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THE TIME COMPLEXITY OF MAXIMUM MATCHING BY SIMULATED ANNEALING[1]

by

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Abstract

The random, heuristic search algorithm called simulated annealing is considered for the problem of finding a maximum cardinality matching in a graph. A basic form of the algorithm is shown to produce matchings with nearly maximum cardinality such that the average time required grows as a polynomial in the number of nodes in the graph. In contrast, it is also shown that for a certain family of graphs, neither the basic annealing algorithm, nor any other algorithm in a fairly large related class of algorithms, can find maximum cardinality matchings in polynomial average time.

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1. INTRODUCTION

1.1 Motivation

Simulated annealing is a Monte Carlo search heuristic which can be used to solve minimization (or maximization) problems. The simulated annealing method has received much attention from researchers since it was introduced in [Cer82, KiGV83, VeKi83], but there are apparently no concrete theoretical results regarding the average time complexity of the algorithm as the size of the problem instance tends to infinity, for a nontrivial problem.

In this paper we consider simulated annealing applied to maximum matching, a fundamental problem in combinatorial optimization. An instance of the maximum matching problem is a simple graph $G = (V,E)$, where V denotes the set of nodes of G and E denotes the set of (undirected) edges of G. A matching M in G is a subset of E such that no two edges in M share a node. The maximum matching problem for instance G is to find a matching in G with maximum cardinality.

The maximum matching problem is easy in the sense that there is a known deterministic algorithm which solves the problem in $O(|V|^3)$ steps (see [PaSt82]), where $|V|$ is the cardinality of V. However, we do not consider maximum matching to be trivial since the deterministic algorithm is somewhat subtle.

1.2 The Basic Annealing Algorithm for Maximum Matching

We will here describe what is perhaps the most obvious way to apply simulated annealing to search for the maximum matching of a graph $G = (V,E)$. Let λ_1, λ_2,... be a nonincreasing sequence of numbers in the interval $(0,1]$.

(λ_k will play the role of $\exp(-1/T_k)$ where T_k is the "temperature" at time k [KiGV83].) We say that an edge e is matchable relative to a matching M if $e \varepsilon M$ and if M+e is a matching (here M+e is our notation for $M \cup \{e\}$, which we use only if $e \notin M$).

To begin the algorithm, choose an arbitrary matching X_0 in G -- for example X_0 could be the empty set. Having selected $X_0, X_1, \ldots, X_k$, choose X_{k+1} as follows. Choose an edge e at random, all edges in E being equally likely.

If e is matchable relative to X_k, let $X_{k+1} = X_k + e$.

$$\text{If } e \varepsilon X_k, \text{ let } X_{k+1} = \begin{cases} X_k - e & \text{with probability } \lambda_k \\ X_k & \text{with probability } 1-\lambda_k \end{cases} .$$

Else, let $X_{k+1} = X_k$.

The sequence of states visited by the algorithm, $X_0, X_1, \ldots,$ forms a Markov chain.

1.3 Convergence in Probability

We begin by giving some standard notation [PaSt82]. Given a matching M in G, a node v is <u>exposed</u> if no edge in M is incident to v. A <u>path</u> p in G is a sequence of nodes $p = [v_1, v_2, \ldots, v_k]$ where $k \geq 1$, the nodes $v_1, v_2, \ldots, v_k$ are distinct, and $[v_i, v_{i+1}] \varepsilon E$ for $1 \leq i \leq i-1$. The length of such a path is k-1. The path is <u>augmenting</u> for M if its length is odd (so k is even), if v_1 and v_k are exposed and if $[v_i, v_{i+1}] \varepsilon M$ for even values of i with $2 \leq i \leq k-2$. It is fairly easy to show that a matching M does not have maximum cardinality if and only if there exists an augmenting path for M.

Let M_0 be a matching which does not have maximum cardinality, and let $[v_1,v_2,\ldots,v_k]$ be an augmenting path for M_0. Starting from M_0, it is possible for the basic annealing algorithm to reach a higher cardinality matching by passing through the sequence of matchings $M_1,M_2,\ldots,M_{k-1}$ given by

$$
\begin{array}{ll}
M_1 = M_0 - [v_2, v_3] & M_2 = M_1 + [v_1,v_2] \\
M_3 = M_2 - [v_4,v_5] & M_4 = M_3 + [v_3,v_4] \\
\quad\vdots & \quad\vdots \\
M_{k-3} = M_{k-4} - [v_{k-2}, v_{k-1}] & M_{k-2} = M_{k-3} + [v_{k-3}, v_{k-2}]
\end{array}
$$

and finally

$$M_{k-1} = M_{k-2} + [v_k, v_{k-1}].$$

The matchings in the sequence have cardinality at least as large as the cardinality of M_0 minus one. In the terminology of [Haj85b], the depths of the local maxima for the matching problem are at most one. The following theorem is thus an immediate consequence of [Haj85b, Thm. 1]. A matching M is said to be maximal if no edge is matchable relative to M. Let S^* denote the set of matchings with maximum cardinality.

Theorem 0. Let $G = (V,E)$ be a graph with a nonempty set of edges E. If all maximal matchings of G are in S^* then

$$\lim_{k \to \infty} P[X_k \in S^*] = 1 \quad \text{if and only if} \quad \lim_{k \to \infty} \lambda_k = 0$$

If some maximal matching is not in S^* then

$$\lim_{k \to \infty} P[X_k \in S^*] = 1 \quad \text{if and only if} \quad \lim_{k \to \infty} \lambda_k = 0 \text{ and } \sum_{k=0}^{\infty} \lambda_k = +\infty.$$

Theorem 0 gives a large-time asymptotic result for each fixed instance G, and the conditions do not depend on the size of G. In contrast, our goal in this paper is to give asymptotic results as $|V|$ tends to infinity.

1.4 Organization of the Paper

In Section 2 we show how to use the basic simulated annealing algorithm to produce matchings with nearly maximum cardinality using average time upper-bounded by a polynomial in $|V|$. In contrast, we show in Section 3 that for a certain family of graphs, the basic simulated annealing algorithm or any other algorithm in a fairly large related class, cannot find maximum cardinality matchings using average time upper-bounded by a polynomial in $|V|$. Sections 2 and 3 can be read independently.

2. NEAR MAXIMUM MATCHING IN POLYNOMIAL TIME

Let d^* denote the maximum node degree of the graph G and let m^* denote the maximum of the cardinalities of matchings in G. Let $\lfloor a \rfloor$ be the integer part of a.

<u>Theorem 1</u>. Let $\beta > 1$. Consider the basic simulated annealing algorithm run with $\lambda_k = \lambda$ for all k, where λ is given by

$$\lambda = \frac{1}{3|V|\omega^*} \quad \text{where} \quad \omega^* = (1+\beta)|V|(2d^*)^\beta$$

Let R denote the random time

$$R = \min\{k : |X_k| \geq \lfloor m^*(1 - \tfrac{1}{\beta}) \rfloor\}$$

Then

$$ER \leq 24(1+\beta)^2|V|^5(2d^*)^{2\beta}$$

<u>Remarks</u>.

(1) If β and d^* are bounded as $|V| \to \infty$, then $ER = O(|V|^5)$. In the proof below we see that three of these five factors of $|V|$ arise from our upper-bound D_1 on the mean time it takes the algorithm to make a single move. These can be reduced by using a more clever implementation of the simulated annealing algorithm.

(2) Since $2d^* \leq 2|V|$, we have with no restriction on d^* that

$$ER \leq 24(1+\beta)^2 2^{2\beta} |V|^{5+2\beta}$$

<u>Proof</u>. Define a random process Y by

$$Y_i = X_{J_i} \qquad i = 0,1,2,\ldots$$

where $J_0=0$ and for $i \geq 0$,

$$J_{i+1} = \min\{k>J_i : X_k \neq X_{k-1}\}$$

and define

$$R_Y = \min\{i: |Y_i| \geq \lfloor m^*(1 - \tfrac{1}{\beta})\rfloor\}$$

Note that $|Y_{i+1}| - |Y_i| \in \{-1,1\}$ with probability one for each i.

Next, define a random process Z by

$$Z_i = Y_{S_i} \qquad i = 0,1,\ldots$$

where $S_0 = 0$, $S_1 = 1$ and for $i \geq 1$

$$S_{i+1} = \min\{j: j>S_i, |Y_j| - |Y_{j-2}| \in \{-2,2\}\}$$

and define

$$R_Z = \min\{i: |Z_i| \geq \left\lfloor m^*(1 - \frac{1}{\beta}\right\rfloor\}.$$

Define constants D_1, D_2 and D_3 by

$$D_1 = 6|V|^2\omega^* \quad D_2 = 2\omega^* \quad D_3 = 2|V|$$

Lemma 2.1.

$$E[J_{i+1}-J_i|J_i,(X_0,X_1,\ldots,X_{J_i})] \leq D_1$$

Lemma 2.2a.

$$E[(S_{i+1}-S_i)I_{\{i<R_Z\}}|S_i, (Y_0,Y_1,\ldots,Y_{S_i})] \leq D_2$$

Lemma 2.3.

$$ER_Z \leq D_3$$

We will next prove Theorem 1 assuming the lemmas are true. We have

$$ER = EJ_{R_Y} = E\sum_{i=0}^{\infty} (J_{i+1}-J_i)I_{\{i<R_Y\}}$$

$$= \sum_{i=0}^{\infty} E[J_{i+1}-J_i \,|\, i<R_Y]P[i<R_Y].$$

Now the outcome of the event $\{i<R_Y\}$ is determined by

$$(J_i,\ (X_0,\ldots,X_{J_i}))$$

so we can apply Lemma 2.1 to get

$$ER \leq \sum_{i=0}^{\infty} D_1P[i<R_Y] = D_1ER_Y.$$

Similarly, the fact

$$R_Y = S_{R_Z},$$

and Lemma 2.2a imply that $ER_Y \leq D_2ER_Z$. So, also using Lemma 2.3, we conclude that

$$ER \leq D_1D_2D_3.$$

This will establish the theorem once we prove the three lemmas above.

$\square$

Proof of Lemma 2.1. By the strong Markov property of X,

$$E[J_{i+1} - J_i | J_i, (X_0, X_1, \ldots, X_{J_i})] = E[J_{i+1} - J_i | Y_i].$$

Since the transition probabilities of X are time invariant,

$$E[J_{i+1} - J_i | Y_i = M] = P[X_{k+1} \neq X_k | X_k = M]^{-1}.$$

Now, fix a matching M. One of two cases is true:

Case 1: Some edge in E is matchable with respect to M. Then

$$P[X_{k+1} \neq X_k | X_k = M] \geq \frac{1}{|E|}.$$

Case 2: No two of the $|V| - 2|M|$ exposed nodes are connected by an edge in the graph. Then

$$|E| \leq \binom{|V|}{2} - \binom{|V| - 2|M|}{2} = |M|(2|V| - 2|M| - 1) \leq 2|M||V|$$

so that

$$P[X_{k+1} \neq X_k | X_k = M] = \frac{\lambda |M|}{|E|} \geq \frac{\lambda}{2|V|}.$$

Hence, in either case,

$$E[J_{i+1}-J_i \mid Y_i=M] \le \max(|E|,\ 2|V|/\lambda) = D_1.$$

□

<u>Proof of Lemma 2.2a</u>. Lemma 2.2a is trivial for i=0 so we fix i with $i \geq 1$. Let m be an integer with $1 \le m \le \lfloor m^*(1 - 1/\beta)\rfloor$. Define a set of matchings B by

$$B = \{M: |M| = m-1 \text{ or } |M| = m\}$$

and let M_o be a fixed matching in B. Consider the event

$$F = \{Y_{S_i} = M_o,\ Y_{S_i-1} \in B \text{ and } R_Z > i\}.$$

The outcome of F is determined by

$$(S_i,\ (Y_0,Y_1,\ldots,Y_{S_i})),$$

and the union of events of the form of F as M_o and m vary as above is equal to the event $\{R_Z > i\}$. Hence, it suffices to prove that

$$E[S_{i+1}-S_i \mid S_i,(Y_0,Y_1,\ldots,Y_{S_i})]I_F \le D_2$$

for arbitrary fixed values of m and M_o as above.

Now, on the event F, we have

$$S_{i+1} = \min\{j>S_i:\ Y_j \not\in B\}$$

Using this and the Markov property of Y we have

$$E[S_{i+1}-S_i \,|\, S_i,(Y_0,\ldots,Y_{S_i})]I_F = E[\min\{j>S_i:Y_j\not\in B\} - S_i \,|\, Y_{S_i} = M_o]I_F$$

$$= E[S \,|\, Y_0 = M_o]I_F, \qquad (2.1)$$

where S denotes the stopping time

$$S = \min\{j\geq 1:\ Y_j \not\in B\}.$$

Let $\bar{B}$ be the set of matchings

$$\bar{B} = \{M:\ |M| \geq m-1\}.$$

Note that $B\subset\bar{B}$. We let $\bar{Y}$ denote a stationary-transition Markov chain with state space $\bar{B}$ and one-step transition probabilities determined by conditioning Y to stay in $\bar{B}$ for each consecutive jump:

$$P[\bar{Y}_{k+1} = M' \,|\, \bar{Y}_k = M] = P[Y_{k+1} = M' \,|\, Y_k = M]/\Delta(M)$$

for M, M' in B, where

$$\Delta(M) = P[Y_{k+1} \in B \,|\, Y_k = M].$$

Define a stopping time $\bar{S}_+$ by

$$\bar{S}_+ = \min\{k \geq 1: \bar{Y}_k \in \bar{B}-B\}$$

Let $\bar{S}_-$ denote a random variable on the same (or possibly enlarged) probability space as $(\bar{Y}_0, \bar{Y}_1, \ldots)$ such that

$$P[\bar{S}_- > k | \bar{Y}_0, \bar{Y}_1, \ldots] = \prod_{j=0}^{k-1} \Delta(\bar{Y}_j)$$

Let $\bar{S} = \min(\bar{S}_+, \bar{S}_-)$. Then, if we impose the conditions $\bar{Y}_0 = M_o$ and $Y_0 = M_o$

$$(\bar{S}, (\bar{Y}_k: 0 \leq k < \bar{S})) \text{ and } (S, (Y_k: 0 \leq k < S))$$

have the same distribution. Since $\bar{S} \leq \bar{S}_+$, it follows that

$$E[S | \bar{Y}_0 = M_o] \leq E[\bar{S}_+ | \bar{Y}_0 = M_o]. \tag{2.2}$$

Lemma 2.2a is implied by (2.1), (2.2) and Lemma 2.2b, which is stated and proved next.

$\square$

Lemma 2.2b. Under the conditions given in the proof of Lemma 2

$$E[\bar{S}_+ | \bar{Y}_0 = M_o] \leq 2\omega^*$$

Proof of Lemma 2.2b. Either $|M_o| = m$ or $|M_o| = m-1$. We will prove that if $|M_o| = m$, then

$$E[\bar{S}_+ | \bar{Y}_0 = M_o] \leq 2\omega^* - 1. \tag{2.3}$$

This will imply the lemma in general. Hence, we assume that $|M_o| = m$ for the rest of the proof of Lemma 2.2b.

For any matching M, let f(M) denote the length of the shortest augmenting path for M. The function f(M) is well defined if M is not a maximum matching (in particular if $|M| = m$) and $f(M) \in \{1,3,5,...\}$. Let $\bar{L}$ denote the maximum of f(M) over all M with $|M| = m$.

Claim 1. $\bar{L} \leq 2\beta$.

Proof of Claim 1. Let M be a matching with $|M| = m$ and let M^* be a maximum cardinality matching in G. Let G' denote the graph with set of nodes V and set of edges $M \Delta M^*$, where $M \Delta M'$ denotes the symmetric difference of M and M^*. Each node in G' has at most one edge from M and one edge from M^* incident to it. Thus, all maximal connected components of G' are paths or cycles, and all cycles have even length. The odd length paths have one more edge from M^* than from M so there are at least m^*-m paths. Since G' has fewer than $2m^*$ edges, the average length of the paths is less than $2m^*/(m^*-m)$ which, since $m \leq m^*(1-1/\beta)$, is at most 2β. Thus, at least one of the paths has length at most 2β. Such a path is an augmenting path for M, so $f(M) \leq 2\beta$.

□

Claim 2. Suppose $|M| = m$ and define p_0 and p_1 by

$$p_0 = \frac{1}{2m} \text{ and } p_1 = \min(\frac{d^*}{m}, 1-p_0)$$

Then

(a) $P[f(\bar{Y}_{2k+2}) \leq f(M) - 2 \,|\, \bar{Y}_{2k} = M] \geq p_0$ if $f(M) \geq 3$

(b) $P[\,|\bar{Y}_{2k+1}| \geq m+1 \,|\, \bar{Y}_{2k} = M] \geq p_0$ if $f(M) = 1$

(c) $P[f(\bar{Y}_{2k+2}) > f(M) + 2 \,|\, \bar{Y}_{2k} = M] = 0$

(d) $P[f(\bar{Y}_{2k+2}) = f(M) + 2 \,|\, \bar{Y}_{2k} = M] \leq p_1$

Proof of Claim 2. We will first prove (a) under the assumption that $f(M) \geq 5$. Choose an augmenting path p of length $f(M)$ and label some of the nodes and edges of it as indicated in Fig. 2.1. No neighbor of u_1, except possibly node v_1, can be an exposed node. Also, if u_1 and v_1 are neighbors, then w_1 and v_2 are not. Thus, there are at most two choices for an edge e', namely e_1 and possibly either $[u_1,v_1]$ or $[w_1,v_2]$, such that

$$f(M-e_1 + e') \geq f(M).$$

There is also at least one choice of e', namely $e' = \bar{e}_1$, such that

$$f(M-e_1+e') = f(M) - 2.$$

Thus,

$$P[f(\bar{Y}_{2k+2}) = f(M) - 2\,|\,\bar{Y}_{2k+1} = M-e_1] \geq 1/3.$$

This is true with e_1 replaced by e_2 as well, so

$$P[f(\bar{Y}_{2k+2}) = f(M) - 2\,|\,\bar{Y}_{2k} = M] \geq P[\bar{Y}_{2k+1} = M-e_1 \text{ or } \bar{Y}_{2k+1} = M-e_2\,|\,\bar{Y}_{2k} = M]/3$$

$$= \frac{2}{3|M|} \geq \frac{1}{2|M|} = p_0.$$

This establishes (a) if $f(M) \geq 5$. We will now complete the proof of (a) by considering the case $f(M) = 3$.

Let $[v_1, w_1, w_2, v_2]$ be an augmenting path of length 3 and let $e = [w_1, w_2]$. Then e is in M, and nodes v_1 and v_2 are not neighbors. Now, if e' is an edge such that

$$f(M-e+e') = 3 \tag{2.4}$$

then e' must be incident to either v_1 or w_1 and to either v_2 or w_2.

Moreover, if $e' = [v_1, w_2]$ is such an edge, then v_2 and w_1 must not be neighbors. Thus, there are at most two choices of e' such that (2.4) is true, namely e and possibly one of $[v_1, w_2]$ or $[v_2, w_1]$. There are also at least two values of e' such that $f(M-e+e') = 1$, namely $[v_1, w_1]$ and $[v_2, w_2]$. Thus,

$$P[f(\bar{Y}_{2k+2}) = f(M) - 2\,|\,\bar{Y}_{2k} = M] \geq \frac{2}{4}\, P[\bar{Y}_{2k+1} = M-e\,|\,\bar{Y}_{2k} = M] = p_0.$$

Part (a) is proved.

Turning to part (b), assume that $f(M) = 1$. Then some edge e is matchable relative to M. Hence,

$$P[|\bar{Y}_{2k+1}| \geq m+1 | \bar{Y}_{2k} = M] \geq (1+m\lambda)^{-1} \geq (1+m)^{-1} \geq p_0$$

so that part (b) is proved.

Choose a minimum length augmenting path p for M. Let

$$\Gamma_+ = \{(e_1,e_2)\colon e_1 \varepsilon M,\ e_2 \text{ is matchable relative to } M-e_1,$$

$$\text{and } f(M-e_1+e_2) \geq f(M) + 2\}.$$

Suppose $(e_1, e_2)\ \varepsilon\ \Gamma_+$. Then e_1 and e_2 are incident to a common node (otherwise $f(M-e_1+e_2) = f(M) = 1$) and $e_1 \neq e_2$. Since p is not an augmenting path for $M-e_1+e_2$, at least one of e_1 or e_2 is incident to a node of p. This means that either e_1 is an edge of p or e_2 is incident to one of the exposed nodes on the ends of p. Thus, we have narrowed down the possibilities to one of the four cases shown in Fig. 2.2. We can rule out the first three of these cases since in these cases there is an augmenting path for $M-e_1+e_2$ with length at most the length of p. We have thus shown that if $(e_1,e_2)\ \varepsilon\ \Gamma_+$, then e_2 is incident to an exposed node of p, e_1 and e_2 are incident to a common node, and e_1 is not in the path p. It follows that

$$f(M-e_1+e_2) = f(M)+2,$$

for any (e_1,e_2) in Γ_+, which proves part (c).

Define

$$W = \{e_2 \colon (e_1,e_2) \in \Gamma_+ \text{ for some } e_1\}.$$

If $e \in W$ there is exactly one edge, call it $w(e)$, such that $(w(e), e) \in \Gamma_+$. Each edge in W is incident to an exposed node of p so that $|W| \lesssim 2d^*$. Thus,

$$P[f(\bar{Y}_{2k+2}) = f(M)+2 \,|\, \bar{Y}_{2k}=M] = \sum_{(e_1,e_2)\in\Gamma_+} P[\bar{Y}_{2k+2}=M-e_1+e_2,\ \bar{Y}_{2k+1}=M-e_1 \,|\, \bar{Y}_{2k} = M]$$

$$= \sum_{e\in W} P[\bar{Y}_{2k+2}=M-w(e)+e \,|\, \bar{Y}_{2k+1} = M-w(e)]P[\bar{Y}_{2k+1}=M-w(e) \,|\, \bar{Y}_{2k} = M]$$

$$\lesssim |W| \frac{1}{2} \frac{1}{|M|} \lesssim \frac{d^*}{|M|} .$$

Together with part (a), this proves part (d) so that Claim 2 is completely proved.

□

We will now complete the proof of Lemma 2.2b using Claims 1 and 2. Define a process $U = (U_0,U_1,\ldots)$ by

$$U_k = \frac{1}{2} (1+f(\bar{Y}_{2k}))I_{\{2k \lesssim \bar{S}_+\}}.$$

Note that U_k takes values in $\{0,1,\ldots,L\}$ where $L = (1+\bar{L})/2$. Claim 1 implies that $L \lesssim 1+\beta$ and Claim 2 implies that

$$P[U_{k+1} = i \,|\, U_k=j,\ U_{k-1},\ldots,U_0] \begin{cases} \geq p_0 & \text{if} \quad i=j-1 \\ = 0 & \text{if} \quad i \geq j+2 \\ \leq p_1 & \text{if} \quad i=j+1 \end{cases} \tag{2.5}$$

Let $W = (W_0, W_1, \ldots)$ denote the Markov chain with one-step transition probabilities shown in Fig. 2.3. From (2.5) it follows that if $W_0 = U_0$, then the chain W stochastically dominates the process U. Hence

$$\begin{aligned} E[\bar{S}_+ | \bar{Y}_0=M] + 1 &= 2E[\min\{j:U_j=0\} \,|\, \bar{Y}_0=M] \\ &\leq 2E[\min\{j:W_j=0\} \,|\, W_0=f(M)] \\ &\leq 2E[\min\{j:W_j=0\} \,|\, W_0=L] \\ &= \frac{2}{p_0} \sum_{j=0}^{L-1} (L-j)\left(\frac{p_1}{p_0}\right)^j \\ &\leq \frac{2L}{p_0}\left(\frac{p_1}{p_0}\right)^{L-1} \leq \frac{2(1+\beta)}{p_0}\left(\frac{p_1}{p_0}\right)^{\beta} \leq 2\omega^* . \end{aligned}$$

This establishes Ineq. (2.3), so the proof of Lemma 2.2b, and hence also the proof of Lemma 2.2a, is complete.

□

Proof of Lemma 2.3. In the first part of the proof, we will refer to the set-up in the proof of Lemma 2.2a. By the reasoning there, we see that for $i \geq 1$,

$$P[|Y_{S_{i+1}}| > |Y_{S_i}| \,|\, S_i, (Y_0, Y_1, \ldots, Y_{S_i})] I_F = P[|Y_S| = m+1 \,|\, Y_0 = M] I_F$$

$$= P[\bar{S}_- > \bar{S}_+ \,|\, \bar{Y}_0 = M] I_F$$

$$= E[\prod_{j=0}^{\bar{S}_+ - 1} \Delta(\bar{Y}_j) \,|\, \bar{Y}_0 = M] I_F .$$

Now $\Delta(\bar{Y}_j)=1$ for at least half of the values of j with $0 \leq j \leq \bar{S}_+ - 1$, and for all j

$$\Delta(\bar{Y}_j) \geq (1+(m-1)\lambda)^{-1} \geq (1 + \frac{\lambda|V|}{2})^{-1} .$$

Thus

$$E[\prod_{j=0}^{\bar{S}_+ - 1} \Delta(\bar{Y}_j) \,|\, \bar{Y}_0 = M] \geq E[(1 + \frac{\lambda|V|}{2})^{-S_+/2} \,|\, \bar{Y}_0 = M]$$

$$\geq (1 + \frac{\lambda|V|}{2})^{-\omega^*}$$

$$\geq \exp(-\lambda|V|\omega^*/2) = \exp(-1/6) \geq 5/6$$

where for the second inequality we used Lemma 2.2b and Jensen's inequality, and for the last two inequalities we used the inequality $\exp(u) \geq 1+u$. Therefore, for $i \geq 1$,

$$P[|Z_{i+1}| > |Z_i| \,|\, Z_0,\ldots,Z_i] \geq 5/6.$$

Now $|Z_{i+1}| - |Z_i| \in \{-2,1,1,2\}$ so that

$$E[|Z_{i+1}| - |Z_i| \,|\, Z_0,\ldots,Z_i] \geq \frac{5}{6} - 2 \cdot \frac{1}{6} = \frac{1}{2}$$

which implies that the process

$$|Z_i| - \frac{i-1}{2}, \quad i=1,2,\ldots$$

is a submartingale uniformly bounded from above. Thus, by a version of Doob's optional sampling theorem [ChTe78, Thm. 7.4.6.ii]

$$E\left[|Z_{R_Z}| - \frac{R_Z-1}{2}\right] \geq E[|Z_1| - \frac{1-1}{2}] \geq 0$$

which yields

$$ER_Z \leq 2E|Z_{R_Z}| + 1 = 2\lfloor m^*(1 - \frac{1}{\beta})\rfloor + 1 \leq D_3.$$

Lemma 2.3, and hence Theorem 1, are completely proved.

$\square$

3. THE IMPOSSIBILITY OF MAXIMUM MATCHING IN POLYNOMIAL AVERAGE TIME USING CERTAIN ANNEALING TYPE ALGORITHMS

Certain local search algorithms for the maximum matching problem will be considered in this section. The algorithms will not be restricted much in an attempt to include several implementations of simulated annealing. Both the basic simulated annealing algorithm given in Section 1 when $X_0 = \phi$ and a particular multistart algorithm will be included. Nevertheless, it will be proven that the algorithms cannot reach a maximum matching in average time bounded by a polynomial in $|V|$, for a particular family of graphs.

First, we allow the "temperature" to depend on both time and the current and past states of the algorithm. Second, we assume that the type of each move can be specified from among the three possibilities whenever they exist: addition of an edge, deletion of an edge, no change. The key restriction we do impose is that given the type of a move, the location of the edge to be added or deleted is uniformly distributed over the possible locations.

We thus view the sequence X_0, $X_1, \ldots$ of states generated by the algorithm as a controlled Markov process. Suppose for each t that "controls" a_t and d_t are given such that

C.1 $a_t, d_t, a_t + d_t \in [0,1]$ with probability one, and a_t and d_t are functions of $(X_0,\ldots,X_t)$.

C.2 $$P[X_{t+1}=M' \mid X_t=M, X_{t-1},\ldots,X_0] = \begin{cases} 1-a_t-d_t & \text{if } M = M' \\ d_t/|M| & \text{if } e \in M \text{ and } M' = M-e \\ a_t/|\mathring{M}| & \text{if } e \in \mathring{M} \text{ and } M' = M+e \\ 0 & \text{if } |M \Delta M'| \geq 2 \end{cases}$$

where $\mathring{M}$ is the set of edges matchable relative to M.

Clearly, if we choose the controls appropriately we can use this controlled Markov process to mimic the basic simulated annealing process of Section 1. We can also control the Markov process to mimic a multistart algorithm (although only at half speed). To do this we assume that $X_0 = \phi$. We then let $a_t = 1$ for $0 \leq t < S_1$ where S_1 is the first time that a maximal matching is reached. Then we let $d_t = 1$ for $S_1 \leq t < 2S_1$, which guarantees that $X_t = \phi$ for $t = 2S_1$. We then keep repeating this process.

Define the graph $G_n = (V_n, E_n)$ for $n \geq 1$ by letting the set V_n of nodes be

$$V_n = \{u_{ij}: 1 \leq i,j \leq n+1\} \cup \{v_{ij}: 1 \leq i,j \leq n+1\}$$

and letting the set E_n of edges be

$E_n = H \cup B$, where

$$H = \bigcup_{j=1}^{n+1} H_j, \quad B = \bigcup_{j=1}^{n} B_j,$$

$$H_j = \{(u_{ij}, v_{ij}): 1 \le i \le n+1\} \qquad 1 \le j \le n+1$$

and

$$B_j = \{(u_{ij}, v_{kj+1}: 1 \le i,\ k \le n+1\} \qquad 1 \le j \le n.$$

Graph G_3 is sketched in Fig. 3.1. Graph G_n is a bipartite graph with $2(n+1)^2$ nodes and $(n+1)^3$ edges.

Theorem 2. There exist positive constants A and B such that the following is true. For any $n \ge 1$, let (X,a,d) be a controlled process for finding the maximum matching of G_n satisfying conditions C.1 and C.2. Define R^* by

$$R^* = \min\{k: X_k \text{ is a maximum matching}\}.$$

Then

$$E[R^* | X_0 = \emptyset] \ge A \exp(Bn).$$

The proof of Theorem 2 will be given after several lemmas are proved. Given a matching M, let $U_j(M)$ (resp. $V_j(M)$) denote the number of nodes in

$$\{u_{ij}\colon 1 \leq i \leq n+1\} \text{ (resp. } \{v_{ij}\colon 1 \leq i \leq n+1\})$$

which are exposed relative to M for $1 \leq j \leq n+1$, and let $V_0(M) = U_{n+2}(M) = 0$. Let

$$g(M) = c\,|B \quad M| + \sum_{j=1}^{n} \varphi(V_j(M),\ U_{j+1}(M))$$

where c=18 and

$$\varphi(x,y) = 2\min(x,y) + I_{\{x>0,\ y>0,\ x\neq y\}}.$$

Note that H is the unique maximum cardinality matching for G_n, that $g(M) \geq 0$ for all matchings M and $g(M) = 0$ if $M = H$.

Lemma 3.1. Suppose $x,\ y \geq 0$. Then

(a) $$\varphi(x+1,\ y) - \varphi(x,y) \begin{cases} \varepsilon\ \{1,2,3\} & \text{if } y \geq \min(x,1) \\ 0 & \text{otherwise} \end{cases}$$

(b) $$\varphi(x,\ y+1) - \varphi(x,y) \begin{cases} \varepsilon\ \{1,2,3\} & \text{if } x \geq \min(y,1) \\ = 0 & \text{otherwise} \end{cases}$$

(c) $$\varphi(x+1,\ y+1) - \varphi(x,y)\ \varepsilon\ \{2,3\}$$

Proof. Easy by inspection of Table 1.

$\square$

<u>Lemma 3.2</u>.

(a) Suppose $e \varepsilon M \cap H_j$. Then $g(M-e)-g(M) \geq 1$ if and only if

$U_{j+1}(M) \geq \max(1, V_j(M))$ or $V_{j-1}(M) \geq \max(1, U_j(M))$

(b) $g(M-e)-g(M) \varepsilon \{0,1,\ldots,6\}$ for $e \varepsilon M \cap H$

(c) $g(M-e)-g(M) \varepsilon \{-c+2, -c+3\}$ for $e \varepsilon M \cap B$

<u>Proof</u>. It is easy to see that for $e \varepsilon M \cap H_j$,

$$g(M-e) - g(M) = \varphi(V_j(M) + 1, U_{j+1}(M)) - \varphi(V_j(M), U_{j+1}(M))$$

$$+ \varphi(V_{j-1}(M), U_j(M)+1) - \varphi(V_{j-1}(M), U_j(M))$$

and for $e \varepsilon M \cap B_j$

$$g(M-e) - g(M) = -c + 2 + I_{\{U_j(M) = 0 \text{ or } V_j(M) = 0\}}.$$

Lemma 3.2 can be easily deduced from these equations and Lemma 3.1.

□

Define

$A(M) = \{e \text{ is matchable relative to } M \text{ and } g(M+e) \neq g(M)\}$

$D(M) = \{e \in M: g(M-e) \neq g(M)\}$

$A_+(M) = \{e \in A(M): g(M+e) > g(M)\}$ $A_-(M) = \{e \in A(M): g(M+e) < g(M)\}$

$D_+(M) = \{e \in D(M): g(M-e) > g(M)\}$ $D_-(M) = \{e \in D(M): g(M-e) < g(M)\}$

Lemma 3.3. Let M be a matching and let $0<\delta<1$.

(a) $D_+(M) \subset M \cap H$, $D_-(M) = M \cap B$, $A_-(M) \subset H$ and

$$A_+(M) = \{e \in B-M: M+e \text{ is a matching}\}$$

(b) $|A_-(M)| \leq 2|A_+(M)|$

(c) $|D_-(M)| \leq n\delta$ if $g(M) < nc\delta$

(d) $|D_+(M)| \geq n(1-\delta(\frac{c}{2}+1))$ if $0<g(M) < nc\delta$

Proof. Part (a) is a consequence of (b) and (c) of Lemma 3.2 and the fact that $c>3$.

Let $e \in A_-(M)$. Then $e \in H_j$ for some j by part (a) and moreover at least one of $U_{j+1}(M)$ or $V_{j-1}(M)$ is strictly positive by (a) of Lemma 3.2. Thus, there is at least one edge e' in $B_j \cup B_{j-1}$ which is matchable relative to M, and which shares a node with e. Hence, every edge in $A_-(M)$ shares a node with an edge in $A_+(M)$. On the other hand, since $A_-(M) \subset H$ and $A_+(M) \subset B$, each edge in $A_+(M)$ is incident to at most two edges in $A_-(M)$. These two facts imply (b).

By (a),

$$|D_-(M)| \leq |B \cap M| \leq g(M)/c < n\delta$$

which proves (c).

Finally, let M be a matching with $0<g(M)<nc\beta$. The fact that $g(M) > 0$ implies that M is not equal to the unique perfect matching H, which implies that there exists at least one exposed node. Since $g(M) < nc$, M contains fewer than n edges from $B_1 \cup B_2 \cup \ldots \cup B_n$. Hence $M \cap B_k = \emptyset$ for some k. Now, the set of nodes

$$Z = \{v_{ij}: 1 \leq i \leq n+1,\ 1 \leq j \leq k\} \cup \{u_{ij}: 1 \leq i \leq n+1,\ k+1 \leq j \leq n\}$$

contains exactly half of the nodes of the graph. Since $M \cap B_k = \emptyset$, each edge in M is incident to a node in Z and a node not in Z. Thus, Z contains half, and therefore at least one, of the exposed nodes, so at least one of the 2n numbers

$$V_1(M), \ldots, V_n(M),\ U_2(M), \ldots, U_{n+1}(M)$$

is nonzero. By the symmetry between the U's and V's, we can restrict attention to the case that for some j with $1 \leq j \leq n$, $U_{j+1}(M)$ is as least as large as any of the other 2n−1 numbers. Then $U_{j+1}(M) \geq \max(1, V_j(M))$ so $M \cap H_j \subset D_+(M)$ by part (a) of Lemma 3.2. Hence

$$|D_+(M)| \geq |M \cap H_j|$$

$$= n+1 - V_j(M) - |M \cap B_j|$$

$$= n+1 - \min(V_j(M), U_{j+1}(M)) - |M \cap B_j|$$

$$\geq n - g(M)/2 - g(M)/c \geq n(1 - \delta(\tfrac{c}{2} + 1))$$

which proves (d) and completes the proof of Lemma 3.3.

$\square$

Lemma 3.4. Now set $\delta = 1/43$. If $0 < g(M) < nc\delta$, then

$$|A(M)|^{-1} \sum_{e \varepsilon A(M)} [g(M+e) - g(M)] \geq 1 \qquad \text{if } A(M) \neq \emptyset \tag{3.1}$$

and

$$|D(M)|^{-1} \sum_{e \varepsilon D(M)} [g(M-e) - g(M)] \geq \frac{1}{2} \qquad \text{if } D(M) \neq \emptyset \tag{3.2}$$

Proof. By Lemma 3.3 and (a) and (b) of Lemma 3.2, we have

$$g(M+e) - g(M) \geq \begin{cases} c-3 & \text{if } e \ \varepsilon \ A_+(M) \\ -6 & \text{if } e \ \varepsilon \ A_-(M) \end{cases}$$

which, together with (b) of Lemma 3.3, yields edge.

Suppose M is a matching with $0<g(M)<nc\delta$. Then

$$E[g(X_{t_{k+1}}) - g(X_{t_k}) \mid X_{t_{k+1}-1} = M,\ \theta_k = \theta,\ X_{t_{k+1}-2}, \ldots, X_0] = s(M,\theta) \geq \frac{1}{2}\ .$$

Averaging over appropriate values of θ_k and $(X_i\colon t_k < i < t_{k+1})$, it follows that

$$E[g(X_{t_{k+1}}) - g(X_{t_k}) - \tfrac{1}{2},\ g(X_{t_k}) < nc\delta,\ R^* > t_k \mid X_0, \ldots, X_{t_k}] \geq 0$$

Also, by Lemmas 3.2b and 3.2c, the magnitudes of the increments of $g(X_t)$ are bounded by c−2. Thus, Theorem 2.3 of [Haj82] is in force if we define $(Y,\ \varepsilon_0,\ a,\ b)$ by

$$Y_k = -g(X_{t_k}),\ \varepsilon_0 = \frac{1}{2},\quad a = -n\delta c \text{ and } b=0. \tag{3.3}$$

Using the fact that $Y_0 \leq a$, this produces constants $\eta>0$, $\rho\varepsilon(0,1)$ and $D>0$ such that

$$P[R^*=t_k \mid X_0 = \phi] \leq P[g(X_{t_k}) = 0,\ R^* \geq t_{k-1} \mid X_0=\phi] \leq u$$

where

$$|A(M)|^{-1} \sum_{e \varepsilon A(M)} [g(M+e) - g(M)] \geq |A(M)|^{-1}[(c-3)|A_+(M)| - 6|A_-(M)|]$$

$$\geq 1$$

Similarly, by Lemma 3.3 and (b) and (c) of Lemma 3.2, we have

$$g(M-e) - g(M) \begin{cases} \geq 1 & \text{if } e \varepsilon D_+(M) \\ \geq -c+2 & \text{if } e \varepsilon D_-(M) \end{cases}$$

which, together with (c) and (d) of Lemma 3.3, yields (3.2).

$\square$

<u>Proof of Theorem 2</u>.

Let t_1, t_2,... denote the jump times for the process $(g(X_k): k \geq 0)$. We can and do assume that $P[R^* < +\infty | X_0 = \phi] = 1$. It follows that, with probability one, $R^* \varepsilon \{t_1, t_2, \ldots\}$, which implies that

$$P[t_{k+1} < \infty | R^* > t_k, X_0 = \phi] = 1.$$

Given a matching M, define s(M,1) and s(M,-1) to be the normalized sums appearing in Ineqs. (3.1) and (3.2) respectively, whenever they are well-defined. By Lemma 3.4, $s(M,\theta) \geq 1/2$ if $0 < g(M) < nc\delta$ and if $s(M,\theta)$ is well defined.

Let $\theta_k = 1$ if the jump at time t_{k+1} is caused by the addition of an edge, and let $\theta_k = -1$ if the jump at time t_{k+1} is caused by the deletion of an

$$u = \frac{D\ \exp(-\eta n c \delta)}{1-\rho}$$

Since $R^* \varepsilon \{t_1, t_2, \ldots\}$ and since $t_k \geq k$, we have

$$P[R^* > k \,|\, X_0 = \phi] \geq P[R^* > t_k \,|\, X_0 = \phi] \geq \max(0,\ 1-ku).$$

Hence

$$E[R^* \,|\, X_0 = \phi] = \sum_{k=0}^{\infty} P[R^* > k \,|\, X_0 = \phi]$$

$$\geq \sum_{k=0}^{\infty} \max(0,\ 1-ku) \geq \frac{1}{2u}\ .$$

Thus, taking $A = (1-\rho)/2D$ and $B = \eta c \delta$, Theorem 2 is proved.

□

Remark. Some extra work shows that Conditions D.1 and D.2 of [Haj82] are satisfied for Y, a and b given in (3.3) and $\eta = .0033683$, $\rho = .9998$, $a = -n\delta c$, $b=0$ and $D=1$. This shows that Theorem 2 above is true for $A = .0001$ and $B = .0014$.

4. SPECULATIONS

We believe that ER is significantly smaller than the upper bound given in Theorem 1, and that Theorem 2 is true for constants A and B much larger than what we provided in the proof. Moreover, we conjecture that for $0 < r < 1$, the average time needed for the controlled processes described in Section 3 to reach a matching having cardinality at least the maximum possible minus $|V|^r$ is not upperbounded by a polynomial in $|V|$, for some sequence of graphs. The key to proving stronger statements may be to keep track of the progress of many augmenting paths, instead of concentrating as we have on just one.

The upper bound on ER given in Theorem 1 is valid for all graphs. Perhaps one can find a much smaller bound on ER by restricing attention to graphs G that are "typical" in some sense, or by considering a random graph.

Our methods of analyzing simulated annealing, like the deterministic methods known for solving the maximum matching problem, don't easily carry over to "industrial strength" variations of the problem or to other problems. More work will be needed to evaluate the average time complexity of simulated annealing and other search heuristics for a wide range of problems.

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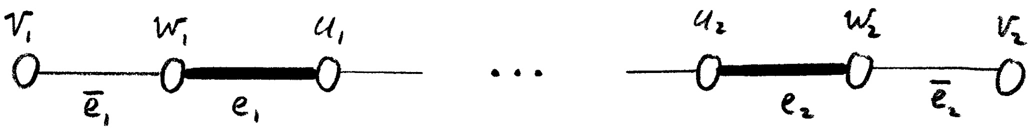

Figure 2.1

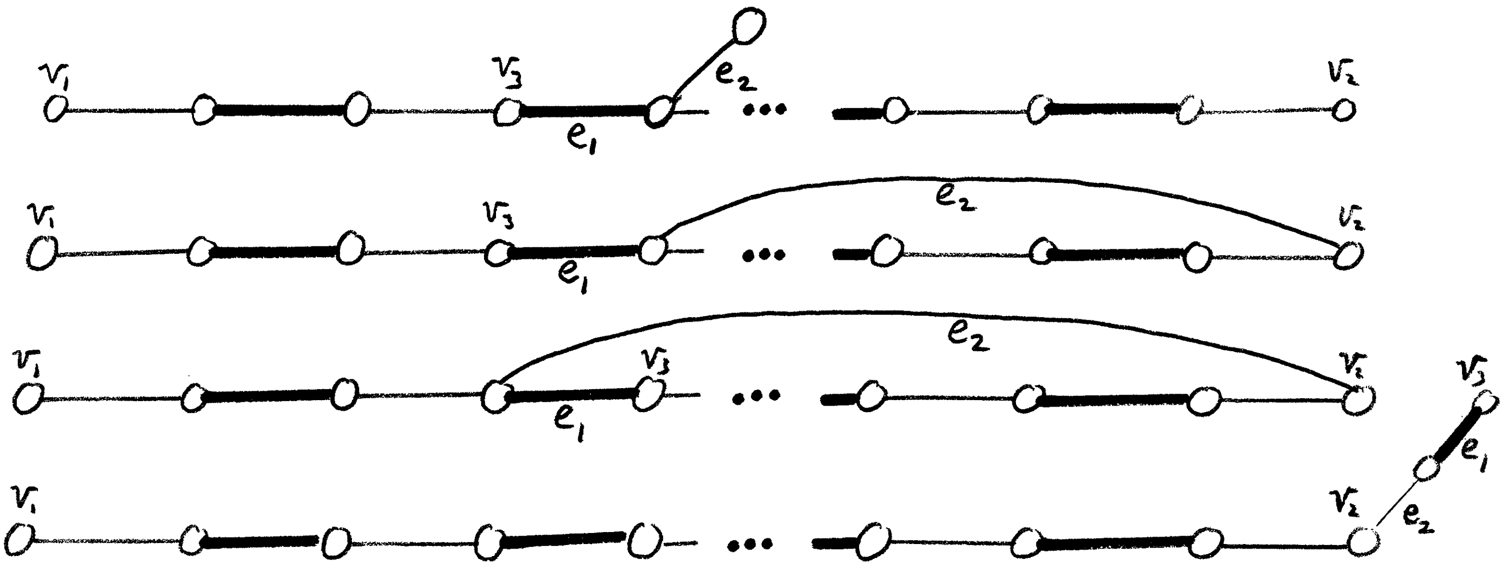

Figure 2.2

Four possibilities for (e_1, e_2) are pictured. Edges in the path $\bar{p}$ are drawn straight and horizonally. Edges in M are bold. Nodes v_1 and v_3 are the end nodes of an augmenting path for $M-e_1+e_2$. Only the fourth possibility can really occur.

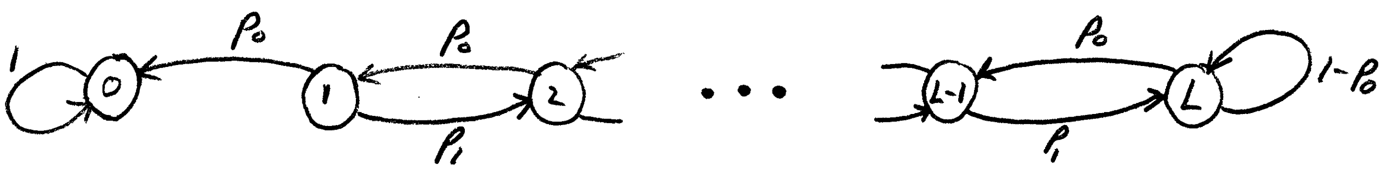

Figure 2.3

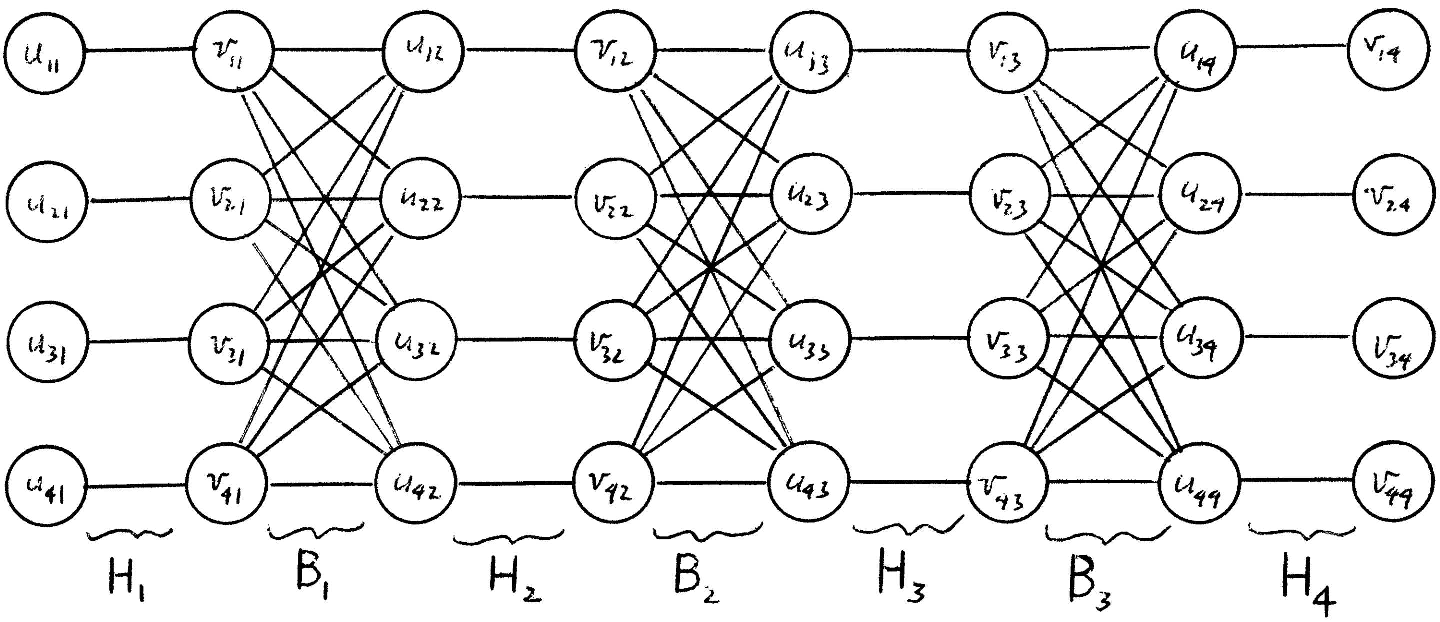

Figure 3.1 Sketch of G_3.

Table 1. Values of $\psi(x,y)$

$x \backslash y$	0	1	2	3	4	5	6	...
0	0	0	0	0	0	0	0	
1	0	2	3	3	3	3	3	
2	0	3	4	5	5	5	5	...
3	0	3	5	6	7	7	7	
4	0	3	5	7	8	9	9	
5	0	3	5	7	9	10	11	
6	0	3	5	7	9	11	12	
⋮			⋮					