

SOME RESULTS ON THE PROBLEM OF EXIT FROM A DOMAIN

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ABSTRACT

The problem of exit from a domain of attraction of a stable equilibrium point in the presence of small noise is considered for a restricted class of two dimensional systems. It is shown that for this class of systems, the exit measure is "skewed" in the sense that if S denotes the saddle point in the quasipotential towards which the exit measure collapses as the noise intensity goes to zero, then there exists an ϵ dependent neighborhood Δ of S such that $\lim P(\text{exit in } \Delta)/|\Delta| = 0$. Thus, the most probable exit point is not S but is rather skewed aside by ϵ^γ for some γ . The existence of such skewness, which was predicted by asymptotic expansions, depends on the ratio of normal to tangential forces around the saddle point.

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1. Introduction

The problem of exit from a domain D for dynamical systems in the presence of small white noise has received a lot of attention in the literature. Basically, two approaches have been used:

- a) Large deviations approach- initiated by Freidlin and Wentzell [7] and pursued by Kifer, Azencott, Kushner, Dupuis and Kushner, among others ([1,6,10,11]).
- b) Asymptotic expansions approach- initiated by Matkowsky and Schuss [12] and extended by them and others ([4,5,9,13,14]).

Typically, the large deviations approach has been fully rigorized, whereas the asymptotic expansion approach, though yielding sharper information, depends usually on a regularity assumption of the quasi-potential which is typically hard to check [4]. In many applications, one is interested in a domain D which is the basin of attraction of some stable point. The boundary of such a basin is called a "characteristic boundary". Note that for such a boundary, the dynamics on the boundary do not have a normal component. To such systems, one can associate a "quasipotential" ([7],[12]) which measures the cost to exit from a point on the boundary. Assume now that the quasipotential has a unique global minimum on the boundary which we denote by S . Under a controllability hypothesis, one can then show using large deviations theory that as the noise intensity converges to zero, the exit measure concentrates on any fixed neighborhood of S ([7],[6]). However, in the analysis of several such two dimensional systems, Bobrovsky and Schuss[2,3] have noticed that, using asymptotic expansions, one predicts that in the presence of "small but not too small" noise, the exit measure is not centered around the saddle point, but rather the most probable exit point is skewed aside. This prediction, which has been corroborated by numerical evidence [2,3], is due to the fact that in those systems, the asymptotic expansion yielded:

$$p_x(s) = \exp(-\psi(s)/\epsilon^2)(c_0(s) + \epsilon c_1(s) + \dots) \quad (1)$$

where s denotes the arclength along the boundary, $\psi(s)$ is the quasi-potential, $c_0(s), c_1(s), \dots$ are continuous functions (where $c_0(s)$ can be explicitly computed) and $p_x(s)$ is the density of the exit measure at s . In both examples treated in [2,3], and in many other examples, $c_0(S) = 0$. In those cases, the asymptotic expansions method predicts that the "most probable exit point" will be $O(\epsilon^\gamma)$, some $\gamma > 0$, away from the saddle point, in the sense that there is an ϵ dependent neighborhood N_ϵ of S from which the exit is much less probable than from the same neighborhood M_ϵ centered around some $s_\epsilon \neq S$ such that $N_\epsilon \cap M_\epsilon = \emptyset$. Eventhough this fact does not contradict the large deviations theory per se, it has lead to some controversy in recent years, due to the fact that these sharper results predicted by the asymptotic expansion method have never, to our knowledge, been rigorously proved, whereas the large deviations approach is not refined enough to show (or even hint at) these results. Note that if $c_0(S) \neq 0$, the most probable exit point predicted by the

asymptotic expansion method coincides with the large deviations limit.

In this paper, we examine a (very) restricted class of two dimensional systems for which $c_0(S) = 0$ and show rigorously, by probabilistic arguments, that for appropriate $\Delta(\epsilon)$, s_ϵ with $|s_\epsilon - S| > \Delta(\epsilon)$,

$$\lim_{\epsilon \rightarrow 0} \frac{\text{Probability of exit in a neighborhood } \Delta(\epsilon) \text{ around } S}{\text{Probability of exit in a neighborhood } \Delta(\epsilon) \text{ around } s_\epsilon} = 0 \quad (2)$$

for those systems (c.f. below theorem 3.1 for the exact statement), thus yielding the "skewing" of the exit measure alluded to earlier: the "most probable" exit point cannot therefore be S . Note that we do not claim that the density at S is zero: indeed, in principle for each $\epsilon > 0$ one could have a Dirac measure at S , however the total mass of such measure decreases to zero as $\epsilon \rightarrow 0$ faster than in other places on the boundary. Thus, we are able to show only a weaker statement than that of the full asymptotic expansion approach, for we do not compute explicitly where the "most probable exit point" lies (and we don't even show that in the sense of maximizing densities it is not S). We merely point out the skewing property described in (2).

We remark that the method used undoubtly can be generalized to a wider class of models than we consider here, however it will not be done here.

We conjecture a critical behavior the skewing in (2): only above a certain threshold associated with the ratio of normal to tangential forces around the saddle point S are we able to show that (2) holds.

We finally mention that a different (and more general) approach to rigorizing the asymptotic expansions approach is presented in [4,5]. Unfortunately, it seems that the result of both those papers do not yield the kind of tight estimates we seek.

The organization of this paper is as follows: in section 2 below we describe our model. In section 3 we prove our main result and establish a somewhat stronger version of it (theorem (3.1) and corollary (3.1)). A proof of a one dimensional auxiliary result is deferred to the appendix.

2. Model Description

We consider the two dimensional diffusion model:

$$dx_t = f(x_t)dt + \epsilon Bdw_t \quad (3)$$

where (w_t^1, w_t^2) are two independent Brownian motions and f is Lipschitz continuous. We assume that the function f and the matrix B satisfy the following restrictive conditions:

- A-1) 0 is a stable point of (3), and its domain of attraction, denoted D , has a smooth boundary, denoted ∂D . In general, this domain may be unbounded.
- A-2) Along ∂D , f can be decomposed to tangential and normal coordinates. The tangential flow along the boundary has a unique stable point denoted by S .

A-3) In local coordinates along ∂D , (3) may be rewritten in a strip of width δ around ∂D (which we denote by $\partial_L D$) as:

$$ds_t = g(s_t, n_t)dt + \epsilon d\tilde{w}_t^1 \quad (\text{tangential component}) \quad (4)$$

$$dn_t = kn_t dt + \epsilon d\tilde{w}_t^2 \quad (\text{normal component}) \quad (5)$$

with \tilde{w}^1, \tilde{w}^2 being two independent Brownian motions, and

$$g(0, 0) = 0$$

$$g(s, n) \geq -gs, \quad g > 0, \quad s > 0, \quad \text{at least in a small neighborhood of } (0, n_t)$$

$$\text{sign}(g(s, n)) = - \text{sign}(s), \quad \text{at least in a small neighborhood of } (0, n_t)$$

$$k > 0$$

i.e., the tangential flow has $s = 0$ as a stable point with attracting force, for $s > 0$, smaller than gs , and the normal flow has 0 as an unstable equilibrium with repulsive force kn .

Note that (4),(5) are strong restrictions: the noises in the normal and tangential directions have to be independent in the neighborhood of the boundary and the normal drift near the boundary is linear. The removal of those restrictions is subject to further study.

Following [7], let us define the quasipotential $V_a(y)$ as:

$$V_a(y) = \inf_{(\phi \in W^{1,2} | \phi_0 = a, \phi_T = y)} \int_0^T (\dot{\phi} - f(\phi))^T (BB^T)^\# (\dot{\phi} - f(\phi)) dt$$

where $(BB^T)^\#$ denotes the pseudo inverse of the matrix BB^T . As usual, $V_a(\partial D) \triangleq \inf_{y \in \partial D} V_a(y)$, and we will use $V_0(y)$ to denote the quasipotential starting from $a = 0$. Let now ∂_s denote the boundary of a strip of width δ_1 inside ∂D , with $\delta_1 \leq 0.1\delta$, $|V_0(\partial D) - V_0(\partial_s)| < 0.05V_0(\partial D)$ and $k\delta_1 < 0.05V_0(\partial D)$ (c.f. fig. 1). We further assume that:

A-4) $V_0(y)$ has a unique nonzero minimum for $y \in \partial_s$ with local coordinates (s_0, δ_1) such that $s_0 \neq 0$. W.l.o.g., we assume $s_0 > 0$. In addition, $V_0(y)$ has as a unique minimum on ∂D , which by A-3) must be, in local coordinates, $S = (0, 0)$.

Note that (A-4), unlike (A-1)-(A-3), is a global assumption, for it involves the behavior of the system inside the whole domain D but not around the boundary. In many problems of interest, (A-4) is satisfied - c.f. e.g. [2], [6].

Remark: In some examples, the quasi-potential has a few separated global minima and the system is symmetric w.r.t. those global minima. The PLL's in [2] and [6] are examples of such situations. Those examples can be treated along the lines described here, although technically they do not satisfy (A-4) for they have

multiple minima.

Finally, we need two one dimensional result , which we state below as two lemmas:

Lemma 2.1 Consider the following one dimensional linear stochastic equation having an unstable equilibrium at $\theta = 0$:

$$d\theta_t = k\theta_t dt + \epsilon dw_t, \theta(0) = \theta_0 < 0, k > 0 \quad (6)$$

and define

$$\tau(\theta_0) \hat{=} (\inf q \mid \theta_q = 0) \quad (\text{first origin hitting time starting at } \theta_0) \quad (7)$$

$$P(t, \theta_0) \hat{=} \text{Prob}(\tau(\theta_0) \leq t) \quad (\text{exit time distribution}) \quad (8)$$

$$\mu(t) \hat{=} \frac{(2k)^{1/2}}{\epsilon} \left(\frac{e^{2kt}}{e^{2kt} - 1} \right)^{1/2} \quad (9)$$

Then

$$P(t, \theta_0) = 2\Phi(-\theta_0\mu) \quad (10)$$

where

$$\Phi(x) \hat{=} (1/\sqrt{2\pi}) \int_x^\infty \exp(-\theta^2/2) d\theta$$

Proof of Lemma 2.1 see the appendix.

Lemma 2.2 Let \tilde{s}_t be an Ohrenstein-Uhlenbeck process starting at $\tilde{s}_0 = s(x) > 0$ and stopped at 0, governed by the generator:

$$\mathcal{L} = \frac{\epsilon^2 \partial^2}{2\partial x^2} - gx \frac{\partial}{\partial x}$$

Then, for all $y > 0$,

$$\begin{aligned} p(\tilde{s}_t \in dy) &= \frac{\sqrt{2g}}{\sqrt{2\pi(1 - \exp(-2gt))}\epsilon} \\ &\quad \left[\exp\left(-\frac{g}{\epsilon^2(1 - \exp(-2gt))}(y - s(x)\exp(-gt))^2\right) - \right. \\ &\quad \left. \exp\left(-\frac{g}{\epsilon^2(1 - \exp(-2gt))}(y + s(x)\exp(-gt))^2\right) \right] dy \quad (11) \end{aligned}$$

Proof of Lemma 2.2 By substitution. Note that $p_{\tilde{s}_t}(0) = 0$ and thus satisfies the absorbing boundary conditions.

3. Exit measure bounds

In this section we will prove the main result of this paper, namely:

Theorem 3.1 Assume that $k > 3g/2$. Then, for any $\alpha > 0$ such that $1 > \alpha > \frac{2}{2k/g-1}$, and for $\Delta = ((-d, 0), (\epsilon^\alpha, 0))$, where d is some positive constant independent of ϵ ,

$$\lim_{\epsilon \rightarrow 0} P(x_{\tau_0} \in \Delta) / \epsilon^\alpha = 0$$

Moreover, there exists a $s_\epsilon \in \partial D$ such that $s_\epsilon > 2\epsilon^\alpha$ and

$$\lim_{\epsilon \rightarrow 0} \frac{P(|x_{\tau_0} - s_\epsilon| < \epsilon^\alpha)}{P(|x_{\tau_0}| < \epsilon^\alpha)} = \infty$$

Finally, $P(|x_{\tau_0} - 2\epsilon^\alpha| > |x_{\tau_0}|) \rightarrow_{\epsilon \rightarrow 0} 0$

Note that theorem 3.1 implies that the most probable exit point (in the sense of an exit from a ϵ^α neighborhood of it) cannot occur at S , in agreement with the predictions of [2,3].

Proof of theorem 3.1: Let $\partial_L D$ be as defined above in (A-3). Let E denote a neighborhood of 0, whose radius is such that

$$\sup_{y \in \partial E} V_0(y) < \min(0.1 \inf_{y \in \partial_L D} V_0(y), 0.1(V_0(0, \delta_1) - V_0(s_0, \delta_1)))$$

Such a neighborhood always exists since $V_0(y)$ is Lipschitz and $V_0(0) = 0$. We define the following quantities:

$$\begin{aligned} \tau_x &\triangleq \inf_{t>0} (t | x(t) \in \partial D, x(0) = x) && \text{first exit time from } \partial D \\ \tau_x^E &\triangleq \inf_{t>0} (t | x(t) \in \partial E, x(0) = x) && \text{first exit time from } \partial E \\ \tau_x^L &\triangleq \inf_{t>0} (t | x(t) \in \partial_L D, x(0) = x) && \text{first exit time from } \partial_L D \\ \tau_x^s &\triangleq \inf_{t>0} (t | x(t) \in \partial_s, x(0) = x) && \text{first exit time from } \partial_s \end{aligned}$$

Define further

$$\begin{aligned} A_x &\triangleq \text{Prob}(x_{\tau_x} \in \Delta) \\ B_x &\triangleq \text{Prob}(x_{\tau_x} \in \Delta \wedge \tau_x < \tau_x^L) \\ C_x &\triangleq \text{Prob}(\tau_x^E < \tau_x) \\ D_x &\triangleq \text{Prob}(x_{\tau_x} \in \Delta \wedge \tau_x < \tau_x^E | \tau_x > \tau_x^L) \end{aligned}$$

We will use $A_x - D_x$ above for $x \in \partial_s$. Let $P_\mu^\epsilon(s)$ denote the exit measure from ∂_s starting from $\mu \in \partial E$, i.e. $P_\mu^\epsilon(s) \triangleq \text{Prob}(s(x_{\tau_\mu}) < s)$. Define

$$A \triangleq \sup_{\mu \in \partial E} \int_{\partial_s} dP_\mu^\epsilon(x) A_x$$

We claim that:

Lemma 3.1 There exists a constant c_1 such that

$$A \leq \exp(-c_1\delta_1/2\epsilon^2) + \frac{\sup_{\mu \in \partial E} \int_{\partial_s} B_x dP_\mu^\epsilon(x)}{\inf_x P(\tau_x \leq \tau_x^E)} \quad (12)$$

Proof of lemma 3.1: Note first that for $x \in \partial_s$,

$$\begin{aligned} A_x &= B_x + P(x_{\tau_x} \in \Delta \mid \tau_x > \tau_x^L)P(\tau_x > \tau_x^L) \\ &= B_x + P(x_{\tau_x} \in \Delta \mid \tau_x > \tau_x^E)P(\tau_x > \tau_x^E \mid \tau_x > \tau_x^L)P(\tau_x > \tau_x^L) \\ &\quad + P(x_{\tau_x} \in \Delta \wedge \tau_x < \tau_x^E \mid \tau_x > \tau_x^L)P(\tau_x > \tau_x^L) \\ &= B_x + P(x_{\tau_x} \in \Delta \mid \tau_x > \tau_x^E)P(\tau_x > \tau_x^E) + D_x P(\tau_x > \tau_x^L) \end{aligned} \quad (13)$$

Integrating w.r.t. P_μ^ϵ , one obtains:

$$P(x_{\tau_x} \in \Delta \mid \tau_x > \tau_x^E) \leq A \quad (14)$$

Substituting (14) in (13), one obtains:

$$\begin{aligned} A(1 - \sup_{x \in \partial_s} P(\tau_x > \tau_x^E)) &\leq \sup_{\mu \in \partial E} \int_{\partial_s} B_x dP_\mu^\epsilon(x) \\ &\quad + \sup_{\mu \in \partial E} \int_{\partial_s} D_x P(\tau_x > \tau_x^L) dP_\mu^\epsilon(x) \end{aligned} \quad (15)$$

Using the estimates of [7], one has:

$$1 - \sup_{x \in \partial_s} P(\tau_x > \tau_x^E) \geq c_2 \exp(-1.1k\delta_1/\epsilon^2) \quad (16)$$

Also, note that the last term in (15) is bounded by $\sup_{x \in \partial_L D} P(\tau_x < \tau_x^E)$, which in turn is bounded by $c_3 \exp(-0.9k\delta/\epsilon^2)$. Combining the above, one has the lemma.

Lemma (3.1) enables us to reduce the computations essentially to computations related to the linear one dimensional system (6), for B_x depends only on the dynamics in the strip δ . Note also that $P(\tau_x \leq \tau_x^L)$ is bounded above by $P(\infty, \delta_1)$ in lemma (2.1).

We turn now to the computation of the integral in (12). Note that by a standard comparison theorem [8, ch. 6.1], one has that $P(s_t \in \Delta) \leq P(\tilde{s}_t \in \Delta)$, where \tilde{s}_t was defined in lemma 2.2 and we define $\tilde{s}_t = 0$ once \tilde{s}_t had been stopped. In the sequel, \tilde{B}_x will denote the expression for B_x with $\tilde{s}_t < \epsilon^\alpha$ replacing $s_t \in \Delta$. Clearly, $\tilde{B}_x \geq B_x$ for $s(x) \in (s_0/2, 3s_0/2)$. One has therefore for such $s(x)$:

$$B_x \leq \int_0^\infty dP(t, \delta_1) P(\tilde{s}_t < \epsilon^\alpha \mid s_0 = s(x)) \quad (17)$$

For any ξ , one also has

$$\begin{aligned} \int_0^\infty dP(t, \delta_1) P(\tilde{s}_t < \epsilon^\alpha \mid s_0 = s(x)) &\leq \\ (P(\infty, \delta_1) - P(\xi, \delta_1)) + P(\xi, \delta_1) P(\tilde{s}_\xi < \epsilon^\alpha \mid s_0 = s(x)) &\end{aligned} \quad (18)$$

Take $\xi = \ln(s(x)/\epsilon^\beta)/g$, where $\beta < \alpha$ will be chosen later. Substituting ξ in (10), one obtains, for ϵ small enough,

$$\begin{aligned} P(\infty, \delta_1) - P(\xi, \delta_1) &= \frac{2}{\sqrt{2\pi}} \int_{\mu\delta_1}^{\mu\delta_1(\frac{\eta}{\eta-1})^{1/2}} e^{-\theta^2/2} d\theta \\ &\leq \frac{2}{\sqrt{2\pi}} \mu\delta_1 \left(\left(\frac{\eta}{\eta-1} \right)^{1/2} - 1 \right) e^{-\mu^2\delta_1^2/2} \end{aligned} \quad (19)$$

where

$$\mu = \sqrt{2k}/\epsilon \quad \text{and} \quad \eta = \exp(2k\xi) = \left(\frac{s(x)}{\epsilon^\beta} \right)^{2k/g}$$

and

$$\begin{aligned} P(\infty, \delta_1) &= \frac{2}{\sqrt{2\pi}} \int_{\mu\delta_1}^{\infty} e^{-\theta^2/2} d\theta \\ &\geq \frac{1}{\sqrt{2\pi}(1 + \mu\delta_1)} e^{-\mu^2\delta_1^2/2} \end{aligned} \quad (20)$$

Therefore, for ϵ small enough,

$$\begin{aligned} \frac{P(\infty, \delta_1) - P(\xi, \delta_1)}{P(\infty, \delta_1)} &\leq c_2(1 + \mu\delta_1)\mu\delta_1 \left(\left(\frac{\eta}{\eta-1} \right)^{1/2} - 1 \right) \\ &\leq c_3\mu^2\delta_1^2 \left(\left(\frac{\eta}{\eta-1} \right)^{1/2} - 1 \right) \\ &\leq c_4\mu^2\delta_1^2/\eta \end{aligned} \quad (21)$$

On the other hand,

$$\begin{aligned} \frac{P(s_\xi \in \Delta \mid s_0 = s(x))P(\xi, \delta_1)}{P(\infty, \delta_1)} &\leq P(s_\xi \in \Delta \mid s_0 = s(x)) \\ &\leq P(\tilde{s}_\xi < \epsilon^\alpha \mid \tilde{s}_0 = s(x)) \end{aligned}$$

Let $\mu_1 = \sqrt{2g}/\epsilon$. Using lemma 2.2, one has that

$$\begin{aligned} P(\tilde{s}_\xi < \epsilon^\alpha \mid \tilde{s}_0 = s(x)) &= P(0 < \tilde{s}_\xi < \epsilon^\alpha \mid \tilde{s}_0 = s(x)) + P(\tilde{s}_\xi = 0 \mid \tilde{s}_0 = s(x)) \\ &= \frac{\mu_1}{\sqrt{2\pi}(1 - \frac{\epsilon^{2\beta}}{s(x)^2})^{1/2}} \int_0^{\epsilon^\alpha} \left(\exp\left(\frac{-\mu_1^2(y - \epsilon^\beta)^2}{2(1 - \epsilon^{2\beta}/s(x)^2)}\right) - \exp\left(\frac{-\mu_1^2(y + \epsilon^\beta)^2}{2(1 - \epsilon^{2\beta}/s(x)^2)}\right) \right) dy \\ &\quad + \left(1 - \frac{\mu_1}{\sqrt{2\pi}(1 - \frac{\epsilon^{2\beta}}{s(x)^2})^{1/2}} \int_0^{\infty} \left(\exp\left(\frac{-\mu_1^2(y - \epsilon^\beta)^2}{2(1 - \epsilon^{2\beta}/s(x)^2)}\right) - \exp\left(\frac{-\mu_1^2(y + \epsilon^\beta)^2}{2(1 - \epsilon^{2\beta}/s(x)^2)}\right) \right) dy \right) \\ &\leq 3 \frac{\mu_1}{\sqrt{2\pi}(1 - \frac{\epsilon^{2\beta}}{s(x)^2})^{1/2}} \int_{\epsilon^\beta - \epsilon^\alpha}^{\infty} \exp\left(\frac{-\mu_1^2 y^2}{2(1 - \epsilon^{2\beta}/s(x)^2)}\right) dy \end{aligned} \quad (22)$$

Combining (17),(18), (21) and (22), one obtains:

$$\frac{\int_{\partial_s} B_x dP_\mu^\epsilon(x)}{P(\infty, \delta_1)} \leq c_5 \epsilon^{\frac{2k\beta}{g} - 2} + c_6 \exp(-c_7 \epsilon^{2\beta - 2}) + P_\mu^\epsilon(x_{\tau_0^\epsilon} \notin (s_0/2, 3s_0/2)) \quad (23)$$

Finally, note that $P(\tau_x \leq \tau_x^E)/P(\infty, \delta_1)$ converges to 1 as $\epsilon \rightarrow 0$, and that by the usual Freidlin-Wentzell estimates, there exist constants c_8, c_9 such that

$$P_\mu^\epsilon(x_{\tau_0} \notin (s_0/2, 3s_0/2)) \leq c_8 \exp(-c_9/\epsilon^2) \quad (24)$$

Combining (12), (23) and (24) and taking $1 > \alpha > \beta > \frac{2}{2k/g-1}$ (assuming $2k/g > 3$), one obtains:

$$A \leq c_{10} \exp(-c_{11}/\epsilon^2) + c_{12} \epsilon^{2k\beta/g}/\epsilon^2 + c_{13} \exp(-c_{14} \epsilon^{2\beta-2}) \quad (25)$$

We conclude that if $k/g > 3/2$ then

$$\lim_{\epsilon \rightarrow 0} A/\epsilon^\alpha = 0$$

which proves the first part of theorem 3.1. The last part of the theorem follows from the fact that from the large deviations results, the exit occurs in a fixed neighborhood of S with probability approaching 1. Let this neighborhood be taken arbitrarily as $(-d, d)$. Then there exists a point $s_\epsilon \in (-d, d)$ such that $P(x_{\tau_0} \in (s_\epsilon - \epsilon^\alpha, s_\epsilon + \epsilon^\alpha))/\epsilon^\alpha \geq 1$. By the first part of the theorem, noting that the theorem holds for $c\epsilon^\alpha$, any positive fixed c , this s_ϵ must satisfy, for ϵ small enough, $s_\epsilon > 2\epsilon^\alpha$, from which the ratio result follows. Finally, note that

$$P(|x_{\tau_0} - 2\epsilon^\alpha| > |x_{\tau_0}|) \leq P(x_{\tau_0} \notin (-d, d)) + P(x_{\tau_0} \notin (-d, \epsilon^\alpha)) \rightarrow_{\epsilon \rightarrow 0} 0$$

which concludes the proof of the theorem.

We also obtain the following corollary from theorem 3.1:

Corollary 3.1 Assume that in (4) $\tilde{w}_t^1 = 0$, i.e. that there is no noise in the tangential flow in a strip around ∂D . Then the conclusion of theorem 3.1 still holds if $k/g > 1/2$ for all $\alpha > \frac{2}{\frac{2k}{g}-1}$.

Proof of the Corollary To see corollary 3.1, note that if noise is not present in the tangential flow, the last term in (25) disappears. Choose now $\alpha > \frac{2}{\frac{2k}{g}-1}$, and the corollary follows.

Remark: We can use corollary 3.1 in the analysis of the tracking system described in [3]: Consider the system analyzed there:

$$dx_t = (-\beta y_t - H(x_t))dt - \epsilon dw_t \quad (26)$$

$$dy_t = (-a + H(x_t))dt + \epsilon dw_t \quad (27)$$

where

$$H(x) = \begin{cases} 3x & \text{if } |x| < 1/3 \\ \frac{3}{2}(1-x) & \text{if } 5/3 > x > 1/3 \\ \frac{3}{2}(1+x) & \text{if } -5/3 < x < -1/3 \end{cases}$$

We take $a = 0$ throughout. Note that $S = (-1, 0)$, and that around S , ∂D is a straight line.

Let now

$$\alpha = \frac{3}{4\beta}(1 + \sqrt{1 + 8\beta/3})$$

Using the change of coordinates

$$\begin{pmatrix} s \\ n \end{pmatrix} = \frac{1}{\sqrt{1 + \alpha^2}} \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} y \\ x + 1 \end{pmatrix}$$

one can check that n, s are, respectively, the normal and tangential coordinates around S . Choosing $\beta = 3$, which corresponds to a damping factor of $1/2$ for the system (26,27), one obtains a system of the type (4),(5) with $g(s, n) = -\frac{3s}{2} + \frac{3n}{2}$, $k = 3$ and $\tilde{w}_t^1 = 0$. Thus, corollary 3.1 applies and yields results which agree with [3].

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Appendix

Proof of lemma 2.1 Let $p(x, dy, t)$ denote the transition kernel of the diffusion (6). Note that $p(0, dy, t) = p(0, -dy, t)$. The proof is deduced from this symmetry and the D.Andre principle as follows:

Let $P_c(t)$ denote the probability of a zero level crossing in $(0, t)$, and let $P_{nc}(t) = 1 - P_c(t)$. Note that

$$p(\theta_0, dx, t) = p(\theta_0, dx, t | nc)P_{nc}(t) + p(\theta_0, dx, t | c)P_c(t)$$

By the symmetry of the kernel and the strong Markov property, one has, for $\text{sign}(\theta_0) = \text{sign}(x)$,

$$P_c p(\theta_0, dx, t | c) = p(\theta_0, -dx, t)$$

Combining the above equations, one has:

$$p(\theta_0, dx, t) = p(\theta_0, dx, t | nc)P_{nc}(t) + p(\theta_0, -dx, t)$$

Let $P_+ \hat{=} \text{Prob}(\theta_t > 0 | \theta(0) = \theta_0)$. Assuming $\theta_0 < 0$, one has by integrating the above equality

$$P_c = 2P_+$$

Substituting the kernel $p(\theta_0, dx, t)$, one obtains readily the lemma.

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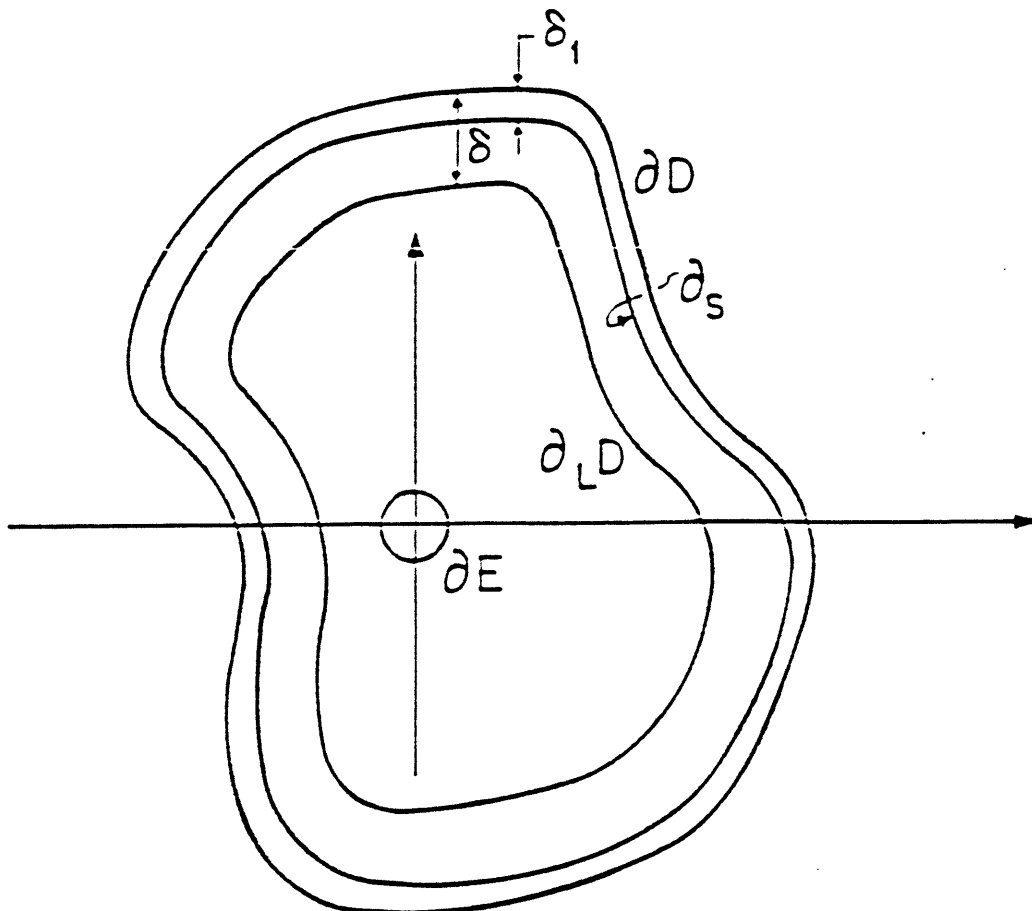


Figure 1. Neighborhoods definitions.

Abbreviated Title: Exit Problem