NON-LINEAR LARGE DEFLECTION BOUNDARY VALUE PROBLEMS OF RECTANGULAR PLATES

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Cambridge, Massachusetts
October 9, 1944

Professor George W. Swett
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Dear Sir:

I hereby submit a thesis entitled, "Non-Linear Large Deflection Boundary Value Problems of Rectangular Plates", in partial fulfillment of the requirements for the degree of Doctor of Science in the Department of Aeronautical Engineering.

Respectfully,

Chi-Teh Wang

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ABSTRACT

This thesis presents a theoretical analysis of an initially flat, rectangular plate with large deflections under either normal pressure or combined normal pressure and side thrust. As small deflections of a flat plate are governed by a single linear equation, large deflections introduce nonlinear terms into the conditions of equilibrium and are governed by two fourth-order second-degree partial differential equations. These so-called von Karman's equations are studied in this thesis by the finite differences approximation. The differences equations are solved by two methods, namely, the method of successive approximations and the relaxation method. Neither of these methods is new, but their applications to non-linear problems require new techniques.

The problem of a uniformly loaded square plate with boundary conditions which approximate the riveted sheet-stringer panels is solved by the method of successive approximations. The theoretical center deflections show good agreement with the recent experimental results obtained at Calofornia Institute of Technology when the deflections are of the order of the plate thickness. This perhaps suggests the range in which these von Karman's equations are to be applied.

Other problems of thin plates with large deflections are discussed from the point of view of an aeronautical engineer.

The boundary conditions which approximate the various cases are formulated and the methods for solving these problems are outlined.

Since the method presented in this thesis is general, it may be applied to solve the bending and the combined bending and buckling problems with practically any boundary conditions, and the results may be obtained to any degree of accuracy required. Furthermore, the same method may be applied to solve the membrane theory of the plate which applies when the deflection is very large in comparison with the thickness of the plate.

NOMENCLATURE

a, b	=	length and width of the plate, respectively
h	=	thickness of the plate
x, y, z	=	coordinates of a point in the plate
u, v	=	horizontal displacements in directions x, y of points in the middle surface; their non-dimensional forms are ua/h^2, va/h^2, respectively
w	=	deflection of middle surface out of its initial plane; its non-dimensional form is w/h
p	=	normal load on plate per unit area; its non-dimensional form is pa^4/Eh^4
E, μ	=	elastic constants, Young's modulus and Poisson's ratio
D	=	$Eh^3/12(1-\mu^2)$, flexural rigidity of the plate
∇^2	=	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^4	=	$\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
σ_x', σ_y', τ_{xy}'	=	membrane stresses in middle surface; their non-dimensional forms are $\sigma_x' a^2/Eh^2$, $\sigma_y' a^2/Eh^2$, and $\tau_{xy}' a^2/Eh^2$, respectively
σ_x'', σ_y'', τ_{xy}''	=	extreme-fiber bending and shearing stresses; their non-dimensional forms are $\sigma_x'' a^2/Eh^2$, $\sigma_y'' a^2/Eh^2$, and $\tau_{xy}'' a^2/Eh^2$, respectively

ϵ_x', ϵ_y', γ_{xy}'	=	membrane strains in middle surface; their non-dimensional forms are $\epsilon_x' a^2/h^2$, $\epsilon_y' a^2/h^2$, and $\gamma_{xy}' a^2/h^2$, respectively.
ϵ_x'', ϵ_y'', γ_{xy}''	=	extreme-fiber bending and shearing strains; their non-dimensional forms are $\epsilon_x'' a^2/h^2$, $\epsilon_y'' a^2/h^2$, and $\gamma_{xy}'' a^2/h^2$, respectively
F	=	stress function; its non-dimensional form is F/Eh^2
Δ, Δ^2,... Δ^n	=	the difference of first, second, ..., and n-th order, respectively
Δ_x, Δ_y	=	the difference of first order in x and y directions, respectively

CHAPTER I

INTRODUCTION

The classical theory of the bending of a thin elastic plate expresses the relation between the transverse deflection of the middle surface of the plate and the lateral loading of intensity p by the equation (1)

$$D \nabla^4 w = p$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$ is the flexural rigidity of the plate. It is known that the theory is restricted in application, for on the one hand its basic assumptions can be questioned unless the plate is thin, and on the other hand it neglects an effect which must be appreciable when w has values comparable with the thickness. This is the "membrane effect" of curvature, whereby tension or compression in the middle surface tends to oppose or to reinforce p. The effect is negligible when w is very small, provided that no stresses act initially in the plane of the middle surface; but even so it operates when w is small because stretching the middle surface is a necessary consequence of the transverse deflection. When the deflection gets larger and larger, the "membrane effect" becomes more and more prominent until for very large w the "membrane effect" is predominant while the bending stiffness is comparatively negligible. (2), (5), (6)

As small transverse displacements of a flat elastic plate are governed by a single linear equation, large displacements entail stretching of the middle surface and consequent tensions which, interacting with the curvatures, introduce non-linear terms into the conditions of equilibrium and so make those equations no longer independent.

The large-deflection theory of flat plates is due to A. Föppl (3) and the second order terms were formulated by Th. von Kármán (4) in 1910.

The amended (large-deflection) equations have been solved, however, only in a few cases (5)-(21) and then with considerable difficulty.

Essentially there are three problems concerning flat plates with large deflections. They are:

1. The Bending Problems, where the flat plates are subjected to lateral loading perpendicular to the plane of the plates but to no side thrust which is applied in the plane of the plates.
2. The Buckling Problems, where the plates are subjected to side thrust applied in the plane of the plates, but to no lateral loading.
3. The Combined Bending and Buckling Problems, where the plates are subjected to both lateral loading and side thrust.

In the case of metal airplanes, where weight is of primary importance, the metal sheets used must be thin and the deflections of the plates are usually large in comparison with their thickness. To obtain the design formulae or charts for proportioning such plates the large deflection theory must be used.

The bending problem is important in the design of seaplanes. Seaplanes are subjected to a severe impact during landing and take-off, especially on rough water. The impact must be withstood first by the bottom plating and then by a system of transverse and longitudinal members, to which the bottom plating is attached, before it is carried into the body of the structure. The bottom should be strong enough not to dish in or "washboard" permanently under these impact pressures. Such "washboarding" is undesirable, both because of the increased friction between the float bottom and the water and because of the increased aerodynamical drag in flight.

The bottom plating of seaplanes is, as a rule, subdivided into a large number of nearly rectangular areas by the transverse and longitudinal supporting ribs. Each of these areas will behave subtantially like a rectangular plate under normal pressure. Bending of rectangular flat plates may therefore be used to study the "washboarding" of seaplane bottoms, provided the boundary conditions at the edges can be formulated just as in the seaplane.

The buckling problem is important in determining the strength of sheet-stringer panels in end-compression. The use of stiffened sheet to carry compressive loads is increasingly popular in the box beams for airplane wings and in other types of monocoque construction.

Inasmuch as the sheets used as aircraft structural elements are generally quite thin, the buckling stresses of these sheet elements are necessarily low. The designer is therefore confronted with the problem of using sheet metal in the buckled or wave state, and of determining the stress distribution and allowable stresses in such buckled plates.

The combined bending and buckling problem has become a problem of importance with the increasing use of stressed-skin type wings and the pressurized fuselage construction for high altitude flying. During flying the wing is subjected to a pressure difference between the two sides which gives the lift. The normal pressure acts directly on the sheet covering and is then distributed to ribs and spars. At the same time the sheet panels are also subjected to a side thrust due to bending of the wing. In an airplane of pressurized fuselage construction an attempt is made to keep the pressure inside the cabin at a comfortable level for the passengers, no matter how high the ship may be flying.

Thus, there is a pressure differential across the fuselage skin with an internal pressure higher than that outside. The fuselage skin is usually subdivided into a number of rectangular curved panels by longitudinal stringers and rings. These panels are subjected to the pressure difference and also side thrust resulting from bending of the fuselage. As pointed out by Niles and Newell,[57] the strength of curved sheet-stringer panels can be determined approximately from the flat sheet-stringer panels. The problem is then essentially that of determining the strength of flat plates under combined lateral loading and side thrust.

Levy[21] has shown that the effective width of a square plate with simply-supported edges decreases with the addition of lateral pressure, and the reduction is appreciable for $pa^4/Eh^4 > 2.25$. Therefore, a panel is unsafe in strength if the design is based upon side thrust only and the study of combined loading is of great significance.

A great number of authors[(22)-(54)] have studied the buckling problems and considerable experimental work has been carried out. As a result, design formulae are available and seem to be of good accuracy for practical usage. The bending problems, however, have been studied only by a few persons,[(5)-(21)] and test results[(55),(56),(65)] are too few to reach any conclusion. The combined bending and buckling problem has been studied, so far as the author knows, only in one case[21] and yet the results are incomplete.

Among the solutions of the large deflection problems of rectangular plates under bending or combined bending and compression, Levy's solutions are the only ones of the theoretically exact nature. However, his solutions are limited to a few boundary conditions and the numerical results are tremendously laborious to carry out.

The purpose of this thesis is to investigate whether it is possible to develop a simpler and yet sufficiently accurate method to solve the bending and the combined bending and buckling problems for engineering uses. It is quite fortunate that the author has been able to develop such a method by means of the finite differences approximation.

Solving the partial differential equations by finite differences equations is nothing new. However, to solve the resulting differences equations is still a problem. In the case of linear differences equations, solutions by successive approximation are always convergent (58) and the work is only tedious. Besides, one may apply Southwell's Relaxation Method without too much difficulty. But, to solve the non-linear differences equations, the successive approximation method cannot be always applied because it does not always give a convergent solution. The Relaxation Method, since it is nothing but intelligent guessing, can be applied in a few cases and then with great difficulties. (18)

A study of the finite differences expressions of the large deflection theory reveals that although the successive approximation is actually divergent, the mean of certain quantities of two consecutive cycles is convergent. The system of non-linear difference equations can then be solved by successive approximation using Crout's (62) method to solve a system of linear simultaneous equations with rapid convergence.

A square plate under uniform normal pressure with boundary conditions approximating the riveted sheet-stringer panel is studied by this method as an illustration. Non-dimensional deflections and stresses are given under various normal pressures. The results are consistent with Levy's (21) approximate numerical solution for ideal simply-supported plates and Way's[17] approximate solution for ideal clamped edges, and the center deflections check closely with the test results by Head and Sechler[65] for pa^4/Eh^4 ratio up to 120. The deviation above $pa^4/Eh^4 = 120$ is probably due to the approximate assumptions used in the derivation of the governing differential equations.

While the method is general, it may be applied to solve the problems of rectangular plates of any length-width ratio with various boundary conditions under either normal pressure or combined normal pressure and side thrust.

CHAPTER II

REVIEW OF PREVIOUS WORK

1. Introduction

The large deflection theory of flat plates is due to A Föppl,[3] and the difficulty of solving non-linear equations has been noted by Th. von Kármán.[4] The earliest work to deal with these differential equations is perhaps by H. Hencky[5],[6] who devised an approximate method to solve the cases of circular and square plates when the deflection is very large and then the bending stiffness is negligible. Following the same procedure, Kaiser[7] solved the case of a simply-supported plate with zero edge compression under lateral loading. His theoretical results have checked closely with those from his experiment.

In the case of circular plates with large deflections, because of the radial symmetry, the two governing partial differential equations which contain the linear biharmonic differential operator and quadratic terms in the second derivatives can be reduced to a pair of ordinary non-linear differential equations, each of the second order. For both the bending and buckling problems, "exact" solutions are available.[10]-[14]

The bending problem has been solved approximately by Nadai[8] and Timoshenko,[9] and exactly by Way[10] when the plate is under lateral pressure and edge moment. Way presented his solution in terms of power series for a rather large range of applied load.

The buckling problem has been solved by Federhofer and Friedrichs[11] and Stoker.[12] [13] [14] Federhofer gave the solution for both simply-supported and clamped edges, which yields accurate results up to N about 1.25, where N is the ratio of the pressure applied at the edge to the lowest critical or Euler's pressure at which the buckling just begins. Friedrichs and Stoker gave a complete solution for the simply-supported circular plate for N up to infinity. To cover this range they employ three methods which are different, but which interlock. Each of the three methods is suitable for a particular range of values of N; namely, perturbation method for low N, power series method for intermediate N, and the asymptotic solution for N approaching infinity.

There is no solution, however, for the case of circular plates under combined lateral pressure and edge thrust.

The exact solution for a thin infinitely long rectangular strip with clamped or simply-supported edges was obtained by Boobnoff[(15)] and the other cases were discussed by Prescott,[(16)] Way,[(17)] Green and Southwell,[(18)] Levy,[(19),(21)] Levy and Greenman.[(20)]

Prescott gives an approximate solution for the simply-supported plate with no edge displacement. While Prescott's approximation is rather rough, Way presented a better approximate solution by Ritz' Energy Method for the clamped plates. Kaiser[(7)] transformed the differential equations into finite differences equations and solved them by the cut and trial method. Green and Southwell extended the finite differences study into finer divisions and solved the differences equations by a technique based on relaxation method.

Levy[(21)] gives a general solution for the simply-supported plates and numerical solutions are given for the square and rectangular plates with a width-span ratio of 3 to 1, under sonic combined lateral and side loading conditions. Levy and Greenman[(19),(20)] also extended the solution for simply-supported edges to clamped edges. However, their conditions are limited to the case where the edge supports are assumed to clamp the plate rigidly against rotations and displacements normal to the edge, but to permit displacements parallel to the edge. They presented a numerical solution for square and rectangular plates with a width-span ratio of 3 to 1, under lateral pressure.

In short, the problem of rectangular plates with large deflections has been solved by three methods; namely, the energy method, the finite difference equations method, and by means of Fourier series. They are briefly outlined as follows:

2. The Energy Method

The method of attack used by Way[17] is the Ritz Energy Method. Expressions are assumed for the three displacements in the form of algebraic polynomials satisfying the boundary conditions, then by minimizing the energy with respect to the coefficients, a system of simultaneous equations is obtained to solve for these constants.

The energy expression for the plates with large deflection is

$$V = \iint \Big\{ \frac{(\nabla^2 w)^2}{2} - qw + 6\Big[u_x^2 + u_x w_x^2 + v_y^2 + v_y w_y^2 + \tfrac{1}{4}(w_x^2 + w_y^2) + 2\mu\left(u_x v_y + v_y \frac{w_x^2}{2} + u_x \frac{w_y^2}{2}\right) + \frac{1-\mu}{2}\left(u_y^2 + 2u_y v_x + v_x^2 + 2u_y w_x w_y + 2v_x w_x w_y\right)\Big]\Big\}\, d_x d_y \quad \text{...(2-1)}$$

where u, and v, w are the dimensionless horizontal and vertical displacements $q = pa^4/16Dh$, and the subscripts indicate partial differentiation.

For u, v, and w, to satisfy the boundary conditions for clamped edges, Way assumes (Figure II-1):

$$u = (1-x^2)(\beta^2-y^2)x(b_{00}+b_{02}y^2+b_{20}x^2+b_{22}x^2y^2)$$

$$v = (1-x^2)(\beta^2-y^2)y(c_{00}+c_{02}y^2+c_{20}x^2+c_{22}x^2y^2)$$

$$w = (1-x)^2(\beta^2-y^2)^2(a_{00}+a_{02}y^2+a_{20}x^2) \quad \text{.................(2-2)}$$

where $\beta = b/a$

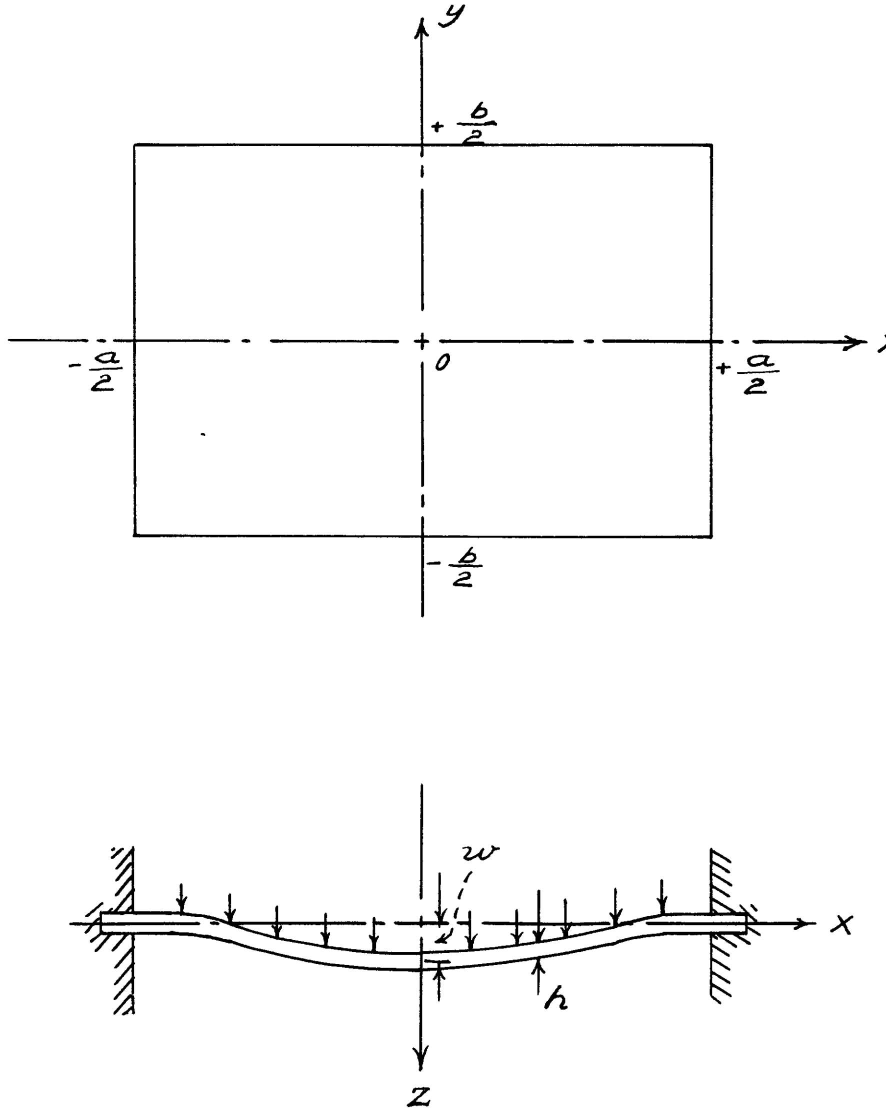


Fig. II - 1

Minimizing V with respect to a_{ij}, b_{ij} and c_{ij}, eleven simultaneous equations are obtained, corresponding to the eleven constants. They are:

$$\frac{\partial V}{\partial a_{00}} = 0; \ \frac{\partial V}{\partial a_{02}} = 0; \ \frac{\partial V}{\partial a_{20}} = 0 \qquad (2\text{-}3)$$

$$\frac{\partial V}{\partial b_{00}} = 0; \ \frac{\partial V}{\partial b_{02}} = 0; \ \frac{\partial V}{\partial b_{20}} = 0; \ \frac{\partial V}{\partial b_{22}} = 0 \qquad (2\text{-}4)$$

$$\frac{\partial V}{\partial c_{00}} = 0; \ \frac{\partial V}{\partial c_{02}} = 0; \ \frac{\partial V}{\partial c_{20}} = 0; \ \frac{\partial V}{\partial c_{22}} = 0 \qquad (2\text{-}5)$$

These equations are not linear in the constants. The first three, Eq. (2-3), will contain terms of the third degree in the a's. The equations (2-4) and (2-5) are linear in the b's and c's and quadratic in the a's. Way solved equations (2-4) and (2-5) for b's and c's in terms of a's and then substituted these expressions in Eqs. (2-3). There then are left three equations of third degree involving the a's alone. These were solved by Way by successive approximations.

Way gives the numerical solutions for the cases $\beta = 1$, 1.5 and 2 for $\mu = .3$ up to q = 210.

Since he assumed the displacements by finite terms of algebraic polynomials, the solution is essentially an approximate one. By comparing with Boobnoff's exact solution for infinite plate, Way estimated that the error of his solution for $\beta = 2$ is about 10% on the conservative side.

3. The Finite Difference Solutions

Kaiser[7] writes the non-dimensionized Kármán's equations in the form of five expressions, as follows:

$$\nabla^2 S = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = -K$$

$$\nabla^2 F = S$$

$$2\,\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} = p_G \qquad (2\text{-}6)$$

$$\frac{1}{12(1-\mu^2)}\nabla^2 M = p - p_G$$

$$\nabla^2 w = M$$

and then transforms them into finite differences equations.

His procedure is to assume w's at all the points and then solve for S's, F's, M's and w's. If the calculated w's do not check with the assumed ones, he assumes a new set of w's and repeats the process. In doing so, the work is tedious and elaborate. In fact, if one does it by successive approximation, the process is actually divergent as will be pointed out later.

Kaiser solved the simply-supported square plate with zero edge compression under a uniform lateral pressure of $pa^4/Eh^4 = 118.72$. His numerical solutions checked with his experimental results with good accuracy.

(18)

Southwell and Green solved four examples of the problem with a technique based on the former's "Relaxation Method". The fundamental requirements to work with the relaxation technique are a simple finite differences pattern of the variables and a simple expression of the boundary conditions. In doing so, they expressed the differential equations in terms of the displacements u, v and w, which then gave simple boundary conditions. Instead of using "exact" relaxation patterns they worked with the patterns which are given by the linear terms of the differential equations, the non-linear terms being combined with the "residue" and making corrections from time to time.

$$\nabla^4 w = \frac{p}{D} + \frac{3}{h^2}\Big[\frac{\partial^2 w}{\partial x^2}\Big\{\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y} + \tfrac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2 + \tfrac{1}{2}\mu\Big(\frac{\partial w}{\partial y}\Big)^2\Big\} +$$

$$\frac{\partial^2 w}{\partial y^2}\Big\{\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x} + \tfrac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2 + \tfrac{1}{2}\mu\Big(\frac{\partial w}{\partial x}\Big)^2\Big\} + (1-\mu)\frac{\partial^2 w}{\partial x\,\partial y}\Big(\frac{\partial u}{\partial y} +$$

$$\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big)\Big]$$

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{1-\mu}{1+\mu}\nabla^2 u + \tfrac{1}{2}\frac{\partial}{\partial x}\Big\{\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big\} +$$

$$\frac{1-\mu}{1+\mu}\frac{\partial w}{\partial x}\nabla^2 w = 0$$

$$\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{1-\mu}{1+\mu}\nabla^2 v + \tfrac{1}{2}\frac{\partial}{\partial y}\Big\{\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big\} + \qquad (2\text{-}7)$$

$$\frac{1-\mu}{1+\mu}\frac{\partial w}{\partial y}\nabla^2 w = 0$$

It is readily seen that in order to obtain a simple expression for boundary conditions, not only the number of the partial differential equations is increased from two to three, but the form of their terms is more complicated and the number of terms is increased. It is again tedious and very difficult to operate.

Equations (2-7), being conditions of equilibrium, could have been derived by minimizing the total potential energy V, which is given by the expression

$$\frac{L^2}{h^2}\frac{V}{D} = I_1 + I_2 + I_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2\text{-}8)$$

where $I_1 = \frac{1}{2} \iint (\nabla^2 w)^2 \, dx \, dy,$

$$I_2 = \frac{3}{2} \iint (e_{xx}^{\;2} + e_{yy}^{\;2} + 2\mu e_{xx}e_{yy} + \frac{1-\mu}{2} e_{xy}^{\;2}) \, dx \, dy,$$

and $I_3 = -\alpha \iint w \, dx \, dy,$

α being the lateral loading.

The relaxation technique consists of assuming a set of answers first and then changing them according to the relaxation pattern and boundary conditions. In performing the operation they found the process convergent. To obtain a more rapid convergence, they multiplied the given values of w by k, and substituted them into the energy expression to obtain

$$\frac{L^2}{h^2} \frac{V}{D} = k^2 I_1 + k^4 I_2 + \alpha k I_3 \quad \text{........................} \quad (2\text{-}9)$$

which was then minimized with respect to k, namely, to put $\frac{V}{k} = 0$, to give

$$2kI_1 + 4k^3 I_2 - \alpha \; I_3 = 0 \quad \text{..............................} \quad (2\text{-}10)$$

From the third order Eq. (2-10), k can be solved and would give a set of w values which are closer to the true values.

4. The Fourier Series Solutions.

Levy[19],[21] and Levy and Greenman[20] obtained general solutions of the rectangular plates under combined bending and side thrust with large deflections by means of Fourier Series. The approach of the solution is outlined as follows, (Fig. II-2).

(a) Simply-supported rectangular plates.

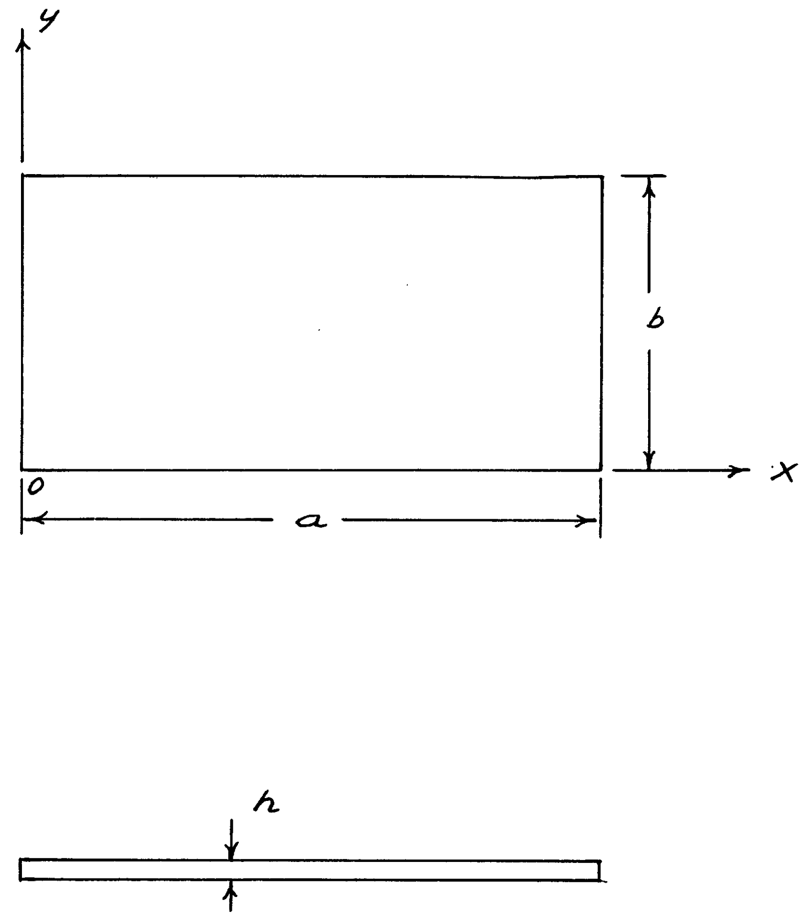


Fig. II - 2

To satisfy the boundary conditions, w can be expressed by the Fourier Series

$$w = \sum_{m=1,2,3}^{\infty} \sum_{n=1,2,3}^{\infty} w_{m,n} \sin m \frac{\pi x}{a} \sin n \frac{\pi y}{b} \quad \text{.................} \quad (2\text{-}11)$$

The normal pressure may be expressed as a Fourier Series

$$p_z = \sum_{r=1,2,3}^{\infty} \sum_{s=1,2,3}^{\infty} p_{r,s} \sin r \frac{\pi x}{a} \sin s \frac{\pi y}{b} \quad \text{.................} \quad (2\text{-}12)$$

To satisfy the compatability equation, F is found to be

$$F = -\frac{\overline{p}_x y^2}{2} - \frac{\overline{p}_y x^2}{2} + \sum_{p=0,1,2}^{\infty} \sum_{q=0,1,2}^{\infty} b_{p,q} \cos p \frac{\pi x}{a} \cos q \frac{\pi y}{b} \quad (2\text{-}13)$$

Where $\overline{p}_x$, $\overline{p}_y$ are constants equal to the average membrane pressure in the x- and y- direction and where

$$b_{p,q} = \frac{E}{4\left[p^2 \frac{b}{a} + q^2 \frac{a}{b}\right]^2} (B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9)$$

and $$B_1 = \sum_{k=1}^{p-1} \sum_{t=1}^{q-1} \left[kt(p-k)(q-t) - k^2 (q-t)^2\right] w_{k,t}\, w_{(p-k),(q-t)}$$

if $q \neq 0$ and $p \neq 0$. $B_1 = 0$ if $q = 0$ or $p = 0$.

$$B_2 = \sum_{k=1}^{\infty} \sum_{t=1}^{q-1} \left[kt(k+p)(q-t) + k^2 (q-t)^2\right] w_{k,t}\, w_{(k+p),(q-t)}$$

if $q \neq 0$. $B_2 = 0$ if $q = 0$.

$$B_3 = \sum_{k=1}^{\infty} \sum_{t=1}^{q-1} \left[(k+p)\, kt\, (q-t) + (k+p)^2 (q-t)^2\right] w_{(k+p),t}\, w_{k,(q-t)}$$

if $q \neq 0$ and $p \neq 0$. $B_3 = 0$, if $q = 0$ or $p = 0$.

$$B_4 = \sum_{k=1}^{p-1} \sum_{t=1}^{\infty} \left[kt(p-k)(t+q) + k^2(t+q)^2\right] w_{k,t}\, w_{(p-k),(t+q)}$$

if $p \neq 0$. $B_4 = 0$, if $p = 0$.

$$B_5 = \sum_{k=1}^{p-1} \sum_{t=1}^{\infty} \left[kt(t+q)(p-k) + k^2t^2\right] w_{k,(t+q)}\, w_{(p-k),t}$$

if $p \neq 0$ and $q \neq 0$. $B_5 = 0$, if $p = 0$ or $q = 0$.

$$B_6 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[kt\,(k+p)(t+q) - k^2(t+q)^2\right] w_{k,t}\, w_{(k+p),(t+q)}$$

if $q \neq 0$.

$$B_7 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[kt(t+q)(k+p) - k^2t^2\right] w_{k,(t+q)}\, w_{(k+p),t}$$

if $q \neq 0$ and $p \neq 0$. $B_7 = 0$, if $p = 0$ or $q = 0$.

$$B_8 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[kt(k+p)(t+q) - (k+p)^2(t+q)^2\right] w_{(k+p),t}\, w_{k,(t+q)}$$

if $q \neq 0$ and $p \neq 0$. $B_8 = 0$, if $p = 0$ or $q = 0$.

$$B_9 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[(k+p)(t+q)\,kt - (k+p)^2t^2\right] w_{(k+p),(t+q)}\, w_{k,t}. \quad \ldots\ldots (2\text{-}14)$$

The equilibrium equation is satisfied if

$$P_{r,s} = D\, w_{r,s} \left(r^2 \frac{\pi^2}{a^2} + s^2 \frac{\pi^2}{b^2}\right)^2 - \bar{p}_x\, h\, w_{r,s} r^2 \frac{\pi^2}{a^2}$$

$$- \bar{p}_y\, h\, w_{r,s}\, s^2 \frac{\pi^2}{b^2}$$

$$+ \frac{h\pi^4}{4a^2\, b^2} \Bigg\{ - \sum_{k=1}^{r} \sum_{t=1}^{s} \quad (s-t)\,k - (r-k)t \quad ^2\, b_{(r-k),(s-t)}\, w_{k,t}$$

$$- \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \left[t(k+r) - k\,(t+s)\right]^2 b_{k,t}\, w_{(k+r),(t+s)}$$

$$+ \sum_{k=0}^{\infty} \sum_{t=1}^{\infty} \left[(k+r)(t+s) - kt\right]^2 b_{k,(t+s)}\, w_{(k+r),\, t}$$

$$+ \sum_{k=1}^{\infty} \sum_{t=0}^{\infty} \left[tk - (k+r)(t+s)\right]^2 b_{(k+r),t}\, w_{k,(t+s)}$$

$$- \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[(t+s)\,k - (k+r)t\right]^2 b_{(k+r),(t+s)}\, w_{k,t}$$

$$- \sum_{k=1}^{r} \sum_{t=0}^{\infty} \left[tk + (r-k)(t+s)\right]^2 b_{(r-k),t}\, w_{k,(t+s)}$$

$$+ \sum_{k=1}^{r} \sum_{t=1}^{\infty} \left[(t+s)\,k + (r-k)t\right]^2 b_{(r-k),(t+s)}\, w_{k,t}$$

$$- \sum_{k=0}^{\infty} \sum_{t=1}^{s} \left[(s-t)(k+r) + tk\right]^2 b_{k,(s-t)}\, w_{(k+r),t}$$

$$+ \sum_{k=1}^{\infty} \sum_{t=1}^{s} \left[(s-t)\,k + t(k+r)\right]^2 b_{(k+r),(s-t)}\, w_{k,t} \Bigg\} \quad \ldots\ldots (2\text{-}15)$$

When the lateral pressure is given, $p_{r,s}$ can be determined. Equation (2-15) represents a doubly infinite family of equations. In each of the equations of the family, the coefficients b p, q may be replaced by their values as given by Equation (2-14). The resulting equations will involve the known normal pressure coefficients $p_{r,s}$, the cubes of the deflection coefficients $w_{m,n}$, and the average membrane pressures in the x- and the y-directions, $\overline{p}_x$ and $\overline{p}_y$, respectively. $\overline{p}_x$ and $\overline{p}_y$ can be determined from the conditions that the plates are either subjected to known edge compressions or known edge displacements. The number of these equations is equal to the number of unknown deflection coefficients, $w_{m,n}$.

Now, the procedure is with the known values of $p_{r,s}$ to assume $w_{1,1}$ and solve the other coefficients by successive approximation. The work in doing so, however, is tremendous and it is very easy to make mistakes. As in the case illustrated by Levy, for a relative simple case or square plates if six deflection coefficients are used, each equation contains sixty third-order terms. And for each normal pressure applied one has to solve these six sixty-term, third-order equations once by successive approximation. The method is too laborious to be of practical use.

(b) Clamped Rectangular Plates.

Levy and Greenman solved the case of the clamped rectangular plate by assuming that the edges are clamped rigidly against rotations and displacements normal to the edge but are permitted to move freely parallel to the edge.

The required edge moments $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are replaced by an auxiliary pressure distribution $p_a(w,y)$ near the edges of the plate. The auxiliary pressure can be expressed in terms of a Fourier Series as follows.

$$p_a(x,y) = \sum_{r=1,3,5\cdots}^{\infty} \frac{4\pi \mathfrak{m}_y}{a^2} \; r \sin \frac{r\pi x}{a}$$

$$+ \sum_{s=1,3,5\cdots}^{\infty} \frac{4\pi \mathfrak{m}_x}{b^2} \; s \sin \frac{s\pi y}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-}16)$$

Express $\mathfrak{m}_x$ and $\mathfrak{m}_y$ by a Fourier Series, where k_s and k_r are coefficients to be determined,

$$mx = \frac{4a^2}{\pi^3} p \sum_{r=1,3,5\,..}^{\infty} k_r \sin \frac{r\pi x}{a}$$
$$my = \frac{4b^2}{\pi^3} p \sum_{s=1,3,5\,..}^{\infty} k_s \sin \frac{s\pi y}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-}17)$$

Inserting equation (2-17) in equation (2-16) gives

$$p_a(x,y) = \left(\frac{4}{\pi}\right)^2 p \sum_{r=1,3,5\,..}^{\infty} \sum_{x=1,3,5\,..}^{\infty} (rk_s + sk_r) \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b} \ldots . (2\text{-}18)$$

Combining with the normal pressure p_z, Eq. (2-12), we have

$$p_c(x,y) = \sum_{r=1,2,3\,..}^{\infty} \sum_{s=1,2,3\,..}^{\infty} p_{r,s} \sin r \frac{\pi x}{a} \sin s \frac{\pi y}{b} \ldots . (2\text{-}19)$$

where $$p_{r,s} = \left(\frac{4}{\pi}\right)^2 (rpk_s + spk_r) + p'_{r,s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-}20)$$

Since the edge moments mx and my have been replaced by the auxiliary pressure distribution p_a (x,y), the general solution for the simply-supported rectangular plate (Eqs. 2-11 to 2-25) can be applied to the clamped ones and the only thing one has to do is to determine the values of k_s and k_r. This is obtained by the condition of zero slope at the edges of the plate.

Setting the slope perpendicular to the edges x = 0, x = a, y = 0 and y = b, to zero gives

$$\left(\frac{\partial w}{\partial x}\right)_{x=0,\ x=a} = 0 = \sum_{m=1,3,5\,..}^{\infty} \sum_{n=1,3,5\,..}^{\infty} \frac{m\pi}{a} w_{m,n} \sin \frac{n\pi y}{b}$$
$$\left(\frac{\partial w}{\partial y}\right)_{y=0,\ y=b} = 0 = \sum_{m=1,3,5\,..}^{\infty} \sum_{n=1,3,5\,..}^{\infty} \frac{n\pi}{b} w_{m,n} \sin \frac{m\pi x}{a} (2\text{-}21)$$

Equations (2-21) are equivalent to the family of equations

$$0 = w_{1,1} + 3\,w_{1,3} + 5\,w_{1,5} + \ldots\ldots$$
$$0 = w_{3,1} + 3\,w_{3,3} + 5\,w_{3,5} + \ldots\ldots \qquad (2\text{-}22).$$
$$0 = w_{5,1} + 3\,w_{5,3} + 5\,w_{5,5} + \ldots\ldots$$

Now the deflection coefficients $w_{m,n}$ must be solved from the family of equation (2-15) for the linear term in terms of the cubic terms and the pressure coefficients $p_{r,s}$. The $w_{m,n}$ thus obtained are now substituted into Equation (2-22), and for the pressure coefficients $p_{r,s}$, are substituted their values by Equation (2-20). The resulting family of equations contains linear terms of pk_r and pk_s and the cubes of the deflection functions $w_{m,n}$.

The method of obtaining the required values of the deflection coefficients $w_{m,n}$ and the edge moment coefficients pk_r, pk_s consists of assuming values for $\frac{w_{1,1}}{h}$ and then solving for $\frac{pa^4}{Eh^4}$, $\frac{w_{1,3}}{h}$, ..., pk_s, pk_r ..... by successive approximations from the simultaneous equations.

The procedure is even more laborious than that for simply-supported plates. Two numerical solutions are given, namely, the bending problem of a square plate and a rectangular plate with length-width ratio of 1.5.

CHAPTER III

THE GOVERNING DIFFERENTIAL EQUATIONS

1. Bending of Thin Sheet Plates

A cylinder is called a plate if its height is small compared with the linear dimensions of its cross-section. We shall designate the height of the cylinder by h and shall call it the thickness of the plate. The middle plane of the plate will be taken to coincide with the xy-plane of the coordinate system, and as a plane of elastic symmetry. After bending, the points of the middle plane are displaced and lie on some surface which we shall call the middle surface of the plate. The displacements of the points of the middle plane in the direction of the z-axis will be denoted by w and will be termed deflection of the points of the plate.

We consider the case where the deflections are large in comparison with the thickness but are at the same time small enough to justify the following assumptions:

(1) Sections of the plate cut by planes normal to the middle plane before deformation remain plane and normal to the deformed middle surface.

(2) The normal stress σ_z perpendicular to the faces of the plate is negligible in comparison to the other normal stresses.

To investigate the state of strain in a bent plate we suppose that the middle surface is actually deformed, with but slight extension of any linear element, so that it becomes a surface differing only slightly from one or other of the surfaces which are applicable upon the unstrained middle surface.

The first hypothesis is equivalent to saying that points originally on a normal to the undeformed middle surface remain approximately on a normal to this surface in its deformed state. That is, the shear strains γ_{xz}, and γ_{yz} or $\partial u/\partial z + \partial w/\partial x$ and $\partial v/\partial z + \partial w/\partial y$ are negligible. This is based on the behavior of beams subjected to pure bending. It is the consequence of this hypothesis that the middle surface of the plate undergoes no extension, and hence is a neutral surface.

The second hypothesis disregards in effect the contribution to the deflection of the compressive stress σ_z produced by the load p.

Since, for z = 0, u and v are zero, and by the first assumption, the displacements for any point in the plate are given by $u = z(\partial u/\partial z)$ and $v = z(\partial v/\partial z)$. Again by the first assumption,

$$\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

or

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}$$

$$\frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y}$$

therefore

$$u = -z\frac{\partial w}{\partial x}$$

$$v = -z\frac{\partial w}{\partial y}$$

These are the displacements at any point in the plate due to bending alone, the effect due to the extension of the middle surface being neglected.

But the components of the strain tensor are related to the displacements u and v by the formulas $\epsilon_x = \partial u/\partial x$, $\epsilon_y = \partial v/\partial y$, $\gamma_{xy} = 1/2(\partial v/\partial x + \partial u/\partial y)$ so that by making use of these relations we can write

$$\epsilon_x = - z \frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = - z \frac{\partial^2 w}{\partial y^2} \qquad (3\text{-}1)$$

$$\gamma_{xy} = - z \frac{\partial^2 w}{\partial x \partial y}$$

By Hook's law, since $\sigma_z = 0$, we have

$$\sigma_x = \lambda(\epsilon_x + \epsilon_y) + 2\nu\,\epsilon_x$$

$$\sigma_y = \lambda(\epsilon_x + \epsilon_y) + 2\nu\,\epsilon_y \qquad (3\text{-}2)$$

$$\tau_{xy} = \nu\,\gamma_{xy}$$

Substituting Eq. (1) into Eq. (2), we obtain

$$\sigma_x = - z(\lambda \nabla^2 w + 2\nu \frac{\partial^2 w}{\partial x^2})$$

$$\sigma_y = - z(\lambda \nabla^2 w + 2\nu \frac{\partial^2 w}{\partial y^2}) \qquad (3\text{-}3)$$

$$\tau_{xy} = - 2z\nu \frac{\partial^2 w}{\partial x \partial y}$$

where λ and ν are elastic constants and ∇^2 is $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Consider an element of the plate (Fig. III-1) and take the moment of the forces about the x-axis. Neglecting third order terms, the equilibrium conditions require that

$$\text{(a)} \qquad \tau_{yz}\, dx\, h\, dy + \int_{-h/2}^{+h/2} z \frac{\partial \sigma_y}{\partial y}\, dy\, dx\, dz - \int_{-h/2}^{+h/2} z \frac{\partial \tau_{xy}}{\partial x}\, dx\, dy\, dz = 0$$

Similarly, take the moment of the forces about the y-axis and set it equal to zero.

$$\text{(b)} \qquad \tau_{xz} dx h dy + \int_{-h/2}^{+h/2} z \frac{\partial \sigma_x}{\partial x}\, dx dy dz - \int_{-h/2}^{+h/2} z \frac{\partial \tau_{xy}}{\partial y}\, dx dy dz = 0$$

By rearranging terms, we have

$$\text{(c)} \qquad \begin{aligned} h\,\tau_{yz} &= \int_{-h/2}^{+h/2} z \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right) dz \\ h\,\tau_{xz} &= \int_{-h/2}^{+h/2} z \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) dz \end{aligned}$$

Substituting Eq.(3-3) into Eq.(c), and integrating, we have

$$h\tau_{yz} = -\int_{-h/2}^{+h/2} z^2 dz \left[\lambda \frac{\partial}{\partial y} \nabla^2 w + 2\nu \frac{\partial^3 w}{\partial y^3} + 2\nu \frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

$$= -\frac{2}{3}\left(\frac{h}{2}\right)^3 \left[\lambda \frac{\partial}{\partial y} \nabla^2 w + 2\nu \frac{\partial}{\partial y} \nabla^2 w\right]$$

$$\therefore \qquad \begin{aligned} \tau_{yz} &= -\frac{h^2}{12} (\lambda + 2\nu) \frac{\partial}{\partial y} \nabla^2 w \\ \tau_{xz} &= -\frac{h^2}{12} (\lambda + 2\nu) \frac{\partial}{\partial x} \nabla^2 w \end{aligned} \qquad (3\text{-}4)$$

$$\therefore \qquad \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} = -\frac{h^2}{12} (\lambda + 2\nu) \nabla^4 w \qquad (3\text{-}5)$$

The normal stresses distributed over the lateral sides of the element can be reduced to couples, the magnitudes of which

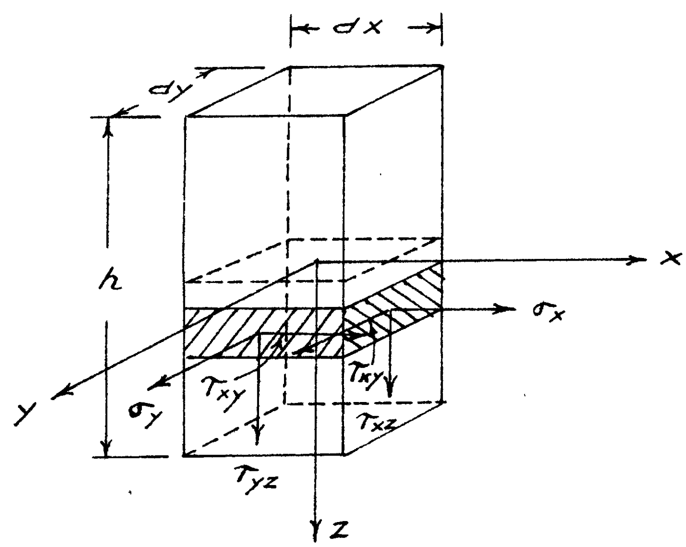


Fig. III-1

per unit length evidently must be equal to the external moments M_x and M_y. In this way, we obtain

$$M_x dy = \int_{-h/2}^{h/2} \sigma_x \, z \, dy \, dz$$

$$M_y dx = \int_{-h/2}^{h/2} \sigma_y \, z \, dx \, dz$$

Substituting expressions for σ_x and σ_y and integrating, we have

$$M_x = -\frac{h^2}{12} (\lambda + 2\nu) \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = -\frac{h^2}{12} (\lambda + 2\nu) \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \qquad (3\text{-}6)$$

2. Differential Equation of the Deflection Surface

Consider the equilibrium of a small element cut from the plate by two pairs of planes parallel to the xz- and yz-coordinate planes. (Fig. III-2). Let the magnitudes of the lateral forces be S_x, S_y and note that $S_{xy} = S_{yx}$.

Projecting these forces on the x- and y-axes and assuming that there are no body forces or tangential forces acting in those directions at the faces of the plate, we obtain the following equations of equilibrium:

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_{xy}}{\partial y} = 0$$

$$\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_y}{\partial y} = 0 \qquad (3\text{-}7)$$

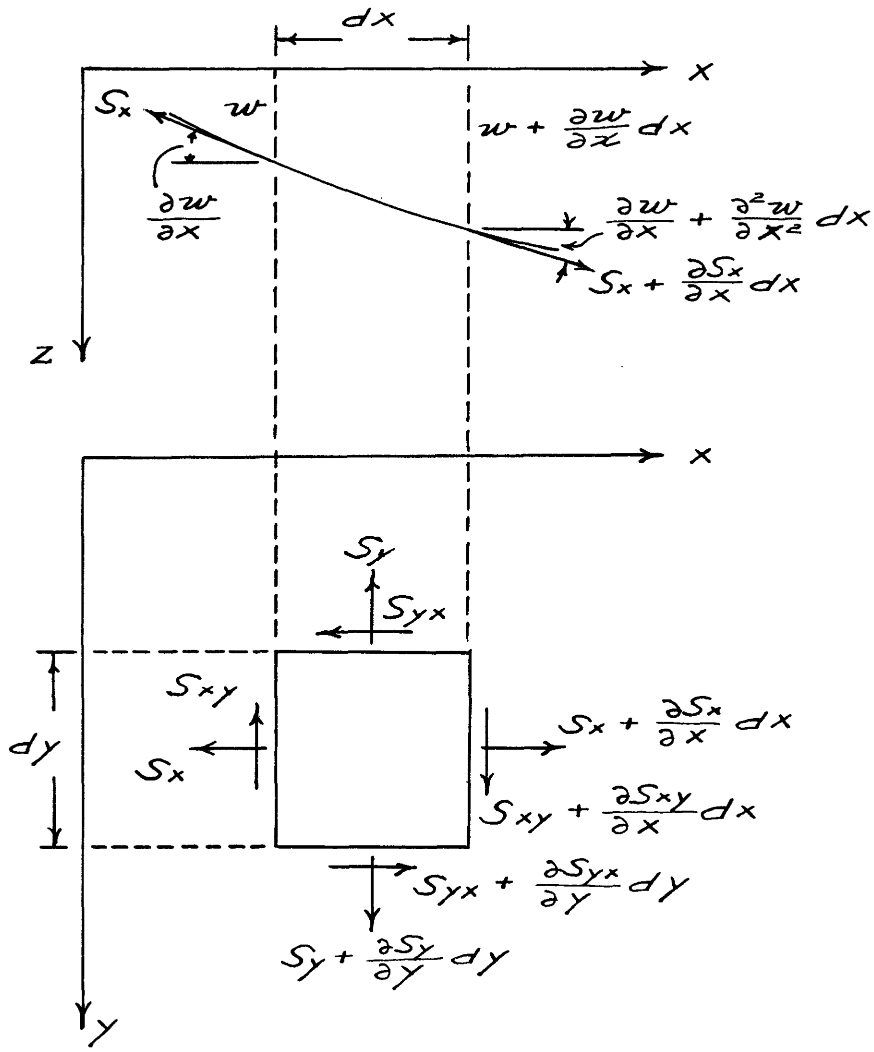

dx
x
Sx
w
w + ∂w/∂x dx
∂w/∂x
∂w/∂x + ∂²w/∂x² dx
Sx + ∂Sx/∂x dx
z
x
Sy
Syx
Sxy
Sx
dy
Sx + ∂Sx/∂x dx
Sxy + ∂Sxy/∂x dx
Syx + ∂Syx/∂y dy
Sy + ∂Sy/∂y dy
y

Fig. III-2

In considering the projection of the forces on the z-axis, we must take into account the bending of the plate and the resulting small angles between the forces S_x and S_y that act on the opposite sides of the element. As a result of this bending the projection of the normal forces S_x on the z-axis gives

$$\text{(a)} \qquad -S_x dy \frac{\partial w}{\partial x} + \left(S_x + \frac{\partial S_x}{\partial x} dx\right)\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx\right) dy$$

$$= S_x \frac{\partial^2 w}{\partial x^2} dxdy + \frac{\partial S_x}{\partial x} \frac{\partial w}{\partial x} dxdy$$

The last step was obtained by neglecting small quantities of higher than the second order.

Similarly the projection of the normal forces S_y on the z-axis gives

$$\text{(b)} \qquad S_y \frac{\partial^2 w}{\partial y^2} dxdy + \frac{\partial S_y}{\partial y} \frac{\partial w}{\partial y} dxdy$$

Regarding the projection of the shearing forces S_{xy} on the z-axis, we observe that the slope of the deflection surface in the y-direction on the two opposite sides of the element is $\partial w/\partial y$ and $\partial w/\partial y + (\partial^2 w/\partial x \partial y) dx$. Hence the projection of the shearing forces on the z-axis then can be written as

$$S_{xy} \frac{\partial 2w}{\partial x \partial y} dxdy + \frac{\partial S_{xy}}{\partial x} \frac{\partial w}{\partial y} dx\, dy$$

An analogous expression can be obtained for the projection of the shearing forces $S_{yx} = S_{xy}$ on the z-axis. The final expression for the projection of all the shearing forces on the z-axis then can be written as

$$S_{xy} \frac{\partial^2 w}{\partial x \partial y} \ dx\ dy + \frac{\partial S_{xy}}{\partial x} \frac{\partial w}{\partial y} \ dx\ dy$$

An analogous expression can be obtained for the projection of the shearing forces $S_{yx} = S_{xy}$ on the z-axis. The final expression for the projection of all the shearing forces on the z-axis then can be written as

$$\text{(c)} \qquad 2 S_{xy} \frac{\partial^2 w}{\partial x \partial y} dx\ dy + \frac{\partial S_{xy}}{\partial x} \frac{\partial w}{\partial y} dx\ dy + \frac{\partial S_{xy}}{\partial y} \frac{\partial w}{\partial x} dx\ dy$$

Adding expressions (a), (b) and (c) to the load p dx dy acting on the element and equating the z-component of forces to zero, we obtain the equation of equilibrium:

$$\text{(d)} \qquad S_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial S_x}{\partial x} \frac{\partial w}{\partial x} + S_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial S_y}{\partial y} \frac{\partial w}{\partial y}$$

$$+ 2 S_{xy} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial S_{xy}}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial S_{xy}}{\partial y} \frac{\partial w}{\partial x}$$

$$+ h \frac{\partial \tau_{xz}}{\partial y} + h \frac{\partial \tau_{yz}}{\partial x} + p = 0$$

where τ_{xz} and τ_{yz} are the average values over the height h.

From eqs. (3-7), we have

$$\frac{\partial S_x}{\partial x} = - \frac{\partial S_{xy}}{\partial y}$$

$$\frac{\partial S_y}{\partial y} = - \frac{\partial S_{xy}}{\partial x}$$

Substituting into equation (d), we have

$$\text{(e)} \quad h\left(\frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{yz}}{\partial x}\right) + p + S_x \frac{\partial^2 w}{\partial x^2} + 2\, S_{xy} \frac{\partial^2 w}{\partial x \partial y} + S_y \frac{\partial^2 w}{\partial y^2} = 0$$

Assume that the deflections are large in comparison with the thickness but are at the same time small enough to justify the application of simplified formulas for curvatures of a plate. Then by the condition resulted from pure bending, we have

$$\frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{yz}}{\partial x} = -\frac{h^2}{12}(\lambda + 2\nu)\ \nabla^4 w$$

$$= -\frac{h^2}{12}\ \frac{E}{1-\mu^2}\ \nabla^4 w;$$

and eq. (e) becomes

$$\text{(f)} \quad -\frac{h^3}{12}\ \frac{E}{1-\mu^2}\ \nabla^4 w + p + S_x \frac{\partial^2 w}{\partial x^2} + 2\, S_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

$$+ S_y \frac{\partial^2 w}{\partial y^2} = 0$$

where E is the Young's Modulus and μ is the Poisson's Ratio.

Introducing a stress function F defined by

$$S_x = h\frac{\partial^2 F}{\partial y^2}\ ; \quad S_y = h\frac{\partial^2 F}{\partial x^2}\ ; \quad S_{xy} = -h\frac{\partial^2 F}{\partial x \partial y}$$

the resulting equilibrium equation is

$$-\frac{h^3}{12}\ \frac{E}{1-\mu^2}\ \nabla^4 w + p + h\frac{\partial^2 F}{\partial y^2}\ \frac{\partial^2 w}{\partial x^2} - 2\,h\frac{\partial^2 F}{\partial x \partial y}\ \frac{\partial^2 w}{\partial x \partial y}$$

$$+ h\ \frac{\partial^2 F}{\partial x^2}\ \frac{\partial^2 w}{\partial y^2} = 0$$

or $$\nabla^4 w = h/D \left(p/h + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{3-8}$$

where $D = \dfrac{Eh^3}{12(1-\mu^2)}$ and is called the flexural rigidity of a plate.

3. Compatability Equation.

Considering the strain in the middle surface of the plate during bending, (Fig. III-3)

$$\alpha \cong \tan\alpha \cong \frac{\partial w}{\partial x}$$

$$\cos\alpha \cong 1 - \frac{\alpha^2}{2} \cong 1 - 1/2\,(\partial w/\partial x)^2$$

Take an element of length dx. After deformation, its length is

$$dS \cong \frac{dx + \partial u/\partial x \; dx}{\cos\alpha}$$

$$\cong \frac{dx\,(1 + \partial u/\partial x)}{1 - 1/2\,(\partial w/\partial x)^2}$$

$$\cong dx\left[1 + \partial u/\partial x + 1/2\,(\partial w/\partial x)^2\right].$$

The x-component of the strain is therefore

$$\epsilon_x \cong \partial u/\partial x + 1/2\,(\partial w/\partial x)^2 \tag{a}$$

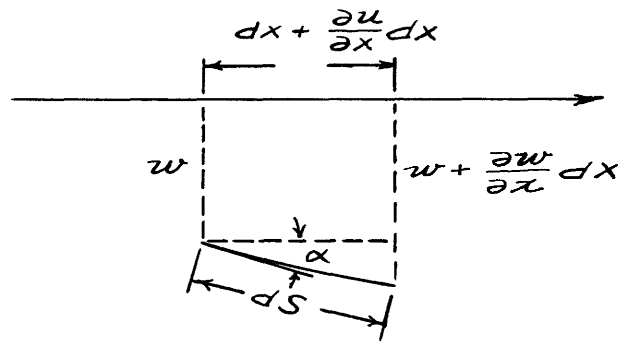


Fig. III - 3

Similarly the y-component of the strain is

$$\epsilon_y \cong \partial v/\partial y + 1/2\ (\partial w/\partial y)^2 \qquad \text{(b)}$$

For the determination of shearing strain due to displacements w, we take two infinitely small linear elements OA and OB in the x- and y-directions, as shown in the following figure. Because of the displacements w, these elements come to the positions $O_1\ A_1$ and $O_1\ B_1$. The difference between the angle $A_1O_1B_1$ and $\pi/2$ is the shearing strain corresponding to the displacements w, (Fig. III-4).

$$O_1A_1 = \left[1 + (\partial w/\partial x)^2 + (\partial v/\partial x)^2\right]^{1/2} dx$$

$$\cong \left[1 + 1/2\ (\partial w/\partial x)^2 + 1/2\ (\partial v/\partial x)^2\right] dx$$

The directional cosines of O_1A_1 are

$$dx/O_1A_1 \cong 1 - 1/2\ (\partial w/\partial x)^2 - 1/2\ (\partial v/\partial x)^2 \quad \text{with x-axis}$$

$$\frac{\partial v/\partial x\ \ dx}{O_1A_1} \cong \partial v/\partial x \quad \text{with y-axis}$$

$$\frac{\partial w/\partial x\ \ dx}{O_1A_1} \cong \partial w/\partial x \quad \text{with z-axis}$$

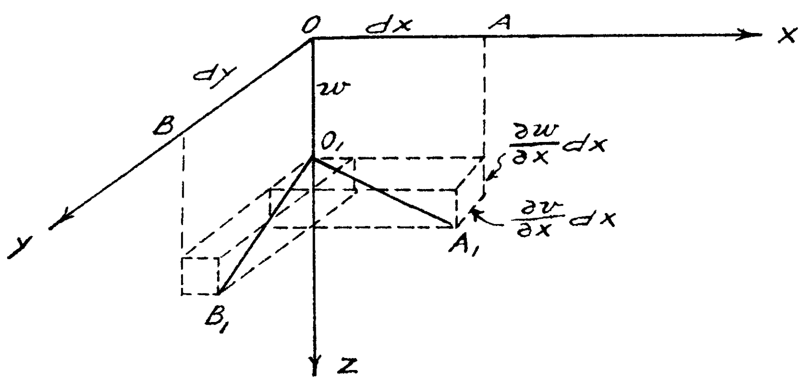
Similarly, the directional cosines of O_1B_1 are

$$\frac{\partial u}{\partial y} \quad \text{with x-axis}$$

$$1 - 1/2\ (\partial w/\partial y)^2 - 1/2\ (\partial u/\partial y) \quad \text{with y-axis}$$

$$\partial w/\partial y \quad \text{with z-axis}$$

$$\therefore\ \pi/2 - \angle A_1O_1B_1 \cong \cos A_1O_1B_1$$



O
dx
A
X
dy
w
B
O₁
$\frac{\partial w}{\partial x}dx$
$\frac{\partial v}{\partial x}dx$
A₁
Y
B₁
Z

Fig. III - 4

$$= \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$$

$$\cong \partial u/\partial y + \partial u/\partial x + (\partial w/\partial x)(\partial w/\partial y) \qquad \text{(c)}$$

The strain components are therefore

$$\epsilon_x = \partial u/\partial x + 1/2\ (\partial w/\partial x)^2$$

$$\epsilon_y = \partial v/\partial y + 1/2\ (\partial w/\partial y)^2 \qquad \text{(3-9)}$$

$$\gamma_{xy} = \partial u/\partial y + \partial v/\partial x + (\partial w/\partial x)(\partial w/\partial y)$$

By differentiating these expressions, it can be shown

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \qquad \text{(3-10)}$$

Hook's Law is

$$\epsilon_x = 1/E\ (\sigma_x - \mu\sigma_y)$$

$$\epsilon_y = 1/E\ (\sigma_y - \mu\sigma_x) \qquad \text{(d)}$$

$$\gamma_{xy} = 1/G\ \tau_{xy}$$

Noting that

$$\sigma_x = S_x/h = 1/h\ \frac{\partial^2 F}{\partial y^2}$$

$$\sigma_y = S_y/h = 1/h\ \frac{\partial^2 F}{\partial y^2} \qquad \text{(e)}$$

$$\tau_{xy} = S_{xy}/h = 1/h\ \frac{\partial^2 F}{\partial x \partial y}$$

Therefore,

$$\epsilon_x = 1/hE \left(\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2}\right)$$

$$\epsilon_y = 1/hE \left(\frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2}\right) \qquad \text{(f)}$$

$$\gamma_{xy} = -\frac{2(1+\mu)}{E} \frac{\partial^2 F}{\partial x \partial y}$$

Substituting these expressions in Eq. (3-10), we obtain

$$\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$$

or $$\nabla^4 F = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}\right] \qquad \text{(3-11)}$$

4. Summary of Equations for the Deformation of Thin Plates.

The fundamental equations governing the deformation of thin plates developed in Sections 2 and 3 may be summarized as follows: the partial differential equations are:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p/D + p/D\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right.$$

$$\left. + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)$$

where the median-fiber stresses are

$$\sigma'_x = \frac{\partial^2 F}{\partial y^2}$$

$$\sigma'_y = \frac{\partial^2 F}{\partial x^2}$$

$$\tau'_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

and the median-fiber strains are

$$\epsilon'_x = 1/E\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)$$

$$\epsilon'_y = 1/E\left(\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}\right)$$

$$\gamma'_{xy} = -\frac{2(1+\mu)}{E}\frac{\partial^2 F}{\partial x \partial y}$$

The extreme-fiber bending and shearing stresses are:

$$\sigma_x'' = -\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y'' = -\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy}'' = -\frac{Eh}{2(1+\mu)}\frac{\partial^2 w}{\partial x \partial y}$$

5. Non-dimensional Form of the Equations.

Writing

$$F' = \frac{F}{h^2 E} \qquad x' = x/a$$

$$w' = w/h \qquad y' = y/a$$

$$p' = \frac{p\,a^4}{E\,h^4} \qquad \epsilon' = \epsilon(a/h)^2$$

$$\sigma' = \frac{\sigma}{E}(a/h)^2$$

where a is the smaller side of the rectangular plate and h is the thickness, we have the following differential equations:

$$\frac{\partial^4 F'}{\partial x'^4} + 2\frac{\partial^4 F'}{\partial x'^2 \partial y'^2} + \frac{\partial^4 F'}{\partial y'^4} = \left(\frac{\partial^2 w'}{\partial x' \partial y'}\right)^2 - \frac{\partial^2 w'}{\partial x'^2}\frac{\partial^2 w'}{\partial y'^2}$$

$$\frac{\partial^4 w'}{\partial x'^4} + 2\frac{\partial^4 w'}{\partial x'^2 \partial y'^2} + \frac{\partial^4 w'}{\partial y'^4} = 12(1-\mu^2)\,p'$$

$$+ 12(1-\mu^2)\Big[\partial^2 F'/\partial y'^2 \;\; \partial^2 w'/\partial x'^2$$

$$+ \frac{\partial^2 F'}{\partial x'^2}\frac{\partial^2 w'}{\partial y'^2} - \frac{\partial^2 F'}{\partial x'\partial y'}\frac{\partial^2 w'}{\partial x'\partial y'}\,2\Big]$$

Letting $\mu^2 = 0.1$, which is characteristic of aluminum alloys, and drop the primes, we have the partial differential equations in non-dimensional form as

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \qquad (3\text{-}12)$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 10.8\ p + 10.8\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right] \qquad (3\text{-}13)$$

The median-fiber stresses in non-dimensional forms are

$$\sigma_x' = \frac{\partial^2 F}{\partial y^2}$$

$$\sigma_y' = \frac{\partial^2 F}{\partial x^2} \qquad (3\text{-}14)$$

$$\tau'_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

and the median-fiber strains are

$$\epsilon'_x = \frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}$$

$$\epsilon'_y = \frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2} \qquad (3\text{-}15)$$

$$\gamma'_{xy} = -2(1+\mu)\frac{\partial^2 F}{\partial x \partial y}$$

The extreme-fiber bending and shearing stresses are

$$\sigma''_x = - \frac{1}{2(1-\mu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma''_y = - \frac{1}{2(1-\mu^2)} \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \qquad (3\text{-}16)$$

$$\tau_{x,y}'' = - \frac{1}{2(1+\mu^2)} \frac{\partial^2 w}{\partial x \partial y}$$

CHAPTER IV

FORMULATION OF BOUNDARY CONDITIONS

1. General Discussion of Boundary Conditions

The governing differential equations as derived in Chapter III are two two-dimensional fourth order simultaneous partial differential equations. To obtain an unique solution, in the case of rectangular plates, there must be four given boundary conditions at each edge.

Before proceeding to the actual case, two theoretical boundary conditions may be mentioned.

(1) Simply-supported plates, where the edges can rotate freely about the supports and can move freely along the supports.

(2) Clamped or built-in plates, where the edges are clamped rigidly against rotation about the supports and at the same time they are prevented from having any displacements along the supports.

Actually, it is to be expected that neither of these conditions will be fulfilled exactly in a structure.

Now let us proceed to the bending problem where the bottom plating of a seaplane is to be studied. The behavior of the sheet approximates that in an infinite sheet supported on a homogeneous elastic network with rectangular fields of the same rigidity as the supporting framework of the seaplane.

The displacement in the plane of the sheet and the slope of the sheet relative to the plane of the network must be zero from symmetry wherever the sheet passes over the center line of each supporting beam. Each rectangular field will therefore behave as a rectangular plate clamped along its four edges on supports that are rigid enough in the plane of the sheet to prevent their displacement in that plane. At the same time these supports must have a rigidity normal to the plane of the sheet equal to that of the actual supports in the seaplane bottom.

The rigidity of the supports will lie somewhere between the unattainable extremes of zero rigidity and infinite rigidity. The extreme of infinite rigidity normal to the plane of the sheet is one that may be approximated in actual designs. It is probable that the stress distribution in such a fixed edge plate will, in most cases, be less favorable than the stress distribution in the elastic-edge plate. The strength of plates obtained from the theory will therefore be on the safe side if applied in seaplane design. Reference might be made in this connection to a paper by Mesnager[63] in which it is shown that a rectangular plate with elastic edges of certain flexibility will be less highly stressed than a clamped-edge plate. It may also be seen clearly by comparing the extreme fiber-stress calculations by Levy[21] and Way[17] for simply-supported plates and clamped plates.

However, the impact pressure on a flying boat bottom in actual cases is not even approximately uniform over a portion of the sheet covering several rectangular fields. Usually one rectangular panel of the bottom plating would resist a higher impact pressure than the surrounding panels, and the sheet is supported on beams of torsional stiffness insufficient to develop large moments along the edges. The high bending stresses at the edges characteristic of rigidly clamped plates would then be absent. To approximate this condition the plate may be assumed to be simply supported so that it is free to rotate about the supports. At the same time the riveted joints prevent it from moving in the plane of the plate along and perpendicular to the supports. According to the same considerations as in the case of rigidly clamped edges, the result would be on the safe side. This case has never been discussed before, and it seems to be of importance to study such a problem.

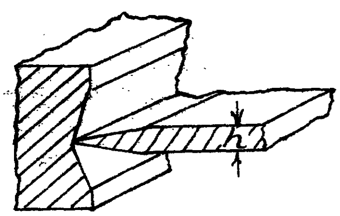
For the combined bending and buckling problems the same considerations will be true. It is evident, however, that as soon as the side thrust is applied, there are displacements perpendicular to the supported edges in the plane of the plate. Gall[64] has found that a stiffener attached to a flat sheet carrying a compressive load contributed approximately the same elastic support to the sheet as was required to give a simply-supported edge. In combined bending and compression problems, therefore, it seems also important to study the ideal simply-supported plates.

The analytical expressions for all these boundary conditions are formulated in the following sections.

2. Analytical Expressions of the Boundary Conditions

(1) Simply Supported Edge

If the edge $y = 0$ of the plate is simply supported, the deflection w along this edge must be zero. At the same time this edge can rotate freely with respect to the x-axis; i.e., there is no bending moment M_y along this edge. This kind of support is represented in the following figure.



The analytical expressions of the boundary conditions in this case are

$$(w)_{y=0} = 0 \qquad (4\text{-}1)$$

$$\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)_{y=0} = 0$$

Similarly, if the edge $x = 0$ of the plate is simply supported, the boundary conditions are

$$(w)_{x=0} = 0 \qquad (4\text{-}2)$$

$$\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)_{x=0} = 0$$

Now since $w = 0$ along $y = 0$, $\partial w/\partial x$ and $\partial^2 w/\partial x^2$ must be zero also. The boundary conditions can therefore be written

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0 \qquad (4\text{-}1a)$$

Similarly, for the edge $x = 0$, we have

$$(w)_{x=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = 0 \qquad (4\text{-}2a)$$

If the plate has ideal simply-supported edges, it must be free to move along the supported edges in the plane of the plate. This is equivalent to saying that the shearing stress along the edges in the plane of the plate is zero. Analytically, it is

$$(\tau'_{xy})_{y=0} = 0$$

or

$$\left(\frac{\partial^2 F}{\partial x \partial y}\right)_{y=0} = 0 \qquad (4\text{-}3)$$

One more boundary condition is required to solve the plate problems uniquely, and this may be obtained by specifying either the normal stresses or the displacements along the edges.

For a plate having zero edge compression, the normal stresses along the edges are zero. It is analytically expressed as

$$(\sigma_x')_{x=0} = 0$$

$$(\sigma_y')_{y=0} = 0$$

or

$$\left(\frac{\partial^2 F}{\partial y^2}\right)_{x=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0 \qquad (4\text{-}4)$$

It is shown in Section 3, Chapter III, that the strain in the median plane is

$$\epsilon_x' = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_y' = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

Therefore, we have

$$\frac{\partial u}{\partial x} = \epsilon_x' - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\frac{\partial v}{\partial y} = \epsilon_y' - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (4\text{-}5)$$

and the displacement of the edges in the x-direction is

$$u = \int_y \left[\epsilon_x' - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx \qquad (4\text{-}5a)$$

while the displacement of the edges in the y-direction is

$$v = \int_x \left[\epsilon_y' - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] dy \qquad (4\text{-}5b)$$

The addition of side thrust may be expressed in terms of the change in displacement of the edges.

Expressing ϵ_x' and ϵ_y' in terms of the stress function F, we have

$$u = \int_y \left[\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] dx$$

$$v = \int_x \left[\frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] dy \qquad (4\text{-}5c)$$

(2) Clamped, or Built-in Edge.

If the edge of a plate is clamped, the deflection along this edge is zero, and the tangent plane to the deflected middle surface along this edge coincides with the initial position of the middle plane of the plate.

If the x-axis coincides with the clamped edge, the boundary conditions are

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=0} = 0 \qquad (4\text{-}6)$$

If the y-axis coincides with the clamped edge, the boundary conditions are

$$(w)_{x=0} = 0$$

$$\left(\frac{\partial w}{\partial x}\right)_{x=0} = 0 \qquad (4\text{-}7)$$

If the edge is clamped rigidly against any displacement along its support, the strain in the median fibers must be zero along that adge. The boundary conditions are therefore

$$(\epsilon_y')_{x=0} = 0$$

$$(\epsilon_x')_{y=0} = 0$$

or
$$\left(\frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2}\right)_{x=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0 \qquad (4\text{-}8)$$

The one additional condition required is again furnished by specifying displacements of edges as in Eq.(4-5c).

(3) Riveted Panel with Normal Pressure Higher than the Surrounding Ones.

As discussed in Section 1, the boundary conditions which would approximate this case are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0 \qquad (4\text{-}9)$$

$$\int_y \left[\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx = 0$$

if $y = 0$ is one of the edges.

The first two expressions are those of simply-supported edges, the third one gives the condition of zero strain along the supports, and the last one specifies that the displacement of the edge is zero.

CHAPTER V

THE FINITE DIFFERENCES EQUATIONS OF THE BOUNDARY VALUE PROBLEMS

1. The Calculus of Finite Differences

Brooks Taylor, in his book "The Method of Increments" (1715-1717), was the first to consider equations of finite differences. They are used more and more frequently in recent years as approximations to differential equations. In fact, the very large deflection problem has been studied by Henky[5],[6] and Kaiser[7] by using finite differences approximations.

Before we proceed to convert our partial differential equations into finite differences expressions, some fundamental concepts about the finite differences approximation may be mentioned.

Let us assume that a function $f(x)$ of the variable x is defined for equidistant values of x. If x is one of the values for which $f(x)$ is defined, $f(x)$ is also defined for the values of $x + k\Delta x$, where Δx is the interval between two successive values of x, and k is an integer. We denote the value of the function $y = f(x)$ for $x + k\Delta x$, for the sake of simplicity, by writing y with a subscript:

$$f(x + k\Delta x) = y_{x+k\Delta x} \qquad (5\text{-}1)$$

We now define the <u>first difference</u> or the <u>difference of the first order</u> Δy_x of y at the point x as the increments of the value of y in going from x to $x + \Delta x$:

$$\Delta y_x = y_{x+\Delta x} - y_x \tag{5-2}$$

It is seen that we have chosen arbitrarily the step in the direction of increasing x; we could also define Δy_x by the difference $y_x - y_{x-\Delta x}$.

Continuing this process, we call the increment of the first difference in going from x to $x + \Delta x$ the difference of second order of y at x; i.e.,

$$\begin{aligned} \Delta^2 y_x &= \Delta y_{x+\Delta x} - \Delta y_x \\ &= y_{x+2\Delta x} - 2y_{x+\Delta x} + y_x \end{aligned} \tag{5-3}$$

In general, we define the <u>difference of order n</u> by

$$\Delta^n y_x = \Delta^{n-1} y_{x+\Delta x} - \Delta^{n-1} y_x \tag{5-4}$$

If we choose Δx equal to unity, then we write

$$y_{x+n\Delta x} = y_{x+n}$$

Using this notation, the sequence of differences becomes

$$\begin{aligned} \Delta y_x &= y_{x+1} - y_x \\ \Delta^2 y_x &= y_{x+2} - 2y_{x+1} + y_x \\ \Delta^3 y_x &= y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x \end{aligned}$$

. . . .

$$\Delta^n y_x = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n-r)!} y_{x+n-r} \tag{5-5}$$

In many physical problems only differences of even order occur. In such cases it is more convenient to define the differences $\Delta^{2m} y_x$ in another way. We shall write

$$\Delta^2 y_x = y_{x-1} - 2y_x + y_{x+1} \tag{5-6}$$

i.e., $\Delta^2 y_x$ is the increment of the first difference taken on the right and left sides of the point x. In general

$$\Delta^{2m}(y_x) = \Delta^2(\Delta^{2m-2} y_x) \tag{5-7}$$

In this case a difference of order 2m represents a linear expression in $y_{x-m}, y_{x-m+1}, \ldots\ldots, y_x, \ldots\ldots, y_{x+m-1}, y_{x+m}$.

2. The Finite Differences Expression of Two-Dimensional Cases.

In replacing the partial differentials by the finite differences expressions, we have to consider the differences corresponding to the changes of both the coordinates x and y. We shall use the following notations for the first differences at a point Amn with coordinates $m\Delta x$ and $n\Delta y$. The notation used in designating adjacent points is shown in the following figure.

Fig. V-1

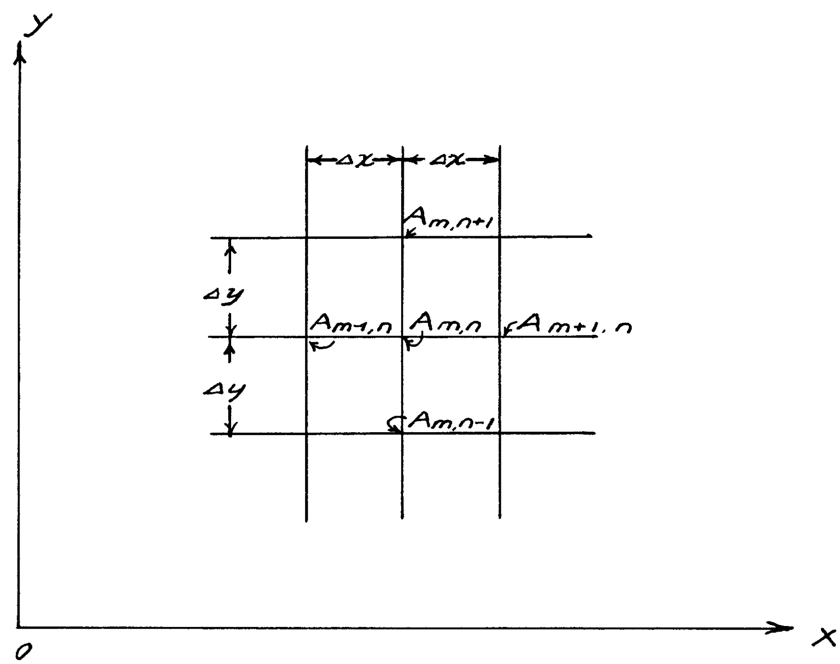


Fig. V-1

$$\Delta x\, w_{m-1,n} = w_{m,n} - w_{m-1,n},$$

$$\Delta x\, w_{m,n} = w_{m+n,n} - w_{m,n},$$

$$\Delta y\, w_{m,n-1} = w_{m,n} - w_{m,n-1},$$

$$\Delta y\, w_{m,n} = w_{m,n+1} - w_{m,n}.$$

Having the first differences, we can form the three kinds of second differences as follows:

$$\begin{aligned}\Delta_{xx} w_{m,n} &= \Delta_x^2 w_{m,n} \\ &= w_{m+1,n} - w_{m,n} - (w_{m,n} - w_{m-1,n}) \\ &= w_{m+1,n} - 2w_{m,n} + w_{m-1,n}\end{aligned}$$

$$\begin{aligned}\Delta_{yy} w_{m,n} &= \Delta_y^2 w_{m,n} \\ &= w_{m,n+1} - w_{m,n} - (w_{m,n} - w_{m,n-1}) \\ &= w_{m,n+1} - 2w_{m,n} + w_{m,n-1}\end{aligned}$$

$$\begin{aligned}\Delta_{xy} w_{m,n} &= \Delta_y w_{m+1,n} - \Delta_y w_{m,n} \\ &= w_{m+1,n+1} - w_{m+1,n} - (w_{m,n+1} - w_{m,n}) \\ &= w_{m+1,n+1} - w_{m+1,n} - w_{m,n+1} + w_{m,n} \qquad (5\text{-}8)\end{aligned}$$

And the three kinds of fourth differences which will be used later on are:

$$\begin{aligned}\Delta_{xxxx} w_{m,n} &= \Delta_x^4 w_{m,n} \\ &= w_{m+2,n} - 4w_{m+1,n} + 6w_{m,n} - 4w_{m-1,n} + w_{m-2,n}\end{aligned}$$

$$\begin{aligned}\Delta_{yyyy} w_{m,n} &= \Delta_y^4 w_{m,n} \\ &= w_{m,n+2} - 4w_{m,n+1} + 6w_{m,n} - 4w_{m,n-1} + w_{m,n-2}\end{aligned}$$

$$\begin{aligned}\Delta_{xxyy} w_{m,n} &= \Delta_{xy}^2 w_{m,n} \\ &= w_{m+1,n+1} - 2w_{m+1,n} + w_{m+1,n-1} - 2w_{m,n+1} + 4w_{m,n} \\ &\quad - w_{m,n-1} + w_{m-1,n+1} - 2w_{m-1,n} + w_{m-1,n-1} \qquad (5\text{-}9)\end{aligned}$$

3. Conversion of the Governing Partial Differential Equations Into Finite Differences Equations.

Partial differentials may be approximated by finite differences as follows:

$$\frac{\partial w}{\partial x} = \frac{\Delta_x w}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{\Delta_y w}{\Delta y},$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\Delta_x^2 w}{\Delta x^2}, \quad \frac{\partial^2 w}{\partial y^2} = \frac{\Delta_y^2 w}{\Delta y^2},$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\Delta_{xy} w}{\Delta x \Delta y},$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{\Delta_x^4 w}{\Delta x^4}, \quad \frac{\partial^4 w}{\partial y^4} = \frac{\Delta_y^4 w}{\Delta y^4},$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{\Delta_{xy}^2 w}{\Delta x^2 \Delta y^2} \tag{5-10}$$

By the above relations, our governing partial differential equations (3-12), (3-13) may be replaced by the following difference equations:

$$\frac{\Delta_x^4 F}{\Delta x^4} + 2\frac{\Delta_{xy}^2 F}{\Delta x^2 \Delta y^2} + \frac{\Delta_y^4 F}{\Delta y^4} = \left(\frac{\Delta_{xy} w}{\Delta x \Delta y}\right)^2 - \frac{\Delta_x^2 w}{\Delta x^2}\,\frac{\Delta_y^2 w}{\Delta y^2}$$

$$\frac{\Delta_x^4 w}{\Delta x^4} = 2\frac{\Delta_{xy}^2 w}{\Delta x^2 \Delta y^2} + \frac{\Delta_y^4 w}{\Delta y^4} = 10.8p$$

$$+ 10.8\left[\frac{\Delta_y^2 F}{\Delta y^2}\,\frac{\Delta_x^2 w}{\Delta x^2} + \frac{\Delta_x^2 F}{\Delta x^2}\,\frac{\Delta_y^2 w}{\Delta y^2} - 2\frac{\Delta_{xy} F}{\Delta x \Delta y}\,\frac{\Delta_{xy} w}{\Delta x \Delta y}\right] \tag{5-11}$$

If we take $\Delta x = \Delta y = \Delta \ell$, and use the relations (5-8), (5-9), Eqs.(5-11) may be written as

$$\begin{aligned} & F_{m+2,n} - 8F_{m+1,n} + 20F_{m,n} - 8F_{m-1,n} + F_{m-2,n} + F_{m,n+2} - 8F_{m,n+1} \\ & \quad - 8F_{m,n-1} + F_{m,n-2} + 2F_{m+1,n+1} + 2F_{m+1,n-1} + 2F_{m-1,n+1} \\ & \quad + 2F_{m-1,n-1} \\ & \quad = (w_{m+1,n+1} - w_{m+1,n} - w_{m,n+1} - w_{m,n})^2 \\ & \quad - (w_{m+1,n} - 2w_{m,n} + w_{m-1,n})(w_{m,n+1} - 2w_{m,n} + w_{m,n-1}) \end{aligned} \qquad (5\text{-}12)$$

and

$$\begin{aligned} & w_{m+2,n} - 8w_{m+1,n} + 20w_{m,n} - 8w_{m-1,n} + w_{m-2,n} + w_{m,n+2} - 8w_{m,n+1} \\ & \quad - 8w_{m,n-1} + w_{m,n-2} + 2w_{m+1,n+1} + 2w_{m+1,n-1} + 2w_{m-1,n+1} \\ & \quad + 2w_{m-1,n-1} \\ & \quad = 10.8(\Delta \ell)^4 p + 10.8 \quad (F_{m,n+1} - 2F_{m,n} + F_{m-1,n}) \\ & \quad (w_{m+1,n} - 2w_{m,n} + w_{m-1,n}) + (F_{m+1,n} - 2F_{m,n} + F_{m-1,n}) \\ & \quad (w_{m,n+1} - 2w_{m,n} + w_{m,n-1}) - 2(F_{m+1,n+1} - F_{m+1,n} - F_{m,n+1} + F_{m,n}) \\ & \quad (w_{m+1,n+1} - w_{m+1,n} - w_{m,n+1} + w_{m,n}) \end{aligned} \qquad (5\text{-}13)$$

Eqs. (5-12) and (5-13) may seem to be very complicated. In actually writing out of these equations the following finite differences patterns or so-called Relaxation Patterns are useful.

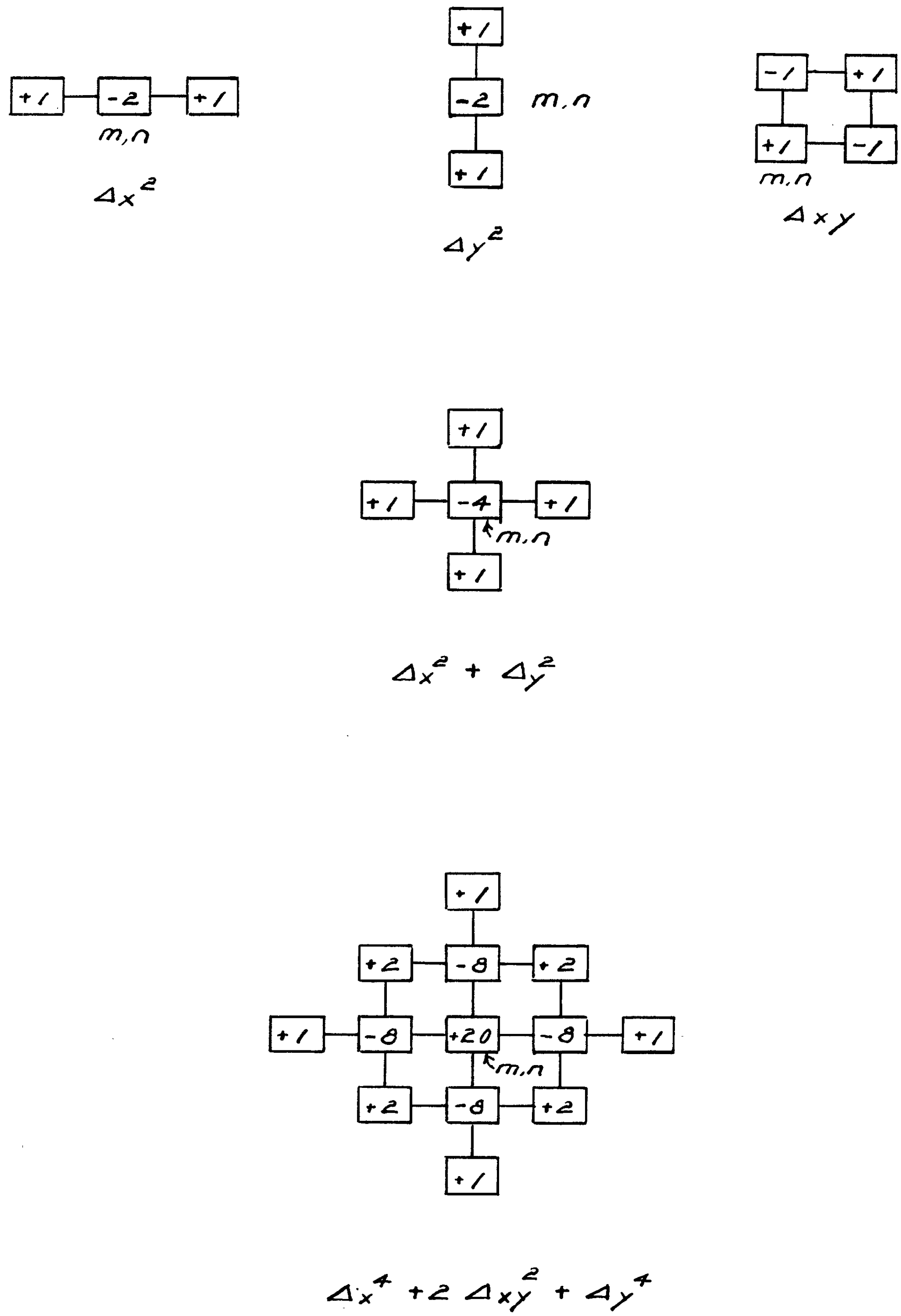


Fig. V-2

The Relaxation Patterns

4. Finite Differences Expressions of the Boundary Conditions

(1) Simply-Supported Edge.

The boundary conditions for the simply-supported edge $y = 0$ are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial x \partial y}\right)_{y=0} = 0$$

and $(\partial^2 F/\partial x^2)_{y=0} = 0$ for plates with zero edge compression, or

$$\int_y \left[\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx = u$$

for plates with zero or known edge displacements.

Let $n = 0$ denote the edge points along $y = 0$. The finite differences expressions of the boundary conditions are:

$$w_{m,o} = 0$$

$$(\Delta_y^2\, w)_{m,o} = 0 \tag{5-14}$$

$$(\Delta_{xy} F)_{m,o} = 0$$

and

$$(\Delta_x^2\, F)_{m,o} = 0 \tag{5-15}$$

for plates with zero edge compression and

$$\sum_{m=o}^{k-1} \left[\Delta_y^2 F - \mu \Delta_x^2 F - \frac{1}{2}(\Delta_x w)^2\right]_{m,i} = u_i \tag{5-16}$$

where $m = o, k$ represents points along the two edges $x = 0$ and x = a, and i is a point along any line y = constant along the plate.

(2) Clamped Edge.

The boundary conditions for the clamped edge y = 0 are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=0} = 0$$

$$\left[\left(\frac{\partial^2 F}{\partial y^2}\right) - \mu \frac{\partial^2 F}{\partial x^2}\right]_{y=0} = 0$$

$$\int_y \left[\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx = u$$

Following the same notations, the finite differences expressions are

$$w_{m,o} = 0$$

$$(\Delta_y w)_{m,o} = 0$$

$$(\Delta_y^2 F - \mu \Delta_x^2 F)_{m,o} = 0 \qquad (5\text{-}17)$$

$$\sum_{m=0}^{k-1} \left[\Delta_y^2 F - \mu \Delta_x^2 F - \frac{1}{2} (\Delta_x w)^2\right]_{m,i} = u_i$$

(3) Riveted Panel with Normal Pressure Higher Than The Surrounding Ones.

The boundary conditions which approximate this case are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0$$

$$\int_y \left[\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx = 0$$

if y = 0 is one of the edges.

Expressed in terms of finite differences, they are:

$$w_{m,o} = 0$$

$$(\Delta_y^2 \, w)_{m,o} = 0$$

$$(\Delta_y^2 \, F - \mu \Delta_x^2 F)_{m,o} = 0 \qquad (5\text{-}18)$$

$$\sum_{m=o}^{k-1} \left[\Delta_y^2 F - \mu \, \Delta_x^2 F - \frac{1}{2}(\Delta_x \, w)^2\right]_{m,i} = 0$$

5. The Boundary Value Problem in Terms of Finite Differences Expressions.

Now the boundary value problem, which approximates the riveted sheet-stringer panel subjected to uniform normal pressure higher than the surrounding ones, may be formulated in terms of finite differences.

To start with a simpler case, the square flat plate will be discussed, because the condition of symmetry requires only one-eighth of the plate to be studied.

The finite differences approximation of any differential equation requires that every point in the domain must satisfy the governing differential equations. If the points to be taken are infinite in number, the solution of the differences equations is the exact solution of the corresponding differential equations, and if the points to be taken are finite in number, the solution will be approximate, and the degree of approximation will increase as the number of points taken is reduced.

Since the diagonals of a square plate are axes of symmetry, if the boundary conditions along the four sides are the same, $w_{i,k} = w_{k,i}$, and $\epsilon_{i,k} = \epsilon_{k,i}$. The conditions for zero edge displacements may be put into different forms, as follows. These conditions are:

$$(\mathcal{E}_x)_{o,i} + (\mathcal{E}_x)_{1,i} + \cdots + (\mathcal{E}_x)_{m,i} + \cdots + (\mathcal{E}_x)_{k,i}$$

$$= \frac{1}{2}\sum_{m=0}^{k-1} (\Delta_x w)^2_{m,i} = \frac{1}{2}\sum_{m=0}^{k-1} (w_{m+1,i} - w_{m,i})^2$$

$$(\mathcal{E}_y)_{i,o} + (\mathcal{E}_y)_{i,1} + \cdots + (\mathcal{E}_y)_{i,n} + \cdots + (\mathcal{E}_y)_{i,k}$$

$$= \frac{1}{2}\sum_{n=0}^{k-1} (\Delta_y w)^2_{i,n} = \frac{1}{2}\sum_{n=0}^{k-1} (w_{i,n+1} - w_{i,n})^2$$

Adding these two equations together and noting that $w_{i,k} = w_{k,i}$, and $(\mathcal{E}_x)_{i,k} = (\mathcal{E}_y)_{k,i}$, we know that

$$(\mathcal{E}_x + \mathcal{E}_y)_{i,o} + (\mathcal{E}_x + \mathcal{E}_y)_{i,1} + \cdots + (\mathcal{E}_x + \mathcal{E}_y)_{i,n} + \cdots$$

$$+ (\mathcal{E}_x + \mathcal{E}_y)_{i,k-1} = \sum_{n=0}^{k-1} (w_{i,n+1} - w_{i,n})^2 \qquad (5\text{-}19)$$

To obtain a better approximation, the left side of Eq.(5-19) may be re-written to give

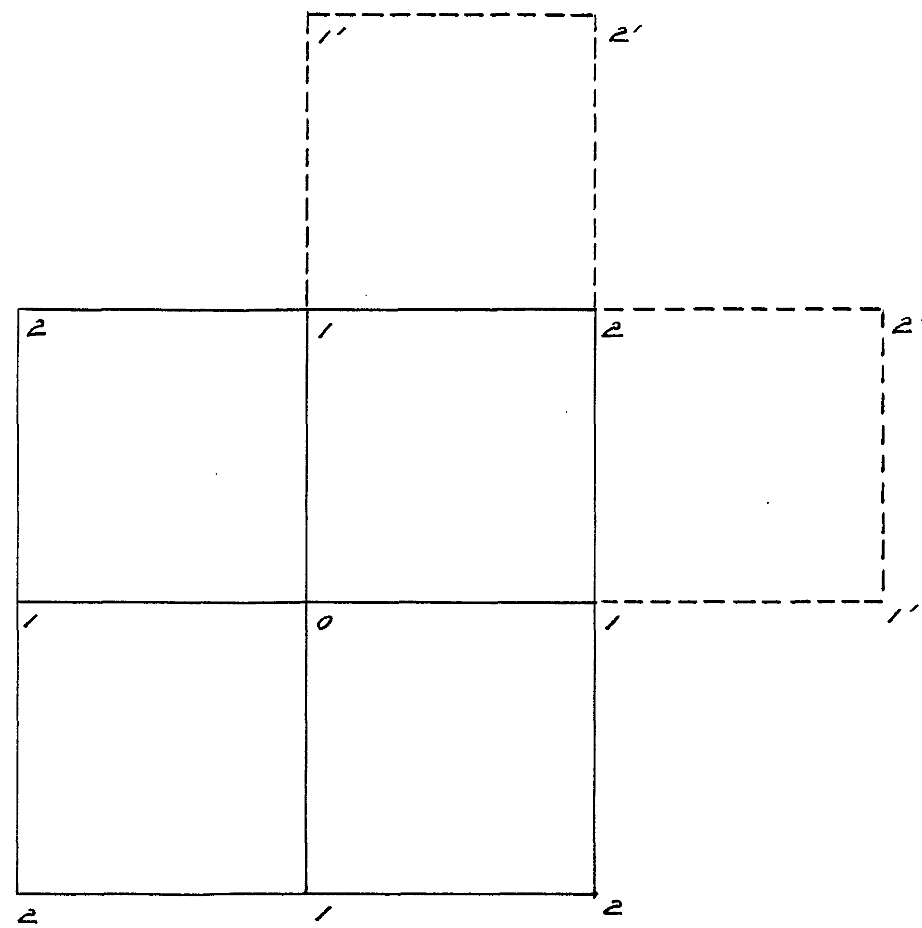


Fig. V-3

$$\frac{1}{2}(\varepsilon_x + \varepsilon_y)_{i,o} + (\varepsilon_x + \varepsilon_y)_{i,1} + \cdots + (\varepsilon_x + \varepsilon_y)_{i,n} + \cdots$$

$$+ (\varepsilon_x + \varepsilon_y)_{i,k-1} + \frac{1}{2}(\varepsilon_x + \varepsilon_y)_{i,k}$$

$$= \sum_{n=0}^{k-1} (w_{i,n+1} - w_{i,n})^2 \qquad (5\text{-}20)$$

Now,

$$\varepsilon_x + \varepsilon_y = \frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2}$$

$$= \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)(1-\mu)$$

$$= (\nabla^2 F)(1-\mu) \qquad (5\text{-}21)$$

Note that here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Eq.(5-20) then becomes

$$(\nabla^2 F)_{i,o} + 2(\nabla^2 F)_{i,1} + \cdots + 2(\nabla^2 F)_{i,n} + \cdots + 2(\nabla^2 F)_{i,k-1}$$

$$+ (\nabla^2 F)_{i,k} = \frac{2}{(1-\mu)} \sum_{n=0}^{k-1} (w_{i,n+1} - w_{i,n})^2 \qquad (5\text{-}22)$$

This simplification is not necessary, but it will be used in applying the Relaxation Method.

(1) n = 1

Referring to Fig. V-3, points 1', 2' are fictitious points outside the plate in order to give a better approximation for the boundary conditions.

Using $\mu^2 = 0.1$ or $\mu = 0.316{,}228$ for aluminum alloy, the compatibility equation is

$$20F_o - 32F_1 + 8F_2 + 4F_1 = K_o \qquad (5\text{-}23)$$

where $K_o = (w_2 - 2w_1 + w_o)^2 - (2w_1 - 2w_o)^2$. Then the equilibrium equation is

$$20\,w_0 - 32\,w_1 + 8\,w_2 + 4\,w_1' = p' + 10.8\left\{2(2F_1 - 2F_0)\right.$$
$$\left.(2\,w_1 - 2\,w_0) - 2(w_2 - 2\,w_1 + w_0)(F_2 - 2F_1 + F_0)\right\} \qquad (5\text{-}24)$$

where $p' = 12(1 - \mu^2)(\Delta \ell)^4 = .675p$, since $\Delta \ell = 1/2$.

The boundary conditions are:

(a) $w_1 = 0 \qquad w_2 = 0$

(b) $w_1' - 2w_1 + w_0 = 0$

(c) $F_0 - 2F_1 + F_1' - \mu(2F_2 - 2F_1) = 0$

(d) $(4F_1 - 4F_0) + (F_0 + 2F_2 + F_1' - 4F_1) = S_1$

where $S_1 = (w_1 - w_0)^2\,2/1-\mu = (w_1 - w_0)^2/.341886$

Now the boundary value problem determines the values of w uniquely, and the values of F with an addition of an unknown constant. Since this constant is irrelevant to the problem, we may define it by letting $F_2 = 0$.

Solving w_1', w_2', F_1' from the boundary conditions, we have

$w_1' = -w_0$

$w_2' = -w_1 = 0$

$F_1' = -F_0 + 2(1 - \mu)F_1$

Substitute these values into Eqs. (5-23), (5-24) and (d), and the resulting equations are

$$16F_0 - 26.529824F_1 = -3w_0^2$$
$$16w_0 = p' + 43.2w_0F_0 \qquad (5\text{-}25)$$
$$-4F_0 + 1.367544F_1 = \frac{w_0^2}{.341886}$$

Note: The nine or ten significant figures used in these equations result from the use of a computing machine having 10 columns. In order to get satisfactory results in subsequent computations it is convenient to retain a number of figures beyond those normally considered justifiable in view of the precision of the basic data.

(2) n = 2

Referring to Fig. V-4, points 3', 4', 5' are again fictitious points. The compatibility equations are:

$$\left.\begin{aligned} &20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0 \\ &-8F_0 + 25F_1 - 16F_2 - 8F_3 + 6F_4 + F_3' = K_1 \\ &2F_0 - 16F_1 + 22F_2 + 4F_3 - 16F_4 + 2F_5 + 2F_4' = K_2 \end{aligned}\right\} \quad (5\text{-}26)$$

where K_0, K_1, K_2 are equal to $\left[(\Delta_{xy}w)^2 - \Delta_x^2 w\, \Delta_y^2 w\right]$ at points 0, 1, 2 respectively.

The equilibrium conditions are:

$$\left.\begin{aligned} &20w_0 - 32w_1 + 8w_2 + 4w_3 = p' + 21.6\left\{(\alpha_0' + \beta_0')(w_1 - w_0) - \gamma'(w_0 - 2w_1 + w_2)\right\} \\ &-8w_0 + 25w_1 - 16w_2 - 8w_3 + 6w_4 + w_3' = p' + 10.8\left\{\alpha_1'(2w_2 - 2w_1) + \beta_1'(w_0 - 2w_1 + w_3) - 2\gamma_1'(w_4 - w_3 - w_2 + w_1)\right\} \\ &+2w_0 - 16w_1 + 22w_2 + 4w_3 - 16w_4 + 2w_5 + 2w_4' = p' + 10.8\left\{(\alpha_2' + \beta_2')(w_4 - 2w_2 + w_1) - 2\gamma_2'(w_5 - 2w_4 + w_2)\right\} \end{aligned}\right\} \quad (5\text{-}27)$$

where α', β', γ' are $\Delta_x^2 F$, $\Delta_y^2 F$, $\Delta_{xy}F$ at the points indicated by the subscripts respectively.

The conditions for zero edge displacements are:

$$\left.\begin{aligned} &-2F_0 - 3F_1 + 4F_2 - 2F_3 + 2F_4 + F_3' = S_1 \\ &F_0 - 5F_2 + 2F_3 + F_5 + F_4' = S_2 \end{aligned}\right\} \quad (5\text{-}28)$$

where $S_1 = \dfrac{1}{.341886}\left\{(w_1 - w_0)^2 + (w_3 - w_1)^2\right\}$

$$S_2 = \frac{1}{.341886}\left\{(w_2 - w_1)^2 + (w_4 - w_2)^2\right\}$$

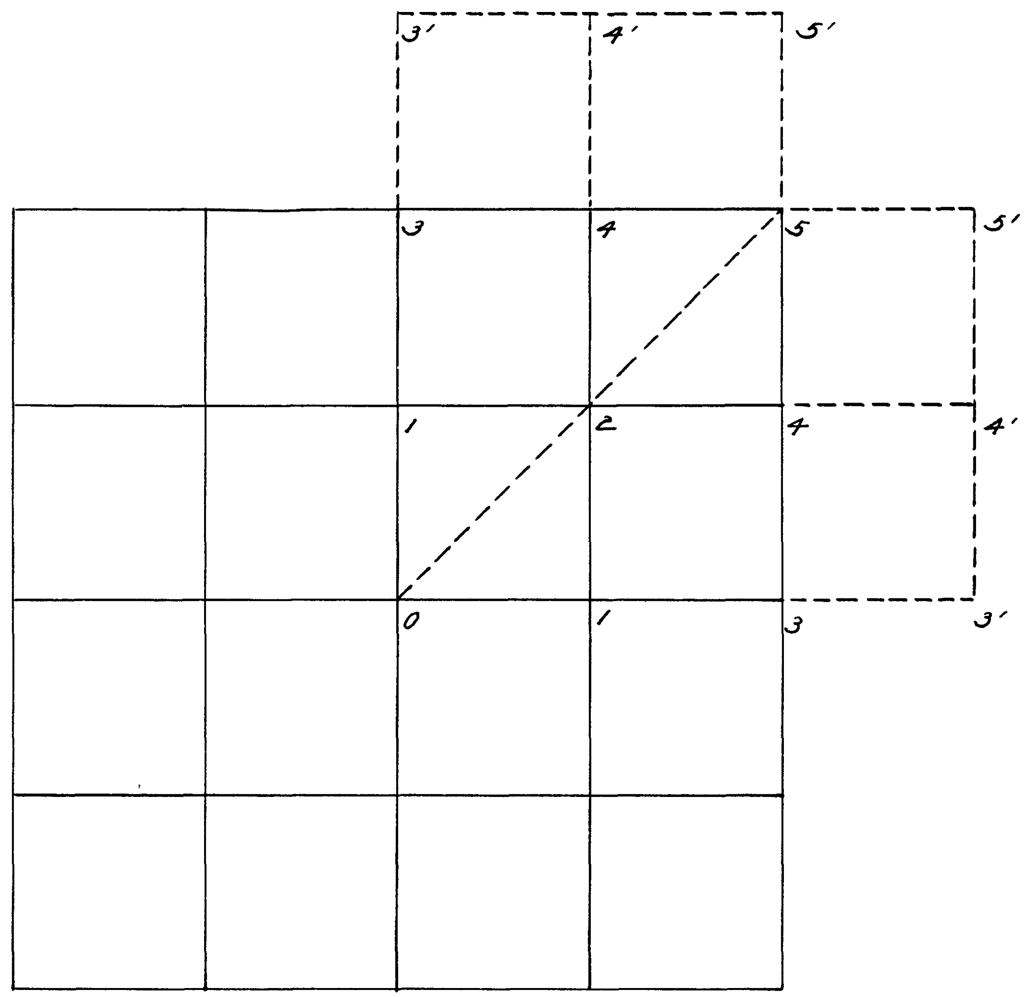


Fig. V-4

The boundary conditions are:

(a) $w_3 = 0, \; w_4 = 0, \; w_5 = 0$

(b) $w_3' - 2w_3 + w_1 = 0$

$w_4' - 2w_4 + w_2 = 0$

$w_5' - 2w_5 + w_4 = 0$

(c) $F_1 - 2F_3 + F_3' - \mu(2F_4 - 2F_3) = 0$

$F_2 - 2F_4 + F_4' - \mu(F_5 - F_4 + F_3) = 0$

$F_4 - 2F_5 + F_5' = 0$

For the same reason as explained in the case of $n = 1$, let $F_5 = 0$. Solving of the boundary conditions equations gives

(d) $w_3' = -w_1$

$w_4' = -w_2$

$w_5' = 0$

(e) $F_5' = -F_4$

$F_4' = 2F_4 + \mu(F_3 - 2F_4) - F_2$

$F_3' = 2F_3 + \mu(2F_4 - 2F_3) - F_1$

Combining Eqs. (d) and (e) with Eqs. (5-26), (5-27) and (5-28), we have

$$\left.\begin{aligned}
&20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0 \\
&-8F_0 + 24F_1 - 16F_2 - 6.632456F_3 + 6.632456F_4 = K_1 \\
&2F_0 - 16F_1 + 20F_2 + 4.632456F_3 - 13.264912F_4 = K_2 \\
&-2F_0 - 4F_1 + 4F_2 - .632456F_3 + 2.632456F_4 = S_1 \\
&F_0 - 6F_2 + 2.316228F_3 + 1.367544F_4 = S_2
\end{aligned}\right\} \quad (5\text{-}29)$$

and

$$[20 + 21.6(\alpha_0' + \beta_0' + \gamma_0')]\, w_0 - 32 + 21.6(\alpha_0' + \beta_0' + 2\gamma_0')]\, w_1 + (8 + 21.6\,\gamma_0')w_2 = p' \quad (5\text{-}30)$$

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$$-(8 + 10.8\beta_1')w_0 + [24 + 21.6(\alpha_1' + \beta_1' + \gamma_1')]\, w_1$$
$$-[16 + 21.6(\alpha_1' + \gamma_1')]\, w_2 = p'$$
$$2w_0 - [16 + 10.8(\alpha_2' + \beta_2')]\, w_1 + [20 + 21.6$$
$$(\alpha_2' + \beta_2' + \gamma_2')]\, w_2 = p'$$

end of (5-30)

Here $p' = 12(1-\mu^2)(\Delta \ell)^4 p = .0421875p$, since $\Delta \ell = 1/4$.

(3) n = 3

Referring to Fig. V-5 and noting that 6', 7', 8', 9' are fictitious points for reasons explained in the case n = 1, we have the compatibility equations as follows:

$$20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0$$
$$-8F_0 + 25F_1 - 16F_2 - 8F_3 + 6F_4 + F_6 = K_1$$
$$2F_0 - 16F_1 + 22F_2 + 4F_3 - 16F_4 + 2F_5 + 2F_7 = K_2$$
$$F_0 - 8F_1 + 4F_2 + 20F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_6' = K_3 \quad (5\text{-}31)$$
$$3F_1 - 8F_2 - 8F_3 + 23F_4 - 8F_5 + 2F_6 - 8F_7 + 3F_8 + F_7' = K_4$$
$$2F_2 + 2F_3 - 16F_4 + 20F_5 + 4F_7 - 16F_8 + 2F_9 + 2F_8' = K_5$$

where K_0, K_1, K_2, K_4, K_5 are equal to $[(\Delta_{xy}w)^2 - \Delta_x{}^2 w \Delta_y{}^2 w]$ at points 0, 1, 2, 3, 4, and 5 respectively.

The equilibrium equations are:

$$20w_0 - 32w_1 + 8w_2 + 4w_3 = p' + 21.6\{(\alpha_0' + \beta_0')$$
$$(w_1 - w_0) - \gamma_0'(w_0 - 2w_1 + w_2)\}$$
$$-8w_0 + 25w_1 - 16w_2 - 8w_3 + 6w_4 = p' + 10.8\{2\alpha_1'(w_2 - w_1)$$
$$+ \beta_1'(w_0 - 2w_1 + w_3) - 2\gamma_1'(w_1 - w_2 - w_3 + w_4)\}$$
$$2w_0 - 16w_1 + 22w_2 + 4w_3 - 16w_4 + 2w_5 = p' + 10.8$$
$$\{(\alpha_2' + \beta_2')(w_1 - 2w_2 + w_4) - 2\gamma_2'(w_2 - 2w_4 + w_5)\}$$

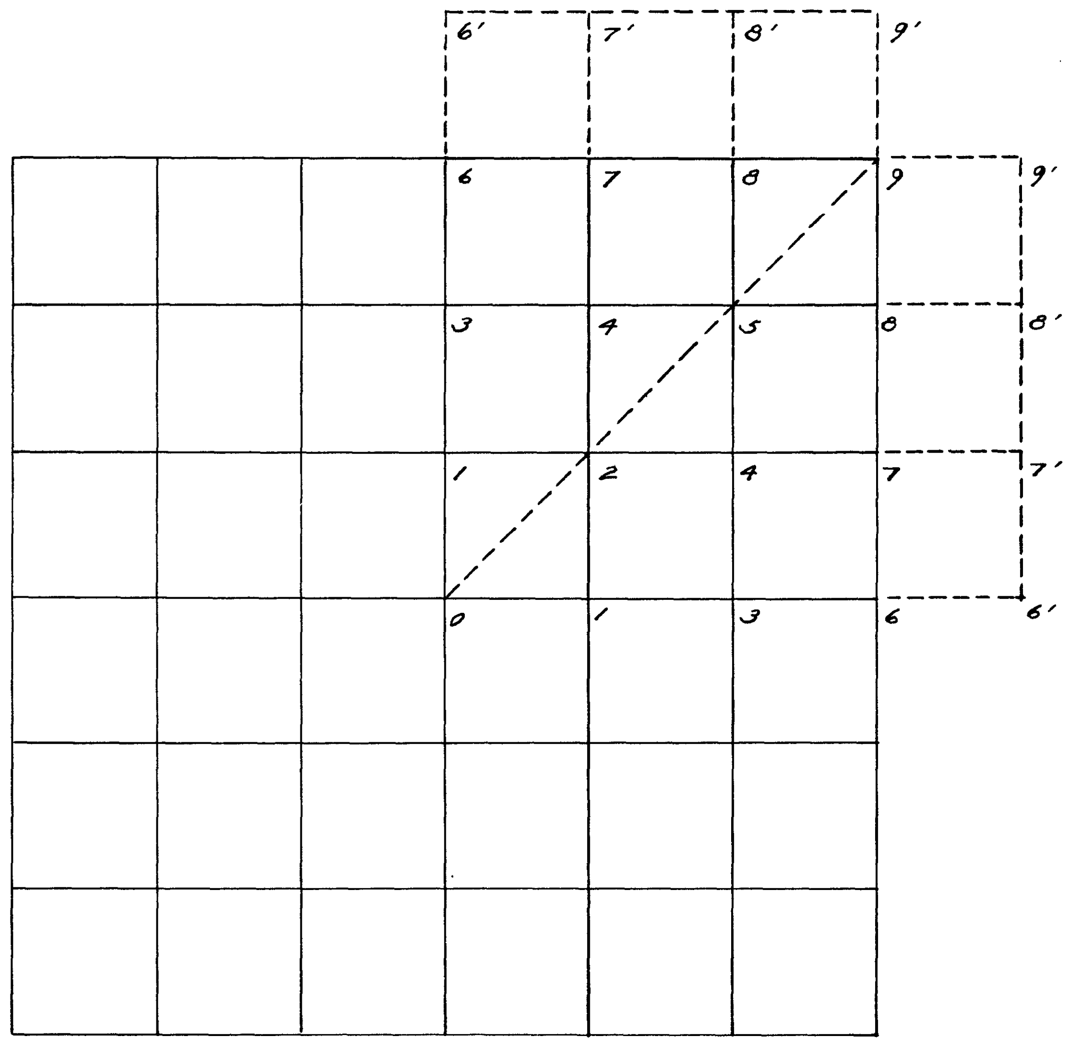


Fig. V-5

$$w_0 - 8w_1 + 4w_2 + 20w_3 - 16w_4 + 2w_5 - 8w_6 + 4w_7 + w_6'$$
$$= p' + 10.8 \left\{\alpha_3'(2w_4 - 2w_3) + \beta_3'(w_1 - 2w_3 + w_6) - 2\gamma_3'(w_3 - w_4 - w_6 + w_7)\right\}$$

$$3w_1 - 8w_2 - 8w_3 + 23w_4 - 8w_5 + 2w_6 - 8w_7 + 3w_8 + w_7'$$
$$= p' + 10.8 \left\{\alpha_4'(w_3 - 2w_4 + w_5) + \beta_4'(w_2 - 2w_4 + w_7) - 2\gamma_4'(w_4 - w_5 - w_7 + w_8)\right\}$$

$$2w_2 + 2w_3 - 16w_4 + 20w_5 + 4w_7 - 16w_8 + 2w_9 + 2w_8'$$
$$= p' + 10.8 \left\{(\alpha_5' + \beta_5')(w_4 - 2w_5 + w_8) - 2\gamma_5'(w_5 - 2w_8 + w_9)\right\} \quad (5\text{-}32)$$

where α', β' and γ' are $\Delta_x^2 F$, $\Delta_y^2 F$, $\Delta_{xy} F$ corresponding to the subscripts respectively, and $p' = 12(1 - \mu^2)(\Delta \ell)^4 p = .008{,}333{,}33p$, since $\Delta\ell = 1/6$.

The conditions for zero edge displacements are:

$$- 2F_0 - 2F_1 + 4F_2 - 5F_3 + 4F_4 - 2F_6 + 2F_7 + F_6' = S_1$$
$$+ F_0 - 4F_2 + 3F_3 - 3F_4 + 2F_5 + F_6 - 2F_7 + F_8 + F_7' = S_2$$
$$+F_1 + 2F_2 - 2F_3 - 2F_4 - 5F_5 + F_6 + 3F_7 + F_9 + F_8' = S_3 \quad (5\text{-}33)$$

where $S = \dfrac{2}{1-\mu} \sum_{m=0}^{k-1} (w_{m+1,i} - w_{m,i})^2$

The boundary conditions are:

(a) $w_6 = 0, \quad w_7 = 0, \quad w_8 = 0, \quad w_9 = 0$

(b) $w_6' - 2w_6 + w_3 = 0$

$w_7' - 2w_7 + w_4 = 0$

$w_8' - 2w_8 + w_5 = 0$

$w_9' - 2w_9 + w_8 = 0$

$$(c)\ F_3 - 2F_6 + F_6' - \mu(2F_7 - 2F_6) = 0$$
$$F_4 - 2F_7 + F_7' - \mu(F_8 + F_6 - 2F_7) = 0$$
$$F_5 - 2F_8 + F_8' - \mu(F_9 + F_7 - 2F_8) = 0$$
$$F_9' - 2F_9 + F_8 = 0$$

Solving the boundary conditions equations gives

$$(d)\ w_6' = -w_3$$
$$w_7' = -w_4$$
$$w_8' = -w_5$$
$$w_9' = 0$$

$$(e)\ F_6' = -F_3 + 1.367544F_6 + .632456F_7$$
$$F_7' = -F_4 + 1.367544F_7 + .316228F_6 + .316228F_8$$
$$F_8' = -F_5 + 1.367544F_8 + .316228F_7$$
$$F_9' = -F_8$$

where we assume $F_9 = 0$ for the same reason as explained in the case of $n = 1$.

Combinining, we have:

$$20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0$$
$$- 8F_0 + 25F_1 - 16F_2 - 8F_3 + 6F_4 + F_6 = K_1$$
$$2F_0 - 16F_1 + 22F_2 + 4F_3 - 16F_4 + 2F_5 + 2F_7 = K_2$$
$$F_0 - 8F_1 + 4F_2 + 18F_3 - 16F_4 + 2F_5 - 6.632456F_6 + 4.632456F_7 = K_3$$
$$3F_1 - 8F_2 - 8F_3 + 22F_4 - 8F_5 + 2.316228F_6 - 6.632456F_7 + 3.316228F_8 = K_4$$
$$2F_2 + 2F_3 - 16F_4 + 18F_5 + 4.632456F_7 - 13.264912F_8 = K_5$$
$$-2F_0 - 2F_1 + 4F_2 - 6F_3 + 4F_4 - .632456F_6 + 2.632456F_7 = S_1$$

$$F_0 - 4F_2 + 3F_3 - 4F_4 + 2F_5 + 1.316228F_6 - .632456F_7 + 1.316228F_8 = S_2$$

$$F_1 + 2F_2 - 2F_3 - 2F_4 - 6F_5 + F_6 + 3.316228F_7 + 1.367,544F_8 = S_3 \qquad (5\text{-}34)$$

and

$$[20 + 21.6(\alpha_0' + \beta_0' + \gamma_0')]\, w_0 - [32 + 21.6(\alpha_0' + \beta_0' + \gamma_0')]\, w_1 + [8 + 21.6\,\gamma_0']\, w_2 + 4w_3 = p'$$

$$-[8 + 10.8\beta_1']\, w_0 + [25 + 21.6(\alpha_1' + \beta_1' + \gamma_1')]\, w_1 - [16 + 21.6(\alpha_1' + \gamma_1')]\, w_2 - [8 + 10.8(\beta_1' + 2\gamma_1')]\, w_3 + (6 + 21.6\,\gamma_1')w_4 = p'$$

$$2w_0 - [16 + 10.8(\alpha_2' + \beta_2')]\, w_1 + [22 + 21.6(\alpha_2' + \beta_2' + \gamma_2')]\, w_2 + 4w_3 - [16 + 10.8(\alpha_2' + \beta_2' + 4\gamma_2')]\, w_4 + (2 + 21.6\,\gamma_2')w_5 = p'$$

$$w_0 - [8 + 10.8\beta_3']\, w_1 + 4w_2 + [19 + 21.6(\alpha_3' + \beta_3' + \gamma_3')]\, w_3 - [16 + 21.6(\alpha_3' + \gamma_3')]\, w_4 + 2w_5 = p'$$

$$3w_1 - (8 + 10.8\,\beta_4')w_2 - (8 + 10.8\,\alpha_4')w_3 + [22 + 21.6(\alpha_4' + \beta_4' + \gamma_4')]\, w_4 - [8 + 10.8(\alpha_4' + 2\gamma_4')]\, w_5 = p'$$

$$2w_2 + 2w_3 - [16 + 10.8(\alpha_5' + \beta_5')]\, w_4 + [18 + 21.6(\alpha_5' + \beta_5' + \gamma_5')]\, w_5 = p' \qquad (5\text{-}35)$$

CHAPTER VI

THE METHOD OF SUCCESSIVE APPROXIMATIONS

1. Outline of the Method.

After we have expressed our boundary-value problems in terms of finite differences equations, the problem now is to solve the systems of non-linear simultaneous equations.

We have now two sets of simultaneous equations. The first set consists of the compatability equations and the equations specifying the conditions of zero edge displacements. These equations contain linear terms of stress function F and the second-order terms of deflection ratio w . They are of the form

$$\begin{aligned} c_{00} F_0 + c_{01} F_1 + \dots + c_{0n} F_n &= K_0 \\ c_{10} F_0 + c_{11} F_1 + \dots + c_{1n} F_n &= K_1 \\ \dots\dots\dots\dots \end{aligned} \qquad (6\text{-}1)$$

and $$c'_{10} F_0 + c'_{11} F_1 + \dots + c'_{1n} F_n = S_1$$

where $K = (\Delta_{xy} w)^2 - (\Delta_x^2 w)(\Delta^2_y w)$ at points 0, 1, etc. corresponding to the subscripts of K; $S_i = \frac{2}{1-\mu} \sum_m (\Delta_x w)^2_{m,i}$;

and c_{00}, c_{01}, ..., c'_{10}, c'_{11}, ... are given constants.

The second set is that of the equilibrium equations, which contain the linear terms of the deflection ratio w with coefficients which contain linear terms of F. They are of the form

$$(a_{00} + b_{00}\alpha'_0 + b'_{00}\beta'_0 + b''_{00}\gamma'_0)\ w_0 + (a_{01}$$

$$+ b_{01}\alpha'_0 + b'_{01}\beta'_0 + b''_{01}\gamma'_0)\ w_1 + \cdots\cdots$$

$$+(a_{0n} + b_{0n}\alpha'_0 + b'_{0n}\beta'_0 + b''_{0n}\gamma'_0)\, w_n = p'$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad (6\text{-}2)$$

where $\alpha' = \Delta_x^2 F$, $\beta' = \Delta_y^2 F$, $\gamma' = \Delta_{xy} F$ at points 0, 1, etc. corresponding to the subscripts of α', β', γ' respectively, and a_{00}, a_{01}, ..., b_{00}, b_{01}, ... b'_{00}, b'_{01}, ... b''_{00}, b''_{01}, ... etc. are given constants.

If we assume a set of values of w at each net point and have the values of K_i and S_i computed, Eq. (6-1) is a system of linear simultaneous equations in F and can therefore be solved exactly by Cront's method(6.2) for solving systems of linear equations. After the values of F have been computed from Eq. (6-1), α', β', γ' can be found without any difficulty. Then Eq. (6-2) becomes another system of linear simultaneous equations and may be solved exactly by Cront's Method again. If the values of w found from Eq. (6-2) check with those assumed, we have the problem completely solved.

In most cases, however, the values of w will not check with each other. Following the usual method of successive approximations, the computed w's will now replace the assumed ones and the cycle of computations will be repeated. If the value of w at the end of the cycle still does not check with the one assumed in the beginning of the cycle, another cycle will be performed. In our problem, however, if we should follow the ordinary method, the results would be oscillatory divergent.

Therefore, a special procedure must be devised if we want to make the process convergent.

A close examination of these equations shows the values of F depend directly upon the values of K's assumed and a plot of the assumed K's and those computed at the end of each cycle by the ordinary method of successive approximations reveals that these values of K oscillate about their true values which is somewhere near the mean of two consecutive cycles. Now, if we use the mean values of the K's from two consecutive cycles and repeat the cycle, the process is convergent, and the rapidity of convergence depends upon the accuracy of the assumed values of K. We will discuss the method of assuming values of K to get better approximation a little later.

2. Crout's Method for Solving Systems of Linear Equations.

To describe the method it is best to do so by an illustrative example. Let the given system of equations be specified by its <u>given matrix</u>, thus

x_1	x_2	x_3	x_4	=
12.1719	27.3941	1.9827	7.3757	6.6355
8.1163	23.3385	9.8397	4.9474	6.1304
3.0706	13.5434	15.5973	7.5172	4.6921
3.0581	3.1510	6.9841	13.1984	2.5393

(6-3)

the first equation being $12.1719\ x_1 + 27.3941\ x_2 + 1.9827\ x_3 + 7.3757\ x_4 = 6.6355$. The solution requires the formation of one matrix and a set of final results; thus we have an <u>auxiliary matrix</u>

x_1	x_2	x_3	x_4	=
12.1719	2.2506	0.16289	0.60596	0.54515
8.1163	5.0720	1.6793	0.0057629	0.33632
3.0706	6.6327	3.9585	1.4193	0.19891
3.0581	-3.7316	12.7526	-6.7332	0.060806

(6-4)

and a <u>final matrix</u>

$$x_1 = 0.15942$$
$$x_2 = 0.14687$$
$$x_3 = 0.11261$$
$$x_4 = 0.060806 \qquad (6\text{-}5)$$

The procedure for obtaining the auxiliary matrix from the given matrix is contained in the following rules.

(1) The various numbers, or <u>elements</u>, are determined in the following order: elements of first column, then elements of first row to right of first column; elements of second column below first row, then elements of second row to right of second column; elements of third column below second row, then elements of third row to right of third column; and so on until all elements are determined.

(2) The first column is identical with the first column of the given matrix. Each element of the first row except the first is obtained by dividing the corresponding element of the given matrix by that first element.

(3) Each element on or below the principal diagonal is equal

to the corresponding element of the given matrix minus the sum of those products of elements in its row and corresponding elements in its column (in the auxiliary matrix) which involve only previous computed elements.

(4) Each element to the right of the principal diagonal is given by a calculation which differs from rule (3) only in that there is a final division by its diagonal element (in the auxiliary matrix).

It might be mentioned here that a <u>matrix</u> is a rectangular array of numbers, or elements. Those elements which have the same row and column index form the <u>principal diagonal</u> which slopes down to the right starting with the element in the upper left corner. The <u>diagonal element</u> of any element to the right of the principal diagonal is that element of this diagonal which lies in the same row as the given element. The diagonal element of any element below the principal diagonal is that element of this diagonal which lies in the same column as the given element.

The procedure for obtaining the one-columned final matrix from the auxiliary matrix is containing in the following rules.

(1) The elements are determined in the following order: last, next to last, second from last, third from last, etc.

(2) The last element is equal to the corresponding element in the last column of the auxiliary matrix.

(3) Each element is equal to the corresponding element of the last column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix and corresponding

elements in its column in the final matrix which involve only previously computed elements.

Mathematically, if we denote the given matrix, the auxiliary matrix, and the final matrix by $\| G_{ij} \|$, $\| A_{ij} \|$, and $\| F_{il} \|$, respectively, the method is contained in the following equations:

$$A_{ii} = G_{ii} - \sum_{k=1}^{i-1} A_{ik} A_{ki}$$

$$A_{ij} = G_{ij} - \sum_{k=1}^{j-1} A_{ik} A_{kj} \quad \text{if } i > j$$

$$A_{ij} = \left[G_{ij} - \sum_{k=1}^{i-1} A_{ik} A_{kj} \right] \frac{1}{A_{ii}} \quad \text{if } i < j$$

and $F_{il} = A_{i,\, n+1} - \sum_{k=i+1}^{n} A_{ik} F_{kl}$ (6-6)

where i, j denote the row and column, respectively, and any whose lower limit exceeds its upper vanishes.

As examples we have the following typical calculations made in obtaining (6-4), the letters R and C representing the words "row" and "column", respectively.

A_{13}, or R1C3: 0.16289 = 1.9827 ÷ 12.1719

A_{22}, or R2C2: 5.0720 = 23.3385 - 8.1163 x 2.2506

A_{42}, or R4C2: -3.7316 = 3.1510 - 3.0581 x 2.2506

A_{25}, or R2C5: 0.33632 = (6.1304 - 8.1163 x 0.54515) ÷ 5.0720

and a typical calculation made in obtaining (6-5) is

F_{21} or R2C1: 0.14687 = 0.33632 - 1.6793 x 0.11261 - .0057629 x .060806

Since a modern computing machine gives in one continuous operation a sum or difference of products with or without a final division, we see that each element of the auxiliary and final matrices is given by a single machine operation.

When the number of equations gets larger and larger, the elements of the given and auxiliary matrices increase with the square of the number of equations. It would save much time if one would write a "check column" and make continuous check on calculations. The "check column" may be written at the right of the given matrix, each column being the sum of the elements of the corresponding row in the matrix. The column is now treated in exactly the same manner as the last column of the given matrix, the calculations being carried along with those for the other columns, and the result being the addition of corresponding "check columns" to the auxiliary matrix and the final matrix. The check columns thus obtained for (6-3), (6-4), and (6-5) are, respectively,

$$\begin{matrix} 55.560 \\ 52.372 \\ 44.421 \\ 28.931 \end{matrix}, \quad \begin{matrix} 4.5646 \\ 3.0214 \\ 2.6182 \\ 1.0608 \end{matrix}, \text{ and } \begin{matrix} 1.1594 \\ 1.1469 \\ 1.1126 \\ 1.0608 \end{matrix} \qquad (6\text{-}7)$$

These columns provide checks at all stages of the computation, because

(1) In the auxiliary matrix any element in the check column is equal to one plus the sum of the other elements in its row which lie to the right of the principal diagonal.

(2) In the final matrix any element in the check column is equal to one plus the sum of the other element in its row.

For example, noting (6-4), (6-5), and (6-7), two of the checks are

$$1 + 1.6793 + .0057629 + .33632 = 3.0214$$

$$1 + .11261 = 1.11261$$

These check columns are not necessary but they are strongly recommended from the author's experience in working with this method. The check column for the final matrix may be omitted, however, and the results may be checked by substituting them in one of the given equations. The satisfaction of the check guarantees the correctness of the solution.

3. Derivation of the Crout's Method.

Define an augmented matrix of a set of equations to be one which contains the coefficients of the unknowns and an additional column composed of the right hand sides of the equations.

Now we want to determine the relations between A_{ij} and G_{ij} if the set of equations having the augmented matrix

$$\begin{Vmatrix} G_{11} & G_{12} \cdots\cdots & G_{1,\, n+1} \\ G_{21} & G_{22} \cdots\cdots & G_{2,\, n+1} \\ & \cdots\cdots\cdots\cdots & \\ G_{n1} & G_{n2} \cdots\cdots & G_{n,\, n+1} \end{Vmatrix} \tag{6-8}$$

And that having the augmented matrix

$$\left\| \begin{array}{ccccccc} 1 & A_{12} & A_{13} & A_{14} \cdots\cdots & A_{1n} & A_{1,\, n+1} \\ 0 & 1 & A_{23} & A_{24} \cdots\cdots & A_{2n} & A_{2,\, n+1} \\ 0 & 0 & 1 & A_{34} \cdots\cdots & A_{3n} & A_{3,\, n+1} \\ & & & \cdots\cdots\cdots\cdots\cdots & & \\ 0 & 0 & 0 & 0 \cdots\cdots & 1 & A_{n,\, n+1} \end{array} \right\| \qquad (6\text{-}9)$$

have the same solution. Since the solution of the equations corresponding to an augmented matrix is not altered if the rows are multiplied by any numbers, or if to any row there is added a linear combination of the other rows, the two sets of equations would have the same solution if the augmented matrix (6-9) can be derived from the augmented matrix (6-8) by such combinations. We see that this is true if

$$[\text{i-th row of } (6\text{-}9)] = 1/A_{ii} \Big\{ [\text{ i-th row of } (6\text{-}8)$$
$$- A_{i1} [\text{ 1st row of } (6\text{-}9)] - A_{i2} [\text{ 2nd row of } (6\text{-}9)$$
$$- \cdots\cdots - A_{i,\, i-1} [\text{ (i-1) th row of } (6\text{-}9)]$$

or more explicitly

$$1 = \Big[G_{ii} - \sum_{k=1}^{i-1} A_{ik} A_{ki} \Big] \frac{1}{A_{ii}} ,$$

$$0 = \Big[G_{ij} - \sum_{k=1}^{j-1} A_{ik} A_{kj} - A_{ij} \Big] \frac{1}{A_{ii}} , \qquad (6\text{-}10)$$

$$j = 1, 2, 3, \ldots\ldots, i - 1$$

$$A_{ij} = \Big[G_{ij} - \sum_{k=1}^{i-1} A_{ik} A_{kj} \Big] \frac{1}{A_{ii}}$$

$$j = i + 1, i + 2, \ldots\ldots, n + 1.$$

Rearranging, we have

$$A_{ii} = G_{ii} - \sum_{k=1}^{i-1} A_{ik} A_{ki}$$

$$A_{ij} = G_{ij} - \sum_{k=1}^{j-1} A_{ik} A_{kj}, \quad j < i \qquad (6\text{-}11)$$

$$A_{ij} = \left[G_{ij} - \sum_{k=1}^{i-1} A_{ik} A_{kj}\right] \frac{1}{A_{ii}}, \quad j > i$$

which gives the rule for determining the elements in the auxiliary matrix and is the exact form as Eq. (6-6).

This proves that if the elements of an auxiliary augmented matrix $\| A_{ij} \|$ are determined by the relations (6-11) from a given augmented matrix $\| G_{ij} \|$, the two sets of equations corresponding to these two matrices are exactly equivalent.

Now, consider another augmented matrix (6-12).

$$\left\| \begin{matrix} 1 & 0 & 0 & \cdots\cdots & 0 & F_{i1} \\ 0 & 1 & 0 & \cdots\cdots & 0 & F_{21} \\ 0 & 0 & 1 & \cdots\cdots & 0 & F_{31} \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots\cdots & 0 & F_{n1} \end{matrix} \right\| \qquad (6\text{-}12)$$

The set of equations corresponding to (6-12) are equivalent to those of (6-9) if

[i-th row of (6-12)] = [i-th row of (6-9)]

$- A_{i,\ i+1}$ [(i + 1)-th row of (6-12)]

$- A_{i,\ i+2}$ [(i + 2)-th row of (6-12)]

$- \cdots\cdots - A_{i},\ i + n$ [(i + n)-th row of (6-12)]

or more explicitly,

$$F_{i1} = A_{i},\ n{+}1 - \sum_{k=i+1}^{n} A_{ik} F_{k1} \qquad \cdots\cdots \qquad (6\text{-}13)$$

which is exactly the same rule for determining the final matrix as given in Eq. (6-6).

This therefore proves that the equations corresponding to the final augmented matrix found before are exactly equivalent to the equations corresponding to the given augmented matrix. In other words, F_{11}, F_{12}, ... F_{1n} are the solutions of the given set of equations.

We shall now see the check columns. If in a matrix the last column is the sum of the other columns, this fact evidently remains true if a row be multiplied by a constant, or if to any row be added a constant times another row. Since (6-9) can be obtained from (6-8), and (6-12) from (6-9) by just such operations, we see that if a check column is annexed to (6-8), the corresponding columns obtained in the auxiliary matrix and the final matrix are the sums of the columns of (6-9) and (6-12), respectively; and hence have the required properties.

4. The Method of Successive Approximations.

Let us examine a simple case first. Consider the boundary-value problem with n = 1 under the normal pressure p = 100. Eq. (5-25) can be easily reduced to the form

$$w_o = \frac{p'}{16 + 37.690,79\, w_o^2}$$

or $\quad w_o^3 + .424,507\, w_o - 1.790,888 = 0 \qquad (6\text{-}14)$

This third order algebraic equation can be easily solved by the standard method, and the roots of this equation are

$$w_o = 1.098{,}254, \text{ and } (-.549{,}127 \pm 1.152{,}878\, i). \qquad (6\text{-}15)$$

For our problem, we are only interested in the real root, because the imaginary roots do not have any physical meaning.

Now let us try to solve Eq. (6-14) by the usual method of successive approximations:

assume $w_o = 1.200{,}000$, $w_o^2 = 1.440{,}000$,

$$w_o^2 = \frac{67.5}{70.27474} = .960{,}516$$

$$w_o^2 = .922{,}591$$

If we assume $w_o^2 = .922{,}591$ for the second cycle, and the value of w_o^2 found from the second cycle as the assumed value for the third cycle, and so on, we have the values of w_o^2 found from various cycles as follows:

1.767,416, .667,554, 2.689,324 etc.

respectively. We see these values are oscillatorily divergent. A plot of these values against cycles shows that they oscillate about the true value, 1.206,161, and the true value is approximately the mean of the values from two consecutive cycles. (Fig. VI-1)

If we now take $w_o^2 = 1/2\,(1.440{,}000 + .922{,}591) = 1.181{,}296$ as the assumed value of w_o^2 for the second cycle, and the mean of this value and the value found from the second cycle as the assumed value for the third cycle, etc., we have the values of w_o^2 found from various cycles as follows:

cycles	2	3	4	5
w_o^2 assumed	1.181,296	1.212,550	1.204,658	1.206,524
w_o^2 found	1.243,805	1.196,766	1.208,390	1.204,526

	6	7
	1.206,075	1.206,182
	1.206,289	1.206,131

The process is convergent and w_o converges to the real root of Eq. (6-14). The value of w_o found at the end of the 7th cycle is 1.098,240, and is accurate to four figures at the end of 5th cycle, where it is found to 1.098,010. The results are plotted against cycles in Fig. (VI-2).

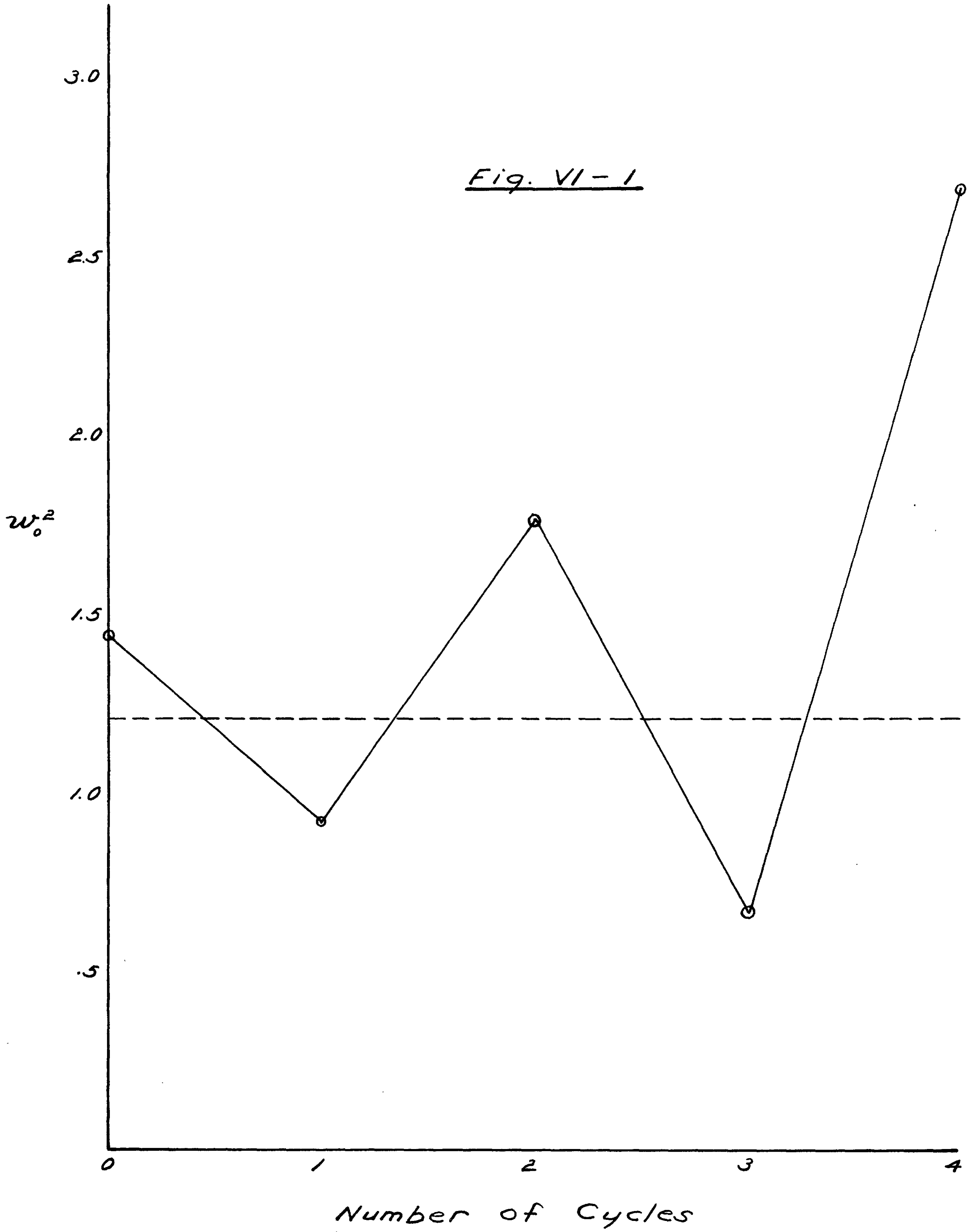
It is to be noted that $K_o = - 3 w_o^2$ in the case of n = 1. The method of successive approximations would converge if we assume K_o to be the mean values of two consecutive cycles. It is found that this property is the same for $n > 1$. If we take the mean of K's or S's found from two consecutive cycles, the process is convergent; but is oscillatorily divergent if we follow the usual way of successive approximation.

It may be pointed out here that for the special case n = 1, if we assume the mean of the values of w_o from two consecutive cycles, the process is also convergent, and if we assume w_o^2 of second cycle to be equal to the sum of .6 times the assumed value for the first cycle and .4 times the value found from the first cycle, and so on, the convergence is much more rapid (Fig. VI-3). But it is not true for the cases with $n > 1$.

The rapidity of the convergence depends upon the accuracy of the assumed values of K's and S's for the first trial. To have a good approximation, the following procedure is found to be valuable.

(1) Always start with less divisions (smaller n) to more divisions and with smaller normal pressure to larger ones. Because when p is small, the values of w determined from the linear small deflection theory would give a good approximation for the

Fig. VI-1



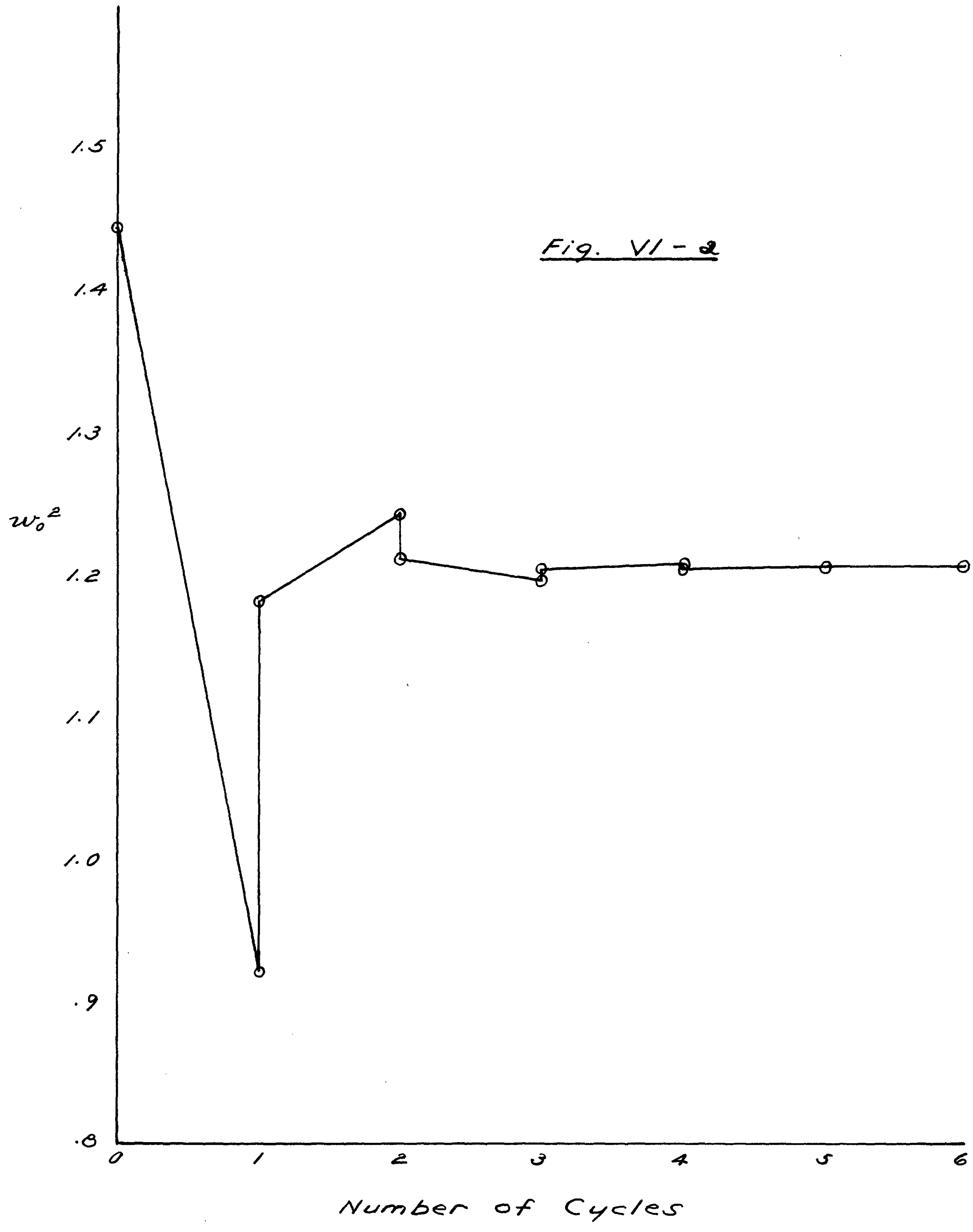


Fig. VI - 2

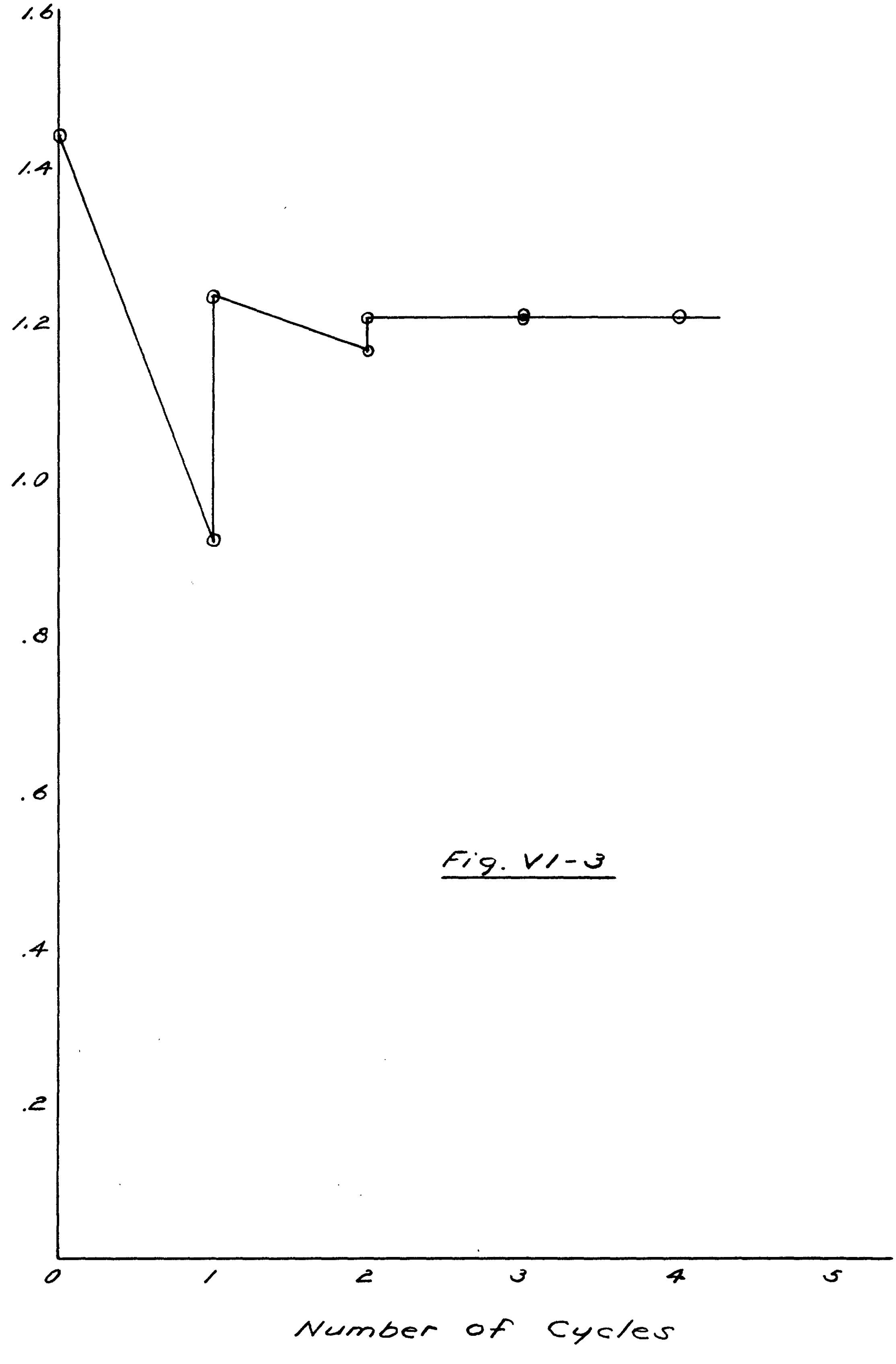


Fig. VI-3

first trial, and the small deflection problem is very easy to solve.

(2) For the case n = 1, we first start our computation with a small p and use the value of w_o determined from linear small deflection theory at the corresponding p as a first trial. Then we can plot w_o against p after we have found the first w_o, because we know that the $w_o \sim p$ curve for the large deflection would become tangent to the $w_o \sim p$ curve for the small deflection curve at the origin. For a larger value of p, w_o can be now estimated by extrapolation.

(3) For the cases $n > 1$, we again start our computations at a small p. For n = 2, we can now assume w_o as found from the case n = 1 as a first trial. w_2, w_3 are still difficult to assume. We may use the values as found from the small deflection theory as a first trial. It is found, however, that we could have better results if we use the ratios (w_2/w_o) and (w_3/w_o) as found from the small deflection theory, and compute the values of w_2 and w_3 by multiplying these ratios by the assumed value of w_o. When the deflections have been assumed at every net point, the values of K and S can be computed. These are the values which may be used as a first trial. By successive approximation, the true values of w's are then determined. The values of w_o and (w_n/w_o)'s are now plotted against p to estimate the corresponding values at a larger p. The values estimated by extrapolation may be used as the trial values corresponding to that p. The process repeats until the maximum p is reached. For n = 3, w_o from n = 2 is used as a first trial, the other procedures are the same.

A sample calculation is given in the next chapter.

CHAPTER VII

THE METHOD OF SUCCESSIVE APPROXIMATIONS: SAMPLE CALCULATIONS

1. The Finite Differences Solutions of the Small Deflection Theory.

Now let us study the small deflection theory of simply-supported square plate first. The differential equation is

$$\nabla^4 w = p/D \tag{7-1}$$

and the boundary conditions are

$$\begin{aligned} w &= 0 \text{ along four edges} \\ \frac{\partial^2 w}{\partial x^2} &= 0 \text{ along } x = \pm a/2 \\ \frac{\partial^2 w}{\partial y^2} &= 0 \text{ along } y = \pm a/2 \end{aligned} \tag{7-2}$$

where a is the length of the sides (Fig. 7-1).

Non-dimensionizing Eqs. (7-1) and (7-2) by letting $w' = w/h$, $p' = p\,a^4/E\,h^4$, $x' = x/a$, $y' = y/b$, where w', p', x', y' are non-dimensional deflection, pressure and length, respectively, and dropping the prime, we have our boundary value problem as follows:

$$\begin{aligned} \nabla^4 w &= 12\,(1 - \mu^2)\,p \\ w &= 0 \text{ at } x = \pm 1/2,\ y = \pm 1/2 \\ \frac{\partial^2 w}{\partial x^2} &= 0 \text{ at } x = \pm 1/2, \\ \frac{\partial^2 w}{\partial y^2} &= 0 \text{ at } y = \pm 1/2 \end{aligned} \tag{7-3}$$

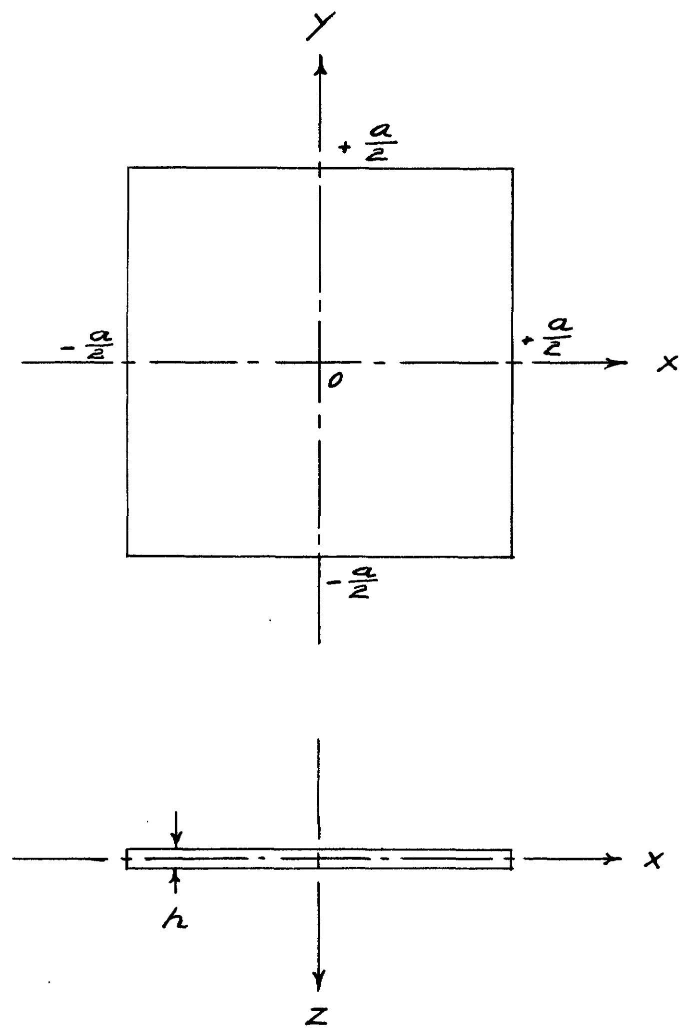


Fig VII - 1

Following the notations as used in Chapter V, the finite differences equations for the problem become

$$\Delta^4{}_x w + 2\Delta^2{}_{xy} w + \Delta^4{}_y w = p'$$

$$(w)_{x = \pm 1/2,\ y = \pm 1/2} = 0$$

$$(\Delta^2{}_x w)_{x = \pm 1/2} = 0 \qquad (7\text{-}4)$$

$$(\Delta^2{}_y w)_{y = \pm 1/2} = 0$$

where $p' = 12\ (1 - \mu^2)(\Delta \ell)^4\ p$.

(1) $n = 1$ (Fig. V-3)

The finite-differences equation after combining with the boundary conditions is

$$16\ w_0 = p' \qquad (7\text{-}5)$$

$$\therefore \quad w_0 = .0625\ p'$$
$$= .042{,}188\ p \qquad (7\text{-}6)$$

for $\mu^2 = .1$.

(2) $n = 2$ (Fig. V-4)

After being combined with the boundary conditions, the finite differences equations are:

$$20\ w_0 - 32\ w_1 + 8\ w_2 = p'$$
$$-8\ w_0 + 24\ w_1 - 16\ w_2 = p' \qquad (7\text{-}7)$$
$$+2\ w_0 - 16\ w_1 + 20\ w_2 = p'$$

Use Crout's Method to solve these equations, and denote the check column by C.C. The given matrix is

w_0	w_1	w_2	$= p'$	C.C.
+20	−32	+ 8	+1	−3
− 8	+24	−16	+1	+1
+ 2	−16	+20	+1	+7

The auxiliary matrix is

+20	−1.600,000	+ .400,000	+.050,000	−.150,000
− 8	+11.200,000	−1.142,857	+.125,000	−.017,857
+ 2	−12.800,000	+4.571,430	+.546,875	+1.546,875

and the final matrix is

+1.031,250

+ .750,000

+ .546,875

The solutions of Eq. (7-7) are therefore

$$\begin{aligned} w_0 &= +1.031,250\ p' = +.043,506\ p \\ w_1 &= +\ .750,000\ p' = +.031,641\ p \qquad (7\text{-}8) \\ w_2 &= +\ .546,875\ p' = +.023,071\ p \end{aligned}$$

where μ^2 is taken to be equal to 0.1. For $\mu = .3$,

$$w_o = +.032,989\ p. \qquad (7\text{-}9)$$

(3) n = 3 (Fig. V-5)

After being combined with the boundary conditions, the finite differences equations are:

$$\begin{aligned} &20\ w_0 - 32\ w_1 + 8\ w_2 + 4\ w_3 = p' \\ &-8\ w_0 + 25\ w_1 - 16\ w_2 - 8\ w_3 + 6\ w_4 = p' \\ &+2\ w_0 - 16\ w_1 + 22\ w_2 + 4\ w_3 - 16\ w_4 + 2\ w_5 = p' \qquad (7\text{-}10) \\ &+\ w_0 - 8\ w_1 + 4\ w_2 + 19\ w_3 - 16\ w_4 + 2\ w_5 = p' \\ &+3\ w_1 - 8\ w_2 - 8\ w_3 + 22\ w_4 - 8\ w_5 = p' \\ &+2\ w_2 + 2\ w_3 - 16\ w_4 + 18\ w_5 = p' \end{aligned}$$

Solving these by Crout's Method, we have the given matrix as follows:

w_0	w_1	w_2	w_3	w_4	w_5	p'	C.C.
+20	-32	+ 8	+ 4	0	0	+1	+1
- 8	+25	-16	- 8	+ 6	0	+1	0
+ 2	-16	+22	+ 4	-16	+ 2	+1	-1
+ 1	- 8	+ 4	+19	-16	+ 2	+1	+3
0	+ 3	- 8	- 8	+22	- 8	+1	+2
0	0	+ 2	+ 2	-16	+18	+1	+7

The auxiliary matrix is as follows:

+20.000,000	- 1.600,000	+ .400,000	+ .200,000	0	0	+ .050,000	+ .050,000
- 8.000,000	+12.200,000	- 1.049,180	- .524,590	+ .491,803	0	+ .114,754	+ .032,787
+ 2.000,000	-12.800,000	+ 7.770,492	- .400,844	- 1.248,945	+ .257,384	+ .304,852	- .087,553
+ 1.000,000	- 6.400,000	- 3.114,754	+14.194,093	- 1.179,548	+ .197,384	+ .185,568	+ .203,404
0	+ 3.000,000	- 4.852,459	- 8.371,308	+ 4.589,774	-1.110,881	+ .803,627	+ .692,746
0	0	+ 2.000,000	+ 2.801,688	-10.197,384	+5.604,145	+1.439,164	+2.439,164

and the final matrix is as follows:

+5.246,672
+4.597,633
+4.031,250
+2.735,207
+2.402,367
+1.439,164

The solutions of Eq. (7-10) are:

$$\begin{aligned} w_0 &= 5.246,672\ p' = .043,722\ p \\ w_1 &= 4.597,633\ p' = .038,314\ p \\ w_2 &= 4.031,250\ p' = .033,594\ p \\ w_3 &= 2.735,207\ p' = .022,793\ p \\ w_4 &= 2.402,367\ p' = .020,020\ p \\ w_5 &= 1.439,164\ p' = .011,993\ p \end{aligned} \qquad (7\text{-}11)$$

if μ^2 is assumed to be 0.1. If μ is assumed to be .3, we have

$$w_0 = .044,208\ p \qquad (7\text{-}12)$$

Timoshenko gives the exact value of w_0 for simply-supported square plate as (Ref. 1, p. 133):

$$w_0 = .0443\ p.$$

Therefore the solution by finite differences with n = 3 has an error of 0.23%. It is seen to be sufficiently accurate for engineering purposes.

The agreement of the finite differences approximation with the more exact results of Timoshenko is sufficiently close to encourage one to apply the finite differences approximation to the problems with large deflections.

2. The Large-Deflections Problem, n = 2.

After combining with the boundary conditions, the two sets of finite differences equations are:

$$20\ F_0 - 32\ F_1 + 8\ F_2 + 4\ F_3 = K_0$$
$$-8\ F_0 + 24\ F_1 - 16\ F_2 - 6.632,456\ F_3 + 6.632,456\ F_4 = K_1$$
$$+2\ F_0 - 16\ F_1 + 20\ F_2 + 4.632,456\ F_3 - 13.264,912\ F_4 = K_2$$
$$-2\ F_0 - 4\ F_1 + 4\ F_2 - .632,456\ F_3 + 2.632,456\ F_4 = S_1$$
$$+ F_0 - 6\ F_2 + 2.316,228\ F_3 + 1.367,544\ F_4 = S_2 \qquad (7\text{-}13)$$

and

$$[20 + 21.6\ (\alpha'_0 + \beta'_0 + \gamma'_0)]\ w_0 - [32 + 21.6\ (\alpha'_0 + \beta'_0 + 2\ \gamma'_0)]\ w_1 + (8 + 21.6\ \gamma'_0)\ w_2 = p'$$

$$-(8 + 10.8\ \beta'_1)\ w_0 + [24 + 21.6\ (\alpha'_1 + \beta'_1 + \gamma'_1)]\ w_1 - [16 + 21.6\ (\alpha'_1 + \gamma'_1)]\ w_2 = p'$$

$$2\ w_0 - [16 + 10.8\ (\alpha'_2 + \beta'_2)]\ w_1 + [20 + 21.6\ (\alpha'_2 + \beta'_2 + \gamma'_2)]\ w_2 = p' \qquad (7\text{-}14)$$

It is to be noted here that the left-hand side terms of Eq. (7-13) do not change if we vary our assumed values of K and S. Eq. (7-13) can be therefore solved uniquely in terms of K's and S's. The given, auxiliary and final matrices are obtained by Crout's Method as in Tables 7-1, 7-2 and 7-3 respectively. More significant figures than required are used to ensure good results. The solutions of Eq. (7-13) are as follows:

$$F_0 = -.048,703\ K_0 - .265,696\ K_1 - .225,111\ K_2 - .304,114\ S_1 - .309,525\ S_2$$
$$F_1 = -.111,203\ K_0 - .307,363\ K_1 - .235,527\ K_2 - .262,447\ S_1 - .288,692\ S_2$$
$$F_2 = -.103,085\ K_0 - .311,962\ K_1 - .221,052\ K_2 - .162,880\ S_1 - .317,642\ S_2$$
$$F_3 = -189,937\ K_0 - .506,498\ K_1 - .316,561\ K_2 - .253,249\ S_1 - .126,624\ S_2$$
$$F_4 = -\ .094,968\ K_o - .316,561\ K_1 - .269,077\ K_2 - .063,312\ S_1 - .221,593\ S_2$$

(7-15)

Table 7-1 - Solutions of Eq. (7-13)

Given Matrix

F_0	F_1	F_2	F_3	F_4	K
+20.000,000,00	-32.000,000,00	+ 8.000,000,00	+4.000,000,00	0	K_o
- 8.000,000,00	+24.000,000,00	-16.000,000,00	-6.632,456,00	+ 6.632,456,000	K_1
+ 2.000,000.00	-16.000,000,00	+20.000,000,00	+4.632,456,000	-13.264,912,00	K_2
- 2.000,000,00	- 4.000,000,00	+ 4.000,000,00	- .632,456,000	+ 2.632,456,00	S_1
+ 1.000,000,00	0	- 6.000,000,00	+2.316,228,00	+1.367,544,00	S_2

Table 7-2 - Solutions of Eq. (7-13)

Auxiliary Matrix

+20.000,000,00	- 1.600,000,000	+ .400,000,000	+ .200,000,000	0
- 8.000,000,00	+11.200,000,00	-1.142,857,14	- .449,326,43	+ .592,183,57
+ 2.000,000,00	-12.800,000,00	+4.571,428,61	- .332,264,25	-1.243,585,49
- 2.000,000,00	- 7.200,000,000	-3.428,571,41	-4.606,798,00	- .571,428,58
+ 1.000,000,00	+ 1.600,000,000	-4.571,428,58	+1.316,228,00	-4.512,781,67

Last Column

K_0	K_1	K_2	S_1	S_2
+ .050,000,00				
+ .035,714,29	+ .089,285,71			
+ .078,125,01	+ .249,999,99	+ .218,750,00		
- .135,669,08	- .325,605,75	- .162,802,88	- .217,070,51	
- .094,968,36	- .316,561,15	- .269,076,99	- .063,312,23	- .221,592,82

Table 7-3 - Solutions of Eq. (7-13)

Final Matrix

K_0	K_1	K_2	S_1	S_2
-.048,702,54	-.265,696,16	-.225,110,74	-.304,113,92	-.309,525,31
-.111,202,54	-.307,362,83	-.235,527,41	-.262,447,25	-.288,691,98
-.103,085,45	-.311,961,99	-.221,052,20	-.162,879,74	-.317,642,40
-.189,936,72	-.506,497,84	-.316,561,16	-.253,248,93	-.126,624,47
-.094,968,36	-.316,561,15	-.269,076,99	-.063,312,23	-.221,592,82

Let us now give a numerical example of the computation.

Let $p = 100$

$p' = .042,1875p = 4.218,750$

From the $w_0 \sim p$, $w_1/w_0 \sim p$, $w_2/w_0 \sim p$ curves (Fig. 9-1, 7-2, 7-3 respectively), it is estimated that

$$w_0 = 1.135$$
$$w_1/w_0 = .7535$$
$$w_2/w_0 = .5775$$

We have, therefore, for the first trial,

$$w_0 = 1.135$$
$$w_1 = .855,222$$
$$w_2 = .655,463$$

Write these values at the right corners below the corresponding net points. Use the finite differences patterns as described in Chapter V (Fig. 5-2), find α, β, γ, $(w_{n+1} - w_n)$ and then K and S at the net points (Fig. 7-4). As an example, we have

$$\alpha_0 = \beta_0 = -2\ (1.135,000 - .855,222) = -\ .559,556$$
$$\gamma_0 = 1.135,000 + .655,463 - 2 \times .855,222 = +\ .080,019$$
$$K_0 = (.080,019)^2 - (-.559,556)^2 = -\ .306,700$$

Similarly, we find

$$K_1 = -\ .189,997$$
$$K_2 = +\ .221,966$$
$$S_1 = +\ 2.368,276$$
$$S_2 = +\ 1.373,368$$

From (7-15), we obtain the values of F's as follows:

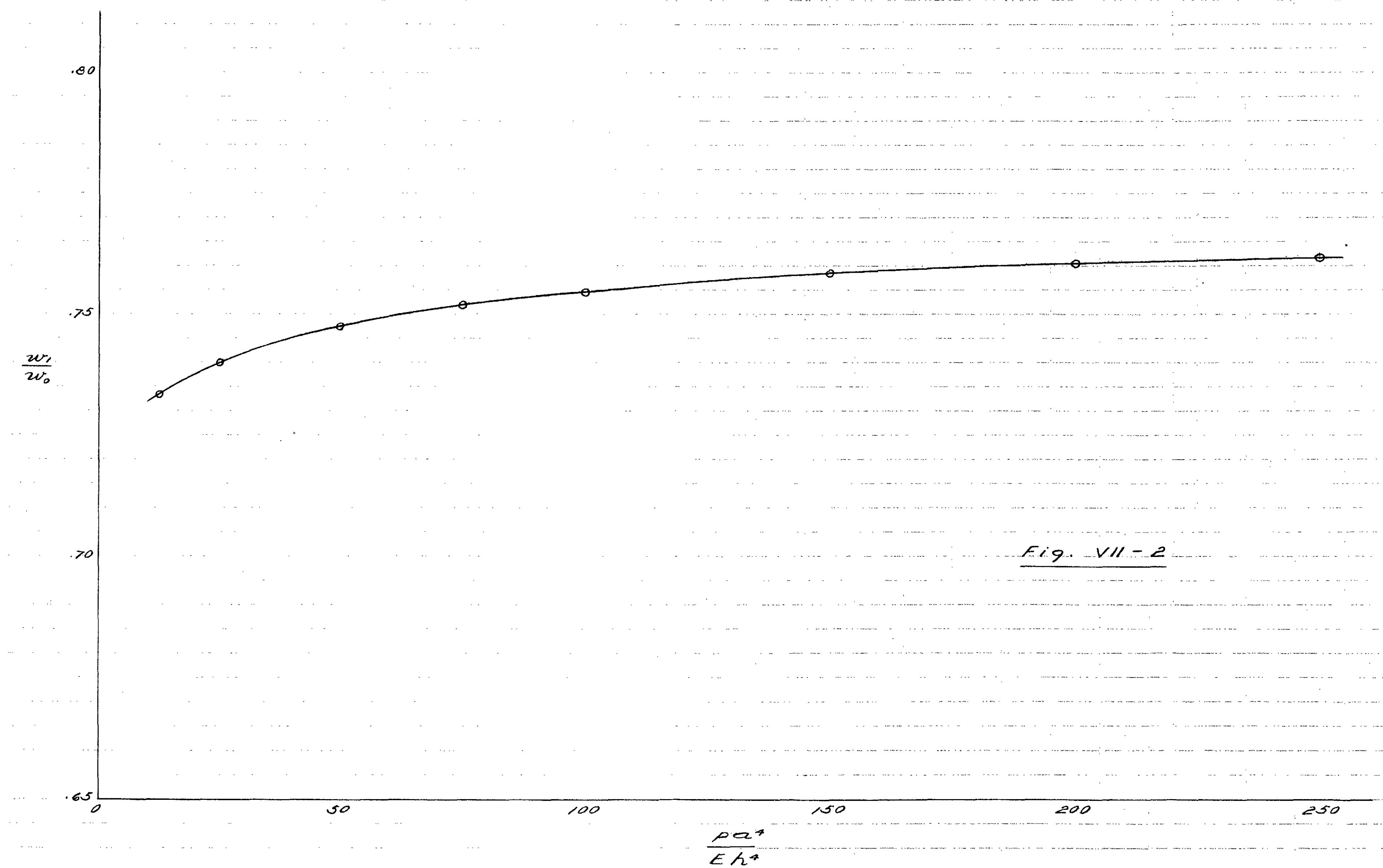


Fig. VII - 2

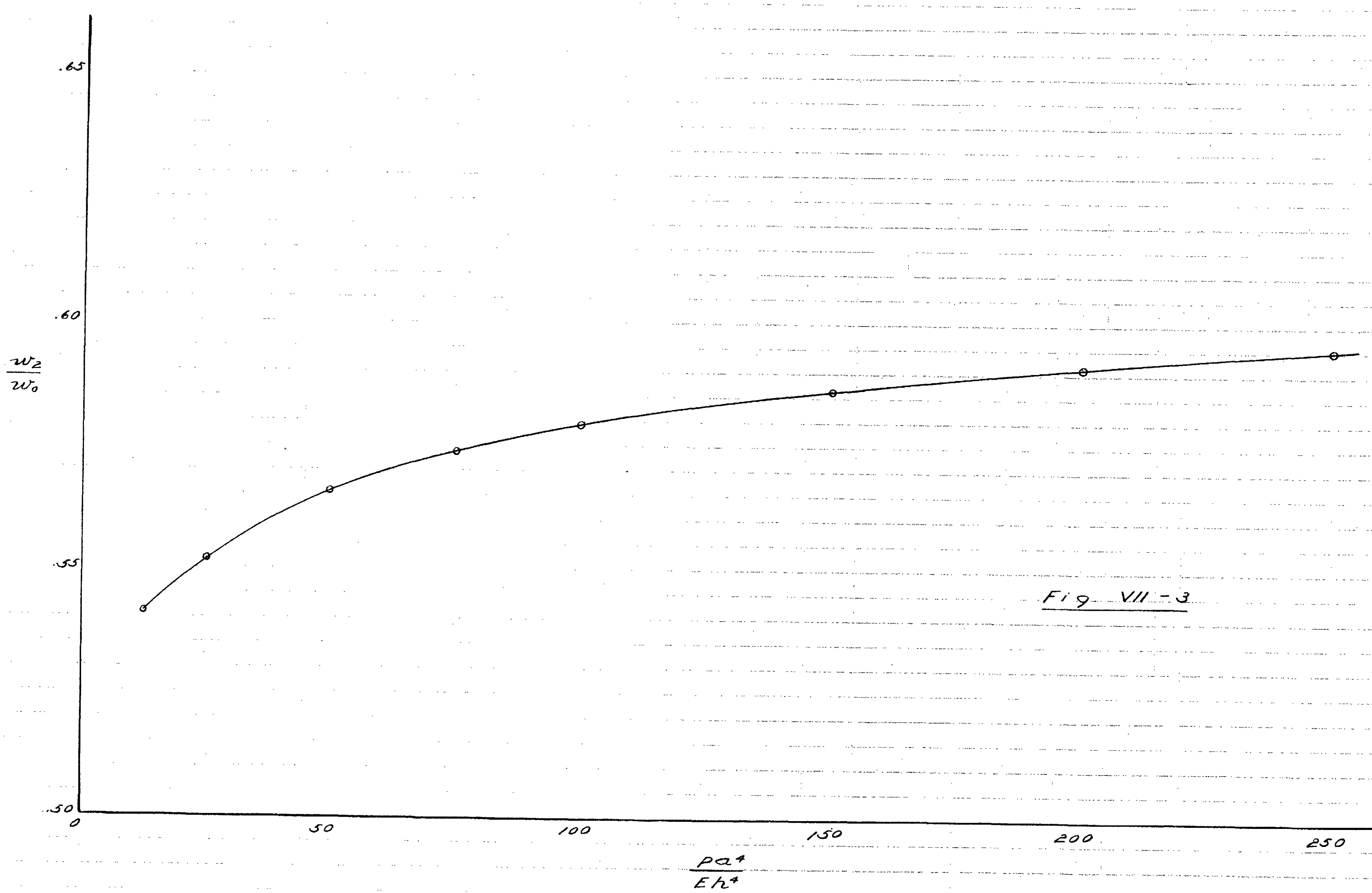


Fig. VII-3

$$F_0 = -\ 1.129,866$$
$$F_1 = -\ .977,802$$
$$F_2 = -\ .780,162$$
$$F_3 = -\ .689,444$$
$$F_4 = -\ .424,723$$

These values are substituted in any one of Eq. (7-13) to ensure correct computation and then are recorded at the net points as on Fig. 7-4. Similarly, we have the values of α', β', γ' recorded below the corresponding values of F.

Now Eq. (7-14) can be written and the given matrix is

w_0	w_1	w_2	= p'	C.C.
+34.122,771	-47.107,213	+ 8.984,442	+4.218,750	+ .218,750
-12.269,024	+36.930,948	-20.392,900	+4.218,750	+ 8.487,774
+ 2.000,000	-19.408,458	+28.313,451	+4.218,750	+15.123,743

The check column can be written by the following relation:

C.C.

$$-4 + p'$$
$$+10.8\,\beta'_1 + p'$$
$$6 + 21.6\,(\alpha'_2 + \beta'_2) + p'$$

The sum of the elements in a row should be equal to the value of the element of the same row in the check column. This procedure provides with a check for the correct substitution made in the given matrix.

The auxiliary and final matrices can now be computed and are as follows:

+34.122,771	- 1.380,5213	+ .263,2975	+.123,6344	+ .006,4107
-12.269,024	+19.993,2990	- .858,4124	+.286,8773	+ .428,4649
+ 2.000,000	-16.647,4154	+13.496,5082	+.648,1118	+1.648,1118

+1.117,078

+ .843,225

+ .648,112

The first approximation gives therefore

$$w_0 = +1.117{,}078$$
$$w_1 = + .843{,}225$$
$$w_2 = + .648{,}112$$

A similar computation as outlined above gives

$$K_0 = - .293{,}781$$
$$K_1 = - .184{,}115$$
$$K_2 = + .214{,}841$$
$$S_1 = +2.299{,}072$$
$$S_2 = +1.339{,}974$$

As a second trial, assume

$$K_0 = 1/2\ (-.306{,}700 - .293{,}781) = -.300{,}241$$

$$K_1 = 1/2\ (-.189{,}997 - .184{,}115) = -.187{,}056$$

$$K_2 = 1/2\ (+.221{,}966 + .214{,}841) = +.218{,}404$$

$$S_1 = 1/2\ (+ 2.368{,}276 + 2.299{,}072) = +2.333{,}673$$

$$S_2 = 1/2\ (+1.373{,}368 + 1.339{,}974) = +1.356{,}671$$

The results of the second trial are shown in Fig. 7-5. Following the same procedure, the results of the third and fourth trials are shown in Fig. 7-6, and 7-7 respectively. The corresponding assumed and computed values of the fourth trial are

	assumed	computed
K_o	-.300,446	-.300,006
K_1	-.187,407	-.187,472
K_2	+.218,738	+.218,786
S_1	+2.337,941	+2.338,531
S_2	+1.360,090	+1.360,631

The first three figures check with each other, and the results corrected to the third figure after the decimal points are

$$w_0 = 1.1269$$
$$w_1 = .8502$$
$$w_2 = .6528$$

and this is our required result.

3. The Large Deflections Problem, n = 3.

When n is taken to be greater than 2, the same procedure of computation as that in the case of n = 2 is still valid. As an example, let us consider the case of n = 3. When the square plate is subjected to a uniform pressure of p = 100.

After combining with the boundary conditions, we obtain the two sets of differences equations (5-34) and (5-35) as given in Chapter V. Eq. (5-34) can be solved in terms of K's and S's, and the solutions are given in Table 7-4.

From the $w_0 \sim p$, $w_1/w_0 \sim p$, $w_2/w_0 \sim p$, $w_3/w_0 \sim p$, $w_4/w_0 \sim p$, $w_5/w_0 \sim p$ curves (Fig. 9-1 , Figs. 7-8 to 7-12), we obtain the following values by extrapolation:

$$w_0 = 1.1247$$
$$w_1/w_0 = .8891$$
$$w_2/w_0 = .7932$$
$$w_3/w_0 = .5516$$
$$w_4/w_0 = .5037$$
$$w_5/w_0 = .3497$$

For a first trial, we assume:

Fig. VII-4

0 0

Assumed:

$S_1 = 2.368,276$

$S_2 = 1.373,368$

Computed:

$S_1 = 2.299,072$

$S_2 = 1.339,974$

$p = 100$

$n = 2$

First Approx.

−.780,162

$\alpha' = \beta' = +.157,799$

$\gamma' = +.069,284$

.655,463

$\alpha = \beta = -.455,704$

$\gamma = +.655,463$

$K_2 = +.214,841$

−.424,723

0

* Note: 1 – assumed

2 – computed

$(\Delta w)_2^* = .273,853$

$(\Delta w)_1^* = .279,778$

−1.129,866

$\alpha' = \beta' = +.304,128$

$\gamma' = +.045,576$

1.135,000

$\alpha = \beta = -.559,556$

$\gamma = +.080,019$

$K_0 = -.306,700$

−.977,802

$\alpha' = +.136,294$

$\beta' = +.395,280$

$\gamma' = +.067,081$

.855,222

$\alpha = -.575,444$

$\beta = -.399,518$

$\gamma = +.199,759$

$K_1 = -.189,997$

−.689,444

0

1.117,078

$\alpha = \beta = -.547,706$

$\gamma = +.078,740$

$K_0 = -.293,781$

.843,225

$\alpha = -.569,372$

$\beta = -.390,226$

$\gamma = +.195,113$

$K_1 = -.184,115$

Fig. VII-5

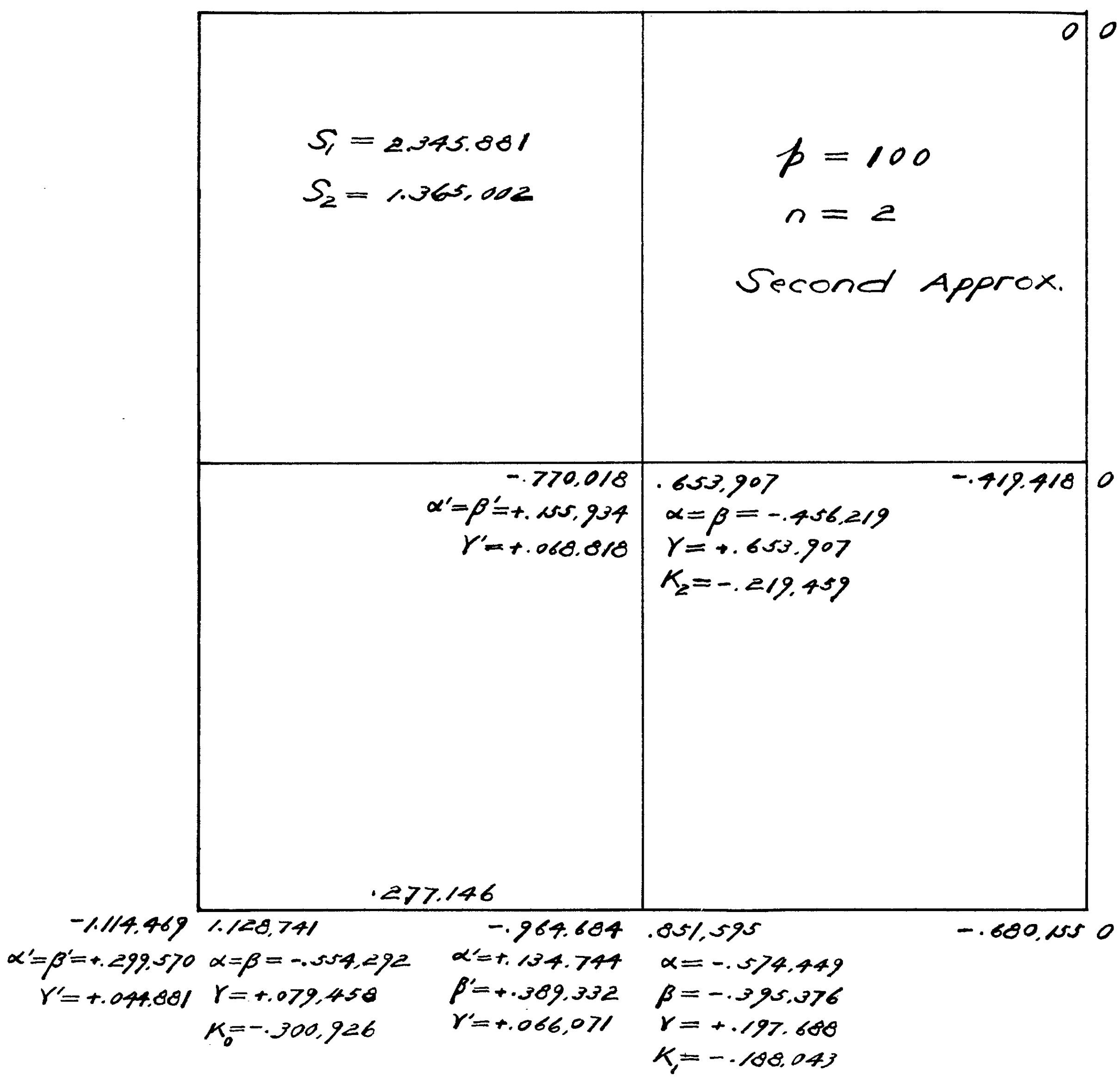
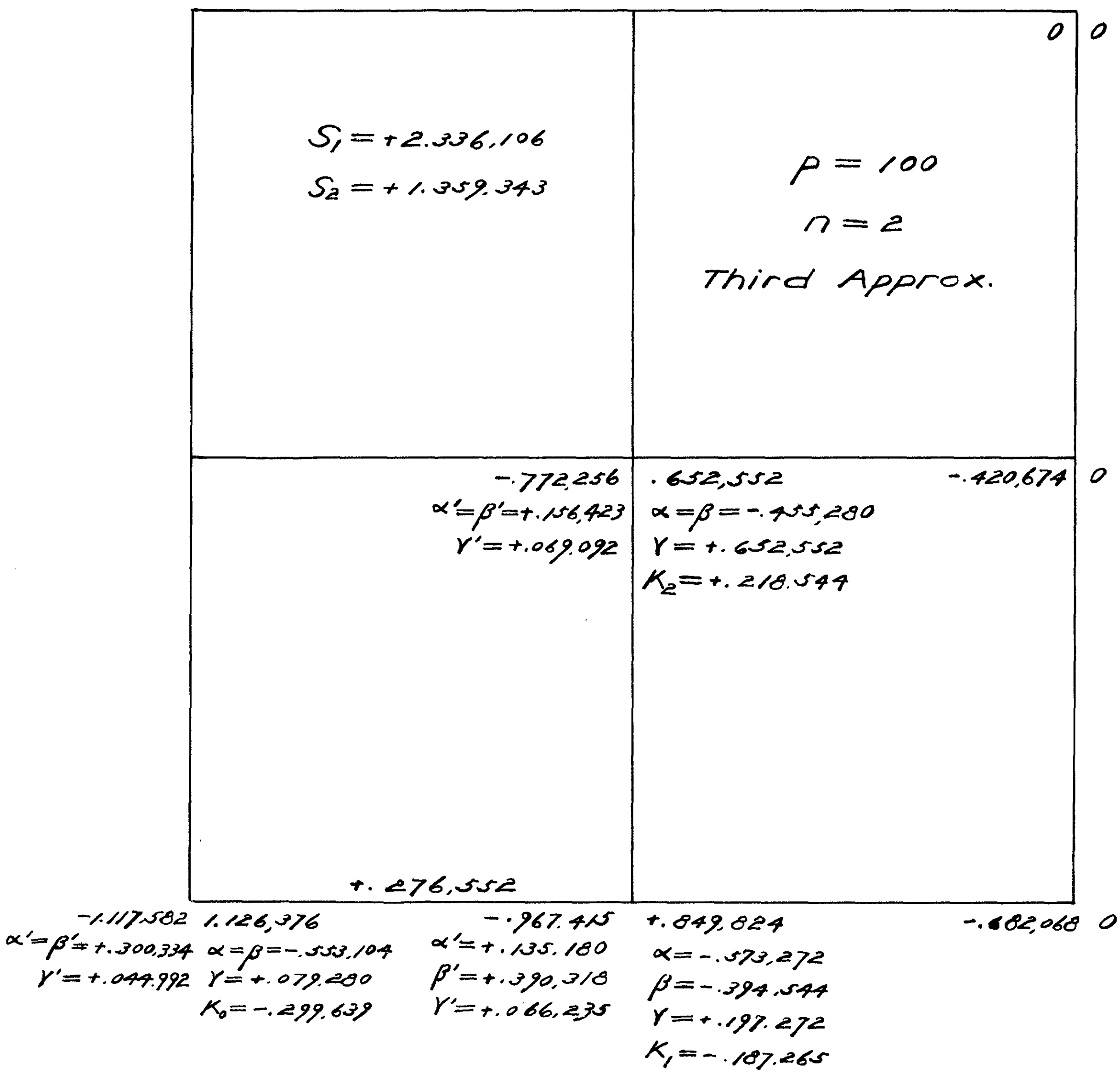
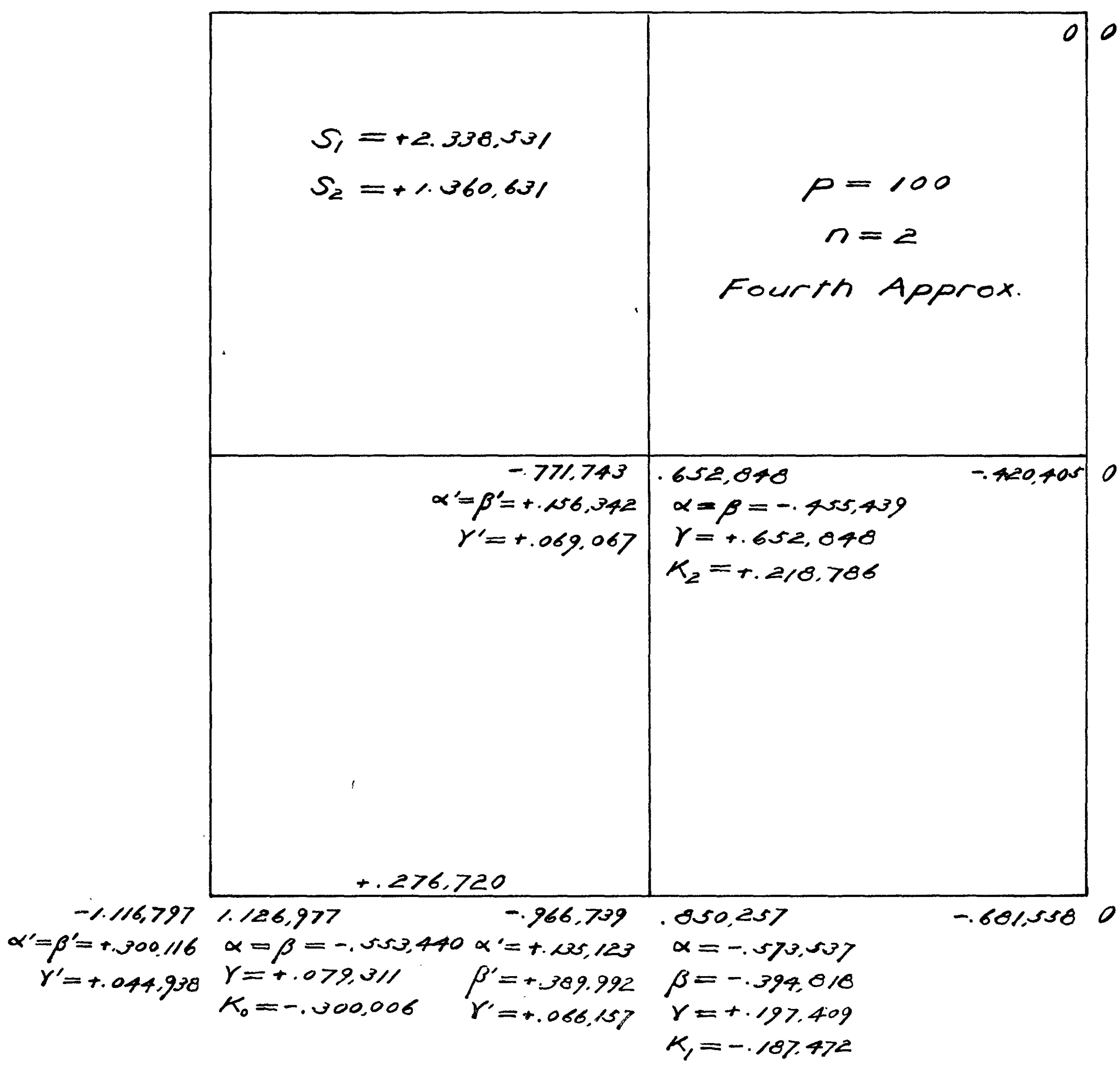

0 0
S_1 = 2.345.881
S_2 = 1.365,002
p = 100
n = 2
Second Approx.
-.770,018
α'=β'=+.155,934
γ'=+.068.818
.653,907
α=β=-.456,219
γ=+.653,907
K_2=-.219,459
-.419.418 0
.277.146
-1.114.469
α'=β'=+.299,570
γ'=+.044.881
1.128,741
α=β=-.554,292
γ=+.079,458
K_0=-.300,926
-.964.684
α'=+.134.744
β'=+.389.332
γ'=+.066,071
.851,595
α=-.574,449
β=-.395,376
γ=+.197.688
K_1=-.188,043
-.680,155 0

Fig. VII-6



0
0
$S_1 = +2.336,106$
$S_2 = +1.359,343$
$p = 100$
$n = 2$
Third Approx.
-.772,256
$\alpha' = \beta' = +.156,423$
$\gamma' = +.069,092$
.652,552
$\alpha = \beta = -.455,280$
$\gamma = +.652,552$
$K_2 = +.218,544$
-.420,674
0
+.276,552
-1.117,582
$\alpha' = \beta' = +.300,334$
$\gamma' = +.044,992$
1.126,376
$\alpha = \beta = -.553,104$
$\gamma = +.079,280$
$K_0 = -.299,639$
-.967,415
$\alpha' = +.135,180$
$\beta' = +.390,318$
$\gamma' = +.066,235$
+.849,824
$\alpha = -.573,272$
$\beta = -.394,544$
$\gamma = +.197,272$
$K_1 = -.187,265$
-.682,068
0

Fig. VII-7


0
0
S₁ = +2.338,531
S₂ = +1.360,631
p = 100
n = 2
Fourth Approx.
-.771,743
α'=β'= +.156,342
γ' = +.069,067
.652,848
-.420,405
0
α = β = -.455,439
γ = +.652,848
K₂ = +.218,786
+.276,720
-1.116,797
α'=β'= +.300,116
γ' = +.044,938
1.126,977
α = β = -.553,440
γ = +.079,311
K₀ = -.300,006
-.966,739
α' = +.135,123
β' = +.389,992
γ' = +.066,157
.850,257
α = -.573,537
β = -.394,818
γ = +.197,409
K₁ = -.187,472
-.681,558
0

$$w_0 = 1.124,700$$
$$w_1 = .999,971$$
$$w_2 = .892,112$$
$$w_3 = .620,385$$
$$w_4 = .566,511$$
$$w_5 = .393,308$$

Again write these values at the right corners below the corresponding net points. With the values of α, β, γ, $\Delta_x w$, $\Delta_y w$ computed, we have

$$K_0 = -.061,945$$
$$K_1 = -.052,063$$
$$K_2 = -.024,186$$
$$K_3 = -.023,043$$
$$K_4 = +.001,252$$
$$K_5 = +.106,245$$
$$S_1 = +1.592,696$$
$$S_2 = +1.282,838$$
$$S_3 = +.548,700$$

By Table 7-4, the values of F's are found to be

$$F_0 = -1.095,495$$
$$F_1 = -1.028,996$$
$$F_2 = -.950,911$$
$$F_3 = -.868,159$$
$$F_4 = -.762,520$$
$$F_5 = -.505,761$$
$$F_6 = -.675,850$$
$$F_7 = -.546,620$$
$$F_8 = -.239,729$$

Write the values of F's at the left corners below the corresponding net points, and compute the values of α', β', and γ'.

Substituting the values of α', β', and γ' into Eq. (5-35) and noting that $p' = 0.008,333,33 \times p = 0.833,333$, we have the given matrix of the equations as in Table 7-5 and the auxiliary matrix as in Table 7-6, and the solutions of Eq. (5-35) given by the final matrix are

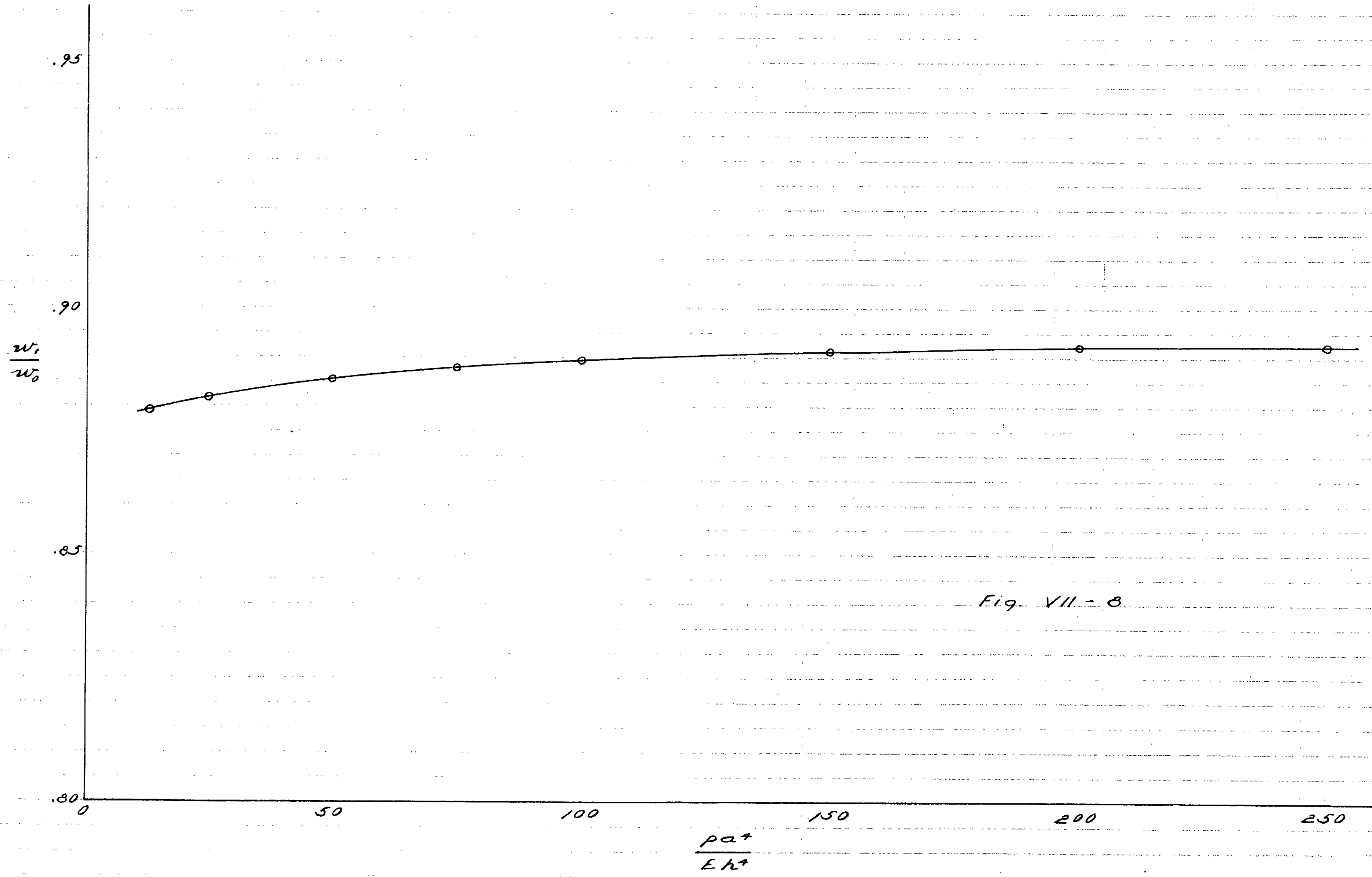


Fig. VII - 8

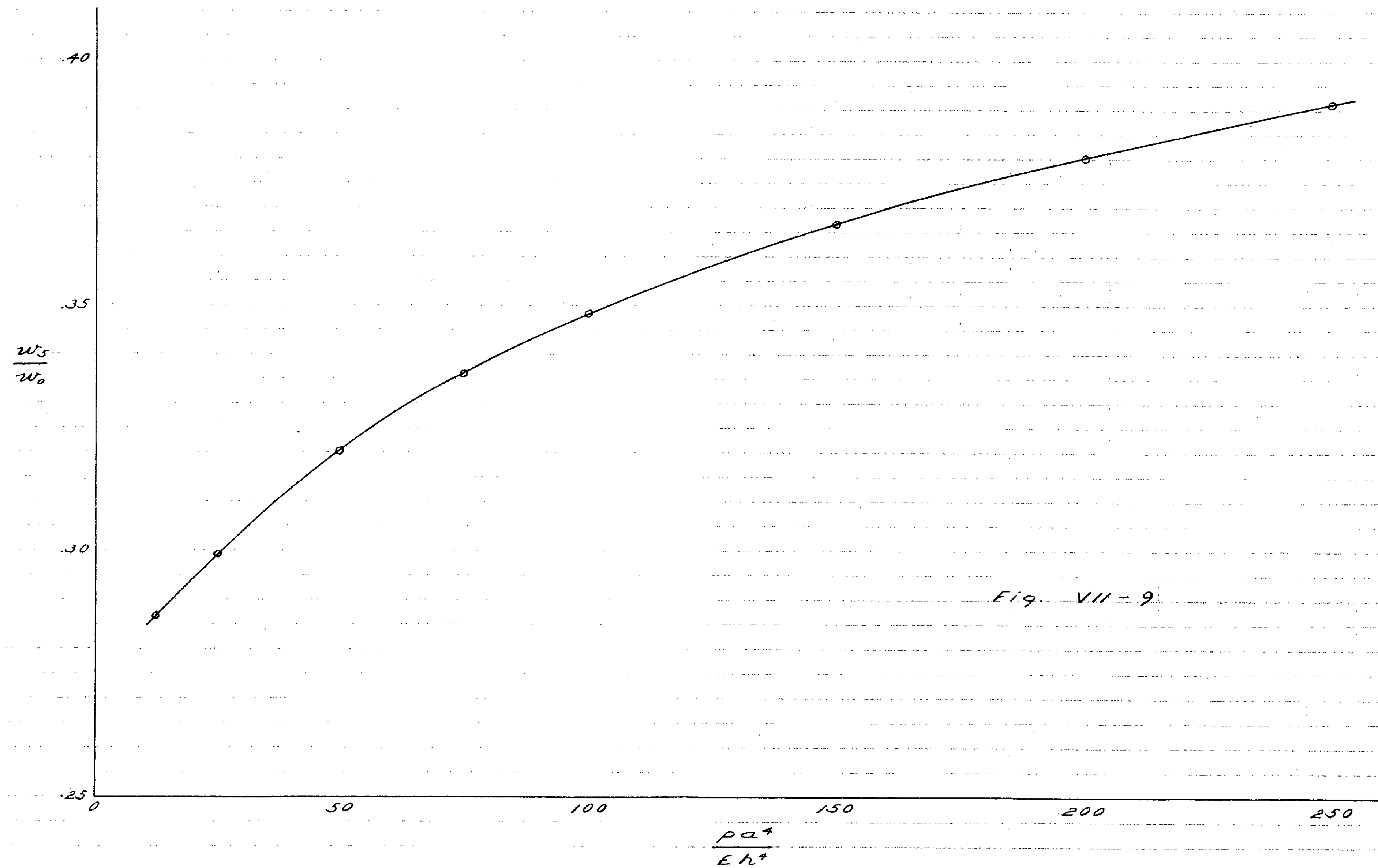


Fig. VII-9

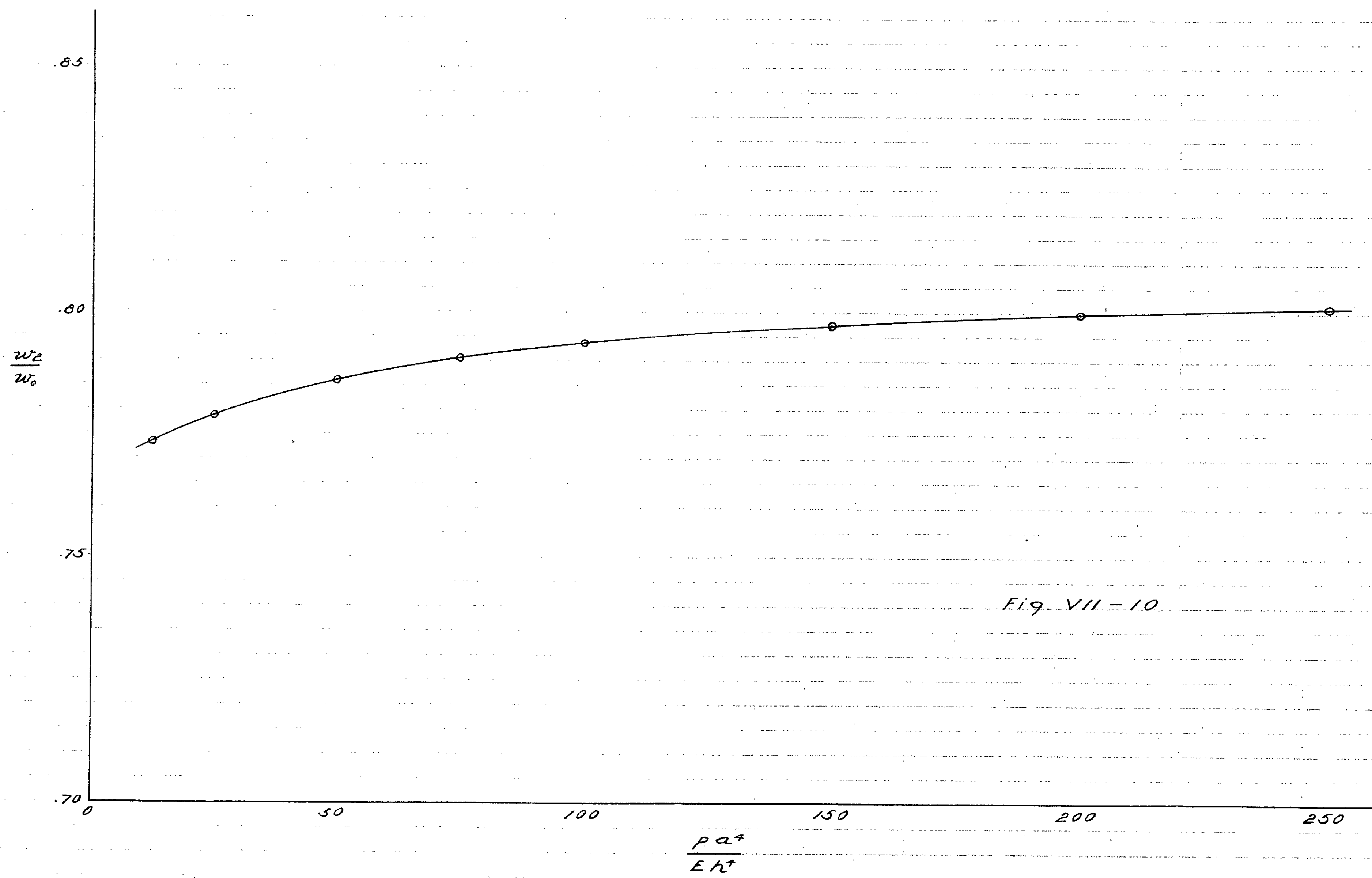


Fig. VII-10

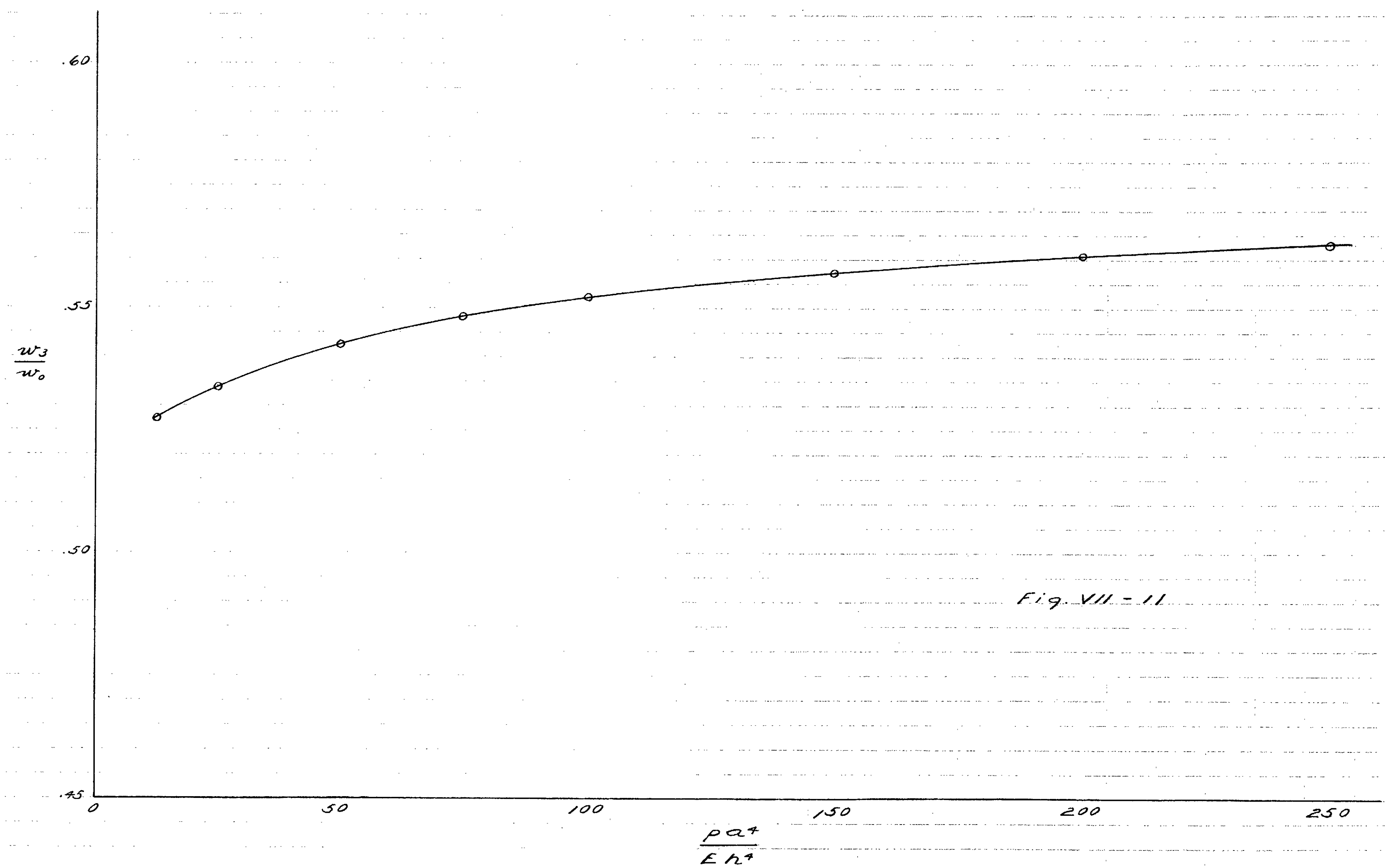


Fig. VII - 11

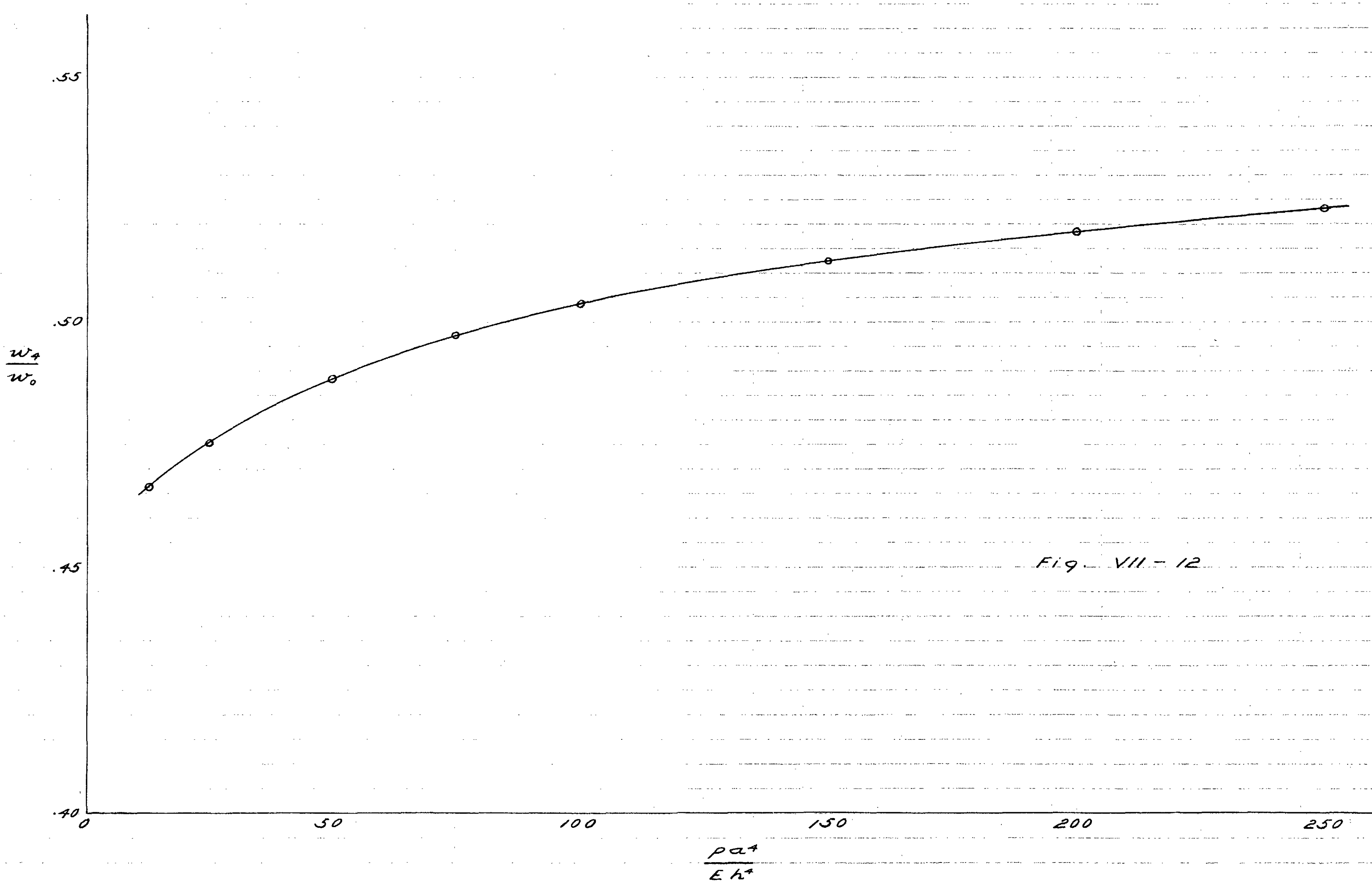


Fig. VII - 12

Table 7-4 Solutions of Eq. (5-34)

	K_0	K_1	K_2	K_3	K_4	K_5	S_1	S_2	S_3
F_0 =	- .118,035	- .622,023	- .632,001	- .496,545	-.904,377	- .288,561	- .292,423	-.439,791	-.182,625
F_1 =	- .199,285	- .714,880	- .673,668	- .514,402	-.923,424	- .290,942	- .274,566	-.420,743	-.177,863
F_2 =	- .224,028	- .787,231	- .673,033	- .525,863	-.916,444	- .288,867	- .233,883	-.413,113	-.182,012
F_3 =	- .306,047	-1.034,464	- .883,271	- .580,764	-1032,622	- .306,995	- .266,646	-.340,766	-.145,756
F_4 =	- .261,272	- .927,438	- .818,231	- .555,182	-.961,845	- .293,340	- .175,344	-.353,101	-.173,066
F_5 =	- .186,116	- .680,999	- .620,292	- .462,889	-.830,524	- .254,089	- .092,310	-.206,823	-.251,569
F_6 =	- .427,358	-1.410,958	-1.139,620	- .841,148	-1.302,423	- .352,740	- .298,472	-.217,071	-.054,268
F_7 =	- .303,899	-1.107,060	-1.012,996	- .651,212	-1.201,124	- .340,077	- .108,535	-.318,370	-.079,593
F_8 =	- .123,459	- .466,702	- .443,186	- .352,740	-.680,154	- .274,956	- .027,134	-.079,593	-.209,835

$$w_0 = 1.123,384$$
$$w_1 = .998,956$$
$$w_2 = .891,465$$
$$w_3 = .620,342$$
$$w_4 = .565,591$$
$$w_5 = .390,999$$

It might be pointed out here that the check column of the given matrix may be obtained by a direct substitution using the following relations:

C.C.

$$p'$$
$$-1 + p'$$
$$-2 + p'$$
$$2 \times 10.8\ \beta'_4 + p'$$
$$1 \times 10.8\ \beta'_5 + p'$$
$$6 + 21.6\ (\alpha'_6 + \gamma'_6) + p'$$

This procedure would give a way of knowing the correct substitution in the given matrix since the sum of the elements in any row should be equal to the element of the same row in the check column.

Find K_0, K_1, K_2, K_3, K_4, K_5, S_1, S_2, S_3 from the computed values of w's. Use the mean values of the K's and S's first assumed and those computed as the trial values for the second cycle and so on. At the end of the third trial, we have the assumed and computed values as follows:

Table 7-5. Solutions of Eq. (5-35)

Given Matrix

w_o	w_1	w_2	w_3	w_4	w_5	p'	C.C.
+25.995,771	-38.246,029	+ 8.250,258	+ 4.000,000	0	0	+.833,333	+ .833,333
- 9.686,636	+31.006,139	-18.632,867	-10.281,802	+ 6.595,166	0	+.833,333	- .166,667
+ 2.000,000	-18.382,610	+28.241,968	+ 4.000,000	-21.336,107	+ 3.476,749	+.833,333	-1.166,667
+ 1.000,000	-10.281,802	+ 4.000,000	+24.752,966	-17.189,361	+ 2.000,000	+.833,333	+5.115,135
0	+ 3.000,000	- 9.632,096	- 8.297,097	+26.941,238	- 9.379,948	+.833,333	+3.465,429
0	0	+ 2.000,000	+ 2.000,000	-16.200,297	+17.832,449	+.833,333	+6.465,485

Table 7-6 Solutions of Eq. (5-35)

Auxiliary Matrix

+25.995,771	- 1.471,2404	+ .317,3692	+ .153,8712	0	0	+.032,0565	+.032,0565
- 9.686,636	+16.754,7688	- .928,6089	- .524,7048	+ .393,6292	0	+.068,2703	+.008,5858
+ 2.000,000	-15.440,1292	+13.269,3882	- .332,2875	- 1.149,8964	+ .262,0128	+.137,4082	-.082,7630
+ 1.000,000	- 8.810,5616	- 4.498,9351	+18.481,2109	- 1.022,3668	+ .172,0006	+.109,3525	+.253,9862
0	+ 3.000,000	- 6.846,2693	- 8.997,9123	+ 8.688,6832	- .694,9835	+.293,8535	+.598,8697
0	0	+ 2.000,000	+ 2.664,5750	-11.176,3312	+9.082,7491	+.390,9987	+1.390,9983

	Assumed	Computed
K_0	- .061,763	- .061,695
K_1	- .051,947	- .051,894
K_2	- .024,660	- .024,799
K_3	- .023,377	- .023,477
K_4	+ .001,614	+ .001,697
K_5	+ .106,177	+ .106,204
S_1	+1.592,106	+1.592,078
S_2	+1.281,878	+1.281,814
S_3	+ .546,560	+ .546,173

These values check with each other to the fourth figures after the decimal points. The deflections at the various net points, accurate to the fourth figure after the decimal points, are

$$w_0 = 1.1240$$
$$w_1 = .9995$$
$$w_2 = .8920$$
$$w_3 = .6207$$
$$w_4 = .5660$$
$$w_5 = .3915$$

The results of various trials are shown in Fig. 7-13 to 7-15.

Fig. VII-13

Row 1

A: $S_1 = 1.592,696$; $S_2 = 1.282,838$; $S_3 = .548.700$

C: $S_1 = 1.590,165$; $S_2 = 1.280,080$; $S_3 = .545,095$

A - assumed

C - computed

$p = 100$

$n = 3$

First Approx.

0 | 0

$\Delta w = \gamma$

Row 2

-.505,761

$\alpha' = \beta' = +.009,273$

$\gamma' = -.026,303$

+.393,308

$\alpha = \beta = -.220,105$

$\gamma = +.393,308$

$K_5 = +.106,245$

-.239,729 | 0

+.390.999

$\alpha = \beta = -.216,407$

$\gamma = +.390.999$

$K_5 = +.106,048$

Δw .107.859

Δw .325.601

$\Delta w = \gamma$

Row 3

-.950,911

$\alpha' = \beta' = +.110,306$

$\gamma' = +.068,368$

.107,491

+.892,112

$\alpha = \beta = -.217.742$

$\gamma = +.152,398$

$K_2 = -.024,186$

-.762,520

$\alpha' = +.027,509$

$\beta' = +.151,120$

$\gamma' = +.050,132$

.325,874

+.891,465

$\alpha = \beta = -.218,383$

$\gamma = +.151,282$

$K_2 = -.024,805$

+.566,511

$\alpha = -.240,910$

$\beta = -.119.329$

$\gamma = +.173,203$

$K_4 = +.001,252$

-.546,620 | 0

+.565,591

$\alpha = -.239,717$

$\beta = -.119,841$

$\gamma = +.174,592$

$K_4 = +.001,754$

Δw .124,729

Δw .379,586

$\Delta w = \gamma$

Row 4

-1.095,495

$\alpha' = \beta' = +.132,998$

$\gamma' = +.011,586$

+1.124.700

$\alpha = \beta = -.249,458$

$\gamma = +.016,870$

$K_0 = -.061,642$

-1.028,996

$\alpha' = +.094,338$

$\beta' = +.156,170$

$\gamma' = +.027,554$

+.999,971

$\alpha = -.254,857$

$\beta = -.215,718$

$\gamma = +.053,985$

$K_1 = -.052,063$

-.868,159

$\alpha' = +.031,472$

$\beta' = +.211,278$

$\gamma' = +.023,591$

+.620,385

$\alpha = -.240,799$

$\beta = -.107,748$

$\gamma = +.053,874$

$K_3 = -.023,043$

-.675,850 | 0

.124,428

+1.123,384

$\alpha = \beta = -.248,856$

$\gamma = +.016,937$

$K_0 = -.061,642$

.378,614

+.998,956

$\alpha = -.254,186$

$\beta = -.214,982$

$\gamma = +.052,740$

$K_1 = -.051,864$

+.620,342

$\alpha = -.241,728$

$\beta = -.109,502$

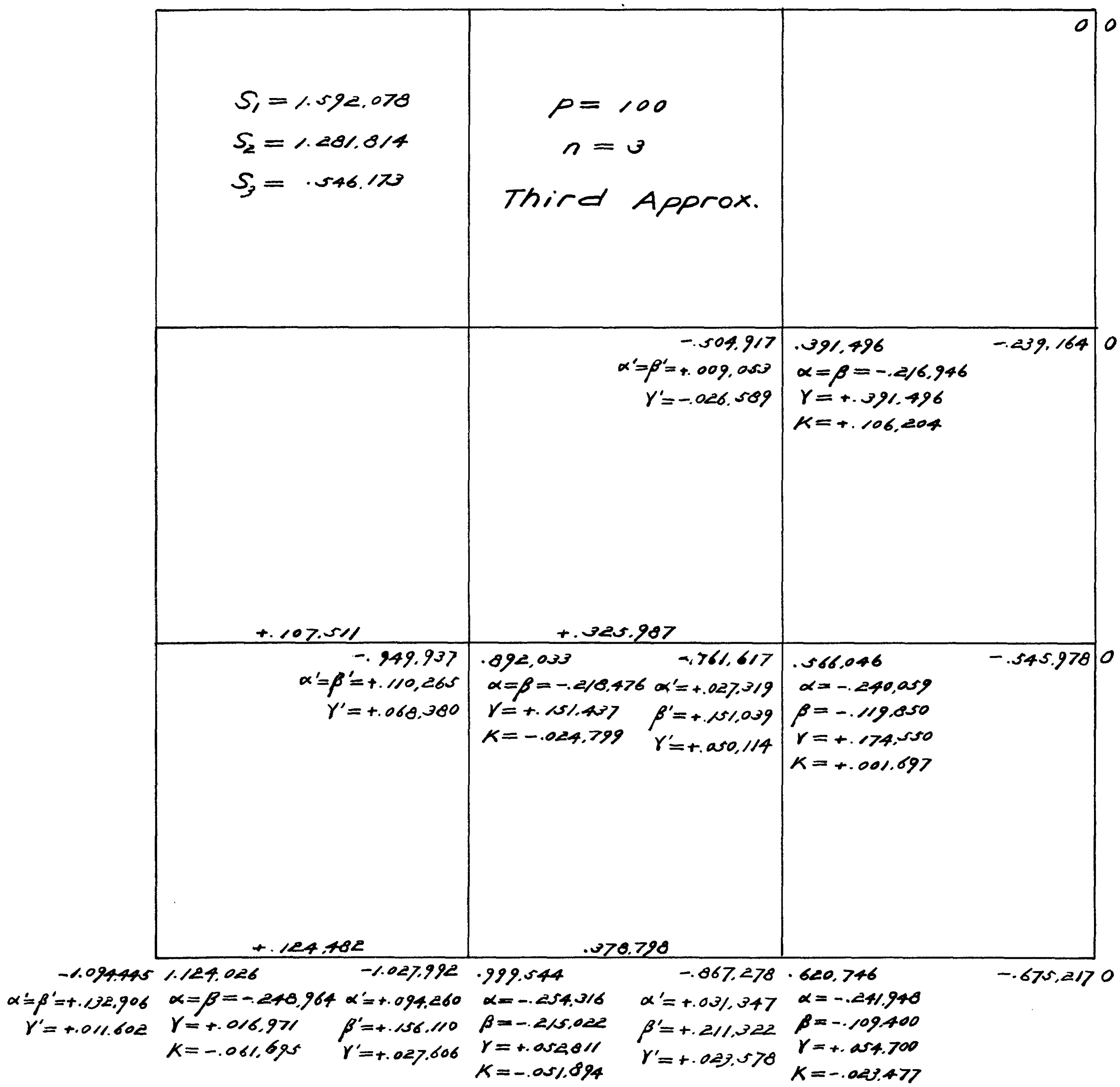
$\gamma = +.054,751$

$K_3 = -.023,472$

Fig. VII - 14

0
0
S_1 = 1.592,781
S_2 = 1.282,296
S_3 = .546,224
ρ = 100
n = 3
Second Approx.
-.504,894
α'=β'=+.009,113
γ'=-.026,488
.391.465
-.239,203
0
α=β=-.216,810
γ=+.391.465
K=+.106,208
.107,554
.326,097
-.949,710
α'=β'=+.110,215
γ'=+.068,340
.892,217
α=β=-.218,543
γ=+.151,442
K=-.024,826
-.761,472
α'=+.027,362
β'=+.150,981
γ'=+.050,091
.566,120
-.545,872
0
α=-.240,023
β=-.119,908
γ=+.174,655
K=+.001,724
.124,521
.378,904
-1.094,165
α'=β'=+.132,864
γ'=+.011,591
1.124,292
α=β=-.249,042
γ=+.016,967
K=-.061,734
-1.027,733
α'=+.094,232
β'=+.156,046
γ'=+.027,574
.999,771
α=-.254,383
β=-.215,108
γ=+.052,807
K=-.051,931
-.867,069
α'=+.031,365
β'=+.211,194
γ'=+.023,571
.620,867
α=-.241,963
β=-.109,494
γ=+.054,747
K=-.023,496
-.675,040
0

Fig. VII-15


0 0
S1 = 1.592.078
S2 = 1.281.814
S3 = .546.173
p = 100
n = 3
Third Approx.
-.504.917
α'=β'=+.009.053
Y'=-.026.589
.391.496
-.239.164 0
α=β=-.216.946
Y=+.391.496
K=+.106.204
+.107.511
+.325.987
-.949.937
α'=β'=+.110.265
Y'=+.068.380
.892.033
α=β=-.218.476
Y=+.151.437
K=-.024.799
-.761.617
α'=+.027.319
β'=+.151.039
Y'=+.050.114
.566.046
-.545.978 0
α=-.240.059
β=-.119.850
Y=+.174.550
K=+.001.697
+.124.482
.378.798
-1.094.445
α'=β'=+.132.906
Y'=+.011.602
1.124.026
α=β=-.248.964
Y=+.016.971
K=-.061.695
-1.027.992
α'=+.094.260
β'=+.156.110
Y'=+.027.606
.999.544
α=-.254.316
β=-.215.022
Y=+.052.811
K=-.051.894
-.867.278
α'=+.031.347
β'=+.211.322
Y'=+.023.578
.620.746
α=-.241.948
β=-.109.400
Y=+.054.700
K=-.023.477
-.675.217 0

CHAPTER VIII

THE RELAXATION METHOD

1. The Method Explained.

When a more accurate result is needed, the plate must be divided into a set of finer nets. The number of simultaneous equations increases as the number of nets is increased, and the labor involved in their solution increases rapidly. In avoiding the solution of simultaneous equations, an ingenious method has been developed by Southwell[59][60][61] which is named by its inventor as the "Relaxation Method".

In the computations by Relaxation Method, as explained by Southwell, "we replace exact differential equations by a finite-difference approximation, then seek values at the nodal points of a regular net .."[60] Although this method has been discussed in a book form[61] and in a number of papers by Southwell and his collaborators, it has never been explained, at least so it seems to the author, to a degree to make it easily understood.

Essentially, the idea behind the treatment by the relaxation method is just the same as that by the Cross's method of moment distribution in the case of bending of continuous beams. It seems, therefore, most easily to explain the relaxation method by a comparison with the moment distribution method, since the latter is well-accepted and is familiar to most structural engineers.

Let us examine the redundant beam as shown in Fig. VIII-1.

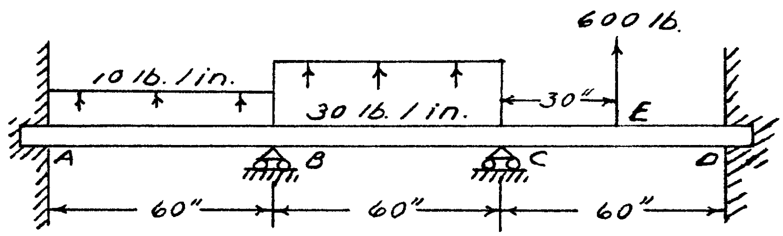


Fig. VIII – 1

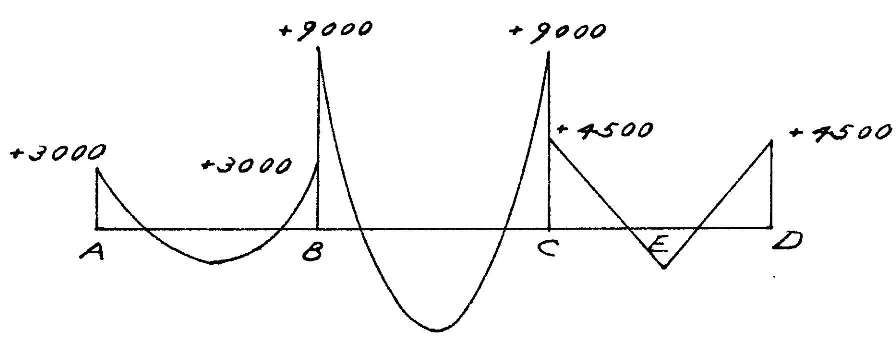


Fig. VIII – 2

The procedure of obtaining the redundant support moments by the moment distribution method is well known. The first step in the moment distribution analysis is to assume that the slope at each of the four supports is zero. By this assumption, the end-moments' at A, B, C, D, can be found without difficulty. The result is shown in Fig. VIII-2. Here we see that the boundary conditions at A and B are satisfied, and the principle of continuity is also satisfied. The condition of equilibrium, however, is not satisfied, since there are unbalanced moments at B and C. The moment distribution method now offers a procedure to balance these unbalanced moments by a relaxation based on consistent deformations.

The analysis by the relaxation method, in this case, would be essentially the same. The moments at A, B, C, and D, are assumed to satisfy the boundary conditions and the condition of continuity. The unbalanced moments at B and C are then distributed by the relaxation based on consistent deformations. The difference lies in that the relaxation method offers more freedom in assuming the end moments and therefore could make the convergence of the operations more rapid. On the other hand, however, it becomes difficult to assume these values.

As the method of moment distributions applies only to redundant structures, the application of the relaxation method goes much further, and its application to the partial differential equations has brought the study of engineering sciences into a new era, because the boundary conditions are now no longer difficult to be described and to be satisfied.

As an example, we may illustrate the procedure by studying the small deflection theory of thin plates. Letting $w = \frac{w'}{p}$, where w' and p are the nondimensional deflection and pressure, the equilibrium equation in terms of the finite difference is

$$\Delta_x^4 w + 2\,\Delta xy^2 w + \Delta y^4 w = 12\,(1-\mu^2)(\Delta \ell)^4 \quad \ldots\ldots\ldots\ldots\ldots \quad (8\text{-}1)$$

To solve the problem, the domain to be investigated is drawn and the net points chosen. Values of w are assumed to satisfy the boundary conditions and are then written in adjacent to each point of the net. From these values of w, the residuals Q at points (m,n)

$$\begin{aligned} Q_{mn} = {} & 20\, w_{m,n} - 8\,(w_{m+1,n} + w_{m-1,n} + w_{m,n+1} \\ & + w_{m,n-1}) + 2\,(w_{m+1,n+1} + w_{m+1,n-1} \\ & + w_{m-1,n+1} + w_{m-1,n-1}) + (w_{m+2,n} \\ & + w_{m-2,n} + w_{m,n+2} + w_{m,n-2}) \\ & - 12\,(1-\mu^2)(\Delta \ell)^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8\text{-}2) \end{aligned}$$

are computed and recorded. The Q, thus computed, can be thought of as an unbalanced force which must be removed from the system. Now, instead of setting up a specific iteration process, it is merely observed that if the deflection at one point (m,n) is altered, all others remaining fixed, the residuals will change according to the pattern, Fig. V-2, the "relaxation pattern". Each change of w at any point effects a redistribution of the residuals, Q, among the net point, and such changes of w are desired as will move all the unbalanced forces to the boundary.

These "operation instructions" may appear to be very vague. Indeed, they are vague. Their vagueness is a disadvantage of the method but it is also the source of the great power of the method because it permits the computor to alter the procedure to attain more rapid approach to the final answer (of no residuals) without violating the requirements of a fixed routine. There is only one way to appreciate fully the meaning of these remarks and that is to do a problem.

2. The Small Deflection Problems of Rectangular Plates.

The non-dimensional equilibrium equation of a thin plate in bending with small deflections is

$$\nabla^4 w = p \qquad (8\text{-}3)$$

and the boundary conditions for a simply-supported plate are

$$w = 0 \text{ along } x = \pm\frac{1}{2},\ y = \pm\frac{1}{2}\left(\frac{b}{a}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ along } x = \pm\frac{1}{2}$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \text{ along } y = \pm\frac{1}{2}\ \frac{b}{a} \qquad (8\text{-}4)$$

Since $w = 0$ along $x = \pm\frac{1}{2}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial^2 w}{\partial y^2}$ must also be zero along those edges. Similarly, $\frac{\partial^2 w}{\partial x^2}$ is zero along $y = \pm\frac{1}{2}\left(\frac{b}{a}\right)$. These properties enable one to write the boundary conditions as

$$w = 0 \text{ along } x = \pm\frac{1}{2},\ y = \pm\frac{1}{2}\left(\frac{b}{a}\right)$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \text{ along } x = \pm\frac{1}{2},\ y = \pm\frac{1}{2}\left(\frac{b}{a}\right) \qquad (8\text{-}5)$$

The differential equation can be written as

$$\nabla^2 (\nabla^2 w) = p.$$

Letting $\nabla^2 w = M$, we have our boundary-valued problems as follows:

$$\nabla^2 M = p$$

$$M = 0 \text{ along the four edges} \qquad (8\text{-}6)$$

and $\nabla^2 w = M$

$$w = 0 \text{ along the four edges} \qquad (8\text{-}7)$$

The problems may now be solved in two steps, i.e., first Eq. (8-6) and then Eq. (8-7). This transformation greatly reduces the labor required in applying the relaxation method, because the relaxation pattern of the harmonic or Laplacian type is much simpler than that of the biharmonic type.

Let us first solve the boundary-valued problem of Eq. (8-6) when the plate is a square one. As an illustrative example, let us study the process with n = 4. From the previous results as found from the calculations with n = 3 (Chapter VII) the values of w at all the net points can be assumed. By Eq. (8-7),

$$M_{m,n} = w_{m+1,n} + w_{m-1,n} + w_{m,n+1} + w_{m,n-1} - 4\, w_{m,n} \qquad (8\text{-}8)$$

The values of $M_{m,n}$ are then recorded at the right of the corresponding net point, and the residuals

$$Q_{m,n} = M_{m+1,n} + M_{m-1,n} + M_{m,n+1} + M_{m,n-1} - 4\, M_{m,n} - 12(1-\mu^2)(\Delta \ell)^4 \qquad (8\text{-}9)$$

are computed and are recorded at the left of these net points. The results are shown in Fig. VIII-3 and VIII-4. As an example, we have

$$M_o = 4\, w_1 - 4 w_o = 4 \times .0406 - 4 \times .0437$$

$$= -.0124$$

$$M_4 = w_2 + w_3 + w_5 + w_7 - 4 w_4$$

$$= .0377 + .0316 + .0231 + .0163 - 4 \times .0295$$

$$= -\,.0093$$

$$Q_o = 4M_1 - 4M_o - .002,617$$
$$= 4 \times (-.0117) - 4x\ (0.0124) -.002,617$$
$$= + .000,163$$

$$Q_4 = M_2 + M_3 + M_5 + M_7 - 4\ M_4 - .002,617$$
$$= -.0106 - .0093 - .0078 - .0054 - 4 \times (-.0093) - .002,617$$
$$= + .001,463.$$

Where $.002,617 = 12\ (1 - \mu^2)(\Delta \ell)^4$, since $\mu^2 = 0.1$ and $\Delta \ell = \frac{1}{8}$.

The largest counter-balanced M occurs in the vicinity of the greatest deviation of the assumed values from the correct solution, so changes are first made at this point. An examination of Fig. VIII-4 shows that the greatest residual occurs at point 2. Since

$$Q_2 = 2M_1 + 2M_4 - 4M_2 - .002,617$$

a change of M_2 would change Q_2 by an amount equal to four times $(-\Delta M_2)$. Mathematically

$$\Delta Q_2 = -4\ \Delta M_2$$

Where Δ denotes the amount of change. Adding -.0004 to M_2, while assuming all the other values of M to remain unchanged, $\Delta Q_2 = + .0016$ and Q_2 is now equal to -.000,637. If we use a nomenclature similar to that in the method of moment distribution, this process can be called "balancing the unbalanced Q". A symbol (bl) is put at the side of the value to indicate this is the first balancing. Now, we see that

$$Q_1 = M_o + 2M_2 + M_3 - 4M_1 - .002,617$$

and

$$Q_4 = M_2 + M_3 + M_5 + M_7 - 4M_4 - .002,617.$$

A change of M_2 with all the other M's fixed would change Q_1 and Q_4 by the relations as follows:

$$\Delta Q_1 = 2 \Delta M_2$$

$$\Delta Q_4 = \Delta M_2$$

Now, by "relaxing" the nets,

$$\Delta Q_1 = 2 \times (-.0004) = -.0008$$

$$\Delta Q_4 = -.0004$$

and

$$Q_1 = + .001,263 - .0008 = + .000,463$$

$$Q_4 = + .001,463 - .0004 = + .001,063$$

These operations may be called "carrying-over" and be denoted by (cl).

The whole process consists of 20 "balancing" and "carrying-over" operations by similar calculations. The detail operations of the computations are shown in Fig. VIII-4. After the values of M's computed, the residuals

$$Q'_{m,n} = w_{m+1,n} + w_{m-1,n} + w_{m,n+1} + w_{m,n-1} - 4 w_{m,n} - M_{m,n}$$

are computed and we may determine the values of w by a similar series of calculations. The detail operations are computations are shown in Fig. VIII-3. The whole process consists of eleven "balancing" and "carrying-over" operations.

The center deflection ratio thus obtained is

$$w_o = .043790 \, p$$

for $\mu = 0.316,228$. For $\mu = .3$, we have

$$w_o = .043790 \times \frac{.91}{.9} \; p$$
$$= .0443 \; p$$

which checks exactly with the "exact" analytical solution.

For thin plates with clamped edges, the boundary conditions are

$$w = 0$$

$$\frac{\partial w}{\partial x} = 0 \text{ along } x = \pm \frac{1}{2}$$

$$\frac{\partial w}{\partial y} = 0 \text{ along } y = \pm \frac{1}{2}$$

The relaxation pattern of the biharmonic type must be used in this case. Although the pattern is more complicated, the process is essentially the same.

After the essential idea of the relaxation method is graphed, other problems may be solved by rather obvious steps. It may be noted that no question of convergence can occur in the general relaxation process since no specific instructions are given. If, after some steps, the residuals get worse the intelligent computor goes back and makes changes in the opposite direction. These remarks, however, oversimplify the problem somewhat because of two facts; the computor may become confused as to whether or not the residuals are really better, and secondly there is always a question of whether or not a solution with zero residuals exists.

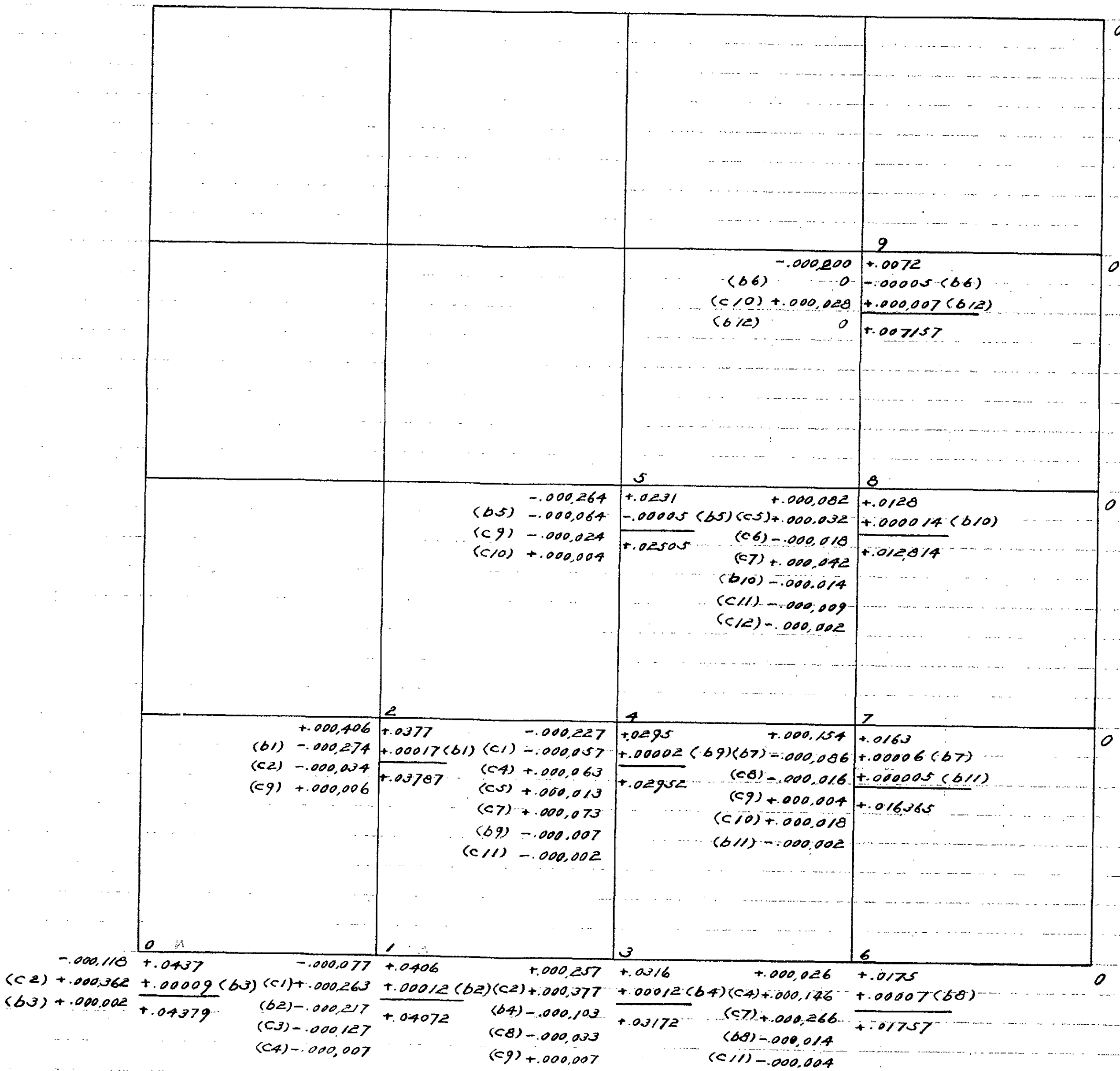


Fig. VIII - 3

Solution of the Small Deflection Theory by the Relaxation Method

$$\nabla^2 w = M$$

$$Q' = \nabla^2 w - M$$

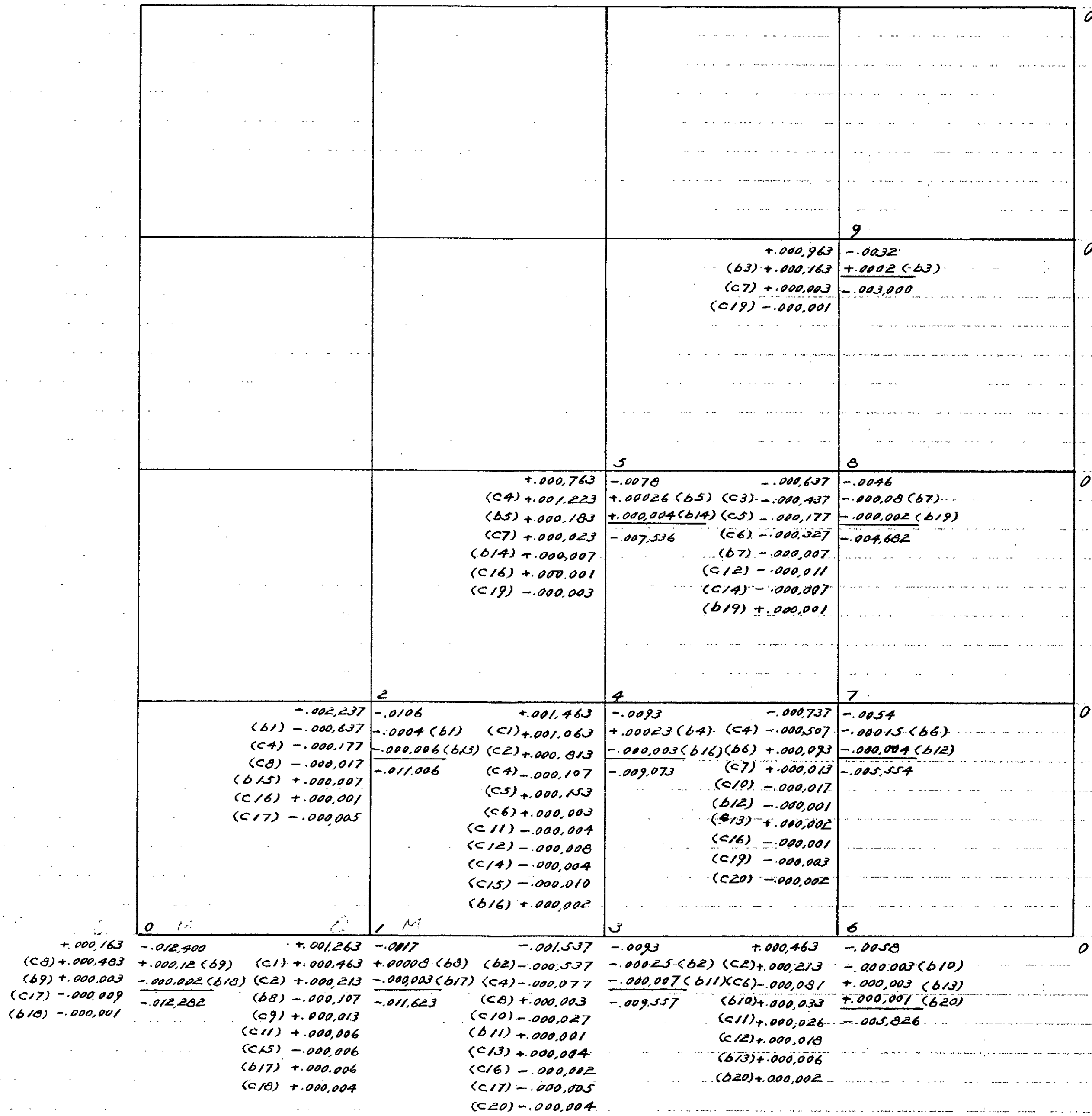


Fig. VIII - 4

Solution of the Small Deflection Theory by the Relaxation Method

$$\nabla^2 M = 12(1-\mu^2)(\Delta l)^4$$
$$= +.002,617$$

$$Q = \nabla^2 M - .002,617$$

3. The Large Deflection Problems of the Rectangular Plates.

The solution of the general case of the large deflection problems of the rectangular plates by the relaxation method has been studied by Green and Southwell[18] and the method was outlined in Chapter II. They worked with the three complicated equilibrium equations in terms of u, v, and w. The author has found, however, that it would be equally simple to use the two much simpler equations in terms of the stress function, F, and the deflection, w.

Our governing differential equations are

$$\nabla^4 F = K$$

$$\nabla^4 w = 10.8\ p + 10.8\ k' \quad \text{.................................} \quad (8\text{-}10)$$

where $$k = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \quad ;$$

and $$k' = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}$$

These equations are non-linear, and therefore the "exact" relaxation pattern becomes rather complicated. Instead of using it, we may use the relaxation pattern of the biharmonic type if the non-linear terms are taken care of by making corrections from time to time.

The boundary conditions for F are usually the source of trouble. But it is always possible to solve the boundary values of F's in terms of those of interior points. Here, the boundary values of F's are no longer constants but they vary from time to time as the interior values change.

The domain of the problem to be solved is first drawn, and the net points chosen and then a set of solutions of w and F guessed. K then can be computed and the residuals

$$Q = \Delta x^4 F + 2\,\Delta xy^2\,F + \Delta y^4\,F - k$$

are computed at each point. By the relaxation process carried out exactly as in the linear problems, the residuals Q are reduced somewhat. Before bothering to eliminate the Q completely, use the values of F thus obtained and the values of w assumed to compute k'. On another sheet, values of

$$Q' = \Delta x^4\,w + 2\Delta xy^2\,w + \Delta y^4\,w - 10.8\,p - 10.8\,k'$$

are computed at each point. By the relaxation process, the residual Q' are reduced somewhat. New values of k are computed and hence corrected values of the residuals attached to each net point. This process of reduction and correction is continued until sufficient accuracy has been attained.

In the case of square plates with given edge displacements, the biharmonic type compatability equation can be transformed into two equations of the harmonic type. They are

$$\nabla^2 T = K$$

$$\nabla^2 F = T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8\text{-}11)$$

The condition of given edge displacements

$$\frac{1}{2} T_{o,i} + T_{1,i} + \ldots\ldots + T_{m-1,\,i} + \frac{1}{2} T_{m,i} = s_i \quad \ldots\ldots\ldots\ldots\ldots \quad (8\text{-}12)$$

gives the boundary values of $T_{m,i}$, and the condition of zero shearing stress along the edges (when the plate is free to move along the edges) or zero strain along the edges (when the plate is prevented from any displacement along the edges) gives the boundary values of F.

If the edges of the plates are free to rotate about their supports, the biharmonic type equilibrium equation can be transformed into two equations of the harmonic type. They are

$$\nabla^2 M = 10.8\ p + 10.8\ k'$$

$$\nabla^2 w = M \qquad (8\text{-}12)$$

and the boundary conditions are

$$M = 0 \text{ along the edges}$$

$$w = 0 \text{ along the edges} \qquad (8\text{-}13)$$

The problem now is transformed into four equations of the Poisson's type. The relaxation pattern is much simpler at the sacrifice of working with more equations. The process of applying the relaxation method, however, is just the same as that of the general case as described before.

CHAPTER IX

RESULTS AND CONCLUSIONS

1. Discussion of the Results

The bending problem of a square plate under uniform normal pressure, with the edges prevented from displacements along the supports but free to rotate about them, is studied by the finite differences approximation. The differences equations are solved by the method of successive approximation as described in Chapter VI. The computation starts with n = 1 to n = 3, in which case the plate is divided into 36 square nets with 25 inner points. The maximum normal pressure calculated is $pa^4/Eh^4 = 250$.

After the values of w and F have been determined (Chapter VII), the stresses can be found by the following relations:

$$\sigma_x' = \frac{\partial^2 F}{\partial y^2} = \Delta_y^2 F/(\Delta \ell)^2 = \beta'/(\Delta \ell)^2$$

$$\sigma_y' = \frac{\partial^2 F}{\partial x^2} = \Delta_x^2 F/(\Delta \ell)^2 = \alpha'/(\Delta \ell)^2$$

$$\sigma_x'' = -\frac{1}{2(1-\mu^2)}(\Delta_x^2 w + \mu \Delta_y^2 w)\frac{1}{(\Delta \ell)^2}$$

$$= -\frac{1}{2(1-\mu^2)(\Delta \ell)^2}(\alpha + \mu\beta)$$

$$\sigma_y'' = -\frac{1}{2(1-\mu^2)(\Delta \ell)^2}(\beta + \mu\alpha)$$

where σ' and σ'' are the membrane stress and the extreme fiber bending stress respectively. The total stresses, σ, are the sum of the membrane and bending stresses at the section, and are maximum at the extreme fiber of the plate. They are

$$\sigma_x = \sigma_x' + \sigma_x''$$
$$\sigma_y = \sigma_y' + \sigma_y''$$

At the center of the square plate, $\alpha' = \beta'$, and $\alpha = \beta$, and therefore we have

$$\sigma_x' = \sigma_y' = \alpha'/(\Delta\ell)^2 = \beta'/(\Delta\ell)^4$$

$$\sigma_x'' = \sigma_y'' = \frac{\alpha}{2(1-\mu)(\Delta\ell)^2} = \frac{\beta}{2(1-\mu)(\Delta\ell)^2}$$

The deflections at various points determined in the cases n = 1, n = 2, and n = 3 are tabulated in Table IX-1 and Table IX-2. The center deflections are plotted against the normal pressure ratio in Fig. IX-1. The membrane stresses in the center of the plate and at the centers of the edges are tabulated in Table IX-3 and are plotted in Fig. IX-3. The bending and total stresses are tabulated in Table IX-4 and are plotted in Fig. IX-4.

A study of the results shows that as n increases, the center deflection and membrane stresses tend to decrease, whereas the bending stresses and the membrane stresses along the sides tend to increase. The total extreme fiber total stresses at the center of the plate, however, still tend to decrease as n increases.

The maximum error in center deflections is 0.47% for n=2 in comparison with n = 3 and the maximum error in center membrane stresses is 0.44%, both on the conservative side. Both occurred at $pa^4/Eh^4 = 250$. The error in the center bending stresses is 2% at $pa^4/Eh^4 = 12.5$ and is 0.83% at $pa^4/Eh^4 = 250$, both on the unsafe side. The error in the center extreme fiber stresses is 1.6% at $pa^4/Eh^4 = 12.5$ and 0.17% at $pa^4/Eh^4 = 250$, both on the safe side. The error in the membrane stresses at the center of the sides is 12% for both σ_x' and σ_y' at $pa^4/Eh^4 = 12.5$ and 8.9% for both σ_x' and σ_y' at $pa^4/Eh^4 = 250$, all on the unsafe side.

Judging from the above results, the error for n = 3 in comparison with n = 4 would be still smaller. Since in our case only the center deflections and stresses are to be investigated and the errors are sufficiently small for engineering purposes, the case n = 3 is considered to be satisfactory for the final results.

The center deflections obtained by Way,(17) Levy(19)(21) and Head and Sechler(65) are plotted in Fig. IX-2 to compare with the present results. The center membrane, bending, and total stresses are plotted in Fig. IX-5 to compare with the results by Levy.(19)(21) It is seen from these results that the center deflections are in good agreement with C.I.T. test results up to $pa^4/Eh^4 = 120$. The theoretical results seem to be too low at higher pressures.

It is interesting to note that the test results are really for clamped edge plates. The clamping effect seems to be only local, and at the center of the plate the plate behaves just as though it were simply-supported, i.e. the plate is free to rotate about its edges.

From the point of view of the engineer designing the plate, the total stresses at the center of the edges are still much larger in the case of clamped edges than all the other cases, hence a design based on those stresses would give a structure on the conservative side.

The center deflections, however, would give an idea of how much is the "washboarding" of a boat bottom while a seaplane is taxiing or landing.

2. Conclusions

We can draw the following conclusions from this analysis:

(1) The large deflection problems of rectangular plates can be solved approximately by the present method with any boundary conditions and to any degree of accuracy required. Though it is still difficult, the present method is nevertheless simpler than the previously used methods to give the same degree of accuracy.

(2) For the case considered, the case $n = 3$ gives results of good accuracy and the results are consistent with the existing theories.

(3) The clamping effect of a clamped thin plate seems to be only local. At the center, the plate behaves more like a plate with simply-supported edges, i.e. the thin plate is approximately free to rotate about its edges.

(4) The test results show that at $pa^4/Eh^4 > 175$, all the existing solutions of the differential equations give unsafe results for center deflection for a square plate. This suggests

that the differential equations may use some assumptions which neglect some quantities that are large when the deflections are large.

(5) The present results of the center deflections and membrane stresses give good agreement with the test results when $pa^4/Eh^4 < 120$.

3. Suggestions for Further Research

Since the method presented here is general, it may be applied to solve rectangular plates of any length-width ratio with various boundary conditions. The case solved in this thesis is only one of many in which aeronautical engineers are interested. The other problems are as follows:

(1) Clamped rectangular plates. The solution by the present method may serve as a good check to the approximate results obtained by Way.

(2) Simply-supported rectangular plates. Only two cases have been solved by Levy. The present method may be used to solve all the other cases.

(3) Rectangular plates with edges prevented from displacements along the supports, but free to rotate about them. The method gives good agreement in center deflections and membrane-clamped plate in the case of a square plate. Other cases would probably yield similar agreement.

(4) Combined bending and buckling problems can also be solved by this method and they are of interest to aeronautical engineers for reasons discussed in Chapter I .

(5) The membrane problems can also be solved by this method. There are two cases which are to be solved. One is the rectangular plates with edges prevented from any displacements along the supports, and the other is rectangular plates with edges which are free to move along the supports.

TABLE IX-1

CENTER DEFLECTIONS

$\frac{pa^4}{Eh^4}$	w_0/h		
	n = 1	n = 2	n = 3
0	0	0	0
12.5	.3888	.4062	.4055
25	.5844	.6092	.6083
50	.8184	.8474	.8460
75	.9757	1.0052	1.0031
100	1.0980	1.1269	1.1240
150	1.2888	1.3145	1.3104
200	1.4376	1.4616	1.4557
250	1.5623	1.5844	1.5770

TABLE IX-2

DEFLECTIONS AT VARIOUS POINTS

(n = 2)

pa^4/Eh^4	w_0/h	w_1/h	w_2/h
0	0	0	0
12.5	.4062	.2980	.2198
25	.6092	.4508	.3363
50	.8474	.6332	.4791
75	1.0052	.7555	.5766
100	1.1269	.8502	.6528
150	1.3145	.9966	.7713
200	1.4616	1.1116	.8648
250	1.5844	1.2076	.9431

TABLE IX-3

DEFLECTIONS AT VARIOUS POINTS

(n = 3)

pa^4/Eh^4	w_0/h	w_1/h	w_2/h	w_3/h	w_4/h	w_5/h
0	0	0	0	0	0	0
12.5	.4055	.3564	.3136	.2139	.1890	.1159
25	.6083	.5365	.4738	.3249	.2892	.1822
50	.8460	.7494	.6650	.4592	.4131	.2711
75	1.0031	.8905	.7930	.5500	.4986	.3370
100	1.1240	.9995	.8920	.6207	.5660	.3915
150	1.3104	1.1677	1.0450	.7305	.6717	.4804
200	1.4557	1.2988	1.1641	.8164	.7551	.5531
250	1.5770	1.4081	1.2634	.8880	.8249	.6149

TABLE IX-4

MEMBRANE STRESSES

$\frac{pa^4}{Eh^4}$	$\frac{\sigma_{x_\odot}' a^2}{Eh^2} = \frac{\sigma_{y_\odot}' a^2}{Eh^2}$		$\frac{\sigma_{x_1}' a^2}{Eh^2}$		$\frac{\sigma_{y_1}' a^2}{Eh^2}$	
	n = 2	n = 3	n = 2	n = 3	n = 2	n = 3
0	0	0	0	0	0	0
12.5	.6103	.6089	.3338	.3795	1.055	1.200
25	1.384	1.377	.7612	.8574	2.407	2.711
50	2.695	2.683	1.484	1.661	4.693	5.254
75	3.806	3.792	2.096	2.341	6.628	7.401
100	4.802	4.785	2.643	2.943	8.357	9.305
150	6.566	6.542	3.613	4.001	11.43	12.65
200	8.136	8.103	4.473	4.929	14.15	15.59
250	9.575	9.533	5.264	5.778	16.64	18.27

Note: Subscript $_\odot$ denotes the center of the plate.

Subscript $_1$ denotes the center of the sides, $x = \pm a/2$.

TABLE IX-5

EXTREME FIBER BENDING AND TOTAL STRESSES

AT CENTER OF THE PLATE

$\frac{pa^4}{Eh^4}$	Bending Stresses $\frac{\sigma'' a^2}{Eh^2}$		Total Stresses $\frac{\sigma' a^2}{Eh^2} + \frac{\sigma'' a^2}{Eh^2}$	
	n = 2	n = 3	n = 2	n = 3
0	0	0	0	0
12.5	2.530	2.582	3.140	3.191
25	3.708	3.781	5.092	5.158
50	5.010	5.087	7.705	7.770
75	5.845	5.928	9.651	9.720
100	6.475	6.554	11.277	11.339
150	7.439	7.513	14.005	14.055
200	8.191	8.261	16.327	16.364
250	8.817	8.891	18.392	18.424

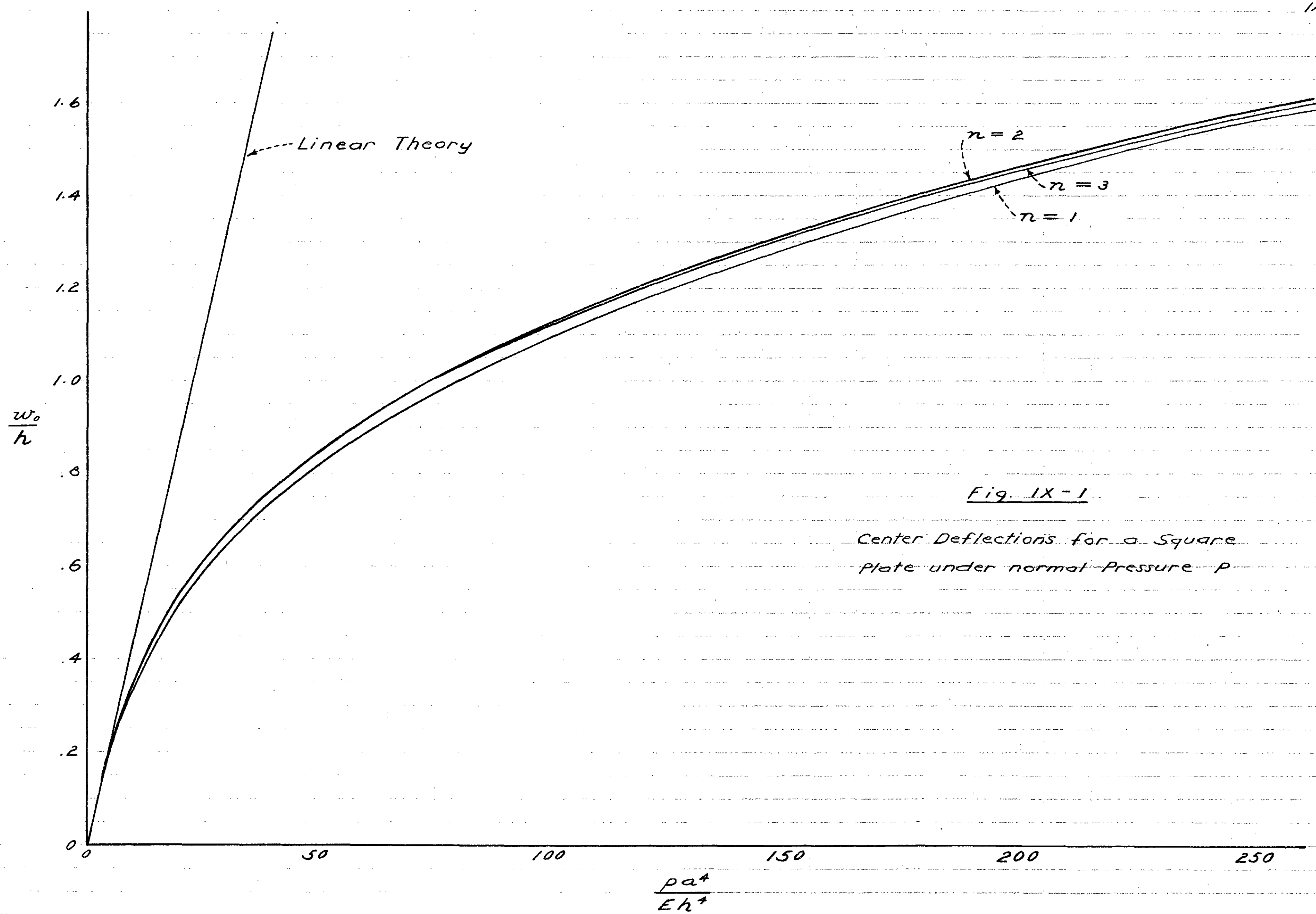


Fig. IX-1

Center Deflections for a Square Plate under normal Pressure p

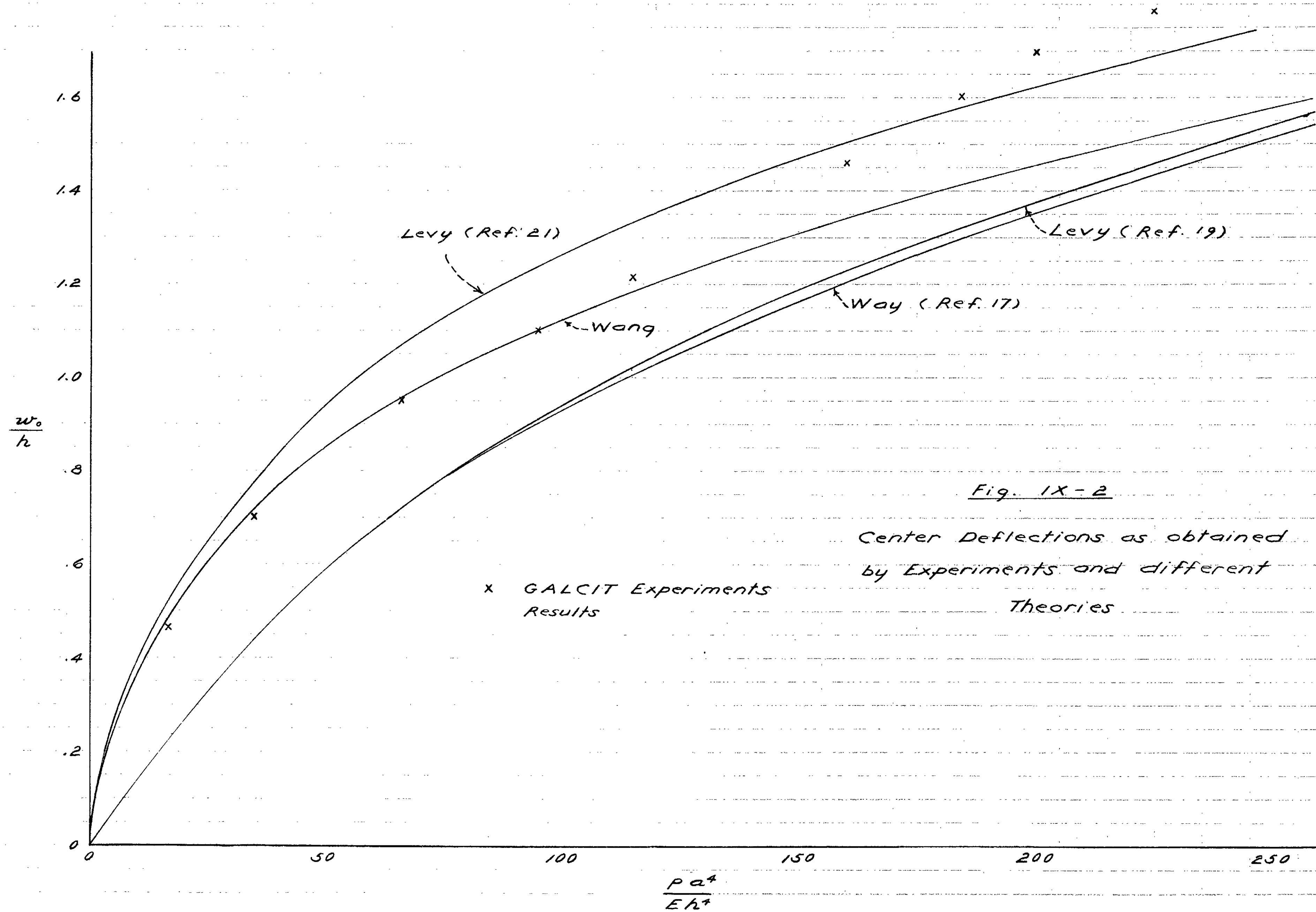


Fig. IX-2

Center Deflections as obtained by Experiments and different Theories

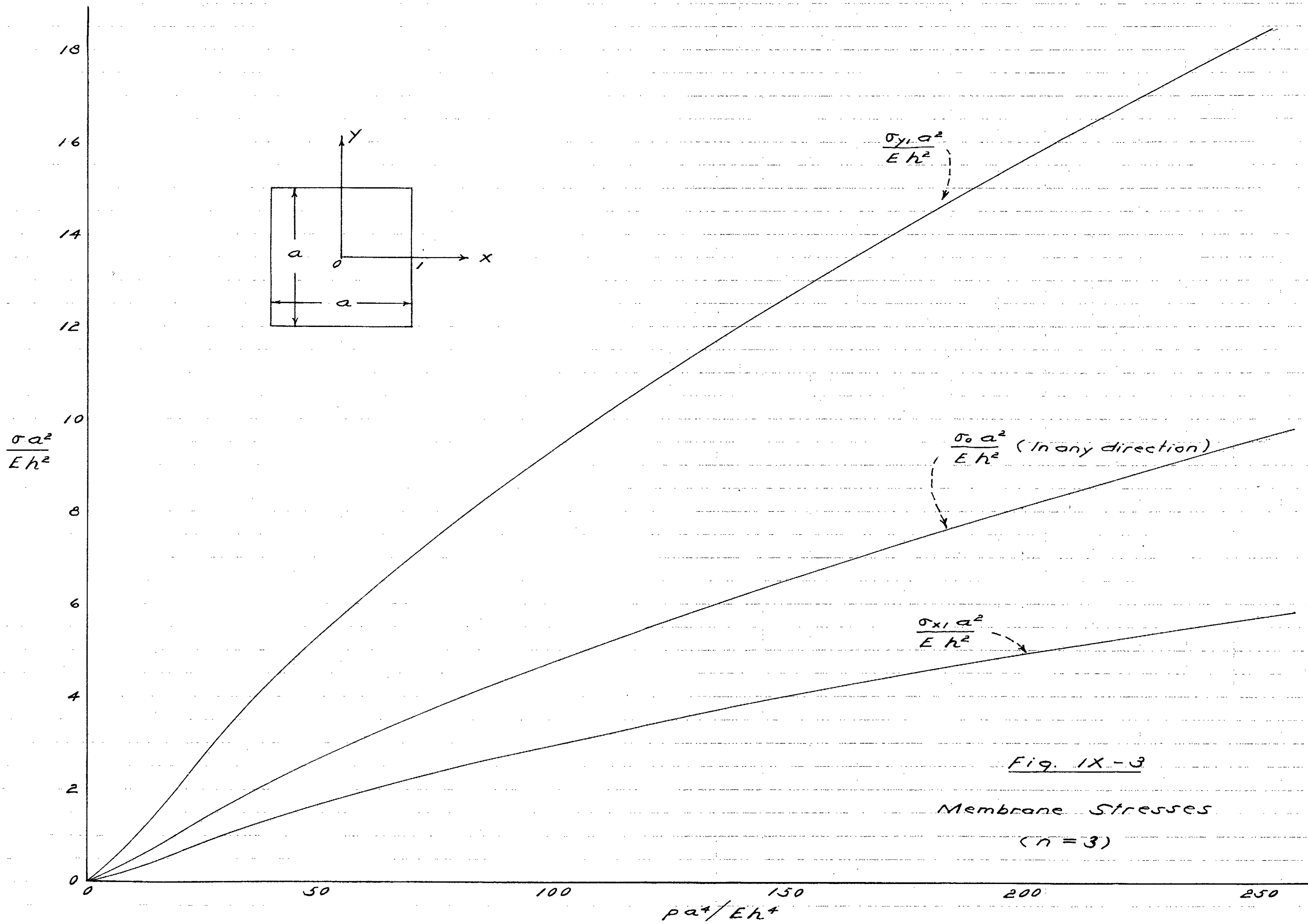


Fig. IX-3

Membrane Stresses

($n = 3$)

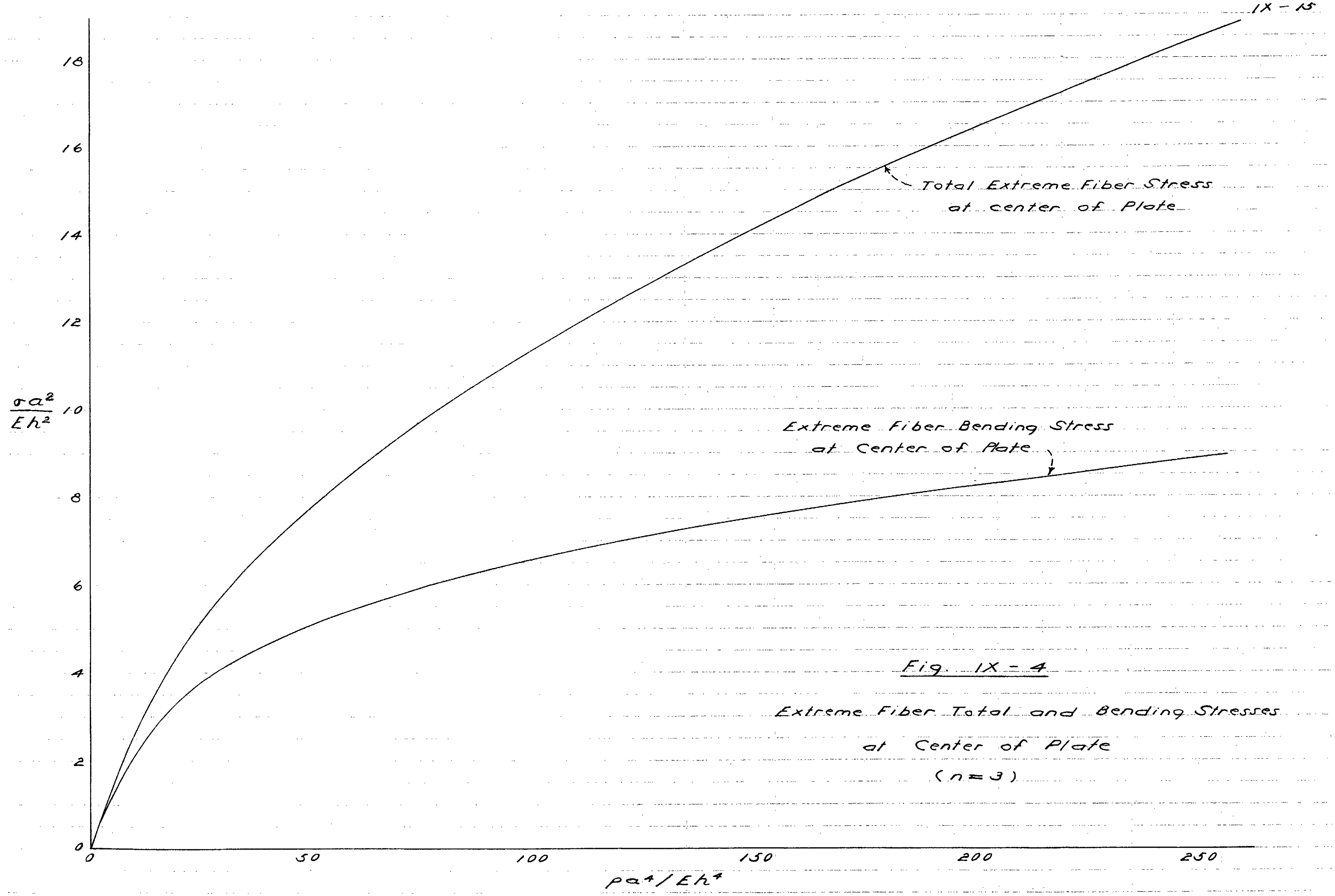


Fig. IX - 4

Extreme Fiber Total and Bending Stresses at Center of Plate ($n = 3$)

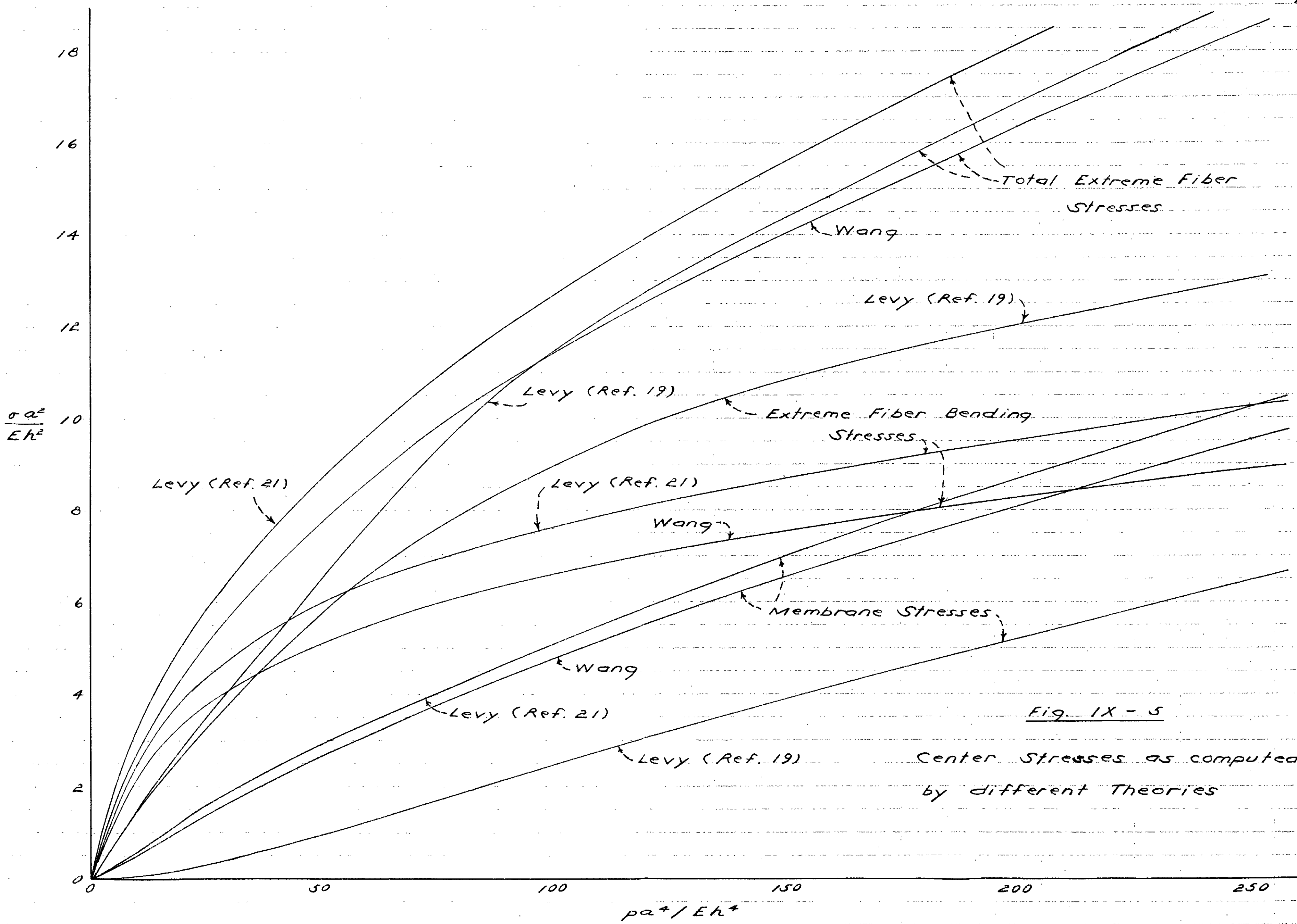


Fig. IX - 5

Center Stresses as computed by different Theories

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BIOGRAPHY

The author, Chi-Teh Wang, was born on May 6, 1918, in Hangchow, China. His father, then a senator from Kiangsu Province, after resigning from a position as a college professor, wished his son to be a statesman and gave him the name Chi-Teh, literally meaning 'developing morality' since he had painfully witnessed the political situation in China at that time to be turbulent and corrupt to a severe degree. At the age of five the author was put into study under a private tutor in Chinese classics and mathematics.

However, the author has, in a way, disappointed his father by showing that he always tends to be more interested in natural sciences than in politics and economics. He chose his career as an engineer and entered National Chiao-Tung University in the Department of Mechanical Engineering in 1936, where he graduated in 1940 with a degree of Bachelor of Science in Mechanical Engineering with an option in Aeronautical Engineering. While he was a sophomore, the Sino-Japanese war broke out and he was entirely cut off from his family. Upon the recommendations of several professors, he was awarded a University Fellowship from 1937 to 1940 which enabled him to complete his undergraduate study. He was one of the five students in the graduating classes who were elected to be members of the Phi Tau Phi Society, the only national honorary society for higher education in China.

After graduation, he joined the staff of the same university to be an assistant in aeronautical engineering. Due to the departure of a professor in aeronautical engineering, he was promoted to be an instructor teaching aerodynamics and propeller design during the academic year of 1940 to 1941.

Upon the recommendation of the faculty of Chiao-Tung University, he was awarded a fellowship from Rensselaer Polytechnic Institute in 1941, which enabled him to come to the United States and to do some graduate work. He received his degree of Master of Aeronautical Engineering in May, 1942, and was elected to be a member of the Sigma Xi Society, the national honorary society for scientific research, and he was fortunate enough to be offered fellowships and scholarships from six different universities.

He studied at Brown University in the Advanced Institute of Mechanics during the summers of 1942 and 1943 on fellowships from Tsing-Hwa University, and Brown University, and received a degree of Master of Science in Applied Mathematics in October, 1943.

He came to this Institute in September, 1942, on a scholarship award from the Institute and a fellowship from Tsing-Hua University during the year 1942 to 1943. From 1943 to 1944 he was awarded a Saltonstall Fellowship and at the same time he worked as a part-time assistant to Professor J. S. Newell.

He has one paper to be published in the October issue of Journal of Aeronautical Sciences, 1944, entitled "Contracting Cones Giving Uniform Throat Speeds" which he wrote jointly with Professor R. H. Smith of M.I.T.