FRACTIONAL STATISTICS

by

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ABSTRACT

Fractional Statistics is introduced as an example in constructing a general theory of quantum statistics. We provide a systematic treatment of a many-body system consisting of N-identical anyons in an external harmonic oscillator potential. A partial set of exact solutions is obtained by using Jacobi coordinates. The multiplicity of the all these linear states are calculated. To complete the full investigation, we employ a perturbative method to compute those spectra which have not been analytically obtained, from both the bosonic and fermionic end. The operator reduction method is developed so that the perturbations can be evaluated in closed form. In the fermionic limit, the first order wave function and the second order energy are obtained for the three anyon ground state. Near the boson limit, we point out that the requirement that bosons are hard core particles is necessary for performing the perturbative calculations. We first show how to use a proper regulation procedure by computing the known N anyon ground state perturbatively so that the method can be tested. Then we use it to compute a few of the interesting missing states for $N = 3$. An important feature of the multi-anyon spectrum is addressed and the nontriviality of the nonlinear missing states seems to depend on the statistical parameter μ exponentially.

Thesis Supervisor: Professor Roman W. Jackiw

Dedicated to my Mother and

the memory of my Father and elder Sister

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Chapter I. Introduction

We all know that there are two different statistics in physics: Bose-Einstein statistics(BES) and Fermi-Dirac statistics(FDS). For a long time, however, people have tried to discover other forms of statistics generalized from the above theories.[1−15] Interest in this problem comes from the hope that there may be different kinds of statistics appearing in fundamental physical phenomena or that a new kind of statistics can be used to describe effectively some interesting physics in a system which is in principle well-understood microscopically. There is also another strategy leading to this investigation. In view of the numerous unresolved problems in the present-day quantum field theory and fundamental particle physics,* improving or completely reforming the foundations of the existing theories will be helpful for classifying or characteristing the difficulties of present theories, as it is generally believed that it is important to know the structure of theory in a context as general as possible. Finally, if it turns out that our known statistics are the only true theories in nature, then this generalization may be able to provide an understanding of why nature makes these particular choices from this family.

Much interesting progress in constructing a general theory of quantum statistics has been made during long periodic investigation,[1,2,3,7,8,9,10,12,13,15] although a complete picture of general quantum statistics remains as an open question. Examples of the new statistics can be constructed by re-examining the common properties sharing by our known theories. Gentile's thermodynamical generalization of

* For example, the spin-statistics connection imposed by local quantum field theories is still not completely understood.

FDS and BES, allowing a finite number of particles in a single quantum state, has not gained much attention.[1] Green's parastatistics(PQS),[2] in which the bilinear quantization for FDS and BES was generalized to the trilinear, quadrilinear, etc, for para-FDS and para-BES, respectively, is related to a symmetric classification (Young pattern) of the symmetric group S_N.[2,3,9] These examples(corresponding to the higher dimensional representations of S_N) however may be viewed as that of ordinary particles with some sort of internal degree of freedom. There also has been considerable interest in the quon statistics,[12] which are based on the q-mutator. It has been observed[16] that this "q deformation" of the Heisenberg algebra is relevant to Quantum Groups and the Yang-Baxter relations.

A most interesting example is the notion of an anyon—a particle obeying fractional quantum statistics(anyonic statistics)(FQS)[7,8]— which has been used to explain successfully the fractional quantum Hall effect(FQHE).[17] The idea of FQS and the techniques developed to study it have proved useful in several quite diverse areas of physics, from cosmic strings and black holes to FQHE and a new mechanism for superconductivity.[18,19,21] Arising from re-examining quantum mechanical features of many identical particle configuration spaces in two spatial dimensions,[7] FQS has a deeper topological origin(Aharonov-Bohm effects),[20] which may be realized as an underlying theory of 2+1 dimensional (topological) Chern-Simons gauge theory.[21,22,34] Anyon is shown to be realized in the notion of charge-flux tube composite[24,25] and also as quasiparticle in FQHE.[17,26] A number of works have examined the field-theory realization of FQS,[27,28] statistical mechanics of anyons,[29,30] and quantum mechanical description of identical anyon systems.[31,32,33] The dynamical symmetries of anyons have also been investigated recently.[34,35] Many numeral studies have also been carried out.[36,37,38] There are also many investigations of anyon systems in the different geometrical situations such as torus and cylinder.[39]

General Quantum Statistics

A central question whether there is a consistent and more general theory of quantum statistics of which ordinary statistics (namely, FDS and BES) may be viewed as special cases is how much of the characteristic feature of ordinary statistics can be kept.[15] It is thus instructive to review some characteristic properties of the ordinary quantum statistics, which is made up of three consistent parts, namely bilinear quantization, quantum mechanics(QM) and statistical mechanics(SM), so that the full picture of a general quantum statistics can be clearly addressed:

Table I: Quantum Statistics: BES and FDS

	QM	Quantization	SM
BES	S	$N=\sum_i a_i^\dagger a_i, \quad [a_i, a_j^\dagger]_- = \delta_{ij}$	$\tilde{n}_l = 0, 1, \cdots$
FDS	A	$N=\sum_i a_i^\dagger a_i, \quad [a_i, a_j^\dagger]_+ = \delta_{ij}$	$\tilde{n}_l = 0, 1$

where the notations in above table have many interpretations as listed in following:

S(A): totally (anti-)symmetric representation of S_N; single(double)-valued;

$[,\ ]_{-(+)}$: (anti-)commutator; single(double)-valued quantization; bilinear quantizations; the Heisenberg deformations.

Note that SM is direct consequence of QM and quantization while QM and the quantization may only be considered as two consistent partners.

In the quantum description of a system of identical particles, the indistinguishability of the particles is usually expressed by symmetry constraints imposed on identical particle wave functions. Namely, the wave functions is either symmetric or antisymmetric under interchanging particle labels. However this postulate, which seems to be in good agreement with our contemporary experiment facts, is based on assumption that identicality can be described by (unphysical) particle labels and concept of the interchange. Unfortunately, these fundamental building blocks of the theory have not been carefully examined in standard textbooks, although many attempts has been made since 1940's.[3,6,9,15] Contrary examples are that there is no meaningful concept of interchange in one spatial dimension quantum systems since particles cannot pass through each other in one spatial dimension, while the operation of the interchange in two spatial dimensions can only be described in a path-dependent way as shown in Fig. 1:

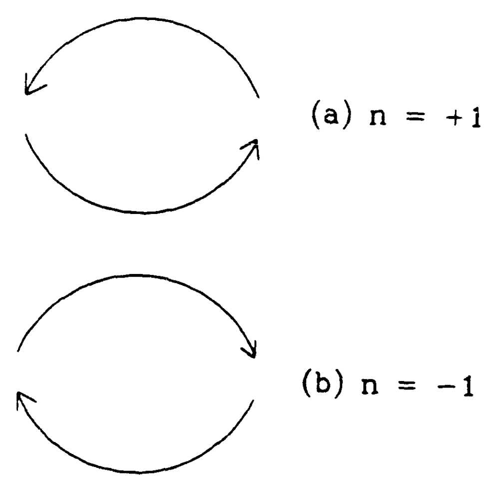

Fig. 1: **Interchanging Two Identical Particles in d=2. (a). The counterclockwise rotation of one particle around another; (b). The counterclockwise rotation of one particle around another. In two spatial dimensions, the two paths in general can not be continuously deformed into a point and they in general belong to different (nontrivial) homotopy class, while they always can be continuously deformed into a point and they are equivalent to each other if $d \geq 3$.**

By improving our understanding of indistinguishability, we may be able to construct a more general theory in which more interesting features of identical particles may shed a light on further development of our quantum systems. There is another concern that may appear in constructing such a new theory. It is whether we can provide the simple enough consistent framework to quantize this theory. Furthermore, ordinary quantum systems have a (linear) superposition property, that ensures that for non-interacting systems (not including statistical interactions) solving a multi-particle problem requires only knowledge of the one-particle sector. Although there are not many such frameworks in the literature, in appendix **B**, such a consistent and simple enough theory of generalized quantum statistics(GQS) is presented.[15]

All known interesting examples of new statistics and their relation to the ordinary statistics are shown in the following table:

Table II: Examples of General Quantum Statistics

	QM	Quantization	SM
PQS	P	Multilinear Quantization	known
Quon	?	$N_i = a_i^\dagger a_i + \sum_j a_j^\dagger N_i a_j, \quad a_i a_j^\dagger = \delta_{ij}$	?
FQS	Mu	?	
GQS	Q	$N = \sum_i a_i^\dagger a_i, \quad [a_i, b_j]_q = \delta_{ij}$	$\tilde{n}_l = 0, 1, \cdots, M-1$

where P, Mu, Q denote the quantum mechanical many-body wave function being the higher dimensional representations (with partial symmetry) of S_N, the braid multivalued(totally symmetric), and the q-generalized determinant,[15] respectively.

SM for PQS is similar to the treatment of ordinary particles with internal degrees of freedom(like spin, color, etc), while for FQS, SM is related to solving the many anyon quantum mechanics and in principle is a well-established problem. The question mark in FQS sector simply denotes that the quantization of anyon in the sense of the traditional framework has not been studied yet and may not exist in the same meaning. Other ?s refer to the unsolved problems with no attention in so far. q, M and b_i are defined in appendix **B**.[15] N_i denotes the number operator for the i-th quon.[12]

Clearly, FQS is only one of the above examples. To see what makes anyon so special and interesting, and also to give a base for this thesis, let's now briefly review the subject of anyons.

Configuration Space and Homotopy Class

The multivalueness of anyon wave functions which describes the path dependent feature of anyon systems is due to the multiconnectivity of the many particle configuration space in two spatial dimensions. We will see that the path integral formalism on multiply connected spaces is a natural tool for studying the this particular problem.[4,5,10]

The wave function $\Psi_I(\mathbf{r},t)$ ($\mathbf{r} \equiv (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$) which is required to be single-valued in the configuration space X_N of N indistinguishable particles propagates as

$$\Psi_I(\mathbf{r}',t') = \int_{X_N} d\mathbf{r}\; K_I(\mathbf{r}'\; t', \mathbf{r}\; t)\; \Psi_I(\mathbf{r},t). \tag{1.1}$$

Since X_N is always multiconnected, the propagator K_I is actually a weighted sum over "partial amplitudes", each being an integration over paths belonging to a distinct homotopy class:

$$K_I(\mathbf{r}'\, t'; \mathbf{r}\, t) = \sum_{\alpha \in \pi_1} \chi(\alpha) \int_{\mathbf{r}(t) \in \alpha} \exp\{i \int_{\mathbf{r}}^{\mathbf{r}'} dt L_0\} \mathcal{D}\mathbf{r}(t), \tag{1.2}$$

where L_0 is the Lagrangian of N identical particles. The homotopy class α of a path from $\mathbf{r}$ to $\mathbf{r}'$ is an element of $\pi_1(X_N)$, the fundamental group of X_N. The identification of such a path is simply to choose a mesh of standard paths from a fixed point $\mathbf{r}_0$ to every point in X_N and adjoin path $\mathbf{r}\mathbf{r}'$ to standard ones $\mathbf{r}_0\mathbf{r}$ and $\mathbf{r}'\mathbf{r}_0$. Note that the complex weight $\chi(\alpha)$ for different homotopy classes has no a priori reason to be same. Invariance under different choices of the standard path mesh and the composition law for the propagator requires that $\chi(\alpha)$ must be a phase factor and form a representation of $\pi_1(X_N)$.

For N indistinguishable particles living in a d-dimensional space R^d, the configuration space $X_N = (R^{d\times N} - D)/S_N$, where $R^{d\times N} \equiv R^d \overbrace{\otimes \cdots \otimes}^{N} R^d$, $D = \{(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}) | \mathbf{r}_i = \mathbf{r}_j,\ \exists i \neq j,\ \forall \mathbf{r}_i \in R^d\}$ and S_N is the symmetric group. $R^{d\times N}/S_N$ is locally isomorphic to $R^{d\times N}$, except for its singular points in D. X_N is always multiconnected, for example, in two particle case,[7]

$$X_2 = \begin{cases} \text{double connected,} & \text{for } d \geq 3; \\ \text{infinite connected,} & \text{for } d = 2; \\ \text{more complicated,} & \text{for } d = 1. \end{cases} \tag{1.3}$$

Immediately, we recognize that for $d \geq 3$, $\pi_1(X_N) = S_N$, and for $d = 2$, $\pi_1(X_N)$ is an infinite non-Abelian group, which is isomorphic to the braid group $B_N(R^2)$. The detailed quantum mechanical studies of one dimensional systems and its applications can be found in Refs. 7 and 40. For the symmetric group S_N, there are only two one dimensional representations, known as $\chi_+(\alpha) = 1$ for all α, and

$\chi_-(\alpha) = (-1)^{\delta_\alpha}$ where δ_α denotes the permutations, corresponding to BES and FDS, respectively.

The braid group plays the same basic role for anyons as S_N for usual bosons and fermions. The braid group $B_N(R^2)$ for N particles is much more complicated; it contains $N-1$ generators $\sigma_i (1 \leq i \leq N-1)$ satisfying the braid relations:

$$\begin{aligned} &\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ &\sigma_i \sigma_j = \sigma_j \sigma_i \; (j \neq i \pm 1), \end{aligned} \tag{1.4}$$

where σs generate counterclockwise permutations of adjacent particles with respect to some fixed ordering. The geometrical meaning of σs is explaining in Fig. 2:

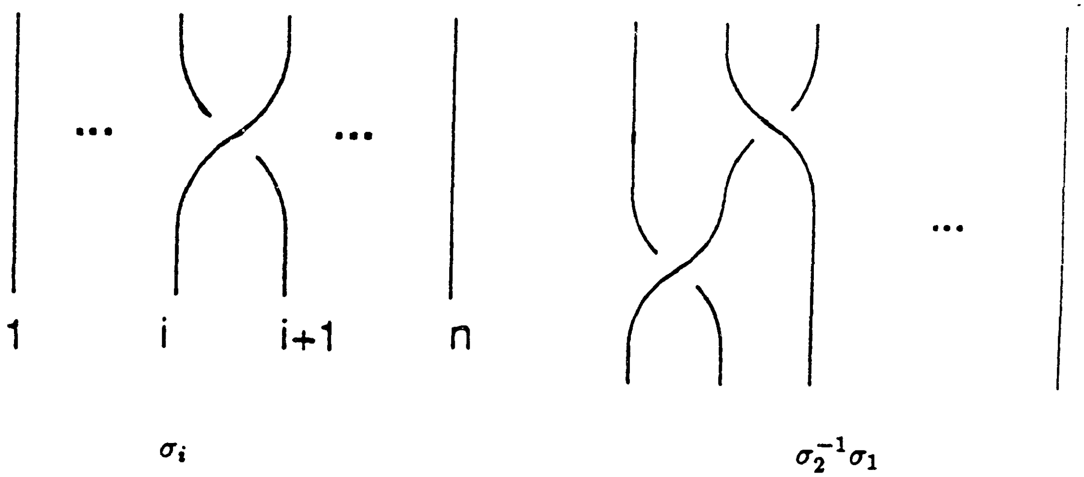

Fig. 2: Geometrical Meaning of Braid Generators σs.

All one dimensional representations (unitary) of $B_N(R^2)$ are labeled by a real parameter μ and have an universal form,

$$\chi_\mu(\sigma_1) = \chi_\mu(\sigma_2) = \cdots = \chi_\mu(\sigma_{N-1}) = e^{-2i\mu\pi}. \tag{1.5}$$

μ plays a role similar to the spin in quantum mechanics. As we recall that physically σ_i denotes an interchange of two particles at $\mathbf{r}_i$, $\mathbf{r}_{i+1}$ along a counterclockwise loop with all other particles being outside, we can rewrite Eq. (1.5) as

$$\chi_\mu(\sigma_i^{\pm}) = e^{\mp 2i\mu\pi} = \exp\{-i\mu \sum_{\text{pairs}} \Delta\theta_{ij}\}, \tag{1.6}$$

where $\Delta\theta_{ij}$ is the change of the azimuthal angle of particle i to particle j. For each σ_i, only one term in above sum is nonvanishing and its value is 2π. Noting that $\alpha \in \pi_1(X_N)$ is a (loop) product sequence of the operations $\sigma_i^{\pm}$. Using Eq. (1.6), we thus have for arbitrary α,

$$\chi_\mu(\alpha) = \exp\left\{-i\mu \int dt \frac{d}{dt} \sum_{\text{pairs}} \theta_{ij}(t)\right\}, \tag{1.7}$$

in which the right-hand side is a homotopy invariant.[10]

Dropping an overall phase related to the contributions from the standard paths $\mathbf{r}_0\mathbf{r}$ and $\mathbf{r}'\mathbf{r}_0$, we finally obtain by inserting Eq. (1.7) into Eq. (1.2),

$$K_I(\mathbf{r}'\ t'; \mathbf{r}\ t) = \int \exp\left\{i \int_{\mathbf{r}}^{\mathbf{r}'} dt \left(L_0 - \mu \frac{d}{dt} \sum_{i \neq j} \theta_{ij}(t)\right)\right\} \mathcal{D}\mathbf{r}(t), \tag{1.8}$$

where $\mathbf{r}(t) \in R^{d\times N} - D$. The total time derivative term in Eq. (1.8) is usually referred to as the statistical Lagrangian, $L_s = -\mu \sum_{i\neq j} \dot{\theta}_{ij}(t)$. Now we introduce the ***single-valued*** wave function $\Psi(\tilde{\mathbf{r}}, t)$ defined on the ***universal covering space*** $\widetilde{X}_N$

of X_N, where $\tilde{\mathbf{r}} \in \widetilde{X}_N$ is chosen such that any closed loop through a point $\mathbf{r} \in X_N$ in different homotopy classes can be identified to a open path in $\widetilde{X}_N$ from point $\tilde{\mathbf{r}}$ to its corresponding point $\tilde{\mathbf{r}}\alpha$ (of course the point $\tilde{\mathbf{r}}\alpha$ is in different sheet).

Similar to Eq. (1.1), $\Psi(\tilde{\mathbf{r}}, t)$ is propagating as,

$$\Psi(\tilde{\mathbf{r}}', t') = \int_{\widetilde{X}_N} d\tilde{\mathbf{r}} \; K(\tilde{\mathbf{r}}' \; t', \tilde{\mathbf{r}} \; t) \; \Psi(\tilde{\mathbf{r}}, t), \tag{1.9}$$

where the propagator K is given by,

$$K(\tilde{\mathbf{r}}' \; t'; \tilde{\mathbf{r}} \; t) = \int \exp\{i \int_{\tilde{\mathbf{r}}\alpha^{-1}}^{\tilde{\mathbf{r}}'} dtL\} \mathcal{D}\tilde{\mathbf{r}}(t). \tag{1.10}$$

Using Eqs. (1.1), (1.8), (1.9) and (1.10), we obtain our final result,

$$\Psi(\tilde{\mathbf{r}}, t) = \exp\{i \int_t dtL_s\} \Psi_I(\mathbf{r}, t). \tag{1.11}$$

Note that the single-valued function on $\widetilde{X}_N$ is in fact multi-valued in the configuration space X_N. Absorbing a constant phase that is related to the contributions of the reference point $\mathbf{r}_0$, we can write Eq. (1.11) as more concrete form,

$$\Psi(\mathbf{r}, t) = \Omega \Psi_{\mathrm{I}}(\mathbf{r}, t), \tag{1.12}$$

with $\Omega = \prod_{n<m} \exp(i2\mu\theta_{nm})$. Eq. (1.12) is often called multivalued boundary conditions.

The Virtual Flux Tubes

We explicitly write down the Lagrange in Eq. (1.8),

$$L = L_0 + L_s = \frac{1}{2} \sum_{i=1}^{N} \dot{\mathbf{r}}_i^2 - \mu \sum_{i \neq j} \dot{\theta}_{ij}. \tag{1.13}$$

The Hamiltonian is

$$H_I = \frac{1}{2}\sum_{n=1}^{N} \left(\mathbf{P}_n - \mathbf{A}_n(\mathbf{r})\right)^2 , \tag{1.14}$$

where $\mathbf{A}_n$ is

$$\begin{aligned} A_n^i(\mathbf{r}) &= 2\mu \sum_{m(\neq n)} \nabla_n \theta_{nm} \\ &= 2\mu \sum_{m(\neq n)} \frac{\epsilon_{ij}(r_n^j - r_m^j)}{|\mathbf{r}_n - \mathbf{r}_m|^2}, \qquad \nabla_n \cdot \mathbf{A}_n = 0, \end{aligned} \tag{1.15}$$

with $\epsilon_{xy} = -\epsilon_{yx} = 1$. $\mathbf{A}_n$ is called the statistical gauge potential seen by the n-th particle.

It is easy to see from the Hamiltonian (1.14) that anyon may be viewed as a charged boson interacting with a infinite thin magnetic flux tube with the flux 2μ.[8,24,25] However one should always remember that anyon may have nothing to do with vortex just as fermions do not need the vortex to exist. Moreover, the vortex systems in general possess an arbitrary family of the self-adjoint extensions[41,42,43,44,45] and the anyon system corresponds only to one of this family (we will explicitly show this for $N = 2$ in next Chapter).[33] Consequently, we have to require the anyon wave function Ψ or $\Psi_{\rm I}$ be regular on the boundary of the N identical anyon configuration space so that they describe the correct statistical generalization, if we really want to view an anyon as a composite formed by a charged boson attaching with a magnetic vortex.

Chern-Simon Gauge Theory

In this subsection, we demonstrate that the notion of anyon can be described alternatively by an (underlying) theory of identical bosons interacting with the Abelian Chern-Simons gauge theory.[22,34,21] Let's write down the total Lagrangian for this theory:

$$
\begin{aligned}
L &= L_0 + L_{CS} + L_{int}, \\
L_{CS} &= \frac{k}{4\pi}\int d\mathbf{r}\ \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma, \\
L_{int} &= \int d\mathbf{r}\ j^\beta A_\beta,
\end{aligned}
\tag{1.16}
$$

where L_0 is defined in Eq. (1.13). The conserved current for the point-like particles depends on the particle dynamical variables,

$$
\begin{aligned}
j^\alpha &= \sum_{n=1}^{N} v_n^\alpha \delta^{(2)}(\mathbf{r} - \mathbf{r}_n) = (\mathbf{j}, \rho), \\
v_n^\alpha &= (\dot{\mathbf{r}}_n, 1).
\end{aligned}
\tag{1.17}
$$

It is easy to recognize that in this non-local formulation A_α are redundant variables and can be eliminated using the equation of motion.[21,34] Namely, A_α depend on only the particle dynamical variables.

The equation of motion for the gauge fields is

$$
\frac{k}{2\pi}\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma + j^\alpha = 0. \tag{1.18}
$$

The Lorentz self-interaction force which is seen by the particles vanishes unambiguously as shown in Ref. 34 using the Chern-Simons field-current identity (1.18). Imposing the Coumlob condition $\nabla \cdot \mathbf{A} = 0$ and taking the vanishing self-interactions into account, we can simply solve the time component of constraint equation (1.18) that gives us Eq. (1.15). In a canonical formulation, we obtain the Hamiltonian for this theory,

$$
H_I = \frac{1}{2}\sum_{n=1}^{N}(\mathbf{P}_n - \mathbf{A}_n)^2 + \int d\mathbf{r}\ A_0(\rho + k\epsilon^{ij}\partial_i A_j), \tag{1.19}
$$

where $\mathbf{A}_n$ is defined in Eq. (1.15) and is only function of the dynamical variables, as $\mu = -1/(2k)$ is identified.[21,34] Note that Weyl gauge $A_0 = 0$ can be imposed since A_0 is the Lagrange multiplier as in the usual case. Therefore we finally reobtain Eq. (1.14) through an abelian Chern-Simons dynamics. As we can see the Lagrangian in Eq. (1.13) is in fact the effective Lagrangian for particle motion in the system which is govern by (1.16) and this explains the vital connection between anyons and Chern-Simons dynamics.

Statistics Transmutations

As we can see from above, the statistical interaction among anyons can be quantitatively described. Therefore, the statistics of particles in two spatial dimensions can be changed arbitrarily into another type of statistics through the above mentioned gauge interaction or Chern-Simons dynamics. The statistics transmutation may be generated from the statistics rule (1.12), which simply means that bosons interacting with a vortex 2μ are changed into anyons with μ-statistics. For example, to get Fermi statistics from the 1/4-statistics, one may use the rule (1.12) as $\Psi_F = \prod_{n\neq m} \exp[2(1/2 - 1/4)\theta_{nm}]\Psi_{1/4}$, where Ψ_F and $\Psi_{1/4}$ are respectively the doublevalued (antisymmetric or fermionic) and quadruplevalued (four-valued) wave functions. Without losing generality, we restrict the statistical parameter μ to the interval $[0, 1/2]$. The non-trivial range $[-1/2, 0]$ is related to $[0, 1/2]$ through the time-reversal operation, as will be discussed below.[32]

★ ★ ★

In this thesis, a systematical study of the multianyon quantum mechanics both analytically and perturbatively is presented. The complete spectrum of anyon quantum mechanics will then be perhaps used to study the statistical (finite temperature) properties of anyon matter, in which the anyon superconducting feature may be clearly understood, as is generally believed. The results may be used to examine the idea of anyon superconductivity at the zero temperature. Especially, the perturbative results may suggest a correct way to find the "missing" anyon states analytically.[33] This thesis is organized as follows. In Chapter II, we introduce the formulation for our system with a remark on anyon wave functions. In Chapter III, a partial set of the exact anyon states is presented by using the Jacobi coordinates and the multiplicity of these states is calculated. We then explicitly identify the "missing" anyon states.[31,32] In particular, we point out that the nonlinearity of the spectra of the missing states is related to the degree of the non-holomorphic homogeneous part of anyon wave functions. In Chapter IV, we employ the conventional perturbative method to compute the anyon states from both the boson and fermion limit. We manage using the operator reduction method[33] all computations in closed form. In the bosonic limit, we present the first order wave functions and the second order energies for the N anyon ground state and a few of the lower-lying anyon states. From the femionic limit, a complete second order perturbative two-anyon spectrum is calculated. Of course, the $N = 2$ results are known exactly so our calculations serve as a check on our perturbation method. We also compute explicitly the three-anyon first order ground state wave functions and the second order ground state energies. The summary is made with one remark in Chapter VI. In Appendix **A**, the related mathematical techniques are collected. They are the Jacobi transformations, a brief discussion on calculating the multiplicity of the quantum states, a universal perturbation theory for our Hamiltonian and the general operator reduction perturbation formulas, and a collection of useful mathematical formulas for our calculations. In Appendix **B**, the notion of genon is introduced.

Natural units are used for the entire thesis, namely, $\hbar$=e=m=c=1.

Chapter II. Multianyon Quantum Mechanics

The multi-anyon quantum mechanics,[31,32,33] which is the central problem in the whole issue of anyon physics, poses a challenging problem in theoretical physics. A complete solution to this problem is thus far out of reach,[31,32,33] and investigations of many-anyon systems[31,32,33,46,47] are still restricted to the mean-field approximation, and computer simulations,[36] except for the clearly understood two-anyon problem in which two-anyon spectra are a linear function of the statistics parameter μ, where μ parameterizes the continuous interpolations of the spectrum degeneracy.[7,8,33] The continuous interpolations of all physical properties as a function of the statistics parameter μ are still lacking. For example, we do not know how the anyonic superfluid liquid is changed into the fermi liquid as the statistics parameter μ varies from 0 to 1/2. However, both the numerical solutions of the three- and four- anyon systems[37,38] and the perturbative studies[33,49,50,51] on the three-anyon systems demonstrate that the anyon states interpolate indeed continuously between the bosonic and fermionic states.

In this Chapter, we briefly review multianyon quantum mechanics.[31,32,33]

Formulation

Let us start with the conventional formulation of the N anyon system as we have outlined in the previous Chapter. The N anyon eigenequation, which comes from solving the Schrodinger equation associated with H_0 in Eq. (1.14) and Ψ in usual fashion, reads,

$$H\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{2.1}$$

where in anyonic representation, the N anyon Hamiltonian in an external harmonic oscillator potential is,

$$H = \frac{1}{2}\sum_{n=1}^{N}[-\nabla_n^2 + \omega^2 \mathbf{r}_n^2]. \tag{2.2}$$

The argument $\mathbf{r}$ in Eq. (2.1) stands for $(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$ and $\mathbf{r}_n = (x_n, y_n)$ represents the coordinate of the n-th anyon. The harmonic oscillator potential is introduced to lift the degeneracy of the single-particle levels.

The statistical boundary conditions (1.12) mimic the dynamical effects of the statistical interaction. The nature of the multivalueness of $\Psi(\mathbf{r})$, defined by (1.12), which we call the "statistical rule", is invariant under the parameter change $\mu \to \mu + 1$. This periodic behavior agrees with the well-known periodicity in the flux of the Aharonov-Bohm effect[20] and with that of the two-anyon spectrum as a function of the statistics parameter μ. As is well known by now, the system described here violates time reversal T and parity P except for $\mu = 0, 1/2$.[31,32] However, this is not obvious for $\mu = 1/2$, but can be understood from the fact that the rule (1.12) is invariant under the non-zero parameter change $\mu \to -\mu$ only for $2\mu =$ integer. This suggests that the conventional statistical equivalence of $-\mu$ and μ, which is related to the time-reversal symmetry of the Hamiltonian, in general is not true.[32]

The long-range statistical interaction associated with the multivalueness of identical anyon wave functions may be explicitly pulled back into the Hamiltonian by using the singular gauge transformations (1.12) and Ψ_I satisfies the eigenequation due to $H_\mu(H_I$ plus the harmonic piece). The statistics strength defined by parallel displacement

$$F_n^{xy} = B_n(\mathbf{r}) = i[D_n^x, D_n^y] = 4\pi\mu \sum_{m(\neq n)} \delta^{(2)}(\mathbf{r}_n - \mathbf{r}_m), \qquad \mathbf{D} = \nabla - i\mathbf{A} \tag{2.3}$$

is locally zero, as it must be for the associated vector potential in Eq. (1.15) to describe statistics.[7] Also Equation (2.3) clearly shows the absence of statistics self-interactions and Lorentz forces for hard-core particles.[34,48]

To understand further the effects of the gauge potential on anyons, we rewrite the Hamiltonian in Eqs. (1.14) and (1.15) as a sum of one-body, two-body, and three body interactions,

$$H_\mu = H + H_2 + H_3, \tag{2.4}$$

where

$$\begin{aligned} H_2 &= \mu \sum_{n\neq m} \frac{1}{|\mathbf{r}_n - \mathbf{r}_m|^2}(2\mu + L_{nm}), \\ H_3 &= \frac{2\mu^2}{3} \sum_{n\neq m\neq k} \left\{ \frac{(\mathbf{r}_n - \mathbf{r}_m)\cdot(\mathbf{r}_n - \mathbf{r}_k)}{|\mathbf{r}_n - \mathbf{r}_m|^2|\mathbf{r}_n - \mathbf{r}_k|^2} + \text{ c. p. of nmk} \right\} , \end{aligned} \tag{2.5}$$

and L_{nm} is the relative angular momentum,

$$L_{nm} = -i\left\{(x_n - x_m)\left(\frac{\partial}{\partial y_n} - \frac{\partial}{\partial y_m}\right) - (y_n - y_m)\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m}\right)\right\}. \tag{2.6}$$

The appearance of the three-body potential in Eq. (2.5) indicates that the statistical potential which sets the boundary conditions is non-trivial. It also explains why solving the problem is difficult.

Using the conventional notation $z = x + iy$ and $z^* = x - iy$, we can rewrite the Hamiltonian in Eqs. (2.4) and (2.5) as,

$$\begin{aligned} H_\mu = &-2\sum_{n=1}^{N} \frac{\partial^2}{\partial z_n \partial z_n^*} + \frac{1}{2}\omega^2 \sum_{n=1}^{N} |z_n|^2 + \mu \sum_{n\neq m} \frac{1}{|z_n - z_m|^2} L_{nm} \\ &+ \mu^2 \sum_{n\neq m, n\neq k} \left(\frac{1}{(z_n - z_m)(z_n^* - z_k^*)} + \text{h.c.} \right), \end{aligned} \tag{2.7}$$

where

$$L_{nm} = (z_n - z_m)\left(\frac{\partial}{\partial z_n} - \frac{\partial}{\partial z_n}\right) - (z_n^* - z_m^*)\left(\frac{\partial}{\partial z_n^*} - \frac{\partial}{\partial z_n^*}\right). \qquad (2.8)$$

The total (canonical) angular momentum operator is $J = \sum_{n=1}^{N}(z_n \frac{\partial}{\partial z_n} - z_n^* \frac{\partial}{\partial z_n^*}) = J_r + J_{cm}$, where J_r, J_{cm} denotes the relative and center-of-mass parts of the angular momentum, respectively.

A Note On Anyon Wave functions

Now we shall show that anyon wave functions must be regular if an anyon is regarded as a composite formed of a charged particle interacting with a point-like vortex. To see this, we consider two charged particles interacting with infinitely thin magnetic flux tubes in two spatial dimensions. We show that this system generally possesses a self-adjoint extension (but we are not going to do self-adjoint extension in this note)[42,43,44,45] other than a logarithmic type one.[44] This new type extension is really due to the cosmic string or Aharonov-Bohm singularities.[42] Since anyon physics deals with a statistical generalization of ordinary bosons or fermions, anyon wave functions will be generated from those of bosons or fermions by applying the boundary conditions (1.12) and are thus regular at the configuration boundaries just like as those for bosons or fermions.

Using the Jacobi coordinates, we can separate out the center-of-mass motion, which is irrelevant in the studies, and the remaining Hamiltonian is equivalent to a single particle in the Aharnov-Bohm background field. To see clearly this new type self-adjoint extension, we write down the radial differential equation,

$$f''(r) + (\frac{1}{r} - 2\omega r)f'(r) + [2(E_r - \omega) - \frac{(l+2\mu)^2}{r^2}]f(r) = 0, \qquad (2.9)$$

where the integer l is even (odd) if the charged particle is a boson (fermion), E_r is the energy eigenvalue associated with the relative wave function $\Psi_r = e^{il\theta}\exp(-\frac{1}{2}\omega r^2)f(r)$ and 2μ = magnetic flux in natural unit. We include a harmonic oscillator for lifting the degeneracy of the single particle level. Two independent solutions to Eq. (2.9) are,

$$f(r) = \begin{cases} e^{-ikr}r^{\pm|l+2\mu|}F(\frac{1}{2} \pm |l+2\mu|,\ 1 \pm 2|l+2\mu|;\ 2ikr) \text{ if } \omega = 0; \\ r^{\pm|l+2\mu|}F(\eta_\pm,\ 1 \pm |l+2\mu|;\ \omega r^2) \text{ if } \omega \neq 0; \end{cases} \tag{2.10}$$

where

$$k = \sqrt{2E_r},\quad \eta_\pm = \frac{1}{2}(1 \pm |l+2\mu|) - \frac{E_r}{2\omega}. \tag{2.11}$$

Since the asymptotic behavior of the above solutions at the configuration boundary($\mathbf{r} = 1/\sqrt{2}(\mathbf{r}_1 - \mathbf{r}_2)$) is explicitly shown in the formula (2.10) (the confluent hypergeometric functions are finite on this boundary), it is thus obvious that the square integrability at $\mathbf{r} = 0$ permits all regularized ones (with plus sign)and the singular solutions (with minus sign), when the condition,

$$|l+2\mu| < 1,\quad (l+2\mu \neq 0), \tag{2.12}$$

is satisfied. From the condition (2.12), we obtain (note $0 \leq 2\mu \leq 1$) that the system has a self-adjoint extension if the vortex carries a non-integer flux. The singular solutions in Eq. (2.10) with the condition (2.12) is thus isolated from $\mu = 0,\ 1/2$ states. Clearly the conclusions are model-independent and purely related to the Aharnov-Bohm or vortex singularities (for $\mu = 0$, the condition (2.12) is no longer satisfied, therefore there is no permitted singular solutions by square integrability).

On the other hand, the regular solutions from Eq. (2.10) yields a set of discrete states ($\eta_+ = -m$, m is a positive integer), which provides a continuous interpolation between $\mu = 0$ and $\mu = 1/2$ as shown in reference [7, 8, 33] (also in Fig.). As in the fermion case, the anyon wave functions can be obtained by shifting the relative angular momentum l to $l + 2\mu$ (this is exactly the statement of the boundary conditions (1.12) for two anyons). It is clear from (2.10) that the regular solutions (with plus sign) may be derived from those for free particles (bosons or fermions) by applying above rule with the exception of the singular solutions. Consequently, the anyon wave functions must be regular if anyons are to be viewed as composite charged particles (either bosons or fermions) attached to magnetic vortex. It is misleading to allow the singular wave functions for anyons without carefully examining its origin, namely anyons are introduced for describing a statistical generalization of bosons and fermions. Finally, we simply point out that the system here does not allow a logarithmic type self-adjoint extension except for $\mu = 0$ and $\omega = 0$ (namely free particle system).

Chapter III. Exact Solutions and Missing States

A set of exact eigenstates for N anyons in the presence of an external harmonic oscillator potential was found in Ref.32 and 33. The spectrum of such eigenstates is linear in the statistical parameter μ. Later it was discovered that the whole states in this set correspond to ones generalized naturally from known three-anyon solutions found by Wu.[31] People then find this exact set of eigenstates in similar systems by using various methods.[46,47] The missing anyon states, which are believed to exist and are not described by the above mentioned the linear states, are identified based on our belief of existing continuous interpolation between the bosonic and fermionic states. Those missing states have been found numerically in $N = 3$ and 4 cases, recently.[37,38] Some suggestions on how to obtain these missing states have been made recently by many people.[35,52,53,54]

In this Chapter, we present this set of linear states using the Jacobi coordinates. The multiplicity of the states is computed. Especially, we display all results explicitly for $N = 2$ and 3 cases and point out the missing states for next Chapter, where the perturbative results for those states will be given.

Jacobi Coordinates

Since the interactions we deal with here are all pairwise, it is convenient to use the measure-invariant Jacobi coordinates,[33,55]

$$\begin{aligned} u_N &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N} z_n, \\ u_{k-1} &= \frac{1}{\sqrt{k(k-1)}}(z_1 + z_2 + \cdots + z_{k-1} - (k-1)z_k), \\ k &= 2, 3, \cdots, N, \end{aligned} \tag{3.1}$$

and their complex conjugates. Using the Jacobi transformations $\nu_{n(m)}$ defined in Appendix (A.1), u_n and z_m can be related in a compact form,

$$\begin{aligned} u_n &= \sum_{m=1}^{N} \nu_{n(m)} z_m \quad (n = 1, 2, \cdots, N), \\ z_m &= \sum_{n=1}^{N} \nu_{n(m)} u_n \quad (m = 1, 2, \cdots, N), \end{aligned} \tag{3.2}$$

and similarly for their complex conjugates. From Appendix (A.2), we easily have,

$$\sum_{n=1}^{N} \frac{\partial^2}{\partial z_n \partial z_n^*} = \sum_{n=1}^{N} \frac{\partial^2}{\partial u_n \partial u_n^*}, \quad \sum_{n=1}^{N} z_n z_n^* = \sum_{n=1}^{N} u_n u_n^*, \tag{3.3}$$

$$\begin{aligned} \sum_{n=1}^{N} z_n \frac{\partial}{\partial z_n} &= \sum_{n=1}^{N} u_n \frac{\partial}{\partial u_n}, \\ z_n - z_m &= \sum_{k=1}^{N-1} (\nu_{k(n)} - \nu_{k(m)}) u_k, \\ \frac{\partial}{\partial z_n} - \frac{\partial}{\partial z_m} &= \sum_{k=1}^{N-1} (\nu_{k(n)} - \nu_{k(m)}) \frac{\partial}{\partial u_k}, \end{aligned} \tag{3.4}$$

and the complex conjugates of Eq. (3.4). It is clearly shown in Eqs. (3.3) and (3.4) that the pairwise interactions are well represented by $2(N-1)$ variables $\{u, u^*\}$, where $\{u, u^*\}$ stands for the relative Jacobi coordinates $\{u_l, u_l^* (l = 1, 2, \cdots, N-1)\}$.

Ansatz and Identities

We make the following *Ansatz* for the energy eigenstates,

$$\Psi_{\mathrm{I}} = \phi(\{z, z^*\}) \varphi(\{z, z^*\}) \Phi, \tag{3.5}$$

where,

$$\phi(\{z, z^*\}) = \prod_{n<m} |z_n - z_m|^{2\mu} \quad , \quad \varphi(\{z, z^*\}) = \exp\left(-\frac{1}{2}\omega \sum_{n=1}^{N} |z_n|^2\right), \tag{3.6}$$

and Φ is an arbitrary function. This type of trial wave function, based on the studies of the singularity behavior of the two-body centrifugal interaction,[56,57] has been successfully used in solving various other problems. The numerical study in Ref. [36] demonstrates that our *Ansatz* represents closely the ground state correlations.

To determine Φ, the following formulas are useful,

$$\begin{aligned}\sum_{n=1}^{N}\frac{\partial^2(\phi\varphi)}{\partial z_n\partial z_n^*} =&\frac{1}{2}\mu^2\phi\varphi\sum_{n\neq m,n\neq k}\left[\frac{1}{(z_n-z_m)(z_n^*-z_k^*)}+\text{h.c.}\right]\\ &+\mu\phi\sum_{n\neq m}\left[\frac{1}{z_n-z_m}\frac{\partial\varphi}{\partial z_n^*}+\text{h.c.}\right]\\ &-\frac{1}{2}\omega N\phi\varphi+\frac{1}{4}\omega^2\sum_{n=1}^{N}|z_n|^2\phi\varphi,\end{aligned}\tag{3.7}$$

$$\sum_{n\neq m}\left[\frac{1}{z_n-z_m}\frac{\partial\varphi}{\partial z_n^*}+\text{h.c.}\right]=-\frac{1}{2}N(N-1)\omega\varphi,\tag{3.8}$$

and

$$\sum_{n\neq m}\frac{1}{|z_n-z_m|^2}L_{nm}\varphi=\sum_{n\neq m}\frac{1}{|z_n-z_m|^2}L_{nm}\phi\equiv 0\ .\tag{3.9}$$

Substituting Eq. (3.5) and (3.6) into the eigenequation of the Hamiltonian (2.7) for the eigenvalue E, and using Eqs. (3.7), (3.8) and (3.9), we find that Φ satisfies,

$$\begin{aligned}\sum_{n=1}^{N}\frac{\partial^2\Phi}{\partial z_n\partial z_n^*}-\frac{1}{2}\omega\sum_{n=1}^{N}(z_n\frac{\partial}{\partial z_n}+z_n^*\frac{\partial}{\partial z_n^*})\Phi+\mu\sum_{n\neq m}\frac{1}{z_n-z_m}(\frac{\partial}{\partial z_n^*}-\frac{\partial}{\partial z_m^*})\Phi\\ +\frac{1}{2}\left(E-\omega(N+N(N-1)\mu)\right)\Phi=0,\end{aligned}\tag{3.10}$$

where three-body interactions disappear. We should also note that except for being singlevalued and totally symmetric, Φ must be chosen such that Ψ_{I} is regular on the configuration boundary. Therefore, Φ is allowed to have some singularities when particles approach each other.

Using Eqs. (3.3), (3.4), (3.10) and the center-of-mass separated solution,

$$\Phi = R^{|L|}e^{iL\phi}F(-n,\ 1+|L|;\ \omega R^2)W, \qquad (L \text{ is a integer}), \tag{3.11}$$

where $R^2 = u_N u_N^*(u_N = Re^{i\phi})$ and $F(-n,\ \alpha;\ x)$ are n-th order confluent hypergeometric polynomials of x,[58] we find the differential equation for W,

$$\sum_{n=1}^{N-1}\left\{\frac{\partial^2 W}{\partial u_n \partial u_n^*} - \frac{1}{2}\omega(u_n\frac{\partial}{\partial u_n} + u_n^*\frac{\partial}{\partial u_n^*})W + 2\mu A_n(\{u\})\frac{\partial W}{\partial u_n^*}\right\} + \frac{1}{2}[E - \omega(2n + |L| + N + N(N-1)\mu)]W = 0. \tag{3.12}$$

Where the homogeneous function $A_n(\{u\})$ with degree -1 is given by

$$A_k(\{u\}) = \sum_{n<m}\frac{\beta_{k(nm)}}{\sum_{j=1}^{N-1}\beta_{j(nm)}u_j}, \tag{3.13}$$

with $\beta_{j(nm)}$ being the j-th component of the positive root vectors for $SU(N)$ defined in Appendix (**A**.3). So far, we have not made any assumptions on W. Therefore solving Eq. (3.12) will in principle give all information in the original problem.

It is easy to verify the identity,

$$\sum_{n=1}^{N-1} A_n(\{u\})u_n \equiv \frac{1}{2}N(N-1). \tag{3.14}$$

In addition, any homogeneous functions of $\{u\}$ are the solutions of Eq. (3.12). Using the identity (3.14), we may naturally make a further *Ansatz* to separate out the homogeneous part of wave functions,

$$W = F(t)G(\{u, u^*\}), \tag{3.15}$$

where $t = \omega r^2 = \omega\sum_{k=1}^{N-1} u_k u_k^*$ (note that r is a totally symmetric function of $\{z, z^*\}$) and G is a homogeneous function of the $\{u, u^*\}$. Plugging the *Ansatz*

(3.15) into Eq. (3.12), and using the identity (3.14) and the homogeneity of the function G, we obtain,

$$
\begin{aligned}
&tF'' + [P + N - 1 + N(N-1)\mu - t]F' \\
&\quad + \frac{1}{2\omega}[E - \omega(2n + |L| + P + N + N(N-1)\mu)]F = 0,
\end{aligned}
\tag{3.16}
$$

and

$$
\sum_{n=1}^{N-1} \left\{ \frac{\partial^2 G}{\partial u_n \partial u_n^*} + 2\mu A_n(\{u\}) \frac{\partial G}{\partial u_n^*} \right\} = 0, \tag{3.17}
$$

where P is the degree of homogeneity of the function G in $\{u, u^*\}$ and is generally a non-trivial function of the statistical parameter μ. Note that the relative Jacobi coordinates $\{u, u^*\}$ (remember they do not include the center-of-mass coordinates u_N, u_N^*) are translation invariant, thus the G in Eq. (3.17) must be also translation invariant.

Although we require G to be a homogeneous function, we can show that this in fact is the only possible case. In other words, there are no eigenstates taken out from Eq. (3.12) by making this assumption. The simplest way to look at this is that when we solve the generalized harmonics of the generalized Laplace equation, the homogeneity is only related to r-degrees of freedom. To show this, let us introduce the homogeneous polar-Jacobi-relative coordinates: r, t_n and t_n^*:

$$
\begin{aligned}
&t_n = \frac{u_n}{r}, \quad t_n^* = \frac{u_n^*}{r}, \\
&\sum_{n=1}^{N-1} t_n t_n^* = 1.
\end{aligned}
\tag{3.18}
$$

Without loss of generality, we choose r, $t_n (n = 1, 2, \cdots, N-1)$ and $t_n^* (n = 1, 2, \cdots, N-2)$ as the independent variables. Using the following formulas:

$$
\begin{aligned}
&\frac{\partial t_n}{\partial u_m} = \frac{1}{r}(\delta_{nm} - \frac{1}{2} t_n t_m^*), \\
&\frac{\partial r}{\partial u_m} = \frac{1}{2} t_m^*, \quad \frac{\partial t_n}{\partial u_m^*} = -\frac{t_n t_m}{2r},
\end{aligned}
\tag{3.19}
$$

and their complex conjugates, we have

$$
\begin{aligned}
\mathcal{N} &= \sum_{n=1}^{N-1}\left(u_n\frac{\partial}{\partial u_n}+u_n^*\frac{\partial}{\partial u_n^*}\right)=r\frac{\partial}{\partial r},\\
J_r &= \sum_{n=1}^{N-1}\left(u_n\frac{\partial}{\partial u_n}-u_n^*\frac{\partial}{\partial u_n^*}\right)=T_N-T_{N-1}^*,\\
\Delta_{2N-2} &= 4\sum_{n=1}^{N-1}\frac{\partial^2}{\partial u_n\partial u_n^*}=\frac{\partial^2}{\partial r^2}+\frac{2N-3}{r}\frac{\partial}{\partial r}+\frac{\Lambda}{r^2},\\
4\sum_{n=1}^{N-1}A_n(\{u_l\})\frac{\partial}{\partial u_n^*} &= 4\sum_{n=1}^{N-2}A_n(\{t_l\})\frac{\partial}{\partial t_n^*}+\frac{N(N-1)}{r}\frac{\partial}{\partial r}-\frac{N(N-1)B}{r^2},
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
T_N &= \sum_{n=1}^{N-1}t_n\frac{\partial}{\partial t_n},\\
T_{N-1}^* &= \sum_{n=1}^{N-2}t_n^*\frac{\partial}{\partial t_n^*},\\
B &= T_N+T_{N-1}^*,\\
\Lambda &= 4\sum_{n=1}^{N-2}\frac{\partial^2}{\partial t_n\partial t_n^*}-B(B+2N-4).
\end{aligned}
\tag{3.21}
$$

Therefore, plugging above result (3.20) into Eq. (3.12), we clearly see that the r degree of freedom can be completely separated from all other $2N-3$ angular variables t_n and t_n^*. This is equivalent to require G being the homogeneous function of $\{u_n, u_n^*\}(n=1,2,\cdots,N-1)$ with the degree P:

$$
\mathcal{N}G(\{u_n,u_n^*\})=PG(\{u_n,u_n^*\}).
\tag{3.22}
$$

Note P in general is the function of μ. An operator method has been used to study the integrability and the missing states of many-anyon system.[54]

Solutions

The basic observation is that the equation (3.17) provides a first integral, $G=\prod_{n<m}(\sum_{j=1}^{N-1}\beta_{j(nm)}u_j)^{-2\mu}g$, and g satisfies $2(N-1)$-dimensional Laplace

equation,

$$\sum_{j=1}^{N-1} \frac{\partial^2 g}{\partial u_j \partial u_j^*} = 0. \tag{3.23}$$

It is rather obvious that we have the following three type solutions to Eq. (3.23),

$$g = \begin{cases} \text{(a). } B_p(\{u\}); \\ \text{(b). } \bar{B}_p(\{u^*\}); \\ \text{(c). } a_{p+1} = \sum_{n=1}^{N-1} \{c_n u_n^* C_p(\{u_k | k \neq n\}) + \bar{c}_n u_n \bar{C}_p(\{u_k^* | k \neq n\})\}, \end{cases} \tag{3.24}$$

where B_p, $\bar{B}_p$, C_p and $\bar{C}_p$ are arbitrary homogeneous functions of degree p, while c_n and $\bar{c}_n$ are constants. Recalling that G must be a totally symmetric, single-valued function, we find that G corresponding to above three type solutions g must have following forms,

$$G = \begin{cases} \text{(a). } s_p(\{u\}) \ (p \text{ is a positive integer}); \\ \text{(b). } \prod_{n<m} |\sum_{j=1}^{N-1} \beta_{j(nm)} u_j|^{-4\mu} \bar{s}_p(\{u^*\}); \\ \text{(c). } \prod_{n<m} (\sum_{j=1}^{N-1} \beta_{j(nm)} u_j)^{-2\mu} a_{p+1} \quad (\mu = 0, \ \frac{1}{2}), \end{cases} \tag{3.25}$$

where s_p, $\bar{s}_p$ are arbitrary totally symmetric homogeneous functions of degree $p(p \geq \mu N(N-1)$ is an integer) and a_{p+1} is defined in Eq. (3.24). Note the identity $\prod_{n<m} |\sum_{j=1}^{N-1} \beta_{j(nm)} u_j| = \prod_{n<m} |z_n - z_m|$, therefore the type (b) solutions can be obtained from the type (a) solutions by performing the combined transformations of $\mu \to -\mu$ and $\{u\} \leftrightarrow \{u^*\}$. It is also worth to note that for $N = 2$, the solutions (3.25) (type (c) solutions are identically zero in this case) solve Eq. (3.17) completely, and the type (c) solutions are single-valued only for $\mu = 0$, $1/2$, as indicated in (3.25)(c).

Exact Eigenstates

Given above solutions and solving Eq. (3.16) by the confluent hypergeometric polynomials, we obtain the exact anyon eigenstates of the Hamiltonian (2.4) from the type (a) solutions in Eq. (3.25),

$$
\begin{aligned}
\Psi_{\mathrm{I}}^{(1)} &= \prod_{k<l} |z_k - z_l|^{2\mu} \exp\left(-\frac{1}{2}\sum_{f=1}^{N} |z_f|^2\right) F(-n,\ 1+|L|,\ \omega R^2) \\
&\quad \times R^{|L|} e^{iL\phi} F(-m,\ N-1+N(N-1)\mu+2p,\ \omega r^2) s_p, \\
E^{(1)} &= \omega(2n+2m+p+|L|+N+N(N-1)\mu), \\
J^{(1)} &= L+p, \\
&\qquad (n, m = 0, 1, 2, \cdots, \ \text{ and } \ L = 0, \pm 1, \pm 2, \cdots),
\end{aligned}
\tag{3.26}
$$

where s_p is an arbitrary homogeneous function of $\{u\}$ with degree p but totally symmetric with respect to $\{z\}$. Note s_p is only a function of the translation invariant Jacobi coordinates. Thus s_p can be generally written as,

$$
s_p = \prod_{l=2}^{N} \left\{ \sum_{m=1}^{N} (z_m - \frac{1}{\sqrt{N}} u_N)^l \right\}^{n_l}, \tag{3.27}
$$

with

$$
p = \sum_{l=2}^{N} l n_l, \tag{3.28}
$$

where $\{n_l(l = 2, 3, \cdots, N)\}$ is a set of non-negative integers. Obviously, s_p is translation invariant and totally symmetric. The combination of transformations $\mu \to -\mu$ and $\{z\} \leftrightarrow \{z^*\}$ is a symmetry of Hamiltonian (2.7) and yields another set of solutions (or from the type (b) solutions in Eq. (3.25)),

$$
\begin{aligned}
\Psi_{\rm I}^{(2)} &= \prod_{k<l}|z_k - z_l|^{-2\mu}\exp\left(-\frac{1}{2}\sum_{f=1}^{N}|z_f|^2\right)F(-n, 1+|L|, \omega R^2) \\
&\qquad \times R^{|L|}e^{iL\phi}F(-m,\ N-1-N(N-1)\mu+2p,\ \omega r^2)\bar{s}_p, \\
E^{(2)} &= \omega(2n+2m+p+|L|+N-N(N-1)\mu), \\
J^{(2)} &= L-p, \\
&\qquad (n, m = 0, 1, 2, \cdots,\ \text{ and }\ L = 0, \pm 1, \pm 2, \cdots).
\end{aligned}
\tag{3.29}
$$

Consider the regularized conditions of the wave functions $\Psi_{\rm I}$, $\bar{s}_p$ may be taken the follow form,

$$
\bar{s}_p = \prod_{n<m}(z_n^* - z_m^*)\bar{a}_{p-1/2N(N-1)},
\tag{3.30}
$$

where $\bar{a}_k$ is a totally antisymmetric, translation invariant, and homogeneous function (with degree k) of $\{z^*\}$.

Using the statistical transformation (1.12), one may combine the solutions (3.26) and (3.29) into a more compact form, which yields solutions of the free Hamiltonian (2.1),

$$
\begin{aligned}
\Psi &= \exp\left(-\frac{1}{2}\sum_{f=1}^{N}|z_f|^2\right)F(-m,\ N-1+\sum_{k'<p'}|l_{k'p'}+2\mu|,\ \omega r^2) \\
&\qquad \times\left\{\prod_{k<j}|z_k - z_j|^{|l_{kj}+2\mu|}\exp\{i(l_{kj}+2\mu)\theta_{kj}\} + \text{c.p. of } 12\cdots \mathrm{N}\right\} \\
&\qquad \times R^{|L|}e^{iL\phi}F(-n,\ 1+|L|,\ \omega R^2),
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
E &= \omega(2n+2m+|L|+\sum_{k<j}|l_{kj}+2\mu|+N), \\
J &= L+\sum_{k<j}l_{kj}+N(N-1)\mu, \\
&\qquad (n, m = 0, 1, 2, \cdots,\ \text{ and }\ L = 0, \pm 1, \pm 2, \cdots),
\end{aligned}
$$

where l_{kj} are integers (either positive or negative for all sub-index kj) such that the wave functions in the brace is non-vanishing and totally symmetric. The cyclic permutation is denoted by $c.p.$ It is easy to show that the complex conjugate of $\Psi_{\mathrm{I}}^{(1,2)}$ is an eigenstate of H_μ with the same energy. We also note that the (3.31) includes a complete set of two anyon solutions.[32,33]

As we pointed out in Ref.32 and 33, the solutions (3.31) do not include non-holomorphic homogeneous boson and fermion states, and therefore anyon states connected with these states are missing from the solutions (3.31). It is worth noting that the type (c) solutions in Eq. (3.24) seems to give us these "missing" anyon states,[33] because one may easily recognize that the type (c) solutions defined in (3.24) and (3.25) gives those non-holomorphic (homogeneous) boson and fermion states. For instance, $a_2 = u_1 u_2^* - u_1^* u_2$ gives three fermion ground states. However as we have pointed out in (3.25), this type of the solutions is not single-valued only except for the bosonic and fermionic cases.

Degeneracy

Using the formula (**A**.12) and result (**A**.13) in the Appendix **A**, we obtain the multiplicity of the type-I, II solutions (3.26) and (3.29) as,

$$\begin{aligned} E_p^{\mathrm{I}} &= \omega(N + N(N-1)\mu + p), \\ d^{\mathrm{I}}(N,p) &= \sum_{m=0}^{p} (1 + \frac{p-m-\delta_m}{2})(1 + \frac{p-m+\delta_m}{2}) a(N,m), \end{aligned} \tag{3.32}$$

$$\begin{aligned} E_p^{\mathrm{II}} &= \omega(N + N(N-1)(\frac{1}{2} - \mu) + p), \\ d^{\mathrm{II}}(N,p) &= \sum_{m=0}^{p} (1 + \frac{p-m-\delta_m}{2})(1 + \frac{p-m+\delta_m}{2}) c(N,m). \end{aligned} \tag{3.33}$$

with

$$\delta_m = \begin{cases} 0, & \text{if } p - m = \text{ even;} \\ 1, & \text{if } p - m = \text{ odd,} \end{cases} \tag{3.34}$$

where $a(N,p)$ and $c(N,p)$ are defined in Appendix (**A**.5) and (**A**.9), respectively. Let $d_B(N,p)$ (or $d_F(N,p)$) be the degeneracies of N boson (or fermion) energy levels $E = \omega(N+p)$ in the external harmonic potential in two spatial dimensions. Obviously, they are the number of the solutions of the equation,

$$p = \sum_{i=1}^{N} (n_i + m_i), \tag{3.35}$$

with the conditions,

(i). For boson, the solutions must be permutationly distinct;

(ii). For fermion, $(\mathrm{n}_1, \mathrm{m}_1) \neq (\mathrm{n}_2, \mathrm{m}_2) \neq \cdots \neq (\mathrm{n}_\mathrm{N}, \mathrm{m}_\mathrm{N})$ and the solutions must be permutationly distinct,

where $\{n_i\}$ and $\{m_i\}$ are sets of non-negative integers.

It is instructive to discuss $N = 2$ and 3 only, since for general N, the anyon spectra are complicated due to the identical particle symmetries.

(a). $N = 2$

From the solutions (3.31), the complete energy spectrum reads $E_{p(2\mu)} = p(2\mu)\omega$ and $p(2\mu) = 2 + 2n + |L| + 2m + |l + 2\mu|$ ($p(2\mu) \geq 2$ and l is a even integer). The degeneracy of these energy levels may be computed by using the formula (**A**.12) and the result (**A**.13) in the Appendix **A**. The result is,

$$
d_{p(2\mu)} = \begin{cases}
\mu = 0, \; d_p^B = \begin{cases} \frac{1}{12}(p-1)p(p+1), & \text{if } p \in N_o; \\ \frac{1}{12}p(p^2+2), & \text{if } p \in N_e; \end{cases} \\[2ex]
\mu = \frac{1}{4}, \; d_{p(\frac{1}{2})} = \begin{cases} \frac{1}{96}(2p-3)(2p+1)(2p+2), & \text{if } p - \frac{1}{2} \in N_o; \\ \frac{1}{96}(2p-2)(2p-1)(2p+3), & \text{if } p - \frac{1}{2} \in N_e; \end{cases} \\[2ex]
\mu = \frac{1}{2}, \; d_p^F = \begin{cases} \frac{1}{12}(p-1)p(p+1), & \text{if } p \in N_0; \\ \frac{1}{12}(p-2)p(p+2)), & \text{if } p \in N_e; \end{cases} \\[2ex]
\mu \neq 0, \frac{1}{4}, \frac{1}{2}, \; \begin{cases} \frac{1}{2}d^F_{p-2\mu-1}, & \text{if } p - 2\mu \text{ is a integer;} \\ \frac{1}{2}d^F_{p+2\mu-3}, & \text{if } p + 2\mu \text{ is a integer;} \end{cases}
\end{cases}
\tag{3.36}
$$

where N_o and N_e represent the odd, even positive integers, respectively. Using (3.36), we plot the spectra as a function of the statistical parameter μ in Fig. 3:

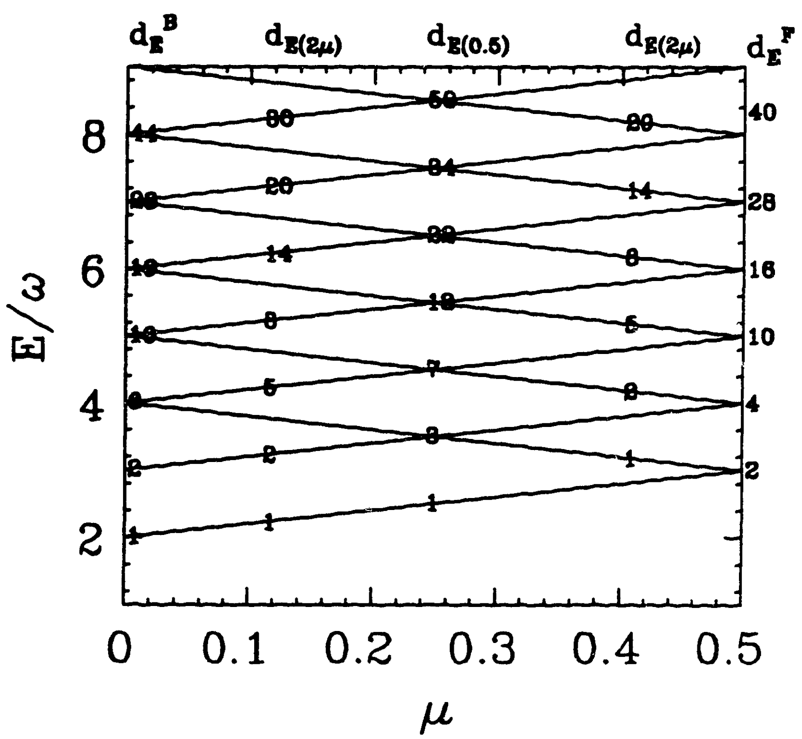

Fig. 3: The energy spectrum for $N = 2$ flows as a function of the statistical parameter μ. The numbers on the lines indicate the multiplicity of this energy flow. The degeneracies for bosonic and fermionic states are the numbers on the right side of the levels. Clearly, anyon degeneracy interpolates continuously between the bosonic ($\mu = 0$) and fermionic ($\mu = 1/2$) values.

(b). $N = 3$

Using (A.7), (A.11), (3.32), (3.33), and (3.35), we explicitly write down $b(3,p)$, $a(3,p)$, $e(3,p)$, $c(3,p)$, $d^{\mathrm{I,II}}(3,p)$, and $d_{B,F}(3,p)$ in **Table III**,

Table III: The Related Multiplicity For N=3.

p	0	1	2	3	4	5	6
$b(3,p)$	1	1	2	3	4	5	7
$a(3,p)$	1	0	1	1	1	1	2
$e(3,p)$	0	0	0	1	1	2	3
$c(3,p)$	0	0	0	1	0	1	1
$d^{\mathrm{I}}(3,p)$	1	2	5	9	16	25	39
$d^{\mathrm{II}}(3,p)$	0	0	0	1	2	5	9
$d_B(3,p)$	1	2	6	14	28	52	93
$d_F(3,p)$	0	0	1	6	14	32	63

Using the figure in **Table III**, we plot the spectra as a function of μ in Fig. 4:

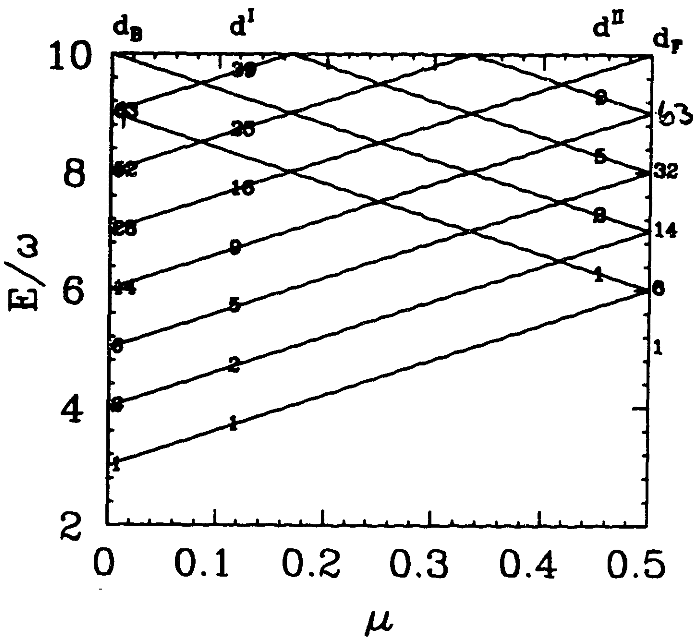

Fig. 4: The exact anyon spectrum we obtain in Eqs. (3.26) and (3.29) for $N = 3$ flows as a function of the statistical parameter μ. The numbers on the lines indicates the multiplicity of this energy flow. The degeneracies for bosonic and fermionic states are the numbers on the right side of the levels. The "missing" states which we do not find in (3.26) and (3.29) are clearly shown by the difference between the bosonic and anyonic multiplicity.

Missing States

From Fig. 3, we see clearly that there is a smooth interpolation between bosonic ($\mu = 0$) and fermionic ($\mu = 1/2$) states. It is easy to identify the missing energy flows from Fig. 4, and they are

(1). one flow connecting to the fermion ground state;

(2). one flow connecting to second boson excited states;

(3). four flows connecting to first fermion excited states;

(4). five flows connecting to third boson excited states;

and etc.

It will be also easy to see from Fig. 5 in the next Chapter that the N anyon ground state for small μ is given by the lowest one in the solutions (3.31),[31,32,33] and the second order three-anyon ground state energy from the fermion limit is not correct for very small μ, because the lowest energy of H_μ in Eq. (2.4) is that of the boson ground state.

Chapter IV. Perturbative Anyon States

As we have shown in the previous Chapter, the missing states correspond to the non-holomorphic homogeneous functions of the generalized Laplace equation (3.17)with the degree $P(\mu)$ being the non-linear function of the statistics parameter μ and have not yet been explicitly obtained. To study the analytic properties of these missing anyon states, which can not be explicitly described in the numerical solutions,[37,38] we compute anyon energies perturbatively to the second order in closed form using the method proposed in Ref.33. From the fermionic end, the second order perturbative energy for the three anyon ground state[50] has obtained in closed form.[33] The closed bosonic perturbative three-anyon energies for some low-lying missing states are also carried out recently.[51] These are the ones connected to the bosonic states $E_0 = 5\omega(J = \pm 2)$ and $E_0 = 6\omega(J = \pm 3)$. The corresponding high-lying flows, differing only by the center-of-mass motions, can also be obtained easily.

In this Chapter, we summarize these closed perturbative results with emphasis the validness of the method and what we learn about the anyon states. We compare our results with the numerical ones. Since there is no difference between three- and many-body systems in this problem, we use the N-body formula for a systematical presentation.

General Feature of Perturbation Theory

We begin to rewrite the N identical anyon Hamiltonian (2.4) in the bosonic representation as follow,

$$H_\mu = H + \mu V_1 + \mu^2 V_2, \tag{4.1}$$

where H is the Hamiltonian for N bosons in the presence of an external harmonic oscillator potential, and

$$
\begin{aligned}
V_1 &= \sum_{n\neq m} \frac{1}{|\mathbf{r}_n - \mathbf{r}_m|^2} L_{nm} \\
V_2 &= \frac{2}{3} \sum_{n\neq m, n\neq k} \left\{ \frac{(\mathbf{r}_n - \mathbf{r}_m)\cdot(\mathbf{r}_n - \mathbf{r}_k)}{|\mathbf{r}_n - \mathbf{r}_m|^2 |\mathbf{r}_n - \mathbf{r}_k|^2} + \text{c.p. of nmk} \right\}.
\end{aligned}
\tag{4.2}
$$

The Hamiltonian in the fermionic representation may be obtained from Eq. (4.1) by changing μ to $\alpha = \mu - 1/2$, and the eigenstates of H are then totally antisymmetric.

Recalling the cyclic permutation symmetries of identical particle Hamiltonian (1.14) or (2.7), we know that all two-body matrix elements are identical, and so are three-body matrix elements. In other words, we have,

$$
\begin{aligned}
&< W'| \sum_{n\neq m} \frac{1}{|\mathbf{r}_n - \mathbf{r}_m|^2} L_{nm} |W >= N(N-1) < W'| \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|^2} L_{12} |W >, \\
&< W'| \sum_{n\neq m\neq k} \left\{ \frac{(\mathbf{r}_n - \mathbf{r}_m)\cdot(\mathbf{r}_n - \mathbf{r}_k)}{|\mathbf{r}_n - \mathbf{r}_m|^2 |\mathbf{r}_n - \mathbf{r}_k|^2} + \text{c.p. of nmk} \right\} |W > \\
&= 3N(N-1)(N-2) < W'| \frac{(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_3)}{|\mathbf{r}_1 - \mathbf{r}_2|^2 |\mathbf{r}_1 - \mathbf{r}_3|^2} |W >,
\end{aligned}
\tag{4.3}
$$

where $|W>$ and $|W'>$ are any N identical particle wave functions.

At this point, we re-write the measure-invariant Jacobi-coordinates in the following way for our convenience: $\{\mathbf{u}_n = (\rho_n, \theta_n)\}$ $(n = 1, 2, \cdots, N)$, namely,

$$
\begin{aligned}
\mathbf{u}_N &= \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \mathbf{r}_l, \\
\mathbf{u}_{k-1} &= \frac{1}{\sqrt{k(k-1)}} (\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_{k-1} - (k-1)\mathbf{r}_k), \\
k &= 2, 3, \cdots, N.
\end{aligned}
\tag{4.4}
$$

In these coordinates, $L_{12} = 2L_{\theta_1} = -2i\frac{\partial}{\partial\theta_1}$. The total angular momentum operator of the relative motions is $J_r = \sum_{n=1}^{N-1} \frac{1}{i}\frac{\partial}{\partial\theta_n}$. The center-of-mass angular momentum operator is $J_{cm} = \frac{1}{i}\frac{\partial}{\partial\theta_N}$. Note both J_r and J_{cm} commute with V_1, and with V_2.

Using the cyclic permutation properties (4.3) of the wave functions, the first order perturbation corrections are related to diagonalize the operator $\frac{1}{\rho_1^2}L_{\theta_1}$. Without loss of generality, one can always choose an unperturbed basis $|d>$ ($d \in D$) (D denotes the degenerate subspace of E_0 with degeneracy d_N) in which the operator $\frac{1}{\rho_1^2}L_{\theta_1}$ has vanishing off-diagonal matrix elements in D. A natural choice for such a basis is a set of the eigenfunctions of the operator J_r, since $[J_r, H\ ] = 0$ and $[L_{\theta_1}/\rho_1^2, J_r] = 0$. Therefore, one may use non-degenerate perturbation theory.

The unperturbed states consist of states with only center-of-mass motions and also with both relative and center-of-mass motions; the latter we call mixed motion states and we denote them as $|E_0 J>$ (they may be degenerate). Using cyclic permutation symmetries of identical particle wave functions, and of V_1, V_2, one has the second order perturbative energy,

$$E_d = E_0 + \mu E_d^{(1)} + \mu^2 (E_{d1}^{(2)} + E_{d2}^{(2)}), \tag{4.5}$$

where

$$\begin{aligned} E_d^{(1)} &= < E_0 J|V_1|E_0 J >= N(N-1) < E_0 J|\frac{L_{\theta_1}}{\rho_1^2}|E_0 J >, \\ E_{d1}^{(2)} &= N(N-1) \sum_{m \notin D} \frac{< E_0 J|V_1|m >< m|(L_{\theta_1}/\rho_1^2)|E_0 J >}{E_0 - E_m}, \\ E_{d2}^{(2)} &= N(N-1) < E_0 J|\frac{1}{\rho_1^2}|E_0 J > + N(N-1)(N-2) < E_0 J|V|E_0 J >, \end{aligned} \tag{4.6}$$

and the three body potential V is defined as

$$V = \frac{1}{\rho_1[\rho_1 + \sqrt{3}\rho_2 \exp\{i(\theta_1 - \theta_2)\}]} + \text{c. c.}, \tag{4.7}$$

is the three-body potential. The first order wave function is,

$$|\Psi_d >= |E_0 J > + \mu |\Psi_d^{(1)} >, \tag{4.8}$$

where

$$|\Psi_d^{(1)}> = \sum_{m \notin D} \frac{< m|V_1|E_0 J >}{E_0 - E_m} |m > . \tag{4.9}$$

In evaluating Eq. (4.9), we have neglected the subspace contributions which vanish due to the facts $< d'|V_2|d >= 0$(for $d' \neq d$, $d, d' \in D$) and $< d|m >= 0$(for $m \notin D$ and $d \in D$).

In general, it is hopeless to carry out the higher order perturbative calculations, unless we can find the C operator defined in Appendix (**A**.21) to evaluate the summations in Eqs. (4.6) and (4.9). From Eqs. (4.3), (**A**.22) and (**A**.23), after using the C operator reduction, all we need are the wave functions for the unperturbed eigenstates. It is fortunately straightforward to establish the C identity for the multi-anyon Hamiltonian (2.4),

$$\begin{aligned} &i \frac{1}{\rho_1^2} L_{\theta_1} = [\theta_1, H], \\ &H L_{\theta_1} |E_0 J >= E_0 L_{\theta_1} |E_0 J > . \end{aligned} \tag{4.10}$$

We know that the θ operator is not a well-defined hermitian operator. Consequently the matrix element $< n|\theta_1|m >$ is not properly defined, and thus the θ operator reduction method is generally not valid for performing the perturbation calculations. However, because of the special form appearing in the second order energy correction formulas $A \frac{Q}{E_0 - H} A$ in Appendix (**A**. 20) (here $A = (1/\rho_1^2) L_{\theta_1} + c.p.$), one can obtain two different reduction formulas as replacing first A or last A by the θ operator identity (4.10). Adding these two results, we find that the correction is related to the matrix element $< n|[L_{\theta_1}, \theta_1]|m >$, which is well-defined. Thus the method established in (4.10) is still valid for computing this special type corrections. Very unfortunately, this situation occurs only in two particle case. Therefore, using Eqs.

(4.3), (4.5), (4.6), (**A**.23) and the θ operator identity (4.10), we obtain the energy correction up to second order for $N = 2$,

$$
\begin{aligned}
E_d^{(1)} &= N(N-1) < E_0 J|\frac{1}{\rho_1^2} L_{\theta_1}|E_0 J >= 2 < E_0 J|\frac{1}{\rho_1^2} L_{\theta_1}|E_0 J >, \\
E_d^{(2)} &= N(N-1)\Big\{< E_0 J|[1-\frac{1}{2}N(\dot{N}-1)]\frac{1}{\rho_1^2} + (N-2)V|E_0 J >\Big\}, \qquad (4.11) \\
&\equiv 0,
\end{aligned}
$$

where $E_d^{(2)} = E_{d1}^{(2)} + E_{d2}^{(2)}$. Without explicit calculation, we find that the second order energy corrections for $N = 2$ vanish identically as we expect. For non-degenerate case, $d_N = 1$ and the N particle wave function $|d>$ is a real (or imaginary) function, thus one immediately obtains from Eq. 11 that all first order energy corrections for non-degenerate fermion states ($d_N = 1$) are zero. Finally we point out that the N ($N \geq 3$) anyon spectra, in general, have a non-linear dependence on the statistics parameter μ, which is absent only for $N = 2$ as is well-known to the literature.

To evaluate the sums in Eqs. (4.6) and (4.9), we have to seek an alternative way. We now summarize the modified C operator reduction method, previously used to obtain the second order perturbative energy of the three anyon ground state from the fermionic end.[33] Actually this method can also be applied to wave functions $|E_0 J>$ satisfying:

$$
\begin{aligned}
L_{12}|E_0 J> &= \sum_{n\neq 0} c_n \rho_1^n e^{-\frac{1}{2}\omega\rho_1^2} f_n, \\
H\rho_1^n e^{-\frac{1}{2}\omega\rho_1^2} f_n &= E_0 \rho_1^n e^{-\frac{1}{2}\omega\rho_1^2} f_n,
\end{aligned} \qquad (4.12)
$$

where f_n is a function of all other variables and c_n are constants. We thus have the C identities,

$$
\begin{aligned}
&\frac{1}{\rho_1^2}|\Phi_1> = (\frac{1}{\rho_1}\frac{\partial}{\partial\rho_1} + \omega)\sum_{n\neq 0}\frac{c_n}{n}\rho_1^n e^{-\frac{1}{2}\omega\rho_1^2} f_n, \\
&\frac{1}{\rho_1}\frac{\partial}{\partial\rho_1} = [C, H] - \pi\delta^{(2)}(\mathbf{u}_1), \quad C = \frac{1}{2}(\ln\omega\rho_1^2 + \gamma - 1),
\end{aligned} \qquad (4.13)
$$

where $|\Phi_1>= L_{12}|E_0 J>$, γ is the Euler constant, and $[J_r, C] = 0$.

Although the most low-lying state wave functions do have the above mentioned properties,* we here are interested only in the simplest case with one of the c_n being non-zero. One can check that the three-fermion ground state $E_0^F = 5\omega(J=0)$ and the three-boson states $E_0^B = 5\omega(J = \pm 2)$ and $E_0^B = 6\omega(J = \pm 3)$ belong to this simplest class. From Eqs. (4.12) and (4.13), we obtain $E_{d1}^{(2)}$ and $|\Psi_d^{(1)}>$ in the closed form,

$$E_{d1}^{(2)} = \frac{N(N-1)}{2n}\{< E_0 J|V_1 C|\Phi_1> - \sum_{d'\in D} < E_0 J|V_1|d'>< d'|C|\Phi_1>\}, \quad (4.14)$$

and

$$\begin{aligned} |\Psi_d^{(1)}> &= \frac{1}{n}\Big(\{C|\Phi_1> + c.p\} - \sum_{d'\in D} < d'|\{C|\Phi_1> + c.p.\}|d'>\Big) \\ &= \frac{1}{n}\Big(\{C|\Phi_1> + c.p\} - \frac{1}{2}N(N-1)\sum_{d'\in D} < d'|C|\Phi_1> |d'>\Big), \end{aligned} \quad (4.15)$$

where $c.p.$ denotes the cyclic permutations of the particle labels. Also, $< m|\Phi_1 >= 0$(for $m \notin D$) and $< m|\delta^{(2)}(\mathbf{u}_1)|\Phi_1 >= 0$ has been used in obtaining Eqs. (4.14) and (4.15). For general case, one can simply replace $\frac{1}{n}|\Phi_1>$ appearing in above and below formulas to $\sum_{n\neq 0}\frac{c_n}{n}\rho_1^2\exp\{-\frac{1}{2}\omega\rho_1^2\}f_n$.

A logarithmic divergence arises generally in the perturbation around the bosonic limit[33,49,51] when matrix elements of V_2 are evaluated. However this difficulty may be overcome by recalling the singular nature of the two-body centrifugal

* In fact all wave functions which are homogeneous respecting with the particle labels except for the common harmonic explential factor do satisfy the conditions (4.12)

interaction[56,57] and the hard core boson conditions. In other words, the centrifugal interaction requires that the wave functions vanish when the two particles are at the same point.[56,57,33,49] Moreover, we note that the conditions of not having two particles at the same point in the boson case will naturally arise as $\mathbf{r}_i \to \mathbf{r}_j$ before explicitly taking the limit $\mu \to 0$. Therefore the definition of the boson should be modified to $\mu = 0^+$ and the $\mu = 0^+$ can be interpreted as the hard core regulator.[33] Therefore, imposing the hard-core boson conditions,[48] removes this difficulty. As shown in Ref. [33], such a term has the form $\frac{1}{\mu}$ (which diverges in the bosonic limit) and the singular part of the second order V_2 will contribute to the first order correction.[33]

We use the regularized wave function $|E_0 J>^{\mathrm{R}} = \rho_1^{\mu}|E_0 J>$ (or its cyclic permutations) to compute the coefficient of $\frac{1}{\mu}$. It is thus necessary to compute one and two higher order perturbations of the singular Hamiltonian $H_{2s} = 2\sum_{n\neq m} 1/|\mathbf{r}_n - \mathbf{r}_m|^2$ to get the full corrections. In our cases, to obtain the complete second order energies, we need to compute the third and fourth order contributions from H_{2s},

$$\begin{aligned} E_s^{(3)} &= \text{singular part of } \left(\sum_{m\notin D} \frac{<E_0 J|H_{2s}|m><m|V_1|E_0 J>}{E_0 - E_m} + \text{c.c.}\right) \\ &= \frac{N(N-1)}{n}\left(<E_0 J|H_{2s}C|\Phi_1>_s - \sum_{d'\in D} <E_0 J|H_{2s}|d'>_s<d'|C|\Phi_1>\right), \end{aligned} \tag{4.16}$$

$$\begin{aligned} E_s^{(4)} &= \text{singular part of } \sum_{m\notin D} \frac{<E_0 J|H_{2s}|m><m|H_{2s}|E_0 J>}{E_0 - E_m} \\ &= \frac{N(N-1)}{\mu}\left(<E_0 J|H_{2s}C|\Phi_0>_s - \sum_{d'\in D} <E_0 J|H_{2s}|d'>_s<d'|C|\Phi_0>\right), \end{aligned} \tag{4.17}$$

where $|\Phi_0>$ is the singular part of the $|E_0 J>$. The subscript s reflects that only the singular terms will be kept.

In principle, we also have to compute the higher order singular wave function corrections to obtain the full first order wave functions. Keeping in mind the special symmetries of the degenerated subspace states, we know that the only relevant term for this thesis work is,

$$\begin{aligned} |\Psi_{ds}^{(2)}> &= \text{singular part of}\left(\sum_{m\notin D}\frac{<m|H_{2s}|E_0 J>}{E_0-E_m}|m>\right), \\ &= \frac{2}{\mu}\left(\{C|\Phi_0>+c.p\}-\sum_{d'\in D}<d'|\{C|\Phi_0>+c.p.\}|d'>\right) \\ &= \frac{2}{\mu}\left(\{C|\Phi_0>+c.p\}-\frac{1}{2}N(N-1)\sum_{d'\in D}<d'|C|\Phi_0>|d'>\right). \end{aligned} \tag{4.18}$$

In our cases, $N=3$,

$$<E_0 J|V_1 C|\Phi_1>=<\Phi_1|\frac{C}{\rho_1^2}|\Phi_1>+8<\Phi_2|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1>, \tag{4.19}$$

with $|\Phi_2>=L_{13}|E_0 J>$. And similarly,

$$<E_0 J|H_{2s}C|\Phi_1>=2<E_0 J|\frac{C}{\rho_1^2}|\Phi_1>+16<E_0 J|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1>, \tag{4.20}$$

$$<E_0 J|H_{2s}C|\Phi_0>=2<E_0 J|\frac{C}{\rho_1^2}|\Phi_0>+16<E_0 J|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_0>. \tag{4.21}$$

The states with $E_0=5(6)\omega$ will be denoted by $|5(6),J>$ and the superscripts I, II in the following refer to $E_0=5\omega, 6\omega$, respectively. Other states will be denoted as presented without further notice. We neglect the subscripts of F or B for simple presentations, since the both bosonic and fermionic perturbations are self-distinct.

Perturbative Anyon States From Fermionic End

In this subsection, we using the method outlined above compute the anyon states perturbatively near the fermion limit. To demonstrate explicitly the non-linear feature of anyon spectra mentioned in the previous subsection, we here examine the multi-anyon ground states around the fermion limit by using the operator reduction formulas defined in Appendix (**A**. 22) and (**A**. 23).

As we point out before, the N identical anyon Hamiltonian in the fermionic representation may be obtained from the Eq. (2.4) by simply changing μ to $\alpha = \mu - 1/2$. Correspondingly, H is a Hamiltonian for N free fermions in the presence of an external harmonic oscillator potential, with a complete set of normalized eigenstates denoted $|n>$ with energy $E_n^{(0)}$. Specially, the perturbations from the fermion ground states are particularly interesting to us. We here use that notations of the N fermion ground states $|d>_g$ as given by,

$$\begin{aligned} &H|d>_g = E_D^{(0)}|d>_g \quad (d = 1, 2, \cdots, d_N = C_{m+2}^k), \\ &E_D^{(0)} = \frac{1}{6}(m+2)\{(m+1)(2m+3) + 6k\}\omega, \end{aligned} \tag{4.22}$$

with the positive integers k, m being solutions of

$$N = \frac{1}{2}(m+1)(m+2) + k \quad (0 \leq k < m+2). \tag{4.23}$$

For the non-degenerate cases $d_N = 1$, $N = 3,\ 6,\ \cdots$, we have $E_0^{(0)} = \frac{1}{3}N\sqrt{8N+1}\omega$.

For $N = 2, 3$, first order wave function corrections can be computed by using the C operator identities (4.13) and the corresponding reduction formula (**A**.22). The explicit first order ground state wave functions and the second order ground state energies for $N = 2, 3$ are presented below.

1. Two – Anyon Ground States

From the solutions (3.31) and the Appendix (**A**.29) and (**A**.30), the two-fold normalized fermion ground state wave functions denoted by $|l_0>$ (l_0 is the relative angular momentum), which is the correct basis for diagonalizing the operator $\frac{1}{\rho_1^2} L_{\theta_1}$, are,

$$|l_0>= \sqrt{\frac{\omega^3}{\pi^2}} \rho_1 e^{i(l_0+1)\theta_1} \exp\{-\frac{1}{2}\omega(\rho_1^2+\rho_2^2)\} \quad (l_0 = 0, -2), \tag{4.24}$$

with energy 3ω. Thus $n = 1$. From Appendix (**A**.26), we find

$$\begin{aligned} &< l_0|C\ |l_0>= 0, \\ &< l_0|1/\rho_1^2\ L_{\theta_1}|l_0>= (l_0+1)\omega. \end{aligned} \tag{4.25}$$

Thus we finally obtain from Eqs. (4.11) and (**A**.22) the second order energies and the first order wave functions,

$$\begin{aligned} E_{l_0} &= \{3 + 2(l_0+1)\alpha + O(\alpha^3)\}\omega, \\ \Psi_{l_0} &= \left\{1 + \alpha(l_0+1)(\ln\omega\rho_1^2 + \gamma - 1) + O(\alpha^2)\right\} |l_0>, \end{aligned} \tag{4.26}$$

which agree exactly with the exact two-anyon ground state in the α expansion,

$$\begin{aligned} E_0^{\text{Exact}} &= (3+2\alpha)\omega, \quad (l_0 = 0) \\ \Psi_0^{\text{Exact}} &= \sqrt{\frac{\omega^{2\mu+2}}{\pi^2\Gamma(2\mu+1)}} \rho_1^{2\mu} \exp\{-\frac{1}{2}\omega(\rho_1^2+\rho_2^2)\} \\ &= \left\{1 + \alpha(\ln\omega\rho_1^2 + \gamma - 1) + O(\alpha^2)\right\} |l_0 = 0> e^{-i\theta_1}. \end{aligned} \tag{4.27}$$

Notice $|l_0 = 0> e^{-i\theta_1}$ is one of two fermion ground states in the bosonic picture and we have used Appendix (**A**.24) to obtain result (4.27).

This result can also be reproduced directly by doing summation on conventional perturbation formulas.

2. Three – Anyon Ground States

The unique real three-fermion ground wave function may be witten as,

$$
\begin{aligned}
&|0> = \frac{\omega^2}{\pi}\sqrt{\frac{2\omega}{\pi}}\rho_1\rho_2\sin(\theta_1-\theta_2)\exp\{-\frac{1}{2}\omega(\rho_1^2+\rho_2^2+\rho_3^2)\}, \\
&|\Phi_1> = L_{12}|0>, \quad |\Phi_2> = L_{13}|0>,
\end{aligned}
\tag{4.28}
$$

with energy 5ω. Thus $n=1$. By using the cyclic permutation symmetries (4.3),

$$
\begin{aligned}
&<0|\frac{L_{\theta_1}}{\rho_1^2}|0> = 0, \quad <0|1/\rho_1^2|0> = \omega, \\
&<\Phi_1|\frac{C}{\rho_1^2}|\Phi_1> = -2\omega, \quad <0|C|\Phi_1> = 0, \\
&<0|V|0> = \omega\ (2-6\ln\frac{4}{3}), \\
&<\Phi_2|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1> = \frac{1}{4}\{9\ln(\frac{4}{3})-2\}\omega,
\end{aligned}
\tag{4.29}
$$

we obtain,

$$
\begin{aligned}
E_0^{(1)} &= 0, \\
E_{01}^{(2)} &= 18\{3\ln(\frac{4}{3})-1\}\omega, \\
E_{02}^{(2)} &= 18\{1-2\ln(\frac{4}{3})\}\omega.
\end{aligned}
\tag{4.30}
$$

Thus the the second order energy and the first order wave function are respectively,

$$
\begin{aligned}
E_0 &= \{5-36\ln\frac{4}{3}\alpha^2+O(\alpha^3)\}\omega, \\
\Psi_0 &= \left\{1+\frac{1}{2}\alpha\sum_{n<m}^{3}(\ln\omega|\mathbf{r}_n-\mathbf{r}_m|^2+\gamma-\ln 2-1)L_{nm}+O(\alpha^2)\right\}|0>.
\end{aligned}
\tag{4.31}
$$

3. The Perturbative Two – Anyon Spectra

As we mention above, θ operator reduction method is very convenience but only good for evaluating the special type matrix elements. We know the two particle case is of this type. Thus, we present here the second order perturbative two-anyon spectra using this method, then compare it with well-known exact solutions (3.31).[7,8,31] For $N=2$, from Eq. (3.31) and the appendix (**A**.29) and (**A**.30), we have a complete set of normalized orthogonal solutions (in the fermionic representation) to

H,

$$H|n,j,m,l>=\omega(2n+|j|+2m+|l+1|+2)|n,j,m,l>,$$

$$|n,j,m,l>=\sqrt{\frac{\omega^{|j|+|l+1|+2}}{\pi^2|j|!\;|l+1|!}C^n_{n+|j|}C^m_{m+|l+1|}}\;\rho_2^{|j|}e^{ij\theta_2}\rho_1^{|l+1|}e^{i(l+1)\theta_1} \tag{4.32}$$
$$\times F(-n,\;1+|j|;\;\omega\rho_2^2)F(-m,\;1+|l+1|;\;\omega\rho_1^2)\exp\{-\frac{1}{2}\omega(\rho_1^2+\rho_2^2)\},$$

with positive integers n, m, integer j, and even integer l. We note that the above solutions (4.32) are also eigenstates of the operator L_{θ_1} with the eigenvalue l. As we have previously mentioned, the result 11 is also good for obtaining perturbation corrections to eigenstates other than ground states. Therefore we have the energy correction up to second order,

$$\begin{aligned}E_{njml}&=\omega(2n+|j|+2m+|l+1|+2)+2\alpha<n,j,m,l|\frac{1}{\rho_1^2}L_{\theta_1}|n,j,m,l>+O(\alpha^3)\\&=\omega\left(2n+|j|+2m+|l+1|+2+2\alpha\frac{l+1}{|l+1|}+O(\alpha^3)\right),\end{aligned} \tag{4.33}$$

which coincides with the exact spectra (3.31) for two anyons.

We note that first order perturbation already gives us the exact solutions for $N=2$.

We plot the second order three anyon ground state energy (4.31) near the fermion limit[33,50] and an exact solution (3.31)[31,33] (which connects boson ground state) in Fig. 5:

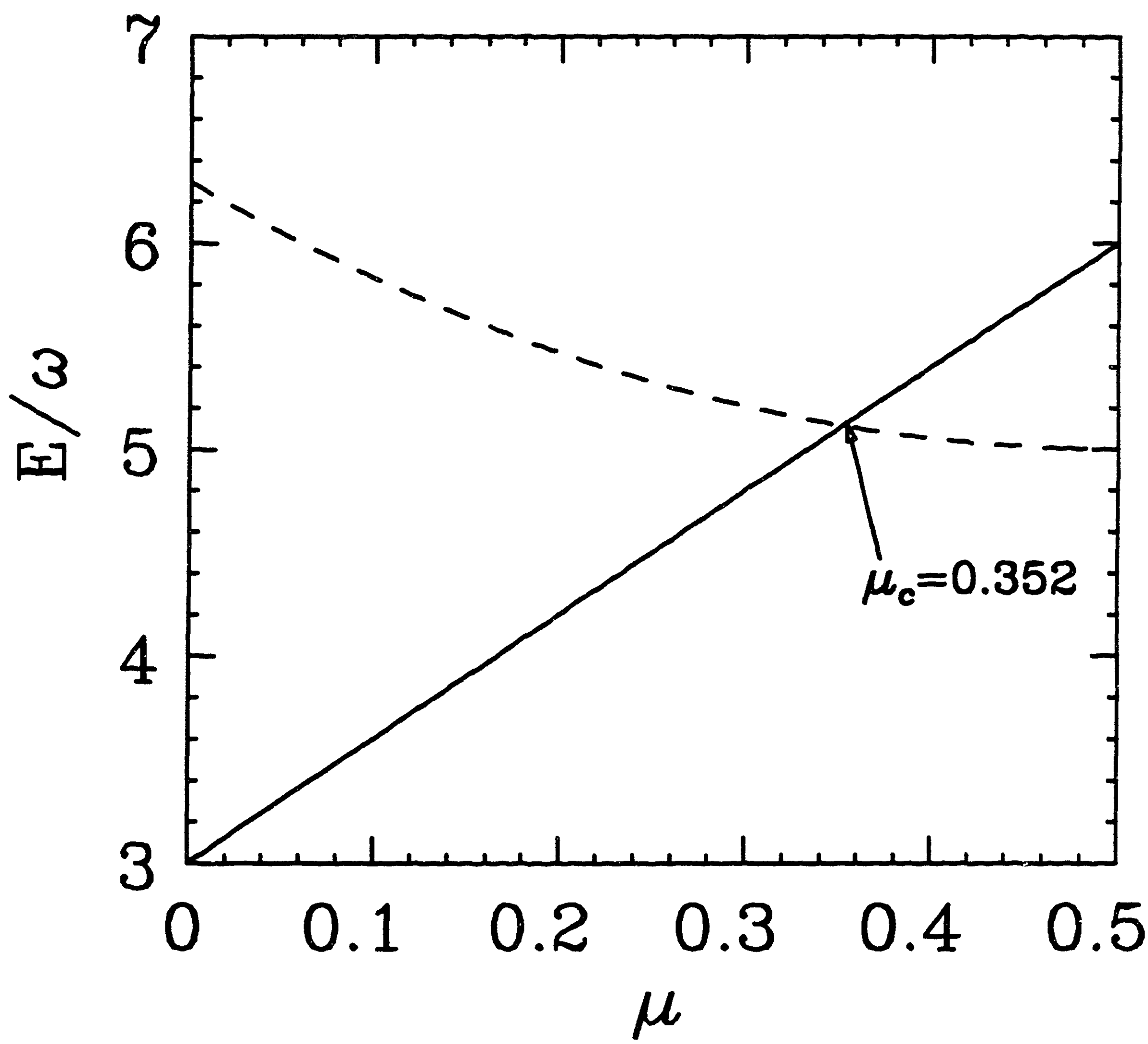

Fig. 5: The second order three anyon ground state energy from the fermion limit. Solid line represents the three anyon exact solution (same as the perturbative first order anyon ground state in Eq. (4.6) from the boson limit) which connects with the boson ($\mu = 0^+$) ground state. The dotdash denotes the third order (from the fermion limit) perturbative energy for $N = 3$.

Perturbative States From Bosonic End

In this subsection, we compute the anyon states from the bosonic limit.

1. N Anyon Ground State

The normalized N boson ground state wave function is,

$$|0_b>=(\frac{\omega}{\pi})^{\frac{N}{2}}\exp\left(-\frac{1}{2}\omega\sum_{n=1}^{N}\rho_n^2\right), \tag{4.34}$$

with the energy $N\omega$. The identity $L_{nm}|0_b>\equiv 0$ implies,

$$|\Phi_1>=0, \quad |\Phi_0>=|0_b>. \tag{4.35}$$

Thus

$$\begin{aligned} E_0^{(1)} &= 0, \\ E_{01}^{(2)} &= 0, \\ E_{0s}^{(3)} &= 0. \end{aligned} \tag{4.36}$$

Namely, the second order anyon ground state correction is simply given by the average of the second order Hamiltonian and the fourth order singular contribution $E_{0s}^{(4)}$:

$$\begin{aligned} E_0^b =& N\omega + N(N-1) < 0_b^{\mathrm{R}}|\frac{1}{\rho_1^2}|0_b^{\mathrm{R}} > \mu^2 \\ &+ N(N-1)(N-2) < 0_b|V|0_b > \mu^2 + E_{0s}^{(4)}\mu^4 + O(\mu^3). \end{aligned} \tag{4.37}$$

It is easy to recognize that this average is logarithmically divergent and this also happens to be the general case in the perturbation calculations near the boson limit as pointed out in the previous section.

From the above considerations, we use the regularized N boson ground state state wave function $|0_b^{\mathrm{R}}>=\rho_1^{\mu}|0_b>$ to compute all related corrections. We give here the relevant results, which lead to the final second order energy and the first order wave function:

$$
\begin{aligned}
&< 0_b^{\mathrm{R}}|\frac{1}{\rho_1^2}|0_b^{\mathrm{R}} >= \frac{1}{\mu}, \quad < 0_b|C|0_b >= -\frac{1}{2}, \\
&< 0_b^{\mathrm{R}}|\frac{C}{\rho_1^2}|0_b^{\mathrm{R}} >_s = -\frac{1}{2\mu}, \quad < 0_b|V|0_b >= 2\ln(\frac{4}{3})\omega, \\
&< 0_b^{\mathrm{R}}|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|0_b^{\mathrm{R}} >_s = -\frac{1}{2\mu}\{1+\ln(\frac{4}{3})\}.
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
&|\Psi_0^{(1)} >=0, \\
&|\Psi_{0s}^{(2)} >=\frac{2}{\mu}\left(\{C|0_b > +c.p.\} + \frac{1}{4}N(N-1)|0_b >\right) \\
&\qquad =\frac{1}{\mu}\sum_{n<m}^{N}\left(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma\right)|0_b > .
\end{aligned}
\tag{4.39}
$$

Using the results in Eqs. (4.14), (4.15), (4.37), (4.38) and (4.39), we obtain the final results, the second order energy,

$$
E_0^b = \omega\{N + N(N-1)\mu + O(\mu^3)\},
\tag{4.40}
$$

and the first order wave function,

$$
|\Psi_0^b >= \left\{1 + \mu\sum_{n<m}^{N}\left(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma\right) + O(\mu^2)\right\}|0_b > .
\tag{4.41}
$$

Note, the exact cancelation of the second order contributions from $< 0_b|V|0_b >$ and the fourth order singular parts is only demonstrated explicitly for $N = 3$. However, one can simply use the cyclic permutation properties of wave functions to show this in general as one more (simple) matrix element is computed. Also, this coincides with the exact result as can be easily checked for $N = 2$. The general check can be done but takes pages due to lack of explicit form for the normalization of the product $\sum_{n<m}|z_n - z_m|^{2\mu}$. Therefore the exact anyon state connected to the N boson ground state in (3.31) is the ground state for small μ.

2. $\underline{E_0 = 5\omega,\ J = \pm 2}$

The bosonic $E = 5\omega$ states are 6-fold degenerate, among which three degenerate relative motions correspond to $J = 0, \pm 2$. The normalized wave functions in Jacobi-coordinates for $E_0 = 5\omega$ and $J = \pm 2$ (the $J = -2$ state is related to the first missing anyon spectra from the bosonic end) are

$$|5,-2> = \frac{1}{2}\sqrt{\frac{\omega^5}{\pi^3}}(\rho_1^2 e^{-2i\theta_1} + \rho_2^2 e^{-2i\theta_2}) \exp\left(-\frac{1}{2}\omega\sum_{n=1}^{3}\rho_n^2\right), \qquad (4.42)$$

and $|5,+2> = |5,-2>^*$. Therefore, we have,

$$\begin{aligned} &|\Phi_1^I, \pm> = L_{12}|5,\pm 2>, \quad |\Phi_2^I, \pm> = L_{13}|5,\pm 2>, \\ &|\Phi_0^I, -> = \frac{1}{2}\sqrt{\frac{\omega^5}{\pi^3}}\rho_2^2 e^{-2i\theta_2} \exp\left(-\frac{1}{2}\omega\sum_{n=1}^{3}\rho_n^2\right), \quad |\Phi_0^I, +> = |\Phi_0^I, ->^*, \end{aligned} \qquad (4.43)$$

and the useful identity,

$$|\Phi_0^I, \pm> \pm \frac{1}{4}|\Phi_1^I, \pm> = |5, \pm 2> . \qquad (4.44)$$

The computation is straightforward,

$$\begin{aligned} &< 5,\pm 2|\frac{L_{\theta_1}}{\rho_1^2}|5,\pm 2> = \pm\frac{1}{2}\omega, \\ &< \Phi_1^I,\pm|\frac{C}{\rho_1^2}|\Phi_1^I,\pm> = 0, \quad < 5,\pm 2|C|\Phi_1^I,\pm> = \pm\frac{1}{2}, \\ &< \Phi_2^I,\pm|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1^I,\pm> = \frac{3}{8}\{1 - 3\ln(\frac{4}{3})\}\omega, \\ &< 5,\pm 2|^{\mathrm{R}}\frac{1}{\rho_1^2}|5,\pm 2>^{\mathrm{R}} = \frac{1}{4}\omega + \frac{1}{2\mu}\omega, \\ &< 5,\pm 2|V|5,\pm 2> = \{4\ln\frac{4}{3} - 1\}\omega, \\ &< 5,\pm 2|\frac{C}{\rho_1^2}|\Phi_1^I,\pm> = 0, \quad < 5,\pm 2|C|\Phi_0^I,\pm> = -\frac{1}{4}, \\ &< 5,\pm 2|^{\mathrm{R}}\frac{C}{\rho_1^2}|\Phi_0^I,\pm>_s^{\mathrm{R}} = -\frac{1}{4\mu}\omega, \\ &< 5,\pm 2|^{\mathrm{R}}\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1^I,\pm>_s^{\mathrm{R}} = \pm\frac{3}{32\mu}\{1 - 2\ln(\frac{4}{3})\}\omega, \\ &< 5,\pm 2|^{\mathrm{R}}\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_0^I,\pm>_s^{\mathrm{R}} = \frac{\omega}{128\mu}\{1 - 2\ln(\frac{4}{3})\}. \end{aligned} \qquad (4.45)$$

Using Eq. (4.45), we obtain,

$$
\begin{aligned}
|\Psi_I^{(1)}, \pm > &= \frac{1}{2}\{C|\Phi_1^I, \pm > + c.p.\} \mp \frac{3}{4}|5, \pm 2 >, \\
|\Psi_{Is}^{(2)}, \pm > &= \frac{2}{\mu}\Big(\{C|\Phi_0^I, \pm > + c.p.\} + \frac{3}{4}|5, \pm 2 >\Big).
\end{aligned}
\tag{4.46}
$$

Combining Eqs. (4.14), (4.19) and (4.45), we have ($n = 2$),

$$
\begin{aligned}
E_{\pm}^{(1)I} &= \pm 3\omega, \\
E_{\pm}^{(2)I} &= \frac{3}{\mu}\omega + \frac{3}{4}\{14\ln(\frac{4}{3}) - 3\}\omega, \\
E_{\pm\ s}^{(3)I} &= \mp\frac{9}{\mu}\ln(\frac{4}{3})\omega, \\
E_{\pm\ s}^{(4)I} &= \frac{3}{4\mu^2}\{3 - 2\ln(\frac{4}{3})\}\omega.
\end{aligned}
\tag{4.47}
$$

Finally, the second order energies for the $J = \pm 2$ states are

$$
\begin{aligned}
E_+^I &= \{5 + 6\mu + O(\mu^3)\}\omega, \\
E_-^I &= \{5 + 18\ln(\frac{4}{3})\mu^2 + O(\mu^3)\}\omega.
\end{aligned}
\tag{4.48}
$$

We using Eqs. (4.15), (4.44) and (4.46) obtain the wave function upto the first order,

$$
\begin{aligned}
|\Psi_I, + > &= \Big\{1 + \mu\sum_{n<m}^{3}\Big(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma - \frac{3}{4}\Big) + O(\mu^2)\Big\}|5, 2 >, \\
|\Psi_I, - > &= \Big\{1 + \mu\sum_{n<m}^{3}\Big(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma - \frac{1}{4}\Big) \\
&\quad + \frac{1}{2}\mu\sum_{n<m}^{3}\Big(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma - 1\Big)L_{nm} + O(\mu^2)\Big\}|5, -2 > .
\end{aligned}
\tag{4.49}
$$

3. $\underline{E_0 = 6\omega,\ J = \pm 3}$

The bosonic $E = 6\omega$ states are 14-fold degenerate, among which 8-fold degenerate states of the mixed motions correspond to $J = 0^2$, $(\pm 2)^2$, $(\pm 3)^1$, respectively (the superscript represents the degeneracy of the states due to the center-of-mass and radial relative motions). The normalized wave functions in Jacobi-coordinates for $E_0 = 6\omega$ and $J = \pm 3$ (the $J = -3$ state is related to the second missing anyon spectra from the bosonic end) are

$$|6,-3>=\frac{\omega^3}{2\sqrt{6\pi^3}}\rho_2 e^{-i\theta_2}(3\rho_1^2 e^{-2i\theta_1}-\rho_2^2 e^{-2i\theta_2})\exp\left(-\frac{1}{2}\omega\sum_{n=1}^{3}\rho_n^2\right), \tag{4.50}$$

and $|6,+3>=|6,-3>^*$. Therefore, we have

$$\begin{aligned}&|\Phi_1^{II},\pm>=L_{12}|6,\pm3>, \quad |\Phi_2^{II},\pm>=L_{13}|6,\pm3>,\\ &|\Phi_0^{II},->=-\frac{\omega^3}{2\sqrt{6\pi^3}}\rho_2^3 e^{-3i\theta_2}\exp\left(-\frac{1}{2}\omega\sum_{n=1}^{3}\rho_n^2\right), \quad |\Phi_0^{II},+>=|\Phi_0^{II},->^*,\end{aligned} \tag{4.51}$$

and the useful identity,

$$|\Phi_0^{II},\pm>\pm\frac{1}{4}|\Phi_1^{II},\pm>=|6,\pm3>. \tag{4.52}$$

Again, we only list the results:

$$\begin{aligned}&<6,\pm3|\frac{L_{\Phi_1}}{\rho_1^2}|6,\pm3>=\pm\frac{3}{4}\omega,\\ &<\Phi_1^{II},\pm|\frac{C}{\rho_1^2}|\Phi_1^{II},\pm>=0, \quad <6,\pm3|C|\Phi_1^{I},\pm>=\pm\frac{3}{4},\\ &<\Phi_2^{II},\pm|\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1^{II},\pm>=\frac{9}{32}\{9\ln(\frac{4}{3})-2\}\omega,\\ &<6,\pm3|^{\mathrm{R}}\frac{1}{\rho_1^2}|6,\pm3>^{\mathrm{R}}=\frac{3}{8}\omega+\frac{1}{4\mu}\omega,\\ &<6,\pm3|V|6,\pm3>=\{\frac{4}{3}-4\ln(\frac{4}{3})\}\omega,\\ &<6,\pm3|\frac{C}{\rho_1^2}|\Phi_1^{II},\pm>=0, \quad <6,\pm3|C|\Phi_0^{II},\pm>=-\frac{1}{8},\\ &<6,\pm3|^{\mathrm{R}}\frac{C}{\rho_1^2}|\Phi_0^{II},\pm>_s^{\mathrm{R}}=-\frac{\omega}{8\mu},\\ &<6,\pm3|^{\mathrm{R}}\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_1^{II},\pm>_s^{\mathrm{R}}=\pm\frac{9}{64\mu}\{\frac{5}{6}-\ln(\frac{4}{3})\}\omega,\\ &<6,\pm3|^{\mathrm{R}}\frac{C}{|\mathbf{u}_1+\sqrt{3}\mathbf{u}_2|^2}|\Phi_0^{II},\pm>_s^{\mathrm{R}}=-\frac{1}{256\mu}\{\frac{5}{6}-\ln(\frac{4}{3})\}\omega.\end{aligned} \tag{4.53}$$

Using Eq. (4.53), we obtain,

$$
\begin{aligned}
&|\Psi_{II}^{(1)},\pm> = \frac{1}{2}\{C|\Phi_1^{II},\pm> + c.p.\} \mp \frac{9}{4}|6,\pm 3>, \\
&|\Psi_{II_s}^{(2)},\pm> = \frac{2}{\mu}\Big(\{C|\Phi_0^{II},\pm> + c.p.\} + \frac{3}{8}|6,\pm 3>\Big).
\end{aligned}
\tag{4.54}
$$

By using Eqs. (4.14), (4.19), and (4.53), we have ($n = 2$),

$$
\begin{aligned}
E_{\pm}^{(1)II} &= \pm\frac{9}{2}\omega, \\
E_{\pm}^{(2)II} &= \frac{3}{2\mu}\omega + \frac{1}{16}\{102\ln(\frac{4}{3}) - 25\}\omega, \\
E_{\pm\ s}^{(3)II} &= \mp\frac{9}{4\mu}\{3\ln(\frac{4}{3}) - 1\}\omega, \\
E_{\pm\ s}^{(4)II} &= \frac{1}{16\mu^2}\{6\ln(\frac{4}{3}) - 11\}\omega.
\end{aligned}
\tag{4.55}
$$

Thus the second order energies for the $J = \pm 3$ states are,

$$
\begin{aligned}
E_{+}^{II} &= \{6 + 6\mu + O(\mu^3)\}\omega, \\
E_{-}^{II} &= \{6 - 3\mu + \frac{9}{2}(3\ln(\frac{4}{3}) - 1)\mu^2 + O(\mu^3)\}\omega.
\end{aligned}
\tag{4.56}
$$

We using Eqs. (4.15), (4.52) and (4.54) obtain the wave function upto the first order,

$$
\begin{aligned}
|\Psi_{II},+> &= \Big\{1 + \mu\sum_{n<m}^{3}\Big(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma - \frac{9}{8}\Big) + O(\mu^2)\Big\}|6,3>, \\
|\Psi_{II},-> &= \Big\{1 + \mu\sum_{n<m}^{3}\Big(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma - \frac{3}{8}\Big) \\
&\quad + \frac{1}{2}\mu\sum_{n<m}^{3}\Big(\ln\omega|z_n - z_m|^2 - \ln 2 + \gamma - 1\Big)L_{nm} + O(\mu^2)\Big\}|6,-3>.
\end{aligned}
\tag{4.57}
$$

As we point out above that the comparing $|\Psi_{I,II},+>$ with the exact results can be done but takes pages to present. However, apparently they are in agreement with each other. $E_{-}^{I,II}$ in Eqs. (4.48) and (4.56) has been plotted in Fig. 6 with the corresponding numerical results[37], while $E_{+}^{I,II}$ coincide with the exact solutions.[31,33]

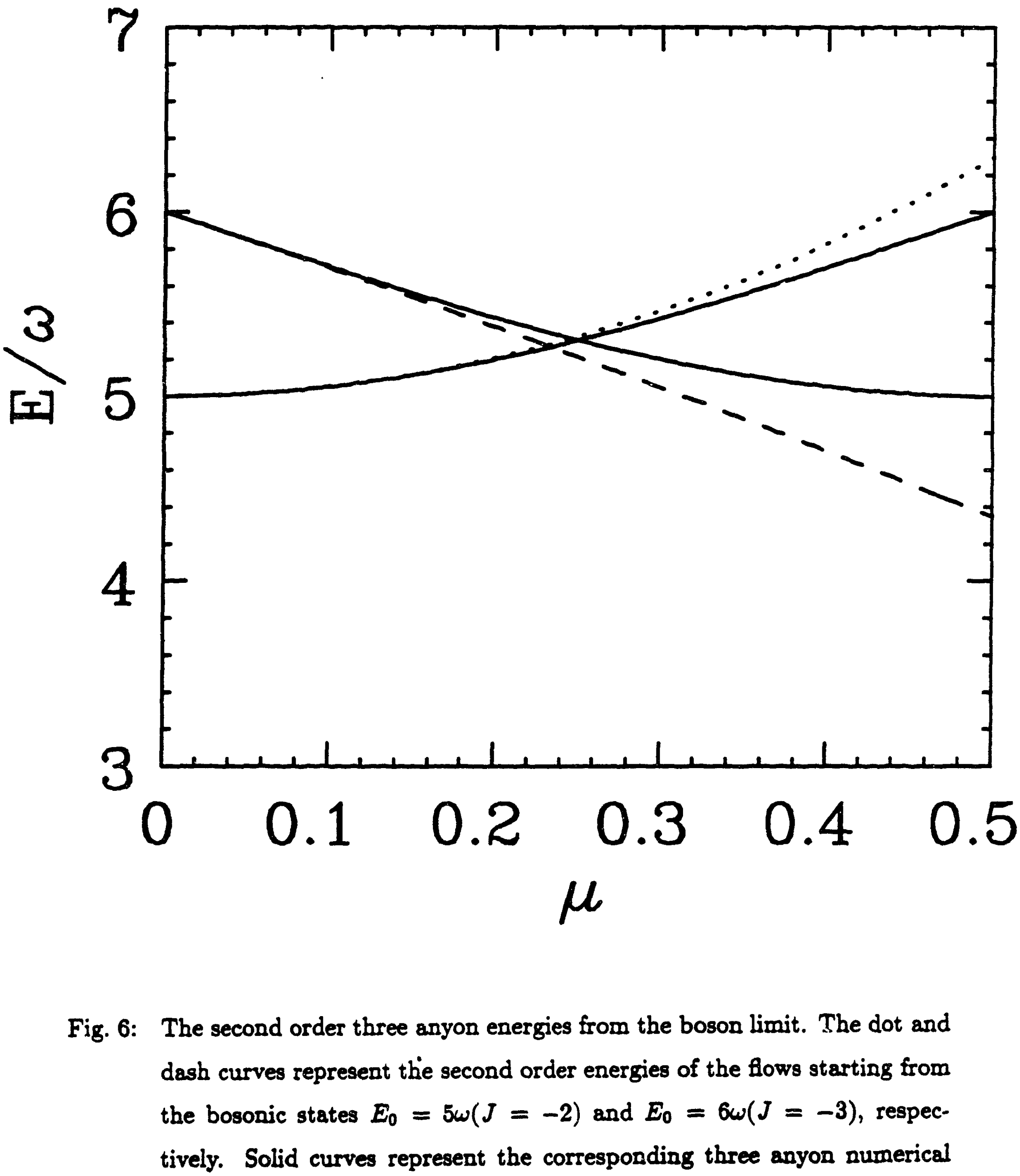

Fig. 6: The second order three anyon energies from the boson limit. The dot and dash curves represent the second order energies of the flows starting from the bosonic states $E_0 = 5\omega(J = -2)$ and $E_0 = 6\omega(J = -3)$, respectively. Solid curves represent the corresponding three anyon numerical solutions.(B.23)

To obtain a clear picture of the three anyon ground state, we plot in Fig. 7 E_-^{II} and the previous perturbative result in Eq. (4.31) from the fermionic ground state:[33,50]

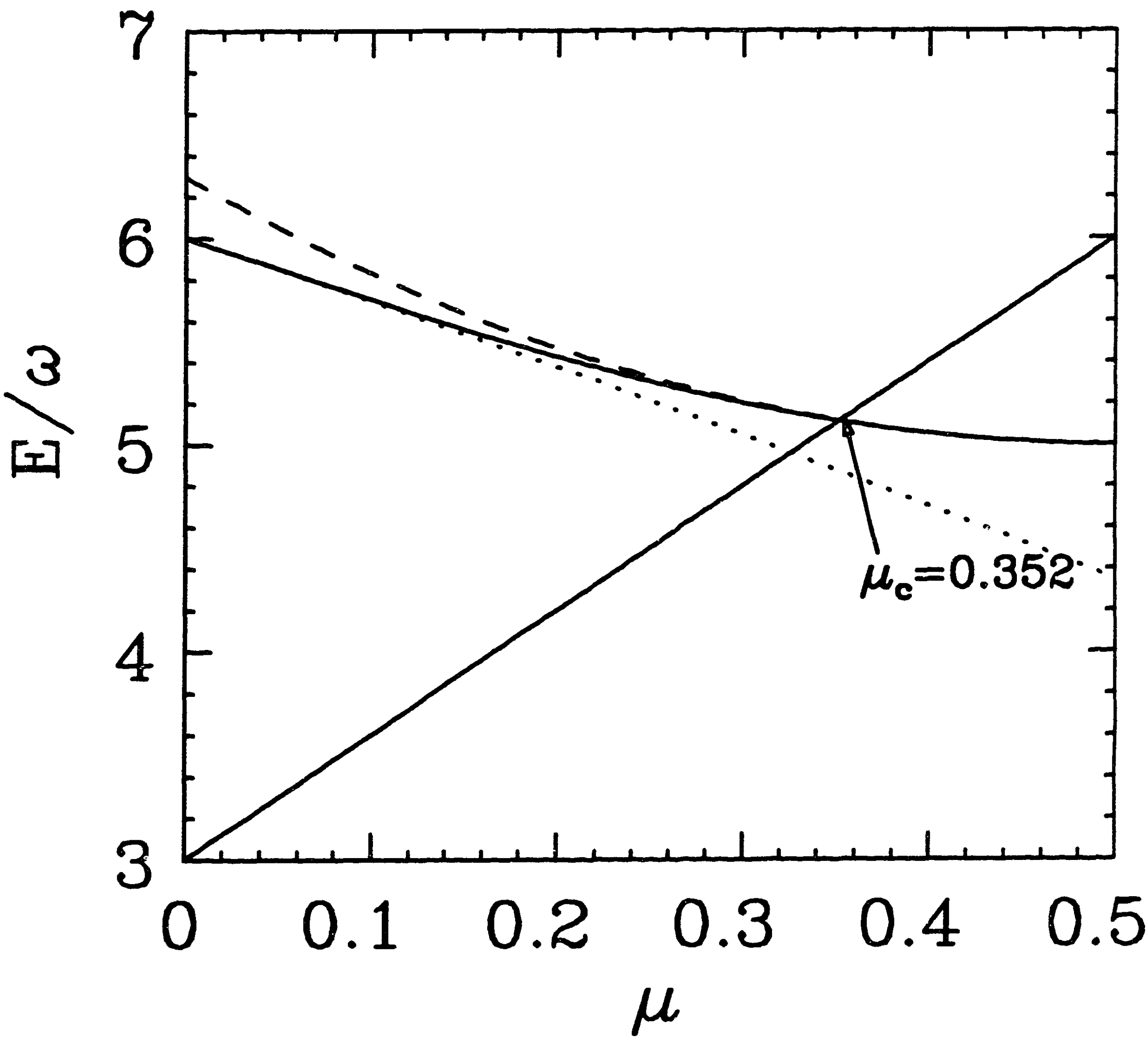

Fig. 7: Three Anyon Ground State. Solid straight line represents the exact three anyon solution that connects with the boson ($\mu = 0$) ground state. The dot and dash curves represent the second order energies of the flows starting from the bosonic state $E_0 = 6\omega(J = -3)$ and the fermion ground state, respectively. The numerical flow $6\omega(\mu = 0) \rightarrow 5\omega(\mu = 1/2)$ is shown by the solid curve.

Summary

We conclude this Chapter by pointing out the following. (1). The semionic symmetry is preserved at least to the second order perturbative calculations: E_-^I is exactly the semionic-mirror of E^F, (2). Comparing with the numerical results, our second order perturbation results in Fig. 1 are good approximations at least for $\mu < 1/4$, at $\mu = 1/2$ the perturbative result underestimates the ground state energy; (3). The flows starting from $|\Psi_{\mathrm{cm}} > |E_0 J >$ have exactly the same curve as for $|E_0 J >$; (4). For the type of pair states we studied above, all we need for the second order energies from the bosonic end are the first order correction and the contributions from the singular parts of the second and third order(In our cases $E_-^{I,II} = E^{(0)I,II} + \mu\{E_-^{(1)I,II} + \mu E_s^{(2)I,II}\} + 2\mu^3 E_{-\ s}^{(3)I,II} + O(\mu^3)$); (5). The C operator reduction method originally introduced in Ref. [33] has been explored in more detail, and further applications of this method to multi-anyon systems are under investigation; (6). Notice that the three-body interaction in Eqs. (4.29), (4.45), and (4.53) is approximately zero and agrees with computer simulations.[36]

Chapter V. Conclusion

We have tried to present a systematic study of many-anyon quantum mechanics, which is a central problem in understanding both the nature of anyon physics and its possible physical applications. A set of the linear-analytic solutions for an N anyon system in a harmonic oscillator potential is obtained. A powerful perturbative method for calculating multi-anyon spectra in closed form near both the boson and fermion limit is outlined and its application for two- and three-anyon problems has been clearly demonstrated. The multi-anyon spectra generally possess a non-linear dependence on the statistics parameter μ in a non-trivial way and some indication on this non-linearality has been learned from the above mentioned closed perturbative calculations. Specially, the closed first order wave functions may provide a hint to eventually solve this many body problem.

To be more specific, we note first that an interesting factor $\ln(\frac{4}{3})$ appears in the both bosonic and fermionic perturbative energies;[33,51] which has been pointed out as the result of the three-body interactions involved.[51] This dependence may suggest that there is a non-trivial factor $(\frac{4}{3})^{c(\mu)}$ in $N(N \geq 3)$ anyon spectra, where $c(\mu)$ is a polynomial of μ. Finally, we note from the first order wave functions in Eqs. (4.31), (4.41), (4.49), and (4.57) that for the missing states, there is an additional contribution $\sum_{n<m}(\ln\omega|z_n - z_m|^2)L_{nm}$* except for $\sum_{n<m}\ln\omega|z_n - z_m|^2$ which is simply due to the usual statistical factor $\prod_{n<m}|z_n - z_m|^{2\mu}$. Therefore, a non-trivial statistical operator factorization in constructing many-anyon wave function may be

* The constant parts which come from the normalization factor have been draped here for simplicitity.

suggested as

$$\Psi_I = \prod_{n<m} |z_n - z_m|^{2\mu} \prod_{k<l} |z_k - z_l|^{2\mu L_{kl}} \Phi, \tag{5.1}$$

where there are still freedoms to define this ambiguous operator product.

We conclude this thesis with the following remark. The notion of identical particles and their statistics plays a key role in the study of quantum systems. The statistics has been realized through the introduction of the Lorentz structure of local field. This Lorentz character is labeled by the spin of the particle or field based on assumption of the spin connection, which is of central importance in formulating a theory of quantum gravity. However, although the spin-statistics theorem has been established a long time ago by examining analyticity of many-point vaccum expectation values of field operator products,[59] this establishment has not covered massless gauge theories[14,60] and some resolution has been made recently.[14] As far as nonrelativistic quantum mechanics is concerned, the spin-statistics connection is not essential as is believed. But, this is not the case for FQS, since FQS is characterized by a mysterious parameter μ. It seems to suggest that the "spin"-statistics theorem may have much more deeper origin than what is seen in local quantum theories.

Appendix A. The Related Mathematical Techniques

In this Appendix, we give some mathematical formulas and techniques used in this thesis. They are the Jacobi transformations, the multiplicity techniques, the universal perturbation theory, and some useful mathematical formulas.

Jacobi Transformations $\nu_{n(m)}$

We summarize here a particularly convenient set of real vectors which is used for generating a compact representation of the Jacobi coordinates. Define the N real vectors (with N components each):[33]

$$\begin{aligned}
\nu_{(1)} &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \cdots, \frac{1}{\sqrt{m(m+1)}}, \cdots, \frac{1}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N}}, \right), \\
\nu_{(2)} &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \cdots, \frac{1}{\sqrt{m(m+1)}}, \cdots, \frac{1}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N}}, \right), \\
\nu_{(3)} &= \left(0, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{12}}, \cdots, \frac{1}{\sqrt{m(m+1)}}, \cdots, \frac{1}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N}}, \right), \\
&\vdots \\
\nu_{(m+1)} &= \left(0, 0, 0, \cdots, -\frac{m}{\sqrt{m(m+1)}}, \cdots, \frac{1}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N}}, \right), \\
\nu_{(N)} &= \left(0, 0, 0, \cdots, 0, \cdots, \frac{1-N}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N}}, \right).
\end{aligned} \tag{A.1}$$

These vectors have the following orthogonal relations:

$$\begin{aligned}
\nu_{(n)} \cdot \nu_{(m)} \equiv \sum_{k=1}^{N} \nu_{k(n)} \nu_{k(m)} = \delta_{nm}, \\
\sum_{k=1}^{N} \nu_{n(k)} \nu_{m(k)} = \delta_{nm},
\end{aligned} \tag{A.2}$$

where $\nu_{n(m)}$ is defined as the n-th component of the vector $\nu_{(m)}$. It is easy to recognize that the Jacobi transformations $\nu_{n(m)}$ transform the orthogonal Cartesian coordinates to the Jacobi coordinates (orthogonal). The positive roots of $SU(N)$ are simply,

$$\beta_{(ij)} = \nu_{(i)} - \nu_{(j)}, \qquad \text{for} \quad i < j. \tag{A.3}$$

Another useful result is

$$\sum_{n=1}^{N} \nu_{(n)} \equiv (0, 0, 0, \cdots, 0, \cdots, 0, \sqrt{N}). \tag{A.4}$$

The Multiplicity

A few of the multiplicity formulas, which are useful for computing the degeneracies of the energy levels in this thesis, are given. We begin with some definitions. Let $a(N, p)$ (or $c(N, p)$) be the number of linearly independent, totally symmetric (or totally antisymmetric) and homogeneous (degree p) polynomials of N variables that are translation invariant. Thus $a(N, p)$ is the number of solutions of Eq. (3.28), and is given by the generating function,[61]

$$\prod_{l=2}^{N} \frac{1}{1 - t^l} = \sum_{p=0}^{\infty} a(N, p) t^p. \tag{A.5}$$

It is helpful if we write $a(N, p) = b(N, p) - b(N, p - 1)$, where $b(N, p)$ satisfies the generating equation,

$$\prod_{l=1}^{N} \frac{1}{1 - t^l} = \sum_{p=0}^{\infty} b(N, p) t^p. \tag{A.6}$$

In fact, $b(N, p)$ is the number of linearly independent and totally symmetric homogeneous (with degree p) polynomials, because the translation invariance equivalently requires that the polynomials $\sum_{i=1}^{N} \partial s_p / \partial z_i$ vanish identically. A few explicit results are obtained,[62]

$$
\begin{aligned}
&b(2,p) = 1 + [\frac{1}{2}p], \\
&b(3,p) = \{\frac{1}{12}(p+2)(p+4)\}, \\
&b(4,p) = \{\frac{1}{144}(p+2)(p^2+13p+37+\frac{9}{2}(1+(-1)^p))\}, \\
&b(5,p) = [(\frac{1}{2880}((p+1)(p+2)(p+3)(p+24)+155p^2+15p(67+3(-1)^p))], \\
&[x] = \text{the integer part of } x, \quad \{x\} = \text{the closest integer of } x.
\end{aligned}
\tag{A.7}
$$

We note that by identifying the residue of the pole of the generating function in Eq. (A.6) at $t = 1$, we obtain the asymptotic expression,

$$
\lim_{p\to\infty} a(N,p) = \frac{p^{N-2}}{N!\,(N-2)!}. \tag{A.8}
$$

Secondly, $c(N,p)$ may be similarly written as $c(N,p) = e(N,p) - e(N,p-1)$, where $e(N,p)$ is the number of linearly independent and totally antisymmetric homogeneous (with degree p) N variable polynomials. Since the totally antisymmetric polynomials can always be written as the Slater determinants, $e(N,p)$ is thus the number of solutions of the following equation,

$$
\begin{aligned}
&\sum_{i=1}^{N} n_i = p, \\
&n_1 < n_2 < \cdots < n_N,
\end{aligned}
\tag{A.9}
$$

where $\{n_i\}$ is a set of non-negative integers. Let $m_i = n_i - (i-1)$ $(i = 1, 2, \cdots, N)$, then $m_i \le m_j$, $\forall i < j$, and $e(N,p)$ is given by the generating function

$$
\prod_{l=1}^{N} \frac{t^{l-1}}{1-t^l} = \sum_{p=0}^{\infty} e(N,p) t^p. \tag{A.10}
$$

Therefore from (**A**.10), we find,

$$e(N,p) = \begin{cases} 0, \text{ if } p \le C_N^2; \\ b(N, p - C_N^2), \text{ if } p > C_N^2, \end{cases} \tag{A.11}$$

where C_N^2 is a binomial coefficient.

Finally, we give an another useful result. Assume that a set p is a sum of two subsets with the known multiplicity d_m^I and d_m^{II}, then the multiplicity d_p of the set p can be computed by the following rather obvious formula,

$$d_p = \sum_{m=0}^{p} d_m^I d_{p-m}^{II}. \tag{A.12}$$

A few examples, which are useful for our discussions, are given as follows,

$$\begin{aligned}
&(1). \begin{cases} p = 2n + |L|, \\ d_p = 1 + p, \end{cases} \\
&(2). \begin{cases} p = 2m + |l|, \\ d_p = \begin{cases} 0, \text{ if } p \text{ is a odd integer;} \\ 1 + p, \text{ if } p \text{ is a even integer,} \end{cases} \end{cases} \\
&(3). \begin{cases} p = 2n + 2m + |L|, \\ d_p = (1 + \frac{p-\delta}{2})(1 + \frac{p+\delta}{2}), \\ \quad = \begin{cases} \frac{1}{4}(p+1)(p+3), \text{ if } p \text{ is a odd integer;} \\ \frac{1}{4}(p+2)^2, \text{ if } p \text{ is a even integer,} \end{cases} \end{cases} \\
&(4). \begin{cases} p = 2n + 2m + |L| + |l|, \\ d_p = \begin{cases} \frac{1}{12}(p+1)(p+2)(p+3), \text{ if } p \text{ is a odd integer;} \\ \frac{1}{12}(p+2)(p^2 + 4p + 6), \text{ if } p \text{ is a even integer,} \end{cases} \end{cases}
\end{aligned} \tag{A.13}$$

where δ=0 (or 1) for p=even (or odd integer), respectively, n, m, and L are defined in Eqs. (3.26) or (3.29) and l is a even integer.

Perturbation Theory Outline

We now outline a universal perturbation theory as in Ref. 63. Consider solving the following eigen-equation,

$$H\Psi = E\Psi \tag{A.14}$$

with $H = H_0 + V$ and unperturbed eigenstates,

$$H_0|n>= E_n^{(0)}|n> . \tag{A.15}$$

By using the conventional projection operator technique, we obtain the effective eigen equation of Eq. (A.14),

$$\tilde{H}\Psi_D = (E - H_0)\Psi, \tag{A.16}$$

with

$$\begin{aligned} \tilde{H} &= V + V\frac{Q}{E - H_0}\tilde{H}, \\ \Psi &= \Psi_D + \frac{Q}{E - H_0}V\Psi, \end{aligned} \tag{A.17}$$

where $\Psi_D = P\Psi$, $P = \sum_{d\in D}|d><d|$, $Q = 1 - P$, and D is a d-dimensional degenerate subspace of the unperturbed state space.

We are particularly interested in perturbation Hamiltonians of the form $V = \alpha A + \alpha^2 B$, where α is a perturbation parameter. The universal perturbation formulas for this Hamiltonian, which result from diagonalizing the corresponding effective Hamiltonian (A.17), are the wave functions to first order,

$$\begin{aligned} \Psi_d =& \Bigg\{1 + \alpha\Bigg(\frac{P_d}{<d|A|d> - A}(B + A\frac{Q}{E_D^{(0)} - H_0}A) \\ &+ \frac{Q}{E_D^{(0)} - H_0}A\Bigg) + O(\alpha^2)\Bigg\}|d>, \end{aligned} \tag{A.18}$$

and the energies to the third order,

$$E_d = E_D^{(0)} + \alpha E_d^{(1)} + \alpha^2 E_d^{(2)} + \alpha^3 E_d^{(3)} + O(\alpha^4), \tag{A.19}$$

where

$$\begin{aligned}
E_d^{(1)} &= < d|A|d >, \quad E_d^{(2)} = < d|B + A\frac{Q}{E_D^{(0)} - H_0}A|d >, \\
E_d^{(3)} &= < d| \left\{ \left(B + A\frac{Q}{E_D^{(0)} - H_0}A \right) \frac{P_d}{< d|A|d > - A} \left(B + A\frac{Q}{E_D^{(0)} - H_0}A \right) \right. \\
&\quad + \left(B\frac{Q}{E_D^{(0)} - H_0}A + A\frac{Q}{E_D^{(0)} - H_0}B + A\frac{Q}{E_D^{(0)} - H_0}A\frac{Q}{E_D^{(0)} - H_0}A \right. \\
&\quad \left. \left. - APA\frac{Q}{(E_D^{(0)} - H_0)^2}A \right) \right\} |d >,
\end{aligned} \tag{A.20}$$

with $P_d = \sum_{d'(\neq d) \in D} |d' >< d'|$ and $P_d \; |d > \equiv 0$.

Without loss of generality, we have assumed in Eqs. (A.18), (A.19), and (A.20) that A is diagonalized in the degenerate subspace D with the basis $|d >$ ($d \in D$) and that the degeneracy of H_0 in this subspace is completely removed at first order. In the case in which the degeneracy is not removed completely, we have to diagonalize further $B + A\frac{Q}{E_D^{(0)} - H_0}A$, and even the higher order part of the effective Hamiltonian $\tilde{H}$, by choosing correct zeroth-order wave function, until the degeneracy is completely lifted. However in this work, first order perturbation completely removes the degeneracy of H_0 and thus we do not need this further procedure. For the non-degenerate case, we simply apply $P_d = 0$ and note that the subspace D then is a "point".

Suppose that A takes the form,

$$iA = [C, H_0] + f(H_0), \tag{A.21}$$

for some non-singular function f and Hermitian operator C. Plugging this identity (**A**.21) into Eqs. (**A**.18) and (**A**.20), we then can rewrite Eqs. (**A**.18), (**A**.19), and (**A**.20) as a single matrix element, which only involves the wave functions in the unperturbed subspace. Eqs. (**A**.18) and (**A**.20) are respectively reduced to

$$\Psi_d = \left\{ 1 + \alpha \left(\frac{P_d}{< d|A|d > -A} \{B - iA(1-P)C\} - i(1-P)C \right) + O(\alpha^2) \right\} |d>, \tag{A.22}$$

and

$$\begin{aligned} &E_d^{(1)} = < d|A|d>, \quad E_d^{(2)} = < d|B + \frac{1}{2} i[C,A]|d>, \\ &E_d^{(3)} = < d| \Big\{ (B + \frac{1}{2} i[C,A]) \frac{P_d}{< d|A|d > -A} (B + \frac{1}{2} i[C,A]) \\ &- i[B,C] + CAC - CPAPC - APC^2 + APCPC \Big\} |d> . \end{aligned} \tag{A.23}$$

We conclude this subsection by pointing out that to find identities similar to Eq. (**A**.21) for both A and B is equivalent to solving H completely. In other words, if we find such identities, by using operator reduction formulas like (**A**.22) and (**A**.23), we can eventually solve H perturbatively up to any order.

The Useful Mathematical Formulas

We collect here a few useful mathematical formulas, which relate to our calculations:

$$\begin{aligned} &\Gamma(1+x) = \sum_{k=0}^{\infty} c_k \, x^k \quad (\text{for small } x), \\ &c_{k+1} = \frac{1}{k+1} \sum_{n=0}^{k} (-1)^{n+1} s_{n+1} c_{k-n}, \quad s_n = \zeta(n) \ (n \geq 2); \quad c_0 = 1, \quad s_1 = \gamma. \end{aligned} \tag{A.24}$$

$$\int_0^{\pi} \frac{\cos nx dx}{1 + a\cos x} = \frac{\pi}{\sqrt{1-a^2}} \left(\frac{\sqrt{1-a^2} - 1}{a} \right)^n \quad (a^2 < 1) \tag{A.25}$$

$$\int_0^\infty e^{-\beta x} \ln x \; dx = -\frac{1}{\beta}(\gamma + \ln \; \beta) \quad (\text{Re } \beta > 0) \tag{A.26}$$

$$\int_0^\infty x^n \; e^{-\beta x} \ln x \; dx = \frac{n!}{\beta^{n+1}}\{\sum_{k=1}^{n} \frac{1}{k} - \gamma - \ln \; \beta\}, \tag{A.27}$$
$$(n = 1, 2, \cdots, \quad \text{and} \quad \text{Re } \beta > 0)$$

$$\int_0^\infty e^{-\beta x}(\ln x)^2 \; dx = \frac{1}{\beta}\{\frac{\pi^2}{6} + (\gamma + \ln \beta)^2\} \quad (\text{Re}\beta \; > 0) \tag{A.28}$$

$$\begin{gathered}\frac{1}{\Gamma(\alpha+1)} C^m_{m+\alpha} \int_0^\infty dx \; e^{-x} \; x^\alpha \; F(-n, \; 1+\alpha; \; x) \; F(-m, \; 1+\alpha; \; x) = \delta_{nm}, \\ \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} C^n_{n+\alpha} \; F(-n, \; 1+\alpha; \; x) \; F(-n, \; 1+\alpha; \; y) = \delta(x-y), \\ (\text{Re } \alpha > 0)\end{gathered} \tag{A.29}$$

where $C^m_{m+\alpha} \equiv \frac{1}{m!}(m+\alpha)(m+\alpha-1)\cdots(\alpha+1)$. Using Eq. (A.29), we have the following the orthogonality relations,

$$\begin{gathered}< n', j', m', l' | n, j, m, l >= \delta_{nn'}\delta_{jj'}\delta_{mm'}\delta_{ll'}, \\ \sum_{njml} < n, j, m, l(\mathbf{u}_1, \mathbf{u}_2) | n, j, m, l(\mathbf{u}'_1, \mathbf{u}'_2) >= \delta(\mathbf{u}_1 - \mathbf{u}'_1)\delta(\mathbf{u}_2 - \mathbf{u}'_2),\end{gathered} \tag{A.30}$$

where $|n, j, m, l>$ is defined in Eq. (4.32).

The following simple trick is useful for computing the singular contributions in the bosonic perturbation theory:

$$\text{singular part of } \int_0^\infty e^{-x} f(x) x^{\mu-1} dx = f(0)\frac{1}{\mu}. \tag{A.31}$$

Appendix B. Generalized Quantum Statistics

In this Appendix, a non-anyonic generalization of quantum statistics is presented, in which Fermi-Dirac statistics(FDS) and Bose-Einstein statistics(BES) appear as two special cases. This new quantum statistics, which is characterized by the dimension of its single particle Fock space, contains three consistent parts, namely the generalized bilinear quantization, the generalized quantum mechanical description and the corresponding statistical mechanics.

Genon Quantization

We begin to study the one particle sector. The multi-genon Fock space is a direct product of the single particle Hilbert spaces, as in the ordinary bosonic and fermionic cases. We define the creation operator b and the annihilation operator a for the genon, with a and b obeying the q-mutator:

$$[a, b]_q = 1, \quad ([A, B]_q = AB - qBA) \tag{B.1}$$

$$\begin{aligned} &a^n \neq 0,\ b^n \neq 0,\ \ \forall n \leq M-1; \\ &a^M = b^M = 0, \end{aligned} \tag{B.2}$$

where M and n are positive integers, $2 \leq M \leq \infty$. The operator b is related to $a^\dagger$ through $b = ca^\dagger$, where the operator c satisfies

$$[c, c^\dagger] = [c, N] = 0, \tag{B.3}$$

with $N = a^\dagger a$ and $[A, B] = AB - BA$(the superscript $\dagger$ denotes the Hermitian conjugates). Eqs. (**B**.1), (**B**.2) and (**B**.3) define a genon. The need to introduce independent creation and annihilation operators will be discussed further below.

As we shall see, M (or q) parameterizes the interpolation between BES($M = \infty$) and FDS($M = 2$) and c becomes an identity for both BES and FDS (in these two cases, $F \equiv ba$, N and $\tilde{N} \equiv bb^\dagger$ will be the same). The relation between M and q is determined by Eqs. (**B**.1) and (**B**.2) and will be given below.

From the definition (**B**.1), we have,

$$[a, F]_q = a, \quad [b, F]_{q^{-1}} = -q^{-1}b, \tag{B.4}$$

$$[a^n, b]_{q^n} = [n]_q a^{n-1}, \quad [b^n, a]_{q^n} = [n]_q b^{n-1}, \tag{B.5}$$

where $[n]_q = 1 + q + \cdots + q^{n-1} (n \neq 0)$ ($[0]_q = 0$ is further assumed for convenience). One may recognize from Eq. (**B**.4) that F plays a role similar to the conventional number operator. Eq. (**B**.3) yields

$$[c, \tilde{N}] = [c, F] = [c^\dagger, F] = [N, F] = 0. \tag{B.6}$$

Using Eq. (**B**.2) and (**B**.5), we obtain,

$$[M]_q = 0, \text{ for } M \neq \infty; \quad q = 1, \text{ for } M = \infty, \tag{B.7}$$

which yields a relation between q and M:

$$q = \exp\{i2\pi(\frac{k}{M} + j)\}, \quad \forall M \geq 2, \tag{B.8}$$

with $1 \leq k \leq M - 1$ and j being an integer. We note that $k/M + j$ plays an analogous role to the total angular momentum of an ordinary particle, and the genon statistics (by means of Eqs. (**B**.1), (**B**.2)[or (**B**.8)] and (**B**.3)) depends only on M and k, as expected (j is irrelevant to the statistics and is thus similar to the orbital angular momentum in the conventional theories). It is clear from Eq. (**B**.8) that an even dimensional Fock space M contains a two-dimensional fermion subspace. Therefore we (conveniently) take the integer k as being relatively prime to M in the following discussion. We simply point out here that if one chooses q in form of Eq. (**B**.8), then Eq. (**B**.2) may be automatically derived from the commutation relation (**B**.1).

We now construct a (M dimensional) Fock space $\{|n>(0 \leq n \leq M-1)\}$ for an genon, which is bounded from both directions

$$a|0>=0, \quad b|M-1>=0. \tag{B.9}$$

From Eq. (**B**.1), we can easily show,

$$F|n>=[n]_q|n>, \quad F^{\dagger}|n>=[n]_{q^*}|n>. \tag{B.10}$$

Without loss of generality, the (one degree of freedom) genon number operator can be realized as N, namely

$$N|n>=n|n>, \tag{B.11}$$

by choosing suitable c as we recall $F=cN$ and Eq. (**B**.10). It is clear that we are also able to choose a different c such that $\tilde{N}=c^{\dagger}F$ is the genon number operator. The reason we have choices is simply because of having more than one analog of the conventional number operator.

Therefore using Eqs. (**B**.3), (**B**.6), (**B**.8) and (**B**.10), we obtain the action of a, b, c, and $\tilde{N}$ on the states $|n>(0 \leq n \leq M-1)$ as follows:

$$\begin{aligned} & a|n>=\sqrt{n}|n-1>, \quad b|n>=\frac{[n+1]_q}{\sqrt{n+1}}|n+1>, \\ & c|n>=\frac{[n]_q}{n}|n>, \quad \tilde{N}|n>=\tilde{n}|n>, \end{aligned} \tag{B.12}$$

where

$$\tilde{n} = \frac{[n]_q [n]_{q^*}}{n} = \frac{1}{n} \frac{\sin^2[n\pi \frac{k}{M}]}{\sin^2[\pi \frac{k}{M}]}, \qquad (\mathbf{B}.13)$$

with $c_0 = 1$ being understood. Therefore, the explicit Fock space is given as follows:

$$\begin{aligned} &|n> = \frac{1}{\sqrt{\tilde{n}!}} b^n |0>, \quad <n| = \frac{1}{\sqrt{\tilde{n}!}} <0|(b^\dagger)^n \quad (\text{or } <n| = \frac{1}{\sqrt{n!}} <0|a^n), \\ &<n|m> = \delta_{nm}, \quad \tilde{n}! \equiv \prod_{m=1}^{n} \frac{1}{m} \frac{\sin^2[m\pi \frac{k}{M}]}{\sin^2[\pi \frac{k}{M}]}. \end{aligned} \qquad (\mathbf{B}.14)$$

The multi-genon case may be easily generated from the above formulation. We leave this for future publication, giving only the basic results here,

$$\begin{aligned} &[a_i, b_j]_q = \delta_{ij}, \quad a_i|0> = 0, \quad i = 1, 2, \cdots, \\ &|n_1, n_2, \cdots> = \frac{1}{\sqrt{\prod_{i=1} \tilde{n}_i!}} b_1^{n_1} b_2^{n_2} \cdots |0>, \\ &<n_1, n_2, \cdots| = \frac{1}{\sqrt{\prod_{i=1} n_i!}} <0| \cdots a_2^{n_2} a_1^{n_1}, \end{aligned} \qquad (\mathbf{B}.15)$$

where $0 \le n_i \le M - 1$, q is given by Eq. ($\mathbf{B}$.8) and $\tilde{n}_i!$ is defined as in Eq. ($\mathbf{B}$.14). The Jacobi identity for the above q-quantization is simply,

$$[A, [B, C]_q] + [B, [C, A]_q] + [C, [A, B]_q] = 0. \qquad (\mathbf{B}.16)$$

From Eqs. (**B**.8) and (**B**.12), we obtain easily that $q = -1$ and c is a two dimensional identity for $M = 2$(namely we have FDS), while for $M = \infty$, $q = 1$ and c is an infinite dimensional identity(namely we have BES). Hence, the (one degree of freedom) genon quantization defined by Eqs. (**B**.1) and (**B**.2)(or (**B**.8)) provides a smooth interpolation between BES and FDS. We conclude this section by making some remarks.

First, note carefully that as Fivel pointed out in Ref. 13, no *bilinear* commutation relations among $\{a_i\}$ or $\{b_i\}$ should be further proposed, otherwise the commutation relations (**B**.15) will not be consistently satisfied except for $q = \pm 1$ (i.e.

BES and FDS). For example, if one assume that $s = b_1 b_2 - \alpha b_2 b_1 = 0$, then using the commutation relation in Eq. (**B**.15), one obtains $0 = a_1 s|0 >= (1 - q\alpha) b_2 |0>$, and $0 = a_2 s|0 >= (q - \alpha) b_1 |0>$, namely, $q = \alpha = \pm 1$. Also one may easily verify that $[a, a^\dagger]|n >= |n>$ only for $n = 0, 1, \cdots, M-2$, but not for $n = M-1$; namely, the commutator $[a, a^\dagger]$ in general is an operator. Therefore there are in general no bilinear commutation relations postulated between a and $a^\dagger$ or b and $b^\dagger$, other than Eqs. (**B**.1) and (**B**.3).

Secondly, it is easy to see that if one requires b to be the Hermitian conjugate of a (namely c is an identity I), then one allows only the following two cases from Eqs. (**B**.1) and (**B**.2): (i) $M = 2$, $q = -1$, i.e. the fermion case; (ii) $M = \infty$, $q = 1$, i.e. the boson case. The commutation relations (**B**.1) and the corresponding consistency conditions do not make room for any other cases, once one defines a number operator as a simple bilinear of a and $a^\dagger$. Thus non-Hermitian quantization (**B**.1) based on independent creation and annihilation operators is essential. It should be noted that the both (i) the "infinite statistics"($q = 0$ and $c = I$) of Greenberg[12] and (ii) the "interpolated statistics" ($-1 < q < 1$ and $c = I$) of Fivel[13] (which are also based on similar q-mutators) are due to the "infinite degree" realization of the number operator and the multi-particle Fock space is not a direct product of the single particle spaces.

Finally, we point out that above bilinear quantization (**B**.1) with its bilinear number operator N can easily be taken as the Hamiltonian of a local quantum field theory without requiring much modification of conventional field theories. It is unlike other theories based on q-mutators which require modification of locality, even for free particles, because of a nonlinear action. For example, in the $q = 0$ case, the number operator of the i-th particle is $N_i = a_i^\dagger a_i + \sum_j a_j^\dagger N_i a_j$; solving one particle problem thus requires knowledge of other particles in the system.

Genon Quantum Mechanics

The indistinguishability of identical particles, which deeply affects the physical nature of the system, is certainly not just the permutation invariance of (unphysical) particle labels. As is well-known, the interchange of particle labels in one spatial dimension is meaningless (and it is not well-defined in two spatial dimensions). Consequently, the symmetrization postulate on the identical particle wave functions, namely the wave functions being either symmetric or antisymmetric, is much stronger than what is implied by the indistinguishability of identical particles. This postulate thus needs to be modified. We adopt here the more general indistinguishability principle due to Greenberg and Messiah,[2,15]

All physical observables are permutation invariant.

We also propose here the corresponding constraints on identical particle wave functions Φ as follows:[11]

i. The individual particles are uniformly distributed over one particle states occurring in Φ,

ii. The states which differ only by a permutation are physically indistinguishable.

Before we continue the discussion, we first introduce a $n \times n$ generalized determinant defined as,[11]

$$
\begin{aligned}
&|a_{ij}|_G = \sum_{\text{all permutations}} q^{\delta(i_1,i_2,\cdots,i_n)} a_{1i_1}a_{2i_2}\cdots a_{ni_n},\\
&\delta(i_1,i_2,\cdots,i_n) = \text{minimal steps} : (i_1,i_2,\cdots,i_n) \longmapsto (1,2,\cdots,n).
\end{aligned}
\tag{B.17}
$$

The definition (**B**.17) yields,

$$
\begin{aligned}
|a_{ij}|_G &= \sum_{p=1}^{n} q^{p-1} a_{1p} A_{1p}^G,\\
|a_{ij}|_G &= |a_{ji}|_G,
\end{aligned}
\tag{B.18}
$$

where A_{1p}^G is the generalized minor with respect to a_{1p}. For example,

$$
\begin{aligned}
n=2,\quad & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_G = a_{11}a_{22} + q a_{12}a_{21} = (12) + q(21),\\
n=3,\quad & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_G \\
& = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_G + q a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}_G + q^2 a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}_G,\\
& = (123) + q(213) + q(132) + q^2(231) + q^2(312) + q^3(321),
\end{aligned}
\tag{B.19}
$$

where $(i_1 i_2 \cdots i_n) \equiv a_{1i_1}a_{2i_2}\cdots a_{ni_n}$. Note that most properties of the ordinary determinant are no longer valid here. For example, exchanging two columns in Eq. (**B**.17) or (**B**.18) makes a non-trivial change in the value of the determinant. Thus we must propose an natural ordering for subsequent discussion. Clearly this generalized determinant is a q analog of the Young patterns for the symmetric group S_n.

From Eqs. (**B**.17), (**B**.18) and (**B**.19) , the proposed general indistinguishability principle, and the quantization rule defined in Eqs. (**B**.1) and (**B**.2), we recognize that a free N identical genon system in which genons are distributed over the states of the configuration $\{n_1, n_2, \cdots, n_N\}$ may be described as,

$$\Phi_q = \frac{1}{\sqrt{\# \text{ of components}}} \begin{pmatrix} \Psi_q(1,2,\cdots,N) \\ P_{12}\Psi_q(1,2,\cdots,N) \\ \vdots \\ \text{independent permutations of } \Psi_q \\ \vdots \end{pmatrix}, \tag{B.20}$$

$$E_q = \sum_{i=1}^{N} \epsilon_{n_i},$$

where $\Psi_q(1,2,\cdots,N)$(the argument denotes the particle labels) is a normalized function, and is given by the $N \times N$ generalized determinant,[11]

$$\Psi_q(1,2,\cdots,N) = \text{normalized factor} \times |\phi_n(m)|_G, \tag{B.21}$$

with $\phi_n(m)$ being the mth single particle wave function with energy ϵ_n and q being given by Eq. (**B**.8). For all different ϕ_n, the normalized factor is $1/\sqrt{N!}$. We adopt, for this determinant, an ordering of states with increasing energy levels, $\cdots \leq \epsilon_{n_1} \leq \epsilon_{n_2} \leq \cdots \leq \epsilon_{n_N} \leq \cdots$. For convenience, we clearly display $|\phi_n(m)|_G$:

$$|\phi_n(m)|_G = \begin{vmatrix} \phi_{n_1}(1) & \phi_{n_2}(1) & \cdots & \phi_{n_N}(1) \\ \phi_{n_1}(2) & \phi_{n_2}(2) & \cdots & \phi_{n_N}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n_1}(N) & \phi_{n_2}(N) & \cdots & \phi_{n_N}(N) \end{vmatrix}_G. \tag{B.22}$$

It should be noted that the matrix notation in Eq. (**B**.20) does not represent a higher dimensional representation, but rather reflects the statement of indistinguishability proposed above: each component in the matrix contributes equally.

The probability density ρ and the physical observable F of identical particle system ($[F, P] = 0$, P is any permutation operator) are defined as

$$\rho = \Phi_q^\dagger \Phi_q, \quad < F >= \int dv \ \Phi_q^\dagger F \Phi_q, \tag{B.23}$$

where the integration is over the N particle configuration space. Clearly, ρ and $< F >$ are permutation invariant and independent of q. Using the basis defined in Eq. (**B**.20), one may easily find the matrix representations for all independent permutation operators. We give here only a few examples,

$$
\begin{aligned}
&N=2, \quad P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi_q(1,2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_q \\ P_{12}\Psi_q \end{pmatrix}, \\
&N=3, \quad P_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
&P_{23} = P_{12}P_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Phi_q(1,2,3) = \frac{1}{2} \begin{pmatrix} \Psi_q \\ P_{12}\Psi_q \\ P_{13}\Psi_q \\ P_{23}\Psi_q \end{pmatrix}.
\end{aligned}
\qquad (\mathbf{B}.24)
$$

Note using Eq. (**B**.24), we have

$$
P_{13}\Psi_q(1,2,3) = \frac{1}{2} \begin{pmatrix} P_{13}\Psi_q \\ P_{23}\Psi_q \\ \Psi_q \\ P_{12}\Psi_q \end{pmatrix}, \qquad (\mathbf{B}.25)
$$

which can be obtained by re-arranging the components of $\Psi_q(1,2,3)$ and is thus physically indistinguishable from $\Psi_q(1,2,3)$.

We now discuss genons with the statistical parameter M. From Eqs. (**B**.8), (**B**.17), (**B**.18), (**B**.19) and (**B**.20), we immediately obtain,

$$
\Phi_q = 0, \quad \text{if any } M \text{ or more } \{\phi_n\} \text{ are same.} \qquad (\mathbf{B}.26)
$$

Proof (one only needs proof for Ψ_q) is straightforward. For $j \geq M$ ($j \leq$ total particle in the system) particles occupying the state ϕ, we have

$$
\begin{aligned}
\Psi_q &= \begin{vmatrix} \cdots & \phi(1) & \cdots & \phi(1) & \cdots \\ \cdots & \phi(2) & \cdots & \phi(2) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \cdots & \phi(N) & \cdots & \phi(N) & \cdots \end{vmatrix}_G \\
&= \left(\sum_{s=0}^{M-1} q^s \right) \times \text{some function}, \quad \text{for } j \geq M, \\
&\equiv 0.
\end{aligned}
\qquad (\mathbf{B}.27)
$$

Also, for a complete orthogonal normalized basis of one particle states ϕ_n, i.e. $< \phi_n|\phi_m >= \delta_{nm}$, we generally have

$$\begin{aligned} &\int dv \Psi_q^* P \Psi_q = 0, \quad \text{unless } P\Psi_q = \pm\Psi_q, \\ &\int dv \Psi_q^* \Psi_q = 1, \end{aligned} \tag{B.28}$$

where P is any permutation operator.

According to a recent discussion,[64] Ψ_q may related to some irreducible representations (q analogy of Young tableaux in the classical cases) of the quantum group $SU(N, q)$.

Statistical Mechanics

From Eq. (**B**.26), we obtain the following Generalized Pauli Exclusion Principle:

Each individual quantum state may only

be occupied by at most M-1 particles.

This is very much like a parafermion with order M-1. But it is not the same, because for a parafermion with order M-1, the quantization is an M-linear relation. Also BES is not simply related to the high order limit $M \to \infty$ of M-para-FDS.[3]

For the energy level ϵ_l with degeneracy ω_l, the ways $\Omega(\omega_l, n_l, M)$ of n_l genons occupying these states is clearly, from the generalized exclusion principle, equal to the number of solutions of the following constrained equation,

$$\sum_{l=1}^{\omega_l} m_l = n_l, \quad (0 \leq m_l \leq M-1). \tag{B.29}$$

Thus, we obtain $\Omega(\omega_l, n_l, M)$ from the generating function,

$$\left(\frac{1-t^M}{1-t}\right)^{n_l} = \sum_{h=0}^{\infty} t^h \Omega(h, n_l, M). \tag{B.30}$$

Namely,

$$\Omega(\omega_l, n_l, M) = \sum_{m=0}^{n_l} (-1)^m C_{n_l}^m C_{\omega_l - Mm + n_l - 1}^{n_l - 1}, \tag{B.31}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

By using the method of either the Lagrange multipliers or the Grand Canonical ensemble, we easily obtain the thermodynamical distribution function for the ideal genon gas as,[11]

$$\tilde{n}_l = \frac{\omega_l}{\exp(\alpha + \beta\epsilon_l) - 1} - \frac{M\omega_l}{\exp\{M(\alpha + \beta\epsilon_l)\} - 1}, \tag{B.32}$$

which yields:

$$\begin{aligned} M = 2, \quad \tilde{n}_l^F &= \frac{\omega_l}{\exp(\alpha + \beta\epsilon_l) + 1}, \\ M = \infty, \quad \tilde{n}_l^B &= \frac{\omega_l}{\exp(\alpha + \beta\epsilon_l) - 1}, \end{aligned} \tag{B.33}$$

where α and β are the two Lagrange multipliers, respectively related to the conventional chemical potential and temperature. We leave the statistical discussion (a rough thermodynamical discussion may be found in Ref. 1) for future publications, in which the genon condensation (similar to Bose-Einstein condensation) and its application will be discussed in detail.

In conclusion, we have established a fully consistent theory of generalized quantum statistics. A smooth statistical interpolation between BES and FDS is explicitly demonstrated. Finally, we point out that this work may provide a natural way of understanding the "spin"-statistics connection within non-relativistic quantum mechanics, since there is unique relation between the quantization rule (**B**.1) and the statistics parameter M(or q) (**B**.8).

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