RECONSTRUCTION OF CONVEX SETS FROM NOISY SUPPORT LINE MEASUREMENTS

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ABSTRACT:

In many applications, measurements of the support lines of a two dimensional set are available. From such measurements, the convex hull of the set may be reconstructed. If, however, the measurements are noisy, then the set of measurements, taken together, may not correspond to any set in the plane — they are inconsistent. This paper describes the consistency conditions for support line measurements when the angles of the lines are precisely known, but the lateral displacements are degraded by noise. We propose three simple algorithms for obtaining consistent support line estimates based on ML and MAP estimation principles, and show examples of the performance of these algorithms.

1.0 INTRODUCTION

There are many applications in which measurements of the support lines of 2-d objects are available. Among these are tactile sensing in robotics [1], robot vision [2], chemical component analysis [3], and silhouette imaging [4]. For example, measurements of the position and angular orientation of a parallel plate robot jaw as it clamps down on a "thick 2-d object" may be interpreted as support line measurements. Given a set of such (noise-free) measurements from different angles, one may reconstruct a convex (2-d) polyhedron which contains the object [11,12].

The area of application which motivated this study is computed tomography (see [5], for example). In computed tomography (CT), the positions of two support lines, supporting the set of points whose density is non-zero, are evident from each projection. If the only information desired is an estimate of the convex hull of this (indicator) set, then the problem is identical to the robot jaw example above. Furthermore, an estimate of this set may be used to reconstruct the density function itself using constraint based reconstruction methods [13].

This paper considers the problem in which there are a finite number of support line measurements for which the angles are known precisely, but the lateral displacements are noisy. In the robot example, this may arise because of backlash in the gears, for example, while in the CT example, the lateral noise arises simply because it is impossible to precisely determine the position of the support lines from (noisy) measured projections. We show that a collection of noisy support line measurements may be inconsistent with any set in the plane, and describe algorithms that exploit the fundamental constraint that is revealed. We also indicate how prior information concerning object shape may be included in the algorithms.

2.0 DISCRETE SUPPORT LINE CONSTRAINTS

The *support line* at angle θ for the closed and bounded (2-d) set S is given by (see Figure 1)

$$L_S(\theta) = \{ x \in \mathbb{R}^2 \mid x^T\omega = h(\theta) \} \qquad (1)$$

where $\omega = [\cos\theta \ \sin\theta]^T$ and

$$h(\theta) = \sup_{x \in S} \{ x^T\omega \} \quad .$$

The function $h(\theta)$ is called the *support function* of the set S; for any particular value of θ we call $h(\theta)$ the *support value* at angle θ. We think of $L_S(\theta)$ as the line with normal ω which just "grazes" the set S, so that any point x in S satisfies $x^T\omega \leq h(\theta)$, i.e., S lies completely in one (particular) closed halfspace determined by $L_S(\theta)$.

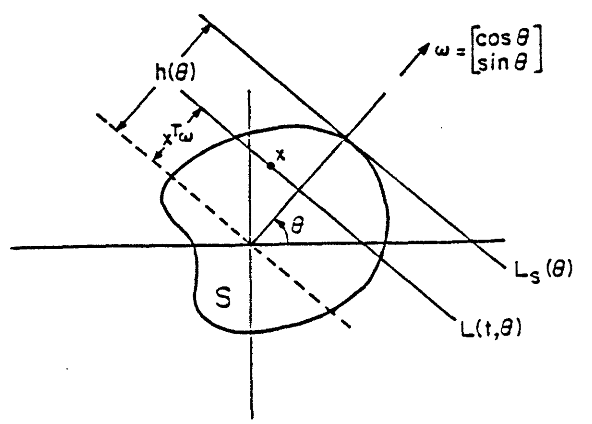

Figure 1. The geometry of support lines.

To see how a finite collection of support line measurements may not correspond to any object in the plane, consider Figure 2. Assume that the two support lines L_{i-1} and L_{i+1} are known perfectly. Then the set D_i must contain the true set S. Now suppose that the line L_i were measured to be to the left of (and parallel to) the dotted line. Then it is possible to construct a set $S \subset D_i$ which touches each of the three lines L_{i-1}, L_{i+1}, and L_i -- these lines are consistent. However, if L_i were measured to be to the right of the dotted line, then it is impossible to construct such a set -- these lines are inconsistent.

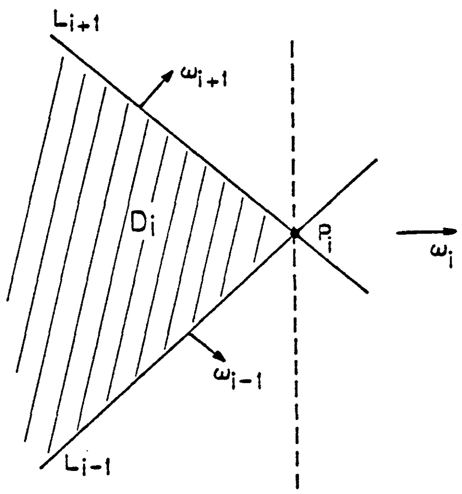

Figure 2. Support line L_i must be positioned to the left of the dotted line for consistency.

2.1 The Support Theorem

We now restrict the measurement angles to be evenly space over 2π, $\theta_i = 2\pi(i-1)/M$, $i=1,\ldots,M$, where $M \geq 5$. Our goal is to precisely characterize the consistency of measurements made at these angles. We will need the following definition in order to more compactly state the main result.

Definition: support vector.

A vector $h = [h_1 \ldots h_M]^T$ is a *support vector* if the lines

$$L_i = \{ u \in \mathbb{R}^2 \mid u^T\omega_i = h_i \} \qquad (2)$$

where $\omega_i = [\cos\theta_i \ \sin\theta_i]^T$ and $\theta_i = 2\pi(i-1)/M$, for $i=1,\ldots,M$, are support lines for some set $S \subset \mathbb{R}^2$.

The following theorem summarizes the consistency relations.

Theorem 1: The Support Theorem.

A vector $h \in \mathbb{R}^M$ $(M \geq 5)$ is a support vector if and only if

$$h^T C \leq [0 \ldots 0] \qquad (3)$$

where C is the MxM matrix given by

$$C = \begin{bmatrix} 1 & -k & 0 & \ldots & -k \\ -k & 1 & -k & & 0 \\ 0 & -k & 1 & & : \\ : & 0 & -k & & 0 \\ 0 & : & & \ldots & -k \\ -k & 0 & 0 & & 1 \end{bmatrix}$$

and $k = 1/2\cos(2\pi/M)$.

It is interesting to point out how Theorem 1 relates to the theory of continuous support functions $h(\theta)$. It is well known that, for $h(\theta)$ twice differentiable, a necessary and sufficient condition for $h(\theta)$ to be a support function is that it satisfy $h''(\theta) + h(\theta) \geq 0$ [6]. One immediately sees the similarity in the continuous support function constraint, $h''(\theta) + h(\theta) \geq 0$, and the discrete support vector constraint, $h^TC \leq 0$. In fact, it can be shown that in the limit as $M \to \infty$ the expression $h^TC \leq 0$ goes to $h''(\theta) + h(\theta) \geq 0$ [7]. The constraint $h^TC \leq 0$ is more fundamental than $h''(\theta) + h(\theta) \geq 0$, however, since the latter requires that the second derivative of h exist, which in turn implies that the set is convex and has continuously turning normals on its boundary. The discrete constraint, instead, applies for any bounded set in the plane.

Sketch of Proof: (See [7] for a complete proof.)

It is relatively straightforward to show the necessity of condition (3). By hypothesis, h is a support vector of some set S. Now consider the set D_i defined by the two support lines L_{i-1} and L_{i+1} as shown in Figure 2[1]. Note that by hypothesis $M \geq 5$, which implies that $\theta_{i+1}-\theta_{i-1} < \pi$. This in turn implies that the two lines L_{i-1} and L_{i+1} have a finite intersection point p_i, and that ω_i may be written as a *positive* combination of ω_{i-1} and ω_{i+1}. These two facts are necessary in order to conclude that the support value at angle θ_i for the set D_i is $p_i^T\omega_i$. Then, since $S \subset D_i$ we must have that $h_i \leq p_i^T\omega_i$. With a bit of algebraic manipulation, this inequality may be shown to be equivalent to the condition given by the i^{th} column of (3). This argument applies to

[1]When indexing support lines, support values, or unit normals, subscripts such as i+1 or i-1 are evaluated modulo M, so that the actual index value remains in the range $1,\ldots,M$.

each column independently, which shows the necessity of (3).

The proof of the sufficiency of (3) is more complicated. Here we assume we have a vector h which satisfies the conditions (3); we must show that h is a support vector. To do this we construct a set S for which h is a support vector. Consider the following two sets constructed from the vector h (see Figure 3):

$$S_B = \{ u \in \mathbb{R}^2 \mid u^T[\omega_1 \,\vdots\, \omega_2 \,\vdots\, \dots \,\vdots\, \omega_M] \leq [h_1\ h_2\ \dots\ h_M] \} \qquad (4)$$

and

$$S_\upsilon = \mathrm{hul}(\upsilon_1, \upsilon_2, \dots, \upsilon_M) \qquad (5)$$

where the υ_i's are points defined as the intersection of two lines (see (2))

$$\upsilon_i = L_i \cap L_{i+1} \qquad (6)$$

and hul(.) denotes the convex hull (of a set of points in this case).

Now suppose that $S_B = S_\upsilon$. Then it must be true that h is the support vector of the set $S = S_B = S_\upsilon$. To see this, first note from the definition of S_B in (4), that we must have

$$\sup_{x \in S} x^T\omega_i \leq h_i \quad .$$

On the other hand, $\upsilon_i \in S_\upsilon = S_B = S$ and $\upsilon_i^T\omega_i = h_i$. Consequently, h_i is the support value at this angle.

What remains to be shown, then is that condition (3) implies that $S_B = S_\upsilon$. The lengthy details of this part of the proof may be found in [7]. □

The immediate use of the support theorem is as a test of consistency: the vector h determines a consistent set of support lines only if $h^TC \leq 0$. From an estimation viewpoint, we see that any estimate $\hat{h}$ must satisfy $\hat{h}^TC \leq 0$.

2.2 The Geometry of the Support Cone

The convex polyhedral cone given by

$$\mathcal{C} = \{ h \in \mathbb{R}^M \mid h^TC \leq [0 \dots 0] \}$$

contains all M-dimensional support vectors. We call $\mathcal{C}$ the *support cone*. It is shown in [7] that, regardless of the dimension M, the matrix C is singular with a 2-dimensional nullspace spanned by the vectors

$$n_1 = [1 \ \ \cos\theta_0 \ \ \cos 2\theta_0 \ \ \dots \ \ \cos(M-1)\theta_0]^T$$

$$n_2 = [0 \ \ \sin\theta_0 \ \ \sin 2\theta_0 \ \ \dots \ \ \sin(M-1)\theta_0]^T \quad .$$

This implies that the support cone $\mathcal{C}$ is not a *proper cone*; i.e., there is a linear subspace (of dimension 2 in this case) contained entirely in $\mathcal{C}$. Thus, any support vector may be written as

$$h = h_p + h_n \quad ,$$

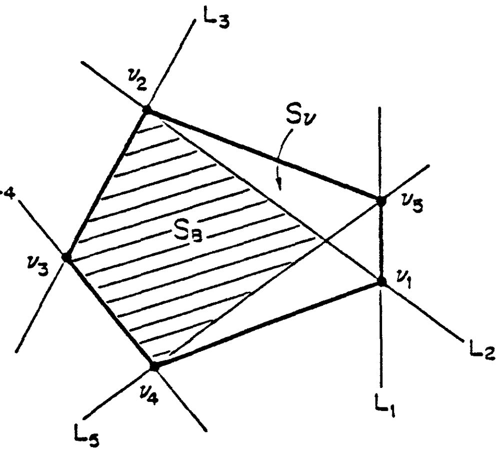

Figure 3. Sets S_B and S_υ and the vertex points $\upsilon_1,\dots,\upsilon_5$ for *inconsistent* lines $L_1,\dots,L_5$.

where h_p is in the proper cone

$$\mathcal{C}_p = \{ h \in \mathbb{R}^M \mid h^TC \leq 0,\ h^T[n_1 \,\vdots\, n_2] = 0\} \quad ,$$

and h_n is in the nullspace of C. We shall see in the following section that the nullspace component of a support vector has a very appealing geometric interpretation.

2.3 The Geometry of Implied Objects

Given a support vector h, there must exist a set S which has h as its support vector. This is true by definition. However, in general, h does not define a unique set S in the plane. But the unique *largest* set implied by h is S_B, defined in (4), which we call the *basic object*. Note that S_B may not be a good approximation to the true set S, but as M gets larger S_B becomes an increasingly better approximation to hul(S).

The extreme points of the basic object are given by the points $\upsilon_1,\dots,\upsilon_M$ in (6). Ordinarily, one would call these points vertices, but since the points υ_i are not necessarily distinct, we call them *vertex points*.

Now we characterize the boundary "smoothness". Suppose that in Figure 2, the line L_i were to pass through the intersection point p_i of L_{i-1} and L_{i+1}. Then one might say that the boundary of S_B is "sharp" at that point. As L_i moves off to the left of that point, the boundary is made "smoother". It can be shown, from the geometry that the distance from p_i to L_i is given by $\rho_i = -h^Tc_i$, where c_i is the i^{th} column of C. We call ρ_i the i^{th} *discrete radius of curvature*. Therefore, the entries of the vector $\rho = -h^TC$ measure the smoothness of the boundary of S_B. We use this idea in Section 3 to specify prior knowledge about object shape.

An interesting parallel exists between the discrete and continuous cases. It can be shown that $h''(\theta) + h(\theta)$ is the radius of curvature at a point on the envelope (curve) defined by the family of lines $h(\theta)$. From this, one is then naturally led to view the vector $-h^TC$ as an anolog of the radius of curvature simply because of the similarity in the statements of the two constraints.

Finally, we characterize the implied geometry with regard to the nullvectors of C. Notice that adding a nullvector h_n to h doesn't change the discrete radii of curvature vector ρ. This suggests that the shape of the basic object doesn't change either. In fact, it is not difficult to show that adding a nullvector $h_n = [n_1 \mid n_2]v$, $v \in \mathbb{R}^2$ to the support vector h, merely shifts the basic object of h by v.

The above paragraph defines a *relative* relationship between the position of two basic objects. We now observe that a useful definition of the *absolute* position of a basic object is the average position of its vertex points, $\bar{v}$. After some algebra we find that the support vector h is related to $\bar{v}$ by

$$\bar{v} \equiv \frac{1}{M}\sum_{i=1}^{M} v_i = \frac{2}{M}[n_1 \mid n_2]^T h$$

Note, in particular, that when h has no nullspace component the basic object is centered on the origin.

3.0 ALGORITHMS

We present three signal processing algorithms based on the ideas developed in Section 2. The basic idea is as follows. Suppose we obtain noisy measurements of M support values at the angles $\theta_i = 2\pi(i-1)/M$ for $i=1,\ldots,M$. It is likely in this case that the support measurement vector is not a feasible support vector. Therefore, a first objective of the following algorithms is to obtain a feasible support vector estimate from the measurements. The second objective of these algorithms is to use *a priori* information to guide the estimate toward "preferable" or "optimal" estimates.

3.1 The Closest Algorithm

Suppose the observed support values are given by

$$y_i = h_i + n_i, \quad i = 1,\ldots,M \qquad (7)$$

where h_i are elements of the true support vector which we are trying to estimate and n_i are samples of independent white Gaussian noise with zero mean and variance σ^2. In the absence of prior probabilistic knowledge concerning h we desire the maximum likelihood estimate of h given the data $y = [y_1 \ldots y_M]^T$ and subject to $h \in \mathscr{C}$. This estimate is given by

$$\hat{h}_{ML} = \underset{\substack{h \\ h^TC \leq 0}}{\operatorname{argmax}} \left(-\frac{1}{2}(y-h)^T(y-h)\right). \qquad (8)$$

We see that $\hat{h}_{ML}$ is the support vector in $\mathscr{C}$ which is closest (in the Euclidean metric) to the observation y; hence, the name Closest estimate. If y is in $\mathscr{C}$ then $\hat{h}_{ML} = y$; otherwise, the solution may be found by (efficient) quadratic programming (QP) methods (see, for example [8]).

3.2 The Mini-Max Algorithm

Suppose, as before, the measurements are given by (7), but in in this case, the n_i are independent, identically distributed noise samples, known to be uniform over the interval $[-\gamma,\gamma]$. Given the vector of measurements $y = [y_i\ldots y_M]^T$ and ignoring the support cone, the true support vector h, has equal probability of being anywhere in the hypercube

$$\mathfrak{B} = \{\, x \in \mathbb{R}^M \mid [-\gamma\ -\gamma\ \ldots\ -\gamma]^T \leq x - y \leq [\gamma\ \gamma\ \ldots\ \gamma]^T \,\}$$

and zero probability of being outside this hypercube. Of course, since we are looking for only feasible h's we insist that the estimate we produce is also in the support cone, hence, the estimate we seek will belong to the set $\mathfrak{B} \cap \mathscr{C}$.

The objective function for the Mini-Max estimate is chosen to reflect our prior knowledge: that the objects of interest tend to have smooth boundaries[2]. One way to achieve this tendency is to maximizes the minimum radius of curvature. Put together (and recalling our definition of discrete radius of curvature), the Mini-Max estimate is defined as

$$\hat{h}_{M-M} = \underset{\substack{h \\ h \in \mathfrak{B} \cap \mathscr{C}}}{\operatorname{argmax}} \min\{-h^Tc_1, -h^Tc_2, \ldots, -h^Tc_M\} \qquad (9)$$

where $c_1,\ldots,c_M$ are the columns of C.

The solution to (9) may be found by linear programming (LP) methods (see [9], for example); it is not necessarily unique, however. In fact, adding any nullvector to the solution does not change the cost, therefore, provided that the constraint is still met, there may be a family of shifted objects, each one corresponding to an optimal solution to (9).

3.3 The Close-Min Algorithm

In section 4 we shall see that the Closest algorithm and the Mini-Max algorithm produce two rather extreme estimates. In a way similar to

[2]Other prior knowledge may be used here to develop alternate algorithms.

maximum *a posteriori* (MAP) estimation [10], the Close-Min algorithm is designed to combine the two criteria, thus producing estimates that will reside somewhere between these two extremes.

The concept is simple: we define a new cost function which is a convex combination of the Closest and Mini-Max objective functions. Relating this to MAP estimation, we see that the Closest objective function plays the role of the measurement density and the Mini-Max objective function plays the role of the *a priori* density. Instead of an optimally defined trade-off between the two objective functions, as is the case in MAP estimation, we use the convexity parameter, α, which has a value between 0 and 1.

Recall that the Closest estimate maximizes

$$f_C(h) = -\frac{1}{2}(y-h)^T(y-h)$$

while the Mini-Max estimate maximizes

$$f_M(h) = \min\{-h^Tc_1, -h^Tc_2, \ldots, -h^Tc_M\} .$$

The Close-Min estimate is therefore defined as

$$\hat{h}_{CM} = \underset{\substack{h \\ h \in \mathfrak{S} \cap \mathfrak{C}}}{\operatorname{argmax}} \quad \alpha f_C(h) + (1-\alpha) f_M(h) \qquad (10)$$

where $0 \leq \alpha \leq 1$. The solution to (10) may be found using a QP algorithm, as before.

4.0 EXPERIMENTAL RESULTS

To show the behavior of the three algorithms, we use noise-corrupted measurements of the 10-dimensional support vector corresponding to a circle with radius 1/2, centered on the origin. The measurements are given by (7) where n_i are independent random variables, uniform over the range $[-\gamma,\gamma]$. To plot the data (for either the feasible support vectors or infeasible observations) we simply connect the vertex points $\{v_1\ v_2 \ldots v_M\}$ in sequence, producing a *vertex plot*. For a (feasible) support vector, this plot produces an outline of the basic object; however, for a (infeasible) measurement, the plot cross itself, clearly demonstrating the infeasibility.

Figures 4(a) and 5(a) show both the true basic object and the vertex plot for the measured vector in the case where $\gamma = 0.2$ and $\gamma = 0.4$, respectively. The set S_B, also shown in these figures, is the set given by (6), but in this case *for the measured vector*. S_B is *not* a basic object in this case (since the measurement is not a support vector); instead S_B is the set one would reconstruct if it were assumed that the measurements were completely accurate. In each case, one sees that there is at least one measured "support line" which does not support S_B, again demonstrating the infeasibility of the measurements.

Figures 4(b) and 5(b) show the basic objects corresponding to the support vector estimates produced by the three algorithms from the measurements shown in the respective (a) figures. The important observation to make here is that the Closest estimate strongly resembles the measurements, the Mini-Max estimate strongly resembles our prior expectations (large circular objects) and the Close-Min estimate "blends" these two outcomes.

It is also important to point out that the set S_B constructed from the raw measurements, is a bad estimate of the true set, in general. This is because the construction of S_B essentially ignores the support lines that are farthest out. Each of the algorithms proposed here use *all* of the measurements to "pull" the inner support lines out, if necessary.

5.0 DISCUSSION

We have seen that knowledge of the basic geometrical constraint, $h^TC \leq 0$, can be used to advantage when recontructing sets from noisy observations of their support lines. We have described and compared three algorithms which utilize this support constraint as well as other criteria and constraints. The primary contribution of this paper is in the formulation of the problem as a constrained optimization problem which includes the fundamental support vector constraint, *a priori* information, and uncertainty in the measurements. Many extensions are possible, both in the inclusion of additional constraints imposed by prior knowledge and in the development of more elaborate objective functions (see [7]).

ACKNOWLEDGEMENTS

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REFERENCES

[1] Schneiter J., "Automated Tactile Sensing for Object Recognition and Localization,", Ph.D. Diss. Mech. Engr., M.I.T., June 1986.

[2] Horn, B.K.P., *Robot Vision*, Cambridge, MA: MIT Press, 1986

[3] Humel, J.M., "Resolving bilinear data arrays", S.M. Thesis, E.E., M.I.T., 1986.

[4] Van Hove P.L. and Verly J.G., "A silhouette-slice theorem for opaque 3-D objects," ICASSP, March 26-29, 1985, pp.933-936.

[5] Herman G.T., *Image Reconstruction From Projections*, New York: Academic Press, 1980.

[6] Kelly P.J., Weiss M., *Geometry and Convexity -- a Study in Mathematical Methods*, New York: Wiley and Sons, 1979.

[7] Prince J.L., and A.S. Willsky, "Estimation algorithms for reconstructing a convex set given noisy measurements of its support

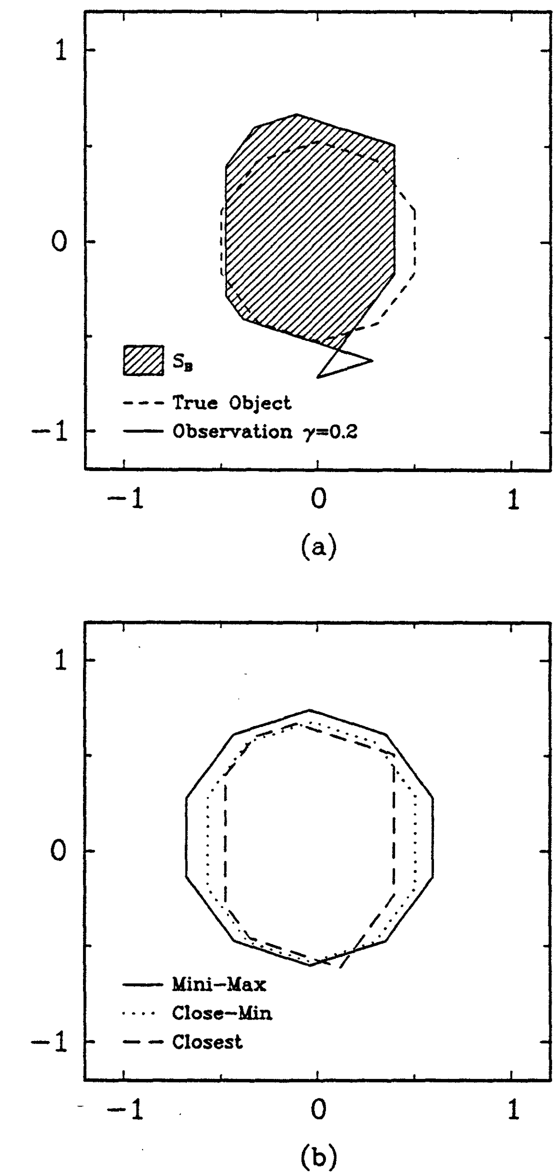

Figure 4. (a) The true object (circle) and the measured support vector (γ = 0.2). (b) The estimates produced by the three basic algorithms.

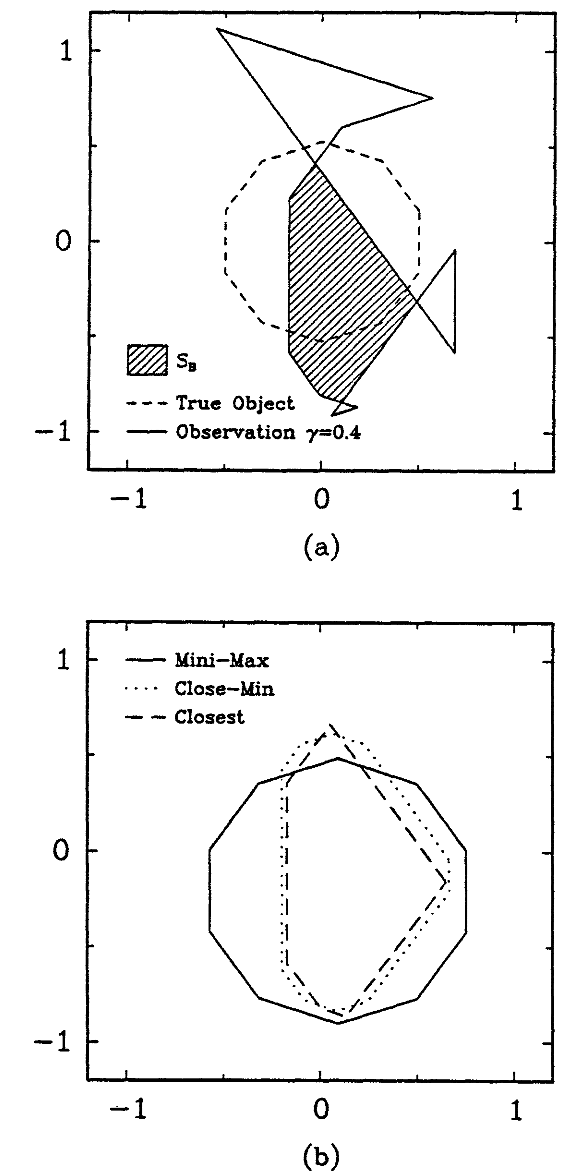

Figure 5. (a) The true object (circle) and the measured support vector (γ = 0.4). (b) The estimates produced by the three basic algorithms.

lines", M.I.T. LIDS Tech. Report, LIDS-P-1638, Jan. 1987.

[8] Land, A.H., and Powell, S., *Fortran Codes for Mathematical Programming*, London: Wiley-Interscience, 1973.

[9] Luenberger D.G., *Linear and Nonlinear Programming*, 2nd edition, Reading Massachusetts: Addison-Wesley, 1984.

[10] Van Trees H.L., *Detection, Estimation, and Modulation Theory, Part I*, New York: John Wiley and Sons, 1968.

[11] Greschak J.P., "Reconstructing Convex Sets", Ph.D. Diss. Elec. Engr., M.I.T., February 1985.

[12] Preparata, F.P. and Shamos, M.I., *Computational Geometry*, New York: Springer-Verlag, 1985.

[13] Leahy R.M. and Goutis, "An optimal technique for constraint based image restoration and reconstruction", IEEE Trans. ASSP, v.34, n.6, Dec. 1986.