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THE EFFECTS OF SMALL NOISE ON IMPLICITLY DEFINED NON-LINEAR DYNAMICAL SYSTEMS

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SECTION 1. INTRODUCTION

The dynamics of a large class of non-linear systems are described implicitly, i.e. as a combination of algebraic and differential equations. These dynamics admit of jump behavior. We extend the deterministic theory to a stochastic theory since (i) the deterministic theory is restrictive, (ii) the macroscopic deterministic description of dynamics frequently arises from an aggregation of microscopically fluctuating dynamics and (iii) to robustify the deterministic theory. We compare the stochastic theory with the deterministic one in the limit that the intensity of the additive white noise tends to zero. We study the modelling issues involved in applying this stochastic theory to the study of the noise behavior of a multivibrator circuit, discuss the limitations of our methodology for certain classes of systems and present a modified approach for the analysis of sample functions of noisy non-linear circuits. The details are in [17].

SECTION 2. DETERMINISTIC CONSTRAINED DYNAMICAL SYSTEMS AND THEIR JUMP BEHAVIOR

The dynamics of a large class of engineering systems, for example, dynamics of non-linear circuits, swing-dynamics of an interconnected power system are not specified explicitly, but rather in the following constrained or implicit form:

$$\dot{x} = f(x, y) \quad (2.1)$$

$$0 = g(x, y) \quad (2.2)$$

Here $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, f and g are smooth maps from $\mathbb{R}^n \times \mathbb{R}^m$ to \mathbb{R}^n and \mathbb{R}^m respectively. The equations (2.1), (2.2) need to be interpreted. Assume that 0 is a regular value of g . We then interpret (2.1), (2.2) as describing implicitly a dynamical system on the n -dimensional manifold

$$M = \{(x, y) | g(x, y) = 0\}$$

The difficulty encountered in interpreting (2.1), (2.2) is as follows: Equation (2.2) is an algebraic equation which requires to be solved for y locally as a smooth function of x -- this is then substituted in equation (2.1) to yield the x

trajectory. At points, where $D_2g(x, y)$ is singular; the implicit function theorem can no longer be invoked to solve equation (2.2) (locally) for y as a smooth function of x .

In several practical problems, the algebraic equation (2.2) is in fact the degenerate limit as $\epsilon \rightarrow 0$ of

$$\epsilon \dot{y} = g(x, y) \quad (2.3)$$

where $\epsilon > 0$ is a small parameter representing parasitics neglected in the course of modelling. Trajectories of (2.1), (2.2) are then interpreted to be the limit as $\epsilon \rightarrow 0$ of those of the well defined system (2.1), (2.3) (provided the limits exist).

A simple example of the application of this procedure is to the degenerate van der Pol oscillator described by

$$\dot{x} = y \quad (2.4)$$

$$0 = -x - y^3 + y \quad (2.5)$$

Figure 1 shows the phase portrait of the system (2.4), (2.5) with the zero in equation (2.6) replaced by $\epsilon \dot{y}$. In the limit that $\epsilon \rightarrow 0$ the closed orbit of the non-degenerate system tends to a "relaxation" oscillation including two jumps as shown in the figure.

The intuition for studying the augmented system is obtained by rescaling (expanding) time t to $\tau = t/\epsilon$ in equations (2.1), (2.3) and taking the formal limit that $\epsilon \rightarrow 0$

$$\left. \begin{aligned} \frac{dx}{d\tau} &= 0 \\ \frac{dy}{d\tau} &= g(x, y) \end{aligned} \right\} S \quad (2.6)$$

$$(2.7)$$

The configuration manifold M of the system (2.1), (2.2) is the set of equilibria of the "sped-up" system S . Further, the set of stable-equilibria of this system S are the portions of the configuration manifold M which are attracting to the parasitic dynamics. Stability of the equilibrium of S in which x is "frozen" is of course determined by the eigenvalues of the linearization, i.e. the eigenvalues of $D_2g(x, y)$. Further, when the equilibria are hyperbolic (no eigenvalues of $D_2g(x, y)$ on the

($j\omega$ -axis) the domains of attraction of these equilibria are their stable invariant manifolds.

Intuition, now, suggests that at the non-hyperbolic points including points of singularity of (2.1), (2.2), the state (x,y) finds itself at the boundary of the domain of attraction of a new equilibrium point (x,\tilde{y}) of S and executes a jump to it. If this new point is unstable, another jump is possible till a stable equilibrium point (x,y^*) of S is reached.

A great deal of the foregoing intuition can be made precise as shown in [11], and may be used to propose a solution concept for the system (2.1), (2.2) allowing for jumps from unstable hyperbolic equilibrium points of S and from singular equilibrium points of S .

However, the deterministic analysis of constrained systems is delicate and requires numerous assumptions on the sped-up system S . Further, the deterministic analysis does not yield a unique solution concept. It would appear then that a probabilistic analysis yielding probabilities of jump, and probabilities of the state lying at certain equilibria of the sped-up system rather than S would yield a more satisfactory solution concept to our constrained system.

SECTION 3. NOISY CONSTRAINED DYNAMICAL SYSTEMS

In the context of several applications, the presence of random fluctuations (which are of very high bandwidth - almost white), prompts us to write a more accurate model than that of the previous section, of the form

$$\dot{x} = f(x,y) + \sqrt{\epsilon} \xi(t) \quad (3.1)$$

$$\dot{y} = g(x,y) + \sqrt{\lambda} \eta(t) \quad (3.2)$$

Here, $\xi(\cdot)$ and $\eta(\cdot)$ are independent \mathbb{R}^n valued and \mathbb{R}^m valued white noise processes and λ, μ scale their variance. Equations (3.1), (3.2) differ from (2.1) and (2.4) in that they both contain additive, non-state dependent additive white noise terms. Note the $\sqrt{\epsilon}$ scaling the variance of the white noise in equation (3.1). This is introduced, so that the sped-up system S (with ϵ then set equal to zero) in the time scale $\tau = t/\epsilon$, is meaningful; namely

$$\frac{dx}{d\tau} = 0 \quad (3.3)$$

$$\frac{dy}{d\tau} = g(x,y) + \sqrt{\lambda} \eta(\tau) \quad (3.4)$$

For each $\epsilon, \lambda, \mu > 0$ the evolution of the probability density $p_{\mu,\epsilon}^\lambda(x,y)$ is governed by the forward-equation of Kolmogorov (or Fokker Planck equation)

$$\frac{\partial}{\partial \tau} p_{\mu,\epsilon}^\lambda = (L_0^* + \frac{1}{\epsilon} L_1^*) p_{\mu,\epsilon}^\lambda \quad (3.5)$$

where L_0^* and L_1^* are formal adjoints of the operators

L_0 and L_1 given by

$$L_0 p = \sum_{i=1}^n \left[\frac{\mu}{2} \frac{\partial^2}{\partial x_i^2} + f_i \frac{\partial}{\partial x_i} \right] p$$

and

$$L_1 p = \sum_{j=1}^m \left[\frac{\mu}{2} \frac{\partial^2}{\partial y_j^2} + g_j \frac{\partial}{\partial y_j} \right] p.$$

Proceeding, formally, from (3.5) we expect that in the limit that $\epsilon \rightarrow 0$ $p_{\mu,\epsilon}^\lambda$ should satisfy

$$L_1^* p_{\mu,0}^\lambda = 0 \quad (3.6)$$

It follows by inspection of equations (3.4) that any solution to this equation is (up to a multiplicative function of x), the invariant density of the diffusion of equation (3.4) of the sped-up system S with x frozen. Thus,

$$p_{\mu,0}^\lambda(x,y) = \tilde{p}_\mu^\lambda(x) \bar{p}^\lambda(x,y) \quad (3.7)$$

where $\bar{p}^\lambda(x,y)$ is the invariant density of the diffusion of (3.4) with x frozen (assumed to exist).

Note that $\bar{p}^\lambda(x,y)$ also has the interpretation of being the conditional density of y given x , in the limit that $\epsilon \rightarrow 0$. Now, use (3.7) in (3.5) and note that the operator L_0^* does not involve the y -variable. Integrate both sides of (3.5) with respect to the y -variable to obtain

$$\frac{\partial \tilde{p}_\mu^\lambda}{\partial \tau}(x) = L_0^* \tilde{p}_\mu^\lambda(x) \quad (3.8)$$

where

$$L_0^* p = \sum_{i=1}^n \left[\frac{\mu}{2} \frac{\partial^2}{\partial x_i^2} + (\bar{f}_\lambda)_i \frac{\partial}{\partial x_i} \right] p$$

with $\bar{f}_\lambda(x)$ being $f(x,y)$ averaged (integrated) over the invariant density of y given x , i.e.

$$\bar{f}_\lambda(x) = \int_{\mathbb{R}^m} f(x,y) \bar{p}^\lambda(x,y) dy \quad (3.9)$$

The foregoing manipulations suggest that the intuition for studying (3.1), (3.2) in the limit that $\epsilon \rightarrow 0$, is similar to that involved in the study of (2.1), (2.4) as $\epsilon \rightarrow 0$. Equilibria of S with x frozen are replaced by the invariant density of the y process of (3.4) given x ; and the drift in x instead of being $f(x,y)$ evaluated at a specific equilibrium of $g(x,y)$ is $f(x,y)$ averaged over the invariant density of y - given x .

Conditions for the existence of an invariant density of (3.4), given x are far less restrictive than the condition that $g(x,y)$ be gradient like (needed for the deterministic analysis) - see for example Bhattacharya [2] or Papanicolaou, et al. [10].

Theorem 3.1 [10] (Weak convergence for (3.1), (3.2)) Given any $T > 0$, the first component $t \rightarrow x(t)$ of the solution to (3.1), (3.2) converges weakly as $\varepsilon \rightarrow 0$ in $C([0, T]; \mathbb{R}^n)$ to the unique diffusion $t \rightarrow x_{\lambda, \mu}(t)$, governed by \bar{L}_0 .

Remarks: (1) Weak convergence of a sequence of diffusions is convergence of their measures on $C([0, T]; \mathbb{R}^n)$ in the weak topology induced by endowing $C([0, T]; \mathbb{R}^n)$ with the Skorokhod topology.

(2) While it is true that the y -process of (3.1), (3.2) has no weak-convergent limit; the conditional density of y given x in the limit that $\varepsilon \rightarrow 0$ is given by $\bar{p}^\lambda(x, y)$ of equation (3.7).

3.1 Constrained Dynamical Systems in the Presence of Small Driving Noise

Frequently the model (3.1), (3.2) of noisy constrained systems differs from the deterministic model (2.1), (2.3) in only a small way - i.e. the variances μ, λ scaling ξ and η in equations (3.1) and (3.2) respectively are small. In the circuit context, as we shall show in the next section, μ and λ are of order of kT (where k is the Boltzmann constant and T the temperature in degrees Kelvin), a quantity that is small at room temperatures. Thus, we compare the behavior of noisy constrained dynamical systems with that of the deterministic constrained systems of Section 2 in the limit that λ, μ the variances of the driving noise go to zero.

3.2 The Case of a Gradient Constraint Equation

Assume that $g(x, y)$ is the gradient with respect to y of a function $S(x, y)$, i.e.

$$g(x, y) = -\frac{1}{2} \text{grad}_y S(x, y)$$

for some function $S: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$. Then, the results of Papanicolaou, et al. [10] yield that provided the derivatives of S with respect to y grow rapidly enough at ∞ , the density of the diffusion generated by (3.4) converges exponentially to

$$\bar{p}^\lambda(x, y) = C^\lambda(x) \cdot \exp\left(-\frac{S(x, y)}{\lambda}\right) \quad (3.10)$$

where $C^\lambda(x)$ is chosen such that

$$\int_{\mathbb{R}^m} \bar{p}^\lambda(x, y) dy = 1$$

Note that for all $\lambda > 0$ and $x \in \mathbb{R}^n$ the critical points (with respect to y of $\bar{p}^\lambda(x, y)$) are the equilibrium points of the deterministic system (2.4) with x frozen given in this instance by

$$\dot{y} = -\frac{1}{2} \text{grad}_y S(x, y) \quad (3.11)$$

Further, if for some $x_0, S(x_0, y)$ is a Morse function

(of y), then for all $\lambda > 0$ every local maximum of $\bar{p}^\lambda(x, y)$ is a stable equilibrium of (3.11).

Theorem 3.2 (Laplace's Method)

Let for each $x \in \mathbb{R}^n$, $S(x, y)$ have global minima at $y_1^*(x), y_2^*(x), \dots, y_N^*(x)$, where N may depend on x . Let them all be non-degenerate.† Further, let $S(x, y)$ have at least quadratic growth (in y) as $y \rightarrow \infty$. Then, in the limit that $\lambda \rightarrow 0$, $\bar{p}^\lambda(x, y)$ converges to

$$\sum_{i=1}^N a_i(x) \delta(y - y_i^*) / \sum_{i=1}^N a_i(x)$$

$$\text{where } a_i(x) = \det(D_2^2 S(x, y_i^*(x)))^{-1/2} \quad (3.12)$$

More precisely, if $\phi(x, y)$ is a smooth function having polynomial growth as $y \rightarrow \infty$, then

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \bar{\phi}_\lambda(x) &= \lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^m} \phi(x, y) \bar{p}^\lambda(x, y) dy \\ &= \sum_{i=1}^N a_i(x) \phi(x, y_i^*(x)) / \sum_{i=1}^N a_i(x) \\ &= \bar{\phi}_0(x) \end{aligned} \quad (3.13)$$

Remark (1) If $S(x, y)$ has a manifold M of global minima, then clearly these global minima cannot be non-degenerate. However, if $D_2^2 S(x, y)$ is non-degenerate in directions orthogonal to M , then a minor modification of the preceding theorem yields

$$\begin{aligned} \int_{\mathbb{R}^m} \exp\left[-\frac{S(x, y)}{\lambda}\right] \phi(x, y) dy \\ = (2\pi\lambda)^{m/2} e^{-\frac{S(\hat{x}, \hat{y})}{\lambda}} \left(\int_M \frac{\phi(x, y)}{H(x, y)} \cdot dy \right)_M + o(1) \end{aligned}$$

where \hat{y} is any point belonging to M , $H^2(x, y)$ is the determinant of the non-degenerate part of the Hessian and $dy|_M$ is the canonical measure on M .

We can now combine the results of Theorems (3.1) and (3.2):

Theorem 3.3 (Weak convergence of (3.1), (3.2) as $\varepsilon, \lambda \rightarrow 0$)

Given any $T > 0$, in the limit $\varepsilon \rightarrow 0$ and $\lambda \rightarrow 0$ (in that order!), the first component $t \rightarrow x(t)$ of the solution to (3.1), (3.2) converges weakly in $C([0, T]; \mathbb{R}^n)$ to the unique diffusion $t \rightarrow x_\mu(t)$ satisfying in law

$$\dot{x} = \bar{F}_0(x) + \sqrt{\mu} \xi(t)$$

where

$$\bar{F}_0(x) = \sum_{i=1}^N a_i(x) f(x, y_i^*(x)) / \sum_{i=1}^N a_i(x)$$

and the $y_1^*(x), \dots, y_N^*(x)$ are the non-degenerate

† i.e. the Hessian $D_2^2 S(x, y)$ at $y = y_i^*(x)$ is non-singular

We illustrate this with an example - the van der Pol oscillator of (2.4), (2.5) with added noise. Consider

$$\dot{x} = y + \sqrt{\mu} \xi(t)$$

$$\epsilon \dot{y} = -x - y^3 + y + \sqrt{\lambda \epsilon} \eta(t)$$

Here $S(x,y) = -xy - \frac{y^4}{4} + \frac{y^2}{2}$ so that, in the limit that $\epsilon \rightarrow 0$; the x-process converges to one satisfying

$$\dot{x} = \bar{y}^\lambda(x) + \sqrt{\mu} \xi(t)$$

where

$$\bar{y}^\lambda(x) = \frac{\int_{-\infty}^{\infty} y \exp \frac{2}{\lambda} (-xy - \frac{y^4}{4} + \frac{y^2}{2}) dy}{\int_{-\infty}^{\infty} \exp \frac{2}{\lambda} (-xy - \frac{y^4}{4} + \frac{y^2}{2}) dy}$$

In the limit that $\lambda \rightarrow 0$, the conditional density $\bar{p}^\lambda(x,y)$ converges to the delta functions shown in Figure 2. Note that the jump in the conditional density is from one leg of the curve $x = y - y^3$ to the other at $x = 0$, as contrasted with the deterministic behavior shown in Figure 1. $\bar{y}^\lambda(x)$ is plotted for different values of λ in Figure 3: it mirrors this new jump behavior in y - the relaxation oscillation of Figure 1 no longer appears to exist.

SECTION 4. THE EFFECTS OF THERMAL NOISE ON AN EMITTER-COUPLED RELAXATION OSCILLATOR

We study in this section the relevance of the theory developed in Sections 2 and 3 to the study of the effects of thermal noise on a relaxation oscillator. The circuit equations are given by (with $V = V_{e2} - V_{e1}$)

$$\frac{d}{dt} V = \frac{1}{C} (I_0 - i) \quad (4.1)$$

$$0 = V - (2I_0 - 2i)R - V_T \ln \frac{2I_0 - i}{i} \quad (4.2)$$

Equations (4.1), (4.2) form an implicitly defined dynamical system. The solution curve to the algebraic equation (4.2) is plotted in the (V,i) plane in Figure 5. Some of the features of this curve are noted below:

- (i) For $-2I_0R < V < 2I_0R$ the equation (4.2) has three solutions, while for $V > 2I_0R$ and $V < -2I_0R$ the equation has only one solution.
- (ii) As $V \rightarrow \infty$, $i \rightarrow 2I_0$ and as $V \rightarrow -\infty$, $i \rightarrow 0$ asymptotically.
- (iii) The values $V \approx 2I_0R$; $i \approx \frac{V_T}{2R}$ and $V \approx -2I_0R$; $i \approx 2I_0 - \frac{V_T}{2R}$ are the points of bifurcation of equation (4.2) with V treated as the bifurcation parameter, i.e., at these points it is not possible to solve (4.2) for i as a function of V locally and uniquely. These points may be shown to be points of fold bifurcation.

The regularization of this system is accomplished by taking into account the fact that parasitic capacitances present in the transistors, as well as the finite slew rate of the operational amplifiers will prevent i from varying discontinuously and in effect change the description of the circuit dynamics from (4.1), (4.2) to

$$\frac{dV}{dt} = \frac{(I_0 - i)}{C} \quad (4.1)$$

$$\epsilon \frac{di}{dt} = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i \quad (4.3)$$

Equations (4.1) and (4.3) are a gross simplification of all the actual parasitics present in the circuit. The phase portrait of this system shown in Figure 1 includes a single unstable equilibrium point ($V = 0$, $i = I_0$) and a limit cycle. The limit trajectories of (4.1), (4.3) as $\epsilon \rightarrow 0$ exist and include the relaxation oscillation shown in Figure 5 - a limit cycle with two discontinuities - at the points where the trajectory switches from the Q1 on, Q2 off "state" to be Q1 off, Q3 on "state" and vice versa. In such applications, it is important to know the noise characteristics of the oscillator in response to restrictive thermal noise. Experimental observations of Abidi [1] indicate that the actual (noisy) current waveform is as shown in Figure 6. Key features of this figure are as follows:

- (a) the transitions or jumps appear to be noise free
- (b) the noise superimposed on the deterministic waveform of Figure 5 appears to be small (low intensity) immediately following a jump and then appear to build in intensity.

We assume (see e.g. [14]) that all the noise sources in the circuit can be lumped into a single-noisy current source $i_n(t)$ shown dotted in Figure 9: $i_n(t)$ is assumed to be white with intensity λ (with λ small at room temperatures, since it is proportional to kT). It is easy to check that the equation (4.1) is now unchanged, while (4.3) changes to

$$0 = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i + 2R\sqrt{\lambda} i_n(t) \quad (4.4)$$

We regularize the system (4.1), (4.4) as before to obtain

$$\dot{V} = (I_0 - i)/C$$

$$\epsilon \dot{i} = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i + 2R\sqrt{\epsilon \lambda} i_n(t) \quad (4.5)$$

Note that ϵ scales the intensity of the white noise in (4.5) precisely for the same reason as in equation (3.2) of Section 3. The techniques of Section 3.2 may now be used to obtain that as $\epsilon \rightarrow 0$, the V -process converges weakly on $C([0,T];\mathbb{R})$ to one satisfying:

$$\dot{V} = (I_0 - \bar{i}^\lambda(V))/C$$

where $\bar{i}^\lambda(V)$ is i integrated over the conditional density for i given V , in the limit that $\epsilon \rightarrow 0$, $\bar{p}^\lambda(i,v)$. As in the example of Section 3.2, we have in the

limit that $\lambda \rightarrow 0$, $\bar{p}^\lambda(i, v)$ converging to a sequence of delta functions jumping from one leg of the solution curve to (4.2) to the other at $V=0$. Also, choosing the interval of weak convergence to be large it appears that the relaxation oscillation is broken up.

This analysis is contrary to the experimental evidence of Abidi [1]. How does one recover the experimental results of Abidi [1]? The answers to these questions are contained in [17] and will be presented at the conference.

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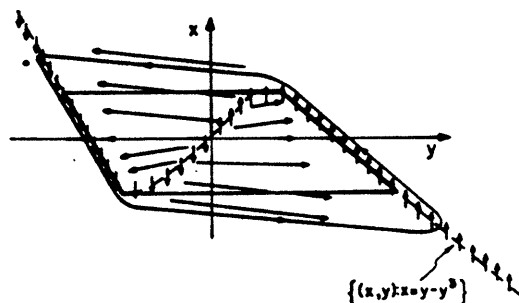


Fig. 1: Showing the dynamics of the Degenerate and Regularized van der Pol Oscillator.

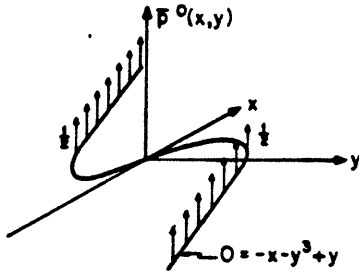


Fig. 2. Showing the Limit as $\lambda \rightarrow 0$ of the Conditional Density $\bar{p}^\lambda(x, y)$.

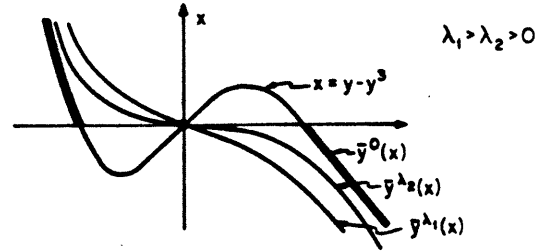


Fig. 3. The Drift $\bar{y}^\lambda(x)$ for the Limit Diffusion of van der Pol Oscillator for Decreasing Values of λ .

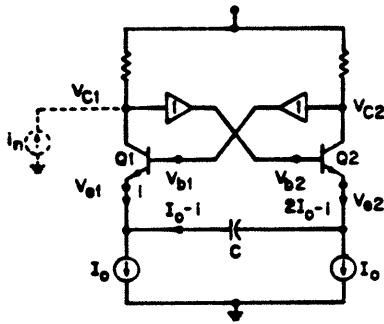


Fig. 4. Simplified Circuit Diagram for the Emitter Coupled Relaxation Oscillator.

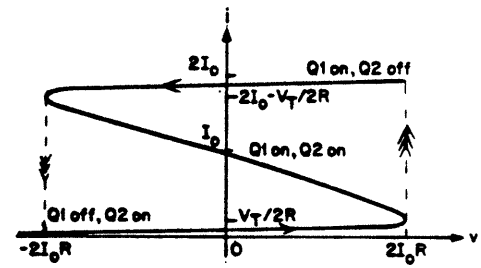


Fig. 5. The Solution Curve to the Algebraic Equation (4.7).

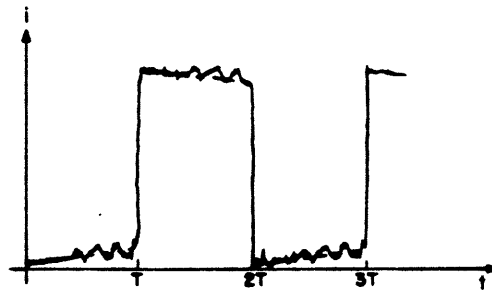


Fig. 6. Experimentally Observed Waveform for $i(t)$ in the Presence of Noise (after Abidi [1]).