K-Theory and Index Theory on Manifolds with Boundary

by

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Abstract

Let X be a manifold with corners and denote by $M_1(X)$ the set of boundary hypersurfaces in X. If $D \subset M_1(X)$ then, by using the calculus of b-pseudodifferential operators, we define quantization maps from the K-cohomology groups $K^*(T^*X, T^*X|_D)$ to the analytic K-homology groups $K_*(X, \complement D)$, where $\complement D = M_1(X) \setminus D$. We prove that these maps are always isomorphisms realizing Poincaré duality.

We then specialize to manifolds with boundary where we extend various results of Atiyah, Patodi and Singer. First we study the asymmetry of the boundary spectrum of an elliptic b-pseudodifferential operator, A, by introducing an appropriate eta function and investigating its meromorphic properties; in particular we prove that $s = 0$, an a priori pole, is a regular point. Next we connect the boundary spectral asymmetry of the operator A to its index as a Fredholm map between r-weighted Sobolev spaces; thus we express the index as the sum of the value at $s = 0$ of the eta function described above and the value at $s = 0$ of the difference of the b-zeta functions associated to A^*A and AA^* respectively :

$$\mathrm{ind}_r(A) = -\eta_r(0, I(A)) + {}^b\zeta(A^*A) - {}^b\zeta(AA^*)$$

For generalized Dirac operators this formula gives the Atiyah-Patodi-Singer index theorem.

Thesis Supervisor: Richard B. Melrose

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Contents

1 INTRODUCTION

"La filosofia e' scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si puo' intendere se prima non s'impara a intendere la lingua e conoscere i caratteri ne' quali e' scritto. Egli e' scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezi e' impossibile a intenderne umanamente parola; senza questi e' un aggirarsi vanamente per un oscuro laberinto"

Galileo Galilei, *Saggiatore*

The goal of this thesis is to employ the theory of b-pseudodifferential operators [**32**],[**33**] to investigate various questions in analytic K-homology and index theory on manifolds with boundary.

It is well known that on a compact manifold without boundary, Y, an elliptic pseudodifferential operator acting between the smooth sections of two vector bundles defines a Fredholm operator. The Atiyah-Singer index theorem computes the index of such an operator in terms of characteristic classes associated to its principal symbol and to the manifold Y. There are two rather different approaches to the index formula of Atiyah and Singer given, respectively, by K-theory and the asymptotics of the heat equation (or, equivalently, the meromorphic properties of the zeta function)

In the original K-theoretic approach [**6**][**7**] the fundamental step is the realization of the analytic index of the elliptic operators as a homomorphism from the K-cohomology ring of the cotangent bundle of Y, $K^0(T^*Y)$, to the integers. The connection between $K^0(T^*Y)$ and pseudodifferential operators is established through the symbolic properties of the latter ; indeed for any $\alpha \in K^0(T^*Y)$ we can find a representative cycle given by $(\pi^* E_1, \pi^* E_2, \sigma_{pr}(P))$ where $\sigma_{pr}(P)$ is the principal symbol of some elliptic pseudodifferential operator, P, acting between the sections of two vector bundles E_1 and E_2 and π denotes the projection from $T^*(Y)$ to Y.

The connection between K-theory and elliptic operator was further clarified by Atiyah in his foundational paper [**1**]. There, an analytic definition of the K-homology groups $K_*(Y)$, defined abstractly as the dual theory to the Atiyah-Hirzebruch K(cohomology)-theory, was proposed. These ideas were later developed by Kasparov, [**28**], and Brown, Douglas and Fillmore, [**16**]. Cycles for this analytic K–homology theory are 'abstract elliptic operators,' i.e. bounded linear operators between Hilbert spaces satisfying additional conditions derived from the properties of elliptic (pseudo–)differential operators. In particular elliptic pseudodifferential operators between sections of vector bundles define such cycles. The homology class of the cycle only depends on the principal symbol and indeed only on the K–cohomology class it represents on the cotangent bundle. In fact it had already been shown, in essence, by Atiyah that the resulting map

$$K^i(T^*Y) \longrightarrow K_i(Y) \tag{1.1.1}$$

is an isomorphism, realizing Poincaré duality. We think of (1.1.1) as a 'quantization map' since it arises by representing a K–cohomology class as a symbol and then quantizing the symbol, i.e. taking an operator with this symbol. This result was discussed in detail by Baum and Douglas in [**12**]. Different proofs, relying on the full power of the bivariant KK–theory of Kasparov, were also given in [**29**] and [**18**].

More recently the K–homology groups of a compact manifold with boundary have been considered by various authors [**9**], [**13**], [**23**] and [**24**]. The analogue of (1.1.1) for the absolute K–homology groups is the isomorphism

$$K^i(T^*X, T^*_{\partial X}X) \longrightarrow K_i(X) \tag{1.1.2}$$

which is realized as a 'quantization map' through pseudodifferential operators which are trivial, i.e. bundle isomorphisms, near the boundary.

In the second chapter of this thesis we show that the quantization map for b–pseudodifferential operators similarly gives an explicit realization of Poincaré duality both for (1.1.2) and the relative K–homology of a compact manifold with boundary (or corners)

$$K^i(T^*X) \longrightarrow K_i(X, \partial X). \tag{1.1.3}$$

This is joint work with R.Melrose [**36**].

The definition of (1.1.3) is based on the properties of the space of b–pseudodifferential operators on X, acting between sections of any two smooth vector bundles E and F :

$$A \in \Psi_b^m(X; E, F) \Longrightarrow A : C^\infty(X; E) \longrightarrow C^\infty(X; F).$$

The subscript b identifies these operators as obtained by 'microlocalization' of the Lie algebra, similarly denoted $\mathcal{V}_b$, of C^∞ vector fields on X tangent to the boundary. There is an associated vector bundle ${}^bTX$ which is isomorphic (but not naturally isomorphic) to TX of which $\mathcal{V}_b$ forms the space of all sections. The symbol map for these pseudodifferential operators

$$\sigma_m : \Psi_b^m(X; E, F) \longrightarrow S^{[m]}({}^bT^*X; \mathrm{Hom}(\pi^*E, \pi^*F))$$

identifies the symbol as an equivalence class of symbols on the dual (compressed cotangent) bundle with values in the homomorphisms of the lifted bundles. Ellipticity has the usual meaning, namely invertibility of the symbol outside the zero section. This "small" calculus of b-pseudodifferential operators can be extended to any manifold with corners X and we will in fact prove that for each boundary hypersurface B in X the quantization map (1.1.3) induces isomorphisms

$$K^i(T^*X, T_B^*X) \longrightarrow K_i(X, \complement B) \tag{1.1.4}$$

where $\complement B = M_1(X) \backslash B$, $M_1(X)$ denoting the set of all boundary hypersurfaces.

It follows from (1.1.3) that any elliptic differential operator on the manifold with boundary X defines a relative K-class. This generalizes and simplifies results of Baum, Douglas and Taylor, [**13**], who showed that an elliptic boundary problem defines a relative K-homology class independent of the boundary conditions. Indeed, by (1.1.3), all elements in $K_*(X, \partial X)$ can be represented by cycles defined by elliptic b-pseudodifferential operators on X.

One useful consequence of this identification is that the boundary maps in the long exact sequence in K-homology for a compact manifold relative to its boundary:

$$\begin{array}{ccccc} K_0(\partial X) & \xrightarrow{\iota_*} & K_0(X) & \longrightarrow & K_0(X, \partial X) \\ {\scriptstyle \partial_1}\uparrow & & & & \downarrow{\scriptstyle \partial_0} \\ K_1(X, \partial X) & \longleftarrow & K_1(X) & \xleftarrow{\iota_*} & K_1(\partial X). \end{array} \tag{1.1.5}$$

can also be realized analytically. The inward pointing half of the normal bundle to the boundary, $N_+\partial X$, is a trivial $\mathbf{R}^+$-bundle over ∂X. It has a natural compactification as a bundle, $\overline{N_+\partial X} \cong [-1,1] \times \partial X$ of closed intervals such that the $\mathbf{R}^+$-action, and inversion, extend smoothly. For any bundles E and F the process of 'freezing coefficients' at the boundary gives a map

$$I : \Psi_b^m(X; E, F) \longrightarrow \Psi_{b,I}^m(\overline{N_+\partial X}; E_{\partial X}, F_{\partial X})$$

where the subscript I denotes the $\mathbf{R}^+$-invariance of the resulting 'indicial operator.' The map ∂_0 in (1.1.5) is just given by identifying, for an elliptic $A \in \Psi_b^m(X; E, F)$,

$$\partial_0([A]) = [I(A)] \in K_0([-1,1] \times \partial X, \partial([-1,1] \times \partial X)) \cong K_1(\partial X) \tag{1.1.6}$$

where we use Bott periodicity. The map ∂_1 can be identified similarly.

On a closed compact manifold, Y, K-theory provides the right tool to prove the index theorem; besides the original K-cohomological proof of Atiyah and Singer we mention the approach of Baum and Douglas [**10**], which is based on the Poincaré isomorphism (1.1.1) and the equivalence between analytic and topological K-homology, as well as those of Kasparov [**28**], and Connes and Skandalis [**18**] which exploit the functoriality properties of KK-theory in an essential way.

On a manifold with boundary, X, the index problem for elliptic operators is more subtle and K-theory is, in general, not enough. If the operator $D : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F)$ admits a Lopatinski boundary condition $B : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(\partial X; G)$ then, by a classical result (see [**25**]), it defines a Fredholm operator. In this case it is possible to pull back the class $[\sigma(D)] \in K^0(T^*X)$ to a class $[\sigma(D, B)] \in K^0(T^*X, T^*_{\partial X}X)$; once this has been achieved, something which is not at all obvious, the proof of the index theorem is reduced to the one for closed manifolds by doubling X. This theorem, due to Atiyah and Bott [**2**], has been generalized by Boutet de Monvel [**15**] to an algebra of pseudodifferential operators on X naturally containing such classical boundary value problems.

However for an elliptic differential operator D to admit Lopatinski boundary conditions it is necessary that the K-cohomology class defined by its symbol lies in the kernel of the restriction homomorphism $K^0(T^*X) \longrightarrow K^0(T^*_{\partial X}X)$. Using Poincaré duality this is equivalent to requiring that $[D] \in \operatorname{Ker} \partial_0 \subset K_0(X, \partial X)$ where ∂_0 is the boundary map in K-homology (1.1.5). It turns out that the differential operators naturally arising in Riemannian geometry do not satisfy this condition. As an example consider a Dirac operator, D_X, on a even dimensional spin$_{\mathbb{C}}$-manifold with boundary X; then (1.1.6) together with the definition of Bott periodicity in analytic K-homology yields $\partial_0[D_X] = [D_{\partial X}]$, and this is in general non zero in $K_1(\partial X)$.

In the fundamental paper [4], Atiyah, Patodi and Singer introduced non-local boundary conditions for which these classical first order differential operators become Fredholm. If we assume X to be isometric near the boundary ∂X to a product $\partial X \times I$ then any of these operators splits in a collar neighbourhood of the boundary as

$$D = \sigma(\frac{\partial}{\partial x} + D_0) \tag{1.1.7}$$

where x is the normal variable, $\sigma = \sigma(D)|_{\partial X}(dx)$ and D_0 is a self-adjoint elliptic operator $C^\infty(\partial X, E_{\partial X}) \longrightarrow C^\infty(\partial X; F_{\partial X})$; the analytic index of the Atiyah-Patodi-Singer boundary problem can be expressed as the sum of two terms ([4][**17**]): the usual Atiyah-Singer integral plus a (global) correction term coming from the boundary

$$\xi(0) = -\frac{\eta(0) + h}{2}.$$

Here h is the dimension of the kernel of D_0 and $\eta(0)$ is the value at 0 of the meromorphic extension of the function

$$\eta(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)|\lambda|^{-s} \qquad \operatorname{Re} s >> 0 \tag{1.1.8}$$

where $\lambda \in \operatorname{spec}(D_0)$. The value of $\eta(s)$ at $s = 0$ measures the asymmetry of the spectrum of the self-adjoint operator D_0 with respect to the origin. One can introduce this eta invariant for any self adjoint elliptic operator on ∂X; the proof of the regularity at $s = 0$ in this general case is quite subtle ([**5**], [**22**]).

In the third and fourth chapter of this thesis we generalize the results of Atiyah, Patodi and Singer to an arbitrary elliptic element $A \in \Psi_b^m(X; E, F)$. The Fredholm properties of A, already established by Melrose and Mendoza, are obtained by analyzing the indicial family associated to the operator A. This is the entire family of pseudodifferential operator on ∂X defined by

$$I_A(z) = (x^{-iz} A x^{iz})_\partial \in \Psi^m(\partial X; E_{\partial X}, E_{\partial X})$$

where x is a boundary defining function and

$$(\ )_\partial : \Psi_b^m(X; E, F) \longrightarrow \Psi^m(\partial X; E_{\partial X}, F_{\partial X})$$

is the restriction homomorphism. The indicial family $I_A(z)$ defines a discrete set in the complex plane, the boundary spectrum of A;

$$\text{spec}_b(A) = \{z \in \mathbb{C}; I_A(z) \text{ is not invertible }\}. \tag{1.1.9}$$

The points in $\text{spec}_b(A)$ are also called the indicial roots of the operator A. The Fredholm property of A, as a map between weighted b-Sobolev spaces is controlled by $\text{spec}_b(A)$, to be more precise one has the following

THEOREM [**35**]. *Let $A \in \Psi_b^m(X; E, F)$. Then*

$$A : x^r H_b^M(X; E) \longrightarrow x^r H_b^{M-m}(X; F) \tag{1.1.10}$$

is Fredholm if and only if

$$A \text{ is elliptic and } r \notin -\Im \,\text{spec}_b(A).$$

Here $\Im z$ denotes the imaginary part of z. The relative index theorem gives the behaviour of the index of A as a function of r :

THEOREM [**35**]. *The index of an elliptic element $A \in \Psi_b^m(X; E, F)$ as above is independent of M, locally constant in r and such that*

$$r, r' \notin -\Im \,\text{spec}_b(A), r > r' \Longrightarrow$$
$$\text{ind}_{r'}(A) - \text{ind}_r(A) = \sum_{\substack{z' \in \text{spec}_b(A) \\ r' < -\Im z < r}} \text{ord}(z')$$

where ord(z') *is the order of z' as a zero of the indicial family.*

It is important to notice that the classical first order differential operators considered above, become b-differential operators when defined in terms of a metric in the compressed tangent bundle and it is not difficult to realize that the index of the Atiyah-Patodi-Singer boundary problem is equal to the index given by the theorem of Melrose and Mendoza; in this case the boundary spectrum (1.1.9) coincides with the spectrum of the tangential operator (multiplied by $\sqrt{-1}$). Bearing in mind this example and the relative index theorem one expects the index of an elliptic element as in (1.1.10) to be expressible as the sum of a symbolic term and a boundary correction term connected with the asymmetry of the set $\text{spec}_b(A)$ with respect to the line $\{z \in \mathbb{C}; \Im z = -r\}$. Thus in Chapter 3 we extend the work of Atiyah, Patodi and Singer and we define an eta function associated to the indicial family $I_A(z)$ of an elliptic b-pseudodifferential operator.

We first introduce two zeta functions $\zeta_r^{\pm}(s, I(A))$ associated to the indicial family; the plus and minus signs stand for our choice of cuts in the complex plane (the positive and negative real axis respectively); the subscript r is associated to the weighting of the Sobolev space in (1.1.10). We study the meromorphic properties of these zeta functions and we define an eta function $\tilde{\eta}(s, I(A))$ by

$$(1.1.11) \qquad \tilde{\eta}_r(s, I(A)) = \left(\frac{1+e^{i\pi s}}{2i \sin \pi s}\right) \zeta_r^-(s, I(A)) - \left(\frac{1+e^{-i\pi s}}{2i \sin \pi s}\right) \zeta_r^+(s, I(A)).$$

We then show that $s = 0$, a priori a simple pole, is a regular point; the proof is based on a K-theoretic argument as in [5]. Although (1.1.11) does measure the asymmetry of the boundary spectrum with respect to $\{\Im z = -r\}$ and extends the eta function of Atiyah-Patodi-Singer for the 'linear' case (1.1.7), we still need to introduce in §3.5 a different extension of (1.1.8); this new eta function, $\eta(s, I(A))$, will be precisely the boundary correction term in the index formula of Chapter 4.

As already remarked, the index of an elliptic element in $\Psi_b^m(X; E, F)$ depends on the boundary spectrum of the operator. Thus it is not a topological invariant and K-theory cannot be used effectively to solve the index problem. In Chapter four we then extend to the b-calculus the zeta function approach to the index formula. It is well known that on a manifold without boundary, Y, the index of an elliptic operator, P, can be expressed as the difference of the traces of the heat kernels of the operators P^*P and PP^* or, equivalently, as the difference of the corresponding zeta functions for $\text{Re}\, s >> 0$. The difference of the values at $s = 0$ of the meromorphic extension of the two zeta functions gives the formula

$$(1.1.12) \qquad \text{ind}\, P = \int_Y U_0(P^*P) - \int_Y U_0(PP^*).$$

where the densities $U_0(P^*P)$ and $U_1(PP^*)$ can be explicitly computed in terms of the complete symbol of P ([**3**] [**37**]).

Our goal in the fourth chapter is to give an analogous treatment for the index of an elliptic b-pseudodifferential operator, A, in which the symbolic contribution from the interior and the global contribution from the boundary are explicitly singled out. There are various problems in trying to extend the zeta function approach. First of all b-pseudodifferential operators of large negative order are not trace class. Thus although we can give a rather explicit treatment of the complex powers of self-adjoint elliptic b-pseudodifferential operators (§4.2) we still cannot take their traces to define the zeta function. We can, however, take their b-trace, an extension of the trace functional due to Melrose [**34**], and hence define the b-zeta function. However after this regularization has been performed the connection with the spectrum of the operator is lost. To get around this second difficulty we introduce a perturbed operator $A(\delta), \delta$ small and positive, which has the same index as A but for which $A(\delta^*)A(\delta)$ and $A(\delta)A^*(\delta)$ have discrete spectrum. Writing the zeta function index formula for $A(\delta)$ and analyzing its behaviour as $\delta \downarrow 0$ we obtain the main result of this chapter

$$\text{ind}_r(A) = -\eta_r(0, I(A)) + {}^b\zeta(A^*A) - {}^b\zeta(AA^*) \tag{1.1.13}$$

for an elliptic b-pseudodifferential operator as in (1.1.8). The contribution coming from the b-zeta functions can be explicitly computed in terms of the (complete) symbol of A; as in the case of closed compact manifolds it reduces to the integral of well known characteristic classes whenever A is a generalized Dirac operator ([**3**] [**20**] [**21**]). Thus (1.1.13) gives a generalization of the Atiyah-Patodi-Singer index theorem.

2. ANALYTIC K-THEORY ON MANIFOLDS WITH CORNERS

2.1 b–pseudodifferential operators: the small calculus

In this section we will describe the algebra of b–pseudodifferential operators on a compact manifold with corners. In the case of a manifold with boundary this algebra was defined in [**32**] were it was called the algebra of totally characteristic pseudodifferential operators (see also [**27**]). The case of manifolds with corners is treated in detail in [**33**]

Recall that on a compact $\mathcal{C}^\infty$ manifold, Y, without boundary the filtered algebra of pseudodifferential operators acting on half–densities

$$\Psi^*(Y;\Omega^{\frac{1}{2}}Y) = \bigcup_{m\in\mathbf{R}} \Psi^m(Y;\Omega^{\frac{1}{2}}Y) \tag{2.1.1}$$

can be defined by reference to local coordinates and oscillatory integrals. Equivalently one can look at the regularity properties of the Schwartz kernel of these operators, namely that they are precisely the distributional half–densities on $Y\times Y$ which are conormal with respect to the diagonal $\Delta\subset Y\times Y$.

If Z is an n-dimensional $\mathcal{C}^\infty$ manifold, W a submanifold of dimension $n-r$ and $\mathcal{V}_W = \{V\in\mathcal{C}^\infty(Z,TZ);\ V \text{ is tangent to } W\}$, one defines the space of conormal distributional half-densities associated to W as

$$\begin{aligned} I(Z,W;\Omega^{\frac{1}{2}}Z) = \{u\in C^{-\infty}(Z,\Omega^{\frac{1}{2}}Z);\ &\text{for some } m,\\ \mathcal{V}_W^k u\subset H^m(Z,\Omega^{\frac{1}{2}}Z)\ \ &\forall\ k\in\mathbf{N}\}\end{aligned} \tag{2.1.2}$$

where $C^{-\infty}(Z,\Omega^{\frac{1}{2}}Z)$ is the space of distributional section of the half-density bundle and the H^m are the usual Sobolev spaces. The same definition applies for manifolds with boundary (or even with corners) and clean submanifolds meeting the boundary transversally.

Conversely every element $u\in I(Z,W;\Omega^{\frac{1}{2}}Z)$ is given, locally, by the Fourier transform of a symbol $a\in S^k(U\times\mathbf{R}^n)$ (some k) where U is open in $\mathbf{R}^{n-r}$. Recall that if $\Omega \subset \mathbf{R}^n$ is open then the space of symbols, $S^m(\Omega \times \mathbf{R}^N) \subset \mathcal{C}^\infty(\Omega \times \mathbf{R}^N)$ consists of the functions satisfying estimates

$$\sup_{x \in K, \xi \in \mathbf{R}^N} (1+|\xi|)^{-m+|\beta|} |D_x^\alpha D_\xi^\beta a(x,\xi)| < \infty$$
$$\forall\, K \subset\subset \Omega,\ \alpha \in \mathbf{N}_0^n,\ \beta \in \mathbf{N}_0^N$$

where $\mathbf{N}_0 = \{0, 1, \dots\}$. These spaces are invariant under linear (even asymptotically homogeneous) transformations in the second variables, so $S^m(V)$ is well–defined for any vector bundle, V.

Defining the subspace $I^m(Z, W, \Omega^{\frac{1}{2}}) \subset I(Z, W, \Omega^{\frac{1}{2}})$ by requiring such a symbol to be of order $M = m + \frac{n}{4} - \frac{r}{2}$ one obtains a well defined symbol map

$$I^m(Z, W; \Omega^{\frac{1}{2}}) \longrightarrow S^M / S^{M-1}(N^*W, \Omega^{\frac{1}{2}}(N^*W)) \tag{2.1.3}$$

which gives rise to an exact sequence
(2.1.4)
$$O \hookrightarrow I^{m-1}(Z, W; \Omega^{\frac{1}{2}}) \longrightarrow I^m(Z, W; \Omega^{\frac{1}{2}}) \longrightarrow S^M / S^{M-1}(NW, \Omega^{\frac{1}{2}}(N^*W)) \longrightarrow 0$$

If $Z = Y \times Y$ and $W = \Delta$, so that $I^m(Z, W, \Omega^{\frac{1}{2}}) \longleftrightarrow \Psi^m(Y, \Omega^{\frac{1}{2}})$, then $N^*\Delta \cong T^*Y$, furthermore the bundle $\Omega^{\frac{1}{2}}(T^*Y)$ has a natural trivialization; thus the short exact sequence (2.1.4) amounts precisely to the well known symbolic properties of pseudodifferential operators.

Consider the space of differential operators acting on half-densities, $\mathrm{Diff}^*(Y, \Omega^{\frac{1}{2}})$ (this space is obtained by considering the enveloping algebra of the Lie algebra $\mathcal{V} = \mathcal{C}^\infty(Y; TY)$ of all smooth vector fields on Y).
The above remarks lead us to regard the extension of filtered algebras

$$\mathrm{Diff}^*(Y, \Omega^{\frac{1}{2}}) \subset \Psi^*(Y, \Omega^{\frac{1}{2}})$$

as the extension of the admissible Schwartz Kernels from all $\mathcal{C}^\infty$ delta sections of the half density bundle over Δ to all conormal sections. This extension is usually referred to as the 'microlocalization' of the Lie algebra $\mathcal{V} = \mathcal{C}^\infty(Y; TY)$.

Now consider a compact manifold with corners, X. Thus X is a compact Hausdorff space covered by compatible coordinate charts where the coordinate maps are to the model spaces

$$\mathbf{R}^n_k = [0, \infty)^k \times \mathbf{R}^{n-k}$$

for fixed n, but variable k. We also demand, as part of the definition of a manifold with corners, that the boundary hypersurfaces of X be embedded. This is equivalent ([**32**]) to demanding that X have an embedding

$$X \hookrightarrow \tilde{X}$$

into a compact manifold without boundary, $\tilde{X}$, of the same dimension as X such that

$$\mathcal{C}^\infty(X) = \mathcal{C}^\infty(\tilde{X}) \restriction X$$

and that there exist J (the number of boundary hypersurfaces) functions $\rho_i \in \mathcal{C}^\infty(\tilde{X})$, $i = 1, \ldots, J$ such that

$$X = \bigcap_{i=1}^{J} \{\rho_i \geq 0\} \tag{2.1.5}$$

and

$$\begin{gathered} \forall I \subset \{1, \ldots, J\} \; d\rho_i(x), i \in I, \text{ are independent in } T_x^*\tilde{X} \\ \text{for all } x \in \partial_I X = \bigcap_{i \in I} \{\rho_i = 0\}. \end{gathered} \tag{2.1.6}$$

Any product of compact manifolds with corner automatically satisfies this condition (with $\tilde{X}$ the product of the doubles).

Of course the condition that the boundary hypersurfaces be embedded can also be stated intrinsically. These two approaches exemplify two attitudes to the boundary — whether it represents a permeable barrier or a closed frontier. The two points of view are widely reflected in the analysis and geometry of the space. Thus there are two obvious generalizations of $\mathcal{V} = \mathcal{C}^\infty(Y; TY)$ to the case of a manifold with corners:

$$\begin{gathered} \mathcal{V}_E(X) = \mathcal{C}^\infty(X; TX) = \mathcal{C}^\infty(\tilde{X}; T\tilde{X}) \restriction X \\ \mathcal{V}_b(X) = \mathcal{C}^\infty(X; {}^bTX) = \{V \in \mathcal{V}_E(X); V \text{ is tangent to the boundary}\}. \end{gathered} \tag{2.1.7}$$

Both are Lie algebras and $\mathcal{C}^\infty(X)$-modules. The bundle $TX = T_X\tilde{X}$ is the usual tangent bundle. The bundle ${}^bTX$ is, by definition, the compressed tangent bundle of X ; its fiber at the point $p \in X$ is given by

$${}^bT_pX = \mathcal{V}_b(X) \Big/ \mathcal{I}_p \cdot \mathcal{V}_b(X)$$

where $\mathcal{T}_p = \{f \in \mathcal{C}^\infty(X); f(p) = 0\}$.

The space of b–pseudodifferential operators is obtained by microlocalizing $\mathcal{V}_b(X)$. The microlocalization of $\mathcal{V}_E(X)$ (the extension algebra) leads to pseudodifferential operators in the ordinary sense, relating to elliptic boundary problems and the transmissions condition [**15**].

Near any point $x \in X$ (a compact manifold with corners) we can introduce the k (possibly $= 0$) of the ρ_i in (2.1.5) which vanish at x as the first k coordinates of a system, $x_1, \dots, x_k, y_1, \dots, y_{n-k}$, based at x. Then $\mathcal{V}_b(X)$ is locally spanned over $\mathcal{C}^\infty$ by

$$x_1\partial_{x_1}, x_2\partial_{x_2}, \dots, x_k\partial_{x_k}, \partial_{y_1}, \dots, \partial_{y_{n-k}}. \tag{2.1.8}$$

These elements give a local basis for ${}^bTX$ in (2.1.7) and allow any $P \in \mathrm{Diff}_b^m(X)$ (the filtration of the enveloping algebra of $\mathcal{V}_b(X)$) to be written locally in the form

(2.1.9)
$$P = \sum_{|\alpha| \le m} a_\alpha(x,y)(x_1\partial_{x_1})^{\alpha_1} \dots (x_k\partial_{x_k})^{\alpha_k} \partial_{y_1}^{\alpha_{k+1}} \dots \partial_{y_{n-k}}^{\alpha_n}, \text{ the } a_\alpha \text{ being } \mathcal{C}^\infty.$$

Notice that (2.1.8) also fixes the following local basis for the compressed cotangent bundle ${}^bT^*X$,

$$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \dots, \frac{dx_k}{x_k}, dy_1, \dots, dy_{n-k}. \tag{2.1.10}$$

Let us consider the space of b–differential operators acting on half–densities, $\mathrm{Diff}_b^*(X, \Omega^{\frac{1}{2}}X)$. The Schwartz kernels of the elements in $\mathrm{Diff}_b^*(X, \Omega^{\frac{1}{2}}X)$ span a subspace of the space of all $\mathcal{C}^\infty$ delta sections of the bundle $\Omega^{\frac{1}{2}}X$ over the diagonal $\Delta \subset X \times X$. The space of b–pseudodifferential operators on a compact manifold with corners is defined by enlarging the class of admissible kernels in $X^2 = X \times X$. In the interior we require the singularities of these kernels to be conormal at the diagonal as in the case $\partial X = \emptyset$. However, as we approach the boundary of X^2, we expect these singularities to be spread along the 'corners' of X^2, that is the submanifolds of codimension two of the form $H^2 \subset (\partial X)^2 \subset \partial(X^2)$ where H is a boundary hypersurface.

To describe the precise class of kernels admitted we first introduce a new manifold with corners, the $\mathcal{V}_b$–stretched product, X_b^2. Denote by $M_1(X)$ the set of boundary hypersurfaces. The b–stretched product X_b^2 is obtained from X^2 by blowing up the subset

$$S = \bigcup_{H \in M_1(X)} H^2 \subset X^2. \tag{2.1.11}$$

Each of the submanifolds, H^2, in S is embedded and they meet normally in the sense that the conormal bundles are independent at intersections. We may think of S as a 'normal' submanifold. Let H_i, $i = 1, \dots, N$ be an enumeration of the boundary hypersurfaces of X and $\rho_i \in \mathcal{C}^\infty(X)$ a family of defining functions. On X^2 let ρ_i and ρ_i', $i = 1, \dots, N$ denote the lifts of these functions from the left and the right factors. Consider the ring of functions on the interior, $\overset{\circ}{X}{}^2$, of X^2 which are of the form

(2.1.12)

$$F(\rho_1 + \rho_1', \frac{\rho_1 - \rho_1'}{\rho_1 + \rho_1'}, \dots, \rho_N + \rho_N', \frac{\rho_N - \rho_N'}{\rho_N + \rho_N'}, x, x'),$$
$$F \in \mathcal{C}^\infty\big(\big([0,\infty) \times [-1,1]\big)^N \times X^2\big).$$

Not only is this space independent of the choice of the ρ_i but it defines a space of functions on the set

$$X_b^2 = [X^2 \setminus S] \sqcup \bigsqcup_{x \in S} \prod_{\{H \in M_1(X); x \in H^2\}} SN_x^+(H^2)$$

turning it into a compact $\mathcal{C}^\infty$ manifold with corners in a natural way. Here $N_x(H^2) = T_xX^2/T_xH^2$ is the fibre at $x \in H^2$ of the normal bundle to H^2 and $N_x^+(H^2) \subset N_x(H^2)$ is the (closed) subset of inward–pointing normal vectors. Then $SN_x^+(H^2) = [N_x^+(H^2)\backslash 0]/\mathbf{R}^+$ is the corresponding projective space. Thus we denote the functions of the form (2.1.12) by $\mathcal{C}^\infty(X_b^2)$. This b–stretched product has, compared to X^2, extra boundary hypersurfaces, one for each $H \in M_1(X)$. These are collectively denoted $\mathrm{ff}(X_b^2)$ the 'front face' of X_b^2.

There is a natural smooth surjective 'blow–down' map $\beta_b^2 : X_b^2 \longrightarrow X^2$. If $\tilde{\rho}_i = (\beta_b^2)^*(\rho_i + \rho_i')$ then

$$\rho_{\mathrm{ff}} = \prod_{i=1}^N \tilde{\rho}_i \in \mathcal{C}^\infty(X_b^2)$$

is a (joint) defining function for $\mathrm{ff}(X_b^2)$. Consider the lifted diagonal, defined as the closure (in X_b^2) of the lift of the interior of the diagonal:

$$\Delta_b = \operatorname{clos}(\beta_b^{-1}[\Delta \setminus S]) \quad (\text{in } X_b^2).$$

This is an embedded C^∞ submanifold such that

$$\beta_b^2 : \Delta_b \longrightarrow \Delta \longleftrightarrow X$$

is a diffeomorphism. Moreover under β_b^2 the C^∞ vector fields in $\mathcal{V}_b(X)$ lift from the left (or the right) factor of X to a Lie subalgebra of $\mathcal{V}_b(X_b^2)$ which is transversal to Δ_b. That is, there is a natural vector bundle map, which turns out to be an isomorphism:

$${}^bTX \longleftrightarrow N\Delta_b.$$

The dual map to this is therefore an isomorphism

$$(2.1.13) \qquad {}^bT^*X \longleftrightarrow N^*\Delta_b.$$

The combination of the lifts of $\mathcal{V}_b(X)$ from the left and the right spans, over $C^\infty(X_b^2)$, the whole of $\mathcal{V}_b(X_b^2)$.

Consider now an operator $P \in \mathrm{Diff}_b^*(X, \Omega^{\frac{1}{2}})$ and its Schwartz kernel $K_P \in C^{-\infty}(X^2, \Omega^{\frac{1}{2}}) = (\dot{C}^\infty(X^2, \Omega^{\frac{1}{2}}))'$ where $\dot{C}^\infty(X, \Omega^{\frac{1}{2}})$ denotes the space of sections of the half-density bundle vanishing of infinite order at $\partial(X^2)$.
It can be proved that:
The blow down map induces an isomorphism

$$(2.1.14) \qquad C^{-\infty}(X_b^2, \Omega_b^{\frac{1}{2}}(X_b^2)) \longleftrightarrow C^{-\infty}(X^2, \Omega^{\frac{1}{2}}(X^2))$$

where $\Omega_b^{\frac{1}{2}}$ is the half density bundle associated to the compressed cotangent bundle. Thus we can lift the kernel K_P to the stretched product X_b^2 and the fundamental observation is that:
As P runs over $\mathrm{Diff}_b^*(X, \Omega^{\frac{1}{2}})$, the lifted kernels run over all C^∞ delta sections of the bundle $\Omega_b^{\frac{1}{2}}$ over Δ_b.
It is then natural to define

$$(2.1.15) \quad \Psi_b^m(X; \Omega_b^{\frac{1}{2}}X) \longleftrightarrow \left\{\kappa \in I^m(X_b^2, \Delta_b; \Omega_b^{\frac{1}{2}}(X_b^2); \kappa \equiv 0 \text{ at } \partial(X_b^2) \setminus \mathrm{ff}(X_b^2)\right\}.$$

The elements are $\mathcal{C}^\infty$ away from Δ_b in particular near $\partial(X_b^2) \setminus \mathrm{ff}(X_b^2)$ where the kernels in (2.1.15) are required to vanish in the sense of Taylor series. This requirement corresponds to the 'small calculus' ; the 'full calculus' will be discussed later in conjunction with the Fredholm properties of the elliptic b–pseudodifferential operators.

The mapping properties of these operators can be discussed using pull–back and push–forward theorems from [**33**]. In particular we have:

$$A \in \Psi_b^*(X, \Omega_b^{\frac{1}{2}}) \Longrightarrow A : \mathcal{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(X, \Omega_b^{\frac{1}{2}})$$

The most fundamental result is the composition law, showing that $\Psi_b^*(X; \Omega_b^{\frac{1}{2}})$ forms a filtered algebra.

Thus the replacement for (2.1.1) on a compact manifold with corners is a filtered ring of operators

$$\Psi_b^*(X, \Omega_b^{\frac{1}{2}}) = \bigcup_{m \in \mathbf{R}} \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \tag{2.1.16}$$

The symbolic properties of these operators are easily obtained from the definition, the symbol map (2.1.3), the short exact sequence (2.1.4) and the isomorphism (2.1.13). Thus if $S^m({}^bT^*X)$ denotes the symbol space, then there is a (non–natural) quantization map

$$q : S^\infty({}^bT^*X) = \bigcup_{m \in \mathbf{R}} S^m({}^bT^*X) \longrightarrow \Psi_b^*(X, \Omega_b^{\frac{1}{2}}) \tag{2.1.17}$$

which is filtered and almost surjective:

$$\begin{gathered} q : S^m({}^bT^*X) \longrightarrow \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \\ q(S^m({}^bT^*X)) + \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}) = \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \ \forall m \in \mathbf{R} \end{gathered} \tag{2.1.18}$$

where

$$\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}) = \bigcap_{m \in \mathbf{R}} \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \tag{2.1.19}$$

is the 'residual space.'

If we use the notation for the quotients

$$S^{[m]}({}^bT^*X) = S^m({}^bT^*X) \,/\, S^{m-1}({}^bT^*X)$$

then q has, for each m, a natural left inverse on $S^{[m]}({}^bT^*X)$:

$$\sigma_m : \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \longrightarrow S^{[m]}({}^bT^*X) \tag{2.1.20}$$

which gives the (exact) symbol sequence:

$$0 \longrightarrow \Psi_b^{m-1}(X, \Omega_b^{\frac{1}{2}}) \hookrightarrow \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^{[m]}({}^bT^*X) \longrightarrow 0. \tag{2.1.21}$$

The spaces $S^{[m]}({}^bT^*X)$ have an obvious (point-wise) product

$$S^{[m]}({}^bT^*X) \cdot S^{[m']}({}^bT^*X) = S^{[m+m']}({}^bT^*X) \tag{2.1.22}$$

and operator composition is consistent with this:

$$\begin{aligned} &\Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \circ \Psi_b^{m'}(X, \Omega_b^{\frac{1}{2}}) \subset \Psi_b^{m+m'}(X, \Omega_b^{\frac{1}{2}}) \\ &\sigma_{m+m'}(A \circ B) = \sigma_m(A) \cdot \sigma_{m'}(B). \end{aligned} \tag{2.1.23}$$

These spaces of operators are also asymptotically complete, i.e.

$$\begin{aligned} &\text{if } A_j \in \Psi_b^{m_j}(X, \Omega_b^{\frac{1}{2}}),\ m_j \longrightarrow -\infty \text{ as } j \longrightarrow \infty \text{ then} \\ &\exists\, A \in \Psi_b^M(X, \Omega_b^{\frac{1}{2}}) \text{ s.t. } A - \sum_{j \le p} A_j \in \Psi_b^{M(p)}(X, \Omega_b^{\frac{1}{2}})\ \forall\, p \end{aligned} \tag{2.1.24}$$

where $M = \max m_j$, $M(p) = \max_{j>p} m_j$. In fact A in (2.1.24) is determined uniquely modulo $\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$.

Summarizing the properties (2.1.17), (2.1.18), (2.1.19), (2.1.20), (2.1.21), (2.1.22), (2.1.23) (2.1.24) and the fact that $\Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is invariant under passage to adjoints we say :

THEOREM 2.1.25. *On any compact manifold with corners the b–pseudodifferential operators form a quantizable asymptotically complete, symbol–filtered algebra closed under conjugation with residual space* $\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$.

Certain other properties of the operators are direct consequences of these results. Thus we say

$$A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \text{ is elliptic if } \exists\, b \in S^{[-m]}({}^bT^*X) \text{ s.t. } \sigma_m(a)b = 1 \in S^{[0]}({}^bT^*X).$$

Then we find

$$A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \text{ is elliptic } \iff \exists\, B \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}}) \text{ s.t.}$$
$$A \circ B - \mathrm{Id} \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}).$$

Moreover B is then uniquely determined modulo $\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$ and $B \cdot A - \mathrm{Id} \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$.

Similarly a direct computation shows that elements of the residual space are bounded on $L^2(X, \Omega_b^{\frac{1}{2}})$. An elegant symbolic argument due to Hörmander ([**26**]) allows us to conclude from this that

(2.1.26) each $A \in \Psi_b^0(X, \Omega_b^{\frac{1}{2}})$ extends to a bounded operator on $L^2(X, \Omega_b^{\frac{1}{2}})$.

So far the difference between the general case and the special case $\partial X = \phi$ (when of course $\Psi_b^m(X, \Omega_b^{\frac{1}{2}}) = \Psi^m(X, \Omega^{\frac{1}{2}})$) appears to be essentially notational. However there is a fundamental divergence, namely:

$$\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}) \text{ consists of compact operators on } L^2(X, \Omega_b^{\frac{1}{2}}) \iff \partial X = \emptyset.$$

To discuss the compactness of b-pseudodifferential operators we need to consider a second (non-commutative) symbol. In a certain sense the fundamental manifold with boundary is the closed half-line, $[0, \infty)$. This has the disadvantage that it is not compact. Consider the one-point compactification

$$[0, \infty) \ni x \longmapsto \frac{x-1}{x+1} \in [-1, 1].$$

Not only does the $(0, \infty)$-action on $[0, \infty)$, $m_s : x \longmapsto sx$, extend smoothly to $[-1, 1]$ but so does the inversion $x \longmapsto 1/x$ $(x \neq 0)$. Thus suppose L is an oriented line bundle over a compact manifold with corners, Y. Then the positive part of L can be compactified to a manifold with corners

$$L_+ \hookrightarrow X = \overline{L_+}$$

so that the $(0, \infty)$-action on the fibres extends smoothly. Of course $\overline{L_+} \cong Y \times [-1, 1]$.

Within the space $\Psi_b^*(\overline{L_+}, \Omega_b^{\frac{1}{2}})$ of b-pseudodifferential operators on $\overline{L_+}$ we can consider those invariant under the $(0,\infty)$-action,

$$A \in \Psi_{b,I}^m(\overline{L_+}, \Omega_b^{\frac{1}{2}}) \Longleftrightarrow A \in \Psi_b^m(\overline{L_+}, \Omega_b^{\frac{1}{2}}),\ m_s^{-1} \circ A \circ m_s^* = A \quad \forall\, s \in (0,\infty).$$

For these spaces we get invariant versions of the quantization and symbol maps, i.e. q can be chosen so

$$q : \{a \in S^m({}^bT^*\overline{L_+}), m_s^* a = a\} \longrightarrow \Psi_{b,I}^m(\overline{L_+}, \Omega_b^{\frac{1}{2}})$$

and if we set ${}^bT_Y^*\overline{L_+} = {}^bT^*\overline{L_+} \restriction Y$, $Y \hookrightarrow \overline{L_+}$ being the zero section; then

$$0 \hookrightarrow \Psi_{b,I}^{m-1}(\overline{L_+}, \Omega_b^{\frac{1}{2}}) \longrightarrow \Psi_{b,I}^m(\overline{L_+}, \Omega_b^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^{[m]}({}^bT_Y^*\overline{L_+}) \longrightarrow 0$$

is exact.

These results extend immediately to the case that

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_p \tag{2.1.27}$$

is a vector bundle which is given as a direct sum of oriented line bundles. Then $\overline{L_+} \cong Y \times [-1,1]^p$.

The reason that this special case is of interest is that if X is a manifold with corners then a boundary face is, by definition, a component of $\partial_I X$ (see (2.1.6)) for some $I \subset \{1, \dots, J\}$. The codimension of the boundary component is $\#(I)$, the number of elements in I. If F is a boundary component of codimension k then the normal bundle to F in X has fibre

$$N_x F = \operatorname{Span}\{\partial_{x_1}, \dots, \partial_{x_k}\} = T_x X / T_x F.$$

Now F is (a component of) the intersection of k boundary hypersurfaces, given by $\{x_i = 0\}$, $i = 1, \dots, k$, and we thus have a well-defined decomposition (2.1.27) of NF as the product of the (oriented) normal bundles to the hypersurfaces. One should think of NF, for F a boundary face of X, as a model for X near F. Indeed there is a natural class of local isomorphisms (normal fibrations) of X near F and NF near F, its zero section. Consider the inward pointing normal bundle N_+F and its compactification $\overline{N_+F}$ as discussed above. The stretched product $(N_+F)^2_b$ has a natural decomposition

$$(N_+F)^2_b \cong (F \times F)_\beta \times [0,\infty)^k \tag{2.2.28}$$

where $(F \times F)_\beta \subset \text{ff}(X^2_b)$ denotes the lift of $F \times F$ under the blow-down map β^2_b. Given an element A in $\Psi^m_b(X, \Omega^{\frac{1}{2}}_b)$ we can thus define an $\mathbf{R}^+$-invariant operator on N_+F, and hence an $\mathbf{R}^+$-invariant operator on $\overline{N_+F}$, by simply restricting the Kernel of A to $(F \times F)_\beta$ and extending it to $(N_+F)^2_b$ using (2.2.28).

This operation defines our non-commutative symbols, the indicial operators:

$$I_F : \Psi^m_b(X, \Omega^{\frac{1}{2}}_b X) \longrightarrow \Psi^m_{b,I}(\overline{N_+F}, \Omega^{\frac{1}{2}}_b F). \tag{2.1.29}$$

where we have used the fact that for any boundary face $F \in M(X)$ there is a natural isomorphism

$$(\Omega^{\frac{1}{2}}_b)_x X \cong (\Omega^{\frac{1}{2}}_b)_x F \qquad \forall\, x \in F.$$

As an example consider the b-differential operator (2.1.9). Then

$$I(P) = \sum_{|\alpha| \le m} a_\alpha(0, y)(x_1 \partial_{x_1})^{\alpha_1} \ldots (x_k \partial_x)^{\alpha_k} \partial^{\alpha_{k-1}}_{y_1} \ldots \partial^{\alpha_n}_{y_{n-k}}.$$

Thus $I(P)$ is obtained by 'freezing the coefficients at the boundary'.
Let $\mathcal{C}^\infty{}_F(X) \subset \mathcal{C}^\infty(X)$ be the ideal of functions vanishing on F, a boundary face of X. Then the map (2.1.29) gives an exact sequence

$$0 \longrightarrow \mathcal{C}^\infty{}_F(X) \cdot \Psi^m_b(X, \Omega^{\frac{1}{2}}_b) \hookrightarrow \Psi^m_b(X, \Omega^{\frac{1}{2}}_b) \xrightarrow{I_F} \Psi^m_{b,I}(\overline{N_+F}, \Omega^{\frac{1}{2}}_b) \longrightarrow 0. \tag{2.1.30}$$

These maps 'passing to the indicial operator' have the obvious consistency property. Namely if $F \subset Y$ are both boundary faces of X and we let F' denote F as a boundary face of $\overline{N_+Y}$ then

$$I_F(A) = I_{F'}(I_Y(A)) \quad \forall\, A \in \Psi^*_b(X, \Omega^{\frac{1}{2}}_b)$$

using the obvious identification of normal bundles.

It is also important to note the consistency condition between the indicial map (2.1.29) and the symbol map (2.1.16). Namely

$$\sigma_m[I_F(A)] = \sigma_m(A) \restriction_{{}^bT^*_F X} \tag{2.1.31}$$

where we use the identification of ${}^bT^*_F X$ with ${}^bT^*_F(\overline{N_+F})$.

The indicial operators allow us to analyze the compactness of b–operators. If X is a compact manifold with corners let $M(X)$ denote the set of all boundary faces then

$$\begin{aligned} &A \in \Psi^*_b(X, \Omega_b^{\frac{1}{2}}) \text{ is a compact operator on } L^2(X, \Omega_b^{\frac{1}{2}}) \\ \Longleftrightarrow\ &A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}}) \text{ with } m < 0 \text{ and } I_F(A) = 0 \quad \forall\, F \in M(X). \end{aligned} \tag{2.1.32}$$

Finally we point out that the spaces $\Psi^m_b(X, \Omega_b^{\frac{1}{2}})$ are local $\mathcal{C}^\infty(X^2)$–modules, $X^2 = X \times X$. This means that we can define the corresponding operators on vector bundles by the simple expedient of setting

$$\Psi^m_b(X; E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}}) = \Psi^m_b(X, \Omega_b^{\frac{1}{2}}) \underset{\mathcal{C}^\infty(X^2)}{\otimes} \mathrm{Hom}_X(E, F).$$

All the results above have immediate extensions to the case of operators on sections of vector bundles.

2.2 K–homology

In this section we recall the basic definitions and most important properties of Kasparov K–theory. We refer to **[14]**, **[28]**, **[30]** for the details.

Let A be a separable C^*–algebra and let J be a two–sided $*$–ideal. We are interested primarily in the case that $A = \mathcal{C}^0(X)$ is the space of continuous (complex–valued) functions on a compact manifold with corners. Let $M_1(X)$ be the set of boundary hypersurfaces of X and suppose $B \subset M_1(X)$. Then for J we take

$$J = \mathcal{C}^0_B(X) = \{f \in \mathcal{C}^0(X); f = 0 \text{ at } F\ \forall\, F \in B\}$$

We shall denote by $\mathcal{L}(H_0, H_1)$ the Banach algebra of all bounded linear operators between two Hilbert spaces and by $\mathcal{K}(H_0, H_1)$ the subalgebra of compact operators. If $H_0 = H_1 = H$ we use the notation $\mathcal{L}(H)$, $\mathcal{K}(H)$.

A cycle defining an element of the space $KK^0(A, \mathbf{C})$ is a triple (H, ϕ, F) where

$$H = H_0 \oplus H_1 \text{ is a } \mathbf{Z}_2\text{–graded separable Hilbert space} \tag{2.2.1}$$

$$\phi = \phi_0 \oplus \phi_1 \text{ with } \phi_j : A \longrightarrow \mathcal{L}(H_j) \text{ a } C^*\text{–algebra homomorphism} \tag{2.2.2}$$

$$F = \begin{pmatrix} 0 & T^{\#} \\ T & 0 \end{pmatrix} \text{ with } T \in \mathcal{L}(H_0, H_1),\ T^{\#} \in \mathcal{L}(H_1, H_0) \tag{2.2.3}$$

satisfying the three additional conditions

$$[\phi(f), F] \in \mathcal{K}(H) \tag{2.2.4}$$

$$\phi(f)(F^2 - Id) \in \mathcal{K}(H) \quad \forall\, f \in A. \tag{2.2.5}$$

$$\phi(f)(F - F^*) \in \mathcal{K}(H) \tag{2.2.6}$$

A cycle is said to be degenerate if all the compact operators in (2.2.4), (2.2.5), and (2.2.6) vanish. If $A = \mathcal{C}^0(X)$ it suffices to check (2.2.4)—(2.2.6) for $f \in \mathcal{C}^\infty(X)$ since these bounded operators depend continuously on $f \in \mathcal{C}^0(X)$, in which $\mathcal{C}^\infty(X)$ is dense, and $\mathcal{K}(H)$ is closed in $\mathcal{L}(H)$.

Elements of $KK^0(A, \mathbf{C})$ are obtained by imposing an equivalence relation amongst these cycles. See [**14**,§17] for a detailed discussion of the various equivalence relations that can be considered. Here we consider only stable homotopy equivalence, i.e. norm continuous homotopy of operators and addition of degenerate cycles. In particular if two cycles have the same Hilbert spaces and representations then (H, ϕ, F_0) and (H, ϕ, F_1) are equivalent if there is a norm continuous family $[0, 1] \ni t \longmapsto F_t$ with (H, ϕ, F_t) satisfying (2.2.4)—(2.2.6) for all $t \in [0, 1]$. Note that if

$$\phi(f)[F_0 - F_1] \in \mathcal{K}(H) \quad \forall\, f \in A \tag{2.2.7}$$

then $F_t = tF_1 + (1 - t)F_0$, $t \in [0, 1]$, gives such an equivalence.

The definition of $KK^1(A, \mathbf{C})$ is similar. The cycles are triples (H, ψ, T) where H is a separable Hilbert space, $\psi : A \longrightarrow \mathcal{L}(H)$ is a $*$–representation, $T \in \mathcal{L}(H)$ and the following conditions are satisfied

$$[\psi(f), T] \in \mathcal{K}(H) \tag{2.2.8}$$

$$\psi(f)(T^2 - Id) \in \mathcal{K}(H) \qquad \forall\, f \in A. \tag{2.2.9}$$

$$\psi(f)(T - T^*) \in \mathcal{K}(H). \tag{2.2.10}$$

The equivalence relation is again stable homotopy.

These descriptions of $KK^i(A, \mathbf{C})$ are usually referred to as the 'Fredholm picture' ([**14;§17.5**]). The groups $KK^i(A, \mathbf{C})$ are special cases of the groups $KK^i(A, B)$, defined by Kasparov for any pair of $\mathbf{Z}_2$-graded C^*-algebras, A and B. We shall use the notation $K^i(A) \equiv KK^i(A, \mathbf{C})$, $i \in \mathbf{Z}_2$.

A C^*-algebra homomorphism $h : A_1 \longrightarrow A_2$ defines a natural group homomorphism

$$h^* : K^i(A_2) \longrightarrow K^i(A_1),\ h^*[(H, \phi, F)] = [(H, \phi \circ h, F]).$$

In particular if X and Z are compact metric spaces and $g : X \longrightarrow Z$ is a continuous map then $g^* : \mathcal{C}^0(Z) \longrightarrow \mathcal{C}^0(X)$, $g^*(f) = f \circ g$, induces a covariant map $g_* = (g^*)^*$:

$$g_* : K^i(\mathcal{C}^0(X)) \longrightarrow K^i(\mathcal{C}^0(Z)).$$

Defining the groups

$$K_i(X) = K^i(\mathcal{C}^0(X))$$

one obtains a homology theory which coincides with the theory defined abstractly by dualizing the K-cohomology theory of Atiyah and Hirzebruch ([**28**]).

One of the main features of Kasparov's K-theory is the existence of an external product ([**30**]):

$$K^i(A_1) \otimes K^j(A_2) \longrightarrow K^{i+j}(A_1 \otimes A_2),\ i, j \in \mathbf{Z}_2.$$

If I is the (closed) unit interval then tensoring with a generator $b \in K^1(\mathcal{C}^0_{\partial I}(I))$ gives a group isomorphism

$$b_i : K^i(A) \longleftrightarrow K^{i+1}(\mathcal{C}^0_{\partial I}(I) \otimes A). \tag{2.2.11}$$

This is the Bott periodicity theorem.

Finally consider the boundary map. If

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is the short exact sequence associated to the C^*-algebra, A, and ideal, J, one can define boundary maps

$$\partial_i : K^i(J) \longrightarrow K^{i+1}(A/J). \tag{2.2.12}$$

Now let X be a compact manifold with boundary. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{C}^0_{\partial X}(X) \hookrightarrow \mathcal{C}^0(X) \stackrel{\restriction \partial X}{\longrightarrow} \mathcal{C}^0(\partial X) \longrightarrow 0 \tag{2.2.13}$$

given by inclusion and restriction to the boundary. This sequence defines the boundary maps

$$\partial_i : K^i(\mathcal{C}^0_{\partial X}(X)) \longrightarrow K^{i+1}(\mathcal{C}^0(\partial X)). \tag{2.2.14}$$

We refer the reader to [23], [24] for some interesting results on this case. It turns out that if $I \times \partial X$ is a collar neighbourhhod of the boundary, $I = [0,1]$, and if $J : \mathcal{C}^0_{\partial I \times \partial X}(I \times \partial X) \longrightarrow \mathcal{C}^0_{\partial X}(X)$ is the obvious extension map, then, in terms of the isomorphism (2.2.11), one obtains ([24])

$$b_{i+1} \circ \partial_i = J^*. \tag{2.2.15}$$

From these maps one easily obtains the standard six–term exact sequence

$$\begin{array}{ccccc} K^0(\mathcal{C}^0(\partial X)) & \xrightarrow{\iota_*} & K^0(\mathcal{C}^0(X)) & \longrightarrow & K^0(\mathcal{C}^0_{\partial X}(X)) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K^1(\mathcal{C}^0_{\partial X}(X)) & \longleftarrow & K^1(\mathcal{C}^0(X)) & \xleftarrow{\iota_*} & K^1(\mathcal{C}^0(\partial X)). \end{array} \tag{2.2.16}$$

It is this exact sequence which we shall discuss more directly in terms of pseudodifferential operators. A different approach to the K–homology groups of a manifold with boundary and to the long exact sequence (2.2.16) is given by Baum and Douglas in [9]; one of their main results is the equivalence of the two approaches. Interesting applications are given in [13].

For the case of a compact manifold with corners it is natural to consider more general sequences obtained by taking $B \subset M_1(X)$ $F \in M_1(X) \setminus B$, $B' = B \cup \{F\}$ and the $*$–ideal $\mathcal{C}^0_{B'}(X) \subset \mathcal{C}^0_B(X)$. Then (2.2.13) is replaced by

$$0 \longrightarrow \mathcal{C}^0_{B'}(X) \hookrightarrow \mathcal{C}^0_B(X) \longrightarrow \mathcal{C}^0_{B(F)}(F) \longrightarrow 0$$

where $B(F) \subset M_1(F)$ is the set of boundary hypersurfaces in F which are contained in elements of B. By the same reasoning as above this leads to the exact sequence:

$$
(2.2.17)\qquad
\begin{array}{ccccc}
K^0(\mathcal{C}^0_{B(F)}(F)) & \xrightarrow{\iota_*} & K^0(\mathcal{C}^0_B(X)) & \longrightarrow & K^0(\mathcal{C}^0_{B'}(X)) \\
\partial_1 \uparrow & & & & \downarrow \partial_0 \\
K^1(\mathcal{C}^0_{B'}(X)) & \longleftarrow & K^1(\mathcal{C}^0_B(X)) & \xleftarrow{\iota_*} & K^1(\mathcal{C}^0_{B(F)}(F)).
\end{array}
$$

2.3 Cycles defined by b–pseudodifferential operators

Suppose that E_1 and E_2 are (complex) vector bundles over a $\mathcal{C}^\infty$ manifold with corners, X and let A be a b-pseudodifferential operator, $A \in \Psi^0_b(X; E_1 \otimes \Omega^{\frac{1}{2}}_b, E_2 \otimes \Omega^{\frac{1}{2}}_b)$. In all this chapter we will be only interested in the topological invariants of b-elliptic operators. We can fix a b-density ν_b on X and let our operators act on sections of vector bundles; thus we use the standard but less natural notation $A \in \Psi^0_b(X; E_1, E_2)$. If $A \in \Psi^0_b(X; E_1, E_2)$ then $A : L^2_b(X; E_1) \longrightarrow L^2_b(X, E_2)$ where $L^2_b(X; E_i)$ is the space of square integrable sections of E_i with respect to the fixed b-half density ν_b. If $F \in M(X)$ is a boundary face then we say that $A \in \Psi^0_b(X; E_1, E_2)$ is trivial over F if

$$
(2.3.1)\qquad I_F(A) \text{ is a vector bundle homomorphism.}
$$

Of course if A is elliptic then, given (2.3.1), $I_F(A)$ is necessarily invertible.

LEMMA 2.3.2. *If $A \in \Psi^0_b(X; E_1, E_2)$ is elliptic and trivial over each $F \in B \subset M(X)$ then A has a parametrix $\tilde{A} \in \Psi^0_b(X; E_2, E_1)$ which is trivial over each $F \in B$.*

PROOF: Since A is elliptic we can find, according to (2.1.18), a parametrix, $A_1 \in \Psi^0_b(X; E_2, E_1)$. This satisfies

$$
(2.3.3)\qquad A \cdot A_1 - \mathrm{Id} \in \Psi^{-\infty}_b(X; E_1, E_1).
$$

For any boundary face $F \in B$ we deduce from (2.3.3) that

$$I_F(A) \cdot I_F(A_1) - \mathrm{Id} \in \Psi_{b,I}^{-\infty}(\overline{N_+F}; E_1).$$

Using the exactness in (2.1.21) we can modify A_1 by a term $A_1'' \in \Psi_b^{-\infty}(X; E_2, E_1)$ so that $A_1' = A_1 + A_1''$ satisfies

$$I_F(A_1') = I_F(A)^{-1}.$$

This construction can be carried out successively at each boundary face in B, proving the lemma. □

Now suppose that E_1 and E_2 are equipped with Hermitian metrics and that $A \in \Psi_b^0(X; E_1, E_2)$ has a unitary symbol, i.e.,

$$A^*A - \mathrm{Id} \in \Psi_b^{-1}(X; E_1, E_1),\ AA^* - \mathrm{Id} \in \Psi_b^{-1}(X; E_2, E_2). \tag{2.3.4}$$

Then let $\phi_i : \mathcal{C}^0(X) \longrightarrow \mathcal{L}(L_b^2(X; E_i))$, $i = 1, 2$, be the $*$-representation given by point-wise multiplication.

PROPOSITION 2.3.5. *If E_1, E_2 are Hermitian vector bundles over X and if A is an elliptic element in $\Psi_b^0(X; E_1, E_2)$ satisfying (2.3.4) and trivial over $B' = M_1(X)\backslash B$, where $B \subset M_1(X)$, then the triple*

$$\left(L_b^2(X; E_1) \oplus L_b^2(X; E_2), \phi_1 \oplus \phi_2, \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix}\right) \tag{2.3.6}$$

where $\tilde{A} \in \Psi_b^0(X; E_2, E_1)$ is a parametrix for A and is trivial over B', defines a cycle for $K^0(\mathcal{C}_B^0(X))$.

PROOF: Since A and $\tilde{A}$ are L_b^2–bounded (see (2.1.26)) we certainly have (2.2.1), (2.2.2), and (2.2.3). Just as noted in §2.2 it suffices to check (2.2.4), (2.2.5), and (2.2.6) for

$$f \in \mathcal{C}^\infty{}_B(X) = \{f \in \mathcal{C}^\infty(X); f \text{ vanishes in Taylor series at each } F \in B\}.$$

In all these cases the operators are pseudodifferential and of order -1. Consider the criterion (2.1.32) for compactness. All three operators are trivial at boundary hypersurfaces in B' so it follows from (2.3.4) that their indicial operators vanish there. On the other hand $f = 0$ on B, so all three indicial operators vanish there too. $\square$

More generally we can extend this result and associate an element of $K^0(\mathcal{C}^0_B(X))$ to an arbitrary elliptic element $A \in \Psi^0_b(X; E_1, E_2)$ which is trivial over $B' = M_1(X)\backslash B$. Indeed introducing metrics on E_1 and E_2 we see that A^*A is elliptic, with asymptotically positive symbol, and trivial over B'. We can therefore construct an approximate inverse square–root, which is trivial over B' :

$$Q^2 \cdot A^*A - \mathrm{Id} \in \Psi^{-\infty}_b(X; E_1, E_2).$$

Then Proposition 2.3.5 applies to $A \cdot Q$. The homotopy invariance in the equivalence classes in $K^0(\mathcal{C}^0_B(X))$ shows that the element

$$[A] = [A \cdot Q] \in K^0(\mathcal{C}^0_B(X)) \tag{2.3.7}$$

is well–defined independent of the choices made in the definition of the cycle.

Recall that the cycles for the K–cohomology space $K^0({}^bTX)$ can be taken as triples $(\pi^* E_1, \pi^* E_2, \sigma)$ where E_1, E_2 are $\mathcal{C}^\infty$ vector bundles over X, $\pi : {}^bT^*X \longrightarrow X$ is the projection and

$$\sigma : \pi^* E_1 \restriction_{{}^bT^*X\backslash 0} \longrightarrow \pi^* E_2 \restriction_{{}^bT^*X\backslash 0} \tag{2.3.8}$$

is a bundle isomorphism which is homogeneous of degree 0. By modifying σ to be smooth near the zero section and using a quantization map as in (2.1.17) we can consider

$$q(\phi\sigma) \in \Psi^0_b(X; E_1, E_2)$$

where $\phi \in \mathcal{C}^\infty({}^bT^*X)$ vanishes near the zero section and is identically one outside a compact set. Now $q(\phi\sigma)$ is certainly elliptic.

The K–group $K^0({}^bT^*X)$ is obtained by imposing the stable–homotopy equivalence relation on the cycles (2.3.8). More generally if $B' \subset M_1(X)$ then the relative groups $K^0({}^bT^*X, {}^bT^*_{B'}X)$ are obtained by restricting the cycles (2.3.8) to be

induced by a bundle isomorphism from E_1 into E_2 over $B' \subset M_1(X)$

and then imposing stable homotopy equivalence within this class. Recalling that q can be chosen to preserve such triviality at the operator level we find:

Proposition 2.3.9. *If $B \subset M_1(X)$ is a set of boundary hypersurfaces and $B' = M_1(X)\backslash B$ then there is a well-defined group homomorphism*

$$K^0({}^bT^*X, {}^bT^*_{B'}X) \longrightarrow K^0(C^0_B(X))$$
$$[(\pi^*E_1, \pi^*E_2, \sigma)] \longmapsto [A]. \tag{2.3.10}$$

We show below that this homomorphism is always an isomorphism, representing Poincaré duality for K-theory.

The same construction works equally well for the K^1 spaces. Thus, following [5] we can identify $K^1({}^bT^*X)$ (or more generally $K^1({}^bT^*X, {}^bT^*_{B'}X)$) with the stable homotopy classes of self-adjoint symbols (2.3.8) (with $E = E_1 = E_2$, trivial over elements of B'.)

Proposition 2.3.11. *With the notation of Proposition 2.3.9 the map*

$$K^1({}^bT^*X, {}^bT^*_{B'}X) \longrightarrow K^1(C^0_B(X))$$
$$\sigma \longmapsto [(q(\phi\sigma), L^2_b(X, E))]$$

is a well-defined group homomorphism.

Proof: Exactly as before it is only necessary to check the compactness conditions (2.2.8), (2.2.9), and (2.2.10) using (2.1.32). □

A different way of using b-pseudodifferential operators to define Kasparov cycles, and hence elements of $K^1(C^0_B(X))$, is to make explicitly use of the Bott isomorphism (2.2.11). Thus, with $I = [0, 1]$, consider an elliptic

$$P \in \Psi^0_b(X \times I; E, F) \text{ with } [\sigma(P)] \in K^0({}^bT^*(X \times I), {}^bT^*_{B' \times I}(X \times I)).$$

Using (2.3.9) this gives an element of $K^0(C^0_B(X) \otimes C^0_{\partial(I)}(I))$ and hence of $K^1(C^0_B(X))$. As $K^0({}^bT^*(X \times I), {}^bT^*_{B' \times I}(X \times I))$ can be identified with $K^{-1}({}^bT^*X, {}^bT^*_{B'}X)$ we obtain a quantization map

$$K^{-1}({}^bT^*X, {}^bT^*_{B'}X) \longrightarrow K^1(C^0_B(X)). \tag{2.3.12}$$

2.4 Cycles defined by pseudodifferential operators

Although, for reasons of naturality, we have used the compressed cotangent bundle the two bundles

$$^bT^*X \cong T^*X \tag{2.4.1}$$

are isomorphic, with the isomorphism natural up homotopy. Thus there is a canonical isomorphism

$$K^i({}^bT^*X, {}^bT^*_{B'}X) \cong K^i(T^*X, T^*_{B'}X). \tag{2.4.2}$$

This allows us to give a rather direct definition of the relative K–homology class associated to an elliptic differential operator on a manifold with boundary (see also [**13**], [**24**]).

Namely let

$$D : \mathcal{C}^\infty(X; E_1) \longrightarrow \mathcal{C}^\infty(X; E_2)$$

be an elliptic differential operator. Then, choosing a metric on X, the symbol of D defines an element $[(\pi^*E_1, \pi^*E_2, \tilde{\sigma}(D)] \in K^0(T^*X)$ by homogeneous extension off the unit sphere. Using (2.4.2) the class of D is well-defined in the relative space

$$[D] \in K^0(\mathcal{C}^0_{\partial X}(X)).$$

If $\sigma^1(D)$ is a self-adjoint bundle isomorphism then $[D] \in K^1(\mathcal{C}^0_{\partial X}(X))$ instead.

For example if X is a spin$_\mathbf{C}$ manifold then (see [**11**]) the Dirac operator D_X is a first order elliptic operator. Thus we find:

Proposition 2.4.3. *Let X be a* spin$_\mathbf{C}$ *manifold with corners then the Dirac operator D_X determines a class $[D_X] \in K^i(\mathcal{C}^0_{\partial X}(X))$, $i \equiv \dim X$, (mod 2).*

There is another form of the quantization map

$$K^i(T^*X) \longrightarrow K_i(X, \partial X) \tag{2.4.4}$$

for a compact manifold with corners. Let V be a $\mathcal{C}^\infty$ vector field on X which is transversal to each boundary hypersurface and inward–pointing. Integration of V gives a 1–parameter family of $\mathcal{C}^\infty$ maps

$$\exp(tV) : X \longrightarrow X,\ t \geq 0 \text{ small}, \tag{2.4.5}$$

which are diffeomorphisms onto their ranges. Thus $X_t = \exp(tV)X$ is an embedding of X as a compact submanifold, with corners, of $\overset{\circ}{X}$. Each diffeomorphism (2.4.5) induces an isomorphism on K-homology.

Let $a = [\pi^*E, \pi^*F, \sigma] \in K^0(T^*X)$ and let $A \in \Psi^0(\overset{\circ}{X}; E, F)$ be the pseudodifferential operator corresponding to $\sigma \restriction_{T^*\overset{\circ}{X}}$. Since

$$L^2(X_t; E) \hookrightarrow L^2_c(X; E)$$

(the space of square-integrable sections of compact support) by extension as zero the operator A defines

$$A : L^2(X_t; E) \longrightarrow L^2_{\mathrm{loc}}(\overset{\circ}{X}; F). \tag{2.4.6}$$

Then restriction to X_t gives

$$A_t : L^2(X_t; E) \longrightarrow L^2(X_t; F).$$

Assuming that E_1 and E_2 have Hermitian metrics and that $\sigma(A)$ is unitary then A has a properly supported parametrix and we can again consider the triple (2.3.6) with A replaced by A_t. This defines a class in $K_0(X_t, \partial X_t)$. Following the proof of Proposition 2.3.5 we need again to check (2.2.4), (2.2.5) and (2.2.6). It suffices to check the compactness for elements $f \in C^\infty{}_c(\overset{\circ}{X}_t)$. Since $\phi(f)$ is just the multiplicative action the commutator in (2.2.4) is of the form $([f, A])_t$. Since $[f, A]$ is of order -1 on X the resulting operator is compact. Similarly in (2.2.6) $F - F^*$ is an operator of the same type with A replaced by $A - A^*$, of order -1. Thus the range of (2.4.6) is contained in $H^1_{\mathrm{loc}}(X; F)$ so after multiplication by f is compactly included in $L^2(X_t; F)$. Finally (2.2.5) is similar except that it involves the composition of F with itself. However, using (2.3.4) we can consider $F\phi(f)F - \phi(f)$ instead. This is given by the restriction of a pseudodifferential operator of order -1 so is again compact.

Thus we have defined, for each $0 < t < \epsilon$,

$$[A] \in K_0(X_t, \partial X_t), \text{ if } A \in \Psi^0(\overset{\circ}{X}; E, F) \text{ is elliptic} \tag{2.4.7}$$

and has unitary symbol. As in (2.3.7) this extends directly to the general case of an elliptic operator. By homotopy invariance each $K_0(X_t, \partial X_t)$, is naturally isomorphic to $K_0(X, \partial X)$ and the class of A does not depend on t, nor does it depend on the choice of transversal vector field V. Thus we have defined the quantization map (2.4.4). If $A \in \Psi^0_b(X; E, F) \subset \Psi(\overset{\circ}{X}; E, F)$ then we can take the limit as $t \to 0$ and conclude that the class of A in (2.4.7) is same as the class obtained in §2.3.

Although we have been discussing the relative space $K_0(X, \partial X)$ the discussion easily extends to the case of a general $B \subset M_1(X)$. If we choose V to be tangent to the boundary hypersurfaces in B and transversal to the others and set the construction above gives a quantization map

$$K^0(T^*X, T^*_{B'}X) \longrightarrow K_0(X, B),\ B' = \complement B.$$

2.5 Poincaré duality

Let X be a compact manifold with corners and suppose, as in §2.1, that $B \subset M_1(X)$ is a set of boundary hypersurfaces. If X' is another manifold with corners then

$$M_1(X' \times X) = M_1(X') \times X \cup X' \times M_1(X).$$

The decomposition of the circle S^1 as the union of two closed intervals with disjoint interiors leads to the exact sequences of K-spaces:

(2.5.1)

$$0 \longrightarrow K^0({}^bT^*(I \times X), {}^bT^*_{\partial I \times X \cup I \times B}(I \times X)) \longrightarrow K^0({}^bT^*(S^1 \times X), {}^bT^*_{S^1 \times B}(S^1 \times X)) \longrightarrow K^0({}^bT^*(I \times X), {}^bT^*_{I \times B}(I \times X)) \longrightarrow 0$$

$$(2.5.2)\quad \begin{aligned} 0 \longrightarrow K_0(I \times X, I \times B) &\longrightarrow K_0(S^1 \times X, S^1 \times B) \\ &\longrightarrow K_0(I \times X, \partial I \times X \cup I \times B) \longrightarrow 0. \end{aligned}$$

In (2.5.1) the first map is induced by the inclusion $I \times X \hookrightarrow S^1 \times X$ and corresponds to extension of a bundle trivial near the boundary, the second map is restriction. In (2.5.2) the first map arises from the restriction of $\mathcal{C}^0_{S^1 \times B}(S^1 \times X)$ to $\mathcal{C}^0_{I \times B}(I \times X)$ and the second from the extension map from $\mathcal{C}^0_{\partial I \times X \cup I \times B}(I \times X)$ into $\mathcal{C}^0_{S^1 \times B}(S^1 \times X)$.

The (compressed) cotangent bundle ${}^bT^*X$ is a manifold with corners, the boundary faces being precisely the subsets

$${}^bT^*_GX = \pi_b^{-1}(G),\ G \in M(X),\ \pi_b : {}^bT^*X \longrightarrow X.$$

We then denote by ${}^bT^*_{\mathcal{B}}X \subset M_1({}^bT^*X)$ the set of boundary faces of ${}^bT^*X$ defined by $\mathcal{B} \subset M(X)$. The following relationship between these sequences is an important tool in the discussion below.

LEMMA 2.5.3. *For any compact manifold with corners, X, and any subset $B \subset M_1(X)$ the diagram, with vertical maps given by* (2.5.1) *(and the equivalence of ${}^bT^*X$ and T^*X) and* (2.5.2) *and horizontal maps the b–quantization maps:*

(2.5.3)

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K^0({}^bT^*(I\times X), {}^bT^*_{\partial I\times X\cup I\times \complement B}(I\times X)) & \xrightarrow[q_b]{} & K_0(I\times X, I\times B) \\
\downarrow & & \downarrow \\
K^0({}^bT^*(S^1\times X), {}^bT^*_{S^1\times \complement B}(S^1\times X)) & \xrightarrow[q_b]{} & K_0(S^1\times X, S^1\times B) \\
\downarrow & & \downarrow \\
K^0({}^bT^*(I\times X), {}^bT^*_{I\times \complement B}(I\times X)) & \xrightarrow[q_b]{} & K_0(I\times X, \partial I\times X\cup I\times B) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

is commutative.

PROOF: The commutativity of the top square is a direct consequence of the definitions of the various spaces and maps. Thus the image in $K_0(S^1\times X, S^1\times B)$ of an element $q_b(a) = [A] \in K_0(I\times X, I\times B)$ being $[A']$ where A' is obtained from A by trivial extension to the complement of I in S^1.

The commutativity of the second square follows from the discussion in §2.4. Indeed suppose an oriented embedding $I \subset S^1$ is chosen and a is an elliptic symbol on $T^*(S^1\times X)$ trivial at $S^1\times \complement B$. Suppose that $A \in \Psi^0_b(S^1\times X; E, F)$ is trivial over $S^1\times \complement B$ and has a as symbol. We need to show that the image of

$[A] \in K_0(S^1 \times X, S^1 \times B)$ in $K_0(I \times X, \partial I \times X \cup I \times B)$ is equal to the class of A' where $A' \in \Psi^0_b(I \times X; E, F)$ is trivial over $I \times \complement B$ and has symbol equal to the image of the restriction of a to $T^*I \times {}^bT^*X$ if ${}^bT^*I$ is identified with T^*I as in (2.4.1) i.e. that the class $[A']$ is equal to $J^*[A]$. Notice that $L^2_b(I \times X; E)$ is a closed subspace of $L^2_b(S^1 \times X; E)$. By adding a degenerate cycle we can thus choose the representative cycles of $[A']$ and $J^*[A]$ to act on the same Hilbert space and with the same $*$-representation. By making use of 2.2.7 we simply have to prove that $\forall\, f \in C^0_{\partial I \times X \cup I \times B}(I \times X)$

$$J(f)(A - (A' \oplus 0)) \in \mathcal{K}$$
$$J(f)(\tilde{A} - (\tilde{A} \oplus 0)) \in \mathcal{K}$$

which follows immediately from the criterior for compactness (2.1.32). □

Now with $B \subset M_1(X)$ suppose $F \in M_1(X) \setminus B$ and set $B' = B \cup \{F\}$. Corresponding to the choice of B and F there is a long exact sequence in K-cohomology with compact supports:

$$\begin{array}{ccc} K^0({}^bT^*X, {}^bT^*_{B'}X) & \longrightarrow & K^0({}^bT^*X, {}^bT^*_{B}X) \\ \uparrow & & \downarrow \\ K^{-1}({}^bT^*_FX, {}^bT^*_{B(F)}X) & & K^0({}^bT^*_FX, {}^bT^*_{B(F)}X) \\ \uparrow & & \downarrow \\ K^{-1}({}^bT^*X, {}^bT^*_{B}X) & \longleftarrow & K^{-1}({}^bT^*X, {}^bT^*_{B'}X). \end{array} \tag{2.5.6}$$

Here $B(F)$ is the set of boundary hypersurfaces of F (hence elements of $M_2(X)$) which are contained in elements of B.

Since ${}^bT^*_FX$ is always a trivial line bundle over ${}^bT^*F$:

$${}^bT^*_FX \cong {}^bT^*F \times \mathbf{R}$$

Bott periodicity gives the natural isomorphisms

$$K^i({}^bT^*_FX, {}^bT^*_{B(F)}X) \cong K^{i+1}({}^bT^*F, {}^bT^*_{B(F)}F) \mod 2 \tag{2.5.7}$$

Using (2.5.7) to replace the appropriate two spaces in (2.5.6) the quantization maps (2.3.10), (2.3.12) can be applied to each space. Recalling the long exact sequence in K-homology, (2.2.17), we obtain the diagram:

$$
\begin{array}{ccc}
K^0({}^bT^*F, {}^bT^*_{B(F)}F) & \longrightarrow & K_0(F, D(F)) \\
\downarrow & & \downarrow \\
K^0({}^bT^*X, {}^bT^*_{B'}X) & \longrightarrow & K_0(X, D) \\
\downarrow & & \downarrow \\
K^0({}^bT^*X, {}^bT^*_{B}X) & \longrightarrow & K_0(X, D') \\
\downarrow & & \downarrow \\
K^{-1}({}^bT^*F, {}^bT^*_{B(F)}F) & \longrightarrow & K_1(F, D(F)) \\
\downarrow & & \downarrow \\
K^{-1}({}^bT^*X, {}^bT^*_{B'}X) & \longrightarrow & K_1(X, D) \\
\downarrow & & \downarrow
\end{array}
\tag{2.5.8}
$$

where $D = M_1(X) \setminus B'$ is the complement of $B' = B \cup \{F\}$ and $D' = D \cup \{F\} = M_1(X) \setminus B$ is the complement of B.

Proposition 2.5.9. *The diagram* (2.5.8) *commutes.*

Before proceeding to the proof we note the main consequence:

Corollary 2.5.10. *For any subset $B \subset M_1(X)$ the quantization maps*

$$
q_b : K^i({}^bT^*X, {}^bT^*_BX) \longrightarrow K_i(X, \complement B) \tag{2.5.11}
$$

are isomorphisms realizing Poincaré duality.

Proof: We proceed by induction over the number of elements $g = \#(\complement B)$. Moreover we only need prove (2.5.11) for $i = 0$ since the first two rows of (2.5.4) are then isomorphisms, hence so is the third, which proves (2.5.11) for $i = 1$. Now, if $g = 0$ then $B = M_1(X)$ and (2.5.11) follows from [29]. Suppose that (2.5.11) is known to hold for all $g \leq g'$. Then for B with $g = g' + 1$ all the rows in (2.5.8) are known to be isomorphisms except the third, since $\#(\complement B') = g'$ and $\#B(F) \leq g'$. Applying the five lemma therefore completes the inductive step since it shows that

(2.5.11) is an isomorphism, and Poincaré duality is the unique isomorphism giving a commutative diagram (2.5.8).

PROOF OF PROPOSITION 2.5.9: Consider first the commutativity of the second and the fifth squares. The maps on the left are just inclusions, of symbols trivial over ${}^bT^*_{B'}X$ into symbols trivial over ${}^bT^*_B X$. Thus the operators inducing classes in $K^i(X, D)$ and $K^i(X, D')$ are the same. The classes then correspond since the map on the right is just restriction of the algebra $C^0_D(X)$ to its ideal $C^0_{D'}(X)$.

Next consider the commutativity of the third square. The map on the left is represented by restriction in (2.5.6) followed by the Bott map. In view of (2.1.31) we need to show that, in terms of (2.2.17), $\partial_i([A]) = b_{i+1}^{-1}[I_F(A)]$. This follows from (2.2.15) since by an argument similar to that in the proof of Lemma 2.5.8 $J^*[A] = [I_F(A)]$.

Finally consider the first and fourth squares in (2.5.8). Here the commutativity reduces to the case $X = \overline{N_+F} = F \times [-1, 1]$ and follows from [**28**]. Indeed the vertical maps on the left are the connecting homomorphisms, δ, in K–theory; it is not difficult to realize that in this case $\delta = i_!$, where $i_!$ is the map induced in the K-theory of the cotangent bundles by the embedding $i : F \hookrightarrow [-1, 1] \times F$. Thus [**28; Lemma 1,** §7] applies.

3 BOUNDARY SPECTRAL ASYMMETRY

3.1 The boundary spectrum of elliptic b-pseudodifferential operators

In §2.1 we reviewed the basic properties of b-pseudodifferential operators on manifolds with corners. In this chapter and in the next one we will only deal with manifolds with boundary and it is worth pausing a moment to see how the various definitions in §2.1 simplify in this context.

As in §2.1 we will state all the results for operators acting on half-densities although everything holds in the more general case of operators acting on sections of vector bundles.

The (small) calculus was defined by reference to a space of conormal distributions on the stretched product X_b^2 (see (2.1.15)). The only hypersurface of a manifold with boundary X is its boundary ∂X; thus X_b^2 is obtained by blowing up $S = \partial X \times \partial X$ in the product $X \times X$. The boundary hypersurfaces of X_b^2 are then the front face and the right and left boundary faces defined respectively by

$$\begin{aligned} ff(X_b^2) &= \beta_b^{-1}(\partial X \times \partial X) = S_+N(\partial X \times \partial X) \\ \ell b(X_b^2) &= c\ell(\beta_b^{-1}(\partial X \times \mathring{X})) \\ rb(X_b^2) &= c\ell(\beta_b^{-1}(\mathring{X} \times \partial X)). \end{aligned}$$

If (x, y) are local coordinates in a neighbourhood of the boundary, with x a normal coordinates, and we denote by (x, y) and (x', y') their lift to X^2 from the left and the right factor respectively, then in a neighbourhood of the front face we obtain the projective coordinate (τ, r, y, y') where $\tau = (x - x')/(x + x'), r = x + x'$ (see 2.1.12).

We will also use the set of coordinates given by (x', s, y, y') where $s = x/x'$; these coordinates are valid in a neighbourhood of $ff(X_b^2)$ disjoint from $rb(X_b^2)$.

For later use we point out that $\text{ff}(X_b^2)$ is a quarter circle bundle over $\partial X \times \partial X$;

$$\begin{array}{ccc} [-1,1] & \longrightarrow & \text{ff}(X_b^2) \\ & & \downarrow \\ & & \partial X \times \partial X \end{array}$$

the fibres have a natural projective structure.

Example 3.1.1. If $X = [0,\infty)$ we obviously have $\partial X \times \partial X = \{0\} \subset [0,\infty) \times [0,\infty)$ so that X_b^2 is just obtained by blowing up the origin.

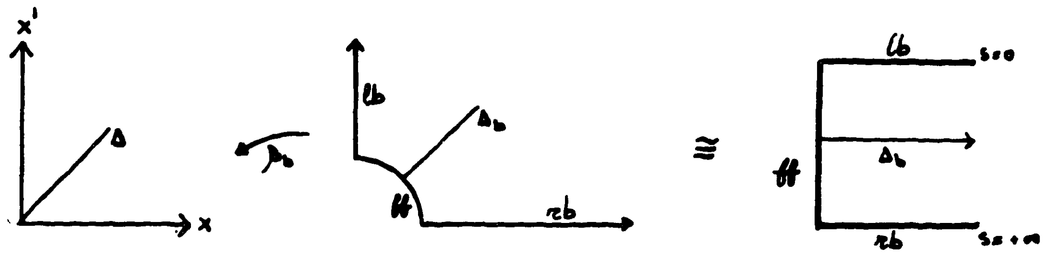

Figure 1

The algebra $\Psi_b^*(X, \Omega_b^{\frac{1}{2}})$ is then defined in terms of the Schwartz kernel of its elements by

$$\Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \longleftrightarrow \{K \in I^m(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}}); K \equiv 0 \text{ at } \ell b \cup rb\} \tag{3.1.2}$$

Thus if $\dim X = (n+1)$ we must have

$$K\Big|_{X_b^2 \backslash \Delta_b} \in C^\infty(X_b^2 \backslash \Delta_b; \Omega_b^{\frac{1}{2}}) \tag{3.1.3}$$

$$K \equiv 0 \text{ at } \ell b \cup rb \tag{3.1.4}$$

$$K(z,z') = (2\pi)^{-n-1} \int e^{i(z-z')\zeta} a(z',\zeta) d\zeta |dz dz'|^{\frac{1}{2}} \text{ near } \Delta_b \text{ iff} \tag{3.1.5}$$

$$K(r,\tau,y,y') = (2\pi)^{-n-1}\int e^{i\tau\lambda+i(y-y')\eta}b(r,y',\lambda,\eta)d\lambda d\eta \tag{3.1.6}$$
$$\times|\frac{dr}{r}d\tau dy dy'|^{\frac{1}{2}} \text{ near } \Delta_b\cap \mathrm{ff}$$

The amplitudes a or b must be symbols of order m; in (3.1.5) we used standard coordinate $(z_0,\dots z_n)$ away from the boundary.

If X is a manifold with boundary there is only one boundary hypersurface and the indicial operator is obtained by restricting the kernel of a b-pseudodifferential operator to the whole front face:

$$I:\Psi_b^m(X,\Omega_b^{\frac{1}{2}})\longrightarrow \Psi_{b,I}^m(\overline{N_+\partial X},\Omega_b^{\frac{1}{2}}). \tag{3.1.7}$$

Recall from (2.1.32) that for operators of negative order, in particular for the elements of the residual space $\Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$, the indicial operator represents the obstruction to their compactness as operators in $L^2(X,\Omega_b^{\frac{1}{2}})$. To understand this fundamental property we recall the Sobolev spaces on which the operators in $\Psi_b^m(X,\Omega_b^{\frac{1}{2}})$, $m\in\mathbb{R}$, act. We already know that 0-order b-pseudodifferential operators are bounded on $L^2(X,\Omega_b^{\frac{1}{2}})$. We thus define

$$H_b^m(X,\Omega_b^{\frac{1}{2}}) = \begin{cases} \{u\in L^2(X,\Omega_b^{\frac{1}{2}}); Au\in L^2(X,\Omega_b^{\frac{1}{2}})\ \forall\ A\in\Psi_b^m\}, m>0 \\ \{u\in C^{-\infty}(X,\Omega_b^{\frac{1}{2}}); u=u'+Au'', u',u''\in L^2(X,\Omega_b^{\frac{1}{2}}) \\ A\in\Psi_b^{-m}(X)\}, m<0\end{cases} \tag{3.1.8}$$

It is also important to introduce weighted Sobolev spaces.
If $\rho\in C^\infty(X)$ is a global boundary defining function then

$$\rho^r H_b^m(X,\Omega_b^{\frac{1}{2}}) = \{u\in C^{-\infty}(X,\Omega_b^{\frac{1}{2}}); \rho^{-r}u\in H_b^m(X,\Omega_b^{\frac{1}{2}})\}. \tag{3.1.9}$$

We then have

PROPOSITION 3.1.10. *If $r,m,M\in\mathbb{R}$ then*

$$A:\rho^r H_b^M(X,\Omega_b^{\frac{1}{2}})\longrightarrow \rho^r H_b^{M-m}(X,\Omega_b^{\frac{1}{2}})$$

is a bounded operator $\forall\ A\in\Psi_b^m(X,\Omega_b^{\frac{1}{2}})$.

The criterion for compactness (2.1.32) is then a consequence of (2.1.30) and the following

PROPOSITION 3.1.11. *For any $r > r'$ and $m > m'$ the inclusion $\rho^r H_b^m(X, \Omega_b^{\frac{1}{2}}) \hookrightarrow \rho^{r'} H_b^{m'}(X, \Omega_b^{\frac{1}{2}})$ is compact. If $m > m' + \dim X$ then it is also nuclear.*

It is now clear that if $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is elliptic the existence of a symbolic parametrix $B_\sigma \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}})$

$$AB_\sigma = 1 - R' \qquad \qquad (3.1.12) \qquad R, R' \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}) \qquad B_\sigma A = 1 - R$$

is not enough to derive the Fredholm property of A as a continuous map between weighted Sobolev spaces. What we need is an operator B for which the error term in (3.1.12) has vanishing indicial operator. In other words we must find a B whose indicial operator inverts the indicial operator of A. To study the invertibility properties of $I(A)$ one considers the associated indicial family $I(A, z) \in \Psi^m(\partial X, \Omega^{\frac{1}{2}})$. In order to define the indicial family we fix a boundary defining function x; we then take the restriction of the kernel of A to $\mathrm{ff}(X_b^2)$ (this restriction defines $I(A)$) followed by the Mellin transform in the fiber variable

$$K(I(A, z)) = \int_0^\infty s^{-iz} K(A)\Big|_{\mathrm{ff}} \frac{ds}{s} \qquad (3.1.13)$$

where $s = x/x'$ as above. Equivalently

$$I(A, z) = (x^{-iz} A x^{iz})_\partial$$

where

$$(\ )_\partial : \Psi_b^m(X, \Omega_b^{\frac{1}{2}}) \longrightarrow \Psi^m(\partial X, \Omega^{\frac{1}{2}})$$

is the restriction homomorphism defined by restriction of the kernel to the front face followed by integration along the fibers.

The indicial family defines an entire map

$$\mathbb{C} \ni z \longrightarrow I(A, z) \in \Psi^m(\partial X, \Omega^{\frac{1}{2}}) \qquad (3.1.14)$$

for each $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$; furthermore

$$\Psi_b^*(X, \Omega_b^{\frac{1}{2}}) \ni A \longrightarrow I(A, z) \in \Psi^*(\partial X, \Omega^{\frac{1}{2}}) \qquad (3.1.15)$$

is an algebra homomorphism $\forall\ z \in \mathbf{C}$.

Example 3.1.16 Let us consider a b-differential operator $P \in \mathrm{Diff}_b^k(X, \Omega_b^{\frac{1}{2}})$. Then, in local coordinates near the boundary

$$P = \sum_{j+|\alpha|\leq k} p_{j,\alpha}(x,y)(xD_x)^j D_y^\alpha.$$

The indicial operator is obtained by 'freezing the coefficients at the boundary'

$$I(A) = \sum_{j+|\alpha|\leq k} p_{j,\alpha}(0,y)(xD_x)^j D^\alpha y$$

and the indicial family is given by

$$I(A,z) = \sum_{j+|\alpha|\leq k} p_{j,\alpha}(0,y)z^j D_y^\alpha$$

Remark 3.1.17. It is not difficult to see from (3.1.13) that for each smoothing operator $R \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$ the indicial family $I(R,z)$ is an entire family of smoothing operators in $\Psi^{-\infty}(\partial X, \Omega^{\frac{1}{2}})$ with kernel rapidly decreasing with all derivatives as $|\operatorname{Re} z| \longrightarrow +\infty$ in any region where $|\Im z|$ is bounded.

Let $A \in \Psi_b^m(X, \Psi_b^{\frac{1}{2}})$ be elliptic. Define

(3.1.18) $$\operatorname{spec}_b(A) = \{z \in \mathbf{C}; I(A,z) \text{ is not invertible on } \mathcal{C}^\infty(\partial X, \Omega^{\frac{1}{2}})\}.$$

As a consequence of (3.1.12), (3.1.15), (3.1.17) and 'analytic Fredholm theory' one can prove the following fundamental

PROPOSITION 3.1.19. *If $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is elliptic then* $\operatorname{spec}_b(A)$ *is a discrete set in the complex plane with the property*

(3.1.20) $$z \in \operatorname{spec}_b(A), |z_j| \longrightarrow +\infty \Longrightarrow |\Im z_j| \longrightarrow +\infty$$

The singular terms in the Laurent series of $I(A,z)^{-1}$ at $z \in \operatorname{spec}_b(A)$ are all finite rank smoothing operators.

The set $\operatorname{spec}_b(A)$ is the boundary spectrum of A; points in $\operatorname{spec}_b(A)$ are also called indicial roots of the operator A.

Remark 3.1.21. Notice that the indicial family does depend on the boundary defining function. However if $\tilde{x} = ax, a \in \mathcal{C}^\infty(X)$ $a > 0$ is another defining function for ∂X then

$$\tilde{I}(A, z) = a^{iz}\Big|_{\partial X} I(A, z) a^{-iz}\Big|_{\partial X} \tag{3.1.22}$$

Thus results in Proposition (3.1.19) are independent of the choice of the boundary defining function. We will briefly denote the indicial family for a fixed boundary defining function as $I_A(z)$.

Making use of Proposition (3.1.19) together with the inverse Mellin transform, it is possible to discuss the invertibility of the indicial operator $I(A)$ and hence the Fredholm property of A itself ;

THEOREM 3.1.23. *(Melrose-Mendoza [35]) Let $A \in \Psi_b^m(X; \Omega_b^{\frac{1}{2}})$. Then*

$$A : x^r H_b^M(X; \Omega_b^{\frac{1}{2}}) \longrightarrow x^r H_b^{M-m}(X; \Omega_b^{\frac{1}{2}})$$

is Fredholm if and only if

$$A \text{ is elliptic and } r \notin -\Im \operatorname{spec}_b(A) \tag{3.1.24}$$

We will comment on the proof of theorem (3.1.23), i.e. on the construction of a true parametrix B_r for each $r \notin -\Im \operatorname{spec}_b(A)$ in the next chapter. For the time being we point out the following theorem, also due to Melrose and Mendoza, on the behaviour of the index of A as a function of r :

THEOREM 3.1.25. *The index of an elliptic element $A \in \Psi_b^m(X; \Omega_b^{\frac{1}{2}})$ as a map*

$$A : x^r H_b^M(X; \Omega_b^{\frac{1}{2}}) \longrightarrow x^r H_b^{M-m}(X, \Omega_b^{\frac{1}{2}})$$

for $r \notin -\Im \operatorname{spec}_b(A)$, is independent of M, locally constant in r and such that

$$r, r' \notin -\Im \operatorname{spec}_b(A), r > r' \Longrightarrow$$
$$\operatorname{ind}_{r'}(A) - \operatorname{ind}_r(A) = \sum_{\substack{z' \in \operatorname{spec}_b(A) \\ r' < -\Im z' < r}} \operatorname{ord}(z').$$

where the order of $z' \in \mathrm{spec}_b(A)$ *is, by definition, the least integer* ℓ *such that* $(z-z')^{\ell} I(A,z)^{-1}$ *is holomorphic in a neighbourhood of* z'.

In other words $\mathrm{ind}_{r'}(A) - \mathrm{ind}_r(A)$ is equal to the sum of the algebraic multiplicities of the indicial roots included in the strip $\{z \in \mathbf{C}; r' < -\Im z < r\}$. The relative index theorem gives important information about the index of A. A formula for $\mathrm{ind}_r(A)$ will have to involve a symbolic term and a global boundary term responsible for the jump in the index as r crosses the imaginary part of an indicial root. Bearing in mind the Atiyah-Patodi-Singer index formula (see the Introduction), we expect such a global boundary term to be connected with the asymmetry of the boundary spectrum of the operator with respect to the horizantal line $\{\Im z = -r\}$ In this chapter we are therefore going to generalize the work of Atiyah, Patodi and Singer [5] by defining an eta function associated to the indicial operator and studing its regularity properties. The index formula will be treated in the next chapter.

3.2 The holomorphic family of operators $\mathbf{Z}_r^{\pm}(\mathbf{s})$

Let E,F be smooth vector bundles over X and let $A \in \Psi_b^m(X; E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}}), m > 0$, be an elliptic b-pseudodifferential operator.
In all this chapter, and in the next one, A will be assumed to be classical.
We also assume, for the time being, that $0 \notin \Im\, \mathrm{spec}_b(A)$ so that there is a strip in the complex plane $\Lambda = \{z \in \mathbf{C}\ ; |\Im z| < \delta\}$ where $I_A(z)$ is invertible in L^2.

Let $\Gamma_\epsilon^{\pm}$ be the following contours in the complex plane:

$$\Gamma_\epsilon^{\pm} = \begin{cases} re^{i\theta_\pm} & \infty \geq r \geq \epsilon \\ \epsilon e^{i\phi} & \theta_\pm \geq \phi \geq \theta_\pm - 2\pi \\ re^{i(\theta_\pm - 2\pi)} & \epsilon \leq r \leq \infty \end{cases}$$

where $\theta_+ = 2\pi$, $\theta_- = \pi$ and $0 < \epsilon < \delta$.

Our goal in this section is to prove that for $s \in \mathbf{C}$, $\mathrm{Re} s < 0$, the following integral

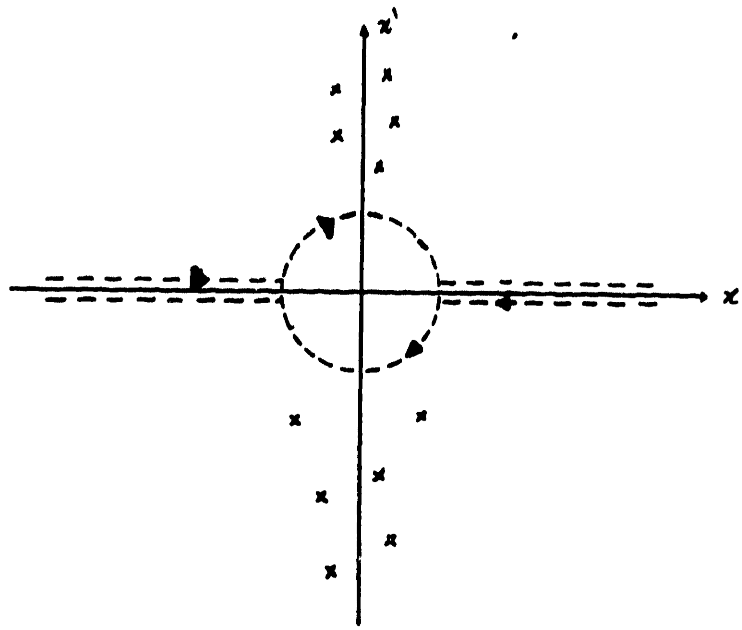

Figure 2

defines a pseudodifferential operator of order s on ∂X :

$$Z_0^{\pm}(s) = \frac{i}{2\pi} \int_{\Gamma_\epsilon^{\pm}} \lambda^s I_A(\lambda)^{-1} \cdot \frac{d}{d\lambda} I_A(\lambda) d\lambda \tag{3.2.1}$$

Here λ^s is determined by the values of $arg\lambda$ given in the description of $\Gamma_\epsilon^{\pm}$.

In order to show that the above integral converges to define an operator from $L^2(\partial X; E_{\partial X} \otimes \Omega^{\frac{1}{2}})$ to $L^2(\partial X; F_{\partial X} \otimes \Omega^{\frac{1}{2}})$ it suffices to prove the following

LEMMA 3.2.2. *For $\lambda \in \Gamma_\epsilon^{\pm}, \lambda$ large, the following estimates hold*

$$\|I_A(\lambda)^{-1} \frac{d}{d\lambda} I_A(\lambda)\|_{\ell,\ell} \leq C_\ell |\lambda|^{-1} \tag{3.2.3}$$

where $\|\cdot\|_{\ell,\ell}$ denotes the operator norm in the Sobolev spaces H^ℓ.

PROOF: To simplify the notation we assume that E and F are the trivial one dimensional vector bundle $X \times \mathbf{C}$. Using (3.1.12), (3.1.15) and Remark (3.1.17) we deduce that, for $z \in \Lambda_\delta$,

$$I_A(z)^{-1} = I_{B_\sigma}(z) + G(z) \tag{3.2.4}$$

with $G(z)$ smoothing and rapidly decreasing with all derivatives as $|\operatorname{Re} z| \longrightarrow +\infty$. Thus it suffices to prove that for $\lambda \in \Gamma_\epsilon^\pm$, λ large,

$$\|I_{B_o}(\lambda)\frac{d}{d\lambda}I_A(\lambda)\|_{\ell,\ell} \leq C_\ell|\lambda|^{-1} \tag{3.2.5}$$

Let $K_A(x', s, y, y')|\frac{dx'}{x'}\frac{ds}{s}dydy'|^{\frac{1}{2}}$ be the Schwartz kernel of A in the projective coordinates (x', s, y, y') near the front face of X_b^2. The kernel of $I_A(z)$ is then given, in local coordinates by

$$K(I_A(z))(y, y')|dydy'|^{\frac{1}{2}} = \int_0^\infty s^{-iz}K_A(0, s, y, y')\frac{ds}{s} \cdot |dydy'|^{\frac{1}{2}} \tag{3.2.6}$$

Making the change of variables $\log s = r$ in (3.2.6) we get

$$K(I_A(z)(y, y') = \int_{-\infty}^{\infty} e^{-iRez\cdot r}\left[e^{\Im z\cdot r}K_A(0, e^r, y, y')\right]dr.$$

Since the kernels of the elements in Ψ_b^m are assumed to vanish of infinite order at $\ell f(X_b^2)$ and $rb(X_b^2)$ we know that , $\forall\ z \in \Lambda_\delta$,

$$\tilde{K}_A(x', s, y, y') \equiv K_A(x', s, y, y')s^{\Im z}$$

belongs to Ψ_b^m. Thus $\forall\ z \in \Lambda_\delta$

$$K(I_A(z)(y, y') = \int e^{i(y-y')\cdot\eta}\tilde{a}(y; Rez, \eta)d\eta \tag{3.2.7}$$

where $\tilde{a} \in S^m(U \times \mathbb{R}^{1+n})$, U being our fixed coordinate chart. In particular, if $\Im z = 0$, $\operatorname{Re} z = \lambda$, we obtain,

$$K(I_A(\lambda))(y, y') = \int e^{i(y-y')\cdot\eta}a(0, y; \lambda, \eta)d\eta \tag{3.2.8}$$

a being the symbol of A in local coordinates. Similarly we get

$$K(\frac{d}{d\lambda}I_A(\lambda))(y, y') = \int e^{i(y-y')\cdot\eta}(\frac{d}{d\lambda}a)(0, y; \lambda, \eta)d\eta. \tag{3.2.9}$$

By (3.2.8), applied to B_σ and (3.2.9), we have for $\lambda \in \{z \in \mathbf{C}; \Im z = 0\}$.

$$K(I_B(\lambda) \cdot \frac{d}{d\lambda} I_A(\lambda))(y, y') = \int e^{i(y-y')\cdot\eta} \beta(y, \eta; \lambda) d\eta \tag{3.2.10}$$

with $\beta(y, \eta; \lambda) \in S^{-1}(U \times \mathbf{R}^{n+1}_{(\eta,\lambda)})$.

Standard arguments then show that for $\lambda \in \Gamma^\pm$, λ large, the following estimate holds ([**39**]):

$$\|I_B(\lambda) \frac{d}{d\lambda} I_A(\lambda)\|_{\ell,\ell-p} \leq C_p |\lambda|^{-(1+p)} \qquad 0 \geq p \geq -1$$

where $\| \quad \|_{\ell,\ell-p}$ is the operator norm between the Hilbert spaces H^ℓ and $H^{\ell-p}$. For $p = 0$ this is nothing but (3.2.5). □

Remarks 3.2.11 Notice that we actually proved that, for Re$s < 0$, $Z_0^\pm(s)$ defines an operator from H^ℓ into H^ℓ $\forall$ ℓ, so that $Z_0^\pm(s)$ maps $C^\infty(\partial X; E_{\partial X} \otimes \Omega^{\frac{1}{2}})$ into $C^\infty(\partial X; F_{\partial X} \otimes \Omega^{\frac{1}{2}})$. Clearly $Z_0^\pm(s)$ does not depend on ϵ, for $\epsilon < \delta$.

Next we want to prove that $Z_0^\pm(s)$ are pseudodifferential operators of order s. Since we are assuming A to be classical we certainly have, in local coordinates and for $\lambda \in \{\Im z = 0\}$, an asymptotic expansion

$$\beta(y, \eta; \lambda) \sim \sum_{J=0}^{\infty} \beta_{-1-j}(y, \eta; \lambda) \tag{3.2.12}$$

with β_{-1-j} homogeneous of degree $(-1-j)$ in (η, λ) for $|(\eta, \lambda)| \geq 1$. This asymptotic expansion of the symbol of $I_B(\lambda)\frac{d}{d\lambda}I_A(\lambda)$, $\lambda \in \Gamma^\pm$, holds only for λ in the straight part of $\Gamma^\pm$. In order to exhibit a symbol for $Z_0^\pm(s)$ we then introduce an *almost analytic extension* of $\beta_{-1-j}(y, \eta; \lambda)$, $\lambda \in \{z \in \mathbf{C}$ s.t. $\Im z = 0\}$ ([**31**]).

Such an extension, that we denote by $\hat{\beta}_{-1-j}(y, \eta, z)$, is now defined for every $z \in \mathbf{C}$, is still homogeneous of degree $(-1 - j)$ for $|(\eta, z)| \geq 1$ and satisfies the almost analyticity condition:

(3.2.13) $\qquad \overline{\partial}\hat{\beta}_{-1-j}$ vanishes of infinite order at $\{\Im z = 0\}$.

Let us now consider the following contour integral

$$c^{\epsilon,\pm}_{-1-j}(s)(y, \eta) = \frac{i}{2\pi} \int_{\Gamma^\pm_\epsilon} \lambda^s \hat{\beta}_{-1-j}(y, \eta; \lambda) d\lambda \tag{3.2.14}$$

where Re$s < 0$.

Remark. When our arguments will apply simultaneously to Γ_ϵ^+ and Γ_ϵ^- we will drop the subscripts $\pm$. Furthermore we will write $\hat{\beta}_{-1-j}(z)$ when the point (y,η) does not play any role.

It is easily checked that $c_{-1-j}^\epsilon(s)$ are in fact symbols of order $s-j$ on $U \times \mathbf{R}^n$. We define

$$c_{-1-j}(s)(y,\eta) = \lim_{\epsilon \longrightarrow 0} c_{-1-j}^\epsilon(s)(y,\eta). \tag{3.2.15}$$

In order to prove that this limit exists we notice that by Stoke's theorem, for $\epsilon' < \epsilon$,

$$\begin{aligned} \mid c_{-1-j}^{\epsilon}(s)(y,\eta) - c_{-1-j}^{\epsilon'}(s)(y,\eta) \mid &= \mid \frac{i}{2\pi}\int_{\Gamma_\epsilon} \lambda^s \hat{\beta}_{-1-j} d\lambda - \frac{i}{2\pi}\int_{\Gamma_{\epsilon'}} \lambda^s \hat{\beta}_{-1-j} d\lambda \mid \\ &\leq \frac{1}{2\pi} \int_{\Omega(\epsilon,\epsilon')} |z|^s |\bar{\partial}\hat{\beta}_{-1-j}(y,\eta,z)| dz \wedge d\bar{z} \end{aligned}$$

where $\Omega(\epsilon,\epsilon')$ is the shaded region in Figure 3 .

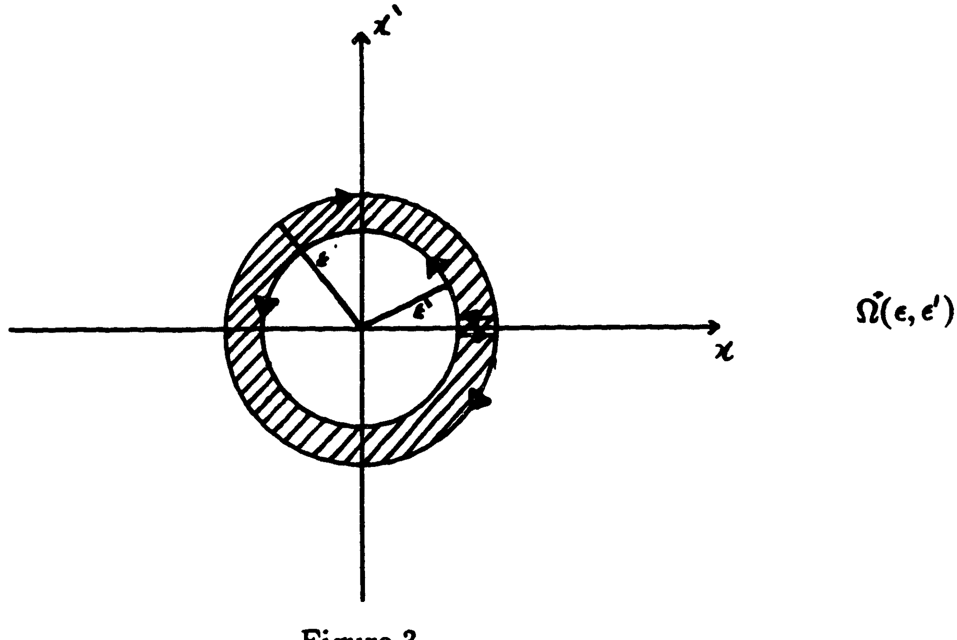

Figure 3

Since $\hat{\beta}_{-1-j}$ is almost analytic this last integral is bounded by $c_N \epsilon^N$ with $N \in \mathbf{Z}^+$ arbitrarily large. This means that $c_{-1-j}^\epsilon(s)(y,\eta)$ is a Cauchy sequence and it therefore converges as $\epsilon \downarrow 0$. Notice that for $|\eta| \geq 1, t \geq 1, c_{-1-j}(s)$ is homogeneous of degree $s - j$. In fact,

$$\begin{aligned} c_{-1-j}(s)(y,t\eta) &\equiv \lim_{\epsilon \longrightarrow 0}\Big[\frac{i}{2\pi}\int_{\Gamma_\epsilon}(\frac{\lambda}{t})^s t^{s+1}\hat{\beta}_{-1-j}(y,t\eta;t\frac{\lambda}{t})d(\frac{\lambda}{t})\Big] \\ &= \lim_{\epsilon \longrightarrow 0} t^{s-j}\Big[\frac{i}{2\pi}\int_{\Gamma_{\frac{\epsilon}{t}}}\mu^s\hat{\beta}_{-1-j}(y,\eta;\mu)d\mu\Big] \\ &= t^{s-j}c_{-1-j}(s)(y,\eta) \end{aligned}$$

THEOREM 3.2.16. *For each* $s \in \mathbf{C}, \mathrm{Re}s < 0$, *the operator* $Z_0^{\pm}(s)$ *is a classical pseudodifferential operator of order* s *on* ∂X *with symbol given, in local coordinates, by*

$$\sigma(Z_0^{\pm}(s)) \sim \sum_{j=0}^{\infty} c_{-1-j}^{\pm}(s)$$

PROOF: Let $\partial X = \bigcup_\gamma V_\gamma$ be a finite covering by charts, ϕ^γ a partition of unity and the functions $\psi^\gamma \in {C^\infty}_0(V_\gamma)$ be such that $\psi^\gamma \equiv 1$ in a neighborhood of $\mathrm{supp}\phi^\gamma$. Consider the pseudodifferential operator B_{s-j} on ∂X which is constructed out of the local symbols $c_{-1-j}^{\gamma}(s)$; by this we mean that

$$B_{s-j} = \sum_\gamma \phi^\gamma B_{s-j}^\gamma \psi^\gamma$$

where B_{s-j}^γ in the pseudodifferential operator on V_γ corresponding to $c_{-1-j}^{\gamma}(s)$. Let $B_{(N)}^{(s)} = \sum_{J=0}^{N-1} B_{s-j}$. It suffices to show that $Z(s) - B_{(N)}^{(s)} \equiv R_{(N)}^{(s)}$ is a pseudodifferential operator of order $-N$. For this we introduce the pseudodifferential operator $B_{(N)}(\lambda) = \sum_{J=0}^{N-1} B_{-1-j}(\lambda)$ where $B_{-1-j}(\lambda) = \sum_\gamma \phi^\gamma Op(\hat{\beta}_{-1-j}^\gamma(\lambda))\psi^\gamma$. Here we depart from our notation in §2.1 and we call Op (instead of q) the quantization map for pseudodifferential operators.

We claim that

$$I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda) - B_{(N)}(\lambda) \equiv R_{(N)}(\lambda) \in \Psi^{-1-N} \text{ for } \lambda \in \Gamma^{\pm} \qquad (3.2.17)$$

with $R_{(N)}(\lambda)$ locally corresponding to a $(-1-N)$ symbol $r_N(\lambda)(y,\eta)$ which is almost analytic in $\lambda, |\lambda| \le \epsilon$, and a joint symbol in (η,λ) for $\lambda \in \Gamma^{\pm}, |\lambda| > \epsilon$.

Assuming this claim for a moment we can complete the proof of the theorem. First we notice that because of Fubini's theorem

$$\frac{i}{2\pi}\int_{\Gamma_\epsilon} \lambda^s B_{(N)}(\lambda)d\lambda (\equiv \sum_{J=1}^{N-1} \frac{i}{2\pi}\int_{\Gamma_\epsilon} \lambda^s B_{-1-j}(\lambda)d\lambda)) = \sum_{J=0}^{N-1} B^{\epsilon}_{s-j} \qquad \text{mod } \Psi^{-\infty}$$

where $B^{\epsilon}_{s-j} = \sum_{\gamma} \phi^{\gamma} Op(c^{\epsilon,\gamma}_{-1-j}(s))\psi^{\gamma}$. On the other hand on each coordinate chart

$$\lim_{\epsilon\longrightarrow 0} Op(c^{\epsilon}_{-1-j}(s)) = Op(c_{-1-j}(s)). \tag{3.2.18}$$

This is because the sequence $\{c^{\epsilon}_{-1-j}(s)\}_{0<\epsilon<\delta}$ is bounded in $S^{s-j}(V\times \mathbf{R}^n)$; thus the limit defining the symbol on the right hand side actually converges in the symbol topology so that (3.2.18) is a consequence of standard continuity properties of the quantization map ([**27**]). We thus have,

$$\lim_{\epsilon\longrightarrow 0} \frac{i}{2\pi}\int_{\Gamma_\epsilon} \lambda^s B_{(N)}(\lambda)d\lambda = B^{(s)}_{(N)} \qquad \text{mod } \Psi^{-\infty}$$

so that we only have to prove that

$$Z(s) - \lim_{\epsilon\longrightarrow 0} \frac{i}{2\pi}\int_{\Gamma_\epsilon} \lambda^s B_{(N)}(\lambda)d\lambda \equiv R'^{(s)}_{(N)} \in \Psi^{-N}.$$

Bearing in mind that $Z(s)$ is independent of ϵ this is in turn implied by the claim if we use Fubini's Theorem and an argument analogous to (3.2.18) to conclude that $R'^{(s)}_{(N)}$ is the pseudodifferential operator locally corresponding to the symbol of order $(-N)$

$$r^{(s)}_{(N)}(y,\eta) = \lim_{\epsilon\longrightarrow 0} \frac{i}{2\pi}\int_{\Gamma_\epsilon} \lambda^s r_N(\lambda)(y,\eta)d\lambda.$$

We still have to prove claim (3.2.17).
We consider the difference $\beta(\lambda) - \sum_{J=0}^{N-1} \hat{\beta}_{-1-j}(\lambda) \equiv r_N(\lambda)$. For $\lambda \in \Gamma^{\pm}$, $|\lambda| > \epsilon$ there is nothing to prove: the claim follows from the definitions.

For $|\lambda| \leq \epsilon$ it is still true that $r_N(A)$ is an element of $S^{-1-N}(U \times \mathbf{R}^n_\eta)$. In order to see this we differentiate N times $r_N(\lambda)$ with respect to $\mu = \Im\lambda$; since $\beta(\lambda)$ is holomorphic in λ, for $\lambda \in \Lambda = \{z \in \mathbf{C} \text{ s.t. } |\Im z| < \delta\}$, we certainly have $\partial_\mu\beta = -i\partial_\tau\beta$ (where $\lambda = \tau + i\mu$); on the other hand β is a joint symbol in (η, τ) on every line parallel to the real axis which implies that $\partial^N_\mu\beta_{-1}$ is bounded by $c(1+|\eta|+|\lambda|)^{-1-N}$. If we now use the fact that the $\hat{\beta}_{-1-j}$'s are symbols also in μ and if we integrate back N times from 0 to λ we can conclude that the difference $|r_N(\lambda)|$ is bounded by a constant times $(1 + |\eta|)^{-1-N}$ for every $\lambda \in \mathbf{C}, |\lambda| \leq \epsilon$. The same argument gives the required estimate for $|\partial^\alpha_y\partial^\beta_\eta r_N(\lambda)(y,\eta)|$. This proves the claim and hence the theorem. □

Recall that we assumed at the beginning of the section that $0 \notin \Im\,\mathrm{spec}_b(A)$. We now want to deal with an arbitrary $r \notin -\Im\,\mathrm{spec}_b(A)$ and define the operators $Z^\pm_r(s)$. In order to do this we observe that if x is a boundary defining function

$$\Psi^m_b(X, \Omega^{\frac{1}{2}}_b) \ni A \longrightarrow x^r A x^{-r} \in \Psi^m_b(X, \Omega^{\frac{1}{2}}_b)$$

is an isomorphism. In fact in local coordinates near the front face

$$K(x^r A x^{-r})(x, t, y, y') = t^{-r} K(A)(x, t, y, y') \quad t = x'/x$$

which implies that $(x^r A x^{-r}) \in \Psi^m_b(X, \Omega^{\frac{1}{2}}_b)$ by (3.1.4). We also obtain

$$I(x^{-r} A x^r, z) = I(A, z - ir) \tag{3.2.19}$$

Thus if $r \notin -\Im\,\mathrm{spec}_b(A)$ we define

$$Z^\pm_r(s, I(A)) = Z^\pm_0(s, I(x^{-r} A x^r)) \tag{3.2.20}$$

3.3 Zeta and eta functions associated to the indicial family.

Let $s \in \mathbf{C}, \operatorname{Re} s > 0$, by Theorem (3.2.16) the operator $Z_r^{\pm}(-s, I(A))$ is a classical pseudodifferential operator of order $-s$, for each $r \notin -\Im \operatorname{spec}_b(A)$. If $\operatorname{Re} s > n$ it is thus a trace class operator; its Schwartz kernel $K_r^{\pm}(-s)$, restricted to the diagonal in $\partial X \times \partial X$, is now a continuous section of the bundle $\operatorname{End}(E) \otimes \Omega(\partial X)$ over ∂X and we have:

$$\operatorname{Tr}(Z_r^{\pm}(-s, I(A))) = \int_{\partial X} \operatorname{tr}_{E_p}(K_r^{\pm}(-s)(p,p)) \qquad \operatorname{Re} s > n \tag{3.3.1}$$

DEFINITION 3.3.2. *Let $s \in \mathbf{C}, \operatorname{Re} s > n$. For each $r \notin -\Im \operatorname{spec}_b(A)$ we define two zeta functions associated to the indicial family $I_A(Z)$ as follows:*

$$\zeta_r^{\pm}(s, I(A)) = \operatorname{Tr}(Z_r^{\pm}(-s, I(A))).$$

We will often denote these zeta functions simply by $\zeta_r^{\pm}(s)$.

Following [37] we want to meromorphycally extend $\zeta_r^{\pm}(s)$ to the whole complex plane. There is no loss of generality assuming that $0 \notin \Im \operatorname{spec}_b(A)$. To extend $\zeta_0^{\pm}(s)$ we first need a lemma.

Let us fix a coordinate chart and let us consider the local symbols $c_{-1-j}(s)$ defined in (3.2.15). These symbols are defined for every $s \in \mathbf{C}$ such that $\operatorname{Re} s < j$.

LEMMA 3.3.3. *For $|\eta| \leq 1$ the symbols $c_{-1-j}^{\pm}(s)(y,\eta)$ admit a holomorphic extension from $\operatorname{Re} s < j$ to the whole complex plane.*

PROOF: We write

$$c_{-1-j}^{\epsilon,\pm}(s)(y,\eta) = \frac{i}{2\pi} \int_{\substack{\lambda \in \Gamma_\epsilon^{\pm} \\ |\lambda|<1}} \lambda^s \hat{\beta}_{-1-j}(y,\eta,\lambda) d\lambda$$
$$+ \frac{i}{2\pi}\left(e^{i(\theta_{\pm}-2\pi)s} - e^{i\theta_{\pm} s}\right) \int_1^{\infty} r^s \beta_{-1-j}(y, \eta \pm r) dr.$$

The first term on the right hand side extends holomorphically to the whole complex plane and the same is true for its limit as $\epsilon \longrightarrow 0$. Let us consider the second and choose for example the cut $\theta_+ = 2\pi$; making the change of variables $t = \frac{1}{r}$ we obtain $\frac{i}{2\pi}(1 - e^{2\pi i s}) \int_0^1 t^{-s+j-1} \beta_{-1-j}(y, t\eta, 1) d\lambda$. Notice that $\beta_{-1-j}(y; t\eta, 1)$ is a smooth function of $t \in [0, 1]$. In order to extend the definition of the above integral, which we will denote by $I(s)$, from $\operatorname{Re} s < j$ to arbitrary $s \in \mathbf{C}$ we choose a $k \in \mathbf{Z}^+$ s.t. $\operatorname{Re}[(-s + j - 1) + k] > -1$ and we define

$$I(s) = \frac{i}{2\pi}(1 - e^{2\pi i s})[((-1)^k \prod_{\ell=1}^{k} \frac{1}{((-s+j-1)+\ell)} \int_0^1 t^{-s+j-1+k} \partial_t^k \beta_{-1-j}(y, t\eta, 1) d\lambda)$$
$$+ (\sum_{m=1}^{K} ((-1)^{m-1} \prod_{\ell=1}^{m} \frac{1}{(-s+j-1)+\ell}) \delta_1^{m-1}(\beta_{-1-j}(y, t\eta, 1))]$$

where δ_1^k denotes the derivative of order k of the delta function at $t = 1$. By integration by parts the two definitions agree when $\operatorname{Re} s < j$. Because of the coefficient $(1 - e^{2\pi i s})$ at the numerator, $I(s)$ is holomorphic in s; this proves the lemma. □

PROPOSITION 3.3.4. *The function $\zeta_0^{\pm}(s)$ defined by (3.3.2) for $\operatorname{Re} s > n$ can be continued to a meromorphic function in the entire complex s-plane with at most simple poles at $s_j = n - j$, $j = 0, 1, 2, \ldots$.*
The residue of $\zeta_0^{\pm}(s)$ at $s = s_j$ is given by the formula

$$\text{(3.3.5)} \qquad \operatorname{Res}_{s=s_j} \zeta_r^{\pm}(s) = \frac{1}{(2\pi)^n} \int_{\partial X} \int_{|\eta'|=1} \operatorname{tr} c_{-1-j}(-s_j)(y, \eta') d\eta' dy.$$

The residues at $s = 0, -1, -2, \ldots$ vanish.

PROOF: The proof proceeds as in the classical case ([**37**] [**39**]). The main step consists of extending the kernel of $Z_0^{\pm}(s)$ restricted to the diagonal. Since, for $\operatorname{Re} s - N < -n$, the kernel of $R_{(N)}^{(s)} \equiv Z(s) - \Sigma \phi^{\gamma} O_p(c_{-1-j}^{\gamma}(s)) \psi^{\gamma}$ is a holomorphic function of s with values in the continuous sections of the bundle $\operatorname{End}(E) \longrightarrow \partial X$, one is reduced to analyzing the singularities of the extension of the kernel of $O_p(c_{-1-j}(s))$ in a fixed coordinate chart U. Using Lemma (3.3.3) and the homogeneity of $c_{-1-j}(s)$ for $|\eta| \geq 1$ it is not difficult to see that the kernel of $O_p(c_{-1-j}(s))$ computed at $(y, y) \in U \times U$, extends from $\operatorname{Re} s < -n$ to the whole complex plane with one simple pole at the point $s = j - n$ and residue equal to

$-1/(2\pi)^n \int\limits_{|\eta'|=1} c_{-1-j}(s)(y,\eta')d\eta'|dy|$. From the above statement all the first part of the theorem follows (using Definition (3.3.2)) and we are left with proving that $\mathrm{Res}_{s=-\ell}\,\zeta_0^{\pm}(s) = 0$ for $\ell \in \mathbb{N}$. This in turn is obvious from (3.3.5) and the fact that, for $j = k+n, k \in \mathbb{Z}^+$:

$$\begin{aligned} c^{\pm}_{-1-j}(j-n)(y,\eta') &\equiv \lim_{\epsilon \longrightarrow 0}\Big(\frac{i}{2\pi}\int\limits_{\Gamma_\epsilon^{\pm}} \lambda^{j-n}\hat{\beta}_{-1-j}(y,\eta',\lambda)d\lambda\Big) \\ &= \frac{i}{2\pi}\big(e^{i(\theta_{\pm}-2\pi)(j-n)} - e^{i\theta_{\pm}(j-n)}\big)\int_0^\infty r^{j-n}\beta_{-1-j}(y;\eta' \pm r)dr \\ &= 0 \end{aligned}$$

□

By (3.2.20) we finally obtain :

THEOREM 3.3.6. *Let $A \in \Psi_b^m(X,\Omega_b^{\frac{1}{2}})$ be elliptic and $r \notin -\Im\,\mathrm{spec}_b(A)$. The function $\zeta_r^{\pm}(s,I(A)) defined by (3.3.2) for$ $\mathrm{Re}s > n$, can be continued to a meromorphic function in the entire complex s-plane with at most simple poles at $s_j = n-j$ $j = 0,1,\ldots$ The residues at $s = 0,-1,-2,\ldots$ vanish.*

We can now introduce our eta function.

DEFINITION 3.3.7.. *Let $s \in \mathbb{C}, \mathrm{Re}s > n$. and let $r \notin -\Im\,\mathrm{spec}_b(A)$. The eta function associated to the indicial operator of A is defined by*

$$\tilde{\eta}_r(s,I(A)) = \Big(\frac{1+e^{i\pi s}}{2i\sin\pi s}\Big)\zeta_r^-(s,I(A)) - \Big(\frac{1+e^{-i\pi s}}{2i\sin\pi s}\Big)\zeta_r^+(s,I(A)) \tag{3.3.8}$$

Remarks. (3.3.9) Because of Proposition (3.3.4), $\tilde{\eta}_r(s,I(A))$ has a meromorphic continuation to the whole s-plane with only simple poles at the points $s_j = n-j, j \in \mathbb{N}$. Notice however that, because of the denominator in (3.3.8), $s=0$ is now a possible pole.

(3.3.10) A straightforward computation shows that for $s \in \mathbb{C}, \mathrm{Re}s > n$ we also have

$$(3.3.11)\qquad \tilde\eta_r(s, I(A)) = \operatorname{tr}\Big(\frac{i}{2\pi}\int_{\Delta_r^-} \lambda^{-s} I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda)d\lambda - (-1)^{-s} \int_{\Delta_r^+} \lambda^{-s} I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda)d\lambda\Big)$$

where $\Delta_r^- = \{z \in \mathbb{C}; -\Im z = r + \epsilon\}$ oriented in the sense of $\operatorname{Re} z$ decreasing, $\Delta_r^+ = \{z \in \mathbb{C}; \Im z = -r+\epsilon\}$ oriented in the sense of $\operatorname{Re} z$ increasing and where the complex power is taken with respect to the cut $0 < \arg\lambda < 2\pi$.

We want to prove that the meromorphic extension of $\tilde\eta_r$ is regular at $s = 0$.

However, before doing this, we want to give a rather fundamental example, showing how our definition (3.3.8) generalizes the one given by Atiyah, Patodi and Singer in [4].

Example 3.3.12 We fix a conic (or b-) metric on the manifold with boundary X. This is, by definition, a metric on the compressed tangent bundle ${}^bTX$.
In local coordinates near the boundary we can write a b-metric in the form

$$(3.3.13)\qquad g = a_{00}\Big(\frac{dx}{x}\Big)^2 + 2\sum_{j=1}^{n-1} a_{0j}\frac{dx}{x}dy^j + \sum_{j,k=1}^{n-1} a_{jk}dy^j dy^k$$

where the coefficient are $\mathcal{C}^\infty$ and the quadratic form

$$(3.3.14)\qquad a_{00}\lambda^2 + 2\sum_{j=1} a_{0j}\lambda\eta^j + \sum_{j,k=1}^{n-1} a_{jk}\eta^j\eta^k$$

is positive definite.
As in [4] we will require such a metric to be of the form

$$(3.3.15)\qquad g = \Big(\frac{dx}{x}\Big)^2 + h$$

near the boundary, where h is a metric on ∂X.
Thus let F_0, F_1 be hermitian vector bundles and let A be an elliptic element in $\operatorname{Diff}_b^1(X; F_0, F_1)$ (the density bundles have been trivialized using the metric g). We assume that

$$(3.3.16)\qquad I(A) = {}^b\sigma(xD_x + \frac{1}{i}E)$$

where ${}^b\sigma = {}^b\sigma(A)(\frac{dx}{x})$ is the isomorphism $F_0 \longrightarrow F_1$ given by the symbol of A restricted to ${}^bT^*_{\partial X}X$ and $E \in \mathrm{Diff}^1(\partial X; F_0 |_{\partial X}, F_1 |_{\partial X})$ is invertible and self-adjoint with respect to h and the induced hermitian structures on $F_0 |_{\partial X}, F_1 |_{\partial X}$. From (3.3.16) we obtain

$$I(A, z) = {}^b\sigma(z + \frac{1}{i}E) \tag{3.3.17}$$

so that $z \in \mathrm{spec}_b(A) \iff \frac{1}{i}z \in \mathrm{spec}(E)$. Since E is invertible $0 \notin -\Im\,\mathrm{spec}_b(A)$. Thus

$$\zeta_0^-(s, I(A)) = \mathrm{tr}(\frac{i}{2\pi}\int_{\Gamma_-} \lambda^{-s} i(E - \frac{1}{i}\lambda)^{-1} d\lambda) = e^{-\frac{\pi}{2}is}\,\mathrm{tr}\,\frac{1}{2\pi i}\int_{\Gamma_\uparrow} \mu^{-s}(E-\mu)^{-1}d\mu$$

$$\equiv e^{-\frac{\pi}{2}is}\zeta_\uparrow(s, E)$$

where $\zeta_\uparrow(s, E)$ denotes the classical zeta function of E with respect to the cut in the positive imaginary axis. Similarly $\zeta_-^+(s, I(A)) = e^{-\frac{\pi}{2}is}\zeta_\downarrow(s, E)$ whereas $\tilde{\eta}_0(s, I(A)) = e^{-\frac{\pi}{2}is}\eta_E(s)$ where

$$\eta_E(s) = \sum_{\lambda \neq 0} (\mathrm{sign}\,\lambda)|\lambda|^{-s} \tag{3.3.18}$$

denotes the eta function of E as introduced by Atiyah, Patodi and Singer [4]. More generally we could consider $A \in \mathrm{Diff}_b^k(X; F_0, F_1)$ with the property that

$$I(A) = {}^b\sigma((xD_x)^k + \frac{1}{i}E)$$

E being a self adjoint and invertible differential operator of order k on ∂X acting between the sections of $F_0 |_{\partial X}$ and $F_1 |_{\partial X}$. A routine computation then shows

$$\tilde{\eta}_0(s, I(A)) = e^{-\frac{\pi}{2}i\frac{s}{k}}\eta_E(\frac{s}{k}).$$

Notice, in particular, that for such elliptic b-differential operators

$$\mathrm{Res}_{s=0}\,\tilde{\eta}_0(s, I(A)) = k\,\mathrm{Res}_{s=0}\,\eta_E(s). \tag{3.3.19}$$

3.4 The regularity at 0 of $\tilde{\eta}_r$

The goal of this section is to prove the following

THEOREM 3.4.1. . *Let* $A \in \Psi_b^m(X, F_0 \otimes \Omega_b^{\frac{1}{2}}, F_1 \otimes \Omega_b^{\frac{1}{2}})$ *be elliptic and let* $r \notin -\Im \operatorname{spec}_b(A)$. *Then the meromorphic extension of* $\tilde{\eta}_r(s, I(A))$ *is regular at* $s = 0$.

Before going into the proof we notice some consequences of Theorem (3.4.1). If $r' < r$ we then have, using (3.3.11),

$$\begin{aligned}\tilde{\eta}_{r'}(s) - \tilde{\eta}_r(s) = \operatorname{tr}\Big(\frac{i}{2\pi} \sum_{\substack{p \in \operatorname{spec}_b(A) \\ r' < -\Im p < r}} \Big(\oint_{C_p^+} \lambda^{-s} I_A(\lambda)^{-1} \cdot \frac{d}{d\lambda} I_A(\lambda) d\lambda - \\ (-1)^s \frac{i}{2\pi} \oint_{C_r^-} \lambda^{-s} I_A(\lambda)^{-1} \frac{d}{d\lambda} I_A(\lambda)\Big)\Big)\end{aligned} \tag{3.4.2}$$

where $C_p = \{z \in \mathbb{C};\ |z - p| = \epsilon_p\}$, ϵ_p is so small that no indicial roots, beside p itself, are contained in the contour C_p and we use $+$ and $-$ for the orientation of the contour ($-$ is clockwise). If we assume the regularity of $\tilde{\eta}_r$ of $s = 0$ we then have

$$\begin{aligned}\tilde{\eta}_r(0) - \tilde{\eta}_{r'}(0) &= 2\Big(\sum_{\substack{p \in \operatorname{spec}_b(A) \\ r' < -\Im p < r}} \Big(\operatorname{tr} \frac{1}{2\pi i} \oint_{C_p^+} I_A(\lambda)^{-1} \frac{d}{d\lambda} I_A(\lambda) d\lambda\Big)\Big) \\ &= 2\Big(\sum_{\substack{p \in \operatorname{spec}_b(A) \\ r' < -\Im p < r}} (ord(p))\Big).\end{aligned} \tag{3.4.3}$$

Recalling the relative index theorem for elliptic b-pseudodifferential operator we obtain the following corollary of (3.4.1)

COROLLARY 3.4.4. *Let* $A \in \Psi_b^m(X; F \otimes \Omega_b^{\frac{1}{2}}, F_1 \otimes \Omega_b^{\frac{1}{2}})$ *be elliptic and let* $r \notin -\Im \operatorname{spec}_b(A)$. *Then* $\operatorname{Ind}_r(A) + \frac{\tilde{\eta}_r(0, I(A))}{2}$ *is independent of* r.

Let us prove Theorem (3.4.1). We want to show that $\operatorname{Res}_{s=0} \tilde{\eta}_r(s, I(A)) = 0$. We point out that since the right hand side of (3.4.2) is holomorphic in s, $\operatorname{Res}_{s=0} \tilde{\eta}_r(s, I(A))$ does not depend on r and will be therefore denoted by $R(I(A))$. Our eta function, and thus its residue at 0, is defined for every elliptic element $B \in \Psi_{b,I}^m(\overline{N_+ \partial X}, V_0 \otimes \Omega_b^{\frac{1}{2}}, V_1 \otimes \Omega_b^{\frac{1}{2}})$ where V_0, V_1 are smooth vector bundles over ∂X. For arbitrary vector bundles V_0, V_1, the stable-homotopy classes of the symbols of the elliptic elements in these spaces of operators, form a group naturally isomorphic to $K^0({}^bT^*(\overline{N_+ \partial X})) = K^{-1}(T^* \partial X)$ and the main step of the proof is to show that $\mathrm{Res}_{s=0}\,\tilde{\eta}_r(s,B) = R(B)$ factors through a homomorphism $K^{-1}(T^*\partial X) \longrightarrow \mathbf{C}$. The only delicate point is to show that $R(B)$ is a homotopy invariant. Let us now consider a smooth 1-parameter family of elliptic b-pseudodifferential operator, B_u, on $\overline{N^*\partial X}$. Notice that for u small enough we can always assume that there exists $r \in \mathbf{R}$ s.t. $r \notin -\Im\,\mathrm{spec}_b(B_u)\ \forall\ u$. Let us assume, without loss of generality, that $r = 0$.

LEMMA 3.4.5. *For Res large and B_u as above*

$$\frac{d}{du}\zeta_0^{\pm}(s,B_u) = -s(\mathrm{tr}\,\frac{1}{2\pi i}\int_{\Gamma_\pm} \lambda^{-(s+1)} I_{B_u}(\lambda)^{-1}\frac{d}{du}I_{B_u}(\lambda)d\lambda).$$

PROOF: We want to prove that for Res >> 0

$$\frac{d}{du}(\mathrm{tr}(\frac{i}{2\pi}\int_{\Gamma^\pm} \lambda^{-s} I_{B_u}(\lambda)^{-1}\frac{d}{d\lambda}I_{B_u}(\lambda)d\lambda))$$

$$= \mathrm{tr}\,\frac{i}{2\pi}\int_{\Gamma_\pm} \lambda^{-1}\partial_\lambda(I_{B_u}(\lambda)^{-1}\frac{d}{du}I_{B_u}(\lambda))d\lambda \tag{3.4.6}$$

Since Res >> 0 we can rewrite the left hand side of (3.4.6) as

$$\frac{d}{du}\,\mathrm{tr}(\frac{i}{2\pi}\int_{\Gamma_\pm}(\prod_{k=1}^{N}\frac{1}{(k-s)})\partial_\lambda^N \lambda^{-s+n}\left(I_{B_u}(\lambda)^{-1}\frac{d}{d\lambda}I_{B_u}(\lambda)\right)d\lambda);$$

Our arguments in §3.2, and in particular (3.2.10), show that for N large enough the operator $\partial_\lambda^N(I_{B_u}(\lambda)^{-1}\frac{d}{d\lambda}I_{B_u}(\lambda))$ is trace class and bounded in λ. Thus, after integrating by parts, we can interchange integration over $\Gamma^\pm$ with taking the trace, obtaining for the left hand side of (3.4.6).

$$\frac{i}{2\pi}\int_{\Gamma_\pm}(\prod_{k=1}^{N}\frac{\lambda^{-s+N}}{(k-S)})\,\mathrm{tr}(\frac{d}{du}\partial_\lambda^N(I_{B_u}(\lambda)^{-1}\frac{d}{d\lambda}I_{B_u}(\lambda)))$$

Applying the same argument to the right hand side of (3.4.6) we are reduced to showing that

$$\mathrm{tr}[\frac{d}{du}(\partial_\lambda^N(I_{B_u}(\lambda)^{-1}\frac{d}{d\lambda}I_{B_u}(\lambda)))] = \mathrm{tr}[\partial_\lambda^{N+1}(I_{B_u}(\lambda)^{-1}\frac{d}{du}I_{B_u}(\lambda))]. \tag{3.4.7}$$

This is established by making repeated use of the trace identity (i.e. the fact that $tr[A,B]=0$ if A is bounded and B is trace class). □

Remark. A different way of proving (3.4.6), and hence (3.4.5), is to use the holomorphic functional calculus to express $I_{B_u}(\lambda)^{-1}$ as a contour integral. After some straightforward computations one is reduced once again to an application of the trace identity. The details of this approach as well as the somewhat lengthy computation establishing (3.4.7) are left to the reader.

PROPOSITION 3.4.8. *Let B be an elliptic b-pseudodifferential operator on the collar neighborhood $\overline{N^+\partial X}$ and let $R(B)=\operatorname{Res}_{s=0}\tilde{\eta}_r(s,B)$. Then $R(B)$ is constant for continuous variation of B.*

PROOF: Using (3.4.2) and a standard approximation argument we can consider a smooth 1-parameter family B_u as in Lemma (3.4.5). We thus want to prove that

$$\frac{d}{du}\Big|_{u=0}\operatorname{Res}_{s=0}\tilde{\eta}_0(s,B_u)=0$$

or, equivalently, that $\operatorname{Res}_{s=0}(\frac{d}{du}\tilde{\eta}_0(s,B_u)\big|_{u=0})=0$. Using Lemma (3.4.5) we have, for large $\operatorname{Re}s>0$,

$$\frac{d}{du}\tilde{\eta}_0(s,B_u)=-s\Big(\frac{1+e^{-i\pi s}}{2i\sin\pi s}\Big)\Big(\operatorname{tr}\frac{i}{2\pi}\int_{\Gamma_-}\lambda^{-(s+1)}I_{B_u}(\lambda)^{-1}\frac{d}{du}I_{B_u}(\lambda)d\lambda\Big)+$$
$$s\Big(\frac{1+e^{i\pi s}}{2i\sin\pi s}\Big)\Big(\operatorname{tr}\frac{i}{2\pi}\int_{\Gamma_+}\lambda^{-(s+1)}I_{B_u}(\lambda)^{-1}\frac{d}{du}I_{B_u}(\lambda)d\lambda\Big)$$

and we only have to show that the difference

$$\begin{aligned}&\operatorname{Res}_{s=0}\Big(\operatorname{tr}\frac{i}{2\pi}\int_{\Gamma_\pm}\lambda^{-(s+1)}I_{B_0}(\lambda)\frac{d}{du}\Big|_{u=0}I_{B_u}(\lambda)d\lambda\Big)+\\ &-\operatorname{Res}_{s=0}\Big(\operatorname{tr}\frac{i}{2\pi}\int_{\Gamma_-}\lambda^{-(s+1)}I_{B_u}(\lambda)^{-1}\frac{d}{du}\Big|_{u=0}I_{B_u}(\lambda)d\lambda\Big)=0\end{aligned}\tag{3.4.9}$$

In order to prove this we just have to modify our analysis of the meromorphic properties of the zeta function associated to $I(A)$.

The operator $I_{B_u}(\lambda)^{-1}\frac{d}{du}\Big|_{u=0} I_{B_u}(\lambda)$ is a 0-order pseudodifferential operator on ∂X locally corresponding to a classical symbol $\gamma(y,\eta;\lambda)\in S^0(U\times \mathbf{R}^{n+1}_{(\eta,\lambda)}; End(V_0))$. For $\lambda\in\{z\in\mathbf{C}, \Im z=0\}$ we certainly have

$$\gamma(y,\eta;\lambda)\sim\sum_{j=0}^{\infty}\gamma_{-j}(y,\eta;\lambda)$$

Following §3.2 we denote by $\hat{\gamma}_{-j}$ an almost analytic extension of $\gamma_{-j}(\lambda), \lambda\in\{z\in\mathbf{C};\Im z=0\}$ and we define

$$d^{\pm}_{-j}(w)(y,\eta)=\lim_{\epsilon\to 0}\Big(\frac{i}{2\pi}\int_{\Gamma^{\pm}_{\epsilon}}\lambda^{w-1}\hat{\gamma}_{-j}(y,\eta;\lambda)d\lambda\Big) \tag{3.4.10}$$

for $w\in\mathbf{C}, Re w<j$.

Then, with a proof analogous to the one given for (3.3.4), we can explicitly write down the left hand side of (3.4.9) as

$$\left(\frac{1}{2\pi}\right)^n\int_{\partial X}\int_{|\eta'|=1} tr\left(d^{+}_{-n}(0)(y,\eta')-d^{-}_{-n}(0)(y,\eta)\right)d\eta' dy.$$

Since we are considering $w=0$ in (3.4.10), we certainly have

$$d^{+}_{-n}(0)=d^{-}_{-n}(0).$$

Thus (3.4.9) holds and the proposition is proved. □

Since $R(\ )$ is additive and since $R(\Phi)=0$ whenever Φ is a bundle isomorphism, we conclude, by Proposition (3.4.8), that $R(B)$ only depends on the stable homotopy class of the principal symbol of $B\in\Psi^m_{b,I}$ so that $R(\ )$ induces a well defined homomorphism

$$\rho: K^{-1}(T^*(\partial X))\longrightarrow\mathbf{C}$$
$$[\sigma_m(B)]\longrightarrow R(B)$$

By **[5]** in the case in which $\dim\partial X$ is odd and by **[22]** in the case in which $\dim\partial X$ is even, there exist sets of elliptic self adjoint differential operators on ∂X (respectively $\{B^{ev}_V\}_{V\in \mathrm{Vect}(\partial X)}$ and $\{P(\Psi)\}$ where $\Psi:\partial X\longrightarrow S^j, 0<j<n, j$ odd) whose principal symbols generate $K^1(T^*X)$ over $\mathbf{Q}$ and such that, respectively, $\mathrm{Res}_{s=0}\,\eta_{B^{ev}_V}(s)=$

0 and $\mathrm{Res}_{s=0}\,\eta_{P(\Psi)}(s) = 0$. Using the example given in §3 (and in particular (3.3.19)) and the periodicity isomorphism $K^1(T^*\partial X) \longrightarrow K^{-1}(T^*\partial X)$ we then have that $\rho \equiv 0$ on a set of rational generators for $K^{-1}(T^*\partial X)$ so that ρ is the zero homomorphism. In particular $\rho[\sigma_m(I(A))] = R(I(A)) = \mathrm{Res}_{s=0}\,\tilde{\eta}_0(s, I(A)) = 0$ and we are done. □

§3.5 A different regularization

In the last three sections we introduced an eta invariant, $\tilde{\eta}_r(s, I(A))$, associated to the indicial family of an elliptic b-pseudodifferential operator $A \in \Psi_b^m(X; E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}})$. We can think to the value $\tilde{\eta}_r(0, I(A))$ as a measurement for the asymmetry of the boundary spectrum of A with respect to the line $\{z \in \mathbf{C}; \Im z = -r\}$. Our definition can be viewed as a way of regularizing the expression

$$\frac{tr}{2\pi i}\int_{\Im z=-r} I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda)d\lambda \tag{3.5.1}$$

A different regularization has been proposed by R. Melrose*; in this section we will introduce yet another way of regularizing (3.5.1). As we will show in the next chapter it is exactly this last definition that enters in a natural way as the boundary correction term in the index formula.

Thus let A be an elliptic element in $\Psi_b^m(X; E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}})$ and let Res $> n/m$ where $n = \dim \partial X$. Assume for the moment that $0 \notin -\Im\,\mathrm{spec}_b(A)$.
We define our new eta function as

$$\eta_0(s, I(A)) = \frac{tr}{2\pi i}\int_{\Im z=0} I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda)(I_{A^*A}(\lambda))^{-\frac{s}{2}}d\lambda \tag{3.5.2}$$

where we choose the cut in the complex plane defined by $0 < \arg\lambda < 2\pi$.

*Private communication

As in §3.3 we extend the definition to any $r \notin -\Im \operatorname{spec}_b(A)$ by

$$\eta_r(s, I(A)) = \eta_0(s, I(x^{-r}Ax^r)). \tag{3.5.3}$$

By making use of (3.1.15), (3.2.19) and

$$I_{A^*}(z) = (I_A(\bar{z}))^* \tag{3.5.4}$$

we obtain right away

$$\eta_r(s, I(A)) = \frac{tr}{2\pi i} \int\limits_{\Im\lambda=-r} I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda)(I_A(\lambda)^* I_A(\lambda))^{-\frac{s}{2}}d\lambda. \tag{3.5.5}$$

In order to prove that (3.5.2) is well defined we proceed as in §3.2 and §3.3. Assume, for simplicity, that $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$. From (3.1.15) and (3.5.4) we see that $I_{A^*A}(\lambda)$ is a positive self-adjoint operator $\forall\, \lambda \in \{z \in \mathbb{C}; \Im z = 0\}$. Thus

$$\operatorname{spec}(I_{AA^*}(\lambda)) \cap \Gamma^- = \emptyset \quad \forall\, \lambda \in \{\Im z = 0\}$$

where Γ^- is the path in the complex plane described in §3.2.
Let $s \in \mathbb{C}, \operatorname{Re} s > 0$. By the work of Seeley we know that for each $\lambda \in \{z \in \mathbb{C}; \Im z = 0\}$

$$I_{A^*A}(\lambda)^{-\frac{s}{2}} = \frac{1}{2\pi i}\int\limits_{\Gamma^-} \mu^{-\frac{s}{2}}(I_{A^*A}(\lambda) - \mu)^{-1}d\mu$$

is a pseudodifferential operator of order ms with symbol given, in local coordinates, by

$$\sigma(I_{A^*A}(\lambda)^{-\frac{s}{2}}) \sim \sum_{j=0}^{\infty} \frac{1}{2\pi i}\int\limits_{\Gamma^-} \mu^{-\frac{s}{2}} b_{-2m-j}(\lambda, \mu)d\mu \tag{3.5.6}$$

where the b_{-2m-j} are fixed locally by

$$\begin{gathered} b(\lambda,\mu) \sim \sum_{j=0}^{\infty} b_{-2m-j}(\lambda,\mu) \\ b(\lambda,\mu)\sigma(I_{A^*A}(\lambda) - \mu) = 1 \end{gathered} \tag{3.5.7}$$

From (3.2.7), applied to AA^*, it is then easy to show that $\sigma(I_{A^*A}(\lambda)^{-\frac{s}{2}}) \in S^{-ms}(U\times \mathbf{R}^{n+1}_{(\eta,\lambda)})$, U being our fixed coordinate chart. The usual estimates then show that

$$\|I_{AA^*}(\lambda)^{-\frac{s}{2}}\|_{\ell,\ell} \leq C_\ell |\lambda|^{-ms} \quad \lambda \text{ large.}$$

Together with Lemma (3.2.2) this proves that for Re $s < 0$ the integral

$$\frac{1}{2\pi i}\int\limits_{\Im\lambda=0} I_A(\lambda)^{-1}\frac{d}{d\lambda}I_A(\lambda)(I_{A^*A}(\lambda))^{-\frac{s}{2}}d\lambda$$

converges to define an operator $W_0^s : H^\ell \longrightarrow H^\ell\ \forall\ \ell \in \mathbf{R}$. Since the integrand is a pseudodifferential operator of order $-ms-1$, it is certainly trace class for Res >> 0; the corresponding Schwartz kernel is a continuous half-density whose absolute value is bounded by $|\lambda|^{-ms-1}$ for λ large. We can thus interchange integration over $\{\Im z = 0\}$ with taking traces, this proves that (3.5.2) is well defined.

Example 3.5.8

Consider a first order elliptic b-differential operator $A \in \mathrm{Diff}^1_b(X; F_0, F_1)$ as in Example (3.3.12). Then

$$\begin{aligned}
\eta_0(s, I(A)) &= \frac{tr}{2\pi i}\int\limits_{\Im\lambda=0}(\lambda + iE)(I_A(\lambda)I_{A^*}(\lambda))^{-\frac{s}{2}-1} \\
&= \frac{tr}{2\pi i}\int\limits_{\Im\lambda=0}(\lambda + iE)\left(\frac{1}{\Gamma(\frac{s}{2}+1)}\int_0^\infty t^{\frac{s}{2}+1}e^{-t(\lambda^2+E^2)}\frac{dt}{t}\right)d\lambda \\
&= \frac{1}{\Gamma(\frac{s}{2}+1)}\int_0^\infty t^{\frac{s}{2}+1}\left(\frac{tr}{2\pi i}\int_0^\infty(\lambda + iE)e^{-t(\lambda^2+E^2)}d\lambda\right)\frac{dt}{t} \\
&= \frac{1}{\Gamma(\frac{s}{2}+1)}\int_0^\infty t^{\frac{s}{2}+1}\left(\frac{tr}{2\pi}\int\limits_{\Im z=0}(Ee^{-tE^2})e^{-t\lambda^2}d\lambda\right)\frac{dt}{t} \\
&= \frac{1}{\Gamma(\frac{s}{2}+1)}\int_0^\infty t^{\frac{s}{2}+1}\frac{t^{-\frac{1}{2}}}{2\sqrt{\pi}}tr(Ee^{-tE^2})\frac{dt}{t} \\
&= \left(\frac{\Gamma(\frac{s}{2}+\frac{1}{2})}{2\sqrt{\pi}\Gamma(\frac{s}{2}+1)}\right)\frac{tr}{\Gamma(\frac{s}{2}+\frac{1}{2})}\int_0^\infty t^{(s-1)/2}Ee^{-tE^2}dt \\
&= \left(\frac{\Gamma(\frac{s}{2}+\frac{1}{2})}{2\sqrt{\pi}\Gamma(\frac{s}{2}+1)}\right)\eta_E(s)
\end{aligned}$$

Hence, in this particular case, $\eta_0(s, I(A))$ admits a meromorphic continuation to the whole s-plane which is regular at $s = 0$ and, furthermore,

$$\eta_0(0, I(A)) = \frac{1}{2}\eta_E(0) \tag{3.5.9}$$

Of course we could go on as in §3.3 and §3.4 studying the meromorphic continuation of $\eta_r(s, I(A))$ and proving in particular that $s = 0$ is a regular point. However since these properties will be a consequence of our main formula in the next chapter we simply state the result :

THEOREM 3.5.10. *Let $r \notin -\Im\,\mathrm{spec}_b(A)$ and $\mathrm{Re}\, s > \frac{n}{m}$. The function $\eta_r(s, I(A))$ admits a meromorphic extension to the whole complex plane with only simple poles. The point $s = 0$ is regular.*

Assuming this theorem we look at the difference $\eta_r(0, I(A)) - \eta_{r'}(0, I(A))$ for $r > r$, $r, r' \notin -\Im\,\mathrm{spec}_b(A)$. From (3.5.5) we get

$$\eta_r(0, I(A)) - \eta_{r'}(0, I(A)) = \sum_{\substack{p \in \mathrm{spec}_b(A) \\ r' < -\Im z < r}} \left(\oint_{C_\epsilon^+} I_A(\lambda)^{-1} \frac{d}{d\lambda} I_A(\lambda) d\lambda \right)$$

or, in other words,

$$\eta_r(0, I(A)) - \eta_{r'}(0, I(A)) = \sum_{\substack{p \in \mathrm{spec}_b(A) \\ r' < -\Im z < r}} (ord(p)).$$

This shows that

$$\mathrm{ind}_r(A) + \eta_r(0, I(A)) = C(A) \tag{3.5.11}$$

i.e. it is independent of r. In the next chapter we will relate this sum to the meromorphic extension of the (regularized) trace of the complex powers of A^*A and AA^*.

4. AN ANALYTIC INDEX FORMULA

4.1. b-pseudodifferential operators: the full calculus.

In this section (which follows closely [**33**]) we resume our discussion on the Fredholm property of b-pseudodifferential operators.
In §3.1 we stated the main result (Theorem 3.1.23); if $A \in \Psi_b^m(X; E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}})$ and $r \notin -\Im \operatorname{spec}_b(A)$ then as a map

$$(4.1.1) \qquad A : x^r H_b^M(X, E \otimes \Omega_b^{\frac{1}{2}}) \longrightarrow x^r H_b^{M-m}(X; F \otimes \Omega_b^{\frac{1}{2}})$$

the operator A is Fredholm

In the sequel we will assume E and F to be trivial bundles $X \times \mathbf{C}$; it will be clear how to modify the statements in the general case. As explained in §3.1 (see in particular the discussion following Proposition (3.1.11)) the fundamental step in the proof of Theorem (3.1.23) is the inversion of the indicial operator

$$(4.1.2) \qquad I(A) : x^r H_b^m(\overline{N_+\partial X}, \Omega_b^{\frac{1}{2}}) \longrightarrow x^r H_b^{M-m}(\overline{N_+\partial X}, \Omega_b^{\frac{1}{2}}).$$

This is possible only if $r \notin \Im \operatorname{spec}_b(A)$. To construct the inverse we consider the indicial family $I_A(z)$ and its inverse $I_A(z)^{-1}$. The proof of Proposition (3.1.19) shows that

$$(4.1.3) \qquad I_A(z)^{-1} = I_{B_\sigma}(z) + G(z)$$

where B_σ is a symbolic parametrix of A. The first term on the right hand side is holomorphic in z (being the indicial family of an element in $\Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}})$); the second is a meromorphic family of smoothing operators having residues of finite rank at the points of $\operatorname{spec}_b(A)$ and decaying rapidly as $|\operatorname{Re} z| \longrightarrow +\infty$ when $|\Im z|$ is bounded.

The Schwartz kernel corresponding to the inverse of (4.1.2) is obtained by considering the inverse Mellin Transform of (4.1.3). Thus if $r \notin -\Im\,\mathrm{spec}_b(A)$ we have

$$
\begin{aligned}
K(I(A)^{-1})(s,y,y') &= \frac{1}{2\pi}\int_{\Im\lambda=-r} s^{i\lambda} I_A(\lambda)^{-1}\,d\lambda \\
&= I(B_\sigma)(s,y,y') + G(s,y,y')
\end{aligned}
\tag{4.1.4}
$$

Here, as in §3.1, we are using the local coordinates induced on the front face by the projective coordinates $(x', s, y, y'), s = x/x'$ in an open neighbourhood of $\mathrm{ff}(X_b^2)$ in X_b^2. The meromorphic properties of $G(z)$ transflate, under the inverse Mellin transform, into an asymptotic expansion

$$
G(s,y,y') \sim \begin{cases} \sum_{\substack{z\in\mathrm{spec}_b(A)\\ k\le \mathrm{ord}(z)\\ \Im(z)<-r}} s^{iz}(\log(s))^k A_{z,k}(y,y') & \text{as } s\downarrow 0 \\ \sum_{\substack{z\in\mathrm{spec}_b(A)\\ k\le \mathrm{ord}(z)\\ \Im z>-r}} \left(\frac{1}{s}\right)^{-iz}\left(\log\left(\frac{1}{s}\right)\right)^k A_{z,k}(y,y') & \text{as } s\uparrow +\infty \end{cases}
\tag{4.1.4}
$$

where $\mathrm{ord}(z)$ denotes the order of $z \in \mathrm{spec}_b(A)$ as a pole of the inverse indicial family and where the coefficients correspond to finite rank smoothing operator; the asymptotic expansion means that the difference

$$
G(s,y,y') - \sum_{-N\le -\Im z<-r} s^{iz}(\log(s))^k A_{z,k}(y,y') \in \dot{C}^N([0,1)\times(\partial X)^2)
$$

i.e. it is N times continuously differentiable with all N derivatives vanishing at $s = 0$. Thus the inverse of $I(A)$, as a map (4.1.2), is the sum of an element in the calculus, $I(B_\sigma)$, and an operator whose kernel in X_b^2 is smooth up to the front face but with singularities, described explicitly by (4.1.4), at the left ($s = 0$) and right ($s = +\infty$) bounderies.

This suggest that we extend the small b-calculus to allow for these kernels; to do this properly we must specify the exponents that we allow in the asymptotic expansions (see **[33]** for more details on what follows). We thus recall the notion of index set.

An index set is a subset

$$
E \subset \mathbb{C}\times\mathbb{N}
$$

with the following properties

$$(4.1.5) \qquad E \text{ is discrete}$$

$$(4.1.6) \qquad (z,k) \in E \Longrightarrow (z,\ell) \in E \qquad 0 \leq \ell \leq k$$

$$(4.1.7) \qquad (z,k) \in E \Longrightarrow (z-i\ell,k) \in E \qquad \forall\, \ell \in \mathbb{N}$$

$$(4.1.8) \qquad (z_j,k_j) \in E, |(z_j,k_j)| \longrightarrow +\infty \Longrightarrow \Im z_j \longrightarrow -\infty$$

Then if H is a boundary hypersurface in a manifold with corners X and ρ is a defining function for H, we define the space of polyhomogeneous conormal distributions with index set E as

$$(4.1.9) \qquad \begin{aligned} & u \in \mathcal{A}^E_{phg}(X) \Longleftrightarrow \exists\; a_{z,k} \in \mathcal{C}^\infty(X)\ \forall\,(z,k) \in E \\ & \text{s.t. } \forall\, N, u - \sum_{\substack{(z,k)\in E \\ \Im z \geq -N}} \rho^{iz}(\log\rho)^k a_{z,k} \in \dot{C}^N(X) \end{aligned}$$

where $\dot{C}^N(X)$ is the space of functions on X which are N times differentiable on X and vanish at $\rho = 0$ with all derivatives up to order N.

For example for the index set $E = -i\mathbb{N} \times \{0\}$ we have $\mathcal{A}^E_{\text{phg}}(X) = \mathcal{C}^\infty(X)$. Notice that condition (4.1.7) ensures that $\mathcal{A}^E_{\text{phg}}(X)$ is a $\mathcal{C}^\infty(X)$-module. One can extend this notion to allow asymptotic expansions at different hypersurfaces $H_1, \ldots, H_k$. The corresponding notion is the one of index family $\mathcal{E} = (E_1, \ldots, E_k)$ where each E_j is an index set associated to H_j. If $\mathcal{E}$ is an index set, then

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \subset \mathcal{A}(X)$$

where $\mathcal{A}(X) = \{u \in C^{-\infty}(X); Pu \in H^s(X) \text{ for some } s = s(u),\ \forall\, P \in \text{Diff}^*_b(X)\}$, is the space of all extendable distributions conormal at the boundary of X.

Let $E_{\ell b}$ and E_{rb} be index sets associated to the right and left boundary and denote by 0 the index set $-i\mathbb{N} \times \{0\}$. Consider the index family $\mathcal{E} = (E_{\ell b}, E_{rb}, 0)$ associated to the ordered triple of boundary hypersurfaces $(\ell b, rb, \text{ff})$. We define

$$\tilde{\Psi}_b^{-\infty,\mathcal{E}}(X, \Omega_b^{\frac{1}{2}}) \equiv \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X^2_b, \Omega_b^{\frac{1}{2}})$$

and

$$\tilde{\Psi}_b^{-m,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}}) \stackrel{\text{def}}{=} \Psi_b^m(X,\Omega_b^{\frac{1}{2}}) + \tilde{\Psi}_b^{-\infty,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}}) \tag{4.1.10}$$

One can prove that for any index family as above, $\tilde{\Psi}_b^{-\infty,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}})$ is a two-sided module over $\Psi_b^*(X,\Omega_b^{\frac{1}{2}})$. The exact sequence (2.1.21) yields the symbol sequence

$$0 \longrightarrow \tilde{\Psi}_b^{m-1,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}}) \longrightarrow \tilde{\Psi}_b^{m,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}}) \longrightarrow S^{[m]}({}^bT^*X) \longrightarrow 0$$

to which the extra term (being $\mathcal{C}^\infty$ near Δ_b) does not contribute.

Finally restriction to the front face defines once again the indicial operator and we have

$$0 \longrightarrow \rho_{\mathrm{ff}}\tilde{\Psi}_b^{m,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}}) \longrightarrow \tilde{\Psi}_b^{m,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}}) \longrightarrow \tilde{\Psi}_{b,I}^{m,\mathcal{E}}(\overline{N_+}\partial X,\Omega_b^{\frac{1}{2}}) \longrightarrow 0 \tag{4.1.11}$$

Let A be an elliptic operator in $\Psi_b^m(X,\Omega_b^{\frac{1}{2}})$ and let B_σ a symbolic parametrix

$$\begin{aligned} B_\sigma A &= 1 - R \\ &R, R' \in \Psi_b^{-\infty} \\ AB_\sigma &= 1 - R'. \end{aligned} \tag{4.1.12}$$

Recall ((2.1.32), (3.1.11)) that the indicial operator of R represents the obstruction to the compactness of R as a map on weighted Sobolev spaces. To remove this obstruction we construct a parametrix in the extended calculus. Thus let $r \notin -\Im\,\mathrm{spec}_b(A)$ and let $\mathcal{E}(r) = (E^+(r), E^-(r), 0)$ be associated to $(\ell b, rb, \mathrm{ff})$ with

$$\begin{aligned} &E_\pm(r) = \text{ smallest index set containing } F^\pm(r) \\ &\text{where} \\ &F^\pm(r) = \Big\{(w_{j,k}^\pm, m(w_{j,k}^\pm)) \text{ where} \\ &\quad (i)\quad w_{j,k}^\pm = \pm z_j - ik \text{ with } z_j \in \ spec_b(A), k \in \mathbb{N} \\ &\quad (ii)\quad m(w_{j,k}^\pm) = \sum_{\substack{p = w_{j,q}^\pm \\ 0 \le q \le k}} \mathrm{ord}(p) \\ &\quad (iii)\quad \pm\Im z_j < \mp r\Big\} \end{aligned} \tag{4.1.13}$$

It's easy to see that from (2.1.26) $\forall\ m, M, r \in \mathbf{R}$,

$$P \in \tilde{\Psi}_b^{m,\mathcal{E}(r)}(X,\Omega_b^{\frac{1}{2}}) \Longrightarrow P : x^r H_b^M(X,\Omega_b^{\frac{1}{2}}) \Longrightarrow x^r H_b^{M-m}(X,\Omega_b^{\frac{1}{2}})$$

is continuous.

Then we can obtain a left parametrix $B_r \in \tilde{\Psi}_b^{-m,\mathcal{E}(r)}$ for A as a map $x^r H_b^M \longrightarrow x^r H_b^{M-m}$ by considering

$$B_r = B_\sigma + B_r'$$

where $B_r' \in \tilde{\Psi}_b^{-\infty,\mathcal{E}(r)}(X,\Omega_b^{\frac{1}{2}})$ is obtained from (4.1.11) by solving the indicial equation

$$I(B_r')I(A) = I(R) \tag{4.1.14}$$

i.e. by considering the inverse Mellin Transform of $I_R(z)I_A(z)^{-1}$.

This completes our sketch of the proof of (3.1.13) as we now have an operator $B_r : x^r H_b^{M-m} \longrightarrow x^r H_b^M$ with the property

$$\begin{aligned} &B_r A = 1 + R \\ &\qquad\qquad R, R' \in \rho_{\mathrm{ff}}\tilde{\Psi}_b^{-\infty,\mathcal{E}}(X,E) \\ &AB_r = 1 + R' \end{aligned} \tag{4.1.15}$$

i.e., by Proposition (3.1.11), an inverse modulo a compact error.

We end this section by (briefly) introducing the full calculus $\Psi_b^{*,\mathcal{E}}(X,\Omega_b^{\frac{1}{2}})$. One can improve (4.1.15) by constructing an inverse modulo finite rank operators. The kernel of the corresponding parametrix, still denoted by B_r, will be the sum of three terms

$$B_r = B_\sigma + B_r' + B_r'' \tag{4.1.16}$$

where the new term, B_r'', corresponds to a kernel in the ordinary product X^2 which is smooth in the interior $\mathring{X}^2$ and with complete asymptotic expansions at $\partial X \times X$ and $X \times \partial X$. In fact B_r'' belongs to the space $\mathcal{A}_{phg}^{\mathcal{E}(r)}(X^2,\Omega_b^{\frac{1}{2}})$ where the index family $\mathcal{E}(r) = (E^+(r), E^-(r))$ is now associated to the ordered pair $(\partial X \times X, X \times \partial X)$. Thus the full calculus $\Psi_b^{*,\mathcal{E}(r)}(X,\Omega_b^{\frac{1}{2}})$ is defined as

$$\Psi_b^{*,\mathcal{E}(r)}(X,\Omega_b^{\frac{1}{2}}) = \Psi_b^*(X,\Omega_b^{\frac{1}{2}}) + \tilde{\Psi}_b^{-\infty,\mathcal{E}(r)}(X,\Omega_b^{\frac{1}{2}}) + \Psi^{-\infty,\mathcal{E}(r)}(X\Omega_b^{\frac{1}{2}}) \tag{4.1.17}$$

where

$$\Psi^{-\infty,\mathcal{E}(r)}(X,\Omega_b^{\frac{1}{2}}) \longleftrightarrow \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}(r)}(X^2,\Omega_b^{\frac{1}{2}})$$

is, by definition, the residual calculus. Notice, in particualar, that the kernel of the inverse of an invertible element $P \in \Psi_b^m(X,\Omega_b^{\frac{1}{2}})$ as a map $x^r H_b^M \longrightarrow x^r H_b^{M-m}$ will belong to the full calculus:

$$P^{-1} \in \Psi_b^{-m} + \tilde{\Psi}_b^{-\infty,\mathcal{E}(r)} + \Psi^{-\infty,\mathcal{E}(r)}. \tag{4.1.18}$$

4.2. Complex powers of b-elliptic operators

Let $A \in \Psi_b^m(X,\Omega_b^{\frac{1}{2}})$ be a classical elliptic b-pseudodifferential operator. For the time being we assume

$$0 \notin -\Im\, \mathrm{spec}_b(A). \tag{4.2.1}$$

Thus we work with the unweighted Sobolev spaces $H_b^M(X,\Omega_b^{\frac{1}{2}})$; the general case $r \notin -\Im\, \mathrm{spec}_b(A)$ will be reduced to this by conjugation as in (3.2.20), (3.5.3). We make the following assumtpion on A

$$A \text{ is formally self-adjoint } : A = A^* \tag{4.2.2}$$

$$A \text{ is positive } : (A\phi,\phi) > 0 \quad \forall\, \phi \in H_b^m. \tag{4.2.3}$$

We fix the following contour in the complex plane

$$\Gamma_\epsilon^- = \begin{cases} re^{i(-\pi)} & \infty \geq r \geq \epsilon \\ \epsilon e^{i\phi} & \pi > \phi > -\pi \\ re^{i\pi} & \epsilon \leq r \leq \infty. \end{cases}$$

Let $s \in \mathbf{C}, \mathrm{Re}\, s < 0$. We define a continuous operator on $L^2(X,\Omega_b^{\frac{1}{2}})$ by

$$A_s = \frac{1}{2\pi i} \int_{\Gamma^-} \lambda^s (A-\lambda)^{-1} d\lambda \tag{4.2.4}$$

where λ^s is determined by the value of $arg\lambda$ given in the definition of Γ^-. Our first goal is to show that the integral on the right hand side converges in the $L^2(X,\Omega_b^{\frac{1}{2}})$ operator norm. In order to do this we use the symbolic properties of b-pseudodifferential operator and we construct a first approximation $B_\sigma(\lambda)$ to $(A-\lambda)^{-1}$ as in the classical case.

Thus if ${}^b\sigma(A) \sim \sum_{j=0}^{\infty} a_{m-j}$ in some local coordinates with

$$a_{m-j}(x,t\xi) = t^{m-j}a_{m-j}(x,\xi) \qquad |\xi| \geq 1 \quad t \geq 1$$

we define a local symbol

$$b(\lambda) \sim \sum_{j=0}^{\infty} b_{-2m-j}(\lambda)$$

by solving

$$b(\lambda)^b\sigma(A-\lambda) = 1 \tag{4.2.5}$$

All the estimates in [**37**], being purely symbolic, carry over.

In particular in our fixed coordinate chart, we have

$$|D_r^\beta D_\xi^\alpha b_j(\lambda)| \leq C(1+|\xi|+|\lambda|^{\frac{1}{m}})^{-m}(1+|\xi|)^{-|\alpha|+m} \tag{4.2.6}$$

Using a partition of unity and the local symbols $b(\lambda)$'s we can construct a family of b-pseudodifferential operators $B_\sigma(\lambda) \in \Psi_b^{-m}$ satisfying the estimates

$$\|B_\sigma(\lambda)(A-\lambda) - \|_{\ell,\ell+q-k-2m} \leq C|\lambda|^{-1+q/m} \qquad |\lambda| \text{ large} \tag{4.2.7}$$

where $0 \leq q \leq m$ and $(-k) \in \mathbb{N}$ is arbitrarily large. The operator norm in (4.2.6) refers of course to the b-Sobolev spaces (3.1.8). The constant C may depend on ℓ, q and k but not on λ. As in [**37**] we obtain two important corollaries from (4.2.7). The first one states that for $\lambda \in \Gamma^-$, $|\lambda|$ large

$$\|(A-\lambda)^{-1}\|_{\ell,\ell+p} \leq C|\lambda|^{-1+p/m} \quad 0 \leq p \leq m. \tag{4.2.8}$$

The second, an easy consequence of (4.2.7) and (4.2.8), gives us the estimates

$$\|B_\sigma(\lambda) - (A-\lambda)^{-1}\|_{\ell,\ell+q-k-2(m)} \leq C|\lambda|^{-2+q/m} \quad 0 \leq q \leq 2m. \tag{4.2.9}$$

From (4.2.8) we obtain

PROPOSITION 4.2.10. *Let $s \in \mathbb{C}, \operatorname{Re} s < 0$. The integral*

$$A_s = \frac{1}{2\pi i} \int_{\Gamma^-} \lambda^s (A - \lambda)^{-1} d\lambda$$

defines, for each $\ell \in \mathbb{R}$, a holomorphic family of operators on $H_b^\ell(X, \Omega_b^{\frac{1}{2}})$. $A_s A_t = A_{s+t}$ and for each $j \in \mathbb{N}$, $A_{-j} = A^{-j}$.

As usual we can define A^s $\forall\, s \in \mathbb{C}$ by

$$(4.2.11) \qquad \begin{aligned} A^s &= A_s && \operatorname{Re} s < 0 \\ A^s &= A^k A_{s-k} && k \in \mathbb{Z}, \operatorname{Re} s < k. \end{aligned}$$

The definition does not depend on $k \in \mathbb{Z}$, provided $\operatorname{Re} s < k$. Furthermore

$$A^s A^t = A^{s+t} \qquad\qquad s, t \in \mathbb{C}.$$

The fact that these properties hold is not at all surprising. They are, in fact, purely functional analytic statements and rest ultimately on the estimates (4.2.8) and well known properties of the contour integral in (4.2.4).

Slightly more subtle is the investigation of the structure of the kernel of A^s. This is due to the fact, explained in the previous section, that the inverse of an invertible b-pseudodifferential operator is not in the small calculus. Recall that A is supposed to be positive and self-adjoint. Thus for $\mu \in \{z \in \mathbb{C}; \Im z = 0\}$

$$I_A(\mu) = (I_A(\mu))^*$$
$$I_A(\mu) > 0$$

which implies that if $\lambda \in \Gamma^-$, then $\lambda \notin \operatorname{spec}(I_A(\mu))$ $\forall\, \mu \in \{z \in \mathbb{C}; \Im z = 0\}$. The fact that, for μ real, $I_A(\mu)$ is positive is proved by observing that for each $\phi \in H^m(\partial X, \Omega^{\frac{1}{2}})$ and each $\mu \in \{z \in \mathbb{C}; \Im z = 0\}$,

$$(I_A(\mu)\phi, \phi) = \lim_{\epsilon \to 0} (A x^{i\mu} \phi \psi_\epsilon, x^{i\mu} \phi \psi_\epsilon)$$

where x is a boundary defininig function and $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})|\frac{dx}{x}|^{\frac{1}{2}}$, ψ being a smooth non-negative function with support in [0,1] and with the property that

$$\int_0^\infty \psi^2 \frac{dx}{x} = 1$$

Since

$$\mathrm{spec}_b(A-\lambda) = \{z \in \mathbf{C}: \lambda \in \mathrm{spec}(I_A(z)\}$$

we conclude that points on the real axis are never in $\mathrm{spec}_b(A-\lambda)$ for $\lambda \in \Gamma^-$ i.e.

$$\{\Im z = 0\} \cap \mathrm{spec}_b(A-\lambda) = \emptyset \qquad \forall\, \lambda \in \Gamma^- \tag{4.2.12}$$

Thus the inverse of $(A-\lambda)$ i.e. the resolvant family $R(A,\lambda)$

$$R(A,\lambda): L^2(X,\Omega_b^{\frac{1}{2}}) \longrightarrow L^2(X^2,\Omega_b^{\frac{1}{2}})$$

has a kernel in the full calculus $\Psi_b^{-m,\mathcal{E}_\lambda(0)}$ for each $\lambda \in \Gamma^-$. The point here is that in (4.1.13) we can choose $r=0$ uniformly in $\lambda \in \Gamma^-$. The following lemma gives us some more control on the indicial roots of $(A-\lambda)$ as $|\lambda| \longrightarrow +\infty$.

LEMMA 4.2.13. *Let $d \in (0,\infty)$ and $\Lambda_d = \{z \in \mathbf{C}; |\Im z| < d\}$ Then there exists $\tilde{\lambda}(d) \in \Gamma^-$ such that*

$$\mathrm{spec}_b(A-\lambda) \cap \Lambda_d = \emptyset$$

for each $\lambda \in \Gamma^-, |\lambda| > |\tilde{\lambda}(d)|$.

PROOF: Consider the parametrix with parameter $B_\sigma(\lambda)$ constructed above.

$$B_\sigma(\lambda)(A-\lambda) = 1 + R(\lambda) \quad R(\lambda) \in \Psi_b^{-\infty}(X). \tag{4.2.14}$$

Taking the corresponding indicial families we obtain

$$I(B_\sigma(\lambda), z) I((A-\lambda), z) = 1 + I(R(\lambda), z). \tag{4.2.15}$$

If $z \in \Lambda_d$ then (3.1.10), (3.2.6) together with Seeley's estimates (4.2.6) imply that

$$\|I(R(\lambda), z)\|_{0,0} \leq C_d(1+|\lambda|)^{-1} \quad \forall\, z \in \Lambda_d$$

with C independent of z and λ. Thus there $\exists$ a $\tilde{\lambda}(d) \in \Gamma^-$ s.t. $\forall\, z \in \Lambda_d$

$$\|I_{R(\lambda)}(z)\|_{0,0} \leq \frac{1}{2} \quad \forall\, \lambda \in \Gamma^-, \ |\lambda| > |\tilde{\lambda}|.$$

For these λ's the right hand side of (4.2.15) can be inverted showing that no indicial roots of $(A-\lambda)$ are located in Λ_d. □

From Lemma 4.2.13 and the remarks preceeding it, we learn that there $\exists$ a $\delta_1 \in (0,\infty)$ with the property

$$\text{spec}_b(A-\lambda) \cap \{z \in \mathbb{C}; |\Im z| < \delta_1\} = \emptyset \qquad \forall\, \lambda \in \Gamma^-. \tag{4.2.16}$$

By (4.1.13) we can write the resolvant family $R(A,\lambda) = (A-\lambda)^{-1}$, as a map between 0-weighted (i.e. unweighted!) Sobolev spaces as

$$R(A,\lambda) = B_\sigma(\lambda) + B_0'(\lambda) + B_0''(\lambda). \tag{4.2.17}$$

Here $B_0'(\lambda) \in \tilde{\Psi}_b^{-\infty,\mathcal{E}_\lambda(0)}(X,\Omega_b^{\frac{1}{2}})$ is obtained from (4.1.10) and the inverse Mellin transform of $I_{R(\lambda)}(z)(I_{(A-\lambda)}(z))^{-1}$; the index family $\mathcal{E}_\lambda(0) = (E_\lambda^+(0), E_\lambda^-(0), 0)$ associated to $(\ell b, rb, \mathrm{ff})$ is defined as in (4.1.13) but with $z_j^\pm \in \text{spec}_b(A-\lambda)$ and the operator $B_0''(\lambda)\cdot$ corresponds to a kernel $\in \mathcal{A}_{\text{phg}}^{\mathcal{E}_\lambda(0)}(X^2,\Omega_b^{\frac{1}{2}})$, thus a conormal distribution on X^2 with specified singularities (only) at $\partial X \times X$ and $X \times \partial X$.

We are now going to analyze the three contour integrals defined by $B_\sigma(\lambda), B_0'(\lambda)$, $B_0''(\lambda)$. By Fubini's theorem and the estimates (4.2.6) we easily obtain

PROPOSITION 4.2.18. *The operator B_σ^s defined for* $\operatorname{Re} s < 0$ *by the contour integral*

$$B_\sigma^s = \frac{1}{2\pi i}\int_{\Gamma^-} \lambda^s B_\sigma(\lambda)\,d\lambda$$

is a b-pseudodifferential operator of order ms with symbol given in local coordinates by

$${}^b\sigma(B_\sigma^s) \sim \sum_{J=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma^-}\lambda^s b_{-m-j}(\lambda)\,d\lambda\right).$$

Let us consider the other two contour integrals. Directly from (4.2.16) and the definiitons of these spaces of distributions (see in particular (4.1.13) (4.1.9)) we see that $B_0'(\lambda)$ is, for each $\lambda \in \Gamma^-$, a continuous b-half density in X_b^2 which is smooth up to the front face, and vanishes of order at least δ_1 at ℓb and rb. Similarly $B_0''(\lambda)$ is for each $\lambda \in \Gamma^-$ a continuous b-half density in X^2 which is smooth in the interior and vanishes of order at least δ_1^2 at $\partial(X^2)$. By Lemma (4.2.13), (4.2.16) and (4.2.9) the same is therefore true for the contour integrals

$$\frac{1}{2\pi i}\int_{\Gamma^-} \lambda^s B_0'(\lambda)d\lambda \tag{4.2.19}$$

$$\frac{1}{2\pi i}\int_{\Gamma^-} \lambda^s B_0''(\lambda)d\lambda. \tag{4.2.20}$$

Notice that estimates (4.2.9) actually show that the above integrals converge even for $\mathrm{Re}\, s < 1$. Thus

PROPOSITION 4.2.21. *The Schwartz kernel of the operator defined for each* $\mathrm{Re}\, s < 1$ *by* (4.2.19) *is a continuous b-half density on* X_b^2 *which is smooth in the interior and up to the front face and vanishes of order at least* δ_1 *at* $\ell b, rb$

PROPOSITION 4.2.22. *The Schwartz kernel of the operator defined by* (4.2.20) *for each* $\mathrm{Re}\, z < 1$ *is a continuous b-half density on* X^2 *which is smooth in the interior and vanishes of order at least* δ_1^2 *at* $\partial(X^2)$.

Notice that the operator corresponding to (4.2.20) is, by Proposition (3.1.11), a trace class operator on $L^2(X, \Omega_b^{\frac{1}{2}})$.

The kernel corresponding to A^s is thus the sum of three conormal distributions. We can lift the kernel corresponding to (4.2.20) to X_b^2 where it becomes a continuous b-half density which is smooth in the interior and vanishes on $\ell b, rb, \mathrm{ff}$. We denote by $K_0^s(A)$ the kernel

$$K_0^s(A) = K(B_\sigma^s) + K_1(s) + \beta_b^* K_2(s)$$

where $K_1(s)$ and $K_2(s)$ corresponds to (4.2.19), (4.2.20). By Propositions (4.2.18), (4.2.21), and (4.2.22), $K_0^s(A)$ defines a holomorphic map.

$$\{\mathrm{Re}\, s < -\frac{\dim X}{m}\} \ni s \longrightarrow K_0^s(A) \in C^0(X_b^2, \Omega_b^{\frac{1}{2}}). \tag{4.2.23}$$

We are interested in the restriction of $K_0^s(A)$ to the lifted diagonal.
Notice that $K_0^s(A)|_{\Delta_b} \in C^0(X, \Omega_b)$ using the natural identification $\Delta_b \longleftrightarrow X$, and the isomorphism $\Omega_b^{\frac{1}{2}}(X_b^2)|_{\Delta^b} \cong \Omega_b(X)$.

THEOREM 4.2.24.
(i) Let $p \in \Delta_b$ and fix a coordinate chart around p. Then $s \longrightarrow K_0^s(A)(p)$ extends from $\operatorname{Re} s < -\frac{\dim X}{m}$ to a meromorphic function on the complex plane with poles at $s = \frac{k-(n+1)}{m}$, $k = 0, 1, \ldots$. The pole at $s_k = k - (n+1)/m$ is simple and its residue is given by

$$(4.2.25) \qquad \left(\frac{1}{(2\pi)^{n+1} im} \int_{|\xi|=1} \int_{\Gamma^-} \lambda^{s_k} b_{-m-k}(\lambda) d\lambda \right) |\frac{dx}{x} dy|.$$

(ii) The residue at $s = 0$ vanishes and the value of $K_0^0(A)(p)$ is given by

$$(4.2.26) \qquad K_0^0(A)(p) = \left(\frac{1}{m(2\pi)^n} \int_{|\xi'|=1} \int_0^\infty b_{-m-(n+1)}(x, \xi', te^{i\theta}) dt d\xi' \right) |\frac{dx}{x} dy|.$$

PROOF: First consider the meromoprhic extension up to $Res < 1$. Then by Propositions (4.2.21), (4.2.22) we only have to deal with the symbolic part

$$B_\sigma^s = \frac{1}{2\pi i} \int_{\Gamma^-} \lambda^s B_\sigma(\lambda) d\lambda$$

The claim then follows by repeating verbatim the arguments in [**37; Theorem 8**]. The same is true for the extension to $k \leq \operatorname{Re} s < k+1$ which is based on the representation

$$A^s = A^k A^{s-k} = \frac{1}{2\pi i} \int_\Gamma A^k \lambda^{s-k} (A - \lambda)^{-1} d\lambda.$$

(here we use the fact that $\tilde{\Psi}_b^{-\infty,\mathcal{E}}$ and $\Psi^{-\infty,\mathcal{E}}$ are two sided modules over Ψ_b^* for each index family $\mathcal{E}$).
To prove (ii) we follow [**38; p. 918**]. That the residue vanishes follows from (4.2.25). Let us prove (4.2.26). Consider q in the same coordinate chart of $p, q \notin \Delta_b$. The following statements are proved as in Seeley's paper.

i) The analytic continuation of $B_\sigma^s(q)$ to $s = 0$ vanishes.
(ii) The same is true for $K_0^s(q)$.
(iii) The analytic continuation of $B_\sigma^s(p)$ to $s = 0$ is given by the right hand side of formula (4.2.29).

Then we remark that

(iv) The continuation of $K_0^s(q) - B_\sigma^s(q)$ is continuous even at $q = p$.

This is again a consequence of Propositions (4.2.21), (4.2.22). From (i) and (ii) we see that the analytic continuation $K_0^0(q) - B_\sigma^0(q)$ vanishes, hence, by (iv) it vanishes also at p, thus by (iii) formula (4.2.26) holds. □

Remark 4.2.27 We can consider the indicial operator associated to the kernel $K_0^s(A)$, defined, as usual by the restriction to the front face. The residual term does not contribute at all, thus

$$I(A^s) = I(B_0^s)$$

where

$$B_0^s = \frac{1}{2\pi i}\int_{\Gamma^-} \lambda^s(B_\sigma(\lambda) + B_0'(\lambda))d\lambda.$$

We can clearly commute the operations of restricting to the front face and integrating over Γ^- since the integral is absolutely convergent ; thus

$$I(B_0^s) = \frac{1}{2\pi i}\int_{\Gamma^-} \lambda^s(I(B_\sigma(\lambda)) + I(B_0'(\lambda))d\lambda$$

Observe that the Mellin transform of the right hand side is equal to the integral of the Mellin transform since, again, everything converges absolutely. If we now take the Mellin transform on both sides and recall that

$$I(B_0'(\lambda), z) = I(R(\lambda), z)I((A-\lambda), z)^{-1} \quad z \in \Lambda_{\delta_1}$$
$$I(B_\sigma(\lambda), z) = I((A-\lambda), z)^{-1} - I(R(\lambda), z)I((A-\lambda), z)^{-1}$$

with $R(\lambda)$ defined, as before, by (4.2.14), we obtain

(4.2.28) $$I(A^s, z) = (I(A, z))^s \qquad \mathrm{Re}\, s << 0$$

for $z \in \Lambda_{d_1}$. Notice that $I(A^s, z)$ depends holomorphycally on z for z in such a region.

4.3. The b-zeta function of an elliptic b-pseudodifferential operator.

Let $A \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}})$ be a b-pseudodifferential operator of order $m, m > \dim X$. By the continuity properties (3.1.10)

$$A : L^2(X, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^m(X, \Omega_b^{\frac{1}{2}}).$$

For this choice of $m \in \mathbf{R}$, the indicial operator of A represents, by Proposition (3.1.11), the obstruction to A being a trace class operator on $L_b^2(X, \Psi_b^{\frac{1}{2}})$ (or any other weighted b-Sobolev space). Let us assume $I(A) \equiv 0$. Then we can consider the kernel of A in the ordinary product (we still denote it by $K(A)$) where it becomes a continuous b-half-density vanishing at $\partial(X_b^2)$. By Lidskii's theorem

$$\operatorname{tr} A = \int_X K(A)|_\Delta$$

where we used the identification $\Delta \longleftrightarrow X$, and the isomorphism $\Omega_b^{\frac{1}{2}}(X^2)|_\Delta \cong \Omega_b(X)$ so that $K(A)|_\Delta \in C^\infty(X, \Omega_b)$. Using the natural identification $\beta_b : \Delta_b \longleftrightarrow \Delta$ we obtain

PROPOSITION 4.3.1. *Let $A \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}}), m > \dim X$. If $I(A) \equiv 0$, then A is trace class in $L^2(X, \Omega_b^{\frac{1}{2}})$ with*

$$\operatorname{tr} A = \int_X K(A)|_{\Delta^b}$$

For arbitrary $A \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}}), m > \dim X$, the integral in (4.3.1) is not convergent. This is due to the fact that $K(A)|_{\Delta^b}$ defines a continuous b-density on X; thus in local coordinates (r, τ, y, y') near the front face

$$K(A)|_{\Delta^b} = K(A)(r, 0, y, y) = \frac{1}{(2\pi)^{n+1}} \int a(r, y, \lambda, \eta) d\lambda d\eta \times |\frac{dr}{r} dy| \tag{4.3.2}$$

which is clearly not integrable.

Following Melrose [34] we can however introduce a regularized trace for arbitrary elements in $\Psi^{-m}(X, \Omega_b^{\frac{1}{2}}), m > \dim X$. In order to do this we fix a normalization ν of the normal bundle $N\partial X$ and we denote by x a corresponding normal variable.

DEFINITION 4.3.3. *The b-trace of an element* $A \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}})$ $m > \dim X$ *is defined as*

$$b - \mathrm{Tr}_\nu(A) = \lim_{\epsilon \to 0} \left(\int_{r>\epsilon} K(A)|_{\Delta^b} + \ell n\epsilon \int_Y K(I(A))|_{\Delta^b \cap \mathrm{ff}} \right)$$

This definition simply removes the degeneracy in the integral (4.3.1). If the indicial operator of A vanishes, so that A is trace class, then clearly

$$b - \mathrm{Tr}_\nu(A) = \mathrm{Tr}(A)$$

Consider now an elliptic operator $A \in \Psi_b^m$ as in the previous section. Thus A satisfies

$$\mathrm{spec}_b(A) \cap \{|\Im z| < \delta_0\} = \emptyset \tag{4.3.4}$$

and is positive and self-adjoint.

Consider the operator on $L^2(X, \Omega_b^{\frac{1}{2}})$ defined by $A^s, \mathrm{Re}\, s < -\frac{\dim X}{m}$. This operator is not in the small calculus, however by (4.2.23) and (4.2.27), we can still define the b-trace.

DEFINITION 4.3.5. *The b-zeta function of an elliptic b-pseudodifferential operator* A *as above is defined as*

$${}^b\zeta_{\nu,0}(A, s) = b - \mathrm{Tr}_\nu(A^{-s}) \qquad \mathrm{Re}\, s > \frac{\dim X}{m}.$$

Recall that the subscript 0 stands for our choice of weight ($r = 0$). Directly from Theorem 4.2.20 we obtain

THEOREM 4.3.6.
(i) Let $A \in \Psi_b^m$ *be self-adjoint positive and satisfying* (4.3.4). *Then* ${}^b\zeta_{\nu,0}(A, s)$ *extends from* $\mathrm{Re}\, s > \frac{\dim X}{m}$ *to the whole complex plane with simple poles at the points* $s_k = \frac{\dim X - k}{m}$ $k = 0, 1, \ldots.$
(ii) The residue at 0 *vanishes and the value* ${}^b\zeta_{\nu,0}(A, 0)$ *is obtained by considering the regularized integral, in the sense of* (4.3.3), *of the b-density* (4.2.26).

By (4.2.28) we can express the b-zeta function as

$${}^b\zeta_{\nu,0}(A, s) = \lim_{\epsilon \to 0} \left(\int_{r>\epsilon} K_0^{-s}(A)|_{\Delta_b} + \ell n\epsilon\, \mathrm{tr} \int_{\Im z = 0} (I(A, z))^{-s} dz \right) \tag{4.3.7}$$

where $\operatorname{Re} s > (n+1)/m$.

We end this section by discussing the complex powers of an elliptic operator $A \in \Psi_b^m(X, \Psi_b^{\frac{1}{2}})$ which satisfies (4.2.1) (4.2.2) but is only non negative. This will be important in the next section when we will connect the b-zeta function to the index.

We first collect some useful information on the null space of an arbitrary elliptic element A in $\Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ as a map on $x^r H_b^M(X, \Omega_b^{\frac{1}{2}})$, $r \notin -\Im \operatorname{spec}_b(A)$.
We denote by

$$\operatorname{null}_r^M(A) = \{u \in x^r H_b^M(X, \Omega_b^{\frac{1}{2}}); Au = 0\}.$$

Using the Mellin transform one can prove quite directly that

$$\{u \in x^r H_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}); Au \in \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})\} \subset \mathcal{A}_{\text{phg}}^{E^+(r)} \tag{4.3.8}$$

where $E^+(r)$ is defined as in (4.1.13) and $H_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}) = \bigcup_{m \in \mathbf{R}} H_b^m(X, \Omega_b^{\frac{1}{2}})$. This implies that $\operatorname{null}_r^M(A)$ is independent of M and locally constant in $r \in \mathbf{R} \backslash (-\Im \operatorname{spec}_b(A))$. Thus if $H_b^\infty(X, \Omega_b^{\frac{1}{2}}) \equiv \bigcap_{m \in \mathbf{R}} H_b^m(X, \Omega^{\frac{1}{2}})$ we can write

$$\operatorname{null}_r^M(A) = \operatorname{null}_{r+\epsilon}^\infty(A) \equiv \operatorname{null}_{r+\epsilon}(A) \tag{4.3.9}$$

for ϵ small enough.

Let $0 \notin -\Im \operatorname{spec}_b(A)$ and let $\mathcal{H}$ be the orthogonal projection onto $\operatorname{null}_0(A)$. We can complete a basis $\{\phi_j\}_{j=1}^\infty$ for $\operatorname{null}_0(A)$ to an orthonormal basis of $L_b^2(X, \Omega_b^{\frac{1}{2}})$

$$\{\phi_j\}_{j=1}^\infty, \qquad \phi_j \in H_b^\infty(X, \Omega_b^{\frac{1}{2}}). \tag{4.3.10}$$

Then from(4.3.8) we see that the kernel of $\mathcal{H}$ in X^2 is smooth in the interior and admits complete asymptotic expansions at the left and right bounderies; more precisely $\mathcal{H}$ belongs to the residual calculus

$$\mathcal{H} \in \mathcal{A}_{\text{phg}}^{\mathcal{E}(0)}(X^2, \Omega_b^{\frac{1}{2}}) \tag{4.3.11}$$

where $\mathcal{E}(0) = (E^+(0), E^-(0))$ as in (4.1.13). This implies that the inverse of $(A+\mathcal{H})$, being the sum of $\mathcal{H}$ with a parametrix modulo finite rank, is again an element in the full calculus $\Psi_b^{-m,\mathcal{E}(0)}(X, \Omega_b^{\frac{1}{2}})$. We apply these remarks as follows.

Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ satisfy

(i) $0 \notin -\Im\,\mathrm{spec}_b(A)$

(ii) $A = A^*$

(iii) $A \geq 0$ (as a map on $L^2(X, \Omega_b^{\frac{1}{2}})$ with domain $\mathcal{D}(A) = H_b^m(X, \Omega_b^{\frac{1}{2}})$)

The inverse of $(A + \mathcal{H})$ is thus an element in the full calculus $\Psi_b^{-m,\mathcal{E}(0)}(X, \Omega_b^{\frac{1}{2}})$ with Schwartz kernel

$$K(A + \mathcal{H})^{-1} = B_\sigma + B_0' + C_0$$

where B_σ and B_0' are defined as in §4.1 and C_0 is a residual term in $\Psi^{-\infty,\mathcal{E}(0)}(X, \Omega_b^{\frac{1}{2}})$. By (ii) and (iii) we can consider the resolvant family $R(A + \mathcal{H}, \lambda), \lambda \in \Gamma^-$, which will be an element in $\Psi_b^{-m,\mathcal{E}_\lambda(0)}(X, \Omega_b^{\frac{1}{2}})$. If we define

$$(A + \mathcal{H})^s = \frac{1}{2\pi i} \int_{\Gamma^-} \lambda^s R(A + \mathcal{H}, \lambda) d\lambda \qquad \mathrm{Re}\, s < 0$$

then all our arguments extend in a straightforward way; in particular we can define the b-zeta function ${}^b\zeta_{\nu,0}(A + \mathcal{H}, s)$ as in (4.3.5) and obtain the analogue of theorem (4.3.6). Notice that the value of ${}^b\zeta_{\nu,0}(A + \mathcal{H}, s)$ at $s = 0$ does not depend on $\mathcal{H}$; this is a consequence of theorem (4.3.6) (ii) and formula (4.2.26).

Remark 4.3.12 We will apply these results to the operators AA^* and A^*A with A an arbitrary elliptic operator in $\Psi_b^m(X; \Omega_b^{\frac{1}{2}})$. To deal with the general case $r \notin -\Im\,\mathrm{spec}_b(A)$ we use conjugation. We can define an invertible operator from $x^r H_b^{2m}(X, \Omega_b^{\frac{1}{2}}) \subset x^r L^2(X, \Omega_b^{\frac{1}{2}})$ to $x^r L^2(X, \Omega_b^{\frac{1}{2}})$ by considering $A^*A + \mathcal{H}$ where $\mathcal{H}$ is defined as $x^r \mathcal{H}(r) x^{-r}, H(r)$ being the orthogonal projection onto null(A_{-r}). We then define

$$(4.3.13) \qquad {}^b\zeta_{\nu,r}(A^*A + \mathcal{H}, s) = {}^b\zeta_{\nu,0}((A_{-r})^* A_{-r} + \mathcal{H}(r), s).$$

Observe that for $\mu \in \{z \in \mathbb{C}; \Im z = 0\}$,

$$I((A_{-r}^* A_{-r}), \mu) = I(A, \mu - ir)^* I(A, \mu - ir)$$

which implies that

$$\mathrm{spec}_b((A_{-r}^* A_{-r} - \lambda) \cap \{\Im z = 0\} = \emptyset \qquad \forall\, \lambda \in \Gamma^-,$$

as in §4.2. Hence we can study the meromorphic properties of the right hand side by applying exactly the same analysis as before.

Remark 4.3.14 Once again the value of the meromorphic extension of ${}^b\zeta_{\nu,r}(A^*A+\mathcal{H},s)$ at $s=0$, being purely symbolic, does not depend on $\mathcal{H}$.

§4.4. An analytic index formula.

Let X be a smooth manifold with boundary and let E,F be smooth vector bundles over X. We fix once for all a trivialization ν of the normal bundle of the boundary and we denote by x a corresponding normal variable. The goal of this section is to prove the following

THEOREM 4.1.1. *Let $A\in\Psi_b^m(X;E\otimes\Omega_b^{\frac12},F\otimes\Omega_b^{\frac12})$ be an elliptic b-pseudodifferential operator and let $r\notin-\Im\operatorname{spec}_b(A)$, so that*

$$A: x^rH_b^M(X,E\otimes\Omega_b^{\frac12})\longrightarrow x^rH_b^{M-m}(X,F\otimes\Omega_b^{\frac12})$$

is Fredholm. The following formula holds $\forall\ s\in\mathbb{C}$, $\operatorname{Re}s>(n+1)/m$:

$$\operatorname{ind}_r(A)=-\eta_r(s,I(A))+{}^b\zeta_r(A^*A+\mathcal{H},s)-{}^b\zeta_{-r}(AA^*+\mathcal{K},s). \tag{4.4.2}$$

We refer to §3.5 and Remark (4.3.12) for the definition of the terms appearing on the right hand side of (4.4.2). Recall that in the notation of §4.3, see in particular (4.3.9),

$$\begin{aligned}\operatorname{ind}_r(A)&=\dim\operatorname{null}_r^M(A)-\operatorname{codim}\operatorname{ran}_r^M(A)\\&=\dim\operatorname{null}_r(A)-\dim\operatorname{null}_{-r}(A^*)\end{aligned}$$

where $\operatorname{ran}_r^M(A)=A(x^rH_b^M(X,E\otimes\Omega_b^{\frac12}))\subset x^rH_b^M(X,F\otimes\Omega_b^{\frac12})$ and where we used the fact that $(x^rH_b^{M-m})'=x^{-r}H_b^{m-M}$. Thus for each $s\in\mathbb{C},\operatorname{Re}s>(n+1)/m$,

$$\operatorname{ind}_r(A)+\eta_r(s,I(A))={}^b\zeta_r(A^*A+\mathcal{H},s)-{}^b\zeta_{-r}(AA^*+\mathcal{K},s).$$

By theorem (4.3.6) the right hand side admits a meromorphic continuation to the whole complex plane which is regular at $s = 0$. Thus the same is true for the left hand side. This proves theorem (3.5.10) and gives us the formula

$$\operatorname{ind}_r(A) = -\eta_r(0, I(A)) + {}^b\zeta_r(A^*A + \mathcal{H}, 0) - {}^b\zeta_{-r}(AA^* + \mathcal{K}, 0).$$

Notice however that by (3.5.11) the difference of the two b-zeta functions does not depend on r and it is in fact given by the regularized integral, in the sense of (4.3.3), of the difference of the two b-densities given by theorem (4.2.24). We denote this common value by ${}^b\zeta(A^*A) - {}^b\zeta(AA^*)$. We then obtain the main result of this chapter:

THEOREM 4.4.3. *Let A be an elliptic b-pseudodifferential operator as in* (4.4.1) *and let $r \notin -\Im\,\mathrm{spec}_b(A)$. Then*

$$\operatorname{ind}_r(A) = -\eta_r(0, I(A)) + {}^b\zeta(A^*A) - {}^b\zeta(AA^*). \tag{4.4.4}$$

Remark 4.4.5 On a closed compact manifold, Y, the index of an elliptic operator $P \in \Psi^m(Y, E \otimes \Omega^{\frac{1}{2}}, F \otimes \Omega^{\frac{1}{2}})$ can be expressed as

$$\begin{aligned} \operatorname{ind} P &= \zeta(P^*P + \mathcal{H}, 0) - \zeta(PP^* + K, 0) \\ &= \int_Y U_0(P^*P) - U_0(PP^*) \end{aligned} \tag{4.4.6}$$

where $\mathcal{H}$ and K are the orthogonal projections onto $\operatorname{null} P$ and $\operatorname{null} P^*$ and where the densities $U_0(P^*P)$, $U_1(PP^*)$ are obtained from the analogue of Theorem (4.3.6). Our formula should be seen as a generalization of (4.4.6) to the case of manifolds with boundary. The densities appearing in (4.4.6) depend of course on the complete symbol of P; however for Dirac-type operators formula (4.4.6) does give the local index formula of Atiyah and Singer ([3] [21] [20]); thus we expect formula (4.4.4) to specialize, for Dirac-type operators, to the Atiyah-Patodi-Singer formula. This is indeed the case. A Dirac operator on an even dimensional spin-manifold with boundary equipped with a b-metric g which splits near the boundary as

$$g = \left(\frac{dx}{x}\right)^2 + h, h \text{ a metric on } \partial X$$

is of the form (3.3.16), thus by example (3.5.8) the eta invariant appearing in (4.4.4) is exactly the one given in the Atiyah-Patodi-Singer theorem (we assume, for simplicity, that 0 is not in the spectrum of $D_{\partial X}$). To prove that the symbolic term reduces to the familiar integral of the $\hat{A}$-roof polynomial one has to pass from the b-zeta function to the b-heat equation and use Getzler's rescaling. The details will appear in [34] where yet a different proof of the Atiyah-Patodi-Singer theorem from the point of view of the b-calculus is presented.

Let us go to the proof of (4.4.2). We first assume that $0 \notin -\Im \operatorname{spec}_b(A)$. Thus there $\exists \delta_0 > 0$ such that for $\Lambda_{\delta_0} = \{z \in \mathbb{C}; |\Im z| < \delta_0\}$ we have

$$\operatorname{spec}_b(A) \cap \Lambda_{\delta_0} = \emptyset. \tag{4.4.7}$$

Our starting point is formula (4.4.6) on a closed compact manifold Y. It is well known that this formula is a consequence of two facts

$$\begin{gathered} \forall\, Q \in \Psi^m(Y, V \otimes \Omega^{\frac{1}{2}}); Q = Q^*, \quad Q > 0 \\ \operatorname{Tr}(Q^{-s}) = \zeta_Q(s) = \sum_{\lambda \in \operatorname{spec}(Q)} \lambda^{-s} \dim E_Q(\lambda) \equiv \sum_{\lambda \in \operatorname{spec}(Q)} \lambda^{-s} \end{gathered} \tag{4.4.8}$$

where $E_Q(\lambda) = \{\phi \in L^2(X; V \otimes \Omega^{\frac{1}{2}}); Q\phi = \lambda\phi\}$

$$\text{If } \lambda \neq 0 \text{ then } P \text{ maps } E_{P^*P}(\lambda) \text{ isomorphically onto } E_{PP^*}(\lambda) \tag{4.4.9}$$

In dealing with b-pseudodifferential operators several problems arise; first of all we have to regularize the trace in (4.4.8) since elements in Ψ_b^{-ms}, $\operatorname{Re} s > (n+1)/m$, are not trace class; furthermore the connection with the spectrum (which is anyway non-discrete) is completely lost. We can overcome both of these problems by perturbing the operator A and using the stability (4.3.9) of the null space with respect to the weighting of the Sobolev space.

DEFINITION 4.4.10. *Let $A \in \Psi_b^m(X; E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}})$ be elliptic and fix δ_0 as in (4.4.7). For $0 < \delta < \delta_0$ we define a linear operator $x^\delta H_b^M(X; E \otimes \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^{M-m}(X; F \otimes \Omega_b^{\frac{1}{2}})$ by*

$$A(\delta) = x^{-\delta} A \tag{4.4.11}$$

x denoting our fixed boundary defining function.

Directly from (4.3.9) we deduce that

$$\text{(4.4.12)} \qquad \begin{aligned} \operatorname{null}(A(\delta)) &= \operatorname{null}_\delta^M(A) = \operatorname{null}_0(A) \\ \operatorname{null}(A(\delta)^*) &= \operatorname{null}_{-\delta}^{-M+m}(A^*) = \operatorname{null}_0(A^*) \end{aligned}$$

Thus $A(\delta)$ is Fredholm and

$$\text{(4.4.13)} \qquad \operatorname{ind} A(\delta) = \operatorname{ind}_0(A) \qquad 0 \le \delta < \delta_0.$$

Let us now consider the operator

$$\text{(4.4.14)} \qquad \begin{aligned} &A(\delta)^* A(\delta) : L^2(X, E \otimes \Omega_b^{\frac12}) \longrightarrow L^2(X, E \otimes \Omega_b^{\frac12}) \\ &\text{with domain} \\ &\mathcal{D}(A(\delta)^* A(\delta)) = x^{2\delta} H_b^{2m}(X, E \otimes \Omega_b^{\frac12}). \end{aligned}$$

PROPOSITION 4.4.15. *The operator (4.4.14) is a closed linear operator with dense domain. There exists a complete orthonormal system $\{\phi_j\}, j = 1, \dots$ of eigenfunctions of $A(\delta)^* A(\delta)$; here $\phi_j \in H_b^\infty(X, E \otimes \Omega_b^{\frac12})$, $A(\delta)^* A(\delta)\phi_j = \lambda \phi_j$ and the eigenvalues λ_j are real, with $\lambda_j \longrightarrow +\infty$ as $j \longrightarrow +\infty$. The spectrum $\operatorname{spec}(A(\delta) * A(\delta))$ coincides with the set of all eigenvalues.*

PROOF: We can write the operator $A(\delta)^* A(\delta)$ as $x^{2\delta} A_{2\delta}^* A$ where, as usual, $x^z A x^{-z} \equiv A_z$, $\forall\, z \in \mathbb{C}$. To prove that $A(\delta)^* A(\delta)$ is closed we need to show that

$$\lim_{L^2} u_n = u, \qquad \lim_{L^2} (A(\delta)^* A(\delta)) u_n = f \Longrightarrow \begin{cases} \text{(4.4.16)} & u \in x^{2\delta} H_b^{2m} \\ \text{(4.4.17)} & A(\delta)^* A(\delta) u = f \end{cases}$$

(4.4.17) follows immediately from the mapping properties of b-pseudodifferential operators (in particular their continuity on extendable distributions). To prove (4.4.16) we use the full calculus. Let B_δ be a full parametrix for $A_{2\delta}^* A$. Thus $B_\delta \in \tilde{\Psi}_b^{-m, \mathcal{E}_\delta(0)}$ with $\mathcal{E}_\delta(0) \longleftrightarrow (E_\delta^+(0), E_\delta^-(0), 0)$ as in §4.1 and $E_\delta^\pm(0)$ defined as in (4.1.13) (but of course with $z_j \in \operatorname{spec}_b(A_{2\delta}^* A)$).
Since δ is small we certainly have

$$\text{(4.4.18)} \qquad \sup_{z_j \in E_\delta^+(0)} \Im z_j < 0 \qquad \inf_{z_j \in E_\delta^-(0)} \Im z_j > 0.$$

Consider $B_\delta x^{2\delta}$; from $A^*(\delta)A(\delta)u = f$ we then obtain

$$u + R_\delta u = B_\delta x^{2\delta} f$$

where $R_\delta \in \rho_{ff}\tilde{\Psi}_b^{-\infty,\mathcal{E}_\delta(0)}$. From the continuity properties of B_δ and R_δ on weighted Sobolev spaces we therefore obtain (4.1.16).

Clearly $\mathrm{spec}(A(\delta)^*A(\delta)) \subseteq \mathbb{R}$. If $\lambda_0 \in (-\infty, 0)$, then $R(A(\delta)^*A(\delta), \lambda_0)$ defines a self-adjoint compact operator on $L^2(X; E \otimes \Omega_b^{\frac{1}{2}})$; this follows from the argument above and the compactness of the inclusion

$$x^{2\delta} H_b^{2m}(X, E \otimes \Omega_b^{\frac{1}{2}}) \hookrightarrow L^2(X, E \otimes \Omega_b^{\frac{1}{2}}) \quad \delta > 0, m > 0.$$

The rest of the proof is standard (see for example [**39**]). □

In other words the operator (4.4.11) enjoys all the familiar properties of self-adjoint elliptic pseudodifferential operators on closed compact manifolds. It is then not surprising that the following proposition holds.

PROPOSITION 4.4.19. *Let* $A \in \Psi_b^m(X, E \otimes \Omega_b^{\frac{1}{2}}, F \otimes \Omega_b^{\frac{1}{2}})$ *be elliptic and assume* (4.4.7). *Then for* $\mathrm{Re}\, s > (n+1)/m$

$$\text{(4.4.20)} \qquad \mathrm{ind}\, A(\delta) = \mathrm{Tr}(A(\delta)^*A(\delta) + \mathcal{H})^{-s} - \mathrm{Tr}(A(\delta)A^*(\delta) + K)^{-s}$$

where $\mathcal{H}$ *and* K *denote, respectively, the orthogonal projections onto* $\mathrm{null}(A(\delta)^*A(\delta)) = \mathrm{null}_0(A)$ *and* $\mathrm{null}(A(\delta)A^*(\delta)) = \mathrm{null}_0(A^*)$.

PROOF: First of all we have to make sense of the right hand side of (4.4.20). We are considering $(A(\delta)^*A(\delta) + \mathcal{H})$ and $(A(\delta)A^*(\delta) + K)$ as operators on $L^2(X, E \otimes \Omega_b^{\frac{1}{2}})$ and $L^2(X, F \otimes \Omega_b^{\frac{1}{2}})$ respectively as in (4.4.11).

We define

$$\text{(4.4.21)} \qquad (A(\delta)^*A(\delta) + \mathcal{H})^{-s} = \frac{1}{2\pi i} \int_{\Gamma^-} \lambda^{-s} R(A(\delta)^*A(\delta) + \mathcal{H}, \lambda)\, d\lambda \quad \mathrm{Re}\, s > 0$$

with Γ^- as in §4.2. Writing $A(\delta)^*A(\delta) = x^{2\delta} A_{2\delta}^* A$ and using (4.3.9) we have

$$\text{(4.4.22)} \qquad R(A(\delta)^*A(\delta) + H, \lambda) = R(A_{2\delta}^* A + \mathcal{H}, x^{2\delta}\lambda) x^{2\delta} \quad \lambda \in \Gamma^-$$

as operators on $L^2(X, E\otimes\Omega_b^{\frac{1}{2}})$. The inverse of $(A_{2\delta}^* A + \mathcal{H} - x^{2\delta}\lambda)$ is an element in the full calculus $\Psi_b^{-2m,\mathcal{E}_\delta(0)}(X, E\otimes\Omega_b^{\frac{1}{2}})$; thus exactly the same analysis as in §4.3 applies proving that if $\operatorname{Re} s > \frac{\dim X}{m} + k$ the kernel of (4.4.21) will be a continuous section of the bundle $E\boxtimes E^*\otimes\Omega_b^{\frac{1}{2}}$ over X^2 which is C^k in $\overset{\circ}{X}{}^2$. Furthermore, because of (4.4.18) and the factor $x^{2\delta}$ in (4.4.22) it will vanish on ∂X^2. Observe also that if $\operatorname{Re} s > \frac{\dim X}{2m}$ such a kernel defines a Hilbert-Schmidt operator on $L^2(X, E\otimes\Omega_b^{\frac{1}{2}})$; using the group properties of complex powers ((4.2.10)) this implies that $(A(\delta)^* A(\delta) + \mathcal{H})^{-s}$ is trace class for $\operatorname{Re} s > \dim X/m$. Applying Proposition (4.4.15) and Parceval identity we easily obtain

$$\operatorname{Tr}(A(\delta)^* A(\delta) + \mathcal{H})^{-s} = \sum_{\substack{\lambda_j \in \operatorname{spec}(A(\delta^*)A(\delta)) \\ \lambda_j > 0}} \lambda_j^{-s} + \dim\operatorname{null}(A(\delta)^* A(\delta))$$

$$\operatorname{Tr}(A(\delta) A^*(\delta) + K)^{-s} = \sum_{\substack{\mu_j \in \operatorname{spec}(A(\delta)A^*(\delta)) \\ \mu_j > 0}} \mu_j^{-s} + \dim\operatorname{null}(A(\delta) A^*(\delta)).$$

This proves the proposition once we observe that (4.4.9) holds for $A(\delta)$. □

We already know that $\operatorname{ind}_0(A) = \operatorname{ind} A(\delta)$ for δ small enough. Formula (4.4.2) will be obtained by letting $\delta \downarrow 0$ in the right hand side of (4.4.20).

PROOF OF THEOREM 4.4.1: The proof will occupy the rest of this section. We first establish the formula under assumption (4.4.7). By Proposition (4.4.19) we can write

$$\operatorname{Tr}(A(\delta)^* A(\delta) + \mathcal{H})^{-s} = \int_X \operatorname{tr}_{E_p} K_\delta^s(p,p) \qquad \operatorname{Re} s > (n+1)/m \tag{4.4.23}$$

where K_δ^s denotes the kernel of $(A(\delta)^* A(\delta) + \mathcal{H})^{-s}$, a continuous section of the bundle $E\boxtimes E^*\otimes\Omega_b^{\frac{1}{2}}(X^2)$ over X^2 which therefore restricts over Δ to a section of $\operatorname{End}(E)\otimes\Omega_b$. To simplify the notation we assume that $E = F = X\times \mathbb{C}$. Thus

$$\operatorname{Tr}(A(\delta)^* A(\delta) + \mathcal{H})^{-s} = \int_X K_\delta^s|_\Delta = \int_X \beta_b^*(K_\delta^s)|_{\Delta_b} \tag{4.4.24}$$

using the natural identifications $\beta_b : \Delta_b \longleftrightarrow \Delta, \Delta \longleftrightarrow X$. To study the behaviour of (4.4.24) as $\delta \downarrow 0$ we need some additional information about K_δ^s.

We fix a collar neighbourhood of the boundary through our choice of boundary defining function. Thus we fix $\mathcal{U} = \{p \in X; p = (x,p'), p' \in \partial X, x \leq 1\}$. We will assume that $x \equiv 1$ on $\complement(\mathcal{U})$, the complement of $\mathcal{U}$.

By (4.4.22) we can rewrite the kernel K^s_δ in local coordinates near the boundary as

(4.4.25)

$$\begin{aligned} K^{\delta}_s(x,y,x',y') &= \frac{1}{2\pi i}\int_{\Gamma^-} \lambda^{-s} R(A^*_{2\delta}A + \mathcal{H}, x^{2\delta}\lambda) x'^{2\delta}\lambda d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^-} (\lambda x'^{2\delta})^{-s} R(A^*_{2\delta}A + \mathcal{H}, (\frac{x}{x'})^{2\delta}(x'^{2\delta}\lambda)) d(\lambda x'^{2\delta}) x'^{2\delta s} \\ &= \frac{1}{2\pi i}\int_{\Gamma} \mu^{-s} R(A^*_{2\delta}A + \mathcal{H}, (\frac{x}{x'})^{2\delta}\mu) d\mu \cdot x'^{2\delta s}. \end{aligned}$$

Notice that the contour Γ depends, a priori, on δ and on $x' > 0$. On the other hand the kernel $R(A^*_{2\delta}A + \mathcal{H}, (\frac{x}{x'})^{2\delta}\mu)$ is holomorphic in μ, for μ ranging in an open neighbourhood of the negative real axis $\{\Im z = 0, \operatorname{Re} z \leq 0\}$; thus by Cauchy theorem we can choose Γ to be independent of x' and δ and in fact equal to the original Γ^-. From (4.4.25) we infer that for $\operatorname{Re} s > \frac{(n+1)}{m} + k$

$$(A(\delta)^* A(\delta) + \mathcal{H})^{-s} = L^s_\delta x^{2\delta s} \tag{4.4.26}$$

as operators on $L^2(X, \Omega_b^{\frac{1}{2}})$, with L^s_δ defined by

$$L^s_\delta(x,y,x',y') = \frac{1}{2\pi i}\int_{\Gamma^-} \mu^{-s} R(A^*_{2\delta}A + \mathcal{H}, (\frac{x}{x'})^{2\delta}\mu) d\mu.$$

We can lift this kernel to the stretched product X^2_b. Since we are ultimately interested in traces we consider its restriction to a small neighbourhood V of the lifted diagonal Δ_b. Let us fix projective coordinate $(x,t,y,y'), t = \frac{x}{x'}$ near the front face in V. We can fix a neighbourhood of $\mathrm{ff} \cap \Delta_b$ in X^2_b by $U = V \cap \{x \leq 1\}$ and consider local coordinates (x,t,y,y') with $x \leq 1, |t-1| < \epsilon$. The restriction of the lift of $R(A^*_{2\delta}A + \mathcal{H}, (\frac{x}{x'})^{2\delta}\mu)$ to U will be the sum of two terms; a conormal distribution of order $2m$ associated to the diagonal Δ_b and a residual term which is smooth in the interior, continuous in $\bar{V}$ and vanishing on the front face ((4.4.18)).

Away from the front face it will simply be a conormal distribution of order $2m$ in V associated to Δ_b. By applying the same analysis given for the kernel of the complex powers of b-elliptic operators, we conclude that, for $\mathrm{Re}\, s > \frac{(n+1)}{m} + k$, $\beta_b^* L_\delta^s|_V$ will be the sum of a C^k b-half density on V, $L_1(\delta, s)$, and a residual term $L_2(\delta, s)$ which is a continuous b-half-density on V vanishing on the front face.
Finally we observe that the resolvant kernel $\beta_b^*(R(A_{2\delta}^* A + \mathcal{H}, (\frac{x}{x'})^{2\delta}\mu)$ restricted to V depends smoothly on δ, for $0 \leq \delta < \delta_0$, as a conormal distribution on V.
As a consequence we obtain

$$\lim_{\delta \downarrow 0} \beta_b^* L_\delta^s = K(A^* A + \mathcal{H})^{-s} \quad \mathrm{Re}\, s > \frac{(n+1)}{m} \tag{4.4.27}$$

uniformly as continuos functions over X_b^2, as well as the smoothness of the maps

$$[0, \delta_0) \ni \delta \longrightarrow L_1(\delta, s) \in C^k(V, \Omega_b^{\frac{1}{2}}) \quad \mathrm{Re}\, s > \frac{(n+1)}{m} + k$$
$$[0, \delta_0) \ni \delta \longrightarrow L_2(\delta, s) \in C^0(V, \Omega_b^{\frac{1}{2}}) \quad \mathrm{Re}\, s > \frac{(n+1)}{m}$$

We apply these remarks as follows. By Lidskii's theorem

$$\begin{aligned} \mathrm{Tr}(A(\delta)^* A(\delta) + \mathcal{H})^{-s} &= \int_X K_\delta^s|_\Delta = \int_X \beta_b^*(K_\delta^s)|_{\Delta^b} \\ &= \int_{X \cap \mathcal{U}} \beta_b^*(K_\delta^s)|_{\Delta^b} + \int_{X \cup \complement(\mathcal{U})} \beta_b^*(K_\delta^s)|_{\Delta^b}. \end{aligned} \tag{4.4.28}$$

We can consider the indicial operator associated to $L_1(\delta, s)$ which is defined exactly as in §3.1. Then, for $\mathrm{Re}\, s > (n+1)/m + 1$ we have

$$L_1(\delta, s)|_{\Delta^b \cap \mathcal{U}} = I(L_1(\delta, s)|_{\Delta^b \cap \mathrm{ff}} \frac{dx}{x} + x\tilde{L}(\delta, s)|_{\Delta^b \cap \mathcal{U}}$$

where $\tilde{L}(\delta, s)$ has the same regularity properties of $L_1(\delta, s)$. Thus
(4.4.29)

$$\begin{aligned} \mathrm{Tr}(A(\delta)^* A(\delta) + \mathcal{H})^{-s} &= \int_0^1 \int_Y I(L_1(\delta, s))|_{\Delta^b \cap \mathrm{ff}} x^{2\delta s} \frac{dx}{x} + \int_0^1 \int_Y x\tilde{L}(\delta, s)|_{\Delta^b \cap \mathcal{U}} x^{2\delta s} \\ &+ \int_0^1 \int_Y L_2(\delta, s)|_{\Delta^b} x^{2\delta s} + \int_{\complement(\mathcal{U})} L_\delta^s|_{\Delta^b} \end{aligned}$$

Denote by $Z(\delta, s)$ the sum of the last three integrals in (4.4.29):

$$(4.4.30) \qquad Z(\delta, s) = \int_0^1 \int_Y x\tilde{L}(\delta, s)|_{\Delta^b} x^{2\delta s} + \int_0^1 \int_Y L_2(\delta, s)|_{\Delta^b} x^{2\delta s} + \int_{C(\mathcal{U})} L_\delta^s|_{\Delta^b}.$$

LEMMA 4.4.31. *For each* $\operatorname{Re} s > \frac{n+1}{m} + 1$,

$$\lim_{\delta \downarrow 0} Z(\delta, s) = {}^b\zeta_0(A^*A + \mathcal{H}, s)$$

in $C^0(\{\operatorname{Re} s > \frac{n+1}{m} + 1\})$.

PROOF: To prove (4.4.31) recall how ${}^b\zeta_0(A^*A + \mathcal{H}, s)$ is defined

$${}^b\zeta_0(A^*A + \mathcal{H}, s) = \lim_{\epsilon \to 0} \left(\int_{x > \epsilon} K(A^*A + \mathcal{H})^{-s}|_{\Delta^b} + \ell n \epsilon \int_Y (I(A^*A + \mathcal{H})^{-s}|_{\Delta^b \cap \mathrm{ff}} \right).$$

We rewrite the right hand side as

(4.4.32)

$$\lim_{\epsilon \to 0} \left(\int_\epsilon^1 \int_Y K(A^*A + \mathcal{H})^{-s}|_{\Delta^b} + \int_{C(\mathcal{U})} K(A^*A + \mathcal{H})^{-s}|_{\Delta^b} + \ell n \epsilon \int_Y (I(A^*A + \mathcal{H})^{-s}|_{\Delta^b \cap \mathrm{ff}} \right).$$

Let $\operatorname{Re} s > \frac{n+1}{m} + k$. Then from §4.3 we know that the kernel $K(A^*A + \mathcal{H})^{-s}$ on X_b^2 is the sum of two terms: a continuous b-half density $L_1(s)$ on X_b^2 which is C^k on $\mathring{X}_b^2$ and up to the front face and a continuous b-half density $L_2(s)$ vanishing on ff, ℓb and rb. Thus the indicial operator of $(A^*A + \mathcal{H})^{-s}$ is equal to $I(L_1(s))$, the indicial operator associated to $L_1(s)$ (see also Remark (4.2.23)) for this).

For $\operatorname{Re} s > (n+1)/m + 1$

$$L_1(s)|_{\Delta^b \cap \mathcal{U}} = I(L_1(s))|_{\Delta^b \cap \mathcal{U}} \frac{dx}{x} + x\tilde{L}(s)|_{\Delta^b \cap \mathcal{U}}.$$

Thus (4.4.32) is equal to

$$\lim_{\epsilon \to 0} \left[\int_\epsilon^1 \int_Y I(L_1(s))|_{\Delta^b \cap \mathrm{ff}} \frac{dx}{x} + \int_\epsilon^1 \int_Y L_2(s)|_{\Delta^b} + \int_\epsilon^1 \int_Y x\tilde{L}(s)|_{\Delta^b} + \right.$$
$$\left. + \int_{C(\mathcal{U})} K(A^*A + \mathcal{H})^{-s}|_{\Delta_b} + \ell u \epsilon \int_Y (I(A^*A + \mathcal{H})^{-s})|_{\Delta^b \cap \mathrm{ff}} \right].$$

Since $I(L_1(s)) = I(A^*A+\mathcal{H})^{-s}$ we obtain after integrating in x

(4.4.33)

$$^b\zeta_0(A^*A+\mathcal{H},s) = \int_0^1\int_Y x\tilde{L}(s)|_{\Delta^b\cap u} + \int_0^1\int_Y L_2(s)|_{\Delta^b\cap u} + \int_{\mathbb{C}(u)} K(A^*A+\mathcal{H})^{-s}|_{\Delta^b}.$$

That $Z(\delta,s)$ converge as $\delta\downarrow 0$ to the right hand side of (4.4.31) now follows from (4.4.26), (4.4.27), (4.4.30). □

Similarly we can write

$$\mathrm{Tr}(A(\delta)A(\delta)^*+K)^{-s} = \int_0^1\int_Y I(J_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}}\,x^{2\delta s}\frac{dx}{x} + W(\delta,s)$$

where

$$\lim_{\delta\downarrow 0} W(\delta,s) = {}^b\zeta_0(AA^*+K,s) \tag{4.4.34}$$

Integrating in x we obtain

$$\begin{aligned}&\mathrm{Tr}(A(\delta)^*A(\delta)+\mathcal{H})^{-s} - \mathrm{Tr}(A(\delta)A(\delta)^*+\mathcal{K})^{-s}\\ &= \frac{1}{2\delta s}\left[\int_Y I(L_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}} - \int_Y I(J_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}}\right]\\ &+ Z(\delta,s) - W(\delta,s)\end{aligned} \tag{4.4.35}$$

and we are left with the task of proving that

$$\lim_{\delta\downarrow 0}\frac{1}{2\delta s}\left[\int_Y I(L_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}} - \int_Y I(J_1(\delta,s))|_{\Delta_b\cap\mathrm{ff}}\right] = -\eta_0(s,I(A)). \tag{4.4.36}$$

Recall that the kernels $I(L_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}}$ and $I(J_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}}$ are $\mathcal{C}^\infty$ in δ. We consider their Taylor expansion in δ near $\delta=0$. By (4.4.25), (4.4.27) we obtain, for δ small,

$$\begin{aligned}I(L_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}} &= I\left((A^*A+\mathcal{H})^{-s}\right)|_{\Delta^b\cap\mathrm{ff}}\\ &+ \delta\frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{1}{2\pi i}\int_\Gamma \mu^{-s}R(A^*_{2\delta}A+\mathcal{H},\mu)d\mu\right)|_{\Delta^b\cap\mathrm{ff}} + O(\delta^2)\end{aligned}$$

We compute

$$\frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{1}{2\pi i}\int\limits_{\Gamma}\mu^{-s}\cdot R(A^*_{2\delta}A+\mathcal{H},\mu)d\mu\right)\Big|_{\Delta^b\cap\mathrm{ff}}$$

$$=\frac{i}{2\pi}\int\limits_{\Gamma}\mu^{-s}\frac{d}{d\delta}\Big|_{\delta=0}(A^*_{2\delta}A)R(A^*A+\mathcal{H},u)^2d\mu\Big|_{\Delta^b\cap\mathrm{ff}}$$

$$+\frac{i}{2\pi}\int\limits_{\Gamma}\mu^{-s}\left[R(A^*A+\mathcal{H},\mu),\frac{d}{d\delta}\Big|_{\delta=0}(A^*_{2\delta}A)\cdot R(A^*A+\mathcal{H},\mu)\right]d\mu\Big|_{\Delta^b\cap\mathrm{ff}}$$

$$=\frac{d}{d\delta}\Big|_{\Delta=0}(A^*_{2\delta}A)\left(\frac{-s}{2\pi i}\int\limits_{\Gamma}\mu^{-s-1}R(A^*A+\mathcal{H},\mu)d\mu\right)\Big|_{\Delta^b\cap\mathrm{ff}}$$

$$+\frac{i}{2\pi}\int\limits_{\Gamma}\mu^{-s}\left[R(A^*A+\mathcal{H},\mu),\frac{d}{d\delta}\Big|_{\delta=0}(A^*_{2\delta}A)\cdot R(A^*A+\mathcal{H},\mu)\right]d\mu\Big|_{\Delta^b\cap\mathrm{ff}}.$$

Expressing the kernels as the inverse Mellin transform of their indicial family and recalling Remark (4.2.23) and, in particular, (4.2.24) we obtain

$$\frac{1}{2\delta s}\int\limits_{Y}I(L_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}}$$

$$=\frac{1}{2\delta s}\left[\frac{\mathrm{Tr}}{2\pi}\int\limits_{\Im z=0}I_{A^*A}(z)^{-s}dz+\frac{2\delta s}{2\pi i}\mathrm{Tr}\int\limits_{\Im z=0}\frac{d}{dz}\cdot I_{A^*}(z)(I_A(z)I_{A^*A}(z)^{-s-1})dz\right.$$

$$+\frac{\mathrm{Tr}}{2\pi}\left(\frac{i\delta}{2\pi}\int\limits_{\Gamma-}\int\limits_{\Im z=0}\mu^{-s}\left[(I_{A*A}(z)-\mu)^{-1},2i\left(\frac{d}{dz}I_{A*}(z)\right)I_A(z)(I_{A*A}(z)-\mu)^{-1}\right]dzd\mu\right)$$

$$\left.+\,0(\delta^2)\right]$$

An analogous expression can be written for

$$\frac{1}{2\delta s}\int\limits_{Y}I(J_1(\delta,s))|_{\Delta^b\cap\mathrm{ff}}.$$

Let us consider the difference of these two expressions. The crucial observation is that

$$\mathrm{Tr}\,I_{A^*A}(z)^{-s}-\mathrm{Tr}\,I_{AA^*}(z)^{-s}=\mathrm{ind}\,I_A(z)=0$$

for each $z \in \{\Im z = 0\}$.
Thus the singular terms in the difference cancel out; since the terms involving the commutators are identically zero, we can rewrite the left hand side of (4.4.36) as

$$\lim_{\delta\downarrow 0}\frac{1}{2\delta s}\left[2\delta s\cdot\frac{\mathrm{Tr}}{2\pi i}\int\limits_{\Im z=0}\frac{d}{dz}I_A^*(z)\cdot I_A^*(z)(I_{A^*A}(z))^{-s-1}dz-\right.$$
$$\left.2\delta s\cdot\frac{\mathrm{Tr}}{2\pi i}\int\limits_{\Im z=0}\frac{d}{dz}I_{A^*A}(z)\cdot(I_{A^*A}(z))^{-s-1}dz+O(\delta^2)\right]$$

which is equal to

$$=\frac{\mathrm{Tr}}{2\pi i}\int\limits_{\Im z=0}\frac{d}{dz}I_{A^*}(z)I_{A^*}(z)^{-1}(I_{A^*A}(z))^{-s}-\frac{\mathrm{Tr}}{2\pi i}\int\limits_{\Im z=0}\frac{d}{dz}I_{AA^*}(z)(I_{AA^*}(z))^{-s-1}dz.$$

The first term is just $-\eta_0(s, I(A))$ as defined in §3.5. As far as the second is concerned we observe that it is equal to

$$\frac{1}{s}\lim_{x\to+\infty}\left(\mathrm{Tr}(I_{AA^*}(x)^{-s})-\mathrm{Tr}(I_{AA^*}(-x)^{-s})\right)$$

which is 0 because of our estimates in §3.5. This establishes (4.4.36). Hence by (4.4.36), (4.4.35), (4.4.30), (4.4.20), and (4.4.11) the proof of theorem (4.4.1) is completed in the case $0\notin -\Im\,\mathrm{spec}_b(A)$ and $E=F=X\times\mathbf{C}$.
The general case $r\notin-\Im\,\mathrm{spec}_b(A)$ is obtained by conjugation. Finally, for arbitrary vector bundles we just need to take the fiber traces tr_{E_p}, tr_{F_p} in the integrals above as in (4.4.23).
The theorem is proved. □

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