tor71], and on ultrametric spaces by physicists [BO89, MP85, OS85, Par93]. Brownian motion is the most fundamental diffusion process, and its definition has been extended to such exotic spaces as the Sierpinski gasket [BP88] and other nested fractals [Lin90].

Wiener measure is normally required to be supported on continuous paths, and therefore cannot be defined on a totally disconnected space like $k_{\mathfrak{p}}$. We can, however, define a measure - which we call \mathfrak{p} -Wiener measure - that is analogous in the sense of having independent increments and satisfying appropriate re-scaling conditions.

A natural analog on $k_{\mathfrak{p}}$ of the Laplacian is the operator $H_{\mathfrak{p}} = \mathcal{F}^{-1} \mathcal{H}_{\mathfrak{p}} \mathcal{F}$, where \mathcal{F} is the Fourier transform and $\mathcal{H}_{\mathfrak{p}}$ is the (unbounded) operator of multiplication by a suitable function $h_{\mathfrak{p}} : k_{\mathfrak{p}} \rightarrow \mathbf{R}$. We will use $h(x) = |x|_{\mathfrak{p}}^2$ to define \mathfrak{p} -Wiener measure, but the following lemma applies to a different class of functions $h(x)$.

Lemma 4.0. *Assume that $h_{\mathfrak{p}}(x)$ depends only on $|x|_{\mathfrak{p}}$, that it is an increasing function of $|x|_{\mathfrak{p}}$, that $\lim_{x \rightarrow 0} h_{\mathfrak{p}}(x) = 0$, and e^{-th} is L^2 for all $t > 0$.*

Then condition (3.1) of §3 is satisfied, and we have $(3.2) \iff \sum_{\mathfrak{p} \in \mathfrak{P}} h_{\mathfrak{p}}(1) < \infty$.

Proof. The heat kernel $k_t^{\mathfrak{p}}(x, y)$ of $e^{-tH_{\mathfrak{p}}}$ is the Fourier transform of $f_t(z) = e^{-th_{\mathfrak{p}}(z)}$

evaluated at $x-y$. It is non-negative by the monotonicity of $h(|x|_{\mathfrak{p}})$. Furthermore,

$$\int_{k_{\mathfrak{p}}} k_t^{\mathfrak{p}}(x, y) dy = \int_{k_{\mathfrak{p}}} \hat{f}_t(x-y) dy = \lim_{x \rightarrow 0} f_t(x) = 1.$$

Moreover
$$\int_{\mathfrak{o}_{\mathfrak{p}}} k_t^{\mathfrak{p}}(x, 0) dx = \int_{\mathfrak{o}_{\mathfrak{p}}} \hat{f}_t(x) dx = \int_{\mathfrak{o}_{\mathfrak{p}}} f_t(x) dx,$$

so
$$\begin{aligned} \prod_{\mathfrak{p} \in \mathfrak{P}} \int_{\mathfrak{o}_{\mathfrak{p}}} k_t^{\mathfrak{p}}(x, 0) dx > 0 &\iff \prod_{\mathfrak{p} \in \mathfrak{P}} \int_{\mathfrak{o}_{\mathfrak{p}}} e^{-th_{\mathfrak{p}}(x)} dx > 0 \\ &\iff \sum_{\mathfrak{p} \in \mathfrak{P}} \int_{\mathfrak{o}_{\mathfrak{p}}} (1 - e^{-th_{\mathfrak{p}}(x)}) dx < \infty \\ &\iff \sum_{\mathfrak{p} \in \mathfrak{P}} t \int_{\mathfrak{o}_{\mathfrak{p}}} h_{\mathfrak{p}}(x) dx < \infty \iff \sum_{\mathfrak{p} \in \mathfrak{P}} h_{\mathfrak{p}}(1) < \infty. \quad \square \end{aligned}$$

Example 1. $h(x) = |x|_{\mathfrak{p}}^2$. This defines \mathfrak{p} -Wiener measure locally on each $k_{\mathfrak{p}}$. However, the sum $\sum_{\mathfrak{p} \in \mathfrak{P}} h_{\mathfrak{p}}(1)$ does not converge, so condition (3.2) is not satisfied - \mathfrak{p} -Wiener measure cannot be defined globally on the whole adèle ring.

Example 2. $h(x) = \left(\frac{|x|_{\mathfrak{p}}}{N\mathfrak{p}}\right)^2$. This determines a kind of *modified* \mathfrak{p} -Wiener measure which is similar to ordinary \mathfrak{p} -Wiener measure but rescaled in time. In this case,

$$\sum_{\mathfrak{p} \in \mathfrak{P}} h_{\mathfrak{p}}(1) = \sum_{\mathfrak{p} \in \mathfrak{P}} N\mathfrak{p}^{-2} < \infty.$$

So this family of modified \mathfrak{p} -Wiener measures satisfy condition (3.2) and can therefore be combined to define a modified \mathfrak{p} -Wiener measures on \mathcal{O} - an example a conditional global measure, with basepoint $b = 0$ and initial time $t_0 = 0$, on the path space of the adèle ring of k .

Example 3.

$$h_i = \begin{cases} 0, & \text{if } i < 0 \\ i \log(N\mathfrak{p}), & \text{if } i \geq 0 \end{cases}$$

If $|x - y|_{\mathfrak{p}} = N\mathfrak{p}^l$,

$$\begin{aligned} k_t(x, y) &= N\mathfrak{p}^{-l} \left(-N\mathfrak{p}^{tl-t} + (N\mathfrak{p} - 1) N\mathfrak{p}^{l-t} (1 + N\mathfrak{p}^{1-t} + \dots + N\mathfrak{p}^{(-l-1)(1-t)}) + N\mathfrak{p}^l \right) \\ &= (N\mathfrak{p} - 1) N\mathfrak{p}^{-t} \left(\frac{1 - N\mathfrak{p}^{-l(1-t)}}{1 - N\mathfrak{p}^{1-t}} \right) + 1 - N\mathfrak{p}^{tl-t-l} \\ &= \left(\frac{1 - N\mathfrak{p}^{-t}}{1 - N\mathfrak{p}^{1-t}} \right) [1 - N\mathfrak{p}^{(-l+1)(1-t)}] = \left(\frac{1 - N\mathfrak{p}^{-t}}{1 - N\mathfrak{p}^{1-t}} \right) \left(1 - \left(\frac{N\mathfrak{p}}{|x|_{\mathfrak{p}}} \right)^{1-t} \right). \end{aligned}$$

This determines a measure which is supported on paths that a.s. remain inside $\mathfrak{o}_{\mathfrak{p}}$.
i.e.

$$\int_{\mathfrak{o}_{\mathfrak{p}}} k_t^{\mathfrak{p}}(x, 0) dx = 1, \quad \text{for all } \mathfrak{p} \in \mathfrak{P}_f, \quad t > 0.$$

So condition (3.2) is trivially satisfied, and this gives another example of a global path space measure, although in this case the paths a.s.do not range over the whole of the adèle ring, but only over the compact space $\prod_{\mathfrak{p} \in \mathfrak{P}_f} \mathfrak{o}_{\mathfrak{p}}$. Moreover, each operator $H_{\mathfrak{p}}$ of Example 3 has a unique ground state $\Omega_{\mathfrak{p}}$ - namely the characteristic function of $\mathfrak{o}_{\mathfrak{p}}$ - and

$$\int_{\mathfrak{o}_{\mathfrak{p}} \times \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(x) k_t^{\mathfrak{p}}(x, y) \Omega_{\mathfrak{p}}(y) dx dy = 1, \quad \text{for all } \mathfrak{p} \in \mathfrak{P}_f, \quad t \geq 0.$$

So condition (3.2') is satisfied, providing an example of an unconditional measure on the path space of $\prod_{\mathfrak{p} \in \mathfrak{P}_f} \mathfrak{o}_{\mathfrak{p}}$.

Lemma 4.1. *\mathfrak{p} -Wiener measure has the following properties:*

- (i) $\lim_{\substack{t \rightarrow 0 \\ \text{or } t \rightarrow \infty}} \frac{\log k_t^{\mathfrak{p}}(0, 0)}{\log t} = -\frac{1}{2}$
- (ii) $k_t^{\mathfrak{p}}(x', 0) = |u|_{\mathfrak{p}}^{-1} k_t^{\mathfrak{p}}(x, 0), \quad \text{if } x' = u \cdot x, \quad t' = |u|_{\mathfrak{p}}^2 \cdot t$

Proof.

(i) We show equivalently that $A t^{-\frac{1}{2}} \leq k_t^{\mathfrak{p}}(0, 0) \leq B t^{-\frac{1}{2}}$, for some $B > A > 0$.

Now
$$k_t^{\mathfrak{p}}(0, 0) = (1 - N\mathfrak{p}^{-1}) \sum_{j \in \mathbf{Z}} N\mathfrak{p}^j e^{-tN\mathfrak{p}^{2j}}.$$

For the upper bound, we approximate this sum by the integral

$$\int_{-\infty}^{\infty} N\mathfrak{p}^x e^{-tN\mathfrak{p}^{2x}} dx = \frac{\pi^{\frac{1}{2}} t^{-\frac{1}{2}}}{2 \log N\mathfrak{p}}$$

The integrand has a unique local maximum of $(2et)^{-\frac{1}{2}}$ at $x = x_0$, where $2tN\mathfrak{p}^{2x_0} = 1$, so

$$k_t^{\mathfrak{p}}(0, 0) \leq B t^{-\frac{1}{2}}, \quad \text{where } B = \frac{\pi^{\frac{1}{2}}}{2 \log N\mathfrak{p}} + (2e)^{-\frac{1}{2}}.$$

To get the lower bound, we note that for $j = \lfloor x_0 \rfloor$, $N\mathfrak{p}^j e^{-tN\mathfrak{p}^{2j}} > N\mathfrak{p}^{-1} (2et)^{-1/2}$, so

$$k_t^{\mathfrak{p}}(0, 0) \geq A t^{-1/2}, \quad \text{where } A = N\mathfrak{p}^{-1} (1 - N\mathfrak{p}^{-1}) (2e)^{-\frac{1}{2}}.$$

$$\begin{aligned} \text{(ii)} \quad k_t^{\mathfrak{p}}(x', 0) &= \int_{\mathbf{Q}_{\mathfrak{p}}} e^{-t|\xi|_{\mathfrak{p}}^2} \chi(x', \xi) d\xi \\ &= |u|_{\mathfrak{p}}^{-1} \int_{\mathbf{Q}_{\mathfrak{p}}} e^{-t|\xi'|_{\mathfrak{p}}^2} \chi(x, \xi') d\xi' = |u|_{\mathfrak{p}}^{-1} k_t^{\mathfrak{p}}(x, 0). \quad \square \end{aligned}$$

4.2 \mathfrak{p} -Wiener measure as a Poisson process

For later computations, it will be convenient to have a more concrete description of the space of paths on which \mathfrak{p} -Wiener measure is supported. Recall that on $k_{\mathfrak{p}}$ we have the ultrametric triangle inequality:

$$|x + y|_{\mathfrak{p}} \leq \max(|x|_{\mathfrak{p}}, |y|_{\mathfrak{p}}).$$

Because of this, we are unable to follow the usual ‘‘bottom-up’’ approach to Wiener measure, since it is not possible to build up a large displacement from a series of small ones [Par88]. Instead we follow a ‘top-down’ approach and give the following alternative model for \mathfrak{p} -Wiener measure, as a compound Poisson process:

Let $\hat{\Omega}$ be the space of triples $\hat{\omega} = (\{n_i\}, \{t_j^{(i)}\}, \{x_j^{(i)}\})$, where $n_i \geq 0$ for $i \in \mathbf{Z}$, with $n_i = 0$ for i sufficiently large, and $t_j^{(i)} \in [0, t]$, $x_j^{(i)} \in k_{\mathfrak{p}}^{(i)} = \{x \in k_{\mathfrak{p}} : |x|_{\mathfrak{p}} \leq N\mathfrak{p}^i\}$, for $1 \leq j \leq n_i$. Every $\hat{\omega} \in \hat{\Omega}$ determines a path $\omega : [0, t] \rightarrow k_{\mathfrak{p}}$ by

$$\omega(s) = \sum_{i \in \mathbf{Z}} \sum_{t_j^{(i)} \leq s} x_j^{(i)}.$$

This is an infinite sum, but in order to calculate $\omega(s)$ to within an accuracy of $N\mathfrak{p}^k$ we need to know only those $x_j^{(i)}$'s for which $i > k$. Given numbers $\lambda_i \geq 0$ such that $s_j = \sum_{i>j} \lambda_i$ is finite for $j \in \mathbf{Z}$, we can define a measure on $\hat{\Omega}$ as follows: If $k \in \mathbf{Z}$ is fixed, composition with the epimorphism $\eta_{(k)} : k_{\mathfrak{p}} \rightarrow k_{\mathfrak{p}}/k_{\mathfrak{p}}^{(k)}$ gives a map $\pi_{(k)}$ from $\hat{\Omega}$ to the space $\hat{\Omega}^{(k)}$ of step functions from $[0, t]$ to the space $k_{\mathfrak{p}}/k_{\mathfrak{p}}^{(k)}$, which has the discrete topology. The image of $\hat{\omega}$ under this map depends only on $\{n_i\}$, $\{t_j^{(i)}\}$ and $\{x_j^{(i)}\}$ for which $i > k$. Let $\hat{\mathcal{B}}(\hat{\Omega})$ be the algebra generated by $\eta_{(k)}^{-1}(\mathcal{B}(\Omega^{(k)}))$. We define a probability measure $\mathfrak{D}^{(k)}\hat{\omega}$ on $\hat{\Omega}^{(k)}$ by

$$\mathfrak{D}^{(k)}\hat{\omega} = \prod_{i>k} e^{-\lambda_i t} (\lambda_i t)^{n_i} |t|^{-n_i} |k_{\mathfrak{p}}^{(i)}|^{-n_i} \prod_{1 \leq j \leq n_i} dt_j^{(i)} dx_j^{(i)}.$$

In other words, the number of "jumps" of maximum size $N\mathfrak{p}^i$ is governed by a Poisson process with parameter λ_i (in the sense that the average number of them in time t is $\lambda_i t$). By the same argument used in the proof of Theorem 2.1, there exists a unique measure $\hat{\mathfrak{D}}\hat{\omega}$ on $\hat{\Omega}$ such that

$$\int_{\pi_{(k)}^* \mathcal{A}_{(k)}} \hat{\mathfrak{D}}\hat{\omega} = \int_{\mathcal{A}_{(k)}} \mathfrak{D}^{(k)}\hat{\omega}, \quad \text{for all } \mathcal{A}_{(k)} \in \mathcal{B}(\hat{\Omega}^{(k)}).$$

The probability spaces $(\Omega, \mathcal{B}, \mathfrak{D})$ and $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{\mathfrak{D}})$ define stochastic processes $(W_s)_{s \in [0, t]}$ and $(\hat{W}_s)_{s \in [0, t]}$, respectively.

Lemma 4.2.

- (a) The processes (W_s) and (\hat{W}_s) have the same finite dimensional distributions.
- (b) They satisfy the properties:
 - (i) $W_0 = 0$,
 - (ii) For any $s_0 < s_1 < \dots < s_n$ in $[0, t]$ the variables $W_{s_i} - W_{s_{i-1}}$ are independent,
 - (iii) For every $\varepsilon > 0$, $s_0 \in [0, t]$ $\lim_{s \rightarrow s_0} \text{Prob}(|W_s - W_{s_0}| > \varepsilon) = 0$.
- (c) (\hat{W}_s) satisfies the additional property
 - (iv) For every $\hat{\omega} \in \hat{\Omega}$ the sample path $s \mapsto \hat{W}_s(\hat{\omega})$ is right continuous with left limits.

Proof. Properties (i) and (ii) follow easily from the definitions of \mathfrak{D} and $\hat{\mathfrak{D}}$.

So, for part (a), we need only show that the heat kernel $k'_t(x, y)$ of the diffusion defined by (\hat{W}_s) is the same as that for (W_s) . We can calculate $k'_t(x, y)$ as follows: At time t , we have an equal probability of being anywhere within $N\mathfrak{p}^j$ of our starting point - i.e. a probability density of $N\mathfrak{p}^{-j}$ within this region - if j is the largest number for which $n_j \neq 0$. The probability of this event is $(1 - e^{-\lambda_j t}) \prod_{i>j} e^{-\lambda_i t}$. So if $|x - y|_{\mathfrak{p}} = N\mathfrak{p}^l$,

$$\begin{aligned} k_t(x, y) &= \sum_{j \geq l} (1 - e^{-\lambda_j t}) N\mathfrak{p}^{-j} \prod_{i>j} (e^{-\lambda_i t}) \\ &= \sum_{j \geq l} (1 - e^{-t\lambda_j}) N\mathfrak{p}^{-j} e^{-ts_j}, \quad \text{where } s_j = \sum_{i>j} \lambda_i \\ &= -N\mathfrak{p}^{-l} e^{-ts_{l-1}} + (1 - N\mathfrak{p}^{-1}) \sum_{\nu \leq -l} N\mathfrak{p}^{\nu} e^{-ts_{-\nu}}, \end{aligned}$$

which is the Fourier transform of e^{-th} evaluated at $x - y$ where, writing $h(x) = h_i$ if $|x|_{\mathfrak{p}} = N\mathfrak{p}^i$, we have $h_i = s_{-i}$ and $\lambda_i = (h_{1-i} - h_{-i})$, completing the proof of (a).

(iii) then follows from the formula for $k_t(x, y)$, and (iv) from the definition of \hat{W}_s .

Lemma 4.3. *There is a map η from Ω to $\hat{\Omega}$ such that $\hat{\omega} = \eta\omega$ agrees with ω on dyadic rationals, and $(W_{s_1}, \dots, W_{s_N}) = (\hat{W}_{s_1 \circ \eta}, \dots, \hat{W}_{s_N \circ \eta})$ for all $s_1 < \dots < s_N$.*

Proof. It follows from a result of J.R. Kinney (see [Tor71], [Bin71] and [Hey77]) that for any stochastic process on a locally compact abelian group satisfying (i) – (iii) above there is an equivalent process which also satisfies (iv). We give only those details of the proof necessary to show that \hat{W}_s is the required equivalent process to W_s . Let $t > 0$ be fixed and let D denote the set of dyadic rationals in the interval $[0, t]$. Consider $i \in \mathbf{Z}^+$ and let $n = n(i) \in \mathbf{Z}^+$ by chosen so that

$$2^{n+1-i} > t^2 \left[\frac{d}{ds} \Big|_{s=0} u_i(s) \right]^2.$$

For each $\omega \in \Omega$, $m \in \mathbf{Z}^+$ and $k \in \mathbf{Z}$ with $1 \leq k \leq 2^n$ let

$$a_{nk}^m = \#\{j \in \mathbf{Z}, 1 \leq j \leq 2^m : |\omega(k2^{-n}t + j2^{-m-n}t) - \omega(k2^{-n}t + (j-1)2^{-m-n}t)| < N\mathfrak{p}^i\}.$$

a_{nk}^m is an increasing function of m . Let $a_{nk}(\omega) = \lim_{m \rightarrow \infty} a_{nk}^m(\omega)$.

[Note: $a_{nk}(\omega)$ may be infinite.] We have

$$\begin{aligned} \text{Prob}(a_{nk} \geq 2) &\leq \lim_{m \rightarrow \infty} 2^{2m-1} u_i(2^{-m-n}t)^2 \\ &\leq 2^{-2n-1}t^2 \left[\lim_{m \rightarrow \infty} 2^{m+n}t^{-1} u_i(2^{-m-n}t) \right]^2 \\ &\leq 2^{-2n-1}t^2 \left[\frac{d}{ds} \Big|_{s=0} u_i(s) \right]^2. \end{aligned}$$

Let $\mathcal{A} = \limsup A_i$. Then

$$\sum_{i \geq 1} \mu(\mathcal{A}_i) \leq \sum_{i \geq 1} 2^{-i} < \infty.$$

So, by the Borel-Cantelli lemma, $\mu(\mathcal{A}) = 0$. For $\omega \in \mathcal{A}$, we put $\eta\omega \equiv 0$, while for $\omega \in \Omega \setminus \mathcal{A}$ we define $\hat{\omega} = \eta(\omega)$ as follows: Since $\omega \notin \mathcal{A}$, there is a subsequence i_l of \mathbf{Z}^+ such that $\omega \notin \mathcal{A}_{i_l}$ for any i_l . Let $i \in \mathbf{Z}$ be fixed and choose an $i_l \geq i$. Put $n = n(i_l)$. For each dyadic subinterval I_{nk} with $a_{nk}(\omega) = 1$, let

$$t_{nk} = \sup\{s \in D \cap I_{nk} : |\omega(s) - \omega(k2^{-n}t)| < N\mathfrak{p}^{i_l}\},$$

and let x_{nk} be the limit of $\omega(s)$ as s approaches t_{nk} from above through dyadic rationals minus the same limit from below. (The existence of these limits follows from our construction.) Then $\{x_j^{(i)}\}$ is defined to be the subset of those x_{nk} for which $|x_{nk}| = N\mathfrak{p}^i$, and $\{t_j^{(i)}\}$ the set of the corresponding t_{nk} 's.

Finally, by approximating each s_i with a sequence of dyadic rationals, we deduce from Lemma 4.1(a) and a standard argument involving convergence of distributions that $(W_{s_1}, \dots, W_{s_N}) = (\hat{W}_{s_1 \circ \eta}, \dots, \hat{W}_{s_N \circ \eta})$. \square

5. Feynman-Kac Formula

Theorem 5.0. Suppose $V : \mathbb{Q}_p \rightarrow \mathbf{R}$ is continuous and bounded below. Then for $f, g \in L^2(k_p)$,

$$(5.0) \quad (f, e^{-t(H+V)}g) = \int f(\hat{\omega}(0))e^{-\int_0^t V(\hat{\omega}(s)) ds}g(\hat{\omega}(t))\hat{\mathcal{D}}\hat{\omega}.$$

Proof. The proof follows that of the usual Feynman-Kac formula (see [Si79]). First consider the case where V is compactly supported. Using the Trotter product formula and Lemma 4.3,

$$\begin{aligned} (f, e^{-t(H+V)}g) &= \lim_{n \rightarrow \infty} \int_{\Omega} f(\omega(0)) \exp\left[-\frac{t}{n} \sum_{j=0}^{n-1} V(\omega(\frac{tj}{n}))\right] g(\omega(t)) \mathcal{D}\omega \\ &= \lim_{n \rightarrow \infty} \int_{\hat{\Omega}} f(\hat{\omega}(0)) \exp\left[-\frac{t}{n} \sum_{j=0}^{n-1} V(\hat{\omega}(\frac{tj}{n}))\right] g(\hat{\omega}(t)) \hat{\mathcal{D}}\hat{\omega}. \end{aligned}$$

Now since the paths in $\hat{\Omega}$ are right continuous with left limits, and V is uniformly continuous, the composition $V \circ \hat{\omega}$ is Riemann integrable, and

$$\frac{t}{n} \sum_{j=0}^{n-1} V(\hat{\omega}(\frac{tj}{n})) \rightarrow \int_0^t V(\hat{\omega}(s)) ds \quad \text{as } n \rightarrow \infty \quad \text{for all } \hat{\omega} \in \hat{\Omega}.$$

Moreover the integral in (5.0) is dominated by $|f(\hat{\omega}(0))||g(\hat{\omega}(t))|e^{-tV_0}$ (where V_0 is a lower bound for V), and this function is L^1 since

$$\int |f(\hat{\omega}(0))||g(\hat{\omega}(t))|\hat{\mathcal{D}}\hat{\omega} < \infty,$$

so an application of the Dominated Convergence Theorem yields (5.0).

For general V , let

$$V_k(x) = \begin{cases} V(x), & \text{if } |x|_{\mathfrak{p}} \leq N\mathfrak{p}^k, \\ 0, & \text{otherwise.} \end{cases}$$

Letting $k \rightarrow \infty$, both sides of (5.0) converge by the Monotone Convergence Theorem for forms and integrals, respectively, completing the proof of the theorem. \square

Corollary 5.1.

Taking the limit as f and g become delta functions centered at x and y , we see that the heat kernel $k_t(x, y)$ of $H + V$ is given by

$$(5.1) \quad k_t(x, y) = \int_{\substack{\hat{\omega}(0)=x \\ \hat{\omega}(t)=y}} e^{-\int_0^t V(\hat{\omega}(s)) ds} \hat{\mathcal{D}}\hat{\omega}.$$

6. Oscillator Process on \mathbb{Q}_p

Given a locally compact abelian group G , together with real-valued functions v and h on G and \hat{G} , respectively, we consider the (unbounded) operator H on $L^2(G)$ given by

$$H = \mathcal{F}^{-1} \mathcal{H}_p \mathcal{F} + V,$$

where \mathcal{F} is the Fourier Transform and V and \mathcal{H}_p are the operators of multiplication by the functions v and h , respectively. If G is self-dual we may consider h to be defined on G instead of \hat{G} . H reduces to the standard harmonic oscillator in the case where $G = \mathbb{R}^n$ and $v(x) = h(x) = |x|^2$.

The case of interest to us is where G is the p -adic field \mathbb{Q}_p , i.e. the completion of \mathbb{Q} in the p -adic norm $|x|_p = p^{-\text{ord}_p(x)}$ (p a prime). We will also assume that the functions $v(x)$ and $h(x)$ are “radially symmetric” in the sense that they depend only on the norm $|x|_p$ of x .

Theorem 6.0.

Suppose that V is continuous, radially symmetric, bounded below, and that e^{-tV} is integrable for all $t > 0$. Then H is θ -summable and therefore has discrete spectrum.

Theorem 6.1. (Uniqueness and positivity of the vacuum.)

In the case where $v(x) = h(x) = |x|_p^2$, the lowest eigenvalue x_0 of H is simple and its eigenfunction Ω_p is positive definite.

Theorem 6.2. *Taking $\tilde{H} = H - x_0 I$ and $\tilde{\Omega}_p = \Omega_p / \|\Omega_p\|_{L^2}$, the kernel k_t^p of $e^{-t\tilde{H}}$ satisfies*

$$\sum_{p \in \mathfrak{P}} \int_{k_p \setminus \mathfrak{o}_p} \tilde{\Omega}_p(y)^2 dy < \infty.$$

Thus, by Theorem 3.1, the family of heat kernels $k_t^{\mathfrak{p}}$ defines a path space measure - analogous to the Ornstein-Uhlenbeck process - on the adèle ring of \mathbb{Q} .

Proof of Theorem 6.0. Put $K = k_t^0(x, x) < \infty$, where k_t^0 is the heat kernel of $\mathcal{F}^{-1}\mathcal{H}_{\mathfrak{p}}\mathcal{F}$, and suppose that V is bounded below by V_0 . Using the Feynman-Kac formula (5.2),

$$\begin{aligned} k_t(x, x) &\leq \int_{\substack{\hat{\omega}(0)=\hat{\omega}(t)=x \\ |\hat{\omega}(s)-x|<|x|, \\ \text{for all } s \in [0, t]}} e^{-\int_0^t V(\hat{\omega}(s)) ds} \hat{\mathcal{D}}\hat{\omega} + \int_{\substack{\omega(0)=\omega(t)=x \\ |\hat{\omega}(s)-x| \geq |x|, \\ \text{for some } s \in [0, t]}} e^{-\int_0^t V(\hat{\omega}(s)) ds} \hat{\mathcal{D}}\hat{\omega} \\ &\leq K.e^{-tV(x)} + \frac{e^{-tV_0}(1 - e^{-th-j})}{|x|_{\mathfrak{p}}(1 - N\mathfrak{p}^{-1})}, \end{aligned}$$

where $|x|_{\mathfrak{p}} = N\mathfrak{p}^j$. So

$$\begin{aligned} \text{Tr } e^{-tH} &= \int_{\mathfrak{o}_{\mathfrak{p}}} k_t(x, x) dx + \int_{\mathfrak{k}_{\mathfrak{p}} \setminus \mathfrak{o}_{\mathfrak{p}}} k_t(x, x) dx \\ &\leq K + K \int e^{-tV(x)} dx + e^{-tV_0} \sum_{j \geq 1} (1 - e^{-th-j}) < \infty. \quad \square \end{aligned}$$

We use the notation $\mathcal{L}_r^2(\mathbb{Q}_{\mathfrak{p}})$ to denote the subspace of $L^2(\mathbb{Q}_{\mathfrak{p}})$ of functions $f(x)$ that depend only on the norm $|x|_{\mathfrak{p}}$ of x . We denote by $\mathcal{L}_0^2(\mathbb{Q}_{\mathfrak{p}})$ its orthogonal complement. We may take as a basis for $\mathcal{L}_r^2(\mathbb{Q}_{\mathfrak{p}})$ the functions

$$\phi_k(x) = \begin{cases} p^{-k/2}, & \text{if } |x|_{\mathfrak{p}} \leq p^k \\ 0, & \text{otherwise.} \end{cases}$$

Note that the subspaces \mathcal{L}_r^2 and \mathcal{L}_0^2 are invariant under V , \mathcal{F} and $\mathcal{H}_{\mathfrak{p}}$, and hence invariant under H . So we may analyse the operator H on each of these spaces separately. Let H_r and H_0 denote the restriction of H to \mathcal{L}_r^2 and \mathcal{L}_0^2 , respectively. We calculate the eigenfunction decomposition of H_0 in §7 and postpone to §8 the analysis of the radial component, which depends crucially on the functions v and h .

7. Orthogonal Component

For $N > 0$ let $\mathcal{S}^{(N)}$ be the p^{2N} -dimensional subspace of $L^2(\mathbb{Q}_p)$ of functions f with the following properties:

- (i) $f(x) = f(y)$ if $-\text{ord}_p(x-y) < -N$,
- (ii) $f(x) = 0$ if $-\text{ord}_p(x) \geq N$.

We denote $\mathcal{S}_r^{(N)} = \mathcal{L}_r^2 \cap \mathcal{S}^{(N)}$, $\mathcal{S}_0^{(N)} = \mathcal{L}_0^2 \cap \mathcal{S}^{(N)}$.

Then $\dim(\mathcal{S}_r^{(N)}) = 2N + 1$, $\dim(\mathcal{S}_0^{(N)}) = p^{2N} - 2N - 1$.

Let $\mathcal{S} = \bigcup_{N>0} \mathcal{S}^{(N)}$. Then \mathcal{S} is dense in $L^2(\mathbb{Q}_p)$. $\mathcal{S}^{(N)}$ is not invariant under H , but $\mathcal{S}_0^{(N)}$ is invariant under H_0 . Indeed with respect to a suitably chosen basis the restriction $H_0^{(N)}$ of H_0 to $\mathcal{S}_0^{(N)}$ will be given by a square matrix of size $p^{2N} - 2N - 1$. When we speak of $H_0^{(N)}$ as an operator on $\mathcal{L}_0^2(\mathbb{Q}_p)$ or $L^2(\mathbb{Q}_p)$, it is understood that we mean $H_0^{(N)} P_0^{(N)}$, where $P_0^{(N)}$ is the orthogonal projection onto $\mathcal{S}_0^{(N)}$. We intend to obtain H_0 and $e^{-t(\log p)H_0}$ as direct limits as $N \rightarrow \infty$ of the operators $H_0^{(N)}$ and $e^{-t(\log p)H_0^{(N)}}$. Recall that a p -adic number c with $-\text{ord}_p(c) = k$ can be written in p -adic notation

$$c = \dots c_{-3}c_{-2}c_{-1}c_0 \cdot c_1c_2 \dots c_k ,$$

where each c_i is one of the digits $\{0, 1, \dots, p-1\}$. The functions $f(c)$ in $\mathcal{S}^{(N)}$ are those that depend only on $\text{ord}_p(c) = k$ and on the digits c_i with $-N \leq i \leq N-1$. As a basis we may take $((b)^\wedge) = ((b_{-N} \dots b_{N-1})^\wedge)$, where

$$(b)^\wedge(c) = \begin{cases} 1 & \text{if } -\text{ord}_p(c) < N \text{ and } c_i = b_i \text{ for } -N \leq i < N, \\ 0 & \text{otherwise.} \end{cases}$$

We now define a family of functions $(\eta_{kl}^{(a)})$, which are parameterized by a pair of integers $k, l \in \mathbb{Z}$, $l \geq 0$, and a sequence $(a) = (a_{k-l} a_{k-l+1} \dots a_k)$ of digits

$a_i \in \{0, 1, \dots, p-1\}$, with $a_k \neq 0$, by $\eta_{kl}^{(a)}(b) = \phi_{k-l-1}(b-a)$, i.e.

$$\eta_{kl}^{(a)}(b) = \begin{cases} p^{(l+1-k)/2} & \text{if } |b|_p = p^k \text{ and } a_i = b_i \text{ for } k-l \leq i \leq k. \\ 0 & \text{otherwise.} \end{cases}$$

$\eta_{kl}^{(a)}$ lies in the space $\mathcal{S}^{(N)}$ if $-N \leq k-l \leq l \leq N-1$. It will be convenient to extend the definition of $\eta_{kl}^{(a)}$ to the case where $l = -1$ and $(a) = ()$ is the empty string, as follows:

$$\eta_{k(-1)}^{()}(b) = \begin{cases} \frac{p^{\frac{1-k}{2}}}{\sqrt{p-1}} & \text{if } |b|_p = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

Given a sequence $(a) = (a_{k-l} a_{k-l+1} \dots a_k)$ of digits $a_i \in \{0, \dots, p-1\}$ with $a_k \neq 0$ we define the sequences (a^0) , (a^1) and (\bar{a}) as follows:

$(a^0) = (0 a_{k-l+1} \dots a_k)$ and $(a^1) = (1 a_{k-l+1} \dots a_k)$ are obtained by replacing the first digit of (a) by 0 and 1, respectively. $(\bar{a}) = (a_{k-l+1} \dots a_k)$ is obtained by removing the first digit of (a) .

Finally we define the orthonormal functions $\psi_{kl}^{(a)}$ as follows:

$$\text{If } l \geq 1, \quad \psi_{kl}^{(a)} = \eta_{kl}^{(a)} - (\sqrt{p}+1)^{-1} (\eta_{kl}^{(a^0)} + \eta_{k(l-1)}^{(\bar{a})})$$

$$\text{If } l = 0, \quad \psi_{k0}^{(a)} = \eta_{k0}^{(a)} - (\sqrt{p-1}+1)^{-1} (\eta_{k0}^{(a^1)} + \eta_{k(-1)}^{()})$$

Notation. Since v and h are radially symmetric, we write $v(x) = v_k$ if $|x|_p = p^k$, and similarly for h_k .

Lemma 7.0. *The functions $(\psi_{kl}^{(a)})$ are a complete set of eigenfunctions for the operator H_0 , and indeed are simultaneous eigenfunctions for V and $\mathcal{F}^{-1}\mathcal{H}_p\mathcal{F}$, with eigenvalues v_k and h_{l+1-k} , respectively.*

Proof. Equivalently, we prove that $\psi_{kl}^{(a)}$ is localized in both position and momentum space, in the sense that

$$\psi_{kl}^{(a)}(b) = 0 \quad \text{if } |b|_p \neq p^k,$$

and

$$\mathcal{F}(\psi_{kl}^{(a)})(c) = 0 \quad \text{if } |c|_p \neq p^{l+1-k}.$$

The first condition follows easily from the definition of $\psi_{kl}^{(a)}$. For the second, we recall that

$$\eta_{kl}^{(a)}(b) = \phi_{k-l-1}(b-a) = \tau_{(a)}(\phi_{k-l-1})(b)$$

where $\tau_{(a)}$ is a shift operator. In the case $l \geq 1$, $\mathcal{F}(\eta_{kl}^{(a)})(c) = \chi(ac)\phi_{l+1-k}(c)$, where χ is the standard character on \mathbb{Q}_p , and

$$\begin{aligned} \mathcal{F}(\psi_{kl}^{(a)})(c) &= [\chi(ac) - (\sqrt{p} + 1)^{-1}\chi(a^0c)]\phi_{l+1-k}(c) - (\sqrt{p} + 1)^{-1}\chi(\bar{a}c)\phi_{l-k}(c) \\ &= [\chi(ac) - (\sqrt{p} + 1)^{-1}(\chi(a^0c) + \sqrt{p}\chi(\bar{a}c))]\phi_{l+1-k}(c) \\ &\quad + (\sqrt{p} + 1)^{-1}\chi(\bar{a}c)(\sqrt{p}\phi_{l+1-k}(c) - \phi_{l-k}(c)) \end{aligned}$$

The second term is clearly zero if $\text{ord}_p(c) \neq l+1-k$.

If $\text{ord}_p(c) > l+1-k$, then $\phi_{l+1-k}(c) = 0$, so RHS = 0.

If $\text{ord}_p(c) < l+1-k$, then $\chi(ac) = \chi(a_0c) = \chi(\bar{a}c)$, so

$$\text{RHS} = \chi(ac) [1 - (\sqrt{p} + 1)^{-1}(1 + \sqrt{p})] \phi_{l+1-k}(c) = 0$$

So $\mathcal{F}(\psi_{kl}^{(a)})(c) = 0$ if $\text{ord}_p(c) \neq l+1-k$, as claimed. The case $l = 0$ is similar.

Finally we show that the functions $\psi_{kl}^{(a)}$ span the whole of $\mathcal{L}_0^2(\mathbb{Q}_p)$. It is enough to show that they span each $\mathcal{S}_0^{(N)}$, since the union $\mathcal{S}_0^{(N)}$ of the $\mathcal{S}_0^{(N)}$'s is dense in $\mathcal{L}_0^2(\mathbb{Q}_p)$.

For a fixed N , k, l and (a) must satisfy the following relations:

- (1) $0 \leq l \leq 2N - 1$,
- (2) $-N + l \leq k \leq N - 1$,
- (3a) If $l = 0$, $a_k \in \{2, \dots, p-1\}$,
- (3b) If $l \geq 1$, $a_i \in \{0, \dots, p-1\}$ for $k-l \leq i \leq k$ and $a_k, a_{k-l} \neq 0$.

The total number of dimensions spanned by these $\psi_{kl}^{(a)}$'s is

$$(2N-1)(p-2) + \sum_{l=1}^{2N-1} (2N-l-1)(p-1)^2 p^{l-2} = p^{2N} - 2N - 1.$$

So they span the whole of $\mathcal{S}_0^{(N)}$. □

We are now ready to calculate the heat kernel of H_0 .

Theorem 7.1. If $|x|_p \neq |y|_p$, the kernel $k_t^0(x, y)$ of e^{-tH_0} is zero. Otherwise

$$k_t^0(x, y) = \frac{e^{-tv(x)}}{|x|_p} \left[-\left(\frac{1}{p-1}\right) e^{-th_{1-k}} + (p-1) \sum_{l=0}^{k-j-1} p^l e^{-th_{(l+1-k)}} - \frac{|x|_p}{|x-y|_p} e^{-th_{1-j}} \right],$$

where $k = -\text{ord}_p(x)$ and $j = -\text{ord}_p(x-y)$.

If the function $f_t = e^{-th}$ is locally integrable, this formula can be rewritten

$$k_t^0(x, y) = e^{-tv(x)} \left[\hat{f}_t(x-y) - \hat{f}_t(x) - \frac{e^{-th_{1-k}}}{(1-p^{-1})|x|_p} \right].$$

Proof. We have

$$\begin{aligned} k_t^0(x, y) &= \sum_{k,l,(a)} \langle x | \psi_{kl}^{(a)} \rangle \langle \psi_{kl}^{(a)} | e^{-tH_0} | \psi_{kl}^{(a)} \rangle \langle \psi_{kl}^{(a)} | y \rangle \\ &= \sum_{k,l,(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) e^{-t(v_k + h_{l+1-k})}. \end{aligned}$$

$\psi_{kl}^{(a)}(x)$ and $\psi_{kl}^{(a)}(y)$ can both be nonzero only if $\text{ord}_p(x) = \text{ord}_p(y) = -k$ and x & y agree with (a) at least in their first l digits. If $x \neq y$, these conditions reduce the above expression to a finite sum, and the formula for $k_t^0(x, y)$ follows from the following

Lemma 7.2. For $k = -\text{ord}_p(x) = -\text{ord}_p(y) > -\text{ord}_p(x-y) > -\infty$ we have

$$\sum_{(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = \begin{cases} (p-2/p-1)p^{1-k}, & \text{if } l=0 & \dots(1) \\ (p-1)p^{l-k}, & \text{if } 0 < l < -\text{ord}_p(x) + \text{ord}_p(x-y) & \dots(2) \\ -p^{l-k}, & \text{if } l = -\text{ord}_p(x) + \text{ord}_p(x-y) & \dots(3) \\ 0, & \text{if } l > -\text{ord}_p(x) + \text{ord}_p(x-y) & \dots(4) \end{cases}$$

For $k = -\text{ord}_p(x) = -\text{ord}_p(y) = -\text{ord}_p(x-y) > 0$

$$\sum_{(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = \begin{cases} \frac{-p^{1-k}}{p-1} & \text{if } l=0 & \dots(5) \\ 0 & \text{if } l \neq 0 & \dots(6) \end{cases}$$

Assuming lemma 7.2, we have (for $k \neq j$)

$$\begin{aligned} k_t^0(x, y) &= \left(\frac{p-2}{p-1}\right)p^{1-k}e^{-t(v_k+h_{1-k})} + (p-1) \sum_{l=1}^{k-j-1} p^{l-k}e^{-t(v_k+h_{l+1-k})} - p^{-j}e^{-t(v_k+h_{1-j})} \\ &= \frac{e^{-tv(x)}}{|x|_p} \left[-\left(\frac{1}{p-1}\right)e^{-th_{1-k}} + (p-1) \sum_{l=0}^{k-j-1} p^l e^{-th_{(l+1-k)}} - \frac{|x|_p}{|x-y|_p} e^{-th_{1-j}} \right]. \end{aligned}$$

If $k = j$,

$$k_t^0(x, y) = -\left(\frac{p^{1-k}}{p-1}\right)e^{-t(v_k+h_{1-k})} = \frac{e^{-tv(x)}}{|x|_p} \left[-\left(\frac{1}{p-1}\right)e^{-th_{(1-k)}} - e^{-th_{1-j}} \right],$$

as required. \square

Proof of Lemma 7.2. We have already remarked that $\psi_{kl}^{(a)}(x)$ and $\psi_{kl}^{(a)}(y)$ can both be nonzero only if $\text{ord}_p(x) = \text{ord}_p(y) = k$ and x & y agree with (a) at least in their first l digits, from which Cases (4) and (6) follow immediately.

Case 1. Since $k = -\text{ord}_p(x) = -\text{ord}_p(y) > -\text{ord}_p(x-y)$, the k^{th} digits of x and y are equal. If the k^{th} digit of x & y is 1, then for each of $(p-2)$ possible choices of (a) we have

$$\psi_{k0}^{(a)}(x) = \psi_{k0}^{(a)}(y) = \frac{-p^{\frac{1-k}{2}}}{\sqrt{p-1}},$$

from which
$$\sum_{(a)} \psi_{k0}^{(a)}(x) \psi_{k0}^{(a)}(y) = \left(\frac{p-2}{p-1}\right)p^{1-k}. \quad \dots(1)$$

If the k^{th} digit of x & y is not equal to 1, then

$$\psi_{k0}^{(a)}(x) = \psi_{k0}^{(a)}(y) = \begin{cases} \left(\frac{p + \sqrt{p-1} - 2}{\sqrt{p-1} + 1}\right) \frac{p^{\frac{1-k}{2}}}{\sqrt{p-1}}, & \text{for } (a) = (k^{\text{th}} \text{ digit of } x \& y), \\ \left(\frac{-1}{\sqrt{p-1} + 1}\right) \frac{p^{\frac{1-k}{2}}}{\sqrt{p-1}}, & \text{for the } (p-3) \text{ other choices of } (a). \end{cases}$$

Squaring and adding gives (1).

Case 2. If the $(k-l)$ th digit of x & y is zero, then for each of the $(p-1)$ possible choices of a_{k-l}

$$\psi_{kl}^{(a)}(x) = \psi_{kl}^{(a)}(y) = \frac{-(p^{\frac{l+1-k}{2}} + p^{\frac{l-k}{2}})}{\sqrt{p} + 1} = -p^{\frac{l-k}{2}},$$

giving
$$\sum_{(a)} \psi_{k0}^{(a)}(x) \psi_{k0}^{(a)}(y) = (p-1)p^{l-k}. \quad \dots(2)$$

If the $(k-l)$ th digit is not zero,

$$\psi_{kl}^{(a)}(x) = \psi_{kl}^{(a)}(y) = \begin{cases} \left(\frac{p + \sqrt{p} - 1}{\sqrt{p} + 1}\right)p^{\frac{l-k}{2}}, & \text{if } a_{k-l} = (k-l)\text{th digit of } x \text{ \& } y, \\ \left(\frac{-1}{\sqrt{p} + 1}\right)p^{\frac{l-k}{2}}, & \text{for the } (p-2) \text{ other choices of } a_{k-l}. \end{cases}$$

Squaring and adding gives (2).

Case 3. Since $l = -\text{ord}_p(x) + \text{ord}_p(x-y)$, the $(k-l)$ th digits of x & y are not equal. Let us first consider the case where one of these digits (w.l.o.g. that of y) is equal to zero. Then

$$\psi_{kl}^{(a)}(x) = \begin{cases} \left(\frac{p + \sqrt{p} - 1}{\sqrt{p} + 1}\right)p^{\frac{l-k}{2}}, & \text{if } a_{k-l} = (k-l)\text{th digit of } x, \\ \left(\frac{-1}{\sqrt{p} + 1}\right)p^{\frac{l-k}{2}}, & \text{for the } (p-2) \text{ other choices of } a_{k-l}, \end{cases}$$

$$\psi_{kl}^{(a)}(y) = -p^{\frac{l-k}{2}}, \quad \text{for all choices of } a_{k-l}.$$

So
$$\sum_{(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = -\left(\frac{p + \sqrt{p} - 1 - (p-2)}{\sqrt{p} + 1}\right)p^{l-k} = -p^{l-k} \quad \dots(3)$$

In the contrary case, where the $(k-l)$ th digits of x & y are both nonzero, then for $a_{k-l} = (k-l)$ th digit of x we have

$$\psi_{kl}^{(a)}(x) = \left(\frac{p + \sqrt{p} - 1}{\sqrt{p} + 1}\right)p^{\frac{l-k}{2}}, \quad \psi_{kl}^{(a)}(y) = \frac{-p^{\frac{l-k}{2}}}{\sqrt{p} + 1}.$$

For $(a) = (k-l)^{\text{th}}$ digit of y we have

$$\psi_{kl}^{(a)}(y) = \left(\frac{p + \sqrt{p} - 1}{\sqrt{p} + 1}\right) p^{\frac{l-k}{2}}, \quad \psi_{kl}^{(a)}(x) = \frac{-p^{\frac{l-k}{2}}}{\sqrt{p} + 1}.$$

For the other $(p-3)$ choices of (a) , $\psi_{kl}^{(a)}(x) = \psi_{kl}^{(a)}(y) = \frac{-p^{\frac{l-k}{2}}}{\sqrt{p} + 1}$.

$$\sum_{(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = \frac{p^{l-k}}{(\sqrt{p} + 1)^2} (-2(p + \sqrt{p} - 1) + (p - 3)) = -p^{l-k} \quad \dots (3)$$

Case 5. The k th digits of x & y are different. Assume first that one of them (w.l.o.g. that of y) is equal to 1. Then

$$\psi_{kl}^{(a)}(x) = \begin{cases} \left(\frac{p + \sqrt{p-1} - 2}{\sqrt{p-1} + 1}\right) \frac{p^{\frac{1-k}{2}}}{\sqrt{p-1}}, & \text{if } (a) = (k^{\text{th}} \text{ digit of } x), \\ \frac{-1}{(\sqrt{p-1} + 1)} \frac{p^{\frac{1-k}{2}}}{\sqrt{p-1}}, & \text{for the } (p-3) \text{ other choices of } (a), \end{cases}$$

$$\psi_{kl}^{(a)}(y) = \frac{-p^{\frac{1-k}{2}}}{\sqrt{p-1}}, \quad \text{for all choices of } (a).$$

$$\sum_{(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = \frac{p^{1-k}}{(p-1)(\sqrt{p-1} + 1)} (p + \sqrt{p-1} - 2 - (p-3)) = \frac{p^{1-k}}{p-1} \quad \dots (5)$$

Now assume the k th digits of x & y are not equal to one.

Then for $(a) = (k^{\text{th}} \text{ digit of } x)$ we have

$$\psi_{kl}^{(a)}(x) = \frac{p + \sqrt{p-1} - 2}{\sqrt{p-1}(\sqrt{p-1} + 1)} p^{\frac{1-k}{2}}, \quad \psi_{kl}^{(a)}(y) = \frac{-1}{\sqrt{p-1}(\sqrt{p-1} + 1)} p^{\frac{1-k}{2}},$$

so
$$\psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = \frac{p^{1-k}(-p - \sqrt{p-1} + 2)}{(p-1)(\sqrt{p-1} + 1)^2}.$$

The same expression holds for $(a) = (k^{\text{th}} \text{ digit of } y)$.

For the other $(p-4)$ choices of (a) ,

$$\psi_{kl}^{(a)}(x) = \psi_{kl}^{(a)}(y) = \frac{p^{\frac{1-k}{2}}}{\sqrt{p-1}(\sqrt{p-1} + 1)}.$$

These combine to give
$$\sum_{(a)} \psi_{kl}^{(a)}(x) \psi_{kl}^{(a)}(y) = \frac{p^{1-k}}{p-1}, \quad \dots (5)$$

which completes the proof of Lemma 7.2. \square

Example 1. $h(x) = v(x) = |x|_p^2$. Then

$$k_t^0(x, y) = \frac{e^{-t|x|_p^2}}{|x|_p} \left[\hat{f}_t(x-y) - \hat{f}_t(x) - \frac{e^{-tp^2/|x|_p}}{(1-p^{-1})|x|_p} \right],$$

where f_t is the Fourier transform of e^{-th} .

Example 2. $h(x) = v(x) = -\text{ord}_p(x)$.

Then the eigenvalue of H_0 for $\psi_{kl}^{(a)}$ is $v_k + h_{l+1-k} = k + l + 1 - k = l + 1$.

We calculate the heat kernel of H_0 . If $|x|_p \neq |y|_p$, the kernel $k_t^0(x, y)$ of $e^{-t(\log p)H_0}$ is zero. Otherwise, from the previous theorem, we have

$$\begin{aligned} k_t^0(x, y) &= \left(\frac{p-2}{p-1}\right)p^{1-k-t} + (p-1)p^{-k-t} \sum_{l=1}^{-\text{ord}_p(x)+\text{ord}_p(x-y)-1} p^{l(1-t)} - p^{-k-t} \left(\frac{p^{-\text{ord}_p(x)}}{p^{-\text{ord}_p(x-y)}}\right)^{1-t} \\ &= \frac{-p^{-t}}{(p-1)|x|_p} + \left(\frac{p-1}{1-p^{1-t}}\right) \frac{p^{-t}}{|x|_p} \left[1 - \left(\frac{|x|_p}{|x-y|_p}\right)^{1-t}\right] - \frac{p^{-t}}{|x|_p} \left(\frac{|x|_p}{|x-y|_p}\right)^{1-t} \\ &= \frac{-p^{-t}}{(p-1)|x|_p} + \frac{(1-p^{-1})}{|x|_p} \left(\frac{p^{1-t}}{1-p^{1-t}}\right) - \frac{(1-p^{-t})}{|x|_p} \left(\frac{p^{1-t}}{1-p^{1-t}}\right) \left(\frac{|x|_p}{|x-y|_p}\right)^{1-t}, \\ &= \frac{-p^{-t}}{(1-p^{-1})|x|_p} + \frac{(1-p^{-t})}{|x|_p} \left(\frac{p^{1-t}}{1-p^{1-t}}\right) \left[1 - \left(\frac{|x|_p}{|x-y|_p}\right)^{1-t}\right]. \end{aligned}$$

8. Radial Component

Suppose Ω is an eigenfunction for H_r with eigenvalue x . Since Ω must be radially symmetric, we write $\omega(y) = \omega_k$ if $|y|_p = p^k$ in the same way as v_k and h_k . Using the following formulas for the Fourier transforms of ω and $v\omega$:

$$\begin{aligned}\widehat{\omega}_l &= (1 - p^{-1}) \sum_{k \leq -l} p^k \omega_k - p^{-l} \omega_{-l+1}, \\ \widehat{v\omega}_l &= (1 - p^{-1}) \sum_{k \leq -l} p^k v_k \omega_k - p^{-l} v_{-l+1} \omega_{-l+1},\end{aligned}$$

and putting $j = -l$ in the relation $h_l \widehat{\omega}_l + \widehat{v\omega}_l = x \widehat{\omega}_l$, we obtain the formula

$$\omega_{j+1}(h_{-j} + v_{j+1} - x) = (1 - p^{-1}) \sum_{k \leq j} p^{k-j} (h_{-j} + v_k - x) \omega_k.$$

If we assume $x \notin \text{Spec } H_0 = \{h_{-j} + v_{j+1} : j \in \mathbf{Z}\}$, we may use the above equation as a recursive formula for ω_j .

$$\text{Let} \quad S_j = \sum_{k \leq j} p^k \omega_k, \quad T_j = \sum_{k \leq j} p^k v_k \omega_k.$$

$$\text{Then} \quad (h_{-j} + v_{j+1} - x) \begin{bmatrix} T_{j+1} \\ S_{j+1} \end{bmatrix} = \begin{bmatrix} h_{-j} + pv_{j+1} - x & (p-1)(h_{-j} - x)v_{j+1} \\ p-1 & p(h_{-j} - x) + v_{j+1} \end{bmatrix} \begin{bmatrix} T_j \\ S_j \end{bmatrix}$$

Lemma 8.0. *If $x \in \text{Spec } H_r \setminus \text{Spec } H_0$ then*

- (i) $\lim_{j \rightarrow \infty} (T_j - xS_j) = 0$,
- (ii) $\lim_{l \rightarrow \infty} \omega_{-l} = \omega_-$ is finite & nonzero and the convergence is $\sim b^{4l}$.
- (iii) If $0 \leq x \leq 2$, $\omega_- > 0$, then $(1 - 2b^{2+4l})\omega_- \leq \omega_{-l} \leq \omega_-$ for $l \geq 0$.

Proof. Since Ω is in the domain of V , $V\Omega \in L^2(\mathbb{Q}_p)$. So $|v_j\omega_j| < Cp^{-j/2}$ for some $C \in \mathbf{R}$. Hence S_j is bounded and $\lim_{j \rightarrow \infty} S_j$ exists. Moreover

$$\begin{aligned} \lim_{j \rightarrow \infty} T_j &= (1 - p^{-1})^{-1} \lim_{y \rightarrow 0} \mathcal{F}(V\Omega)(y) \\ &= (1 - p^{-1})^{-1} \lim_{y \rightarrow 0} (x\mathcal{F}(\Omega)(y) - h(y)\mathcal{F}(\Omega)(y)) = x \lim_{j \rightarrow \infty} S_j, \end{aligned}$$

proving (i). For (ii), note that

$$\omega_{j+1} = (1 - p^{-1}) \sum_{k \leq j} p^{k-j} \left(1 + \frac{v_k - v_{j+1}}{h_{-j} + v_{j+1} - x} \right) \omega_k.$$

$$\begin{aligned} \text{So } \omega_{j+1} - \omega_j &= (1 - p^{-1}) \left(\frac{v_j - v_{j+1}}{h_{-j} + v_{j+1} - x} \right) \omega_j \\ &\quad + (1 - p^{-1}) \sum_{k \leq j-1} p^{k-j} \left(\frac{v_k - v_{j+1}}{h_{-j} + v_{j+1} - x} - \frac{v_k - v_j}{h_{-j+1} + v_j - x} \right) \omega_k \end{aligned}$$

Now for j sufficiently small $|\omega_k| \leq p^{-k/2}$ and

$$\left| \frac{v_k - v_{j+1}}{h_{-j} + v_{j+1} - x} \right| + \left| \frac{v_k - v_j}{h_{-j+1} + v_j - x} \right| < \frac{p^{4j+2}}{1 - p^{-1}}.$$

So $|\omega_{j+1} - \omega_j| < 4p^{4j+2}$. Thus $\{\omega_j\}$ is a Cauchy sequence, and approaches a limit as $j \rightarrow -\infty$. Moreover the limit is nonzero, for otherwise choose small j such that $\omega_k < (1 - p^j)\omega_{j+1}$, for $k \leq j$. Then

$$\omega_{j+1} < \sum_{k \leq j} p^{k-j} (1 - p^{-1} + p^{4j+2})(1 - p^j)\omega_{j+1} < \omega_{j+1},$$

yielding a contradiction. To prove (iii), note: for $0 \leq x \leq 2$, $k \leq j \leq -2$,

$$\left| \frac{v_k - v_{j+1}}{h_{-j} + v_{j+1} - x} \right| < \frac{p^{2j+2}}{p^{-2j} - 2} \leq \frac{p^{4j+2}}{1 - 2p^{-4}}.$$

$$\text{So for } l \geq 1, \quad \frac{\omega_{-l}}{\omega_-} \geq \prod_{k < -l} \left(1 - \frac{p^{4k+2}}{1 - p^{-4}} \right) \geq 1 - 2b^{2+4l}.$$

$$\text{For } l = 0, \quad \frac{\omega_0}{\omega_-} \geq \left(1 - \frac{v_0}{h_1 + v_0 - 2} \right) \prod_{j \leq -2} \left(1 - \frac{p^{4j+2}}{1 - p^{-4}} \right) \geq 1 - 2b^2. \quad \square$$

From now on we choose to normalize Ω so that $\lim_{j \rightarrow -\infty} \omega_j = 1$. Let

$$P_j(b, x) = \begin{cases} b^j T_j R_j(b, x), & j \geq 0 \\ T_j R_j(b, x), & j < 0 \end{cases} \quad Q_j(b, x) = \begin{cases} S_j R_j(b, x), & j \geq 0 \\ b^j S_j R_j(b, x), & j < 0 \end{cases}$$

where $R_0 = 1$ and

$$R_{j+1} = \begin{cases} \left(1 + \frac{h_{-j} - x}{v_{j+1}}\right) R_j, & j \geq 0, \\ \left(1 + \frac{v_{j+1} - x}{h_{-j}}\right) R_j, & j \leq -1. \end{cases}$$

Then

$$\begin{bmatrix} P_{j+1} \\ Q_{j+1} \end{bmatrix} = M_j \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \quad \text{with} \quad M_j = [I + A_j + A'_j],$$

where for $j \geq 0$

$$A_j = \frac{1}{v_{j+1}} \begin{bmatrix} -xb & -xb^j v_{j+1}(1-b) \\ b^{-j-1}(1-b) & -xb^{-1} \end{bmatrix}, \quad A'_j = \frac{1}{v_{j+1}} \begin{bmatrix} bh_{-j} & b^j v_{j+1} h_{-j}(1-b) \\ 0 & b^{-1} h_{-j} \end{bmatrix}$$

while for $j < 0$ we have

$$A_j = \frac{1}{h_{-j}} \begin{bmatrix} -x & -xb^{-j-1} v_{j+1}(1-b) \\ b^j(1-b) & -x \end{bmatrix}, \quad A'_j = \frac{1}{h_{-j}} \begin{bmatrix} b^{-1} v_{j+1} & b^{-j-1} h_{-j} v_{j+1}(1-b) \\ 0 & b v_{j+1} \end{bmatrix}$$

In the case where $h_j = v_j = p^{2j}$ for $j \geq 0$,

$$A_j = \begin{bmatrix} -xb^{2j+3} & -xb^j(1-b) \\ b^{j+1}(1-b) & -xb^{2j+1} \end{bmatrix}, \quad A'_j = \begin{bmatrix} b^{4j+3} & b^{3j}(1-b) \\ 0 & b^{4j+1} \end{bmatrix}$$

$$\text{So} \quad \begin{bmatrix} P_{N+1} \\ Q_{N+1} \end{bmatrix} = \mathcal{U}_N \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix}, \quad \mathcal{U}_N = \prod_{j=0}^N M_j,$$

where the matrices are multiplied from right to left in order of increasing j .

If $h_j = v_j = p^{2j}$ for $j < 0$, we have

$$A_j = \begin{bmatrix} -xb^{-2j} & -xb^{-5j-3}(1-b) \\ b^{-j}(1-b) & -xb^{-2j} \end{bmatrix}, \quad A'_j = \begin{bmatrix} b^{-4j-3} & b^{-3j-3}(1-b) \\ 0 & b^{-4j-1} \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} = \mathcal{U}_{-N} \begin{bmatrix} P_{-N} \\ Q_{-N} \end{bmatrix}, \quad \mathcal{U}_{-N} = \prod_{j=-N}^{-1} M_j,$$

where this time the matrices are multiplied from left to right in order of decreasing j .

Lemma 8.1. *The matrices $\{\mathcal{U}_N\}$ and $\{\mathcal{U}_{-N}\}$ each form a Cauchy sequence and hence approach limits \mathcal{U}_+ and \mathcal{U}_- (resp.) as $N \rightarrow \infty$. Moreover, \mathcal{U}_+ and \mathcal{U}_- are continuous functions of x . If $0 \leq x \leq 2$, then $\mathcal{U}_N, \mathcal{U}_+, \mathcal{U}_-$ are invertible.*

Proof. We treat only \mathcal{U}_+ (the proof for \mathcal{U}_- is similar). For $j \geq 0$,

$$\|A_j\|_{\text{HS}} < |x|b^j + b^{j+1}, \quad \|A'_j\|_{\text{HS}} < b^{j+1}.$$

Replacing each term by a bound on its Hilbert-Schmidt norm, and noting that the convergence of the infinite product

$$\prod_{j=0}^{\infty} (1 + |x|b^j + 2b^{j+1}) \quad \text{follows from that of the sum} \quad \sum_{j=0}^{\infty} (|x|b^j + 2b^{j+1}) = \frac{|x| + 2b}{1 - b},$$

we see that the infinite product converges absolutely. For the derivative,

$$\left\| \frac{d\mathcal{U}_N}{dx} \right\|_{\text{HS}} \leq \sum_{k=0}^N b^k \prod_{j \neq k} (1 + |x|b^j + 2b^{j+1}) \leq 2 \prod_{j=0}^N (1 + |x|b^j + 2b^{j+1}).$$

This sum is uniformly bounded as $N \rightarrow \infty$, so by the dominated convergence theorem, \mathcal{U} is a continuous function of x . Finally, if $0 \leq x \leq 2$, $\|M_0 - I\| < 1$ and $\|M_j - I\| < b^{j-1}$ for all $j \geq 1$. So $\det \mathcal{U}_N \neq 0$ for any N and $\det \mathcal{U}_+ \neq 0$. \square

Let $\mathcal{U}_b^{ij}(x)$ denote the $(i, j)^{\text{th}}$ entry of $\mathcal{U} = \mathcal{U}_+ \cdot \mathcal{U}_-$.

Lemma 8.2. *Suppose $x \leq 2$. Then $x \in \text{Spec } H_r \iff \mathcal{U}_b^{12}(x) = 0$.*

Moreover every eigenvalue in this region is simple.

Proof. If $x \in \text{Spec } H_r$, $x \leq 2$, then $\lim_{j \rightarrow \infty} (T_j - xS_j) = 0$ by lemma 8.0, so

$$\mathcal{U}_b^{12}(x) = \lim_{N \rightarrow \infty} P_N = \lim_{N \rightarrow \infty} \left(\frac{P_N}{b^N Q_N} \right) \cdot \lim_{N \rightarrow \infty} Q_N \cdot \lim_{N \rightarrow \infty} b^N = x \cdot \mathcal{U}_b^{22}(x) \cdot 0 = 0.$$

Conversely, suppose $\mathcal{U}_b^{12}(x) = 0$ and $x \leq 2$. In fact we may assume that $0 < x \leq 2$, because if $x \leq 0$ then each matrix M_j has all positive entries, so $\mathcal{U}_b^{12}(x) > 0$.

\mathcal{U} is invertible by lemma 6.1, so $\mathcal{U}_b^{22}(x) \neq 0$. Moreover $\|M_N^{-1} - I\| < b^{N-1}$, so $\|P_N\| < K b^N$ for some $K \in \mathbf{R}$ (N sufficiently large). Hence

$$\lim_{N \rightarrow \infty} T_N = \left(\lim_{N \rightarrow \infty} R_N \right)^{-1} \lim_{N \rightarrow \infty} (b^{-N} P_N) \quad \text{is finite.}$$

Thus Ω and $V\Omega$ are L^2 functions, so Ω is in the domain of H and is therefore an eigenfunction of H with eigenvalue x . Let Ω be a corresponding eigenfunction, normalized so that $\Omega(0) = 1$.

$$R_- = \lim_{N \rightarrow \infty} R_{-N} = \prod_{j \leq -1} \left(1 + \frac{v_{j+1} - x}{h_{-j}} \right) \quad \text{is finite,}$$

and since $\omega_N \sim 1 + \mathcal{O}(p^{-4N})$ as $N \rightarrow \infty$, we have $\lim_{N \rightarrow \infty} T_{-N} = 0$ and $\lim_{N \rightarrow \infty} b^{-N} S_{-N} = 1$. So

$$\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} = \left(\lim_{N \rightarrow \infty} R_{-N} \right) \left(\lim_{N \rightarrow \infty} \mathcal{U}_{-N} \right) \lim_{N \rightarrow \infty} \begin{bmatrix} P_{-N} \\ Q_{-N} \end{bmatrix} = R_- \mathcal{U}_- \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence Ω is uniquely determined by the following relations ($N \geq 0$):

$$\begin{bmatrix} P_N \\ Q_N \end{bmatrix} = \mathcal{U}_N \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix}, \quad \begin{bmatrix} P_{-N} \\ Q_{-N} \end{bmatrix} = \mathcal{U}_{-N}^{-1} \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix}. \quad \square$$

Lemma 8.3. *The lowest zero x of \mathcal{U}_b^{12} is in the range $2 - 3b < x < 2$.*

Proof. We have already remarked that $\mathcal{U}_b^{12}(x) > 0$ for $x \leq 0$. Suppose $0 < x \leq 2$. Explicit calculations reveal that

$$\begin{aligned} \mathcal{U} &= \prod_{j \geq 5} M_j \cdot \prod_{j=-3}^4 M_j \cdot \prod_{j \leq -4} M_j \\ &= \begin{bmatrix} 1 + \mathcal{O}(b^5) & \mathcal{O}(b^5) \\ \mathcal{O}(b^5) & 1 + \mathcal{O}(b^5) \end{bmatrix} \begin{bmatrix} \mathcal{O}(3) & f_b(x) + \mathcal{O}(b^3) \\ \mathcal{O}(2) & \mathcal{O}(3) \end{bmatrix} \begin{bmatrix} 1 + \mathcal{O}(b^6) & \mathcal{O}(b^6) \\ \mathcal{O}(b^4) & 1 + \mathcal{O}(b^6) \end{bmatrix}, \end{aligned}$$

where $f_b(x) = (2 - 2b + 4b^3) + x(-1 - 4b^2 + 2b^3 - 6b^4) + x^2(2b^2 + 4b^4) - 4x^3b^4$.

From this we can show that $|\mathcal{U}_b^{12}(x) - f_b(x)| < b^2$. The result follows, noting that $f'_b(x) < 0$, $f_b(2 - 3b) \geq b^2$, and $f_b(2) \leq -b^2$.

Proof of Theorem 6.1.

By lemma 7.0, the spectrum of H_0 is bounded below by $p^2 + 1$. By lemmas 8.2 and 8.3, the lowest eigenvalue x of H is in the range $2 - 3b < x < 2$. Let Ω be the corresponding eigenfunction, normalized so that $\Omega(0) = 1$. Now H commutes with the Fourier transform \mathcal{F} , so $\mathcal{F}\Omega = \lambda\Omega$ for some λ with $\lambda^4 = 1$.

$$\text{But} \quad \int_{\mathfrak{o}_{\mathfrak{p}}} \hat{\Omega}(y) dy = \int_{\mathfrak{o}_{\mathfrak{p}}} \Omega(y) dy > 0, \quad \text{so} \quad \lambda = 1.$$

Since $\omega_j > 0$ for $j \leq 0$ and $\omega_k \geq \omega_j$ for $k < j$, it follows that $\Omega = \hat{\Omega} \geq 0$. \square

Proof of Theorem 6.2.

We use lemma 8.0, part (iii). For $l \geq 0$, $\omega_{-l} \geq 1 - 2b^{2+4l}$, so

$$\|\Omega_{\mathfrak{p}}\|_{L^2} \geq \int_{\mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(y)^2 dy \geq (1-b) \sum_{l \geq 0} (1 - 2b^{2+4l})^2 p^{-l} > 1 - 4b^2.$$

For $j \geq 1$, $\omega_j = \hat{\omega}_j = (1-b) \sum_{\nu \leq -j} p^{\nu} \omega_{\nu} - p^{-j} \omega_{-j+1} \leq b^j - b^j(1 - 2b^{4j-2}) \leq 2b^{5j-2}$.

So

$$\int_{k_{\mathfrak{p}} \setminus \mathfrak{o}_{\mathfrak{p}}} \Omega_{\mathfrak{p}}(y)^2 dy \leq (1-b) \sum_{j \geq 1} (2b^{5j-2})^2 p^j < 4b^5.$$

Thus, letting $\tilde{\Omega}_{\mathfrak{p}} = \Omega_{\mathfrak{p}} / \|\Omega_{\mathfrak{p}}\|_{L^2}$,

$$\sum_{\mathfrak{p} \in \mathfrak{P}} \int_{k_{\mathfrak{p}} \setminus \mathfrak{o}_{\mathfrak{p}}} \tilde{\Omega}_{\mathfrak{p}}(y)^2 dy < \sum_{\mathfrak{p} \in \mathfrak{P}} \frac{4p^{-5}}{1 - 4p^{-2}} < \infty,$$

completing the proof of the theorem. \square

BIBLIOGRAPHY

- [BP88] Barlow, M.T. & Perkins, E.A., *Brownian motion on the Sierpinski gasket*, Prob. Th. and Related Fields **79** (1988), 543–624.
- [Bin71] Bingham, M. S., *Stochastic Processes with independent increments taking values in an abelian group*, Proc. London Math Soc. **6** (1971), 507–530.
- [BF93] Brekke, L. & Freund, P.G.O., *p-adic Numbers in Physics*, Physical Reports **233 #1** (1993), 1–66.
- [BO89] Brekke, L. & Olsen. M., University of Chicago preprint EFI-89-23 (1989).
- [CB92] Connes, A. & Bost, J.-B., *Produits eulériens et facteurs de type III*, C. R. Acad. Sci. Paris **315 #3, Série I** (1992), 279–84.
- [FO88] Freund, P.G.O & Olsen, M., *p-adic Dynamical Systems*, Nuclear Physics B **297** (1988), 86–102.
- [FW87] Freund, P.G.O. & Witten, E., *Adelic string amplitudes*, Phys. Lett. B **199 #2** (1987), 191–94.
- [Gel84] Gelbart, S., *An elementary introduction to the Langlands program*, Bull. Am. Math. Soc. **10 #2** (1984), 177–219.
- [Hen97] Hensel, K., see Abrégé d'histoire des mathématiques - Chap V, §V, I (Dieudonné, J., ed.), Hermann, Paris, 1978.
- [Hey77] Heyer, H., *Probability Measures on Locally Compact Groups*, Springer, Berlin, 1977.
- [HK85] Huberman, B.A. & Kerszberg, M., *Ultradiffusion: the relaxation of hierarchical systems*, J. Phys. A **18** (1985), L331–36.
- [Jul90] Julia, B., *Statistical theory of numbers, Number Theory and Physics*, Springer, Berlin, 1990.
- [Ko93] Kochubei, A.N., *A Schrödinger-type equation over the field of p-adic numbers*, J. Math. Phys. **34 #8** (1993), 3420–28.
- [Kr44] Krasner, M., C. R. Acad. Sci. #219, tome II, 433.
- [KT85] Kirkpatrick, S., & Toulouse, G., J. Phys (Paris) #46 (1985), 1277.
- [Lin90] Lindstrøm, T., *Brownian Motion on Nested Fractals*, Memoirs of the AMS **83 #420** (1990).
- [Me89] Meurice, Y., *Quantum Mechanics with p-adic Numbers*, Int.J.of Mod.Phys.A **4 #19** (1989), 5133–47.
- [MP84] Mézard, M., Parisi, G., Sourlas, N., Toulouse, G. & Virasoro, M., *Nature of the Spin-Glass Phase*, Phys. Rev. Lett. **52 #13** (1984), 1156–59; J. Phys. (Paris) #45, 843.
- [OS85] Ogielski, A.T. & Stein, D.L., *Dynamics on Ultrametric Spaces*, Phys.Rev.Lett. **55 #15** (1985), 1634–37
- [Par93] Parisi, G., *On the Branching Structure of the Tree of States in Spin Glasses*, J. Stat. Phys. **72 #5/6** (1993), 857–878.
- [Par88] Parisi, G., *On p-adic functional integrals*, Mod. Phys. Lett. **3 #6** (1988), 639–643.
- [Par80] Parisi, G., *Magnetic properties of spin glasses in a new mean field theory*, J. Phys. A **13** (1980), 1887–95.
- [PV85] Parga, N., & Virasoro, M.A., J. Phys (Paris) (1986).
- [Rao87] Rao, M.M., *Measure Theory and Integration*, John Wiley & Sons, New York, 1987.
- [RTV] Rammal, R., Toulouse, G. & Virasoro, M.A., Rev. Mod. Phys. #58 (1986), 765.
- [RTVW] Ruelle, Ph., Thiran, E., Versteegen, D. & Weyers, J., *Quantum mechanics on p-adic fields*, J. Math. Phys. **30 #12** (1989), 2854–74.
- [Sh84] Shiriyayev, A.N., *Probability*, Springer, New York, 1984.

- [Si79] Simon, B., *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.
- [Sp90] Spector, D., *Supersymmetry and the Möbius inversion function*, *Comm.Math.Phys.* **127** (1990), 239–52
- [St85] Stein, D., *Proc. Natl. Acad. Sci. U.S.A.* #82 (1985), 3670.
- [Tai75] Taibleson, M.H., *Fourier Analysis on Local Fields*, Princeton University Press, 1975.
- [Ta50] Tate, J.T., *Fourier Analysis in Number Fields and Hecke's Zeta-Functions*, Algebraic Number Theory (Cassels, J.W.S. & Fröhlich, A., eds.), Academic Press, London, 1967.
- [Tor71] Tortrat, A., *Calcul des probabilités et introduction aux processus aléatoires*, Masson et Cie, Paris, 1971.
- [Viv92] Vivaldi, F., *Geometry of linear maps over finite fields*, *Nonlinearity* **5** (1992), 133–47.
- [VH92] Vivaldi, F. & Hatjispyros, S., *Galois theory of periodic orbits of rational maps.*, *Nonlinearity* **5** (1992), 961–78.
- [V188] Vladimirov, V.S., *Generalized functions over the field of p -adic numbers*, *Russian Math. Surveys* **43** #5 (1988), 19–64.
- [VV89] Vladimirov, V.S. & Volovich, I.V., *p -adic quantum mechanics*, *Comm.Math.Phys.* **123** (1989), 659–76.
- [We67] Weil, A., *Basic Number Theory*, Springer, Berlin, 1967.
- [Zab89] Zabrodin, A.V., *Non-Archimedean Strings & Bruhat-Tits Trees*, *Commun. Math. Phys.* **123** (1989), 463–83.
- [Zel93] Zelenov, E.I., *p -adic Heisenberg Group and Maslov Index*, *Commun. Math. Phys.* **155** #3 (1993), 489–502.