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Modeling and Analysis of Production Lines With and Without Preventative Maintenance

by

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ABSTRACT

In this paper, two Erlang models of a two-machine, one-buffer production line are discussed. Both are extensions of the exponential production line model and treat random processing, failure, and repair times. In the first, worker intervention occurs only when a failure takes place; in the second maintenance occurs whenever a machine is idle due to starvation or blockage. Numerical results from the first model are indistinguishable from those of the exponential model; a substantial increase in throughput is observed in the second.

I. INTRODUCTION

This paper is concerned with a production line consisting of two machines and one buffer. The buffer has finite capacity, and the machines are unreliable. There has been much research work on two-machine, one-buffer transfer lines and many valuable results have been found.[1][2] In this earlier work, most of authors dealt with exponential service times, repair times and times between failure (t.b.f.). This kind of assumption is very typical because it simplifies analysis. A survey is presented in [1].

Some authors have studied the case in which service time is Erlangian with k phases [3][4][5]. The advantage of this assumption is that very large classes of distributions can be approximated very closely by Erlang distributions. Recently, Altiok [6] considered both processing time and repair time to have phase type distributions (of which the Erlang distribution is a special case) and failure time to be exponential, and performed a numerical analysis of the steady-state equations. This research has provided a very useful background to this paper.

In this paper, the authors consider that MBTF (mean time between failures) of a machine is longer than service time and repair time. This is a realistic situation, because usually a machine does not fail until it processes several workpieces, and the repair can be done in a way in which the whole machine or a part is removed and the replacement can be installed quickly. In this case the repair time and the service time may be assumed exponential, but the time between failures may not be. Consequently we assume the t.b.f. is an Erlangian random variable.

Two models are considered in this paper. The "regular" Erlang failure model is a straightforward extension of the exponential model: at random times (exponentially distributed) the phase of operational machine is advanced by 1. When it reaches a specified value, the machine is considered to have failed. An additional feature is incorporated into the "modified" model: whenever the machine is forced to be idle (due to starvation or blockage) its phase is reset to 1. This represents maintenance which does not interrupt production. Numerical experimentation indicates that the regular Erlang model yields results which are practically indistinguishable from the exponential model. The modified model produces substantially different results.

We describe the regular Erlang model and its assumptions in Section 2, then develop the detailed balance equations in Section 3. In Section 4 we describe measures of performance and in Section 5 we provide some theoretical results. Section 6 describes in detail the calculation of the steady-state probabilities and measures of performance. The modified model is treated in Section 7. The numerical examples appear in Section 8 and the conclusions in Section 9.

2. REGULAR ERLANG MODEL DESCRIPTION AND ASSUMPTIONS

The system consists of two machines that are separated by a finite storage buffer. (fig. 2.1). Workpieces enter machine 1 from outside. Each piece stays there to be operated on for a period of time and is then passed forward to the buffer. It is then transfered to machine 2 whenever it is available. After being operated on in machine 2, the piece leaves the system. It is assumed that a large reservoir of workpieces is available to machine 1 and that a large storage is also available to machine 2. That is, machine 1 is never starved and machine 2 is never blocked.

Figure 2.1

The possible machine states can be divided into 2 main catagories: operational and under repair. The number s_i is defined as the state of machine i (i=1,2). When $s_i=0$ machine i is under repair; $s_i \neq 0$ means machine i is operational. When machine i is operational, the time it operates until its next failure is a random variable with an Erlangian distribution.

Machine i has k_i phases. When it is operational, it is in a state s_i, where $s_i=1,2,\ldots, k_i$. Machine i can change its state from 1 to 2, from 2 to 3,..., from ℓ to $\ell+1$, until $s_i=k_i$. Then s_i can be changed to 0 from k_i. The time that machine i stays in state ℓ (when it is operating) is an exponential random variable whose parameter is the same for all ℓ ($\ell=1,2,\ldots,k_i$).

Assume that this parameter is $k_i p_i$ (i=1,2). We call $k_i p_i$ machine i's aging rate. The mean time that machine i stays in any state $s_i \neq 0$ is $\frac{1}{k_i p_i}$. Its mean operational time, i.e., the time $s_i \neq 0$, is $\frac{k_i}{k_i p_i} = \frac{1}{p_i}$. That is, $\frac{1}{p_i}$ is the mean time between failure (MTBF) of machine i. Here we see that in this model the MTBF of machine i is independent of k_i. The failure and repair behavior of a machine is shown in figure 2.2.

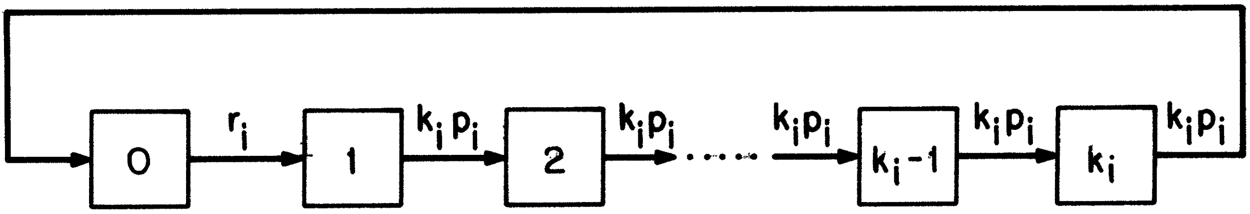


Figure 2.2

Even if a machine is operational (i.e., $s_i \neq 0$), it still cannot process any pieces if no piece is available or if there is no room to put processed pieces. In the former condition, the machine is said to be <u>starved</u>; in the latter it is <u>blocked</u>. Blocked or starved machines, because they are not operating, do not age. They stay in the same phase until the machine begins to work on a piece and then age in the usual aging rate.

It is assumed that the service time for an operational machine i (i=1,2) is an exponential random variable with parameter μ_i regardless of which phase it is in. It is also assumed that repair times are exponential random variables with parameters r_i (i=1,2). Parameters μ_i, r_i (i=1,2) are called processing rate and repair rate respectively.

When a machine is under repair, it remains in this state for a period of time which is exponentially distributed with mean r_i^{-1}. This is a machine's characteristic parameter and is unaffected by the states of the other machine and of the storage.

When a machine is up and not starved or blocked, three kinds of events can happen during the short time interval (t,t+dt): completion of processing on the current piece; machine aging (from phase ℓ to $\ell+1$ (mod k_i+1)); or the machine stays in the same phase and continues processing the piece. These events have probabilities approximately $\mu_i dt$, $k_i p_i dt$, and $1-(\mu_i dt+k_i p_i dt)$, respectively, for small dt.

The amount of material in storage is represented by the integer n, $0 \leq n \leq N$. This is the number of pieces in buffer plus the piece currently in machine 2. When machine 2 finishes the processing of the last piece and the buffer is empty, n=0.

There are thus nine parameters to be used to characterize a two-machine-one-buffer production line. μ_1, μ_2, p_1, p_2, k_1, k_2, r_1, r_2 and N.

A machine's operational time is almost always longer than the time under repair. If the repair is in a parts-change mode and the maintenance is regular, the time period under repair will be short. When the machines are used to process a large amount of standard part, the processing time also will be short. In this situation, the assumption that processing times and repair times are exponential and that operational times are Erlangian does make sense. According to this model, the state of the system can be denoted by

$$s = (n, s_1, s_2)$$

The probability that the system is in this state is written $p(n, s_1, s_2)$. The calculation of these probabilities and of measures of performance are described in the following sections.

3. THE DETAILED BALANCE EQUATIONS

In this section we list the balance equations of the system. The system's state is

$$s = (n, s_1, s_2)$$

with $n=0,1,\dots,N$; $s_1=0,1,\dots,k_1$; $s_2=0,1,\dots,k_2$. Whenever $n=0$, machine 2 is starved and cannot operate, and whenever $n=N$, machine 1 is blocked and cannot operate.

The state transitions can be represented graphically as in figure 3.1.

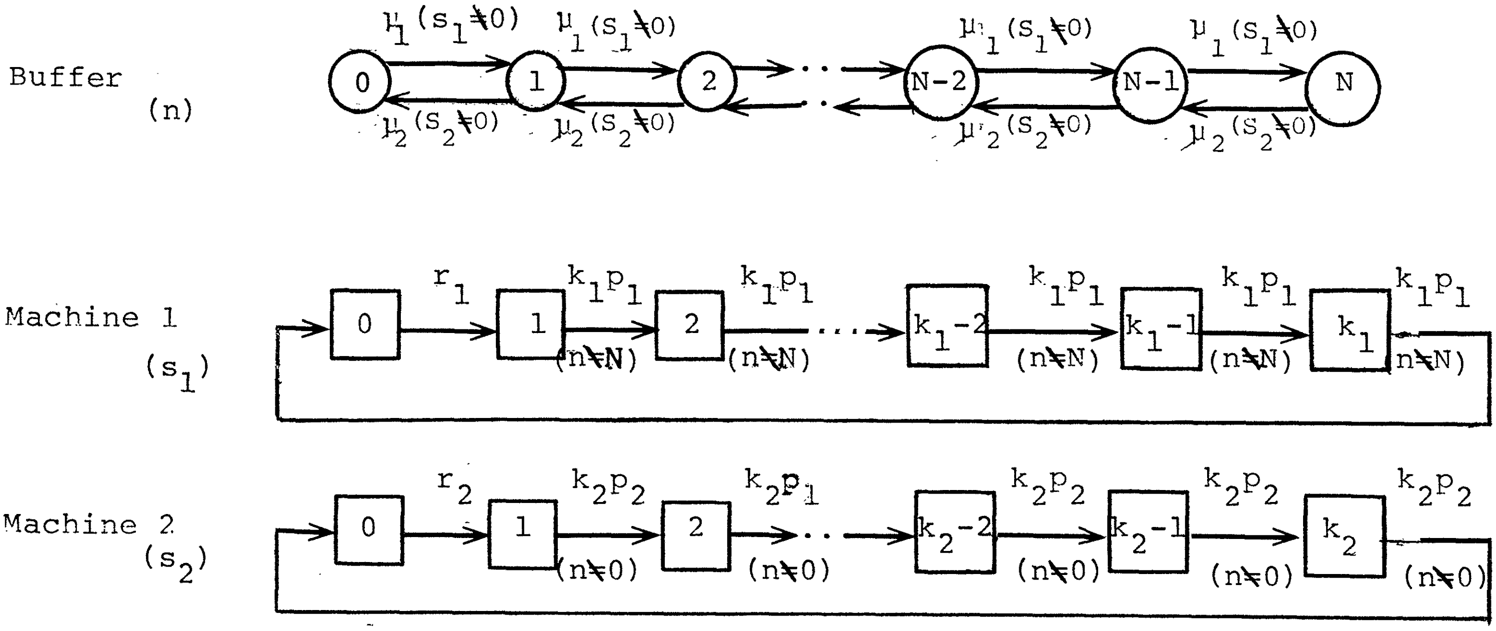


Figure 3.1

(n) state of buffer

$\boxed{s_i}$ state of machine i (i=1,2)

$\longrightarrow$ The rate and direction of state transition. Certain transitions are possible only when indicated conditions in parentheses are true.

This figure makes it easy to write all of the balance equations. When in balance, the transition rate from a state to all others must equal to the transition rate to this same state from all others. The balance equations can be divided into four sets.

For $s_1=0$, $s_2=0$, we have

$$p(n,0,0)(r_1+r_2) = p(n,k_1,0)\ k_1p_1+p(n,0,k_2)\ k_2p_2 \qquad (1\leq n\leq N-1) \tag{3.1}$$

$$p(0,0,0)(r_1+r_2) = p(0,k_1,0)\ k_1p_1 \tag{3.2}$$

$$p(N,0,0)(r_1+r_2) = p(N,0,k_2)\ k_2p_2 \tag{3.3}$$

The system leaves state $(n,0,0)$ only if the repair of one of the two machines has occurred. When one of the machines is under repair and the other is in its oldest operational phase (k_i), the state will change to $(n,0,0)$ according to the aging rate (k_ip_i). In a very short time period it is assumed that those events are exclusive of each other. Consequently, state $(n,0,0)$ can be reached in both ways--from $(n,k_1,0)$ or $(n,0,k_2)$.

The other three sets of balance equations can be explained in a similar way.

For $s_1\neq 0$, $s_2=0$

$$\begin{aligned} p(n,s_1,0)(\mu_1+k_1p_1+r_2) &= p(n-1,s_1,0)\,\mu_1+p(n,s_1-1,0)k_1p_1 \\ &\quad +p(n,s_1,k_2)k_2p_2 \qquad 1\leq n\leq N-1,\ 2\leq s_1\leq k_1 \end{aligned} \tag{3.4}$$

$$p(n,1,0)(\mu_1+k_1p_1+r_2) = p(n-1,\ 1,0)\mu_1+p(n,0,0)r_1+ p(n,1,k_2)k_2p_2 \quad 1\le n\le N-1 \tag{3.5}$$

$$p(0,s_1,0)(\mu_1+k_1p_1+r_2) = p(0,s_1-1,0)k_1p_1 \quad 2\le s_1\le k_1 \tag{3.6}$$

$$p(0,1,0)(\mu_1+k_1p_1+r_2) = p(0,0,0)r_1 \tag{3.7}$$

$$p(N,s_1,0)\ r_2 = p(N-1,s_1,0)\ \mu_1+p(N,s_1,k_2)\ k_2p_2 \quad 2\le s_1\le k_1 \tag{3.8}$$

$$p(N,1,0)\ r_2 = p(N-1,1,0)\ \mu_1+p(N,0,0)\ r_1+p(N,1,k_2)k_2p_2 \tag{3.9}$$

For $s_1=0$, $s_2\neq 0$

$$p(n,0,s_2)(\mu_2+r_1+k_2p_2) = p(n+1,0,s_2)\mu_2 +p(n,k_1,s_2)\ k_1p_1 +p(n,0,s_2-1)k_2p_2, \quad 1\le n\le N-1, \quad 2\le s_2\le k_2 \tag{3.10}$$

$$p(n,0,1)(\mu_2+r_1+k_2p_2) = p(n+1,0,1)\ \mu_2+p(n,k_1,1)k_1p_1 +p(n,0,0)r_2 \quad 1\le n\le N-1 \tag{3.11}$$

$$p(0,0,s_2)r_1=p(1,0,s_2)\mu_2+p(0,k_1,s_2)\ k_1p_1 \quad 2\le s_2\le k_2 \tag{3.12}$$

$$p(0,0,1)r_1 = p(1,0,1)\mu_2+p(0,k_1,1)\ k_{.}p_1+p(0,0,0)r_2 \tag{3.13}$$

$$p(N,0,s_2)(\mu_2+r_1+k_2p_2) = p(N,0,s_2-1)\ k_2p_2 \quad 2\le s_2\le k_2 \tag{3.14}$$

$$p(N,0,1)(\mu_2+r_1+k_2p_2) = p(N,0,0)r_2 \tag{3.15}$$

For $s_1 \neq 0,\ s_2 \neq 0$

$$p(n,s_1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2) = p(n-1,s_1,s_2)\mu_1+p(n+1,s_1,s_2)\mu_2$$
$$+p(n,s_1-1,s_2)k_1p_1+ p(n,s_1,s_2-1)k_2p_2$$
$$1\le n\le N-1,\ 2\le s_1\le k_1,\ 2\le s_2\le k_2 \tag{3.16}$$

$$p(n,1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2) = p(n-1,1,s_2)\mu_1+p(n+1,1,s_2)\mu_2$$
$$+p(n,0,s_2)r_1+p(n,1,s_2-1)k_2p_2$$
$$1\le n\le N-1 \qquad 2\le s_2\le k_1 \tag{3.17}$$

$$p(n,s_1,1)(\mu_1+u_2+k_1p_1+k_2p_2) = p(n-1,s_1,1)\mu_1+p(n+1,s_1,1)\mu_2$$
$$+p(n,s_1-1,1)k_1p_1+p(n,s_1,0)r_2$$
$$1\le n\le N-1 \qquad 2\le s_1\le k_1 \tag{3.18}$$

$$p(n,1,1)(\mu_1+\mu_2+k_1p_1+k_2p_2) = p(n-1,\ 1,1)\mu_1+p(n+1,1,1)\mu_2$$
$$+p(n,0,1)r_1+p(n,1,0)r_2$$
$$1\le n\le N-1 \tag{3.19}$$

$$p(0,s_1,s_2)(\mu_1+k_1p_1) = p(1,s_1,s_2)\mu_2+p(0,s_1-1,s_2)k_1p_1$$
$$2\le s_1\le k_1 \qquad 2\le s_2\le k_2 \tag{3.20}$$

$$p(0,1,s_2)(\mu_1+k_1p_1) = p(1,1,s_2)\mu_2+p(0,0,s_2)r_1$$
$$2\le s_2\le k_2 \tag{3.21}$$

$$p(0,s_1,1)(\mu_1+k_1p_1) = p(1,s_1,1)\mu_2 + p(0,s_1-1,1)k_1p_1 + p(0,s_1,0)r_2 \qquad 2\leq s_1\leq k_1 \tag{3.22}$$

$$p(0,1,1)(\mu_1+k_1p_1) = p(1,1,1)\mu_2 + p(0,0,1)r_1 + p(0,1,0)r_2 \tag{3.23}$$

$$p(N,s_1,s_2)(\mu_2+k_2p_2) = p(N-1,s_1,s_2)\mu_1 + p(N,s_1,s_2-1)k_2p_2 \qquad 2\leq s_1\leq k_1 \quad 2\leq s_2\leq k_2 \tag{3.24}$$

$$p(N,1,s_2)(\mu_2+k_2p_2) = p(N-1,1,s_2)\mu_1 + p(N,0,s_2)r_1 + p(N,1,s_2-1)k_2p_2 \qquad 2\leq s_2\leq k_2 \tag{3.25}$$

$$p(N,s_1,1)(\mu_2+k_2p_2) = p(N-1,s_1,1)\mu_1 + p(N,s_1,0)r_2 \qquad 2\leq s_1\leq k_1 \tag{3.26}$$

$$p(N,1,1)(\mu_2+k_2p_2) = p(N-1,1,1)\mu_1 + p(N,0,1)r_1 + p(N,1,0)r_2 \tag{3.27}$$

This is a set of linear difference equations. The total number of equations is $(N+1)(k_1+1)(k_2+1)$. This number is very large when N, k_1, k_2 are large. After a discussion of some characteristics of these equations in Section 4 and 5, a method of solving these equations is given in Section 6.

4. MEASURES OF PERFORMANCE

Here we consider three measures of performance that are often used as criteria to evaluate the performance of a production system. These are: efficiency E_i of the i-th machine in the system; production rate of the system; and expected in-process inventory.

The definition of E_i is the probability that the i-th machine is operating on a piece, or in other words, the fraction of time during which machine i processes pieces. It can be written as:

$$E_1 = \sum_{n=0}^{N-1} \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(n,s_1,s_2) \tag{4.1}$$

$$E_2 = \sum_{n=1}^{N} \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(n,s_1,s_2) \tag{4.2}$$

In next section it will be shown that

$$\mu_1 E_1 = \mu_2 E_2 \tag{4.3}$$

The quantity $\mu_i E_i$ is the rate that pieces are processed on machine i. The equation above represents a law of flow conservation. Consequently, the definition of production rate of the system is

$$P = \mu_i E_i \tag{4.4}$$

Another important measure of the system performance is the expected in-process inventory. It can be written

$$\bar{n} = \sum_{n=0}^{N} \sum_{s_1=0}^{k_1} \sum_{s_2=0}^{k_2} np(n,s_1,s_2) \qquad (4.5)$$

5. THEORETICAL RESULTS

In this section some theoretical results are presented. They help provide insight into this model and its physical meaning.

Lemma 1a

$$p(0,s_1,0) = 0 \qquad \text{for} \quad 0 \leq s_1 \leq k_1 \tag{5.1}$$

proof

From (3.6), $p(0,s_1,0) = p(0,s_1-1,0)\,\dfrac{k_1 p_1}{\mu_1+k_1p_1+r_2}$

$$2 \leq s_1 \leq k_1 \tag{5.2}$$

From (3.7), $p(0,1,0)=p(0,0,0)\,\dfrac{r_1}{\mu_1+k_1p_1+r_2}$ (5.3)

so that

$$p(0,s_1,0) = \left(\frac{k_1p_1}{\mu_1+k_1p_1+r_2}\right)^{s_1-1}\left(\frac{r_1}{\mu_1+k_1p_1+r_2}\right)p(0,0,0) \tag{5.4}$$

When $s_1=k_1$, we get

$$p(0,k_1,0) = \frac{(k_1p_1)^{k_1-1}\, r_1}{(\mu_1+k_1p_1+r_2)^{k_1}}\, p(0,0,0) \tag{5.5}$$

Substituting (5.5) into (3.2),

$$p(0,0,0)\left[\frac{r_1+r_2}{k_1p_1} - \frac{(k_1p_1)^{k_1-1}\, r_1}{(\mu_1+k_1p_1+r_2)^{k_1}}\right] = 0 \tag{5.6}$$

Since $(r_1+r_2)(\mu_1+k_1p_1+r_2)^{k_1} - (k_1p_1)^{k_1} r_1>0$, the value of expression in bracket is positive. Consequently,

$$p(0,0,0) = 0 \tag{5.7}$$

From (5.2), (5.3) and (5.7)

$$p(0,s_1,0) = 0 \qquad 0\le s_1\le k_1 \tag{5.8}$$

This lemma has an intuitive meaning: as long as the in-process inventory is empty (i.e., n=0), machine 2 cannot be working and therefore it can never fail.

Lemma 1b

$$p(N,0,s_2) = 0 \qquad \text{for } 0\le s_2\le k_2 \tag{5.9}$$

The proof of this lemma is similar to that of lemma 1a. The intuitive meaning of this lemma is worth pointing out: every time the in-process inventory is full (i.e., n=N), machine 1 is not working on any piece so it cannot fail.

Lemma 2a

$$r_1 \sum_{n=0}^{N-1} \sum_{s_2=0}^{k_2} p(n,0,s_2) = k_1p_1 \sum_{n=0}^{N-1} \sum_{s_2=0}^{k_2} p(n,k_1,s_2) \tag{5.10}$$

proof

If equations (3.1) and (3.10)-(3.13) are added, the sum of left hand sides is:

$$\sum_{n=1}^{N-1} p(n,0,0)(r_1+r_2)+\sum_{n=1}^{N-1}\sum_{s_2=1}^{k_2} p(n,0,s_2)(\mu_2+r_1+k_2p_2)$$

$$+\sum_{s_2=1}^{k_2} p(0,0,s_2)r_1 = r_1\sum_{n=0}^{N-1}\sum_{s_2=0}^{k_2} p(n,0,s_2)+$$

$$r_2\sum_{n=1}^{N-1} p(n,0,0)+\sum_{n=1}^{N-1}\sum_{s_2=1}^{k_2} p(n,0,s_2)(\mu_2+k_2p_2)$$

The sum of the right hand sides of these equations is:

$$\sum_{n=1}^{N-1} p(n,k_1,0)k_1p_1 + \sum_{n=1}^{N-1} p(n,0,k_2)k_2p_2$$

$$+\sum_{n=1}^{N-1}\sum_{s_2=1}^{k_2}\left[p(n+1,0,s_2)\mu_2+p(n,k_1,s_2)k_1p_1\right] + \sum_{n=1}^{N-1}\sum_{s_2=2}^{k_2}\left[p(n,0,s_2-1)k_2p_2\right]$$

$$+\sum_{n=1}^{N-1} p(n,0,0)r_2+\sum_{s_2=1}^{k_2}\left[p(1,0,s_2)\mu_2+p(0,k_2,s_2)k_1p_1\right] + p(0,0,0)r_2$$

When these two sides are put in an equation, it can be reduced to

$$r_1\sum_{n=0}^{N-1}\sum_{s_2=0}^{k_1} p(n,0,s_2) = k_1p_1\left[\sum_{n=0}^{N-1}\sum_{s_2=0}^{k_2} p(n,k_1s_2)\right] \qquad (5.11)$$

This lemma asserts that the rate of transition from states where machine 1 is under repair to states where it is operational equals to the transition rate from machine 1 operational to machine 1 under repair.

Lemma 2b

$$r_2 \sum_{n=1}^{N} \sum_{s_1=0}^{k_1} p(n,s_1,0)=k_2p_2\sum_{n=1}^{N} \sum_{s_1=0}^{k_1} p(n,s_1,k_2) \tag{5.12}$$

This lemma is a version of lemma 2a for machine 2. It can be proved in the same way as the proof of lemma 2a.

Lemma 3

$$\mu_1 \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(n,s_1,s_2)=\mu_2 \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(n+1,s_1,s_2) \quad 0\le n\le N-1 \tag{5.13}$$

proof

By induction, first, it can be proved for n=0. Adding (3.6), (3.7), (3.12), (3.13), (3.20) - (3.23), and recalling that $p(0,s_1,0)=0\,(0\le s_1\le k_1)$, the sum of the left hand sides is

$$\begin{aligned}
&\sum_{s_1=1}^{k_1} p(0,s_1,0)(\mu_1+k_1p_1+r_2)+\sum_{s_2=1}^{k_2} p(0,0,s_2)r_1 \\
&+\sum_{s_1=1}^{k_1} \sum_{s_2=1}^{k_2} p(0,s_1,s_2)(\mu_1+k_1p_1) = \mu_1\left[\sum_{s_1=1}^{k_1} p(0,s_1,0)\right. \\
&\left.+\sum_{s_1=1}^{k_1} \sum_{s_2=1}^{k_2} p(0,s_1,s_2)\right] + \sum_{s_1=1}^{k_1} p(0,s_1,0)(k_1p_1+r_2)+\sum_{s_2=1}^{k_2} p(0,0,s_2)r_1 \\
&+\sum_{s_1=1}^{k_1} \sum_{s_2=1}^{k_2} p(0,s_1,s_2)k_1p_1 = \mu_1\sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(0,s_1,s_2) \\
&+ \underline{k_1p_1} \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(0,s_1,s_2)+r_2\sum_{s_1=1}^{k_1} p(0,s_1,0)+r_1\sum_{s_2=1}^{k_2} p(0,0,s_2)
\end{aligned} \tag{5.14}$$

The sum of the right hand sides is

$$\begin{aligned}
&\sum_{s_1=2}^{k_1} p(0,s_1-1,0)k_1p_1+\sum_{s_2=1}^{k_2} p(1,0,s_2)\mu_2+\sum_{s_2=1}^{k_2} p(0,k_1,s_2)k_1p_1\\
&+\sum_{s_1=1}^{k_1}\sum_{s_2=1}^{k_2} p(1,s_1,s_2)\mu_2+\sum_{s_1=2}^{k_1}\sum_{s_2=1}^{k_2} p(0,s_1-1,s_2)k_1p_1\\
&+\sum_{s_2=1}^{k_2} p(0,0,s_2)r_1=\sum_{s_1=2}^{k_1}\sum_{s_2=0}^{k_2} p(0,s_1-1,s_2)k_1p_1+\sum_{s_2=1}^{k_2} p(0,k_1,s_2)k_1p_1\\
&+\sum_{s_2=1}^{k_2} p(0,0,s_2)r_1+\sum_{s_1=0}^{k_1}\sum_{s_2=1}^{k_2} p(1,s_1,s_2)\mu_2 \qquad (5.15)
\end{aligned}$$

The sum of these equations can be reduced to

$$\mu_1\sum_{s_1=1}^{k_1}\sum_{s_2=0}^{k_2} p(0,s_1,s_2)=\mu_2\sum_{s_1=0}^{k_2}\sum_{s_2=1}^{k_2} p(1,s_1,s_2) \qquad (5.16)$$

The second step in the induction method requires that we prove that if the formula holds for $n=m-1$ then the formula also holds for $n=m$ $(1\le m\le N-2)$.

For $n=m$ $(m\le N-2)$ add (3.1), (3.4), (3.5), (3.10), (3.11), (3.16)-(3.19). We get the sum of left hand side

$$p(m,0,0)(r_1+r_2)+\sum_{s_1=1}^{k_1}p(m,s_1,0)(\mu_1+k_1p_1+r_2)$$

$$+\sum_{s_2=1}^{k_2}p(m,0,s_2)(\mu_2+r_1+k_2p_2)+\sum_{s_1=1}^{k_1}\sum_{s_2=1}^{k_2}p(m,s_1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2 \tag{5.17}$$

The sum of the right hand sides is:

$$p(m,k_1,0)k_1p_1+p(m,0,k_2)k_2p_2+\sum_{s_1=1}^{k_1}p(m-1,s_1,0)\mu_1$$

$$+\sum_{s_1=2}^{k_1}p(m,s_1-1,0)k_1p_1+\sum_{s_1=1}^{k_1}p(m,s_1,k_2)k_2p_2+p(m,0,0)r_1$$

$$+\sum_{s_2=1}^{k_2}p(m+1,0,s_2)\mu_2+\sum_{s_2=1}^{k_2}p(m,k_1,s_2)k_1p_1+$$

$$+\sum_{s_2=2}^{k_2}p(m,0,s_2-1)k_2p_2+p(m,0,0)r_2+\sum_{s_1=1}^{k_1}\sum_{s_2=1}^{k_2}p(m-1,s_1,s_2)\mu_1$$

$$+\sum_{s_1=1}^{k_1}\sum_{s_2=1}^{k_2}p(m+1,s_1,s_2)\mu_2+\sum_{s_1=2}^{k_1}\sum_{s_2=1}^{k_2}p(m,s_1-1,s_2)k_1p_1$$

$$+\sum_{s_1=1}^{k_1}\sum_{s_2=2}^{k_2}p(m,s_1,s_2-1)k_2p_2+\sum_{s_2=1}^{k_2}p(m,0,s_2)r_1+\sum_{s_1=1}^{k_1}p(m,s_1,0)r_2 \tag{5.18}$$

The resulting equations can be reduced to

$$\mu_1 \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(m,s_1,s_2) + \mu_2 \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(m,s_1,s_2)$$

$$= \mu_1 \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(m-1,s_1,s_2) + \mu_2 \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(m+1,s_1,s_2) \qquad (5.19)$$

It is assumed that

$$\mu_1 \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(m-1,s_1,s_2) = \mu_2 \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(m,s_1,s_2) \qquad (5.20)$$

Substituting (5.20) into (5.19), we get

$$\mu_1 \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(m,s_1,s_2) = \mu_2 \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(m+1,s_1,s_2) \qquad (5.21)$$

For n=N-1, add all balance equations with n=N. That is add (3.8), (3.9), (3.24)-(3.27). In a similar way we get

$$\mu_2 \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(N,s_1,s_2) = \mu_1 \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(N-1,s_1,s_2) \qquad (5.22)$$

The proof of the formula (5.13) is complete. This lemma asserts that the rate of transition from the set of states with n pieces in the storage and machine 1 operational to the set of states with n+1 pieces in the storage and machine 2 operational equals the rate of transition in opposite direction.

Lemma 4 $\mu_1 E_1 = \mu_2 E_2$ (5.23)

In Section 4, E_1, E_2 are defined as the efficiences of machines 1 and 2 in the transfer production line. Their formulas are (4.1) and (4.2)

Adding lemma 3 from n=0 to n=N-1, we get

$$\mu_1 \sum_{n=0}^{N-1} \sum_{s_1=1}^{k_1} \sum_{s_2=0}^{k_2} p(n,s_1,s_2) = \mu_2 \sum_{n=0}^{N-1} \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(n+1,s_1,s_2)$$

$$= \mu_2 \sum_{n=1}^{N} \sum_{s_1=0}^{k_1} \sum_{s_2=1}^{k_2} p(n,s_1,s_2) \quad (5.24)$$

or

$$\mu_1 E_1 = \mu_2 E_2 \quad (5.25)$$

Lemma 5
$$\left.\begin{aligned} P &= \rho_1 \text{prob}(n \neq N) \\ P &= \rho_2 \text{prob}(n \neq 0) \end{aligned}\right\} \quad (5.26)$$

where

$$\rho_i = \mu_i e_i, \quad (5.27)$$

$$e_i = \frac{r_i}{r_i + p_i} \quad (5.28)$$

The quantities ρ_i and e_i are the isolated production rate and isolated efficiency of machine i, respectively. Lemma 5 can be established in exactly the same manner as the corresponding result in [1]. From this the limiting case results of [1] follow.

6. ANALYSIS OF STEADY-STATE PROBABILITIES

In this section, we analyze the internal balance equations and boundary balance equations. We provide an algorithm to calculate the steady-state probabilities and all the measures of performance of section 4.

6.1 Internal Balance Equation Analysis

We define internal states (n,s_1,s_2) as states with $1\leq n\leq N-1$ and s_1,s_2 taking any possible value $(0,1,2,\ldots,k_1$ and $0,1,2,\ldots k_2$ respectively). We guess that the solution to the internal equations of section 3 has following form:

$$p(n,s_1,s_2) = CX^n Y_{11}^{\gamma_1} Y_{12}^{\beta_1} Y_{21}^{\gamma_2} Y_{22}^{\beta_2} \quad \text{for} \quad 1\leq n\leq N-1 \qquad (6.1)$$

Where

$$\beta_i = \begin{cases} 0 & \text{for } s_i = 0 \\ 1 & \text{for } s_i \geq 1 \end{cases}$$

$$\gamma_i = \begin{cases} 0 & \text{for } s_i = 0 \\ s_i-1 & \text{for } s_i \geq 1 \end{cases}$$

In particular, if $s_1,s_2 >0$,

$$\begin{aligned} p(n,0,0) &= CX^n \\ p(n,s_1,0) &= CX^n Y_{11}^{s_1-1} Y_{12} \\ p(n,0,s_2) &= CX^n Y_{21}^{s_2-1} Y_{22} \\ p(n,s_1,s_2) &= CX^n Y_{11}^{s_1-1} Y_{12} Y_{21}^{s_2-1} Y_{22} \end{aligned} \qquad (6.2)$$

Substituting these expressions into (3.1), (3.4), (3.5), (3.10), (3.11), and (3.16)-(3.19), we get

$$cx^n(r_1+r_2) = cx^n y_{11}^{k_1-1} y_{12}k_1p_1 + cx^n y_{21}^{k_2-1} y_{22}k_2p_2$$

$(1 \leq n \leq N-1)$ (6.3)

$$cx^n y_{11}^{s_1-1} y_{12}(\mu_1+k_1p_1+r_2) = cx^{n-1}y_{11}^{s_1-1} y_{12}\mu_1+cx^n y_{11}^{s_1-2} y_{12}k_1p_1$$
$$+ \; cx^n y_{11}^{s_1-1} y_{12}y_{21}^{k_2-1} y_{22}k_2p_2 \quad (1 \leq n \leq N-1,\; 2 \leq s_1 \leq k_1)$$

(6.4)

$$cx^n y_{12}(\mu_1+k_1p_1+r_2) = cx^{n-1}y_{12}\mu_1 + cx^n r_1+cx^n y_{12}y_{21}^{k_2-1} y_{22}k_2p_2$$

$(1 \leq n \leq N-1)$ (6.5)

$$cx^n y_{21}^{s_2-1} y_{22}(\mu_2+r_1+k_2p_2) = cx^{n+1}y_{21}^{s_2-1} y_{22}\mu_2$$
$$+ \; cx^n y_{11}^{k_1-1} y_{12}y_{21}^{s_2-1} y_{22}k_1p_1 + cx^n y_{21}^{s_2-2} y_{22}k_2p_2$$

$(1 \leq n \leq N-1,\; 2 \leq s_2 \leq k_2)$

(6.6)

$$cx^n y_{22}(\mu_2+r_1+k_1p_1) = cx^{n+1}y_{22}\mu_2+cx^n y_{11}^{k_1-1} y_{12}y_{22}k_1p_1+cx^n r_2$$

$(1 \leq n \leq N-1)$ (6.7)

$$cx^n y_{11}^{s_1-1} y_{12}y_{21}^{s_2-1} y_{22}(\mu_1+\mu_2+k_1p_1+k_2p_2)$$
$$= cx^{n-1}y_{11}^{s_1-1} y_{12}y_{21}^{s_2-1} y_{22}\mu_1+cx^{n+1}y_{11}^{s_1-1} y_{12}y_{21}^{s_2-1} y_{22}\mu_2$$
$$+ \; cx^n y_{11}^{s_1-2} y_{12}y_{21}^{s_2-1} y_{22}k_1p_1 + cx^n y_{11}^{s_1-1} y_{12}y_{21}^{s_2-2} y_{22}k_2p_2$$

$(1 \leq n \leq N-1,\; 2 \leq s_1 \leq k_1,\; 2 \leq s_2 \leq k_2)$

(6.8)

Since (6.8) is a linear combination of other equations it is sufficient to analyze (6.3)-(6.7). These five equations can be simplified as follows:

$$r_1+r_2 = Y_{11}^{k_1-1} Y_{12}k_1p_1 + Y_{21}^{k_2-1} Y_{22}k_2p_2 \tag{6.9}$$

$$\mu_1+k_1p_1+r_2 = X^{-1}\mu_1+Y_{11}^{-1}k_1p_1+Y_{21}^{k_2-1} Y_{22}k_2p_2 \tag{6.10}$$

$$\mu_1+k_1p_1+r_2 = X^{-1}\mu_1+Y_{12}^{-1}r_1+Y_{21}^{k_2-1} Y_{22}k_2p_2 \tag{6.11}$$

$$\mu_2+r_1+k_2p_2 = X\mu_2+Y_{11}^{k_1-1} Y_{12}k_1p_1+Y_{21}^{-1}k_2p_2 \tag{6.12}$$

$$\mu_2+r_1+k_2p_2 = X\mu_2+Y_{11}^{k_1-1} Y_{12}k_1p_1+Y_{22}^{-1}r_2 \tag{6.13}$$

From (6.10) and (6.11)

$$Y_{11}r_1 = Y_{12}k_1p_1 \tag{6.14}$$

From (6.12) and (6.13)

$$Y_{21}r_2 = Y_{22}k_2p_2 \tag{6.15}$$

Substituting (6.14), (6.15) into (6.9)

$$r_1+r_2 = Y_{11}^{k_1}r_1+Y_{21}^{k_2}r_2 \tag{6.16}$$

Multiplying (6.10) by XY_{11}

$$XY_{11}(\mu_1+k_1p_1+r_2) = Y_{11}\mu_1+Xk_1p_1 + XY_{11}Y_{21}^{k_2-1} Y_{22}k_2p_2 \tag{6.17}$$

Multiplying (6.12) by Y_{21}

$$Y_{21}(\mu_2+r_1+k_2p_2) = Y_{21}X\mu_2+Y_{21}Y_{11}^{k_1-1}Y_{12}k_1p_1 + k_2p_2 \tag{6.18}$$

Equations (6.14)-(6.18) will be used to find the solution for the five unknowns X, Y_{11}, Y_{12}, Y_{21}, Y_{22}. Substitute (6.14) to (6.17), (6.15) to (6.18), we get:

$$XY_{11}(\mu_1+k_1p_1+r_2) = Y_{11}\mu_1+Xk_1p_1+XY_{11}Y_{21}^{k_2}r_2 \tag{6.19}$$

$$X^{-1}Y_{21}(\mu_2+r_1+k_2p_2) = Y_{21}\mu_2+X^{-1}Y_{21}Y_{11}^{k_1}r_1+X^{-1}k_2p_2 \tag{6.20}$$

It is enough to solve (6.16), (6.19), and (6.20) to get X, Y_{11} and Y_{21}. Then Y_{12} and Y_{22} can be calculated from (6.14) and (6.15) easily.

Substituting (6.16) into (6.19) and (6.20), and rearranging the terms

$$\begin{cases} Y_{11}^{k_1+1}r_1+Y_{11}(\mu_1+k_1p_1-r_1) - Y_{11}X^{-1}\mu_1-k_1p_1=0 & (6.21) \\ Y_{21}^{k_2+1}r_2+Y_{21}(\mu_2+k_2p_2-r_2) - Y_{21}X\mu_2-k_2p_2=0 & (6.22) \end{cases}$$

From (6.21)

$$X = \frac{Y_{11}\mu_1}{Y_{11}^{k_1+1}r_1+Y_{11}(\mu_1+k_1p_1-r_1) - k_1p_1} \tag{6.23}$$

From (6.16)

$$Y_{21} = \left(\frac{r_1+r_2 - Y_{11}^{k_1}r_1}{r_2}\right)^{1/k_2} \tag{6.24}$$

From (6.22)

$$\left(Y_{21}^{k_2} r_2 + \mu_2 + k_2 p_2 - r_2 - X \mu_2\right)^{k_2} Y_{21}^{k_2} = (k_2 p_2)^{k_2} \tag{6.25}$$

Substituting (6.23), (6.24), into (6.25)

$$\begin{aligned} &\left[(r_1 + \mu_2 + k_2 p_2 - Y_{11}^{k_1} r_1)\,(Y_{11}^{k_1+1} r_1 + Y_{11}(\mu_1 + k_1 p_1 - r_1) - k_1 p_1) \right. \\ &\left. - Y_{11} \mu_1 \mu_2\right]^{k_2} (r_1 + r_2 - Y_{11}^{k_1} r_1) \\ &= (k_2 p_2)^{k_2} \left[Y_{11}^{k_1+1} r_1 + Y_{11}(\mu_1 + k_1 p_1 - r_1) - k_1 p_1\right]^{k_2} r_2 \end{aligned} \tag{6.26}$$

Let

$$\begin{aligned} a &= r_1 \\ b &= r_1 + \mu_2 + k_2 p_2 \\ c &= \mu_1 + k_1 p_1 - r_1 \\ d &= -k_1 p_1 \\ e &= \mu_1 \mu_2 \\ f &= r_1 + r_2 \\ g &= (k_2 p_2)^{k_2} r_2 \end{aligned} \tag{6.27}$$

Then (6.26) becomes

$$\begin{aligned} &\left[(b - aY_{11}^{k_1})(aY_{11}^{k_1+1} + cY_{11} + d) - eY_{11}\right]^{k_2} (f - aY_{11}^{k_1}) \\ &\quad - g(aY_{11}^{k_1+1} + cY_{11} + d)^{k_2} = 0 \end{aligned} \tag{6.28}$$

The left-hand side of (6.28) is a $(2k_1k_2+k_1+k_2)^{th}$ order polynomial, so (6.28) has $\ell=2k_1k_2+k_1+k_2$ roots. Let these roots be denoted Y_{11i} $(i=1,2,\ldots,2k_1k_2+k_1+k_2)$.

Then

$$X_i = \frac{\mu_1 Y_{11i}}{aY_{11i}^{k_1+1} + eY_{11i} + d} \qquad 6.29)$$

$$Y_{21i} = \frac{k_2 p_2}{b - \mu_2 X_i - aY_{11i}^{k_1}} \qquad (6.30)$$

$$Y_{12i} = \frac{r_1}{k_1 p_1} Y_{11i} \qquad (6.31)$$

$$Y_{22i} = \frac{r_2}{k_2 p_2} Y_{21i} \qquad (6.32)$$

The linear combination $\sum_{i=1}^{\ell} C_i X_i^n Y_{11}^{\gamma_1} Y_{12i}^{\beta_1} Y_{21i}^{\gamma_2} Y_{22i}^{\beta_2}$ is a solution of the internal equations of section 3. In section 6.2 we use the boundary conditions to determine the C_i's.

6.2 Analysis of Boundary Equations

The boundary states are defined as those states with n=0 or N. Boundary equations are all those balance equations containing boundary states. From (3.1)=(3.7), the boundary equations are:

$$p(N,s_1,0)r_2 = p(N-1,s_1,0)\mu_1 + p(N,s_1,k_2)k_2p_2 \qquad 1 \le s_1 \le k_1 \qquad (6.33)$$

$$p(0,0,s_2)r_1 = p(1,0,s_2)\mu_2 + p(0,k_1,s_2)k_1p_1 \qquad 1 \le s_2 \le k_2 \qquad (6.34)$$

$$p(0,s_1,s_2)(\mu_1 + k_1p_1) = p(1,s_1,s_2)\mu_2 + p(0,s_1-1,s_2)k_1p_1 \qquad 2 \le s_1 \le k_1 \quad 1 \le s_2 \le k_2 \qquad (6.35)$$

$$p(0,1,s_2)(\mu_1+k_1p_1)=p(1,1,s_2)\mu_2+p(0,0,s_2)r_1 \qquad 1\le s_2\le k_2 \tag{6.36}$$

$$\begin{gathered}p(N,s_1,s_2)(\mu_2+k_2p_2)=p(N-1,s_1,s_2)\mu_1+p(N,s_1,s_2-1)k_2p_2 \\ 1\le s_1\le k_1 \quad 2\le s_1\le k_2\end{gathered} \tag{6.37}$$

$$p(N,s_1,1)(\mu_2+k_2p_2)=p(N-1,s_1,1)\mu_1+p(N,s_1,0)r_2 \qquad 1\le s_1\le k_1 \tag{6.38}$$

$$\begin{gathered}p(1,s_1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(0,s_1,s_2)\mu_1 + p(2,s_1,s_2)\mu_2 \\ +p(1,s_1-1,s_2)k_1p_1 +p(1,s_1,s_2-1)k_2p_2 \quad 2\le s_1\le k_1 \quad 2\le s_2\le k_2\end{gathered} \tag{6.39}$$

$$\begin{gathered}p(1,1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(0,1,s_2)\mu_1+p(2,1,s_2)\mu_2 \\ +p(1,0,s_2)r_1+p(1,1,s_2-1)k_2p_2 \qquad 2\le s_2\le k_2\end{gathered} \tag{6.40}$$

$$\begin{gathered}p(1,s_1,1)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(0,s_1,1)\mu_1+p(2,s_1,1)\mu_2 \\ +p(1,s_1-1,1)k_1p_1+p(1,s_1,0)r_2 \qquad 2\le s_1\le k_1\end{gathered} \tag{6.41}$$

$$\begin{gathered}p(1,1,1)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(0,1,1)\mu_1+p(2,1,1)\mu_2 \\ +p(1,0,1)r_1+p(1,1,0)r_2\end{gathered} \tag{6.42}$$

$$\begin{gathered}p(N-1,s_1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(N-2,s_1,s_2)\mu_1+p(N,s_1,s_2)\mu_2 \\ +p(N-1,s_1-1,s_2)k_1p_1+p(N-1,s_1,s_2-1)k_2p_2 \\ 2\le s_1\le k_1 \quad 2\le s_2\le k_2\end{gathered} \tag{6.43}$$

$$\begin{gathered}p(N-1,1,s_2)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(N-2,1,s_2)\mu_1+p(N,1,s_2)\mu_2 \\ +p(N-1,0,s_2)r_1+p(N-1,1,s_2-1)k_2p_2 \qquad 2\le s_2\le k_2\end{gathered} \tag{6.44}$$

$$p(N-1,s_1,1)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(N-2,s_1,1)\mu_1+p(N,s_1,1)\mu_2$$
$$+p(N-1,s_1-1,1)k_1p_1+p(N-1,s_1,0)r_2 \qquad 2\le s_1\le k_1 \qquad (6.45)$$

$$p(N-1,1,1)(\mu_1+\mu_2+k_1p_1+k_2p_2)=p(N-2,1,1)\mu_1+p(n,1,1)\mu_2$$
$$+p(N-1,0,1)r_1+p(N-1,1,0)r_2 \qquad (6.46)$$

$$p(1,s_1,0)(\mu_1+k_1p_1+r_2)=p(1,s_1-1,0)k_1p_1+p(1,s_1,k_2)k_2p_2$$
$$2\le s_1\le k_1 \qquad (6.47)$$

$$p(1,1,0)(\mu_1+k_1p_1+r_2)=p(1,0,0)r_1+p(1,1,k_2)k_2p_2 \qquad (6.48)$$

$$p(N-1,0,s_2)(\mu_2+r_1+k_2p_2)=p(N-1,k_1,s_2)k_1p_1+p(N-1,0,s_2-1)k_2p_2$$
$$2\le s_2\le k_2 \qquad (6.49)$$

$$p(N-1,0,1)(\mu_2+r_1+k_2p_2)=p(N-1,k_1,1)k_1p_1+p(N-1,0,0)r_2 \qquad (6.50)$$

Lemma 6

All probabilities $p(0,s_1,s_2)$ and $p(N,s_1,s_2)$ (for $1\le s_1\le k_1$, $1\le s_2\le k_2$) are in internal form.

Proof

As indicated previously, the probability $p(n,s_1,s_2)$ can be written as a combination of $\ell=2k_1k_2+k_1+k_2$ terms. That is

$$p(n,s_1,s_2)=\sum_{i=1}^{\ell} c_i\xi_i(n,s_1,s_2) \qquad (6.51)$$

where, for internal states,

$$\xi_i(n,s_1,s_2)=X_i^{n}\,Y_{11i}^{\gamma_1}\,Y_{12i}^{\beta_1}\,Y_{21i}^{\gamma_2}\,Y_{22i}^{\beta_2} \qquad (i=1,\dots,\ell) \tag{6.52}$$

From (6.39) after we get $\xi_i(1,s_1,s_2)$ $\xi_i(2,s_1,s_2)$, $\xi_i(1,s_1-1,s_2)$ and $\xi_i(1,s_1,s_2-1)$ (for $2\le s_1\le k_1$, $2\le s_2\le k_2$, $i=1,\dots,\ell$), $\xi_i(0,s_1,s_2)$ ($2\le s_1\le k_1$, $2\le s_2\le k_2$, $i=1,\dots,\ell$) can be calculated as follows:

$$\begin{aligned}\xi_i(0,s_1,s_2)\,\mu_1 &= X_iY_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}(\mu_1+\mu_2+k_1p_1+k_2p_2)\\ &\quad -X_i^{2}Y_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}\,\mu_2\\ &\quad -X_iY_{11i}^{s_1-2}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}k_1p_1\\ &\quad -X_iY_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-2}Y_{22i}k_2p_2\\ &=X_iY_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}\left[\mu_1+k_1p_1-Y_{11i}^{-1}k_1p_1+Y_{11i}^{k_1}r_1-r_1\right]\end{aligned} \tag{6.53}$$

From (6.20)

$$\mu_2+k_2p_2+r_1=X_i^{-1}\mu_1+Y_{11i}^{k_1}r_1+Y_{21i}^{-1}k_2p_2 \tag{6.54}$$

From (6.19)

$$\mu_1+k_1p_1+r_2=X_i^{-1}\mu_1+Y_{11i}^{-1}k_1p_1+Y_{21i}^{k_2}r_2 \tag{6.55}$$

Substituting (6.54), (6.55) into (6.53), we get

$$\xi_i(0,s_1,s_2)\mu_1=X_iY_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}\left(X_i^{-1}\mu_1+Y_{21i}^{k_2}r_2-r_2+Y_{11i}^{k_1}r_1-r_1\right)$$

$$=Y_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}\mu_1 \tag{6.56}$$

The second equality in (6.56) comes from (6.16). Finally,

$$\xi_i(0,s_1,s_2)=Y_{11i}^{s_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}$$

$$(i=1,\ldots,\ell;\ s_1=2,\ldots,k_1;s_2=2,\ldots,k_2) \tag{6.57}$$

This is in internal form, and $p_i(0,s_1,s_2)$ is in internal form according to (6.51).

Similarly, it can be shown from (6.40)-(6.46) that all the probabilities $p(n,s_1,s_2)$ $(n=0$ and $N;\ s_1=1,\ldots,k_1;\ s_2=1,\ldots,k_2)$ are in internal form.

Lemma 7

The probabilities of boundary states $p(0,0,s_2)$ and $p(N,s_1,0)$ $(s_1=1,\ldots,k_1;s_2=1,\ldots,k_2)$ are in the following forms

$$p(0,0,s_2)=\sum_{i=1}^{\ell}C_iY_{21i}^{s_2-1}Y_{22i}(X_i\mu_2+Y_{11i}^{k_1}r_1)/r_1 \tag{6.58}$$

$$p(N,s_1,0)=\sum_{i=1}^{\ell}C_1X_i^NY_{11i}^{s_1-1}Y_{12i}(X_i^{-1}\mu_1+Y_{21i}^{k_2}r_2)/r_2 \tag{6.59}$$

Proof From (6.34),

$$\xi_i(0,0,s_2)r_1=X_iY_{21i}^{s_2-1}Y_{22i}\mu_2+Y_{11i}^{k_1-1}Y_{12i}Y_{21i}^{s_2-1}Y_{22i}k_1p_1$$

$$=Y_{21i}^{s_2-1}Y_{22i}(X_i\mu_2+Y_{11i}^{k_1-1}Y_{12i}k_1p_1) \qquad (6.60)$$

Substituting (6.14) into (6.60),

$$\xi_i(0,0,s_2)=Y_{21i}^{s_2-1}Y_{22i}(X_i\mu_2+Y_{11i}^{k_1}r_1)/r_1 \qquad (6.61)$$

Similarly, from (6.33) and (6.15), it can be shown that

$$\xi_i(N,s_1,0)=X_i^N Y_{11i}^{s_1-1}Y_{12i}(X_i^{-1}\mu_1+Y_{21i}^{k_2}r_2)/r_2 \qquad (6.62)$$

We now have expressions for all the probabilities of internal and boundary states. They are in the form of (6.51). The coefficients C_i, however, are still unknown. The C's have to satisfy all those remaining boundary equations (6.35)-(6.38) and (6.47)-(6.50). In fact (6.35)-(6.38) and (6.47)-(6.50) are $\ell = 2k_1k_2+k_1+k_2$ equations altogether. As indicated in [2], the rank of this system of equations is $\ell=1$. Consequently, if we use $\ell-1$ of the equations and the normalizing equation

$$\sum_{n=0}^{N}\sum_{s_1=0}^{k_1}\sum_{s_2=0}^{k_2}p(n,s_1,s_2)=1, \qquad (6.63)$$

the C_i's can be determined.

We write (6.35)-(6.38), (6.47)-(6.50) and (6.63) as boundary condition:

$$T\,\Pi = 0 \tag{6.64}$$

where Π is a vector (of dimension $(N+1)(k_1+1)(k_2+1)$):

$$\Pi = p(n,s_1,s_2) \quad \text{(for all states } n,s_1,s_2\text{)} \tag{6.65}$$

and T is a coefficient matrix from the (6.35)-(6.38), (6.47)-(6.50) and (6.63).

Defining the vector $\underline{\xi}_i$ ($(N+1)(k_1+1)(k_2+1)$ rows, 1 column) as

$$\underline{\xi}_i \triangleq \left[\xi_i(n,s_1,s_2)\right] \quad \text{(for all states } n,s_1,s_2\text{)} \quad (i=1,\ldots,\ell) \tag{6.66}$$

and $$\Xi \triangleq \left[\underline{\xi}_1(n,s_1,s_2)\cdot \underline{\xi}_2(n,s_1,s_2),\ldots,\underline{\xi}_\ell(n,s_1,s_2)\right] \;, \tag{6.67}$$

$$C \triangleq \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_\ell \end{bmatrix} \tag{6.68}$$

i.e. know that

$$\Pi = \Xi C. \tag{6.69}$$

From (6.64)

$$T\Xi C = 0 \tag{6.70}$$

This system of equations, along with the normalization equation, determines C. We can summarize these results in an algorithm.

6.3 The algorithm

The algorithm is stated in 3 steps: Step 1: Calculate X_i, Y_{11i}, Y_{12i}, Y_{21i}, Y_{22i} $(i=1,\ldots,\ell)$ using (6.28)-(6.32). Step 2: Solve the linear system of equations (6.70) to obtain C_j $(j=1,\ldots,\ell)$. Step 3: Calculate ξ_i's using (6.52) for all these probabilities in internal forms and using (6.60) and (6.62) for others. Then using (6.69), lemma 1a and lemma 1b, evaluate all probabilities. These probabilities can be used to evaluate the measures of performance of section 4: E_1, E_2, P, $\bar{n}$.

7. MODIFIED ERLANGIAN MODEL

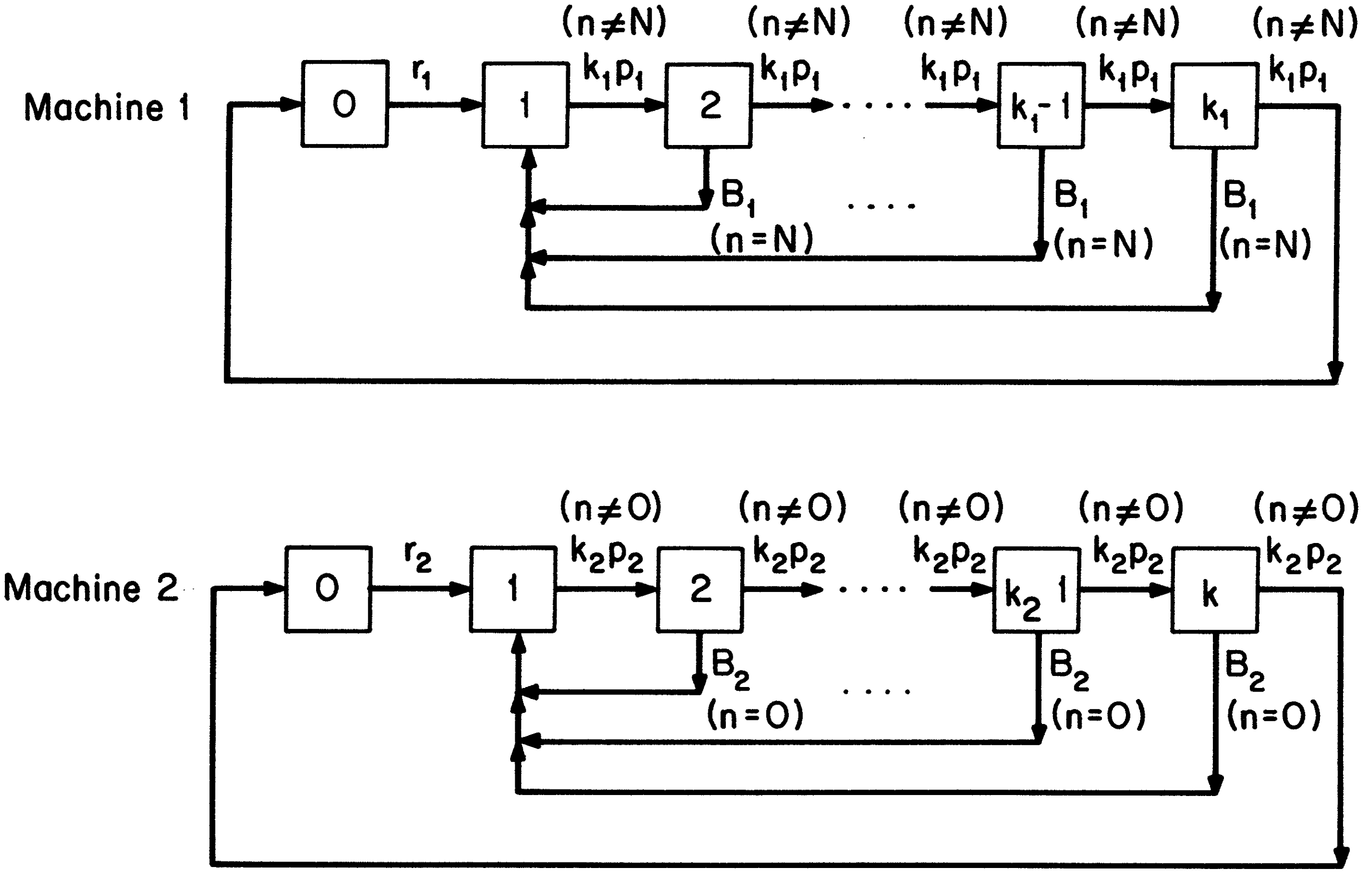
We have discussed a regular Erlangian model in previous sections. "Regular" refers to the assumption that the phase s_i can only increase while a machine is operational. The numerical experiments (in the next section) show that this model does not produce results that are very different from the exponential model (i.e., where $k_i=1$). Consequently, for practical purposes, we can use the methods of [1].

In this section, we present a modification of the previous Erlangian model. We state that the results of Section 5 hold for the modified model. In next section the numerical results demonstrate the effect of the modification.

Figure 7.2 indicates that in a regular Erlangian model, whenever a machine is operational (up and not starved or blocked) it has an aging rate $k_i p_i (i=1,2)$. While a machine is forced down its state remains constant because nothing happens to it.

But this forced down time period may provide an opportunity for maintaining or renewing the machine. Operators might add oil or grease to lubricate the machine, or might change certain tools, and so on. We should consider that everytime a machine is forced down, it not only stops aging, but it is being renewed. That is, the machine will turn back from phase s_i to s_i' $(s_i > s_i',\ i=1,2)$. It is shown in Figure 7.1. To simplify this model we assume $s_i'=1 (i=1.2)$.

FIGURE 7.1: Modified Failure Models



This means that everytime machine gets renewed, it always turns back to phase 1. The rate B_i $(i=1,2)$ is large enough, compared with failure rate (p_i) and repair rate (r_i), that we might consider that the renewal process happens instantly.

7.1 Modified Balance Equations

In Section 3 we have balance equations (3.1)-(3.27). For the present model, we have to modify (3.8), (3.9), (3.12), (3.13), (3.20)-(3.27). These are changed as follows:

$$p(N,s_1,0) = 0 \qquad 2\le s_1\le k_1 \qquad (3.8')$$

$$p(N,1,0)r_2 = p(N-1,1,0)\mu_1 + p(N,0,0)r_1 + p(N,1,k_2)k_2p_2 + \sum_{s_1=2}^{k_2} p(N-1,s_1,0)\mu_1 \qquad (3.9')$$

$$p(0,0,s_2) = 0 \qquad 2\le s_2\le k_2 \qquad (3.12')$$

$$p(0,0,1)r_1 = p(1,0,1)\mu_2 + p(0,k_1,1)k_1p_1 + p(0,0,0)r_2 + \sum_{s_2=2}^{k_2} p(1,0,s_2)\mu_2 \qquad (3.13')$$

$$p(0,s_1,s_2) = 0 \qquad 2\le s_1\le k_1 \quad 2\le s_2\le k_2 \qquad (3.20')$$

$$p(0,1,s_2) = 0 \qquad 2\le s_2\le k_2 \qquad (3.21')$$

$$p(0,s_1,1)(\mu_1+k_1p_1) = p(1,s_1,1)\mu_2+p(0,s_1-1,1)k_1p_1 + p(0,s_1,0)r_2 + \sum_{s_2=2}^{k_2} p(1,s_1,s_2)\mu_2 \qquad 2\le s_1\le k_1 \qquad (3.22')$$

$$p(0,1,1)(\mu_1+k_1p_1) = p(1,1,1)\mu_2 + p(0,0,1)r_1 + p(0,1,0)r_2 + \sum_{s_2=2}^{k_2} p(1,1,s_2)\mu_2 \qquad (3.23')$$

$$p(N,s_1,s_2) = 0 \qquad 2-s_1-k_1 \quad 2\le s_2\le k_2 \qquad (3.24')$$

$$p(N,1,s_2)(\mu_2+k_2p_2) = p(N-1,1,s_2)\mu_1 + p(N,0,s_2)r_1 + p(N,1,s_2-1)k_2p_2 + \sum_{s_1=2}^{k_1} p(N-1,s_1,s_2)\mu_1 \quad 2\le s_2\le k_2 \qquad (3.25')$$

$$p(N,s_1,1) = 0 \qquad 2\le s_1\le k_1 \qquad (3.26')$$

$$p(N,1,1)(\mu_2+k_2p_2) = p(N-1,1,1)\mu_1+p(N,0,1)r_1 + p(N,1,0)r_2 + \sum_{s_1=2}^{k_1} p(N-1,s_1,1)\mu_1 \qquad (3.27')$$

All other equations remain unchanged.

7.2 Theoretical Results

In the modified Erlangian model, lemmas 1-5 hold. These lemmas can be established by the same methods as in Section 5.

7.3 The Algorithm

The algorithm of Section 6 can be modified for this model. This is because the internal equations are unchanged. Some formulas for boundary probabilities must be altered.

8. NUMERICAL RESULTS

In this section we show three graphs. Figure 8.1 shows that for a regular model the production rate of a line changes by an amount too small to be noticed. For the modified model there is an apparent improvement. In Figure 8.2, it is shown that if also k_2's influence is considered, then the production rate will be increased even more. In Figure 8.3, it's shown that when buffer size N increases, the difference between the regular and modified model decreases when $k_1=3$ $k_2=1$. In all the experiments we use $p_1=1$, $p_2=1$, $r_1=10$, $r_2=10$, $\mu_1=100$, $\mu_2=100$. In Figure 8.1, 8.2 N=4.

The buffer level distributions are shown in Table 8.1 and Table 8.2 for the regular and modified model. In the tables we use p(n=0) for the probability of n=0, p(n=1) for probability of n=1, etc. In these two tables the parameters p_1, p_2, r_1, r_2, μ_1, μ_2 are the same as above but N=4.

From Table 8.1, it is shown that the average in-process inventory $\bar{n}$ does not change with k_1 and k_2 for the regular model. Table 8.2 indicates that in-process inventory changes substantially with k_1, k_2 for a modified Erlang model. The first half of Table 8.2 shows the average in-process inventory increases when k_1 increases and k_2 is constant. While the probability of n=0 decreases, all other probabilities increase when k_1 increases. This is intuitively reasonable.

In the lower half of Table 8.2 it is shown that when k_1,k_2 both increase and are equal $\bar{n}$ remains constant at 2 since this

FIGURE 8.1: Production Rate vs. k_1

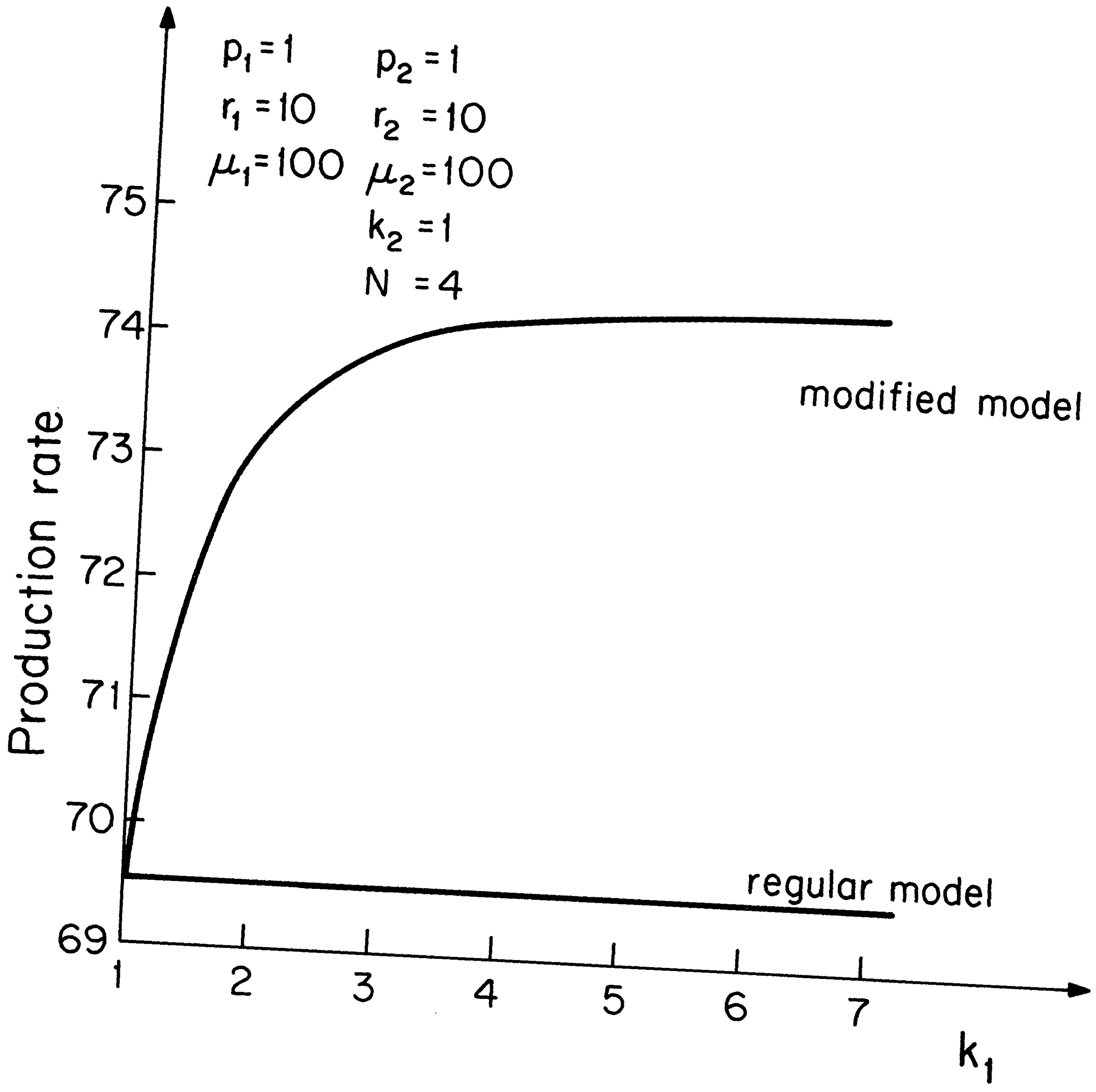


FIGURE 8.2: Production Rate vs. $k_1 = k_2$

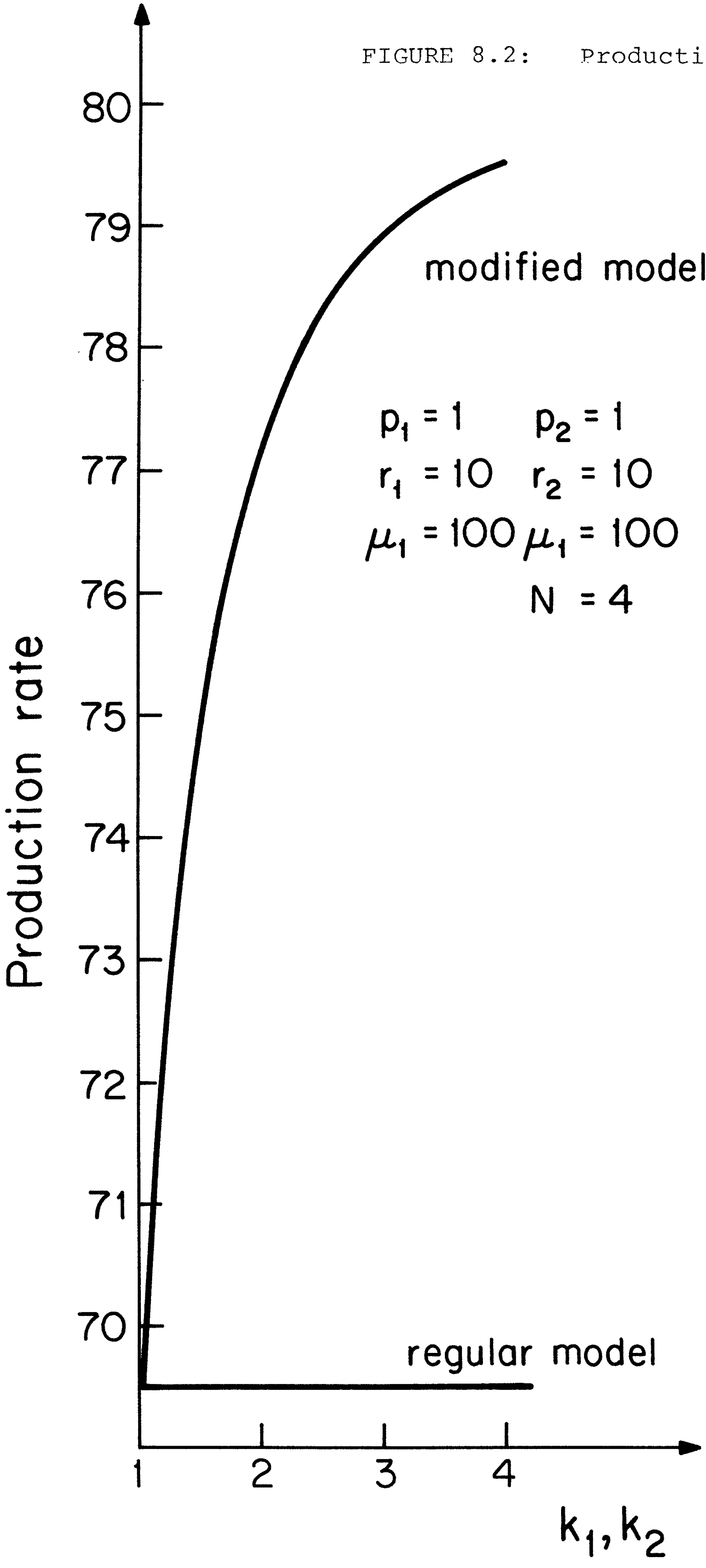
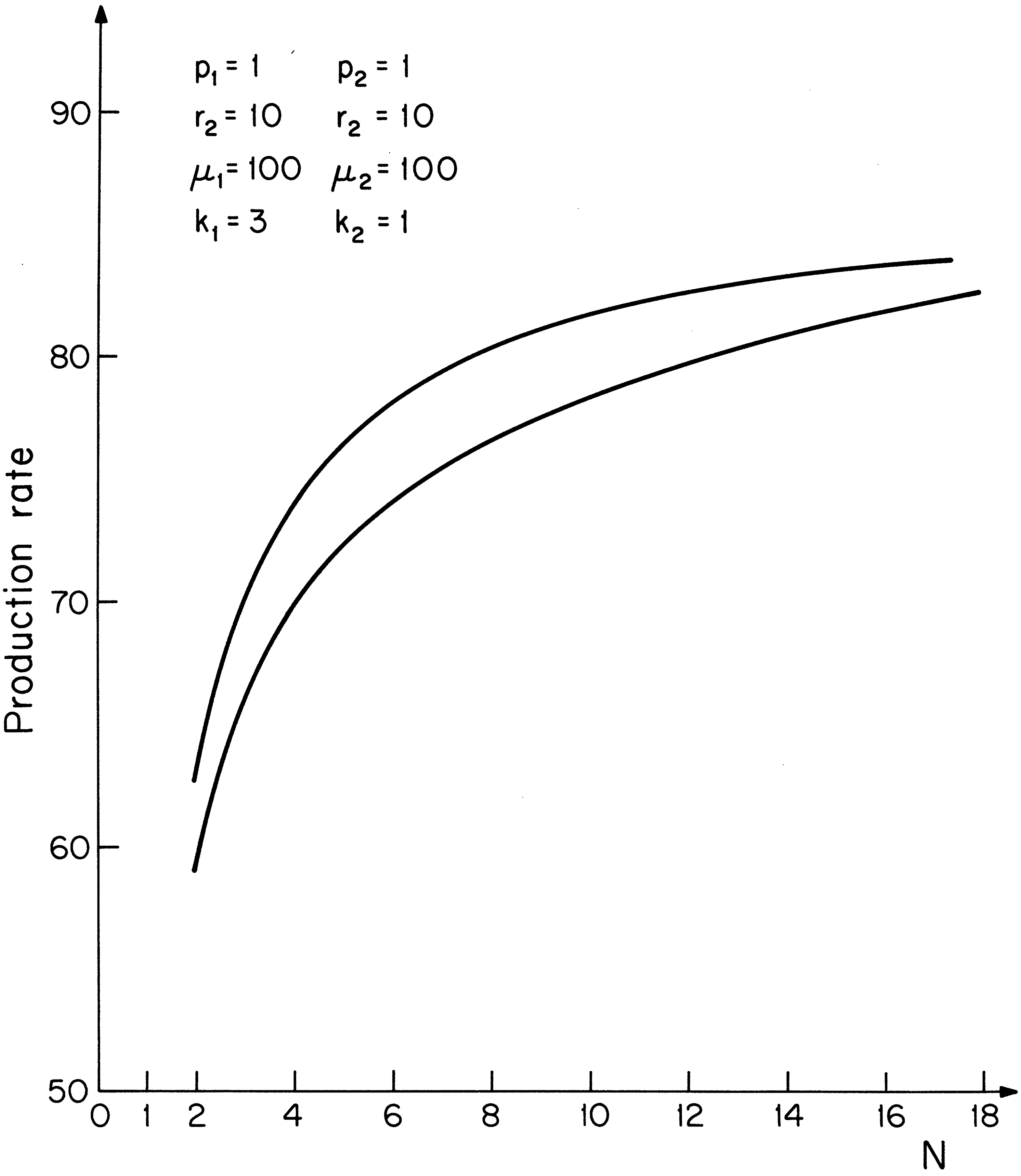


FIGURE 8.3: Production Rate vs. N



k_1	k_2	$\bar{n}$	p(n=0)	p(n=1)	p(n=2)	p(n=3)	p(n=4)
1	1	2	0.235	0.177	0.176	0.177	0.235
2	1	2	0.235	0.177	0.176	0.177	0.235
3	1	2	0.235	0.177	0.176	0.177	0.235
4	1	2	0.235	0.177	0.176	0.177	0.235
5	1	2	0.235	0.177	0.176	0.177	0.235
6	1	2	0.235	0.177	0.176	0.177	0.235
2	2	2	0.235	0.177	0.176	0.177	0.235
3	3	2	0.235	0.177	0.176	0.177	0.235

Table 8.1 Regular Model

k_1	k_2	$\bar{n}$	p(n=0)	p(n=1)	p(n=2)	p(n=3)	p(n=4)
1	1	2.0	0.235	0.177	0.176	0.177	0.235
2	1	2.121	0.194	0.182	0.184	0.188	0.252
3	1	2.143	0.187	0.183	0.186	0.190	0.255
4	1	2.150	0.184	0.183	0.186	0.191	0.256
5	1	2.153	0.183	0.183	0.187	0.191	0.256
6	1	2.154	0.183	0.184	0.187	0.191	0.256
7	1	2.155	0.182	0.184	0.187	0.191	0.256
2	2	2.0	0.209	0.194	0.194	0.194	0.209
3	3	2.0	0.204	0.198	0.197	0.198	0.204
4	4	2.0	0.202	0.199	0.199	0.199	0.202

Table 8.2 Modified Model

case is a balanced production line. (Machine 1 has same parameters as machine 2.) But from Table 8.2 we have both p(n=0) and p(n=4) decreasing while internal probabilities p(n=1), p(n=2), p(n=3) are increasing. That means the probabilities of starvation and blockage decrease. These are also very useful results.

It should be pointed out that the numerical results were not obtained by using the algorithm described above. This was due to the high order of the polynomial. Instead, the steady-state equations of Section 3 were solved by iteration.

9. CONCLUSIONS

In this paper, two Erlang failure models are studied as part of a two-machine, one-buffer production line. The regular model is a simple extension of the earlier exponential model; the modified model is one where preventative maintenance is performed whenever a machine is idle.

Numerical experiments show that there is very little difference between the regular Erlang and the exponential models. On the other hand, there is substantial difference between the exponential and the modified Erlang models.

As a consequence:

1. For a regular Erlang failure model, little is accomplished by setting $k \neq 1$. Therefore, there is no reason to seek data from an actual machine to determine k and no reason to use the relatively complicated algorithm of Section 6.
2. Preventative maintenance can be important. For systems with preventative maintenance, the modified Erlang model may be appropriate and the algorithm may prove to be important. To use the algorithm, however, it is necessary to obtain all the complex roots of a high order polynominal.

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