Existence and regularity of monotone solutions to a free boundary problem

by

Daniela De Silva

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Author ....
Department of Mathematics
April 12, 2005

Certified by ....
David Jerison
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Pavel Etingof
Chairman, Department Committee on Graduate Students

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Abstract

In the first part of this dissertation, we provide the first example of a singular energy minimizing free boundary. This singular solution occurs in dimension 7 and higher, and in fact it is conjectured that there are no singular minimizers in dimension lower than 7. Our example is the analogue of the 8-dimensional Simons cone in the theory of minimal surfaces.

The minimality of the Simons cone is closely related to the existence of a complete minimal graph in dimension 9, which is not a hyperplane. The first step toward solving the analogous problem in the free boundary context, consists in developing a local existence and regularity theory for monotone solutions to a free boundary problem. This is the objective of the second part of our thesis. We also provide a partial result in the global context.

Thesis Supervisor: David Jerison
Title: Professor of Mathematics

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Contents

Abstract

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The minimality of the Simons cone is closely related to the existence of a complete minimal graph in dimension 9, which is not a hyperplane. The first step toward solving the analogous problem in the free boundary context, consists in developing a local existence and regularity theory for monotone solutions to a free boundary problem. This is the objective of the second part of our thesis. We also provide a partial result in the global context.

Chapter 1

Introduction

Let Ω be an open connected subset of $\mathbb{R}^n$, and consider the energy functional

$$J(u,\Omega) = \int_{\Omega} (|\nabla u|^2 + \chi_{\{u>0\}}).$$

For $n \geq 3$, let $t_n > 0$ be the unique constant such that, the positive harmonic function Z in the cone $\Gamma = \left\{ x \in R^n : |x_n| < t_n \sqrt{x_1^2 + ... + x_n^2} \right\}$ (unique up to scalar multiple) which is 0 on $\partial\Gamma$, is homogeneous of degree 1. Denote by Z_ν the inner normal derivative, which by symmetry is homogeneous of degree 0. Then, one can choose a scalar multiple c so that $cZ_\nu = 1$ on $\partial\Gamma \backslash \{0\}$. Let U be the function which equals cZ in Γ and 0 outside of Γ. It follows immediately that U is a critical point for the energy functional $J(\cdot, B)$ for every ball $B \subset \mathbb{R}^n$.

Our first main result is the following (see [DJ]):

Theorem 1.1 *In dimension* $n = 7$*,* U *is a global energy minimizer for the functional* $J(\cdot, B)$*, i.e.* $J(U,B) \leq J(v,B)$ *for all balls* $B \subset \mathbb{R}^7$*, and any function* v *such that* $v = U$ *on* ∂B*.*

Let us briefly motivate this result. In [AC], Alt and Caffarelli analyzed the question of the existence and regularity of a minimizer u of $J(\cdot, \Omega)$. They proved that in two dimensions, the free boundary of u,

$$F(u) = \partial\{u > 0\} \cap \Omega,$$

does not have singularities. They also developed a partial regularity theory in higher dimensions, and showed that in dimension $n = 3$, the singular critical point U is not an energy minimizer.

Subsequently in [W2], Weiss showed that there exists a critical dimension k, $3 \leq k \leq +\infty$, such that energy minimizing free boundaries are smooth for $n < k$.

This draws on a strong analogy with the theory of minimal surfaces, for which it is known that the critical dimension is 8.

In [CJK], the authors proved that there are no singular free boundary minimizers in dimension $n = 3$, which yields $k \geq 4$. They also showed that U is not an energy minimizer in dimension $n \leq 6$. Their proof suggests that $k = 7$, but the problem remains still open.

Theorem 1.1 shows that $k \leq 7$, by providing an example of a singular energy minimizing free boundary in dimension $n = 7$. Analogously, for the theory of minimal surfaces, the Simons cone,

$$S = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 > x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

provides an example of a singular set of minimal perimeter in dimension $n = 8$.

Our proof is inspired by the proof of the minimality of the Simons cone in [BDG], and by the notion of viscosity (weak) solution to the Euler equation for our minimum problem,

$$\Delta u = 0 \quad \text{in} \quad \{x \in \Omega | u(x) > 0\}, \qquad |\nabla u| = 1 \quad \text{on} \quad F(u),$$

introduced by Caffarelli in [C1]. More precisely, for a fixed ball $B \subset \mathbb{R}^n$, centered at the origin, we let u be a minimizer for $J(\cdot, B)$, with data boundary U. In [C2], the author proves that minimizers are weak solutions. We construct a family of weak subsolutions and a family of weak supersolutions which approach U respectively from its positive and its zero phase. We develop comparison techniques for weak solutions, which allow us to trap u between such families, forcing it to coincide with U.

The second objective of our thesis is to pursue even further the analogy between the theory of minimal surfaces, and free boundary regularity. More precisely, let us focus on the theory of minimal graphs, i.e. solutions to the Euler equation associated to the area functional (minimal surface equation)

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

In 1915, Bernstein [B] proved that planes are the only smooth minimal graphs in $\mathbb{R}^3$. Several years later, De Giorgi [D] showed that the existence of non-planar minimal graphs in $\mathbb{R}^{n+1}$ implies the existence of singular minimal cones in $\mathbb{R}^n$. Together with the regularity results of Almgren [A] and Simons [S] about minimal cones, this extended Bernstein result up to dimension 8. In [BDG], the authors proved the minimality of the Simons cone, and, correspondingly, they proved the existence of a non affine minimal graph, one dimension higher. The result in Theorem 1.1, then naturally raises the analogous question for the Euler equation of the energy functional J. More precisely, we consider the following problem in $\mathbb{R}^{n+1}$:

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\}, \\ \partial\{u > 0\} & \text{is a non-planar graph in the } x_{n+1} \text{ direction.} \end{cases} \tag{1.1}$$

In analogy with the minimal surface theory, on the basis of Theorem 1.1, one expects that a global smooth solution to (1.1) exists in dimension 8 or higher.

Our approach to construct such a solution is inspired by the proof of [BDG].

The first step is to develop a local theory which is the analogue of the existence and regularity theory for the minimal surface equation in the ball, when the data boundary is smooth.

Our result is the following:

Theorem 1.2 *Assume that, there exist a strict smooth subsolution V_1 and a strict smooth supersolution V_2 to (1.1) in $\mathbb{R}^{n+1}$, such that*

i. $0 \le V_1 \le V_2$ *on* $\mathbb{R}^{n+1}$;

ii. $\lim_{x_{n+1}\to+\infty} V_1(x', x_{n+1}) = +\infty$, $\partial_{n+1} V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$.

Then, for each $R > 0$, *and* $\overline{x} \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$, *if* h_R *is sufficiently large, there exists* u_R *weak solution to (1.1) in* $C_R(\overline{x}) = \mathcal{B}_R(\overline{x}') \times \{|x_{n+1} - \overline{x}_{n+1}| < h_R\}$, *such that,* u_R *is monotone increasing in the* x_{n+1} *direction, and* $V_1 \le u_R \le V_2$. *Moreover,* $F(u_R)$ *is a Lipschitz graph in the* x_{n+1} *direction.*

We remark that the proof of Theorem 1.1, provides a clear indication of how to construct functions V_1 and V_2 satisfying the assumptions above, when $n \ge 7$.

We also observe that u_R is trapped in between a subsolution and a supersolution. In the minimal surfaces case, this is achieved by ordinary comparison results, which are not available in the free boundary context.

The existence of a local solution u_R is achieved using minimizing techniques. If v is a minimizer of J in an appropriate class of function, then its monotone rearrangement in the vertical direction can be shown to be a weak solution. The main tools to achieve such a result are harmonic replacement and domain variation techniques, together with the maximum principle. Then, using the method of continuity and maximum principle techniques, u_R is compared with a family of subsolutions, which are suprema of vertical translates of u_R over balls (supconvolutions). This yields the desired Lipschitz behavior of the free boundary of u_R.

The second step towards constructing a global solution to (1.1), would be a limiting argument as $R \to +\infty$. In the theory of minimal surfaces, the convergence to a global solution is guaranteed by a very powerful tool, that is the a-priori estimate of the gradient of a solution to the minimal surface equation. In the free boundary context, the analogue of such a tool is not yet available.

A limiting argument allows us to prove the following:

Theorem 1.3 *Assume that, there exist a strict smooth subsolution* V_1 *and a strict smooth supersolution* V_2 *to (1.1), such that:*

i. $0 \le V_1 \le V_2$;

ii. $\partial_{n+1} V_i > 0$ *on* $\overline{\{V_i > 0\}}, i = 1, 2$;

iii. $\lim_{r\to\infty} \dfrac{V_1(rx)}{r} \geq U(x_1, ..., x_n)$.

Then, there exists a global weak solution u *to:*

$$\Delta u = 0 \quad \textit{in} \quad \{u > 0\}, \quad |\nabla u| = 1 \quad \textit{on} \quad F(u),$$

such that u *is monotone increasing in the* x_{n+1} *direction, and* $F(u)$ *is a continuous non-planar graph, with a universal modulus of continuity. Moreover,* $F(u)$ *is locally NTA.*

Here U is the function introduced in Theorem 1.1, interpreted as a function of $n+1$ variables.

The NTA property of $F(u)$ is proved by the means of a monotonicity formula [ACF] for ∇u, together with non-degeneracy properties of u. Then, exploiting the known behaviour of positive harmonic functions in NTA domains [JK], we derive that $F(u)$ cannot contain vertical segments.

Chapter 2

Main Results

2.1 Preliminaries.

2.1.1 Notations.

A point $x \in \mathbb{R}^n$ will be occasionally denoted by (x', x_n), with $x' = (x_1, ..., x_{n-1})$.

A ball of radius r in $\mathbb{R}^{n-1}$, will be denoted by $\mathcal{B}_r$, while a ball of radius r in $\mathbb{R}^n$, will be denoted by B_r. When specifying the center x of the ball, we will use either $B_r(x)$ or $B(x, r)$.

For $a, b > 0$, we set

$$C(a, b) = \mathcal{B}_a(0) \times \{|x_n| < b\}.$$

In particular $C(a) = C(a, a)$.

Let V be a non-negative function on $\mathbb{R}^n$, such that

$$\partial\{V > 0\} = \{(x', \phi(x')), x' \in \mathbb{R}^{n-1}\}, \tag{2.1}$$

with ϕ smooth. For any $\overline{x} \in \mathbb{R}^n$, set

$$d_R(V, \overline{x}) = \max_{\mathcal{B}_R(\overline{x}')} |\phi(x') - \overline{x}_n|. \tag{2.2}$$

In particular, when $\overline{x} = 0$, we let $d_R(V) = d_R(V, 0)$.

Let $V_1 \leq V_2$ be non-negative functions on $\mathbb{R}^n$, satisfying (2.1), and let $0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$. We will denote by

$$C_R = \mathcal{B}_R(0) \times \{|x_n| < h_R\},$$

with

$$h_R \geq \max\{2d_R(V_1), 2d_R(V_2), R\}.$$

2.1.2 Background and Definitions.

Let Ω be an open connected subset of $\mathbb{R}^n, n \geq 3$, with $\partial\Omega$ locally a Lipschitz graph. Consider the energy functional,

$$J(u, \Omega) = \int_\Omega (|\nabla u|^2 + \chi_{\{u>0\}}),$$

and for any given $\phi \in H^1(\Omega)$, ϕ non-negative, set

$$K(\phi) = \{v \in H^1(\Omega) | v = \phi \text{ on } \partial\Omega\}.$$

We recall the following existence and regularity result, for energy minimizers (see [AC]).

Theorem 2.1 *If $J(\phi, \Omega) < +\infty$, then there exists a minimizer u of $J(\cdot, \Omega)$ over $K(\phi)$. Moreover $u \in C^{0,1}(\Omega)$.*

The free boundary of a minimizer u is defined by $F(u) = (\partial\{x \in \Omega | u(x) > 0\}) \cap \Omega$. The following regularity result for energy minimizing free boundaries can be found in [W2].

Theorem 2.2 *There exists a critical dimension k, $3 \leq k \leq +\infty$, such that any energy minimizing free boundary $F(u)$ in dimension $n < k$ is smooth.*

In what follows, we will always denote by k, the critical dimension at which free boundaries may cease to be smooth.

Now, let us introduce the notion of global minimizer to the functional J.

Definition 2.3 *$u \in H^1_{loc}(\mathbb{R}^n)$ is a global minimizer for J, if and only if, for any ball $B \subset \mathbb{R}^n$, and any function $v \in H^1(B)$, such that $u - v \in H^1_0(B)$, $J(v, B) \geq J(u, B)$.*

The following result about global energy minimizers is proved in [CJK].

Theorem 2.4 *In dimension $n = 3$, let $u \geq 0$ be a nonzero global energy minimizer for J, homogeneous of degree 1. Then, after rotation, $u(x) = x_n^+ \equiv \max(x_n, 0)$.*

The significance of this theorem is that it implies classical regularity of energy minimizing free boundaries in dimension 3.

Corollary 2.5 *Any energy minimizing free boundary in dimension $n = 3$ is smooth. In particular $k \geq 4$.*

We now introduce a notion of weak free boundaries, which is related to the notion of minimizing free boundaries, by a result in [C2]. First, for any real-valued function on Ω, we define $\Omega^+(u) = \{x \in \Omega : u(x) > 0\}$ and $\Omega^- = \{x \in \Omega : u \leq 0\}^\circ$.

We consider the one-phase free-boundary problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u), \\ u_\nu = 1 & \text{on } F(u) = (\partial\Omega^+(u)) \cap \Omega, \end{cases} \tag{2.3}$$

where u_ν denotes the inner normal derivative.

Definition 2.6 *Let u be a nonnegative continuous function in Ω. We say that u is a weak (viscosity) solution to (2.3) in Ω, if and only if the following conditions are satisfied:*

i. *$\Delta u = 0$ in $\Omega^+(u)$;*

ii. *If $x_0 \in F(u)$ and $F(u)$ has at x_0 a one-sided tangent ball (i.e. there exists B_ϵ such that $x_0 \in \partial B_\epsilon$ and B_ϵ is contained either in Ω^+ or in Ω^-), then, for ν the unit radial direction of ∂B_ϵ at x_0 into $\Omega^+(u)$,*

$$u(x) = (x - x_0, \nu)^+ + o(|x - x_0|), \quad \text{as } x \longrightarrow x_0.$$

We say that $x_0 \in F(u)$ is a regular point from the positive (resp. zero) side, if $F(u)$ has at x_0 a tangent ball from the positive (resp. zero) side.

Definition 2.7 *Let v be a nonnegative continuous function in Ω. We will say that v is a weak subsolution (resp. supersolution) to (2.3) in Ω, if and only if the following conditions are satisfied:*

i. $\Delta v \geq 0$ *(resp.* ≤ 0*) in* $\Omega^+(v)$*;*

ii. If $x_0 \in F(v)$ and $F(v)$ has at x_0 a tangent ball B_ϵ from the positive (resp. zero) side (i.e. $B_\epsilon \subset \Omega^+(v)$ (resp. $\Omega^-(v)$), $x_0 \in \partial B_\epsilon$), then, for some $\alpha \geq 1$ (resp. $\alpha \leq 1$) and ν the unit inner (resp. outer) radial direction of ∂B_ϵ at x_0,

$$v(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0.$$

We will say that u is a strict weak subsolution (resp. supersolution) if the constant α in Definition 2.7 is strictly greater (resp. smaller) than 1.

The following theorem in [C2] relates the two notions introduced above.

Theorem 2.8 *A minimizer u of $J(\cdot, \Omega)$ over $K(\phi)$, is a weak solution to (2.3).*

Next, we recall a comparison result for weak solution, which can be found in [C1].

Lemma 2.9 *Let v_ρ, $a \leq \rho \leq b$, be a family of weak subsolutions to (2.3) in Ω, continuous in $\overline{\Omega} \times [a, b]$. Let u be a weak solution to (2.3) in Ω, continuous in $\overline{\Omega}$. Assume that*

i. $v_a \leq u$ *in* Ω*;*

ii. $v_\rho \leq u$ *on* $\partial\Omega$*, and* $v_\rho < u$ *in* $[\overline{\Omega^+(v_\rho)} \cap \partial\Omega]$*, for all* $\rho \in [a, b]$*;*

iii. every $x_0 \in F(v_\rho)$ is regular from the positive side;

iv. $\overline{\Omega^+(v_\rho)}$ *is continuous (in the Hausdorff metric)in* ρ.

Then $u \geq v_\rho$ in Ω, for any ρ.

We will also use the following comparison result for a family of supersolutions, which can be proved by similar techniques as Lemma 2.9 (see [DJ]).

Lemma 2.10 *Let* Ω *be a smooth domain and let* u *be a weak solution to (2.3) in* Ω, *continuous in* $\overline{\Omega}$. *Assume that* u *satisfies the following condition:*

C. let $x_0 \in \overline{\Omega^+(u)} \cap \partial\Omega$, *and let* u *be identically* 0 *in a boundary neighborhood of* x_0. *Assume that* $x_0 \in \partial B$ *with* $B \subset \mathbb{R}^n \setminus \Omega^+(u)$, *then there exists* $\alpha \geq 1$ *such that*

$$u(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0.$$

with ν *outward unit normal at* ∂B.

Let w_ρ, $a \leq \rho \leq b$, *be a family of weak strict supersolutions to (2.3) in* $\mathbb{R}^n$, *continuous in* $\mathbb{R}^n \times [a, b]$. *Assume that,*

i. $u \leq w_a$ *in* Ω;

ii. $u \leq w_\rho$ *on* $\partial\Omega$ *for any* ρ, *and* $w_\rho(x_0) > 0$ *at each* $x_0 \in \overline{\Omega^+(u)} \cap \partial\Omega$ *such that* u *is not identically zero in any boundary neighborhood of* x_0;

iii. every $x_0 \in F(w_\rho)$ *is regular from the zero side;*

iv. $\overline{\Omega^+(w_\rho)}$ *is continuous (in the Hausdorff metric) in* ρ.

Then $u \leq w_\rho$ *in* Ω, *for any* ρ.

We conclude this section, by recalling two more notions. The first one is the definition of a variational solution to a free boundary problem, which we will also use occasionally.

Definition 2.11 *We define* $u \in H^1_{loc}(\Omega)$ *to be a variational solution to (2.3), if* $u \in C(\Omega) \cap C^2(\Omega^+(u))$ *and*

$$0 = -\frac{d}{d\epsilon} J(u(x + \epsilon\eta(x)))|_{\epsilon=0} = \int_\Omega (|\nabla u|^2 \mathrm{div}\eta - 2\nabla u D\eta \nabla u + \chi_{\{u>0\}} \mathrm{div}\eta)$$

for any $\eta \in C^1_0(\Omega, \mathbb{R}^n)$.

Finally, we recall the notion of NTA domain.

Let D be a bounded domain in $\mathbb{R}^n$. A M-non-tangential ball in a D, is a ball $B_r \subset D$, such that: $Mr > \operatorname{dist}(B_r, \partial D) > M^{-1}r$.

For $P_1, P_2 \in D$, a Harnack chain from P_1 to P_2 in D is a sequence of $M-$non-tangential balls, such that the first ball contains P_1, the last contains P_2, and such that consecutive balls intersect.

Definition 2.12 *A bounded domain D in $\mathbb{R}^n$ is called NTA, when there exist constants M and $r_0 > 0$ such that:*

i. Corkscrew condition. For any $Q \in \partial D$, $r < r_0$, there exists $A_r(Q) \in D$ such that $M^{-1}r < |A - Q| < r$ and dist$(A, \partial D) > M^{-1}r$;

ii. D^c satisfies the corkscrew condition;

iii. Harnack chain condition. If $\epsilon > 0$ and P_1, P_2 belong to D, dist$(P_j, \partial D) > \epsilon$ and $|P_1 - P_2| < C\epsilon$, then there exists a Harnack chain from P_1 to P_2 whose lenght depends on C, but not on ϵ.

2.2 Main results.

We start by demonstrating the existence of a singular energy minimizer for the functional J in high dimensions.

Let $t_n > 0$ be the unique constant such that the positive harmonic function Z in the cone $\Gamma = \left\{x \in R^n : |x_n| < t_n\sqrt{x_1^2 + ... + x_n^2}\right\}$ (unique up to scalar multiple) which is 0 on $\partial\Gamma$, is homogeneous of degree 1. Denote by Z_ν the inner normal derivative, which by symmetry is homogeneous of degree 0. Then, one can choose a scalar multiple c so that $cZ_\nu = 1$ on $\partial\Gamma\backslash\{0\}$. Let U be the function which equals cZ in Γ and 0 outside of Γ.

Now, let B be a ball centered at the origin, and let u minimize $J(\cdot, B)$, over $K(U)$. The existence of u is guaranteed by Theorem 2.1. The following result is contained in [DJ].

Theorem 2.13 *In dimension $n = 7$, $u = U$. In particular, U is a global energy minimizer for the functional J.*

We immediately deduce the following corollary.

Corollary 2.14 $k \leq 7$.

Our second main result concerns the existence of a free boundary $F(u)$, in a cylinder of $\mathbb{R}^n$, which is a smooth graph in the vertical direction, and it is trapped in between a subsolution and a supersolution. More precisely, we have the following:

Theorem 2.15 *Assume that, there exist a strict smooth subsolution V_1 and a strict smooth supersolution V_2 to (2.3) in $\mathbb{R}^n$, such that*

i. $V_1 \leq V_2$ *on* $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. $\lim_{x_n \to +\infty} V_1(x', x_n) = +\infty$, $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$.

Then, for each $R > 0, h_R$ sufficiently large, there exists u_R weak solution to (2.3) in $C_R = \mathcal{B}_R(0) \times \{|x_n| < h_R\}$, such that, u_R is monotone increasing in the x_n direction, and $V_1 \leq u_R \leq V_2$. Moreover, $F(u_R)$ is a Lipschitz graph in the x_n direction.

Remarks. 1) If $V_i, i = 1, 2$, satisfy the hypotheses of Theorem 2.15, then

$$\partial\{V_i > 0\} = \{(x', \phi_i(x')), x' \in \mathbb{R}^{n-1}\},$$

for ϕ_i smooth, $i = 1, 2$. Hence we can use the notations introduced in Section 2.1.1.
2) The results in [C1] and [KNS], imply that $F(u_R)$ is smooth on $C(R/2, h_R/2)$ (assuming $0 \in F(u_R)$).
3) The hypothesis $0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$, is assumed only for notational simplicity.

As observed in the introduction, Theorem 2.15, is the first step towards exhibiting a global free boundary which is a smooth non-planar graph in the vertical direction. In analogy with the theory of minimal surfaces, on the basis of Theorem 2.13, we expect that such a global solution exists in dimension $n \geq 8$.

Our global result is the following:

Theorem 2.16 *Assume that, there exist a strict smooth subsolution* V_1 *and a strict smooth supersolution* V_2 *to (2.3) in* $\mathbb{R}^n$*, such that*

i. $V_1 \leq V_2$ *on* $\mathbb{R}^n$*;*

ii. $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$*;*

iii. $\lim_{r \to \infty} \dfrac{V_1(rx)}{r} \geq U(x_1, ..., x_n)$.

Then, there exists $u \in C^{0,1}(\mathbb{R}^n)$*, such that* u *is a weak solution to (2.3) in* $\mathbb{R}^n$*, monotone increasing in the* x_n *direction, and* $F(u)$ *is a continuous non-planar graph, with a universal modulus of continuity. Moreover,* $F(u)$ *is locally NTA.*

Comment. Hypothesis (iii) is used to prevent $F(u)$ from being planar. While we could weaken this assumption, its motivation lies in the fact that, in analogy with the minimal surfaces theory, we expect a smooth non-affine free boundary graph u to blow down to an energy minimizing solution. Indeed, we aim to construct functions V_1 and V_2 in $\mathbb{R}^n$, with the property that their blow down is U, which in dimension $n \geq 7$ is an energy minimizer.

The thesis is organized as follows.

In Chapter 3, we provide the proof of Theorem 2.13. This result will be obtained as a consequence of the following theorems, together with the deformation lemmas, Lemma 2.9, and Lemma 2.10.

Theorem 2.17 *In dimension* $n = 7$*, there exists a family* $\{V_\rho\}$*,* $\rho \geq a$*, of weak strict subsolutions to (2.3) in* B*, such that* u *and* V_ρ *satisfy the hypotheses of lemma 2.9. Moreover* V_ρ *converges to* U *on* B*, as* $\rho \longrightarrow +\infty$.

Theorem 2.18 *In dimension* $n = 7$*, there exists a family* $\{W_\rho\}$*,* $\rho \geq a$*, of weak strict supersolutions to (2.3) in* $\mathbb{R}^n$*, such that* u *and* W_ρ *satisfy the hypotheses of lemma 2.10. Moreover* W_ρ *converges to* U *on* B*, as* $\rho \longrightarrow +\infty$.

In Chapter 4, we prove a local existence result for monotone weak solutions . More precisely, we demonstrate the following theorem.

Theorem 2.19 *Assume that, there exist a strict smooth subsolution* V_1 *and a strict smooth supersolution* V_2 *to (2.3) in* $\mathbb{R}^n$*, such that*

i. $V_1 \leq V_2$ *on* $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$*;*

ii. $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$*, for* $i = 1, 2$.

Then, for each $R > 0$*, there exists* u_R *weak solution to (2.3) in* C_R*, such that,* u_R *is monotone increasing in the* x_n *direction, and* $V_1 \leq u_R \leq V_2$.

The techniques used to prove Theorem 2.19, will also yield certain regularity properties of u_R, and $F(u_R)$.

Let u be a non-negative function defined on Ω. Set,

$$d(x) = \operatorname{dist}(x, F(u)).$$

Definition 2.20 *We say that u is non-degenerate, if and only if, for every* $G \Subset \Omega$*, there exists a constant* $K = K(G)$ *such that*

$$u(x) \geq K d(x),$$

for all $x \in G^+(u)$*, with* $B_{d(x)}(x) \subset G$.

Definition 2.21 *We say that u is (I) non-degenerate, if and only if, for every* $G \Subset \Omega$*, there exists a constant* $K = K(G)$ *such that, for any ball* $B_r \subset G$ *centered at a free boundary point,*

$$\int_{B_r} u_R \geq K r |B_r|.$$

Definition 2.22 *We say that* $F(u)$ *satisfies the density property (D) if and only if:*

D. for any $G \Subset \Omega$*, there exists a constant* $c = c(G) < 1$*, such that, for any ball* $B_r \subset G$ *centered at a free boundary point,*

$$c \leq \frac{|B_r \cap \{u > 0\}|}{|B_r|} \leq 1 - c.$$

Proposition 2.23 *u_R satisfies the following:*

a. u_R is Lipschitz continuous on C_R, with universal Lipschitz constant on each $G \Subset G' \Subset C_R$;

b. u_R is non-degenerate, (I) non-degenerate, with local universal constants;

c. u_R satisfies the density property (D), with universal constants on any $G \Subset G' \Subset C_R$.

From hereafter, whenever the assumptions of Theorem 2.19 are satisfied, we will denote by u_R a weak solution to (2.3) in C_R, which satisfies Proposition 2.23. Its existence is guaranteed by Theorem 2.19.

In Chapter 5, we prove a local regularity result. It guarantees that $F(u_R)$ is a smooth graph in the vertical direction, but it does not provide a uniform control on the smoothness, independent of R. More precisely, we show the following:

Theorem 2.24 *Assume that, there exist a strict smooth subsolution V_1 and a strict smooth supersolution V_2 to (2.3) in $\mathbb{R}^n$, such that*

i. $V_1 \leq V_2$ on $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. $\lim_{x_n \to +\infty} V_1(x', x_n) = +\infty$, $\partial_n V_i > 0$ in $\overline{\{V_i > 0\}}$, for $i = 1, 2$.

Then, for each $R > 0$, and h_R, sufficiently large, $F(u_R)$ is a Lipschitz continuous graph in the x_n direction.

Finally, in Chapter 6, we prove Theorem 2.16. A limiting argument, together with local existence and regularity, imply the existence of a global weak solution u, which is monotone increasing in the x_n direction. In order to prove that $F(u)$ does not contain vertical segments, we use the following regularity result, that we obtain using the same techniques as in [ACS].

Theorem 2.25 *Let V_1, V_2 be non-negative functions on $\mathbb{R}^n$, such that* *i.* $V_1 \leq V_2$ *on* $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$*;*

ii. V_i *is smooth,* $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$.

Let u *be a weak solution to (2.3) in* $\mathbb{R}^n$*, such that* $V_1 \leq u \leq V_2$*,* u *is locally Lipschitz continuous and nondegenerate, and* u *satisfies the density property (D). Then,* $F(u)$ *is locally NTA.*

Chapter 3

Existence of a singular minimizer

3.1 Construction of a subsolution

Proof of Theorem 2.17. We start by constructing a subsolution V to (2.3) in $\mathbb{R}^n$, whose positive phase is contained in the set Γ defined in section 2.2. We will obtain V as a homogeneous harmonic perturbation of the function Z defined in that same section.

First, we need to determine the normalizing constant c, such that $cZ_\nu = 1$ on $\partial\Gamma\backslash\{0\}$. Consider $f_n(t)$ the (unique up to scalar multiple) nonzero even function of t, satisfying the Legendre equation

$$(1-t^2)f_n''(t) + (1-n)tf_n'(t) + (n-1)f_n(t) = 0, \quad -1 < t < 1. \tag{3.1}$$

Let t_n be its smallest positive zero, and assume that f_n is positive on the open interval $(-t_n, t_n)$. Then,

$$Z(x) = |x| f_n\left(\frac{x_n}{|x|}\right)$$

We need to compute $|\nabla Z|^2$ on $\partial\Gamma\backslash\{0\}$. We have:

$$\partial_{x_i} Z(x) = \frac{x_i}{|x|} f_n\left(\frac{x_n}{|x|}\right) + |x| f_n'\left(\frac{x_n}{|x|}\right) \cdot \left(\frac{\delta_{i,n}|x| - x_n x_i |x|^{-1}}{|x|^2}\right), i = 1, \ldots n,$$

where $\delta_{i,j}$ is the Kronecker symbol.

Therefore,

$$|\nabla Z|^2 \equiv (1-t_n^2)(f_n'(t_n))^2 \quad \text{on } \partial\Gamma \setminus \{0\}.$$

Set $c_n = (1-t_n^2)(f_n'(t_n))^2$, then the desired constant c equals $1/\sqrt{c_n}$.

Define

$$V(x) = \frac{1}{\sqrt{c_n}}\left(Z(x) - |x|^{\alpha_n} g_{\alpha_n}\left(\frac{x_n}{|x|}\right)\right) \tag{3.2}$$

where $\alpha_n < 1$ is a parameter to be chosen later, and g_{α_n} is the (unique up to scalar multiple) positive and even function of t, satisfying the Legendre equation:

$$(1-t^2)g_{\alpha_n}''(t) + (1-n)tg_{\alpha_n}'(t) + \alpha_n(\alpha_n+n-2)g_{\alpha_n}(t) = 0, \quad \text{on } (-1,1). \tag{3.3}$$

The function $Y(x) = |x|^{\alpha_n} g_{\alpha_n}\left(\frac{x_n}{|x|}\right)$ is a positive harmonic function in the cone Γ such that Y is a constant multiple of $|x|^{\alpha_n}$ on $\partial\Gamma$ and Y is homogeneous of degree α_n. Thus the function V in (3.2) is harmonic in the cone Γ. We want to show that:

$$V_\nu \geq 1 \quad \text{on} \quad \partial\{V>0\}.$$

Then, we can conclude that V^+ is a weak subsolution to problem (2.3) in $\mathbb{R}^n$.

Toward this aim, let us compute $|\nabla V|^2$. For simplicity, we denote $\alpha = \alpha_n$.

We have, for $i = 1, ..., n$:

$$\sqrt{c_n}\partial_{x_i}V(x) = \partial_{x_i}Z - \alpha|x|^{\alpha-2}x_i g_\alpha\left(\frac{x_n}{|x|}\right) - |x|^\alpha g_\alpha'\left(\frac{x_n}{|x|}\right)\cdot\left(\frac{\delta_{i,n}|x| - x_n x_i|x|^{-1}}{|x|^2}\right).$$

Hence,

$$c_n|\nabla V|^2(x) = \left[f_n\left(\frac{x_n}{|x|}\right) - \alpha|x|^{\alpha-1}g_\alpha\left(\frac{x_n}{|x|}\right)\right]^2 + \left(1-\frac{x_n^2}{|x|^2}\right)\left[f_n'\left(\frac{x_n}{|x|}\right) - |x|^{\alpha-1}g_\alpha'\left(\frac{x_n}{|x|}\right)\right]^2$$

from which we deduce the following formula, for x on $\partial\{V>0\}$

$$|\nabla V|^2(x) = \frac{1}{c_n}G_n(\alpha, x_n/|x|)$$

where

$$G_n(\alpha,t) = (1-\alpha)^2 f_n^2(t) + (1-t^2)\left(f_n'(t) - \frac{f_n(t)}{g_\alpha(t)}g_\alpha'(t)\right)^2. \tag{3.4}$$

Whenever this does not create confusion, we will write $G_n(t)$ for $G_n(\alpha, t)$.

In order to conclude that V^+ is a weak subsolution, we have to verify that $|\nabla V|^2 \geq 1$ on $\partial\{V > 0\}$. By definition, $f_n(t_n) = 0$, hence $G_n(t_n) = c_n$. Therefore, the statement $|\nabla V|^2 \geq 1$ on $\partial\{V > 0\}$ is equivalent to requiring that G_n achieves its absolute minimum on $[-t_n, t_n]$ at the boundary points.

Recall that f_n, g_α are even functions of t, hence G_n is also an even function of t. Therefore, 0 is either a local minimum or a local maximum of G_n. In particular, computing $G_n''(0)$, using the properties of f_n and g_α, one gets:

$$\begin{cases} (\alpha+n-1)^2 > n-1 & 0 \text{ is a minimum point,} \\ (\alpha+n-1)^2 < n-1 & 0 \text{ is a maximum point.} \end{cases} \tag{3.5}$$

We study the behavior of the function G_n, for different values of n, and various choices of the parameter α in the ranges above. It turns out that G_n has two other interior critical points. By imposing 0 to be a maximum point, we force G_n to achieve its absolute minimum, smaller than c_n, at those two points. Instead, choosing values of α for which G_n attains a minimum at 0 and $G_n(0) \geq G_n(t_n)$, we preserve the fact that G_n attains a global minimum at the boundary points, at least in high dimensions. In particular, for α such that $G_n(0) = G_n(t_n)$, we can show that G_n attains its absolute minimum on the boundary, for various values of $n \geq 7$. For the numerical computations involved in analysis described above, we have used Mathematica. Let us describe the details for the case n odd and for α such that $G_n(0) = G_n(t_n)$.

In what follows, we will use the notations from [E]. Set $\nu = (n-1)/2$, and $\mu = (n-3)/2$. We have:

$$f_n(t) = (1-t^2)^{-\frac{n-3}{4}} P_\nu^\mu(t).$$

Some numerical values of t_n are listed below:

$t_3 = 0.833557$

$t_5 = 0.623175$

$t_7 = 0.517331$

$t_9 = 0.451615$

$t_{11} = 0.405841.$

In order to compute efficiently the parameter α, we derive an explicit formula for f_n'. Formulas for the derivatives of Legendre functions can be found in [E]. For convenience of the reader we report the two formulas which we have used in this context:

$$\begin{cases} \dfrac{d}{dt}P_\nu^\mu(t) = -\mu t(1-t^2)^{-1}P_\nu^\mu(t) - (1-t^2)^{-\frac{1}{2}}P_\nu^{\mu+1} \\ \\ \dfrac{d}{dt}P_\nu^\mu(t) = \mu t(1-t^2)^{-1}P_\nu^\mu(t) + (\nu+\mu)(\nu-\mu+1)(1-t^2)^{-\frac{1}{2}}P_\nu^{\mu-1}(t). \end{cases} \tag{3.6}$$

Thus, we compute that:

$$\alpha = \alpha_n = 1 - (1-t_n^2)^{-\frac{n-3}{4}}P_\nu^\nu(t_n)/P_\nu^\mu(0).$$

Some numerical values of α are reported below.

$\alpha_3 = -1.71506$

$\alpha_5 = -2.35453$

$\alpha_7 = -3.21122$

$\alpha_9 = -3.91985$

$\alpha_{11} = -4.5382.$

For such values of α, the function g_α has the following representation:

$$g_\alpha(t) = \frac{1}{2}(1-t^2)^{-\frac{n-3}{4}}\left(P_{\alpha+\mu}^{-\mu}(t) + P_{\alpha+\mu}^{-\mu}(-t)\right).$$

Again for efficiency purposes, we use the formula above, together with the formulas in (3.6), to obtain the following explicit formula for G_n:

$$G_n(t) = (1-t^2)^{-\frac{n-3}{2}}\{[P_\nu^\nu(t) + \alpha(\alpha+n-2)\cdot P_\nu^\mu(t)Z_n(t)E_n(t)]^2 \\ + (1-\alpha)^2(P_\nu^\mu(t))^2\}$$

where

$$Z_n(t) = P_{\alpha+\mu}^{-\nu}(t) - P_{\alpha+\mu}^{-\nu}(-t),$$

and

$$E_n(t) = (P_{\alpha+\mu}^{-\mu}(t) - P_{\alpha+\mu}^{-\mu}(-t))^{-1}.$$

Plotting the graph of the function G_n for $n = 7$, we observe that G_n attains its absolute interior minimum at $t = 0$. To prevent numerical errors, we choose a parameter α which is slightly smaller than the one reported above, so that $|\nabla V|^2 > 1$ on $\partial\{V > 0\}$. This does not alter the representation formula for g_α.

Finally, we are ready to exhibit the family of subsolutions $\{V_\rho\}$, in the statement of the theorem. Define,

$$V_\rho(x) = \frac{1}{\rho}V^+(\rho x) = \frac{1}{\sqrt{c_n}}\left(Z(x) - \rho^{\alpha-1}Y(x)\right)^+.$$

V_ρ preserves the subsolution properties of V. Hence V_ρ is a continuous (in ρ) family of subsolutions on any compact interval $[a,b]$, $0 < a < b$. Furthermore, $\inf_B Y$ is positive, therefore we can choose $a > 0$ small enough so that $V_a^+ \equiv 0$ in B.

Then, it is readily seen that u and V_ρ satisfy the hypotheses in Lemma (2.9). Moreover V_ρ converges to U on B, as $\rho \to +\infty$. □

Remark 3.1 *Although we stated Theorem 2.17 for $n = 7$, the calculations above have been carried out for $7 \leq n \leq 20$, which also shows the stability of our method. In the case n even, we proceed as for the case n odd, but using the Legendre function Q.*

3.2 Construction of a supersolution

PROOF OF THEOREM 2.18. We will construct a weak, strict supersolution W to (2.3) in $\mathbb{R}^n$, whose positive phase contains the set Γ. We start by performing a change of variables.

Let $W(x) = w(x_n, |x'|)$, with $w(s,r)$ even function on $\mathbb{R}^2$. Then, W is a weak strict supersolution to (2.3) in $\mathbb{R}^n$, if and only if w solves the following one-phase free boundary problem:

$$\begin{cases} \mathcal{L}w = \dfrac{\partial^2}{\partial s^2}w + \dfrac{\partial^2}{\partial r^2}w + \dfrac{(n-2)}{r}\dfrac{\partial}{\partial r}w \le 0 & \text{on } \{w > 0\}, \\ \\ |\nabla w|^2 < 1 & \text{on } \partial\{w > 0\}. \end{cases} \tag{3.7}$$

In the new coordinate system, the function Z in section 3.1, is given by $Z(x) = z(x_n, |x'|)$, where

$$z(s,r) = \sqrt{s^2+r^2} f_n\left(\frac{s}{\sqrt{s^2+r^2}}\right),$$

while Γ is described by:

$$\Gamma = \{(s,r) \in \mathbb{R}^2 : d_n|s| < |r|\}, \quad d_n = \sqrt{1-t_n^2}/t_n.$$

We proceed to construct W piecewise.

Step 1. For $0 < \beta_n < d_n$, consider the cone $\Gamma' = \{(s,r) \in \mathbb{R}^2 : \beta_n|s| < |r|\}$ and set $\gamma_n = 1/\sqrt{1+\beta_n^2}$. β_n will be chosen later, so that $f_n(\gamma_n) < 0$, where we recall that f_n is defined in (3.1). Define

$$k(s,r) = (s^2+r^2)^{\frac{\tau_n}{2}} g_{\tau_n}\left(\frac{s}{\sqrt{s^2+r^2}}\right),$$

with $\tau_n < 1$, and g_{τ_n} such that:

$$(1-t^2)g''_{\tau_n}(t) - (n-1)tg'_{\tau_n}(t) + \tau_n(\tau_n+n-2)g_{\tau_n}(t) = 0 \quad \text{on } (-1,1), \tag{3.8}$$

g_{τ_n} even, and g_{τ_n} strictly positive on the interval $[-\gamma_n, \gamma_n]$. For simplicity, denote $\tau = \tau_n$. Now, set

$$w_1(s,r) = \frac{1}{\sqrt{c_n}}\{z(s,r) + k(s,r)\}$$

with c_n the normalizing constant from section 3.1. Then equations (3.1) and (3.8) imply that $\mathcal{L}w_1 = 0$ in Γ'. We aim to choose τ and γ_n so that w_1 is a weak strict supersolution to (3.7) away from the singular axis $r = 0$, which can then be extended to a supersolution in the whole plane.

The level set $\{w_1 = 0\}$ intersects $\partial\Gamma'$ at the points $(\pm\bar{s}, \pm\bar{r})$, where

$$\begin{cases} \bar{s} = \gamma_n \left(-\frac{g_\tau(\gamma_n)}{f_n(\gamma_n)}\right)^{\frac{1}{1-\tau}}, \\ \bar{r} = \beta_n \bar{s}. \end{cases} \tag{3.9}$$

Let $A(\gamma_n)$ be the slope of the level curve $w_1(s,r) = 0$ at the point $(\bar{s}, \bar{r})$. We will choose β_n so that $\beta_n - A(\gamma_n) < 0$, which guarantees that for $|r| \geq \bar{r}$, the level set $\{w_1 = 0\}$ is contained in $\overline{\Gamma'} \setminus \overline{\Gamma}$.

Let us denote by $\Omega = \Gamma' \cap \{|r| > \bar{r}\}$. The same computations as in section 3.1, show that on $\partial\{w_1 > 0\} \cap \overline{\Omega}$

$$c_n |\nabla w_1|^2(s,r) = G_n(\tau, s/\sqrt{s^2 + r^2})$$

where we recall that :

$$G_n(\tau, t) = (1-\tau)^2 f_n(t)^2 + (1 - t^2)\left[f_n'(t) - \frac{f_n(t) g_\tau'(t)}{g_\tau(t)}\right]^2.$$

Then, the strict free boundary condition is satisfied if $G_n(t) < c_n$ on $(t_n, \gamma_n]$. Since $c_n = G_n(t_n)$, we are requiring G_n to decrease in a right neighborhood of t_n. As before, for various choices of n and τ, we can write an explicit formula for G_n in Mathematica's language. We can then examine the behavior of G_n near t_n.

Remark. The homogeneity parameter α, used to construct the subsolution V, is not the correct choice for τ. Even if, for $n \geq 7$, it forces G_n to decrease in a neighborhood of t_n, it does not cooperate when linking this supersolution to a supersolution near the origin. A different approach that we have taken in higher dimensional cases (see [DJ]), leads to the right choice of parameter τ.

Step 2. Let us set $h(s) = w_1(s, \overline{r})$, and define:

$$w_2(s,r) = y(r)h\left(\frac{s}{v(r)}\right),$$

where the functions $y(r)$ and $v(r)$ will be chosen positive and even on the real line. Moreover, they must satisfy

$$y(\overline{r}) = v(\overline{r}) = 1 \tag{3.10}$$

and

$$v'(\overline{r}) = \frac{1}{\overline{s}A(\gamma_n)}. \tag{3.11}$$

The latter condition guarantees that the level curves $w_1(s,r) = 0$ and $w_2(s,r) = 0$ have the same slope at the points $(\pm\overline{s}, \pm\overline{r})$. Define

$$w(s,r) = \begin{cases} w_1(s,r) & \text{in } \overline{\Omega}, \\ w_2(s,r) & \text{in } \{|r| \le \overline{r}, |r| \le \overline{s}v(r)\}. \end{cases}$$

We will prove that we can find $y(r), v(r)$ so that $w^+(s,r)$, extended to zero outside its positive phase, is the desired weak strict supersolution to (3.7) and, by construction $\Gamma \subset \{w > 0\}$. Toward this aim we need to verify that:

$$\mathcal{L}w_1 \le 0 \quad \text{in } \Omega^+(w_1) \tag{3.12}$$

$$|\nabla w_1|^2 < 1 \quad \text{on } \partial\{w_1 > 0\} \cap \overline{\Omega} \tag{3.13}$$

$$\mathcal{L}w_2 \le 0 \quad \text{in } \{|r| < \overline{r}, |s| < \overline{s}v(r)\}, \tag{3.14}$$

$$|\nabla w_2|^2 < 1 \quad \text{on } \{|s| = \overline{s}v(r), |r| \le \overline{r}\}, \tag{3.15}$$

$$\frac{\partial}{\partial r}w_1(s,r)|_{\overline{r}} < \frac{\partial}{\partial r}w_2(s,r)|_{\overline{r}} \quad \text{on } (-\overline{s}, \overline{s}). \tag{3.16}$$

We remark that condition (3.16) is needed to guarantee that the piecewise function

w is a supersolution in $\{w > 0\}$, across $\{|r| = \overline{r}\}$.

Condition (3.12) follows already from Step 1. The free boundary condition (3.15) will be obtained as a consequence of condition (3.13). Indeed, we choose $y(r)$ so that $|\nabla w_2|^2|_{s=\overline{s}v(r)} = |\nabla w_1|^2|_{(\overline{s},\overline{r})}$, that is

$$y(r) = \sqrt{\left(1 + \frac{1}{A(\gamma_n)^2}\right)\frac{v(r)}{\sqrt{\overline{s}^2 v'(r)^2 + 1}}}. \tag{3.17}$$

Now we are left with the choice of $v(r)$. Let us compute

$$\begin{aligned}\mathcal{L}w_2(s,r) = \frac{y(r)}{v(r)^2}h''\left(\frac{s}{v(r)}\right)\left\{1 + s^2\frac{v'(r)^2}{v(r)^2}\right\}\\ - \frac{sy(r)}{v(r)}h'\left(\frac{s}{v(r)}\right)M(r) + y(r)h\left(\frac{s}{v(r)}\right)N(r)\end{aligned}$$

where

$$M(r) = \left\{2\frac{y'(r)}{y(r)}\frac{v'(r)}{v(r)} + \frac{v''(r)}{v(r)} - 2\frac{v'(r)^2}{v(r)^2} + \frac{n-2}{r}\frac{v'(r)}{v(r)}\right\}$$

and

$$N(r) = \left\{\frac{y''(r)}{y(r)} + \frac{n-2}{r}\frac{y'(r)}{y(r)}\right\}$$

and we recall that $y(r)$ is expressed as a function of $v(r)$.

In order to determine $v(r)$ so that $\mathcal{L}w_2 \leq 0$, we first study the behavior of the first and second derivative of h. Recall that $h(s) = w_1(s, \overline{r})$, and the formula for w_1 involves the functions f_n and g_τ for which we can compute explicit representation formulas in Mathematica, when varying n and τ.

In the case $n = 7$, we have observed that, for various choices of the parameters τ_7 and γ_7, the functions $\xi h'(\xi)$ and $h''(\xi)$ are both non positive on $[-\overline{s}, \overline{s}]$.

Therefore, if $v(r) \leq 1$, $M(r)$ and $N(r)$ are both positive on $[-\overline{r}, \overline{r}]$, then $\mathcal{L}w_2$ is majorized by

$$K(s,r) = \frac{y(r)}{v(r)^2}\{h''(s/v(r)) - s/v(r)h'(s/v(r))\overline{M} + h(s/v(r))\overline{N}\}$$

with $\overline{M}$ and $\overline{N}$ the maximum values of M and G respectively. We therefore aim to determine $v(r)$ so that,

$$\begin{cases} v(r) \leq 1 & \text{on } [-\overline{r}, \overline{r}] \\ M(r), N(r) \geq 0 & \text{on } [-\overline{r}, \overline{r}] \\ \overline{K}(\xi) = h''(\xi) - \xi h'(\xi)\overline{M} + h(\xi)\overline{N}) \leq 0 & \text{on } [-\overline{s}, \overline{s}] \end{cases}$$

Let $v(r)$ be an even polynomial. Since $v(r)$ must satisfy conditions (3.10) and (3.11), we will assume that $v(r)$ is a fourth degree polynomial so to have a one parameter dependence. Precisely, set

$$v(r) = a_7 r^4 + b_7 r^2 + r_7.$$

We need to determine a_7 so that all of the above are satisfied. In particular, since M and N are both even, we will look for a_7 such that $\overline{M} = M(0)$ and $\overline{N} = N(0)$. Moreover, we also need (3.16) to hold. For efficiency purposes, we have computed explicit formulas for all the derivatives involved, again in function of τ_7 and γ_7. All the required conditions then translate in a set of non linear inequalities which has to be satisfied by a_7. Our purpose is to choose parameters τ_7 and γ_7, compatible with the free boundary condition (3.13), and so that such an a_7 exists. We report the specific numerical values, for which the method described above succeeds. For simplicity, we do not report the formulas in the Mathematica language, for all the functions involved. The main formulas for f_7 and g_{τ_7} are the same as in the previous section. We have,

$\tau_7 = -1.76$

$\gamma_7 = 0.6238$

which yield

$\overline{s} = 0.408906$

$\overline{r} = 0.512334$.

The correspondent value for a_7 is then

$a_7 = -0.3664026$

which implies

$b_7 = 0.717431$

$r_7 = 0.836929.$

Remark. For the reported values, $\mathcal{L}w_2 < 0$, which is necessary to prevent numerical errors.

Finally, we can conclude similarly to the subsolution case, by defining,

$$W_\rho(x) = \frac{1}{\rho} W^+(\rho x).$$

It remains to be shown, that u satisfies condition (C) from Lemma 2.10. This is proved in [DJ]. The techniques used there, will be widely used in the next chapters, therefore we refer the reader to [DJ], for details of the proof.

Chapter 4

Local theory of monotone weak solutions

4.1 Existence

4.1.1 Monotone rearrangements

In this chapter, we will prove Theorem 2.19. We start by introducing the notion of monotone rearrangement.

Let $D \subset \mathbb{R}^n$ be a compact set. For each $x' \in \mathbb{R}^{n-1}$ we introduce the notation

$$D(x') = D \cap \{(x', x_n) | x_n \in \mathbb{R}\}.$$

Assume that D is convex in x_n, i.e. for each $x' \in \mathbb{R}^{n-1}$, $D(x')$ is either empty, or consists of a single closed interval. For a given $b \geq 0$, we define

$$D^*(x') := \begin{cases} \{(x', x_n) \in \mathbb{R}^n | -b \leq x_n \leq |D(x')| - b\} & \text{if } D(x') \neq \emptyset, \\ \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$D^* := \bigcup_{x' \in D'} D^*(x'),$$

where $D' \subset \mathbb{R}^{n-1}$ is the set of those $x' \in \mathbb{R}^{n-1}$ for which $D(x')$ is non empty. Let $C(a,b)$, $a, b > 0$, be a cylinder in $\mathbb{R}^n$, and let u be a Lebesgue measurable function on $\overline{C(a,b)}$. We define the monotone (decreasing) rearrangement u^* of u, in the direction x_n, by the following formula:

$$u^*(x) := \sup\{k \in \mathbb{R} | x \in (C(a,b)_k)^*\}$$

for $x \in \overline{C(a,b)}$. Here we are using the notation $\Omega_k := \{x \in \Omega | u(x) \geq k\}$, for any function u defined on a measurable subset Ω of $\mathbb{R}^n$. The function u^* is monotone decreasing in the x_n-direction, and u and u^* are equimeasurable, that is, for all $k \in \mathbb{R}$

$$|\{u \geq k\}| = |\{u^* \geq k\}|.$$

Moreover, the mapping $u \to u^*$ is order preserving, i.e, $u \leq v$ implies $u^* \leq v^*$. One can define similarly the concept of monotone increasing rearrangement, which we will still denote by u^*. The following proposition holds (see [K]).

Proposition 4.1 *Let* $u \in W^{1,p}(C(a,b))$, $1 < p < \infty$. *Then* $u^* \in W^{1,p}(C(a,b))$ *and we have*

$$\int_{C(a,b)} |\nabla u|^p dx \geq \int_{C(a,b)} |\nabla u^*|^p dx \tag{4.1}$$

4.1.2 Existence of local monotone minimizers

Consider the energy functional $J(\cdot, \Omega)$, introduced in Chapter 2. Let V_1, V_2 be functions on $\mathbb{R}^n$, such that

i. $0 \leq V_1 \leq V_2$, $0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. V_i is smooth and $\partial_n V_i > 0$ on $\overline{\{V_i > 0\}}$, i=1, 2.

Recall that, we are denoting by

$$C_R = \mathcal{B}_R(0) \times \{|x_n| < h_R\},$$

where

$$h_R \geq \max\{2d_R(V_1), 2d_R(V_2), R\},$$

and $d_R(V)$ is defined in section 2.1.1.

For $\Omega = C_R$, we set $J(\cdot, C_R) = J_R(\cdot)$. Denote by K_R the following closed and convex subset of $H^1(C_R)$:

$$K_R := \{v \in H^1(C_R) | V_1 \leq v \leq V_2 \ \text{ a.e on } C_R, v = V_2 \ \text{ on } S_R\},$$

where $S_R := \partial\mathcal{B}_R(0) \times \{|x_n| \leq h_R\}$.

The following existence theorem holds.

Theorem 4.2 *There exists an absolute minimum $u_R \in K_R$ of the functional J_R, which is monotone increasing in the x_n-direction.*

PROOF. Since J_R is non-negative, there exists a minimizing sequence u_m, that is

$$u_m \in K_R, \ J_R(u_m) \to \alpha \equiv \inf_{v \in K_R} J_R(v), \ 0 \leq \alpha \leq J_R(V_2) < \infty.$$

The sequence $\{u_m\}$ is uniformly bounded in $H^1(C_R)$. Indeed,

$$||\nabla u_m||_2^2 \leq J(V_2), \ \ ||u_m||_2 \leq ||V_2||_2.$$

Therefore, we can extract a subsequence, which we will still denote by $\{u_m\}$, such that $u_m \to u \in K_R$, weakly in $H^1(C_R)$. We will show that J_R is lower semicontinuous, with respect to weak H^1 convergence, that is,

$$\liminf_{m \to \infty} J_R(u_m) \geq J_R(u).$$

Indeed,

$$\int_{C_R} |\nabla u_m|^2 \geq \int_{C_R} |\nabla u|^2 + 2\int_{C_R} \nabla(u_m - u) \cdot \nabla u,$$

and the right hand side tends to 0, for $m \to \infty$. Moreover, for each $\epsilon > 0$, up to extracting a subsequence,

$$\begin{aligned} u_m &\to u, \quad \text{a.e. on } C_R \\ u_m &\to u, \quad \text{uniformly on } (C_R \setminus W), \text{ with } |W| < \epsilon. \end{aligned}$$

Thus, for m large, we have

$$\int_{C_R} \chi_{\{u_m>0\}} \geq \int_{C_R\setminus W} \chi_{\{u>\epsilon\}} \geq \int_{C_R} \chi_{\{u>\epsilon\}} - \epsilon,$$

hence

$$\liminf_{m\to\infty} \int_{C_R} \chi_{\{u_m>0\}} \geq \int_{C_R} \chi_{\{u>0\}}.$$

This immediately implies that u is a minimizer for J_R over K_R. Now, let u^* be the monotone increasing rearrangement of u. Then, using Proposition 4.1, together with the equimeasurability of rearrangements, we get that

$$J_R(u^*) \leq J_R(u).$$

Moreover, the order preserving property implies that $u^* \in K_R$. Hence u^* is the desired minimizer, monotone increasing in the x_n direction. □

Given the functions V_1 and V_2, we will henceforth denote by u_R, a minimizer of J_R over K_R, which is monotone increasing in the x_n direction. In the following sections we will prove that u_R satisfies properties (i) and (ii) of Definition 2.6, as long as V_1 and V_2 are respectively a strict subsolution, and a strict supersolution to (2.3) in $\mathbb{R}^n$. Thus, u_R is the desired weak solution of Theorem 2.19.

4.2 Harmonicity and Interior regularity

4.2.1 Continuity and Harmonicity

We start by proving that, under the additional hypothesis that V_1 is a strict subsolution, and V_2 is a strict supersolution, u_R satisfies condition (i) in Definition 2.6. Toward this aim, we will need the following comparison result (see [AH]).

Lemma 4.3 *Let* $f, g \in H^1(\Omega)$, Ω *open bounded subset of* $\mathbb{R}^n$, *and let* $\widetilde{\Omega} \Subset \Omega$. *Assume that* $0 \le f \le g$ *a.e. in* Ω, *and* $g \in H^1_0(\widetilde{\Omega})$, *then* $f \in H^1_0(\widetilde{\Omega})$.

Lemma 4.4 *Assume that, there exist a strict smooth subsolution* V_1 *and a strict smooth supersolution* V_2 *to (2.3) in* $\mathbb{R}^n$, *such that*

i. $V_1 \le V_2$ *on* $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$.

Then, u_R *is continuous in* $C^+_R(V_2)$, *and harmonic in* $C^+_R(u_R)$.

Remark 4.5 *Under the assumptions of Lemma 4.4,* $V_1 < V_2$ *on* $\{V_2 > 0\}$, *and* $F(V_1) \cap F(V_2) = \emptyset$.

PROOF. Assume that there exists $x \in \{V_2 > 0\}$ such that, $V_1(x) = V_2(x)$. Then, since V_1 is subharmonic in $\{V_2 > 0\}$, the maximum principle implies $V_1 \equiv V_2$, which contradicts the fact that V_1 is a strict subsolution and V_2 is a strict supersolution. Analogously, suppose $x \in F(V_1) \cap F(V_2)$, and let $B \subset \{V_1 > 0\}$ be a ball tangent to $F(V_1)$ at x. Then, by Hopf's lemma, $\partial_\nu(V_1 - V_2) < 0$, with ν inner normal derivative to ∂B at x. Again, this contradicts the fact that V_1 is a strict subsolution, and V_2 is a strict supersolution. □

PROOF OF LEMMA 4.4. Let D be a compact subset of $C^+_R(V_2)$, and let B_ρ be a ball of radius ρ in D. Denote by v_ρ the harmonic replacement of u_R on B_ρ, that is the harmonic function in B_ρ which equals u_R on ∂B_ρ. Assume that v_ρ is extended to be u_R outside B_ρ. Since $0 \le u_R \le V_2$ a.e., we have $0 \le (v_\rho - V_2)^+ \le (v_\rho - u_R)^+$. Hence, Lemma 4.3 implies that $(v_\rho - V_2)^+ \in H_0^1(B_\rho)$. Therefore, by the weak maximum principle (see [GT]) we obtain $v_\rho \le V_2$ a.e. on B_ρ. Analogously, we get $V_1 \le v_\rho$ a.e. on B_ρ. Since u_R minimizes $J(\cdot, B_\rho)$ among all competitors v, such that $V_1 \le v \le V_2$, and $v = u_R$ on ∂B_ρ, we get that

$$\int_{B_\rho} (|\nabla u_R|^2 + \chi_{\{u_R>0\}}) \le \int_{B_\rho} (|\nabla v_\rho|^2 + \chi_{\{v_\rho>0\}}).$$

Therefore,

$$\int_{B_\rho} (|\nabla u_R|^2 - |\nabla v_\rho|^2) \le K\rho^n.$$

Here and henceforth, K denotes any dimensional constant.

Since v_ρ is harmonic in B_ρ, it follows that

$$\int_{B_\rho} |\nabla(u_R - v_\rho)|^2 \le \int_{B_\rho} (|\nabla u_R|^2 - |\nabla v_\rho|^2) \le K\rho^n.$$

Analogously, for any $r \le \rho$, let v_r be the harmonic replacement of u_R on B_r. Thus, $\int_{B_{2r}} |\nabla(u_R - v_{2r})|^2 \le Kr^n$, and $\int_{B_{4r}} |\nabla(u_R - v_{4r})|^2 \le Kr^n$, for all $r \le \rho/4$. Hence

$$\int_{B_{2r}} |\nabla(v_{4r} - v_{2r})|^2 \le Kr^n,$$

which implies, by elliptic regularity

$$\max_{B_r} |\nabla(v_{4r} - v_{2r})| \le K, \quad \text{for all } r \le \rho/4.$$

By induction, one obtains,

$$\max_{B_{2^{j-1}r}} |\nabla(v_{2^{j+1}r} - v_{2^j r})| \le K, \quad \text{for } j \ge 0, \quad 2^{j+1}r \le \rho.$$

Therefore, $\max_{B_r} |\nabla(v_{2r} - v_\rho)| \le K\log(\rho/r)$, and for all $r \le \rho/2$, we have,

$$\int_{B_r} |\nabla(u_R - v_\rho)|^2 \le \int_{B_r} (|\nabla(u_R - v_{2r})|^2 + |\nabla(v_\rho - v_{2r})|^2) \le Kr^n[(\log\rho/r)^2 + 1].$$

Thus,

$$\int_{B_r} |\nabla u_R|^2 \leq C(\rho) r^n (\log(\rho/r) + 1)^2,$$

from which the desired continuity follows, as in [M], Theorem 3.5.2.

Now, take $\overline{x} \in C_R^+(u_R)$. By continuity, there exists $r > 0$ such that $B_r(\overline{x}) \subset C_R^+(u_R)$. Let w_r, be the harmonic replacement of u_R on $B_r(\overline{x})$. Since w_r minimizes the Dirichlet integral and $w_r > 0$ on $B_r(\overline{x})$, we get that

$$J(w_r, B_r(\overline{x})) \leq J(u_R, B_r(\overline{x})).$$

As before, the minimality of u_R implies that the reverse inequality holds as well. Hence

$$\int_{B_r(\overline{x})} |\nabla w_r|^2 = \int_{B_r(\overline{x})} |\nabla u_R|^2.$$

By uniqueness of the Dirichlet minimizer we obtain then $u_R = w_r$ on $B_r(\overline{x})$. □

From hereafter, we will assume that V_1 and V_2 satisfy the assumptions in Lemma 4.4.

Lemma 4.4, immediately implies the following corollary.

Corollary 4.6 *u_R is subharmonic in $C_R^+(V_2)$.*

4.2.2 Lipschitz continuity and non-degeneracy

We will now prove Lipschitz continuity of u_R in C_R. In what follows, we set

$$d(x) = \text{dist}(x, F(u_R)).$$

Lemma 4.7 *u_R is Lipschitz continuous in C_R, with universal Lipschitz constant on each $D \Subset D' \Subset C_R$. In particular, for every $D \Subset D' \Subset C_R$, there exists $K > 0$ depending on D, D', V_2 and n, such that, for all $x \in D$,*

$$u_R(x) \leq K d(x).$$

PROOF. Let $x_0 \in D \Subset D' \Subset C_R$, with $u(x_0) > 0$, and let $B_r = B_r(x_0)$ be the maximum ball contained in $D' \cap \{u > 0\}$. If ∂B_r touches $\partial D'$, then $r \geq \operatorname{dist}(D, D')$, and we can apply interior regularity together with the fact that $u_R \leq V_2$, in order to show $|\nabla u_R|(x_0) \leq K$. Otherwise, ∂B_r touches $F(u_R)$ at a point x_1.

We distinguish two cases.

(a) If $d(x_1, F(V_2)) > r/2$, then $B_{r/2}(x_1) \subset C_R^+(V_2)$ and we can proceed as follows. We replace u_R in $B_{r/2}(x_1)$ by the harmonic function v with boundary values u_R. Then, by the maximum principle, $V_1 \leq v \leq V_2$ on $B_{r/2}(x_1)$, and also $v > 0$ in $B_{r/2}(x_1)$. Thus, the minimality of u_R yields

$$\int_{B_{r/2}(x_1)} |\nabla(u_R - v)|^2 \leq \int_{B_{r/2}(x_1)} \chi_{\{u_R=0\}}.$$

The right hand side can be estimated as in [AC], Lemma 3.2. One gets

$$\left(\int_{B_{r/2}(x_1)} \chi_{\{u_R=0\}}\right) (\overline{u_R})^2 \leq K r^2 \int_{B_{r/2}(x_1)} |\nabla(u_R - v)|^2,$$

with $K > 0$ dimensional constant and $\overline{u_R}$ the average of u_R on $B_{r/2}(x_1)$. Combining the two estimates above, and the fact that $x_1 \in F(u_R)$, we obtain

$$\overline{u_R} \leq K r,$$

that is

$$\frac{1}{|B_{r/2}|} \int_{B_{r/2}(x_1)} u_R dx \leq K r.$$

Now, let $\overline{x}$ be on the ray from x_0 to x_1, at distance $r/4$ from x_1. Then, by Harnack inequality, and the mean value property for u_R, we get

$$u_R(x_0) \leq K u_R(\overline{x}) = K \frac{1}{|B_{r/4}|} \int_{B_{r/4}(\overline{x})} u_R \leq K \frac{1}{|B_{r/2}|} \int_{B_{r/2}(x_1)} u_R \leq K r.$$

(b) Assume that $d(x_1, F(V_2)) \leq r/2$. Then,

$$u_R(x_0) \leq V_2(x_0) \leq Kd(x_0, F(V_2)) \leq K|x_0 - x_1| + d(x_1, F(V_2)) \leq Kr.$$

Now, denote by $v(x) = u_R(rx + x_0)/r$. Then, $\Delta v = 0$ and by Harnack inequality, $v(x) \leq K$ on $B_{1/2}(0)$. By interior regularity, $|\nabla u| \leq K'$ on $B_{1/4}(0)$, with K' dimensional constant. Rescaling back to u_R, we obtain $|\nabla u_R| \leq K'$, on $B_{r/4}(x_0)$, which implies the desired Lipschitz continuity. □

Corollary 4.8 *u_R is a Lipschitz continuous subharmonic function in C_R.*

The following result can be found in [C2].

Lemma 4.9 *Let Ω_1 (resp. Ω_2) be such that*

$$\Omega_1 \cap B_1(0) \supset \{x_n > 0\} \cap B_1(0), (\textit{resp. } \Omega_2 \cap B_1(0) \subset \{x_n > 0\} \cap B_1(0)).$$

Assume that u is a Lipschitz positive harmonic function in Ω_1 (resp. Ω_2) vanishing on $\partial\Omega_1$ (resp. $\partial\Omega_2$) and assume that

$$\overline{B_1} \cap \partial\Omega_i \cap \{x_n = 0\} = \{0\}.$$

Then, near zero, u has the asymptotic development

$$u(x) = \alpha x_n^+ + o(|x|) \quad \textit{on } \{x_n > 0\},$$

with $\alpha \geq 0$. Furthermore, $\alpha > 0$ for Ω_1.

Lemmas 4.4, 4.7, and Lemma 4.9, imply that, near a regular free boundary point, u_R has the desired expansion, as in (ii) Definition 2.6, with $\alpha \geq 0$. In particular, $\alpha > 0$ at points where an exterior ball condition is satisfied.

We will now prove a non-degeneracy result. Towards this aim, we will need the following auxiliary lemma, about the behavior of Lipschitz continuous, non-degenerate subharmonic functions. This kind of result can be found in [C3].

Lemma 4.10 *Let v be a Lipschitz non-degenerate function in $\overline{\Omega} \cap B_1(0)$, satisfying $\Delta v \geq 0, v = 0$ on $\partial\Omega \cap B_1(0)$. Assume further that $0 \in \partial\Omega$. Let $v(x_0) \geq Cd(x_0, \partial\Omega)$, for $x_0 \in B_{1/2}(0)$, then, for $\rho \leq 1/4$, we have*

$$\sup_{B_\rho(0)} v \geq C\rho.$$

Lemma 4.11 *u_R is non-degenerate on C_R, i.e., for every compact $D \subset C_R$, there exists $\overline{K} > 0$ depending on D, V_1, and n, such that*

$$\overline{K}d(x) \leq u_R(x),$$

for all $x \in C_R^+(u_R)$, such that $B_{d(x)}(x) \subset D$.

Proof. Let $x_0 \in D^+(u_R)$, and denote by $r = d(x_0)$. Assume $B_r(x_0) \subset D$. We distinguish two cases.

(a) If $d(x_0, F(V_1)) > r/2$, and $x_0 \in C_R^-(V_1)$, then $B_{r/2}(x_0) \subset C_R^-(V_1)$ and we can proceeds as follows. We show the following strongest claim, that is, there exists a positive dimensional constant $\overline{K}$, such that, if $B_{r'} \subset C_R^-(V_1)$, and $\int_{B_{r'}} u_R < \overline{K}r'|B_{r'}|$, then $u_R = 0$ on $B_{r'/2}$. Let v satisfy:

$$\begin{cases} \Delta v = 0 & \text{on } (B_{r'} \setminus B_{r'/2}) \cap C_R^+(u_R), \\ v = 0 & \text{on } B_{r'/2} \cap C_R^+(u_R), \\ v = u_R & \text{on } B_{r'} \cap C_R^-(u_R), \\ v = u_R & \text{on } \partial B_{r'}. \end{cases}$$

The existence of v can be achieved in the following way. Let v_ϵ be a solution to:

$$\begin{cases} \Delta v_\epsilon = 0 & \text{on } (B_{r'} \setminus B_{r'/2}) \cap \{u_R > \epsilon\}, \\ v_\epsilon = \epsilon & \text{on } B_{r'/2} \cap \{u_R > \epsilon\}, \\ v_\epsilon = u_R & \text{on } B_{r'} \cap \{u_R < \epsilon\}, \\ v_\epsilon = u_R & \text{on } \partial B_{r'}, \end{cases}$$

for any ϵ such that $\{u_R = \epsilon\}$ is a smooth surface. v_ϵ is obtained by minimizing the Dirichlet integral over the constraints above. Also v_ϵ is continuous at $\{u_R = \epsilon\} \cap (B_{r'} \setminus \overline{B_{r'/2}})$ and $0 \leq v_\epsilon \leq u_R$. Since ∇v_ϵ is bounded in $L^2(B_{r'})$, the limit $v = \lim_{\epsilon \to 0} v_\epsilon$ exists and $0 \leq v \leq u_R$; hence, since u_R is continuous in $B_{r'}$, v is continuous in $B_{r'}$ and has the desired properties. Moreover, $0 \leq v \leq V_2$, thus $J(u_R, B_{r'}) \leq J(v, B_{r'})$. From this we conclude the proof as in [ACF], Theorem 3.1. Finally, for $r' = r/2$, we get

$$u_R(x_0) = \frac{1}{|B_{r/2}|} \int_{B_{r/2}(x_0)} u_R \geq Kr.$$

If $d(x_0, F(V_1)) > r/2$, and $V_1(x_0) > 0$, then

$$u_R(x_0) \geq V_1(x_0) \geq Kd(x_0, F(V_1)) \geq Kr.$$

(b) If $d(x_0, F(V_1)) = |x_0 - x_1| \leq r/2$, then $B_{r/2}(x_1) \subset B_r(x_0)$. By Lemma 4.10, $\sup_{B_{r/8}(x_1)} V_1 \geq Kr$, hence by Harnack inequality,

$$u_R(x_0) \geq K \sup_{B_{r/4}(x_1)} u_R \geq K \sup_{B_{r/8}(x_1)} V_1 \geq Kr,$$

as desired. □

4.3 Properties of the Free Boundary

4.3.1 Density of free boundary points

We wish to prove a density property for free boundary points. Towards this aim, we will need to reformulate our non-degeneracy property in the following way:

Corollary 4.12 *For any compact $D \subset C_R$, there exist a constant K, such that, for* *any ball $B_r \subset D$ centered at a free boundary point,*

$$\int_{B_r} u_R \geq Kr|B_r|.$$

The corollary above can be deduced by the arguments in the proof of Lemma 4.11. We are now ready to derive the desired density property.

Lemma 4.13 *For any $G \Subset G' \Subset C_R$, there exist a constant $c < 1$, such that, for any ball $B_r \subset G$ centered at a free boundary point,*

$$c \leq \frac{|B_r \cap \{u_R > 0\}|}{|B_r|} \leq 1 - c.$$

PROOF. Assume B_r is centered at 0. By Corollary 4.12, there exists $y \in \partial B_{r/2}$ such that, $u(y) \geq Kr/2$. By Lipschitz continuity, for any $z \in B_{kr}(y)$ we have:

$$u_R(z) \geq u_R(y) - C|z-y| > Kr/2 - Ckr > 0$$

as long as k is sufficiently small. Hence $B_{kr}(y) \subset B_r \cap \{u_R > 0\}$, from which the desired lower bound follows. In order to get the upper bound, we distinguish two cases.

(a) $d(0, F(V_2)) = |x_0| \leq r/2$. Then $B_{r/2}(x_0) \subset B_r$. Hence,

$$|\{u = 0\}^\circ \cap B_r| \geq |\{V_2 = 0\}^\circ \cap B_{r/2}(x_0)| \approx |B_{r/2}(x_0)| \approx |B_r|.$$

(b) $d(0, F(V_2)) = |x_0| > r/2$. Then, $B_{r/2}(0) \subset \{V_2 > 0\}$. Hence we can replace u_R with its harmonic replacement on $B_{r/2}(0)$, and proceed as in [AC], Lemma 3.7. □

4.3.2 Asymptotic expansion around free boundary points

Lemma 4.14 *$u_R < V_2$ in $C_R^+(V_2)$, and $V_1 < u_R$ in $C_R^+(u_R)$.*

PROOF. Assume $u_R(x) = V_2(x)$ at some point $x \in C_R^+(V_2)$, then the strong maximum principle implies that $u_R \equiv V_2$ on $C_R^+(V_2)$, hence $u_R \equiv V_2$ on C_R. We want to show that this contradicts the fact that u_R minimizes J_R on K_R. Let $g \in C_0^\infty(C_R), g \leq 0$. For $\epsilon > 0$, set $y_\epsilon(x) = x + \epsilon g e_n$ and $V_\epsilon(x) = u_R(y_\epsilon(x))$. For ϵ sufficiently small, the monotonicity of V_2 in the x_n-direction and the fact that $V_1 < V_2$ in the positive phase of V_2, imply that $V_\epsilon \in K_R$. Therefore, using that $\mathrm{Det}(y_\epsilon(x)) = 1 + \epsilon\nabla \cdot g e_n + o(\epsilon^2)$, we get that

$$\begin{aligned}0 \leq J_R(V_\epsilon) - J_R(u_R) =& \\ &= \epsilon\left\{\int_{C_R} -(|\nabla u_R|^2 + \chi_{u_R>0})\nabla \cdot g e_n + (2\nabla u_R D g e_n \nabla u_R)\right\} + o(\epsilon^2).\end{aligned}$$

Therefore using Lemma 4.4, we obtain

$$\begin{aligned}0 \geq \int_{\{u_R>0\}} &\nabla \cdot ((|\nabla u_R|^2 + 1) g e_n - 2 g e_n \cdot \nabla u_R \nabla u_R) = \\ &= -\int_{\partial\{u_R>0\}} ((|\nabla u_R|^2 + 1) g e_n - 2 g e_n \cdot \nabla u_R \nabla u_R) \cdot \nu = \\ &\qquad\qquad = -\int_{\partial\{u_R>0\}} (1 - |\nabla u_R|^2) g \nu_n\end{aligned}$$

for all function g as above, and ν the inner unit normal to $\partial\{u_R > 0\}$. Since $u_R \equiv V_2$, this contradicts the strict supersolution property of V_2.

Assuming now, $u_R(x) = V_1(x)$ at some point $x \in C_R^+(u_R)$, then the contradiction follows immediately by the fact that $V_1 < V_2$ on $\{V_2 > 0\}$, and $u_R = V_2$ on S_R. □

In order to prove the next results, we introduce the notion of blow-up.

Let u be a non-negative, Lipschitz continuous function in Ω, open connected subset of $\mathbb{R}^n$. Let $x_0 \in F(u)$, and let $B_{r_k}(x_0) \subset \Omega$ be a sequence of balls with $r_k \to 0$, as $k \to +\infty$. Consider the blow-up sequence:

$$u_k(x) = \frac{1}{r_k} u(x_0 + r_k x).$$

Since for a given $D \Subset \mathbb{R}^n$ and large k the functions u_k are uniformly Lipschitz continuous in D, there exists a function $u_0 : \mathbb{R}^n \longrightarrow R$, such that:

- $u_k \to u_0$ in $C^{0,\alpha}_{loc}(\mathbb{R}^n)$, for all $0 < \alpha < 1$;

- $\nabla u_k \to \nabla u_0$ weakly star in $L^\infty_{loc}(\mathbb{R}^n)$.

Moreover, u_0 is Lipschitz continuous in the entire space. u_0 is called a blow-up of u. Using the same argument as in [F], (see Chapter 3, Lemma 3.6), one can prove the following.

Lemma 4.15 *Let u be a non-negative, Lipschitz continuous and (I) non-degenerate function in Ω. Assume that u satisfies the density property (D). Then the following properties hold:*

a. $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *in the Hausdorff distance;*

b. $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* L^1_{loc}*;*

c. $\nabla u_k \to \nabla u_0$ *a.e.*

We are now ready to prove the following statements, using blowing-up techniques.

Lemma 4.16 *$F(u_R)$ does not intersect $F(V_2)$.*

PROOF. Assume by contradiction that there exists $x_0 \in F(u_R) \cap F(V_2)$. Let $B_r \subset C^+_R(V_2)$ be a ball tangent at x_0 to $F(V_2)$. By Lemma 4.14 and Corollary 4.8, we can apply Hopf's Lemma to the subharmonic function $v_R = u_R - V_2$ and conclude that

$$\liminf_{x \to x_0} (v_R(x)/|x - x_0|) < 0. \tag{4.2}$$

Moreover, $F(u_R)$ has also a tangent ball from the zero side at x_0, hence $u_R(x) = a(x - x_0, \nu)^+ + o(|x - x_0|)$, near x_0, from the positive side of u_R, with ν the inner normal to $\partial\{V_2 > 0\}$ at x_0. Furthermore, by non-degeneracy (Lemma 4.11), $a > 0$. Let $B_{\rho_k}(x_0)$ be a sequence of balls with $\rho_k \to 0$ such that $u_k(x) := \frac{1}{\rho_k} u_R(x_0 + \rho_k x)$ blows up to $U(x)$, and $V_k(x) := \frac{1}{\rho_k} V_2(x_0 + \rho_k x)$ blows up to $\overline{V_2}$. Thus, on the unit ball B, $U(x) = a(x, \nu)^+$ and $\overline{V_2}(x) = b(x, \nu)^+$, and by 4.2, we also have $0 < a < b < 1$. Let us prove that U is an absolute minimum for $J(\cdot, B)$, among all competitors $v \le \overline{V_2}, v < \overline{V_2}$ in $\{\overline{V_2} > 0\}, v = U$ on ∂B. Let v be an admissible competitor and define

$$v_k = v + (1 - \eta)(u_k - U)$$

for $\eta \in C_0^\infty(B), 0 \le \eta \le 1$. Set $w_k = v_k^+$. Then, $w_k = u_k$ on ∂B, and for k large enough $w_k(x) \ge 0 = \frac{1}{\rho_k}V_1(x_0 + \rho_k x)$. Moreover, using that $v_k - V_k$ converges uniformly to $v - \overline{V_2}$, and $u_k \le V_k(x)$, we get that $w_k \le V_k$, for k large enough. Therefore $J(u_k, B) \le J(w_k, B)$ from which we obtain

$$\int_B \nabla((U-v) + \eta(u_k - U)) \cdot \nabla((U+v) + (2-\eta)(u_k - U)) +$$
$$+ \int_B (\chi_{\{u_k>0\}} - \chi_{\{v_k>0\}}) \le 0.$$

Observe that the following inequality holds

$$\chi_{\{w_k>0\}} \le \chi_{\{v>0\}} + \chi_{\{\eta<1\}}$$

Letting $k \to \infty$, we get $J(U, B) \le J(v, B) + |\{\eta < 1\}|$, and an appropriate choice of the function η gives the desired minimality. Now let $g \in C_0^\infty(B), g \ge 0$ and for $\epsilon > 0$, set $U_\epsilon(x) = U(x - \epsilon g \nu)$. Applying the same domain variation technique as in Lemma 4.14, we therefore get $a \ge 1$, which is a contradiction. □

As a consequence of the two lemmas above, we get the following corollary.

Corollary 4.17 *u_R is a variational solution to (2.3) in C_R. In particular, u_R satisfies*

$$\lim_{\epsilon \to 0} \int_{\partial\{u_R > \epsilon\}} (|\nabla u_R|^2 - 1)\eta \cdot \nu = 0, \tag{4.3}$$

for every $\eta \in W_0^{1,\infty}(C_R, \mathbb{R}^n)$.

Proof. Let $\eta \in C_0^\infty(C_R, \mathbb{R}^n)$, and ϵ small. Define $u_\epsilon(x) = u_R(\tau_\epsilon(x))$, where $\tau_\epsilon(x) = x + \epsilon\eta(x)$. Then, Lemmas 4.14 and 4.16, guarantee that $u_\epsilon \in K_R$. By the same computations as in Lemma 4.14, we therefore get the desired limiting equality. □

Using the corollary above, we can now prove the following result.

Lemma 4.18 *$F(u_R)$ does not intersect $F(V_1)$.*

Proof. Assume by contradiction that there exists $x_0 \in F(u_R) \cap F(V_1)$, and let denote by u_0 a blow-up of u_R around x_0. From Lemma 4.15, we deduce that we can pass to the limit in the definition of variational solution, hence u_0 is a variational solution to the one-phase free boundary problem (2.3) on any compact of $\mathbb{R}^n$. Moreover, u_0 is harmonic in its positive phase, hence as in Corollary 4.17, u_0 satisfies the equality:

$$\lim_{\epsilon \to 0} \int_{\partial\{u_0 > \epsilon\}} (|\nabla u_0|^2 - 1)\eta \cdot \nu = 0. \tag{4.4}$$

Since $x_0 \in F(u_R) \cap F(V_1)$, u_R has an asymptotic expansion around x_0, $u_R(x) = a(x - x_0, \nu)^+ + o(|x - x_0|)$, with $a > 0$, and ν the inner unit normal to $F(V_1)$ at x_0. Thus, applying formula (4.4) to the blow-up limit $u_0(x) = a(x, \nu)^+$, we get $a = 1$. Since V_1 is a strict subsolution, Hopf's lemma implies $a > 1$. We have reached a contradiction, hence $F(u_R)$ and $F(V_1)$ cannot touch. □

Finally, set $W_R := C_R^+(V_2) \cap C_R^-(V_1)$, u_R minimizes $J(\cdot, W_R)$ among all competitors which equal u_R on ∂W_R, and by Lemmas 4.16 and 4.18, $F(u_R) \subset W_R$. Hence, Lemma 4.4, together with Theorem 2.8 imply that u_R is a weak (viscosity) solution to (2.3) in C_R.

Chapter 5

Smoothness of local monotone free boundaries

5.1 Preliminaries

In this chapter we will prove Theorem 2.24.

We start by introducing a particular family of weak subsolutions to the free boundary problem (2.3), [C1].

Lemma 5.1 *Let u be a weak solution to (2.3) in Ω. Let $v_t(x) = \sup_{B_t(x)} u(y), t > 0$. Then v_t is a subsolution to (2.3) in its domain of definition. Furthermore, any point of $F(v_t)$ is regular from the positive side.*

We will also need the following result from [C1].

Lemma 5.2 *Let $v \leq u$ be two continuous functions in Ω, $v < u$ in $\Omega^+(v)$, v a subsolution and u a solution. Let $x_0 \in F(v) \cap F(u)$. Then x_0 cannot be a regular point for $F(v)$ from the positive side.*

5.2 Lipschitz continuity of the free boundary

In what follows, we will assume that a solution u_R on C_R, is extended to zero on $\{(x', x_n) : |x'| \leq R, x_n \leq -h_R\}$.

Theorem 2.24 is an immediate corollary to the following Theorem.

Theorem 5.3 *Assume that, there exist a strict smooth subsolution* V_1 *and a strict smooth supersolution* V_2 *to (2.3) in* $\mathbb{R}^n$, *such that*

i. $V_1 \le V_2$ *on* $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. $\lim_{x_n \to +\infty} V_1(x', x_n) = +\infty$, $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$.

For any $R > 0$, *there exist positive constants* c, r_1, r_2, *depending on* V_1, V_2, R, $r_1 < r_2$, *such that if* h_R *is sufficiently large,* $\{u_R = c\} \subset \overline{D_R}$, *with* $D_R = (\mathcal{B}_R \times \{r_1 < x_n < r_2\}) \Subset \{u_R > 0\}$. *Moreover, set* $\Omega = C_R \cap \{u_R < c\}$, *there exists a positive constant* θ, *depending on* V_1, V_2, R, *and on* $\inf_{\overline{D_R}} \partial_n u_R$, *such that, for small* $s > 0$,

$$\sup_{B_{s \sin\theta}(x)} u_R(y - s e_n) \le u_R(x), \tag{5.1}$$

for all $x \in \Omega$.

PROOF. For $R > 0$, denote by η_R the maximum vertical distance between $\partial\{V_1 > 0\}$ and $\partial\{V_2 > 0\}$, over $\mathcal{B}_R$. Now, let r_1 be the maximum vertical distance of $\partial\{V_1(x - 2\eta e_n) > 0\}$ from $\{x_n = 0\}$, over $\mathcal{B}_R$. Set

$$K = \max_{\{|x'| \le R, -2d_R(V_2) \le x_n \le r_1\}} V_2.$$

The strict monotonicity of V_2 in the x_n direction implies $K = \max_{\{|x'| \le R, x_n = r_1\}} V_2$. Since V_1 is strictly increasing in the vertical direction, and $\lim_{x_n \to +\infty} V_1(x', x_n) = +\infty$, we can find $r_2 > r_1$ such that

$$K < \min_{\{|x'| \le R, x_n = r_2\}} V_1 = \widetilde{K}.$$

Now, let $h_R > \max\{R, 2r_2, 2d_R(V_2)\}, C_R = \mathcal{B}_R \times \{|x_n| < h_R\}$, and let u_R be a monotone minimizer of J_R over K_R. Then, u_R is a continuous function on $\overline{C_R}$, such that $u_R \le K$ in $\overline{C_R} \cap \{x_n \le r_1\}$ and $u_R \ge \widetilde{K} > K$ in $\overline{C_R} \cap \{x_n \ge r_2\}$. Thus, for a given c, $K < c < \widetilde{K}$, the level set $\{u_R = c\}$ is contained in $\overline{D_R}$, for $D_R = C_R \cap \{r_1 < x_n < r_2\}$. Finally, set $\Omega = C_R \cap \{u_R < c\}$.

Now, let

$$\overline{s} = \sup\{\lambda > 0 | \ \exists \ \overline{x}, \ \text{ s.t. } \ (\overline{x} + \nu e_n) \in F(u_R), \ \ \forall \ |\nu| \leq \lambda\}.$$

Since $V_1 \leq u \leq V_2$, $0 \leq \overline{s} < +\infty$. Let s be a small positive number, and define

$$u_s(x) = u_R(x - (s + \overline{s})e_n).$$

Now, consider the family of subsolutions $v_t^s(x) = \sup_{B_t(x)} u_s(y)$, $t \geq 0$ and small. The monotonicity of u_R in the x_n direction, guarantees that

$$v_0^s \leq u_R \quad \text{on } C_R. \tag{5.2}$$

Step 1. We will show that, there exists a constant $\tau = \tau(s, \overline{s})$, such that $v_\tau^s < u_R$ on $\overline{\Omega^+(v_\tau^s)} \cap \partial\Omega$, and $\tau(s, \overline{s}) \to \overline{\tau} > 0$, as $s \to 0$. First, observe that, by the definition of r_1,

$$\overline{\Omega^+(v_t^s)} \cap \partial\Omega = \{u_R = c\} \cup (\overline{\Omega^+(v_t^s)} \cap S_R).$$

Now, elliptic regularity guarantees that u_R is smooth in $\overline{D_R}$, and by the maximum principle $\partial_n u_R > 0$ on $\overline{D_R}$. Thus, there exists a θ_1 depending on $\inf_{\overline{D_R}} \partial_n u_R$, such that $v_{\tau_1}^s < u_R$ on $\{u = c\}$, for $\tau_1 = (s + \overline{s}) \sin\theta_1$. Furthermore, there exists θ_2, depending on V_2, R, such that

$$\sup_{B_{\tau_2}(x)} V_2(y - (s + \overline{s})e_n) \leq V_2(x) = u_R(x), \quad \text{for all } x \in S_R, \text{ with} \tau_2 = (s + \overline{s}) \sin\theta_2,$$

and the inequality is strict on $S_R \cap \{V_2 > 0\}$. This implies, $v_{\tau_2}^s < u_R$ on $(\overline{\Omega^+(v_\tau^s)} \cap S_R)$. Finally, $\tau = \min\{\tau_1, \tau_2\}$.

Step 2. Define $A = \{t \in [0, \tau] | v_t^s \leq u_R, \text{in } \Omega\}$. By (5.2) $A \neq \emptyset$, and by the continuity in t of the family v_t^s, A is closed. We want to prove that A is open, hence $A = [0, \tau]$.

Let $t_0 \in A$, then $v^s_{t_0} \leq u_R$ in Ω, and by Step 1 and by the monotonicity of the family v^s_t, we also have $v^s_{t_0} < u_R$ on $\overline{\Omega^+(v^s_{t_0})} \cap \partial\Omega$. Lemma 4.4, together with the fact that $0 \leq u \leq V_2$, imply that u_R is continuous in $\overline{\Omega^+(u_R)}$. Therefore, the strong maximum principle applies and $v^s_{t_0} < u_R$ in $\Omega^+(v^s_{t_0})$. If $t_0 > 0$, then by Lemma 5.1, every point of $F(v^s_{t_0})$ is regular from the positive side, and Lemma 5.2 implies that $F(v^s_{t_0}) \cap F(u_R) = \emptyset$. Hence,

$$\overline{\Omega^+(v^s_{t_0})} \subset \{x \in \overline{\Omega} \mid u_R(x) > 0\}.$$

The inclusion above, for the case $t_0 = 0$ follows from the definition of $\overline{s}$. By the continuity in t, for t close to t_0,

$$\overline{\Omega^+(v^s_t)} \subset \{x \in \overline{\Omega} \mid u_R(x) > 0\}.$$

Thus, $v^s_t < u_R$ on $\partial\Omega^+(v^s_t)$. Since $v^s_t - u_R$ achieves its maximum on the boundary, we then get $v^s_t < u_R$ on $\Omega^+(v^s_t)$, from which we conclude that $t \in A$.

Step 3. From Step 2, we have:

$$\sup_{B_\tau(x)} u_R(y - (s + \overline{s})e_n) \leq u_R(x), \quad \text{in } \Omega. \tag{5.3}$$

From Step 1, we can let $s \to 0$, so to get $\sup_{B_{\overline{\tau}}(x)} u_R(y - \overline{s}e_n) \leq u_R(x)$, in Ω. If $\overline{s} > 0$, then we can choose $x = x_\epsilon + (\overline{s} - \epsilon)e_n$, with $x_\epsilon + \nu e_n \in F(u_R)$, for all $0 \leq \nu \leq \overline{s} - \epsilon$. We therefore get, $\sup_{B_{\overline{\tau}}(x_\epsilon - \epsilon e_n)} u_R(y) = 0$, which contradicts $x_\epsilon \in F(u_R)$, for ϵ sufficiently small. Hence $\overline{s} = 0$, and (5.3), together with the definition of τ, yield the desired estimate (5.1). □

Corollary 5.4 *Assume that the hypotheses of Theorem 5.3 hold. For any* $R > 0$, *there exist positive constants* c, r_1, r_2, r_1 *and* r_2 *large, depending on* V_1, V_2 *and* R, *such that, if* h_R *is sufficiently large, and* u *is a weak solution to (2.3) in* C_R, *satisfying:*

i. $V_1 \leq u \leq V_2$;

ii. u monotone in the x_n direction;

then,

a. $\{u = c\} \subset \overline{D_R} = \mathcal{B}_R \times \{r_1 < x_n < r_2\}$;

b. for all $0 < \delta < R$,

$$\partial_n u \geq M, \quad on \quad \Gamma(R,\delta) = \{|x'| \leq R - \delta, r_1 \leq x_n \leq r_2\},$$

with M depending on δ, V_1, V_2 and R.

PROOF. Following the argument in the proof of Theorem 5.3, (5.4 *a*) is immediate. Furthermore, we have:

$$u(0,r_2) - u(0,r_1) \geq \widetilde{K} - K = M > 0.$$

Hence, there exists $\widetilde{x_n}$, $r_1 < \widetilde{x_n} < r_2$, such that $\partial_n u_R(0,\widetilde{x_n}) \geq M(r_2 - r_1) = \widetilde{M}$. For a small δ, Harnack's inequality implies $\partial_n u_R \geq M'$ on $\Gamma(R,\delta) = \{|x'| \leq R - \delta, r_1 \leq x_n \leq r_2\}$, with M' depending on $\delta, \widetilde{M}, R$. Here we have used that $\partial_n u$ is a non-negative harmonic function in $\mathcal{B}_R \times \{r_1/2 < x_n < 2r_2\}$, for h_R large enough. □

Remark. It follows from the corollary above, that in the proof of Theorem 5.3, the Lipschitz constant of $\{u_R = c\}$ on $\Gamma(R,\delta)$ is controlled by M, with M independent of u_R. If we are able to control the Lipschitz constant of $\{u_R = c\}$ in a δ neighborhood of the fixed boundary S_R, then Theorem 5.3 holds, with constants independent of u_R. More generally, the method of the proof guarantees that it suffices to obtain a uniform control of the Lipschitz constant of all level sets of a weak solution u in a neighborhood of the fixed boundary S_R, to obtain an a-priori control of the Lipschitz constant of the free boundary of u.

Chapter 6

Existence of global monotone free boundaries

6.1 NTA property

We wish to prove Theorem 2.25. Toward this aim, we will need the following known monotonicity formula from [ACF].

Theorem 6.1 *Let v be a continuous function defined on $B = B_R(x_0)$. Suppose that v is harmonic in the open set $\{x \in B | v(x) \neq 0\}$. Let A_1 and A_2 be two different components in B of the set $\{x \in B | v(x) \neq 0\}$. Assume that for some constant $c > 0$, and any $r, 0 < r < R$,*

$$|B_r(x_0) \setminus (A_1 \cup A_2)| \geq c|B_r(x_0)|.$$

Define, for $0 < r < R$,

$$\phi(r) = \left(\frac{1}{r^2} \int_{B_r(x_0) \cap A_1} |\nabla v|^2 \rho^{2-n} dx\right) \left(\frac{1}{r^2} \int_{B_r(x_0) \cap A_2} |\nabla v|^2 \rho^{2-n} dx\right)$$

where $\rho = \rho(x) = |x - x_0|$. Then, for some positive β depending only on the dimension and the constant c, $r^{-\beta}\phi(r)$ is a non-decreasing function of r.

We also need the following result, which can be obtained with the same techniques as in Lemma 4.2 from [ACS]. First we introduce a notation. For any real-valued function u defined on a domain $\Omega \subset \mathbb{R}^n$, and any $d \in \mathbb{R}$, we denote by $\Omega^d(u) = \{x \in \Omega | u(x) > d\}$.

Lemma 6.2 *Let u be a weak solution to (2.3) in $\mathbb{R}^n$, u Lipschitz continuous and non-degenerate. Then, for any compact $D \subset \mathbb{R}^n$, there exists a positive constant τ such that, whenever $x_0 \in F(u) \cap D^\circ, B_R(x_0) \subset D, x \in B_{R/2}(x_0) \cap \{u > 0\}$ and A is a connected component of $D^{\frac{u(x)}{2}}(u) \cap B_R(x_0)$ containing x, then*

$$\int_A |\nabla u| \rho^{2-n} dy > \tau R^2$$

where $\rho = \rho(y) = |y - x_0|$.

Finally, the proof of Theorem 2.25, is obtained combining the density property of u, together with the Harnack chain property from the next Lemma.

We use the notations for cylinders, introduced in Chapter 2. Our proof follows closely the proof in [ACS].

Lemma 6.3 *Let V_1 and V_2 be non-negative functions on $\mathbb{R}^n$, such that:*

i. $V_1 \leq V_2$, $0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. V_i *is smooth and* $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, $i = 1, 2$.

Let u be a weak solution to (2.3) in $\mathbb{R}^n$, such that $V_1 \leq u \leq V_2$, u is locally Lipschitz continuous and nondegenerate, and u satisfies the density property (D). Let $C_{1,1/2} = C(1, h_1 + 1/2)$. Then, there exists constants $M, \overline{\delta}$ such that, for any $\delta > 0$, and for any $x_1, x_2 \in C_{1/2}$, such that $B(x, \delta) \subset C^+_{1,1/2}(u)$ and $|x_1 - x_2| \leq \tilde{c}\delta, \tilde{c}\delta \leq \overline{\delta}$, there exist $y_1 = x_1, ... y_l = x_2$, such that

a. $B_i = B(y_i, \delta/M) \subset C^+_{1,1/2}(u)$, $i = 1, ..., l$

b. $B_i \cap B_{i+1} \neq \emptyset$, $i = 1, ... l - 1$

c. l *independent of* δ, x_1, x_2.

PROOF. Assume that, without loss of generality,

$$\tilde{\delta} = \max\{d(x_1, \partial C^+_{1/2}(u)), d(x_2, \partial C^+_{1/2}(u))\} = d(x_2, \partial C^+_{1/2}(u)).$$

We distinguish two cases.

(a) $\tilde{\delta} \geq 2\tilde{c}\delta$. Then, $x_1 \in B(x_2, \tilde{c}\delta) \subset C^+_{1/2}(u)$, and we can easily find the required chain.

(b) $\tilde{\delta} < 2\tilde{c}\delta$. Then, let $x_0 \in \partial C^+_{1/2}(u)$ be such that $\tilde{\delta} = |x_2 - x_0|$, and let $8\tilde{c}\delta < 1$. Set $r_0 = 4\tilde{c}\delta$, and let $r_0 \leq R \leq 1/2$. Then $x_1, x_2 \in B(x_0, R/2)$, and $B(x_0, R) \subset C_{1,1/2}$. Let $d = \frac{1}{2}\min\{u(x_1), u(x_2)\}$. We will show that, there exists $c \geq 1$, such that if $\tilde{c}\delta \leq 1/(8c)$, and $R = cr_0$, then the connected components A_i of $B(x_0, R) \cap C^d_1(u)$ which contain $x_i, i = 1, 2$, are the same. Indeed, let us suppose that $A_1 \neq A_2$ and let us use Lemmas 6.1, and 6.2 with $v = (u - d)^+$. The density property of u guarantees that the hypotheses of Lemma 6.1 are satisfied, hence, for some exponent $\beta > 0$, the function $r^{-\beta}\phi(r)$ is non-decreasing. By Lemma 6.2 and Schwartz's inequality we obtain

$$\phi(r_0) > \tau^2.$$

Moreover, since u in Lipschitz on $C_{1,1/2}$, we also have the bound

$$\phi(R) \leq c',$$

with c' absolute constant independent of R. Hence,

$$\tau^2 r_0^{-\beta} < r_0^{-\beta}\phi(r_0) \leq R^{-\beta}\phi(R) \leq c'R^{-\beta}$$

or $R < c'r_0$, which is a contradiction if we choose $c = c'$.

We therefore conclude that $A_1 = A_2$. Since A_1 is open and connected we may find a curve Γ inside A_1 having x_1 and x_2 as end point. Denote by m the non-degeneracy constant of u on $C_{1,1/2}$. Then, for each $y \in \Gamma$ we know that

$$u(y) > d = \frac{1}{2}\min\{md(x_1, F(u)), md(x_2, F(u))\} \geq \frac{1}{2}m\delta.$$

Therefore, if K is the Lipschitz constant of u on $C_{1,1/2}$, for any $y \in \Gamma$, we have $d(y, F(u)) > \frac{1}{2}\frac{m\delta}{K}$. Set $\rho = \frac{1}{2}\frac{m\delta}{K}$, so that if $y \in \Gamma$ and $|x-y| < \rho$ then $u(x) > 0$. Since

$$\Gamma \subset \bigcup_{y \in \Gamma} B(y, \rho)$$

we may find a sequence $y_1, ..., y_l$ of points in Γ such that $\Gamma \subset \bigcup_{i=1}^{l} B(y_i, \rho)$, and we may further ask that no y in Γ belong to more than $c(n)$ of the balls $B(y_i, \rho)$. Furthermore, since $\rho = \frac{1}{2}\frac{m\delta}{K}, r_0 = 4\tilde{c}\delta$ and $y_i \in B(x_0, cr_0)$, l must be bounded by a constant depending only on dimension on $c, \tilde{c}$, but independent of x_1, x_2 or δ. □

Comment. It follows from the proof of Lemma 6.3 that M and $\overline{\delta}$ depend on the Lipschitz and non-degeneracy constants of u on $C_{1,1/2}$.

Finally, we recall two fundamental results about NTA domain (see [JK]).

Theorem 6.4 *(Dahlberg Boundary Harnack principle) Let Ω be an NTA domain, and let V be an open set. For any compact set $G \subset V$, there exists a constant C such that for all positive harmonic functions u and v in Ω that vanish continuously on $\partial\Omega \cap V$, $u(x_0) = v(x_0)$ for some $x_0 \in \Omega \cap G$ implies $C^{-1}u(x) < v(x) < Cu(x)$ for all $x \in G \cap \overline{\Omega}$.*

Theorem 6.5 *Let Ω be an NTA domain, and let V be an open set. Let G be a compact subset of V. There exists a number $\alpha > 0$, such that for all positive harmonic functions u and v in Ω that vanish continuously on $\partial\Omega \cap V$, the function $u(x)/v(x)$ is Hölder continuous of order α on $G \cap \overline{\Omega}$. In particular, for every $y \in G \cap \partial\Omega$, $\lim_{x \to y}(u(x)/v(x))$ exists.*

6.2 Global existence and regularity

In this section we prove Theorem 2.16.

We start by deriving the following existence result, which is a direct consequence of the local theory.

Theorem 6.6 *Assume that, there exist a strict subsolution V_1 and a strict supersolution V_2 to (2.3) in $\mathbb{R}^n$, such that*

i. $V_1 \leq V_2$ *on* $\mathbb{R}^n, 0 \in \{V_2 > 0\} \cap \{V_1 = 0\}^\circ$;

ii. V_i *is smooth,* $\partial_n V_i > 0$ *in* $\overline{\{V_i > 0\}}$, *for* $i = 1, 2$.

Then, there exists a global function u, weak solution to (2.3) in $\mathbb{R}^n$, such that u is monotone increasing in the x_n direction. Moreover u is Lipschitz continuous, non-degenerate, (I) non-degenerate, and it satisfies the density property (D).

Proof. Let $\{R_k\}$ be a sequence of radii, $R_k \to +\infty$. Set $u_k := u_{R_k}$; then, by Lemma 4.7, for any compact subset $D \subset \mathbb{R}^n$, and sufficiently large k, the functions $\{u_k\}$ are uniformly Lipschitz continuous on D. Hence, there exists a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, such that (up to a subsequence), $u_k \longrightarrow u$ uniformly on compacts of $\mathbb{R}^n$. Thus, u is locally Lipschitz continuous, and monotone increasing in the x_n direction. Moreover, since the u_k's are Lipschitz continuous, (I) non-degenerate, and satisfy the density property (D), with universal local constant, arguing as in Lemma 4.15, we obtain:

a. $\partial\{u_k > 0\} \to \partial\{u > 0\}$ in the Hausdorff distance;

b. $\chi_{\{u_k>0\}} \to \chi_{\{u>0\}}$ in L^1_{loc};

c. $\nabla u_k \to \nabla u$ a.e.

In particular, u is non-degenerate, (I) non-degenerate, and satisfies the density property (D). Furthermore, u is a variational solution to (2.3), on any compact, and it is harmonic in its positive phase. Arguing as in Lemma 4.18, we conclude that at a regular point x_0, u blows up to the linear function $u_0(x) = (x, \nu)^+$, with ν the radial normal at x_0, pointing towards $\{u > 0\}$. Therefore, $F(u)$ cannot touch neither $F(V_1)$ nor $F(V_2)$. Hence, u is a weak solution to (2.3) in $\mathbb{R}^n$. $\square$

Using the same techniques as in Lemma 4.16, we can then conclude the following:

Corollary 6.7 *u minimizes J among all competitors $v \in H^1_{loc}(\mathbb{R}^n), V_1 \leq v \leq V_2$.*

In what follows, we denote by u, the global weak solution, whose existence is guaranteed by Theorem 6.6. We will use Theorem 2.25, in order to conclude the proof of Theorem 2.16. First, we need to recall the following result from [W1].

Lemma 6.8 *Let v be a variational solution to (2.3) in Ω, and assume that v is Lipschitz continuous and satisfies the density property (D). Then any blow up limit of v is homogeneous of degree 1.*

We are now ready to prove the following:

Theorem 6.9 *$F(u)$ is a continuous graph, with a universal modulus of continuity.*

PROOF. We start by proving that $F(u)$ is a graph. Assume, by contradiction, that $F(u)$ contains a vertical segment.

Let $v(x) = u(x - te_n)$, for some small t. Since u is monotone in the x_n direction, we have $v \leq u$, and $v < u$ in $\{u > 0\}$. Moreover, by the assumption that $F(u)$ contains vertical segments, we have that $F(u) \cap F(v)$ is non-empty, for t sufficiently small. Assume, without loss of generality, that $0 \in F(u) \cap F(v)$. From Lemma 6.8, we obtain that u and v blow up around 0 to functions U and V which are homogeneous of degree 1. Moreover, $U \geq V$, and U, V are harmonic in their positive phase. Hence $U = \lambda V$, for some number λ. Furthermore, since u and v are variational solution to the same free boundary problem, and they are harmonic in their positive phase, the slope condition (see Corollary 4.17) implies $U = V$. Hence, if R_j is a sequence of radii such that $R_j \to 0$ as $j \to +\infty$, then $u_j(x) = u(R_j x)/R_j$ and $v_j(x) = v(R_j x)/R_j$ converge uniformly on compacts to the same function. In particular, since v is (I) non-degenerate, for ϵ small, there is $0 < r \leq 1/2$, such that

$$u - v \leq \epsilon \max_{B_r(0)} v \quad \text{on } B_r(0). \tag{6.1}$$

Now, let w be the solution to the following Dirichlet problem:

$$\begin{cases} \Delta w = 0 & \text{on } B_1(0) \cap \{v > 0\}, \\ w = u - v & \text{on } (\partial B_1(0)) \cap \{v > 0\}, \\ w = 0 & \text{on } B_1 \cap (\partial\{v > 0\}). \end{cases}$$

By the maximum principle, $u - v \geq w > 0$ in $B_1(0) \cap \{v > 0\}$. Hence by the Boundary Harnack inequality, $w \geq Cv$ on $B_{1/2}(0) \cap \{v > 0\}$, for some constant $C > 0$ depending on the ratio of w and v at a fixed scale. On the other hand, by (6.1) we get $w \leq c\epsilon v$ on $B_r(0) \cap \{v > 0\}$. Therefore, we get a contradiction if ϵ is sufficiently small.

Now, let us prove that $F(u)$ has a universal modulus of continuity. We will denote by $v(A, B)$, the vertical distance between two graphs in the x_n direction.

We want to show that for every compact $K \subset \mathbb{R}^n$, and any $\epsilon > 0$, there exists a $\delta > 0$ such that, if $|\eta| < \delta$, then any u global minimizer for J among competitors $v, V_1 \leq v \leq V_2$, satisfies,

$$v(\{u = \eta\}, F(u)) < \epsilon.$$

By contradiction, assume that for some compact $K \subset \mathbb{R}^n$, there exists a positive number ϵ, a sequence $\{\eta_j\}, \eta_j \to 0$, as $j \to +\infty$, and a sequence of energy minimizing solutions $\{u_j\}$, such that

$$u_j(x_j + \epsilon e_n) < \eta_j, \tag{6.2}$$

for some $x_j \in F(u_j) \cap K$.

The uniform Lipschitz continuity of the u_j's, for j large, implies that (up to a subsequence):

$$u_j \to \widetilde{u}, \quad \text{uniformly on compacts,}$$

and

$$x_j \to \overline{x} \in K.$$

Then, using the same techniques as in Theorem 6.6, one obtains that $\widetilde{u}$ is also a Lipschitz continuous minimizing solution, monotone increasing in the x_n direction, and satisfying the (I) non-degeneracy, and the density property (D). Moreover $\widetilde{u}(\overline{x}) = \widetilde{u}(\overline{x}+\epsilon e_n) = 0$. We aim to prove that $\overline{x} \in F(\widetilde{u})$; then by (6.2), we obtain that $F(\widetilde{u})$ contains the vertical segment from $\overline{x}$ to $\overline{x}+\epsilon e_n$, which is a contradiction to what we showed above. Indeed, assume, that $\overline{x}$ does not belong to $F(\widetilde{u})$. Then, there exists $r > 0$ such that, $B_r(\overline{x}) \subset \{\widetilde{u} = 0\}^\circ$. Notice that, since $V_1 \leq u_j$ for all j's, it follows from (6.2) that $\overline{x}$ is not in $F(V_1)$. Therefore we can assume that $B_r(\overline{x}) \subset \{V_1 = 0\}^\circ$. Hence, by non-degeneracy (see proof of 4.11), there exists a constant C such that,

$$u_j(\overline{x}) < Cr \Rightarrow u_j \equiv 0, \text{on } B_{r/2}(\overline{x}).$$

Therefore, since $u_j(\overline{x}) \to 0$, for j sufficiently large,

$$u_j \equiv 0, \text{on } B_{r/2}(\overline{x}).$$

Furthermore, if j is large enough, $x_j \in B_{r/4}(\overline{x}) \cap F(u_j)$, and we get a contradiction. □

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