Symplectic isotopy for cuspidal curves

by

Sandra Francisco

Bachelor of Sciences, Universidade de Lisboa, July 1999
Master of Science, Instituto Superior Técnico de Lisboa, June 2001

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Author .....
Department of Mathematics
April 25, 2005

Certified by .....
Gang Tian
Simons Professor of Mathematics
Thesis Supervisor

Accepted by .....
Pavel Etingof
Chairman, Department Committee on Graduate Students

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Abstract

This work has three purposes. The first one is to prove unobstructedness of deformation of pseudoholomorphic curves with cusps and tacnodes. We show that if the first Chern class of a 4-dimensional symplectic manifold is sufficiently positive then the deformation is unobstructed. We prove this result when the curves have cusps and nodes, not in a prescribed position. We also prove a similar result when the curves have cusps and tacnodes in a prescribed position with a prescribed tangency and in addition nodes, not in a prescribed position.

The second part of this work deals with the local symplectic isotopy problem for cuspidal curves. Let B be the unit ball in $\mathbb{R}^4$ with the standard symplectic form ω_{st}. Let J_0 be a ω_{st}-tame almost complex structure.

Let $C_0 \subset B$ be a connected J_0-holomorphic curve in B with a isolated singularity at $0 \in B$ and without multiple components. Assume in addition that the boundary ∂C_0 is smoothly embedded.

We prove that any two connected, reduced pseudoholomorphic curves in B, with the same number of irreducible components, the same number of nodal points and at most one ordinary cusp point, both sufficiently close to C_0, are symplectic isotopic to each other.

The third part of this work deals with the global symplectic isotopy problem. As an application of unobstructedness of deformation, we show that any irreducible rational pseudo-holomorphic curve in $\mathbb{CP}^2$ of degree d, with only nodes and m ordinary cusps as its singularities, is symplectic isotopic to a holomorphic curve as long as $d > m$.

Thesis Supervisor: Gang Tian
Title: Simons Professor of Mathematics

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Contents

1 Introduction. 9

2 Basic facts and definitions. 11

2.1 The Teichmüller space. 11

2.2 Moduli space of pseudoholomorphic curves. 12

2.3 The Gromov operator. 13

2.4 Transversality and the vanishing theorem. 14

2.5 Moduli space of pseudoholomorphic singular curves. 15

3 Unobstructedness of deformation of cuspidal curves. 17

3.1 Identification of the obstruction. 17

3.2 The normal sheaf with cusps and the induced Gromov operator. . . . 19

3.3 Surjectivity of the induced Gromov operator. 20

3.4 The bundle N^c. 20

3.5 The quotient operator. 22

3.6 Surjectivity of the quotient operator. 23

3.7 Unobstructedness. 24

4 Unobstructedness of deformation of cuspidal curves with fixed position of cusps. 27

4.1 Moduli space of pseudoholomorphic curves with fixed position of cusps. 27

4.2 Unobstructedness. 28

5 Unobstructedness of deformation of curves with fixed position of cusps and tacnodes and fixed tangencies. **31**

5.1 Moduli space of pseudoholomorphic curves with fixed position of cusps and tacnodes and fixed tangencies. 31

5.2 Unobstructedness. 32

6 The cycle topology for pseudoholomorphic curves. **35**

7 Local symplectic isotopy problem for cuspidal curves. **37**

7.1 Local symplectic isotopy problem. 37

7.2 Auxiliary results. 39

7.3 Proof of local symplectic isotopy for cuspidal curves. 40

7.4 Proof of Theorem 11. 41

7.4.1 Extension of the curves to $\mathbb{CP}^2$. 41

7.4.2 Constraints. 43

7.4.3 Deformation of $\widetilde{C}_n$. 44

7.4.4 First limit case. 45

7.4.5 Existence of a partial smoothing. 47

7.4.6 Consequences of the existence of a partial smoothing. 48

7.4.7 Second limit case. 49

8 Global isotopy **51**

Bibliography **53**

Chapter 1

Introduction.

One of the aims of this work is to know under which conditions we have an unobstructed deformation theory for singular pseudoholomorphic curves inside a 4-dimensional symplectic manifold (X, ω).

The results known in this direction are the following. Hofer, Lizan and Sikorav in [Ho-Li-Si] proved unobstructedness in the smooth case. Barraud in [Ba1] proved unobstructedness in the singular case but when the position of the cusps, tacnodes and the complex structure on the Riemann surface are all fixed. In this paper we improve Barraud's result in two directions. First we always allow the almost complex structure on the Riemann surface to vary, leading to an improved result. Second we allow the position of the nodes and cusps to move.

The strategy of the proof is the following. In chapter 2 we recall some basic facts and introduce some notations. In chapter 3 we identify the obstruction to the deformation theory along pseudoholomorphic curves in the homology class $[C]$. We show that if $c_1(X) \cdot [C] > 3m$ the deformation is unobstructed along curves with m ordinary cusps and r nodal points. Here we do not prescribe the position of the cusps or of the nodes. In chapter 4 we do the same but in this case we prescribe the position of the ordinary cusps allowing the nodes to move. We obtain in this case as well that the deformation is unobstructed as long as $c_1(X) \cdot [C] > 3m$. Finally, we also prove that deformation is unobstructed when in addition we ask for the curves to pass through n marked points if $c_1(X) \cdot [C] > 3m + n$, in both cases when

we prescribe the position of the cusps and when we don't. In chapter 5 we prove unobstructedness for the case when we have m cusps points in prescribed positions and prescribed tangencies and in addition ν tacnodes also in prescribed positions and with prescribed tangencies. We show that the deformation is unobstructed as long as $c_1(X) \cdot [C] > 4m + 4\nu$.

In chapter 6 we recall standard facts about the cycle topology.

In chapter 7 we solve the local symplectic isotopy problem for the case of curves with one cusp and nodes, nearby a fixed curve with one isolated singularity. The result known so far are due to Shevchishin, in [Sh1] and [Sh3], where he solves the local symplectic isotopy problem for nodal curves.

We follow a strategy similar to that of Shevchishin in [Sh3], which is the following. First we show that we can reduce to the case when the fixed curve is holomorphic. Then we show that we can deform the nearby curves with one cusp and nodes to holomorphic curves, also with one cusp and nodes. We then use known results for the holomorphic case to conclude that the curves must be symplectic isotopic.

Finally in chapter 8, we discuss the global symplectic isotopy problem. Previous known results are due to Gromov in [Gr] for curves in $\mathbb{CP}^2$ of degree 1 and 2 and for degree 3 curves by Sikorav in [Sk3]. They show that any two such curves in $\mathbb{CP}^2$ of the same degree are symplectically isotopic. In [Sh3], Shevchishin extends these results for smooth curves of degree less or equal than 6. Siebert and Tian in [Si-Ti] show that any smooth curve of degree less or equal than 17 is symplectic isotopic to a holomorphic curve. All this cases deal only with smooth curves. For nodal curves, the known results are due to Shevchishin in [Sh1], for low genus curves in $\mathbb{CP}^2$ or $\mathbb{CP}^2$ blown up in one point or $S^2 \times S^2$. The natural question to ask is what happen when we have worst singularities, namely ordinary cusps.

We use the unobstructedness result to show that any irreducible rational pseudo-holomorphic curve of degree d in $\mathbb{CP}^2$, with only nodes and m ordinary cusps as its singularities is symplectic isotopic to a holomorphic curve, as long as $d > m$.

Chapter 2

Basic facts and definitions.

Let (X,ω) be a symplectic 4-manifold. That is, X is a 4-dimensional manifold equipped with a closed nondegenerated two-form ω, i.e., $d\omega = 0$ and $\omega^2 \neq 0$. From now on we fix (X,ω).

We denote by $\mathcal{J}_X$ the space of ω-tamed almost complex structures on X. From now on, all the almost complex structures on X that we will consider will be of class $C^{\ell,\alpha}$ with ℓ sufficiently large. If $J \in \mathcal{J}$ then

$$T_J\mathcal{J} = \{I \in C^{\ell,\alpha}(X, \mathrm{End}(TX)) : JI + IJ = 0\} = C^{\ell,\alpha}(X, \Lambda^{0,1}X \otimes TX).$$

2.1 The Teichmüller space.

Let S be a connected oriented 2-dimensional manifold of genus g. As before, denote by $\mathcal{J}_S$ the space of (necessarily integrable) complex structures on S of class $C^{\ell,\alpha}$.

Let $\mathbb{T}_g$ denote the Teichmüller space. This is the space of marked complex structures on S. It is well known that this is a complex manifold of dimension

$$\dim_{\mathbb{C}} \mathbb{T}_g = \begin{cases} 0 & \text{if } g = 0; \\ 1 & \text{if } g = 1; \\ 3g-3 & \text{if } g \geq 2. \end{cases}$$

Lemma 1. *The following holds:*

i) There is an immersion $\mathbb{T}_g \hookrightarrow \mathcal{J}_S$;

ii) The tangent space to $\mathbb{T}_g$ *at* $\tau \in \mathbb{T}_g$ *is canonically isomorphic to* $H^1(S, TS)$, *where* S *is equipped with the structure* $J_S(\tau)$.

Remark 1. The product $S \times \mathbb{T}_g$ admits a (non-unique) complex structure $J_{S\times\mathbb{T}_g}$ such that the natural projection $\pi_{\mathbb{T}_g} : S \times \mathbb{T}_g \longrightarrow \mathbb{T}_g$ is holomorphic. Therefore for any $\tau \in \mathbb{T}_g$, the identification $S \simeq S \times \{\tau\}$ induces the complex structure $J_S(\tau) = J_{S\times\mathbb{T}_g}|_{S\times\{\tau\}}$ on S. Moreover, for any complex structure J_S on S there is a unique $\tau \in \mathbb{T}_g$ and a diffeomorphism f of S homotopic to the identity such that $J_S = f^* J_S(\tau)$.

Fix such a structure $J_{S\times\mathbb{T}_g}$. Then we obtain an immersion $\mathbb{T}_g \hookrightarrow \mathcal{J}_S$ given by $\tau \to J_S(\tau)$.

Using this, we identify $\mathbb{T}_g$ with its image in $\mathcal{J}_S$.

Let G be the group of holomorphic automorphisms of $S \times \mathbb{T}_g$ preserving the projection onto $\mathbb{T}_g$. Then

$$G = \begin{cases} \mathrm{PGL}(2,\mathbb{C}) & \text{for } g = 0, \\ \mathrm{Sp}(2,\mathbb{Z}) \ltimes \mathrm{T}^2 & \text{for } g = 1, \\ \text{discrete} & \text{for } g \geq 2; \end{cases}$$

Moreover, $T_e G \simeq H^0(S, TS)$ canonically.

2.2 Moduli space of pseudoholomorphic curves.

Let $J_S \in \mathcal{J}_S$, $J \in \mathcal{J}_X$ and $u : S \longrightarrow X$. We say that u is a (J_S, J)-*holomorphic map* (or simply pseudoholomorphic) if

$$J \circ du = du \circ J_S.$$

Fix a homology class $[C] \in \mathrm{H}_2(X, \mathbb{Z})$ and define

$$\mathcal{C} := \{u \in L^{k,p}(S, X) : u(S) \in [C]\}.$$

Here k, p are fixed such that we have an embedding $L^{k,p} \hookrightarrow C^2$. Define

$$\widehat{\mathcal{M}}_J = \left\{(u, J_S) \in \mathcal{C} \times \mathbb{T}_g | \, du + J \circ du \circ J_S = 0, u \text{ is generically injective}\right\}.$$

There is a natural action of G on $\widehat{\mathcal{M}}_J$ given by

$$(u, J_S) \cdot g := (u \circ g, g^*(J_S)).$$

Define the *total moduli space of parameterized pseudo-holomorphic curves* as

$$\mathcal{M}_J := \widehat{\mathcal{M}}_J / G.$$

2.3 The Gromov operator.

Fix a symmetric connection ∇ on TX, i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$.

Let $u \in \mathcal{C}$. Denote by $E := E_u$ the pull-back tangent bundle u^*TM. By construction, E has a natural complex structure given by u^*J.

Differentiating the equation $du + J \circ du \circ J_S = 0$ we obtain that a vector $(v, \dot{J}_S)$ is tangent to $\widehat{\mathcal{M}}_J$ at the point (u, J_S) if it satisfies the following equation

$$\nabla v + J \circ \nabla v \circ J_S + (\nabla_v J) \circ (du \circ J_S) + J \circ du \circ \dot{J}_S = 0.$$

Define complex Banach bundles $\mathcal{E}$ and $\mathcal{E}'$ over $\mathcal{C} \times \mathbb{T}_g$ by

$$\mathcal{E}_{(u,J_S)} := L^{k,p}(S, E)$$

and

$$\mathcal{E}'_{(u,J_S)} = L^{k-1,p}(S, \Lambda^{0,1} S \otimes E) := L^{k-1,p}_{(0,1)}(S, E).$$

These bundles are $C^{\ell-1,\alpha}$ smooth. Let $D := D_u$ be the *Gromov operator*, i.e, the $C^{\ell-1}$ smooth $\mathbb{R}$-linear homomorphism

$$D_u : L^{k,p}(S, E) \longrightarrow L^{k-1,p}(S, \Lambda^{0,1}S \otimes E),$$

given by the formula

$$D_u(v) = \nabla v + J \circ \nabla v \circ J_S + (\nabla_v J) \circ (du \circ J_S).$$

We can decompose the Gromov operator D_u into its J-linear and J-antilinear parts. Define

$$\bar{\partial}_u v = \frac{1}{2}[D_u v - J \circ D_u(Jv)],$$
$$R_u(v, \cdot) = \frac{1}{2}[D_u v + J \circ D_u(Jv)].$$

Then

$$D_u = \bar{\partial}_u + R_u(\cdot, \cdot),$$

where $\bar{\partial}_u$ is J-linear and $R_u(\cdot, \cdot)$ is J-antilinear.

We call $\bar{\partial}_u$ the $\bar{\partial}$-operator of the J-holomorphic curve u. One can show that R_u is a continuous J-antilinear operator from E_u to $\Lambda^{0,1}S \otimes E_u$ of order 0, satisfying $R_u \circ du = 0$. In fact, $R_u(v, \xi) = 2N(v, du(\xi))$ where N is the torsion vector of J (the Nijenhuis torsion tensor).

It is an easy exercise to show that the Gromov operator D_u does not depend on the choice of the connection ∇, if u is J-holomorphic.

2.4 Transversality and the vanishing theorem.

Let τ be the section of $\mathcal{E}'$ defined by $\tau(u, J_S, J) = du + J \circ du \circ J_S$. Then $\widehat{\mathcal{M}}_J = \tau^{-1}(0)$.

One can show that τ is transverse to the zero section for generic J (see e.g.

[McD-Sa]) and therefore $\widehat{\mathcal{M}}_J$ is a manifold with tangent space at (u, J_S)

$$T_{(u,J_S)}\widehat{\mathcal{M}}_J = \{(v, \dot{J}_S) : D_u v + J \circ du \circ \dot{J}_S = 0\}.$$

Transversality will automatically hold for all J if $c_1(M) \cdot [u(S)] > 0$ as a consequence of the next Theorem. Indeed D induces an operator

$$D^N : L^{k,p}(S, N) \longrightarrow L^{k-1,p}(S, \Lambda^{0,1}N)$$

of the form stated in the theorem, where N is the normal bundle to the curve u. Moreover, one can show that the surjectivity of the linearization of τ is equivalent to the surjectivity of D^N. The result now follows from the adjunction formula.

Theorem 2 (Vanishing theorem). *[Ho-Li-Si] Let S be a closed Riemann surface of genus g and L a holomorphic line bundle over S. Assume L is equipped with a differential operator D of the form $D = \bar{\partial} + R$ with $R \in L^\infty(S, \mathrm{Hom}_{\mathbf{R}}(L, \Lambda^{0,1}S \otimes L))$. If $c_1(L) < 0$ then D is injective. If $c_1(L) > 2g - 2$ then D is surjective.*

2.5 Moduli space of pseudoholomorphic singular curves.

Let $m \in \mathbb{N}$. Consider the generalized diagonal

$$\Delta^m := \{(z_1 \ldots, z_m) \in S^m | \exists i \neq j, z_i = z_j\}.$$

Define

$$\widehat{\mathcal{M}}_{J,\mathrm{cusp}} = \left\{(u, J_S, z) \in \widehat{\mathcal{M}}_J \times (S^m \setminus \Delta^m) | du(z_i) = 0 \, \forall \, i = 1, \ldots, m\right\},$$

the space of pseudo-holomorphic curves with m cusp points. Then

$$T_{(u,J_S,z)}\widehat{\mathcal{M}}_{J,\mathrm{cusp}} = \left\{(v, \dot{J}_S, \dot{z}) \in T_{(u,J_S,J)}\widehat{\mathcal{M}}_J \times T_z S^m | \nabla v(z_i) + \nabla_{\dot{z}_i} du = 0 \, \forall \, i\right\}.$$

There is a natural action of G on $\widehat{\mathcal{M}}_{J,\text{cusp}}$, defined by

$$(u, J_S, z) \cdot g = (u \circ g, g^*(J_S), g^{-1}z).$$

We denote by $\mathcal{M}_{J,\text{cusp}} = \widehat{\mathcal{M}}_{J,\text{cusp}}/G$ the corresponding quotient space.

The projection

$$\pi : \frac{\left\{(u, J_S, z, J) \in \mathcal{C} \times \mathbb{T}_g \times (S^m \setminus \Delta^m) \times \mathcal{J} : (u, J_S, z) \in \widehat{\mathcal{M}}_{J,\text{cusp}}\right\}}{G} \longrightarrow \mathcal{J}$$

is Fredholm of index

$$\text{ind}_{\mathbf{R}} = 2(c_1(E) + g - 1 - m).$$

Let $\sigma(u, J_S, z) = du + J \circ du \circ J_S$ so that $\widehat{\mathcal{M}}_{J,\text{cusp}} = \sigma^{-1}(0)$. Then at all points where the linearization of σ is surjective, $\mathcal{M}_{J,\text{cusp}}$ is a smooth manifold of dimension

$$\dim_{\mathbf{R}} \mathcal{M}_{J,\text{cusp}} = 2(c_1(E) + g - 1 - m).$$

Chapter 3

Unobstructedness of deformation of cuspidal curves.

3.1 Identification of the obstruction.

Our goal is to study under which conditions, given a fixed $J \in \mathcal{J}_X$, we have an unobstructed theory of deformation of J-holomorphic curves with m ordinary cusp points, r nodal points and no other singularities. Any such curve is the image of a map $u : S \longrightarrow X$ which is immersed outside m distinct points $z_1, \ldots, z_m$ where $du(z_i) = 0$. Deformations which preserve the existence of non-immersed points will automatically have ordinary cusps and nodes as their only singularities. This implies that in the discussion that follows we can concentrate exclusively on the cuspidal points and forget about the nodal points.

Let $u : (S, J_S) \longrightarrow (X, J)$ be a pseudoholomorphic map, with $J_S \in \mathbb{T}_g$. Consider a n-tuple $(z_1, \ldots, z_m) \in S^m \setminus \Delta^m$ with $u(z_i)$ an ordinary cusp. What we want to know is when can we deform u to a nearby pseudoholomorphic curve $u' : (S, J'_S) \longrightarrow (X, J)$ and $(z_1, \ldots, z_m)$ to a nearby n-tuple $(z'_1, \ldots, z'_m) \in S^m \setminus \Delta^m$ with $u'(z'_i)$ an ordinary cusp.

Let

$$\Lambda^{0,1}E_{-z} := \{w \in L^{k-1,p}(S, \Lambda^{0,1}E) | w(z_i) = 0\}.$$

This is a bundle over

$$\{(u, J_S, z) \in C \times \mathbb{T}_g \times (S^m \setminus \Delta^m) | du(z_i) = 0\}.$$

Let σ_c be the section of $\Lambda^{0,1}E_{-z}$ defined by $\sigma_c(u, J_S, z) = du + J \circ du \circ J_S$. By definition, $\mathcal{M}_{J,\text{cusp}} = {\sigma_c}^{-1}(0)/G$. As before we want to study the linearization of the section σ_c and show it is surjective.

Denote by

$$\mathcal{N} = u^*(TM)/du(TS)$$

the quotient sheaf. Then

$$\mathcal{N} = N \oplus \mathcal{N}^{\text{sing}},$$

where N is the normal bundle and $\mathcal{N}^{\text{sing}} = \oplus_{z \in S} \mathbb{C}^{\text{ord}_z du}$ is a discrete sheaf, whose support is the set of critical points of u.

Let

$$E^c := \left\{(v, \dot{z}) \in L^{k,p}(S, E) \times T_z S^m | \nabla v(z_i) + \nabla_{\dot{z}_i} du = 0, \ \forall i\right\}.$$

Notice that since $du(z_i) = 0$, the condition $\nabla v(z_i) + \nabla_{\dot{z}_i} du = 0$ does not depend on the choice of connections.

Define

$$\phi_c : E^c \times T_{J_S}\mathbb{T}_g \longrightarrow \Lambda^{0,1}E_{-z}$$

by

$$\phi_c(v, \dot{z}, \dot{J}_S) = Dv + J \circ du \circ \dot{J}_S,$$

where D is the Gromov operator defined previously. Our goal is to show that ϕ_c is surjective. This is the obstruction to the deformation.

The map ϕ_c is well-defined, i.e, $Dv(z_i) + J \circ du \circ \dot{J}_S(z_i) = 0$. In fact,

$$\begin{aligned} Dv(z_i) + J \circ du \circ \dot{J}_S(z_i) &= \nabla v(z_i) + J \circ \nabla v(z_i) \circ J_S \\ &= \nabla v(z_i) + J \circ \nabla v(z_i) \circ J_S + \nabla_{\dot{z}_i} du + J \circ \nabla_{\dot{z}_i} du \circ J_S = 0 \end{aligned}$$

since

$$\nabla_{\dot{z}_i} du + J \circ \nabla_{\dot{z}_i} du \circ J_S = \nabla_{\dot{z}_i}(du + J \circ du \circ J_S) = 0.$$

3.2 The normal sheaf with cusps and the induced Gromov operator.

Let $\widetilde{du}$ denote the map

$$\widetilde{du} : L^{k,p}(S, TS) \longrightarrow E^{\mathrm{c}}$$

defined by

$$\widetilde{du}(v) := (du(v), -v(z_1), \ldots, -v(z_m)).$$

Denote by

$$\mathcal{N}^{\mathrm{c}} := \frac{E^{\mathrm{c}}}{\widetilde{du}(L^{k,p}(S, TS))}$$

the quotient. It follows immediately that

$$\mathcal{N}^{\mathrm{c}} \simeq \frac{\{v \in L^{k,p}(S, E) | \nabla v(z_i) = 0\}}{du(L^{k,p}(S, TS_{-z}))}$$

where TS_{-z} denotes the sheaf whose sections are those $w \in \Gamma(TS)$ which vanish at z_i.

Denote by p both the projections

$$p : E^{\mathrm{c}} \longrightarrow \mathcal{N}^{\mathrm{c}}$$

and

$$p : \Lambda^{0,1} E_{-z} \longrightarrow \Lambda^{0,1} E_{-z} / du(L^{k,p}(S, \Lambda^{0,1} S)).$$

There is an induced operator

$$D^{\mathcal{N}^{\mathrm{c}}} : \mathcal{N}^{\mathrm{c}} \longrightarrow \Lambda^{0,1} E_{-z} / du(L^{k,p}(S, \Lambda^{0,1} S))$$

defined by

$$p \circ Dv = D^{\mathcal{N}^c} \circ p.$$

3.3 Surjectivity of the induced Gromov operator.

Our goal in this section is to show that surjectivity of $D^{\mathcal{N}^c}$ implies surjectivity of ϕ_c.

Lemma 2. *Assume $D^{\mathcal{N}^c}$ is surjective. Then ϕ_c is surjective.*

Proof . Let $w \in \Lambda^{0,1}E_{-z}$. Then there exists $(v, \dot{z}) \in E^c$ such that

$$D^{\mathcal{N}^c}(p(v, \dot{z})) = p(w).$$

This means that $p(Dv) = p(w)$.

If $g = 0$ since $\bar{\partial}$ is surjective on TS then $Dv - w = -J \circ du(\bar{\partial}\xi)$ for some vector field ξ on S. Since $D \circ du = du \circ \bar{\partial}$ then $D(v + du(J_S\xi)) = w$. Therefore $\phi_c((v, \dot{z}, 0) + \widetilde{du}(J_S\xi)) = w$.

If $g \geq 1$ it is no longer true that $\bar{\partial}$ is surjective. But since $T_{J_S}\mathbb{T}_g \simeq H^1(S, TS)$ then $Dv - w = -J \circ du(\dot{J}_S) - J \circ du(\bar{\partial}\xi)$ for some $\dot{J}_S \in T_{J_S}\mathbb{T}_g$ and some vector field ξ on S. We then have $\phi_c((v, \dot{z}, \dot{J}_S) + \widetilde{du}(J_S\xi)) = w$. □

This shows it is enough to prove that $D^{\mathcal{N}^c}$ is surjective.

3.4 The bundle N^c.

Define

$$N^c = N \otimes \mathcal{O}(-2z).$$

Notice that N^c is a holomorphic bundle over S. Moreover, we can identify its local holomorphic sections with those of N which vanish up to order 2 at z_i for all $i = 1, \ldots, m$.

There is an induced operator $D^{N^c} : \Gamma(N^c) \longrightarrow \Gamma(S, \Lambda^{0,1}S \otimes N^c)$, as it follows from next lemma.

Lemma 3. *D induces the operator*

$$D^{N^c} : L^{k,p}(S, N^c) \longrightarrow L^{k-1,p}(S, \Lambda^{0,1}S \otimes N^c)$$

which is of the form $D^{N^c} = \bar{\partial}^{N^c} + R^{N^c}$, where $\bar{\partial}^{N^c}$ the Cauchy-Riemann operator corresponding to the natural holomorphic structure in N^c and

$$R^{N^c} \in L^\infty(\overline{\mathrm{Hom}}_{\mathbb{C}}(N^c, \Lambda^{0,1}S \otimes N^c)).$$

Proof. Fix local holomorphic coordinates ζ_i on S, each centered at the corresponding marked point z_i. Observe that $v \in L^{k,p}(S, N)$ with $j^{(1)}v(z_i) = 0$ belongs to $L^{k,p}(S, N^c)$ if and only if locally near each z_i it has the form $v(\zeta_i) = \zeta_i^2 w_i(\zeta_i)$ for some uniquely defined $L^{k,p}$-section w_i of N. But then

$$\begin{aligned} D_{u,J}v &= D_{u,J}(\zeta_i^2 w_i) = \nabla(\zeta_i^2 w_i) + J \circ \nabla(\zeta_i^2 w_i) \circ J_S + \nabla_{\zeta_i^2 w_i} J \circ du \circ J_S \\ &= d(\zeta_i^2) \otimes w_i + \zeta_i^2 \nabla w_i + d(\zeta_i^2) \circ J_S \cdot Jw_i + \zeta_i^2 J \circ \nabla w_i \circ J_S + \zeta_i^2 \nabla_{w_i} J \circ du \circ J_S \\ &= d(\zeta_i^2) \otimes w_i + id(\zeta_i^2) \otimes Jw_i + \zeta_i^2 D_{u,J} w_i = \zeta_i^2 D_{u,J} w_i. \end{aligned}$$

Hence D_u restricted to $L^{k,p}(S, N^c)$ takes values in $L^{k-1,p}(S, \Lambda^{0,1}S \otimes N^c)$. This yields the operator D^{N^c}.

Moreover D^{N^c} is of order 1 and has the Cauchy-Riemann symbol. Hence it is of the form $D^{N^c} = \bar{\partial}^{N^c} + R^{N^c}$, where $\bar{\partial}^{N^c}$ is the Cauchy-Riemann operator corresponding to the holomorphic structure on N^c and R^{N^c} is the $\mathbb{C}$-antilinear part of D^{N^c}.

Let v_i be a local holomorphic frame of N in a neighborhood of z_i. Then $\zeta_i^2 v_i$ is a local frame of N^c. From the $\mathbb{C}$-antilinearity of R_u we obtain:

$$R_u(\zeta_i^2 v_i(\zeta_i)) = \overline{\zeta_i}^2 R v_i = \frac{\overline{\zeta_i^2}}{\zeta_i^2} R(\zeta_i^2 v_i).$$

Since R_u is a C^0 bundle homomorphism, then R^{N^c} is also C^0 outside the marked points z_i and has singularities of the form $\frac{\overline{\zeta_i^2}}{\zeta_i^2} R_u$ at z_i. Therefore R^{N^c} is of type L^∞. □

Lemma 4. *If $c_1(E) > 3m$ then D^{N^c} is surjective.*

Proof . By the vanishing theorem, Theorem 2, D^{N^c} is onto if $c_1(N^c) > 2g-2$. Since $c_1(N^c) = c_1(N) - 2m$ this is the same as $c_1(N) > 2m + 2g - 2$. Then

$$c_1(E) = c_1(TS) + c_1(N) + m = 2 - 2g + c_1(N) + m.$$

The condition $c_1(N) > 2m + 2g - 2$ is equivalent to $c_1(E) > 3m$. □

3.5 The quotient operator.

Notice that $\Gamma(N^c)$ embeds into $\mathcal{N}^c$ via the map of multiplication by the square of the holomorphic section s of $\mathcal{O}(z)$ which vanishes at z. There is an embedding defined in a similar way from $\Lambda^{0,1}N^c$ to $\Lambda^{0,1}E_{-z}/du(L^{k,p}(S, \Lambda^{0,1}S))$.

Define $Q = \mathcal{N}^c/\Gamma(N^c)$ and $Q' = (\Lambda^{0,1}E_{-z}/du(L^{k,p}(S, \Lambda^{0,1}S)))/\Lambda^{0,1}\Gamma(N^c)$.

We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(N^c) & \xrightarrow{\iota} & \mathcal{N}^c & \xrightarrow{j} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D^{N^c}} & & \downarrow{\scriptstyle D^{\mathcal{N}^c}} & & \downarrow{\scriptstyle D^q} & & \\ 0 & \longrightarrow & \Lambda^{0,1}N^c & \xrightarrow{\iota'} & \Lambda^{0,1}E_{-z}/du(L^{k,p}(S, \Lambda^{0,1}S)) & \xrightarrow{j'} & Q' & \longrightarrow & 0 \end{array}$$

This defines a quotient operator $D^q : Q \longrightarrow Q'$.

Lemma 5. *If $c_1(E) > 3m$ and D^q is surjective then $D^{\mathcal{N}^c}$ is surjective.*

Proof . Let $w \in \Lambda^{0,1}E_{-z}/du(L^{k,p}(S, \Lambda^{0,1}S))$. Then $j'(w) = D^q(j(v))$ for some $v \in \mathcal{N}^c$. Therefore $j'(w) = j'(D^{\mathcal{N}^c}(v))$ and hence $j'(w - D^{\mathcal{N}^c}v) = 0$. This implies that there exists $\varphi \in \Lambda^{0,1}(N^c)$ such that $w - D^{\mathcal{N}^c}v = \iota'(\varphi)$. Since D^{N^c} is surjective, there exists $\widetilde{v} \in \Gamma(N^c)$ such that $D^{N^c}\widetilde{v} = \varphi$. Therefore $w - D^{\mathcal{N}^c}v = \iota' D^{N^c}\widetilde{v} = D^{\mathcal{N}^c}\iota(\widetilde{v})$ and hence $w = D(v + \iota(\widetilde{v}))$. This proves our claim. □

3.6 Surjectivity of the quotient operator.

In this section we will show that D^{q} is surjective.

Lemma 6. *The operator D^q is surjective.*

Proof . We claim that both Q and Q' are finite dimensional vector spaces and we will present a basis for them. Therefore it is enough to show that every element in the basis of Q' is in the image of D^{q}.

The first observation is that both Q and Q' are direct sums of spaces supported at each z_i.

There is a direct sum decomposition $\mathcal{N}^{\mathrm{c}} = \mathcal{N}^{\mathrm{c}N} \oplus \mathcal{N}^{\mathrm{sing}}$ with $\mathcal{N}^{\mathrm{sing}}$ as before. Here $\mathcal{N}^{\mathrm{c}N}$ is the space of all sections of N which arise as the normal component of a section of $\mathcal{N}^{\mathrm{c}}$. Let v_i^N and v_i^T be two local sections of E near z_i, respectively normal and tangent to the curve $u(S)$, which do not vanish at z_i and satisfy $\nabla v_i^N(z_i) = 0$ and $\nabla v_i^T(z_i) = 0$.

Let $w_i^N = d\bar{\zeta}_i \otimes v_i^N$ and $w_i^T = d\bar{\zeta}_i \otimes v_i^T$, where ζ_i is a holomorphic coordinate centered at z_i.

We claim that $\zeta_i^2 v_i^N$ is a local frame of N^{c} in a neighborhood of z_i. Let $w \in N^{\mathrm{c}}$, then $w = a v_i^N$ for some function a. Since $w(z_i) = 0$ holomorphically then $a = \zeta_i b$ for some function b. Then $\nabla w(z_i) = b1 \otimes v_i^N = 0$ holomorphically, implying that $b = \zeta_i c$ for some function c. This shows that $\zeta_i^2 v_i^N$ is a local frame of N^{c}.

Then

$$Q = \oplus_i \operatorname{span}_{\mathbb{C}} \left\{ [f v_i^T], [f v_i^N], f = 1 \text{ or } f = \zeta_i \bar{\zeta}_i^{\nu} \text{ or } f = \bar{\zeta}_i^{\nu+1} \text{ for some } 1 \leq \nu \leq k \right\},$$

$$Q' = \oplus_i \operatorname{span}_{\mathbb{C}} \left\{ [f w_i^T], [f w_i^N], \ f = \zeta_i \bar{\zeta}_i^{\nu} \text{ or } f = \bar{\zeta}_i^{\nu+1} \text{ for some } 0 \leq \nu \leq k \right\}.$$

We will show surjectivity of D^{q} by decreasing induction on ν. This is possible since we know that ν is bounded by k. Assume that for any function f which vanishes to order at least $\nu + 2$ at z_i, $[f w_i^T]$ and $[f w_i^N]$ are in the image of D^{q}.

Let $g = \zeta_i \bar{\zeta}_i^{\nu}$ and consider $[gw_i^N] \in Q'$. Then

$$D^{N^c}\left(\frac{1}{\nu+1}\zeta_i\bar{\zeta}_i^{\nu+1}v_i^N\right) = g \otimes w_i^N + \text{terms which vanish to order } \nu+2 \text{ at } z_i$$

Applying induction, we conclude that $[gw_i^N]$ is in the image of D^q. Similarly for $[gw_i^T]$, $[hw_i^N]$ and $[hw_i^T]$ where $h = \bar{\zeta}_i^{\nu+1}$. □

3.7 Unobstructedness.

We have finally all the tools to show that deformation is unobstructed.

Theorem 3. *Assume $c_1(E) > 3m$. Then ϕ_c is surjective implying that the deformation theory is unobstructed.*

Proof . The result follows immediately from Lemmas 2, 4, 5 and 6. □

Proposition 1. *The deformation of pseudoholomorphic curves in a fixed homology class $[C]$ inside of X, with m cuspidal points, r nodal points and passing through n fixed prescribed points is unobstructed as long as*

$$c_1(X) \cdot [C] > 3m + n.$$

Proof . The proof is very similar to the proof of Theorem 3, we will just point out the differences. Fix n distinct points $x_1, \ldots, x_n \in X$. We want to study deformation of tuples $(u, J_S, z, w) \in \mathcal{M}_{J,\text{cusps}} \times (S^n \setminus \Delta^n)$ such that $u(w_i) = x_i$.

Let E_w^c be the set of $(v, \dot{z}, \dot{w}) \in E^c \times T_w S^n$ such that $v(w_i) + du(\dot{w}_i) = 0$.

Define $\widetilde{du} : L^{k,p}(S, TS) \longrightarrow E_w^c$ by $\widetilde{du}(v) = (du(v), -v(z), -v(w))$ where $v(z) = (v(z_1), \ldots, v(z_m))$ and similarly for $v(w)$.

Quotient E_w^c by $\widetilde{du}(L^{k,p}(S, TS))$, obtaining $\mathcal{N}_w^c$. As before

$$\mathcal{N}_w^c \simeq \frac{\{v \in L^{k,p}(S, E) | \nabla v(z_i) = 0, v(w_i) = 0\}}{\widetilde{du}(L^{k,p}(S, TS_{-z-w}))}.$$

Replace N^{c} with $N_w^{\mathrm{c}} = N \otimes \mathcal{O}(-2z - w)$. Now the argument works in the same way. $\square$

Chapter 4

Unobstructedness of deformation of cuspidal curves with fixed position of cusps.

4.1 Moduli space of pseudoholomorphic curves with fixed position of cusps.

The proof of unobstructedness of deformation of cuspidal curves where now the position of the cusps is prescribed is very similar to the one in Chapter 3. Here we just do a sketch of the proof, pointing out the differences.

Let $m \in \mathbb{N}$ and $(c_1, \ldots, c_m)$ an m-tuple of m distinct points on X.

Define

$$\widehat{\mathcal{M}}_{J,\text{fix cusps}} = \left\{(u, J_S, z) \in \widehat{\mathcal{M}}_J \times (S^m \setminus \Delta^m), u(z_i) = c_i, du(z_i) = 0 \,\forall\, i\right\},$$

the space of pseudo-holomorphic curves with m prescribed cusp points. Then

$$T_{(u,J_S,z)}\widehat{\mathcal{M}}_{J,\text{fix cusps}} = \left\{ \begin{array}{c} (v, \dot{J}_S, \dot{z}) \in T_{(u,J_S)}\widehat{\mathcal{M}}_J \times T_z S^m, \\ v(z_i) = 0, \nabla v(z_i) + \nabla_{\dot{z}_i} du = 0, \ \forall\, i \end{array} \right\}.$$

There is a natural action of G on $\widehat{\mathcal{M}}_{J,\text{fix cusps}}$, defined by

$$(u, J_S, z) \cdot g = (u \circ g, g^*(J_S), g^{-1}z).$$

We denote by $\mathcal{M}_{J,\text{fix cusps}} = \widehat{\mathcal{M}}_{J,\text{fix cusps}}/G$ the corresponding quotient space.

4.2 Unobstructedness.

Our goal is to study under which conditions, given a fixed $J \in \mathcal{J}_X$, we have an unobstructed theory of deformation of J-holomorphic curves with m prescribed ordinary cusp points, r nodal points (not at fixed positions) and no other singularity. That is, given $u : (S, J_S) \longrightarrow (X, J)$ a pseudoholomorphic map, with $J_S \in \mathbb{T}_g$ and $u(z_i) = c_i$ an ordinary cusp, we want to know when can we deform it to a nearby pseudoholomorphic curve $u' : (S, J'_S) \longrightarrow (X, J)$ with $u'(z'_i) = c_i$ an ordinary cusp.

As before, let

$$\Lambda^{0,1}E_{-z} := \{w \in L^{k-1,p}(S, \Lambda^{0,1}E) | w(z_i) = 0\}.$$

This is a bundle over

$$\{(u, J_S, z) \in C \times \mathbb{T}_g \times (S^m \setminus \Delta^m) | u(z_i) = c_i, du(z_i) = 0\}.$$

Let σ_{fc} be the section of $\Lambda^{0,1}E_{-z}$ defined by $\sigma_{\text{fc}}(u, J_S, z) = du + J \circ du \circ J_S$. By definition, $\mathcal{M}_{J,\text{fix cusps}} = \sigma_{\text{fc}}^{-1}(0)/G$. As before we want to study the linearization of the section σ_{fc} and show it is surjective.

Let

$$E^{\text{fc}} := \left\{(v, \dot{z}) \in L^{k,p}(S, E) \times T_z S^m | v(z_i) = 0, \nabla v(z_i) + \nabla_{\dot{z}_i} du = 0, \forall i\right\}.$$

Define

$$\phi_{\text{fc}} : E^{\text{fc}} \times T_{J_S}\mathbb{T}_g \longrightarrow \Lambda^{0,1}E_{-z}$$

by

$$\phi_{\text{fc}}(v, \dot{z}, \dot{J}_S) = Dv + J \circ du \circ \dot{J}_S,$$

Deformation is unobstructed if ϕ_{fc} is surjective.

Let $\widetilde{du}$ denote the induced map $\widetilde{du} : L^{k,p}(S, TS) \longrightarrow E^{\text{fc}}$.

Define

$$\mathcal{N}^{\text{fc}} = \frac{E^{\text{fc}}}{du(L^{k,p}(S, TS))} \simeq \frac{\{v \in E | v(z_i) = 0, \nabla v(z_i) = 0\}}{du(L^{k,p}(S, TS_{-z}))}.$$

Denote by p both the projections

$$p : E^{\text{fc}} \longrightarrow \mathcal{N}^{\text{fc}}$$

and

$$p : \Lambda^{0,1} E_{-z} \longrightarrow \Lambda^{0,1} E_{-z} / du(L^{k,p}(S, \Lambda^{0,1} S)).$$

There is an induced operator

$$D^{\mathcal{N}^{\text{fc}}} : \mathcal{N}^{\text{fc}} \longrightarrow \Lambda^{0,1} E_{-z} / du(L^{k,p}(S, \Lambda^{0,1} S))$$

defined by

$$p \circ Dv = D^{\mathcal{N}^{\text{fc}}} \circ p.$$

Theorem 4. *Assume $c_1(E) > 3m$. Then ϕ_{fc} is surjective.*

Proof . As before one can easily show that if $D^{\mathcal{N}^{\text{fc}}}$ is surjective then ϕ_{fc} is surjective. We will see that this is the case if $c_1(E) > 3m$.

We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(N^{\text{c}}) & \xrightarrow{\iota} & \mathcal{N}^{\text{fc}} & \xrightarrow{j} & \widetilde{Q} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle D^{N^{\text{c}}}} & & \downarrow{\scriptstyle D^{\mathcal{N}^{\text{fc}}}} & & \downarrow{\scriptstyle D^{\text{fq}}} & & \\
0 & \longrightarrow & \Lambda^{0,1} N^{\text{c}} & \xrightarrow{\iota'} & \Lambda^{0,1} E_{-z} / du(L^{k,p}(S, \Lambda^{0,1} S)) & \xrightarrow{j'} & \widetilde{Q}' & \longrightarrow & 0
\end{array}$$

The difference between Q and $\widetilde{Q}$ (see Lemma 6) is that $f = 1$ is no longer allowed.

Moreover, Q' coincides with $\widetilde{Q}'$. Since we never used $f = 1$ in the proof of Lemma 6, then we still have that D^{fq} is surjective. Since $c_1(E) > 3m$, then $D^{N^{\mathrm{c}}}$ is surjective by Theorem 2. This implies that $D^{N^{\mathrm{fc}}}$ is surjective. □

Proposition 2. *The deformation of pseudoholomorphic curves in a fixed homology class* $[C]$ *inside of* X*, with* m *prescribed cuspidal points,* r *nodal points (not prescribed) and passing through* n *fixed prescribed points is unobstructed as long as*

$$c_1(X) \cdot [C] > 3m + n.$$

Proof . Similar to Proposition 1 and Theorem 4. □

Chapter 5

Unobstructedness of deformation of curves with fixed position of cusps and tacnodes and fixed tangencies.

5.1 Moduli space of pseudoholomorphic curves with fixed position of cusps and tacnodes and fixed tangencies.

Let $m, \nu \in \mathbb{N}$ and $(c_1, \ldots, c_m)$ (respectively $(t_1, \ldots, t_\nu)$) an m-tuple (respectively an ν-tuple) of m (respectively ν) distinct points on X. Let $L_i \in T_{c_i}X$, $i = 1, \ldots, m$ and $L_i' \in T_{t_i}X$, $i = 1, \ldots, \nu$, where the L_i's (respectively L_i''s) are non-zero tangent vectors.

Define

$$\widehat{\mathcal{M}}_{J,\text{fix cusps/tac/tan}} = \left\{ \begin{array}{c} (u, J_S, z, x, x') \in \widehat{\mathcal{M}}_J \times (S^m \setminus \Delta^m) \times (S^{2\nu} \setminus \Delta^{2\nu}), \\ u(z_i) = c_i, du(z_i) = 0, d^2u(z_i) = \lambda_i L_i, \\ u(x_i) = u(x_i') = t_i, du(x_i) = \lambda_i' L_i', du(x_i') = \lambda_i'' L_i', \\ \text{for some } \lambda_i \in T^*_{z_i} S^{\otimes 2} \otimes \mathbb{C}, \lambda_i' \in T^*_{x_i} S \otimes \mathbb{C}, \lambda_i'' \in T^*_{x_i'} S \otimes \mathbb{C} \end{array} \right\},$$

the space of pseudo-holomorphic curves with m prescribed cusp points, ν prescribed tacnodes and prescribed tangencies. Its tangent space at (u, J_S, z, x, x') is formed by the set of $(v, \dot{J}_S, \dot{z}, \dot{x}, \dot{x}')$ belonging to the obvious spaces and verifying

$$v(z_i) = 0, \text{pr}^N(\nabla^2 v(z_i) + \nabla_{\dot{z}_i} d^2 u) = 0,$$

where pr^N is the projection $T_{c_i} X \longrightarrow T_{c_i} X / \text{span}\langle L_i, JL_i \rangle$ to the normal direction at the cusp and

$$v(x_i) + du(\dot{x}_i) = v(x_i') + du(\dot{x}_i') = 0,$$

$$\text{pr}^N(\nabla v(x_i) + \nabla_{\dot{x}_i} du) = \text{pr}^N(\nabla v(x_i') + \nabla_{\dot{x}_i'} du) = 0.$$

where pr^N also denotes the projection map $T_{t_i} X \longrightarrow T_{t_i} X / \text{span}\langle L_i', JL_i' \rangle$.

5.2 Unobstructedness.

Let

$$\Lambda^{0,1} E'_{-z-x-x'} := \left\{ \begin{array}{c} w \in L^{k-1,p}(S, \Lambda^{0,1} E) | w(z_i) = 0, \\ \text{pr}^N(\nabla w(z_i)) = 0, \text{pr}^N(w(x_i)) = \text{pr}^N(w(x_i')) = 0 \end{array} \right\}.$$

This is a bundle over

$$\left\{ \begin{array}{c} (u, J_S, z, x, x') \in C \times \mathbb{T}_g \times (S^m \setminus \Delta^m) \times (S^{2\nu} \setminus \Delta^{2\nu}) | u(z_i) = c_i, \\ du(z_i) = 0, d^2 u(z_i) = \lambda_i L_i, u(x_i) = u(x_i') = t_i, \\ du(x_i) = \lambda_i' L_i', du(x_i') = \lambda_i'' L_i', \text{for some } \lambda_i, \lambda_i', \lambda_i'' \end{array} \right\}.$$

Let σ_{fcttan} be the section of $\Lambda^{0,1}E'_{-z-x-x'}$ defined by

$$\sigma_{\text{fcttan}}(u, J_S, z, x, x') = du + J \circ du \circ J_S.$$

By definition, $\mathcal{M}_{J,\text{fix cusps/tac/tan}} = \sigma^{-1}_{\text{fcttan}}(0)/G$. As before we want to study the linearization of the section σ_{fcttan} and show it is surjective.

Let

$$E^{\text{fcttan}} := \left\{ \begin{array}{c} (v, \dot{z}, \dot{x}, \dot{x}') \in L^{k,p}(S, E) \times T_z S^m \times T_x S^\nu \times T_{x'} S^\nu | v(z_i) = 0, \\ \nabla v(z_i) + \nabla_{\dot{z}_i} du = 0, \text{pr}^N(\nabla^2 v(z_i) + \nabla_{\dot{z}_i} d^2 u) = 0, \\ v(x_i) + du(\dot{x}_i) = v(x'_i) + du(\dot{x}'_i) = 0, \\ \text{pr}^N(\nabla v(x_i) + \nabla_{\dot{x}_i} du) = \text{pr}^N(\nabla v(x'_i) + \nabla_{\dot{x}'_i} du) = 0 \end{array} \right\}.$$

Define

$$\phi_{\text{fcttan}} : E^{\text{fcttan}} \times \hat{T}_{J_S} \mathbb{T}_g \longrightarrow \Lambda^{0,1} E'_{-z-x-x'}$$

by

$$\phi_{\text{fcttan}}(v, \dot{z}, \dot{J}_S) = Dv + J \circ du \circ \dot{J}_S,$$

As before, our goal is to show that ϕ_{fcttan} is surjective.

Let $\widetilde{du}$ denote the induced map $\widetilde{du} : L^{k,p}(S, TS) \longrightarrow E^{\text{fcttan}}$, defined by

$$\widetilde{du}(w) = (du(w), -w(z), -w(x), -w(x')),$$

where $w(z) = (w(z_1), \ldots, w(z_n))$ and similarly for $w(x)$ and $w(x')$. Define

$$\mathcal{N}^{\text{fcttan}} = \frac{E^{\text{fcttan}}}{\widetilde{du}(L^{k,p}(S, TS))}.$$

It follows that $\mathcal{N}^{\text{fcttan}}$ is isomorphic to the space of $v \in E$ verifying

$$v(z_i) = v(x_i) = v(x'_i) = 0,$$

$$\nabla v(z_i) = 0, \text{pr}^N(\nabla v(x_i)) = \text{pr}^N(\nabla v(x'_i)) = 0,$$

$$\mathrm{pr}^N(\nabla^2 v(z_i)) = 0$$

quotiented by $du(L^{k,p}(S, TS_{-z-x-x'}))$.

Denote by p both the projections

$$p : E^{\text{fcttan}} \longrightarrow \mathcal{N}^{\text{fcttan}}$$

and

$$p : \Lambda^{0,1}E'_{-z-x-x'} \longrightarrow \Lambda^{0,1}E'_{-z-x-x'}/du(L^{k,p}(S, \Lambda^{0,1}S)).$$

There is an induced operator

$$D^{\mathcal{N}^{\text{fcttan}}} : \mathcal{N}^{\text{fcttan}} \longrightarrow \Lambda^{0,1}E'_{-z-x-x'}/du(L^{k,p}(S, \Lambda^{0,1}S))$$

defined by

$$p \circ Dv = D^{\mathcal{N}^{\text{fcttan}}} \circ p.$$

Let $N^{\text{fcttan}} = N \otimes \mathcal{O}(-3z - 2x - 2x')$.

Theorem 5. *Assume $c_1(E) > 4m + 4\nu$. Then ϕ_{fcttan} is surjective.*

Proof . The proof is similar to the proof of Theorems 3 and 4 and therefore will not be reproduced here. It follows from the fact that if $c_1(E) > 4m + 4\mu$ then $D^{\mathcal{N}^{\text{fcttan}}}$ is surjective by Theorem 2. □

Proposition 3. *The deformation of pseudoholomorphic curves in a fixed homology class $[C]$ inside of X, with m prescribed cuspidal points, ν prescribed tacnodes, prescribed tangencies, r nodal points (not prescribed) and passing through n fixed prescribed points is unobstructed as long as*

$$c_1(X) \cdot [C] > 4m + 4\nu + n.$$

Proof . Proceed as in Proposition 1 and Theorem 5. □

Chapter 6

The cycle topology for pseudoholomorphic curves.

This section is based on section 5 of [Sh3].

Let X be a 4-dimensional symplectic manifold and C a compact nodal curve with smooth boundary. Let $u : C \longrightarrow X$ be a $L^{k,p}$-smooth map up to the boundary. Let J be an almost complex structure on X.

Let $\mathcal{Z}_2(X) = C^\infty(X, \Lambda^2 X)^*$ be the space of 2-currents on X. Here the star denotes the dual space.

Definition 6. The cycle $u[C]$ associated with the map $u : C \longrightarrow X$ is the current whose pairing with a smooth 2-form ϖ is equal to

$$\langle u[C], \varpi \rangle := \int_C u^* \varpi.$$

If u is J-holomorphic for some almost complex structure J on X, we say that $u[C]$ is a J-holomorphic curve in X. We denote by $\mathrm{supp}(u[C]) := u(C)$ the support of $u[C]$.

We say that $u[C]$ is non-multiple if it can be represented by a non-multiple pseudoholomorphic map $v : C \longrightarrow X$.

A sequence of cycles $u_n[C_n]$ converges to a cycle $u_\infty[C_\infty]$ if $\langle u_n[C_n], \varpi \rangle$ converges to $\langle u_\infty[C_\infty], \varpi \rangle$ for all smooth 2-forms ϖ on X. We will briefly recall some standard

facts about the cycle topology.

Facts:

i Let (C, u) and (C', u') be two closed J-holomorphic curves over X where J is an almost complex structure on X. Assume both C and C' are parameterized by the same closed surface S, (C, u) contains no multiple or ghost components (a component where u is constant) and the associated cycles $u[C]$ and $u[C']$ coincide. Then (C, u) and (C', u') are equivalent, that is, there exists a diffeomorphism $\varphi : C \longrightarrow C'$ so that $u = u' \circ \varphi$;

ii) Let J_n be a sequence of continuous almost complex structures on X converging to an almost complex structure J_∞ on X in the C^0-topology. Let (C_n, u_n) be a sequence of stable J_n-holomorphic closed curves over X which converges to (C_∞, J_∞) in the Gromov topology. Then $u_n[C_n]$ also converges to $u_\infty[C_\infty]$ in the cycle topology;

iii) Let J_n, J_∞, (C_n, u_n) and (C_∞, u_∞) as before. Assume that $u_n[C_n]$ converges to $u_\infty[C_\infty]$ in the cycle topology, C_n and C_∞ are all parameterized by the same surface S, (C_∞, u_∞) contains no multiple or ghost components and J_∞ is C^1-smooth. Then (C_n, u_n) converges to (C_∞, u_∞) in the Gromov topology.

Remember the definition of a stable curve. Let $u : C \longrightarrow X$ be J-holomorphic. We say that (C, u) is a stable curve over (X, J) if whenever C' is a closed component of C (i.e., a normalization of a irreducible component of C with marked points corresponding to the nodal points) such that u is constant on C', then there exists only finitely many biholomorphisms of C' which preserve the marked points of C.

Chapter 7

Local symplectic isotopy problem for cuspidal curves.

7.1 Local symplectic isotopy problem.

Let B be the unit ball in $\mathbb{R}^4$ with the standard symplectic form ω_{st}. Let J_0 be a ω_{st}-tame almost complex structure.

Let $C_0 \subset B$ be a connected J_0-holomorphic curve in B with a isolated singularity at $0 \in B$ and without multiple components. Assume in addition that the boundary C_0 is transverse to ∂B. This implies in particular that ∂C_0 is smoothly embedded in ∂B.

Local symplectic isotopy problem. *Let J_1 and J_2 be two ω_{st}-tamed almost complex structures in B sufficiently close to J_0 in the $C^{0,\alpha}$-topology outside a small ball containing 0 and everywhere in the C^0-topology.*

For $i = 1, 2$, let C_i be a J_i-holomorphic curve sufficiently close to C_0 with respect to the cycle topology, with nodal points and ordinary cusp points and no other singularities. Assume C_1 and C_2 are both connected, have no multiple components, have the same number of irreducible components and the same number of cuspidal and nodal points in each irreducible component.

Goal: *Show that C_1 and C_2 are symplectic isotopic.*

Remark 7. Notice that we did not fix the position of the cusps.

Remark 8. If necessary by shrinking the ball B to a smaller ball B' and considering the curve obtained from $C_0 \cap B'$ by taking only the irreducible components of $C_0 \cap B'$ which contain 0, we can assume that all the irreducible components of C_0 are disks. When doing this shrinking, we choose the radius of the ball B' in such a way that C_0 is transverse to its boundary. From now on, we will assume that all components of C_0 are disks and that C_0 is transverse to ∂B.

Theorem 9. *Let C_0, C_1 and C_2 as above. Assume C_1 and C_2 have only one ordinary cusp and nodal points. Then C_1 and C_2 are symplectic isotopic and the isotopy can be chosen to stay sufficiently close to C_0 in the cycle topology.*

Corollary 1. *Let J_0 be an ω_{st}-tamed almost complex structure on B. Let C_0 be a non-multiple J_0-holomorphic curve in B. Let J_1 and J_2 be two ω_{st}-tamed almost complex structures sufficiently close to J_0 in the $C^{\ell,\alpha}$-topology outside any small ball containing any singular point of C_0 and in the C^0-topology everywhere. Let C_i be a J_i-holomorphic curve, for $i = 1, 2$, sufficiently close to C_0 in the cycle topology, with only nodes and cusps and at most one ordinary cusp near each singular point of C_0. Assume that C_1 and C_2 have the same topology and the same number of cusps near each singular point of C_0. Then C_1 and C_2 are symplectic isotopic and the isotopy can be chosen to stay sufficiently close to C_0 in the cycle topology.*

Proof . (of the Corollary) Let $s_1, \ldots, s_\rho$ be the singular points of C_0. Choose r sufficiently small so that s_j is not contained in the ball of radius r, $B_r(s_i)$, centered at s_i, for all $j \neq i$. Moreover choose r generic so that C_0 is transverse to $\partial B_r(s_i)$ for all i. We can furthermore assume that all components of $C_0 \cap B_r(s_i)$ are disks.

Assume C_1 and C_2 have one ordinary cusp (and nodes) near s_i, by Theorem 9, $C_1 \cap B_r(s_i)$ is symplectic isotopic to $C_2 \cap B_r(s_i)$, for all $i = 1, \ldots, \rho$, with an isotopy that stays sufficiently close to $C_0 \cap B_r(s_i)$ in the cycle topology. If C_1 and C_2 are smooth near s_i, the existence of isotopy follows from the result of Shevchishin [Sh3].

Over $U = B \setminus (\bigcup_i B_r(s_i))$, C_0, C_1, C_2 all have the same topology, i.e. they are all parametrized by a same surface with boundary S. Let $u_0 : S \longrightarrow U$ be a parametriza-

tion of C_0. Then by hypothesis it is injective. Therefore any map $u' : S \longrightarrow U$ close enough to u_0 and mapping ∂S to ∂U defines an embedded symplectic curve. Therefore we can interpolate between a parametrization $u_1 : S \longrightarrow U$ of C_1 and a parametrization $u_2 : S \longrightarrow U$ of C_2 in the desired manner. Now use cut-off functions to glue this isotopy to the isotopies constructed inside the $B_r(s_i)$'s. □

7.2 Auxiliary results.

We will now return to the situation of Theorem 9.

Let $C_{0,j}$, $j = 1, \ldots, \tau$ be the irreducible components of C_0. Choose a parametrization $u_j : \Delta_j \longrightarrow B$ for each irreducible component $C_{0,j}$, where Δ_j is the unit disk. Suppose in addition that u_j is $L^{k,p}$-smooth up to the boundaries.

Let Δ be the disjoint union of the Δ_j and define $u : \Delta \longrightarrow B$ as $u|_{\Delta_j} = u_j$. We then obtain a parameterizations of C_0.

We will use the next three theorems to prove Theorem 9.

Theorem 10. *[Mi-Wh] Let X^{2n} be a manifold of $\dim_{\mathbb{C}} X = n$, J an almost complex structure on X of class C^2, $u : S \longrightarrow X$ a J-holomorphic map, $x \in X$. Suppose that for any $y \in X$ sufficiently close to x, $u^{-1}(y)$ is finite. Then there exists a neighborhood U of x, $V \subset \mathbb{C}^n$ of 0 and a C^1-diffeomorphism $\varphi : U \longrightarrow V$ with $\varphi_*(J_x) = J_{\mathrm{st}}$, such that $\varphi(u(S) \cap U)$ is a proper analytic curve in V.*

Theorem 11. *Let C_0 be as before and assume that $J_0 = J_{\mathrm{st}}$ is integrable. Let C_n be a sequence of J_n-holomorphic curves, with the same number of irreducible components, the same number of nodal points and with one ordinary cusp $y_n \in C_n$. Assume that C_n converges to C_0 in the cycle topology and J_n converges to J_0 in the $C^{\ell,\alpha}$-topology outside a small ball around 0 and everywhere in the C^0-topology. For n sufficiently large, C_n is symplectic isotopic to a holomorphic curve C_n^{hol} with one cusp point, at the same point y_n and with the same tangency as the one of C_n.*

Moreover, as we will see in the proof, we can assume C_n^{hol} to be the intersection of B with an algebraic curve $\widetilde{C}_n^{\mathrm{hol}}$ of large degree, such that the components of $\widetilde{C}_n^{\mathrm{hol}} \cap$

$(\mathbb{CP}^2 \setminus B)$ are all disks and intersect each other transversely.

Theorem 12. *[Ka] Let $\mathbb{P}^N$ be the projective space of all degree d plane curves and $V(d,\delta,m)$ the subvariety of $\mathbb{P}^N$ consisting of all reduced and irreducible curves of genus g, with δ nodes and m cusps. Then $V(d,\delta,m)$ is irreducible if*

$$m \leq \frac{1}{2}(d+1) - g.$$

7.3 Proof of local symplectic isotopy for cuspidal curves.

In this section we will prove Theorem 9, assuming Theorem 11 to hold.

By Theorem 10, exists a C^1-diffeomorphism defined on a small neighborhood $B_r(0)$ of 0, $\varphi : B_r(0) \longrightarrow B_r(0)$ with $\varphi(0) = 0$, $\varphi_*(J_0(0)) = J_{\text{st}}$ and such that $\varphi(C_0)$ is a J_{st}-holomorphic curve. If necessary by shrinking the ball B we can therefore assume that J_0 is the standard complex structure J_{st}, by replacing all the objects involved by their image by φ.

Let C_n be a sequence as in Theorem 11. Then assuming Theorem 11, we know that for n sufficiently large, C_n is symplectic isotopic to a holomorphic curve C_n^{hol}. We can assume C_n^{hol} to be the intersection of B with an algebraic curve $\widetilde{C}_n^{\text{hol}}$ of large degree, such that the components of $\widetilde{C}_n^{\text{hol}} \cap (\mathbb{CP}^2 \setminus B)$ are all disks and intersect each other transversely.

Moreover, if necessary by smoothing some nodal points outside B, we can assume $\widetilde{C}_n^{\text{hol}}$ to be irreducible. We therefore conclude by Theorem 12 that for n sufficiently large, all $\widetilde{C}_n^{\text{hol}}$ are isotopic to each other. It follows that all the C_n's are symplectic isotopic for all n sufficiently large.

This implies Theorem 9. Indeed, assume there exists non-isotopic pairs (C_1, C_2) arbitrarily close to C_0. Then there exists a sequence C_n, of holomorphic curves verifying the conditions of Theorem 11, converging to C_0, with C_n not isotopic to C_{n+1} for arbitrarily large n. This contradicts Theorem 11.

This finishes the proof of Theorem 9.

7.4 Proof of Theorem 11.

We will now start the proof of Theorem 11. We will do the proof as long as possible for the case when C_n has m cusp points, with the same number of cusp points in each irreducible component.

Let δ be the nodal number of C_0 in 0. This is the number of nodes of a generic perturbation of C_0 as a holomorphic map from Δ to B. This generic perturbation only has nodes as singularities and the nodal number of C_0 in 0 will be the number of nodes of this generic perturbation.

If $\delta = 0$ then all the C_n are smooth, since they are both close to C_0 in the cycle topology. The result them follows from [Sh3]. Suppose $\delta \geq 1$. We will proceed by induction.

We can therefore assume that the claim of the theorem holds for all curves which satisfy the hypotheses of the theorem and have a nodal number strictly less than δ. We saw that Theorem 11 implies Theorem 9 and Corollary 1. Therefore we can also assume that the claim of Theorem 9 and Corollary 1 holds for all curves satisfying the conditions stated there and with nodal number strictly less than δ.

7.4.1 Extension of the curves to $\mathbb{CP}^2$.

Lemma 7. *Let C_0, C_n, J_0 and J_n as in Theorem 11. Then there exists curves $\widetilde{C}_0$ and $\widetilde{C}_n$ in $\mathbb{CP}^2$ with the following properties:*

i) There exists a large integer d so that $\widetilde{C}_0$ is a holomorphic rational curve with each irreducible component of degree d. Moreover, $\widetilde{C}_0 \cap B$ is $\varepsilon(d)$-close to C_0, where $\varepsilon(d)$ is a function of d that converges to 0 as $d \to \infty$. Furthermore, $\widetilde{C}_0$ has a singular point at 0 with same nodal number δ and extra nodes outside B as its only singularities;

ii) There exists extensions $\widetilde{C}_n$ to $\mathbb{CP}^2$ of C_n such that $\widetilde{C}_n$ converges to $\widetilde{C}_0$ in the

cycle topology and the only singularities of $\widetilde{C}_n$, besides those of C_n, are nodes contained in $\mathbb{CP}^2 \setminus B$;

iii) There exists almost complex structures $\widetilde{J}_n$ $C^{\ell,\alpha}$-smooth, that coincide with J_n in a ball with smaller radius such that $\widetilde{J}_n$ converges to $\widetilde{J}_0$ in the C^0-topology everywhere and in the $C^{\ell,\alpha}$-topology everywhere outside a small ball around 0. Moreover, $\widetilde{C}_n$ is $\widetilde{J}_n$-holomorphic.

Proof . Embed B in $\mathbb{CP}^2$ in the standard way so that J_{st} becomes the standard integrable structure, still denoted by J_{st}. Extend ω to a global symplectic form on $\mathbb{CP}^2$, still denoted by ω, taming J_{st}.

Let $C_{0,j}$, $j = 1, \ldots, \tau$ be the irreducible components of C_0. Consider parameterizations $u_{0,j} : \Delta_j \longrightarrow B$ of $C_{0,j}$ with $u_{0,j}(0) = 0$. We know that such parameterizations do exist from the assumptions made in Remark 8. Consider the Taylor polynomial $u_{0,j,d}$ (at 0) of degree d, with d chosen high enough and always much larger than δ, so that $u_{0,j,d}$ is non-multiple. This is always possible for d sufficiently high.

We can therefore view $u_{0,j,d}$ as an algebraic map $\overline{u}_j$ (dropping 0 and d to simplify the notation) from $\mathbb{CP}^1 = S^2$ to $\mathbb{CP}^2$. Making a (nodal) generic perturbation $\widetilde{u}_j$ of $\overline{u}_j$, which vanishes to high order at 0, we obtain the desired curve $\widetilde{C}_0 \subset \mathbb{CP}^2$, as the union of the images $\widetilde{u}_j(S^2)$. By construction, the extension $\widetilde{C}_0$ has the properties stated in the Lemma.

By the assumptions made in Remark 8 we can assume that C_0 is transverse to the boundary of B, S^3. That is, we assume that the angle between TC_0 and the tangent plane to the sphere S^3, at any point of the boundary of C_0, is bounded from below.

Since C_n is close to C_0 with respect to the cycle topology, for n sufficiently large, then the intersections $C_0 \cap S^3$ and $C_n \cap S^3$ are also close to each other. In particular, this means that the boundaries of C_0 and C_n are all similar. We now construct the extensions $\widetilde{C}_n$ of C_n by gluing C_n with the portion of $\widetilde{C}_0$ outside the ball B, using a cut-off function near the sphere S^3. This requires us to change the almost complex structures J_n near S^3 but this perturbation can be chosen small. We then obtain almost complex structures $\widetilde{J}_n$, defined in all of $\mathbb{CP}^2$, such that $\widetilde{C}_n$ is $\widetilde{J}_n$-holomorphic

and $\widetilde{J}_n = J_n$ in a smaller ball around 0. □

7.4.2 Constraints.

Let $\widetilde{C}_{n,j}$, $j = 1, \ldots, v$ be the irreducible components to $\widetilde{C}_n$. As $n \to \infty$, $\widetilde{C}_{n,j}$ converges in the cycle topology to a union of components of $\widetilde{C}_0$.

By assumption, all the C_n have the same number of irreducible components and the same number of cusp points in each irreducible component. Let y_n be the points where C_n has cusps. Let m_j be the number of cusp points of C_n contained in $C_{n,j}$.

For each $j = 1, \ldots, v$, let d_j be the degree of $\widetilde{C}_{n,j}$. Fix a tuple of $3d_j - 1 - 4m_j$ generic points $\{x_j^0\}$ in the portion of $\widetilde{C}_0$ which is the limit of $\widetilde{C}_{n,j}$. For each j there exists tuples of $3d_j - 1 - 4m_j$ generic points $\{x_j^n\}$ in $\widetilde{C}_{n,j}$, converging to $\{x_j^0\}$. Let $\{x^n\} = \cup\{x_j^n\}$.

By generic we mean the following. Let $S_{n,j}$ be the normalization of $\widetilde{C}_{n,j}$ and $S_n = \bigcup S_{n,j}$. Let S' be any (possibly disconnected) surface with number of components bounded by the number of components of C_0, which either is S_n or can be obtained as a degeneration of S_n. Let D be the total degree of $\widetilde{C}_n$. We require that the moduli space

$$\mathcal{M}_{J_n, x_j^n, \text{cusps, tacnodes}}(S', X, D')$$

of J_n-holomorphic curves of degree D with normalization S', with m prescribed cusp points and/or tacnodes with fixed tangencies and some nodal points, passing through x^n is a (possibly empty) complex manifold of the expected dimension. Similarly for $\widetilde{C}_0$, J_0 and x_0.

We also require that to have a worse singularity at one of the prescribed cusp points or a singularity worse than a node at any other point imposes a positive codimension.

Notice that this is the maximum number of points we can fix and still have unobstructedness of deformation of $\widetilde{C}_{n,j}$ with prescribed position of cusps and tangencies.

7.4.3 Deformation of $\widetilde{C}_n$.

Fix a sequence of generic homotopies of almost complex structures $\widetilde{J}_n(t)$, with $t \in [0,1]$. We will assume that all the almost complex structures in this homotopy are of class $C^{\ell,\alpha}$ and that the homotopy depends $C^{\ell,\alpha}$-smoothly on t. Moreover, we will fix such a homotopy with the following properties:

- $\widetilde{J}_n(0) = \widetilde{J}_n$;
- for some small $\varepsilon_0 > 0$, $\widetilde{J}_n(t) = J_{\text{st}}$ in B, for all $t \in [1-\varepsilon_0, 1]$;
- $\widetilde{J}_n(t)$ converges to $\widetilde{J}_0$ when n goes to infinity, in the C^0-topology, uniformly in $t \in [0,1]$;

Our goal is to deform continuously $\widetilde{C}_n$ inside a family of $\widetilde{J}_n(t)$-holomorphic curves to a curve which is holomorphic over B, in such a way that we preserve the isotopy class. Since in particular we want to control the local symplectic isotopy class, we will impose that the curves in the family all lie sufficiently close to $\widetilde{C}_0$ in the cycle topology. We denote by $\text{dist}_{\text{cycle}}$ the cycle distance.

Let A_n be the set of $t' \in [0,1]$ such that for all $t \leq t'$ there exists a $\widetilde{J}_n(t)$-holomorphic curve $\widetilde{C}_{n,t}$ with the following properties:

a) $\widetilde{C}_{n,t}$ has m singular points, which are either cusps or self-tangencies at y_n and with the same tangencies as the cusps of $\widetilde{C}_n$;

b) $\widetilde{C}_{n,t}$ passes through x^n;

c) $\text{dist}_{\text{cycle}}(\widetilde{C}_{n,t}, \widetilde{C}_0) < \varepsilon$.

d) There exists a curve $\widetilde{C}^{\text{smo}}_{n,t}$, arbitrarily close to $\widetilde{C}_{n,t}$ in the cycle topology, with the following properties. $\widetilde{C}^{\text{smo}}_{n,t}$ has only m ordinary cusp points in the prescribed points and with the prescribed tangencies and r nodes in B. Moreover, $\widetilde{C}^{\text{smo}}_{n,t}$ is symplectic isotopic to $\widetilde{C}_n$.

7.4.4 First limit case.

Let $t_n^+ = \sup A_n$. Our goal is to show that $t_n^+ = 1$ for all $n \gg 1$. Assume $t_n^+ < 1$ for some n.

Let t_ν be an increasing sequence converging to t_n^+. By construction, there exists $\widetilde{J}_{n,t_\nu}$-holomorphic curves $\widetilde{C}_{n,t_\nu}$ with the properties a)-d). In particular all $\widetilde{C}_{n,t_\nu}$ are homologous to $\widetilde{C}_0$.

Passing to a subsequence, if necessary, we may assume that $\widetilde{C}_{n,t_\nu}$ converges to a $\widetilde{J}_n(t_n^+)$-holomorphic curve $\widetilde{C}_n^+$ in the cycle topology, by Gromov's compactness theorem.

By construction, $\mathrm{dist}_{\mathrm{cycle}}(\widetilde{C}_n^+, \widetilde{C}_0) \leq \varepsilon$ and therefore $\widetilde{C}_n^+$ is sufficiently close to $\widetilde{C}_0$, with respect to the cycle topology. Consequently, near each nodal point of $\widetilde{C}_0$ there is exactly one nodal point of $\widetilde{C}_n^+$.

By construction $\widetilde{C}_n^+$ still has cusp points or self-tangencies in the prescribed positions, with prescribed tangencies. We claim that $\widetilde{C}_n^+$ has no singular point with nodal number greater or equal than δ. Assume it does.

By the definition of cycle topology

$$\#(\text{components of } \widetilde{C}_n^+) \leq \#(\text{components of } \widetilde{C}_0).$$

By the genus formula

$$0 = \frac{[\widetilde{C}_0]^2 - c_1 \cdot [\widetilde{C}_0]}{2} + \#(\text{components of } \widetilde{C}_0) - \delta - \#(\text{nodes of } \widetilde{C}_0 \text{ outside } B).$$

Let g_j^+ denote the genus of the j-th irreducible component of $\widetilde{C}_n^+$. Let $\delta(\widetilde{C}_n^+)$ be the total nodal number of $\widetilde{C}_n^+$ minus the number of nodes of $\widetilde{C}_n^+$ outside B. Then

$$\begin{aligned}\sum g_j^+ &= \frac{[\widetilde{C}_0]^2 - c_1 \cdot [\widetilde{C}_0]}{2} + \#(\text{components of } \widetilde{C}_n^+) - \delta(\widetilde{C}_n^+) \\ &\quad - \#(\text{nodes of } \widetilde{C}_n^+ \text{ outside } B) \\ &= \#(\text{components of } \widetilde{C}_n^+) - \#(\text{components of } \widetilde{C}_0) + \delta - \delta(C_n^+) \\ &\quad + \#(\text{nodes of } \widetilde{C}_0 \text{ outside } B) - \#(\text{nodes of } \widetilde{C}_n^+ \text{ outside } B) \leq 0.\end{aligned}$$

Notice that the number of nodes outside B is the same for both curves. Hence $g_j^+ = 0$ for all j. Notice that

$$\#(\text{components of } \widetilde{C}_0) \geq \#(\text{components of } \widetilde{C}_n) \geq \#(\text{components of } \widetilde{C}_n^+) - \sum g_j^+. \tag{7.1}$$

The first inequality is because $\widetilde{C}_0$ is more singular than $\widetilde{C}_n$, the second inequality comes from the genericity. In fact the number of constrain x^n is

$$3D - \#(\text{components of } \widetilde{C}_n) - 4m$$

which has to be less or equal than the dimension of the corresponding moduli space

$$\dim \mathcal{M}(\widetilde{C}_n^+) = 3D - \#(\text{components of } \widetilde{C}_n^+) + \sum g_i^+ - 4m.$$

This implies that the number of components of $\widetilde{C}_n^+$, $\widetilde{C}_n$ and $\widetilde{C}_0$ coincide by (7.1).

Since $\widetilde{C}_n^+$ passes through all the x_n and has prescribed cusps and tangencies, then $\widetilde{C}_n^+$ belongs to a moduli space of dimension 0, since

$$\#\{x^n\} = 3D - \#(\text{components of } \widetilde{C}_n) - 4m.$$

To have a singularity with nodal number greater or equal to δ other than the required ordinary cusps or tacnodes is of positive codimension. This contradicts the genericity of the path $\widetilde{J}_n(t)$.

This argument does not exclude $\widetilde{C}_n^+$ from having an ordinary cusp or an ordinary tacnode at one of the m prescribed points. However, if $\delta > 2$, then all the singularities of $\widetilde{C}_n^+$ are simpler than those of $\widetilde{C}_0$. Therefore we can apply induction to $\widetilde{C}_n^+$.

If $\delta = 1$ then $\widetilde{C}_n^+$ cannot have a tacnode, which has nodal number 2 at one of the prescribed cusp points, since $\delta(C_n^+) = \delta$. So the worst singularity that $\widetilde{C}_n^+$ can have at the prescribed cusp points is an ordinary cusp. By unobstructedness of deformation, we can continue the deformation $\widetilde{C}_{n,t}$ for some $t > t_n^+$, which contradicts the maximality of t_n^+. This means that we don't need to worry about the case of

$\delta = 1$.

Assume $\delta = 2$. The next case to analyze is when $\delta = 2$ and $\widetilde{C}_0$ has a tacnode at the origin. In this case, the domain of C_0 is two disks while the domain of C_n is one cylinder, since it has a cusp by hypothesis. But we have proved before that all $\widetilde{C}_0$, $\widetilde{C}_n$ and $\widetilde{C}_n^+$ should have the same number of components. We conclude therefore that in this case $\widetilde{C}_n^+$ cannot have a singularity with $\delta = 2$. So we can exclude this case.

In the case when C_0 has nodal number $\delta = 2$ at 0 but 0 is not a tacnode, it follows immediately that all the singularities of $\widetilde{C}_n^+$ are simpler than those of $\widetilde{C}_0$.

We therefore conclude that we can apply induction to $\widetilde{C}_n^+$, in all the relevant cases.

7.4.5 Existence of a partial smoothing.

Proposition 4. *There exists a partial smoothing of $\widetilde{C}_n^+$, $\widetilde{C}_n^{+,smo}$, with the same cusps and tangencies than $\widetilde{C}_n$, r nodes and no other singular point inside B. Moreover, $\widetilde{C}_n^{+,smo}$ can be chosen to be pseudoholomorphic with respect to a small perturbation of $\widetilde{J}_{n,t_n}$ and arbitrarily close to $\widetilde{C}_n^+$.*

Proof . By construction, $\widetilde{C}_n^+$ may have self-tangencies or cusp points or one or more of the cusps of $\widetilde{C}_n^+$ may have degenerated to a worst singularity or to a self-tangency.

The first case is when $\widetilde{C}_n^+$ has an extra singular point inside B, not specifically a cusp or a node. Do a local smoothing, by first perturbing it to a collection of nearby nodes and then smoothing the nodes. Notice that we do change the almost complex structure in this process but the change can be as small as we want to.

The second case is when $\widetilde{C}_n^+$ has a worse, non-immersed point at a cusp point of $\widetilde{C}_n$. Then by unobtructedness of deformation, we can deform it to a curve with only one ordinary cusp, if necessary by moving to a nearby generic almost complex structure. The reason why is that curves with only ordinary cusps are dense in the space of curves with cusps, if we allow the almost complex structures to vary.

The third case is when $\widetilde{C}_n^+$ has a tacnode at a cusp point of $\widetilde{C}_n$. In that case we do a "smoothing" back to a cusp, using the following local models. Using Theorem 10, we have that local model for an ordinary tacnode is $y^2 = x^4 + o(x^4)$, in complex

coordinates x, y. Replace this local model by $y^2 = \varepsilon x^3 + x^4 + o(x^4)$ and use a cut-off function to glue this new local model to the rest of the curve. In particular, we need to perturb slightly the almost complex structure but again we can make this perturbation as small as we want.

The fourth and last case is when $\widetilde{C}_n^+$ has an immersed singularity worse then a tacnode at a cusp point of $\widetilde{C}_n$. Again, use a density argument to conclude that we can isotope that self-tangency to a tacnode, by changing slightly the almost complex structure. Then proceed as before.

Using all of this constructions, we obtain $\widetilde{C}_n^{+,\text{smo}}$. □

7.4.6 Consequences of the existence of a partial smoothing.

Let $\widetilde{C}_n^{+,\text{smo}}$ be the curve given by Proposition 4.

We can apply the induction hypothesis and Corollary 1 to the curves $\widetilde{C}_n^+$, $\widetilde{C}_n^{+,\text{smo}}$ and $\widetilde{C}_{n,t_n-h}$ for a small h. We then conclude that the curve $\widetilde{C}_n^{+,\text{smo}}$ is symplectic isotopic to $\widetilde{C}_n$ in B.

Let $u_n^+ : S_n^+ \longrightarrow \mathbb{CP}^2$ be a normal parametrization of $\widetilde{C}_n^+$. This means that S_n^+ is a Riemann surface, possibly not connected, $u_n^+[S_n^+] = \widetilde{C}_n^+$, u_n^+ is $\widetilde{J}_n(t)$-holomorphic and $L^{k,p}$-smooth up to boundary, the restriction of u_n^+ to every connected component of S_n^+ is non-multiple and the number of boundary circles of S_n^+ is as small as possible. Denote by $y_{n,i}$, $i = 1, \ldots, m$, the cusp points of $\widetilde{C}_n$. Then $u_n^+(S_n^+)$ has singularities at $y_{n,i}$, either non-immersed (cusps) or self-tangencies.

Consider the relative moduli space

$$\widehat{\mathcal{M}}_{\text{cusps,self-tangencies}}(S_n^+, \mathbb{CP}^2, \widetilde{J}_n, x^n)$$

of $\widetilde{J}_n(t)$-holomorphic curves which are parameterized by S_n^+, are in the same homology class $[\widetilde{C}_0]$, pass through the fixed points x^n and have either non-immersed points or self-tangencies at $y_{n,i}$. This space is non-empty since it contains $(\widetilde{C}_n^+, t_n^+)$. By unobstructedness of deformation there exists some $t_n^{++} > t_n^+$ such that we can construct a path of $\widetilde{J}_n(t)$-holomorphic curves $\widetilde{C}_{n,t}^+ \in \widehat{\mathcal{M}}_{\text{cusps,self-tangencies}}(S_n^+, \mathbb{CP}^2, \widetilde{J}_n, x^n)$

for $t \in [t_n^+, t_n^{++}]$, starting at $\widetilde{C}_n^+$.

Then there exists curves $\widetilde{C}_{n,t}^{+,\text{smo}}$, constructed as in Proposition 4, from $\widetilde{C}_{n,t}^+$, which will have prescribed ordinary cusp points at $y_{n,i}$ and prescribed tangencies, r nodes and no other singular point in B. This is because the construction of Proposition 4 works for families of curves depending on the parameter t and yields smoothings that are mutually symplectically isotopic.

By construction, these will be symplectic isotopic to $\widetilde{C}_n$ over B.

If additionally we have

$$\text{dist}_{\text{cycle}}(\widetilde{C}_n^+, \widetilde{C}_0) < \varepsilon$$

then also $\text{dist}_{\text{cycle}}(\widetilde{C}_{n,t}^+, \widetilde{C}_0) < \varepsilon$ for all $t \in [t_n^+, t_n^{++}]$, which contradicts the maximality of t_n^+.

7.4.7 Second limit case.

Thus we can assume $\text{dist}_{\text{cycle}}(\widetilde{C}_n^+, \widetilde{C}_0) = \varepsilon$ for every n. Then for every n we can fix $t_n' \in [0, t_n^+]$ and a $\widetilde{J}_{n,t_n'}$-holomorphic curve $\widetilde{C}_{n,t_n'}$ which has the properties a)-d) and satisfies the additional condition

$$\frac{\varepsilon}{2} \leq \text{dist}_{\text{cycle}}(\widetilde{C}_{n,t_n'}, \widetilde{C}_0) \leq \varepsilon.$$

Passing to a subsequence, if necessary, we may assume that $\widetilde{C}_{n,t_n'}$ converges to a $\widetilde{J}_0$-holomorphic curve $\widetilde{C}'$ in the cycle topology. But then

$$\frac{\varepsilon}{2} \leq \text{dist}_{\text{cycle}}(\widetilde{C}', \widetilde{C}_0) \leq \varepsilon.$$

From now on we assume that $m = 1$.

We claim that $\widetilde{C}'$ has no singular point with nodal number greater or equal than δ. Assume it does. Then as before we can conclude that all the components of $\widetilde{C'}$ are rational and in the same number as the components of $\widetilde{C}_0$.

Since $\widetilde{C}'$ passes through x_0 then we have

$$3D - \#(\text{components of } \widetilde{C}_n) - 4$$

constraints.

Notice that the curve $\widetilde{C}'$ is constrained to have a cusp or a self-tangency at the origin, with a prescribed tangency.

By dimension count, $\widetilde{C}'$ has no singular point with nodal number greater or equal than δ as long as $\delta > 2$.

If $\delta = 1$ then $\widetilde{C}'$ has an ordinary cusp at the origin. But then C_n is symplectic isotopic to a holomorphic curve with one cusp, which is exactly what we wanted, by unbstructedness of deformation at $\widetilde{C}'$. Assume that $\delta = 2$ and $\widetilde{C}_0$ has a tacnode at the origin. Since the number of components of $\widetilde{C}_n$ and $\widetilde{C}'$ have to be equal this leads us to a contradiction. In the case when $\widetilde{C}_0$ has nodal number 2 at the origin but it is not a tacnode, we can again apply induction if $\widetilde{C}'$ has a tacnode.

By induction hypothesis and Corollary 1, a holomorphic curve obtained from $\widetilde{C}'$ by smoothing to a curve with one ordinary cusp is symplectic isotopic to $\widetilde{C}_n$ over B.

But $\widetilde{C}' \cap B$ is holomorphic. This concludes the proof of Theorem 11.

Chapter 8

Global isotopy

Let $X = \mathbb{CP}^2$ and J_0 be an ω_{st}-tame almost complex structure.

Theorem 13. *Let C_0 be any irreducible rational J_0-holomorphic of degree d in $\mathbb{CP}^2$ with m ordinary cusp points and nodes as its only singularities such that $d > m$. Then C_0 is symplectic isotopic to a holomorphic curve.*

Proof . Fix a generic path $J(t)$ of ω_{st}-tamed almost complex structures connecting J_0 and J_{st}. Fix $3d-1-3m$ generic points x in C_0. Denote by $y_1, \ldots, y_m$ the ordinary cusp points of C_0.

Let A be the set of $t' \in [0,1]$ so that for all $t \leq t'$ there exists a $J(t)$-irreducible curve C_t with prescribed cusp points at $y_1, \ldots, y_m$ that pass through x and is symplectically isotopic to C_0.

A is open by unobstructedness of deformation. Let's see it is closed. Let $t = \sup A$ and t_n be an increasing sequence converging to t. Fix curves C_n with the properties above. Notice that all C_n are in the same homology class.

Passing to a subsequence if necessary, we can assume that C_n converges to a curve C, which will have prescribed cusps at $y_1, \ldots, y_m$.

Write $C = \sum C_i$ for $i = 1, \ldots, N$, where the C_i's are not necessarily distinct. We claim that $N = 1$. Let d_i be the degree of C_i and m_i the number of cusps in the component C_i. Then $d = \sum d_i$ and $m = \sum m_i$. In fact, none of the cusps can become a tacnode or a worst self-intersection by dimension reasons.

By genericity of $J(t)$ we can have at most

$$k_i \leq 3d_i - 1 - 3m_i$$

fixed points in each component C_i. This is because C_i has genus 0 and by genericity of $J(t)$ the moduli space of degree d_i rational $J(t)$-curves with m_i cusps at prescribed points has dimension $3d_i - 1 - 3m_i$. So overall we can have at most

$$k \leq 3d - N - 3m$$

fixed points. Since C passes through x then we need to have at least $3d - 1 - 3m$ points. Therefore $N = 1$ implying that C is irreducible.

Since the position of the cusps is fixed, they cannot collide and disappear. Moreover, they cannot degenerate to an worst singularity since we are dealing with genus zero curves and we have fixed the maximum number of constraints, in a generic way. There cannot appear any new unwanted singularities, for the same reason. Finally, the number of nodes remains constant by conservation of the nodal number.

Hence A is closed, implying that $A = [0, 1]$. This implies that C_0 is symplectic isotopic to a holomorphic curve. □

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