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REALIZATION THEORY FOR DETERMINISTIC BOUNDARY-VALUE DESCRIPTOR SYSTEMS

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Abstract

This paper examines the realization of acausal weighting patterns with two-point boundary-value descriptor systems (TPBVDSs). We restrict our attention to the subclass of TPBVDSs which are *extendible*, i.e., whose input-output weighting pattern can be extended outwards indefinitely, and *stationary*, so that their weighting pattern is shift-invariant. Then, given an infinite acausal shift-invariant weighting pattern, the realization problem consists in constructing a minimal TPBVDS over a fixed interval, whose extended weighting pattern matches the given pattern. The realization method which is proposed relies on a new transform, the (s,t) transform, which is used to determine the dimension of a minimal realization, and to construct a minimal realization by factoring two homogeneous rational matrices in the variables s and t.

1. Introduction

There exists an extensive literature [1]-[3] on the state-space realization problem for linear time-invariant causal systems, i.e., for systems which admit an input-output description of the form

$$y(k) = \sum_{l=-\infty}^{\infty} W(k-l)u(l) \tag{1.1}$$

where the impulse response (weighting pattern) $W(.)$ satisfies

$$W(m) = 0 \ \text{ for } \ m \leq 0 \ . \tag{1.2}$$

However, for many physical systems, in particular when the independent variable is space rather than time, the causality condition (1.2) does not hold. For example, if we consider the temperature of a heated rod, there is no reason to assume that the temperature at any point of the rod depends exclusively on the applied heat on one side of that point. Weighting patterns that do not satisfy (1.2) are called acausal. In this paper, we develop a realization theory for acausal weighting patterns in terms of two-point boundary-value descriptor systems (TPBVDSs).

The motivation for considering this class of systems is that the dynamics of discrete-time descriptor systems are noncausal, in the sense that they contain components which propagate in both time directions [4]. The boundary conditions are another source of noncausality, since they are expressed symmetrically in terms of the system variables at both ends of the interval of definition. Thus, TPBVDSs have a totally acausal structure [5]-[6], which is ideally suited to model noncausal systems. Motivated by the earlier work of Krener [7], and Gohberg, Kaashoek and Lerer [8] for acausal systems with standard dynamics, a complete system theory of TPBVDSs has been developed recently in [9]-[11], including concepts such as reachability, observability and minimality. In this paper, we restrict our attention to stationary and extendible TPBVDSs, i.e, TPBVDSs whose weighting pattern pattern is shift-invariant, and where the interval of definition of the TPBVDS can be extended outwards indefinitely, without changing the weighting pattern.

The realization problem that we consider can be stated as follows: given an infinite weighting pattern $W(m)$, construct a minimal TPBVDS over a fixed interval, which has $W(m)$ for extended pattern. As for causal time-invariant systems, where the z-transform plays a useful role in transforming the realization problem into a factorization problem for proper rational matrix transfer functions, it is shown that the TPBVDS realization problem can be formulated as a factorization problem. However, instead of using the z-transform, we introduce a new transform, the (s,t) transform, which handles zero and and infinite frequencies symmetrically, and is therefore well adapted to the analysis of descriptor systems. The (s,t) transform is used here to characterize the dimension of TPBVDS realizations in terms of the McMillan degree for rational matrices in s and t, and to formulate the TPBVDS realization problem as a factorization problem for not one, but two homogeneous rational matrices in two variables. Due to space limitations, most results are stated without proof. The reader is referred to [12] for a complete account.

2. Two-point Boundary-value Descriptor Systems

In this section, we review several properties of TPBVDSs, such as stationarity, minimality and extendibility, that will be needed in the development of our TPBVDS realization procedure.

2.1. Model Description

A linear time-invariant TPBVDS is described by the difference equation

$$Ex(k+1) = Ax(k) + Bu(k)\ , \quad 0 \leq k \leq N-1 \tag{2.1}$$

with boundary condition

$$V_i\, x(0) + V_f\, x(N) = v \tag{2.2}$$

and output equation

$$y(k) = Cx(k)\ , \quad 0 \leq k \leq N\ . \tag{2.3}$$

Here, x and v are n-dimensional, u is m-dimensional, y is p-dimensional, and E, A, B and C are constant matrices. We also assume that $N \geq 2n$, so that all modes can be excited and observed. In [9] it was shown that if the system (2.1)-(2.2) is well-posed, we can assume without loss of generality that (2.1)-(2.2) is in *normalized form*, i.e., that there exists scalars α and β such that

$$\alpha E + \beta A = I \tag{2.4}$$

(this is referred to as the *standard form* for the pencil $\{E, A\}$) and in addition

$$V_i E^N + V_f A^N = I \ . \tag{2.5}$$

Then, the map from $\{\mathbf{u}, v\}$ to $\mathbf{x}$ has the form

$$x(k) = A^k E^{N-k} v + \sum_{l=0}^{N-1} G(k,l) B u(l) \ , \tag{2.6}$$

where $G(k,l)$ is the Green's function associated to the descriptor dynamics (2.1) and boundary condition (2.2). The map from inputs $\mathbf{u}$ to outputs $\mathbf{y}$ specifies the weighting pattern W of the system. Setting $v = 0$ in (2.6) yields

$$y(k) = \sum_{l=0}^{N-1} W(k,l) u(l) \ , \tag{2.7}$$

with

$$W(k,l) = C G(k,l) B \ . \tag{2.8}$$

2.2. Stationarity

In contrast with the causal case, where time-invariant state-space models have a time-invariant impulse response, the weighting pattern $W(k,l)$ is not in general a function of the difference $k-l$. TPBVDSs that have this property are called *stationary.*

Theorem 2.1 [10]: The TPBVDS (2.1)-(2.3) is stationary if and only if

$$O_s [V_i, E] R_s = O_s [V_i, A] R_s \tag{2.9a}$$

$$O_s [V_f, E] R_s = O_s [V_f, A] R_s \ , \tag{2.9b}$$

where $[X, Y]$ denotes the commutator product of X and Y

$$[X, Y] = XY - YX \tag{2.10}$$

and

$$R_s = [E^{n-1}B \ AE^{n-2}B \ \cdots \ A^{n-1}B] \tag{2.11a}$$

$$O_s^T = [(E^{n-1})^T C^T \ (AE^{n-2})^T C^T \ \cdots \ (A^{n-1})^T C^T] \ . \tag{2.11b}$$

The matrices R_s and O_s in (2.11) are respectively the *strong reachability* and *strong observability* matrices of the TPBVDS, as discussed in [9]. The stationarity conditions (2.9a) and (2.9b) state that V_i and V_f must commute with E and A, except for parts that are either in the left null space of R_s or the right null space of O_s. Consequently, if R_s and O_s have full rank, i.e., if the TPBVDS is *strongly reachable* and *strongly observable*, V_i and V_f must commute with E and A.

It is shown in [10] that the weighting pattern of a stationary TPBVDS defined over $[0,N]$ is given by

$$W(k) = \begin{cases} CV_i A^{k-1} E^{N-k} B & 1 \le k \le N \\ -CV_f E^{-k} A^{N+k-1} B & 1-N \le k \le 0 \ . \end{cases} \tag{2.12}$$

2.3. Minimality

Since our goal is to realize shift-invariant acausal weighting patterns with stationary TPBVDSs, we need to be able to determine whether a system in this class is minimal or not. This issue was studied in detail in [10]-[11], leading to the following characterization of minimality.

Theorem 2.2: The stationary TPBVDS (2.1)-(2.3) is minimal if and only if

(a) $[V_i R_s \quad V_f R_s]$ has full row rank, (2.13a)

(b) $\begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix}$ has full column rank, (2.13b)

(c) $\mathrm{Ker}(O_s) \subset \mathrm{Im}(R_s)$. (2.13c)

It was also shown in [10, Corollary 5.1] that Theorem 2.2 implies:

Corollary: Let $(C_j, V_i^j, V_f^j, E_j, A_j, N)$ with $j = 1,2$ be two minimal and stationary realizations of the same weighting pattern, where $\{E_j,\ A_j\}$, $j = 1,2$ are in standard form for the same α and β. Then, there exists an invertible matrix T such that

$$B_2 = TB_1 \ , \ C_2 = C_1 T^{-1} \tag{2.14a}$$

$$O_s^1(V_i^1 - T^{-1} V_i^2 T) R_s^1 = O_s^1(V_f^1 - T^{-1} V_f^2 T) R_s^1 = 0 \ , \tag{2.14b}$$

and

$$(A_1 - T^{-1} A_2 T) R_s^1 = (E_1 - T^{-1} E_2 T) R_s^1 = 0 \tag{2.14c}$$

$$O_s^1(A_1 - T^{-1} A_2 T) = O_s^1(E_1 - T^{-1} E_2 T) = 0 \ , \tag{2.14d}$$

where R_s^1 and O_s^1 are the strong reachability and observability matrices for system 1.

2.4. Extendibility

The concept of extendibility for stationary TPBVDSs was introduced in [10].

Definition 2.1: The stationary TPBVDS (2.1)-(2.3) is *extendible* (or input-output extendible) if given any interval $[K,L]$ containing $[0,N]$, there exists a stationary TPBVDS over this larger interval with the same dynamics as in (2.1), but with new boundary matrices $V_i(K,L)$ and $V_f(K,L)$ such that the weighting pattern $W_N(k)$ of the original system is the restriction of the weighting pattern $W_{L-K}(k)$ of the new extended system, i.e.,

$$W_N(k) = W_{K-L}(k) \quad \text{for } 1-N \le k \le N \ . \tag{2.15}$$

Theorem 2.3 [10]: A stationary TPBVDS is extendible if and only if

$$O_s(V_i - V_i E^D E)R_s = 0 \tag{2.16a}$$

$$O_s(V_f - V_f A^D A)R_s = 0 \tag{2.16b}$$

where E^D and A^D denote the Drazin inverses [13, p.8] of E and A.

From conditions (2.16), by using the E-, A-, E^D-, and A^D- invariance of $\text{Im}(R_s)$ [10] and the generalized Cayley-Hamilton theorem for the pencil $\{E,A\}$ [9], it is easy to check that for an extendible stationary TPBVDS, the weighting pattern (2.12) can be rewritten as

$$W(k) = \begin{cases} CV_i E^N E^D (AE^D)^{k-1} B & 1 \leq k \leq N \\ -CV_f A^N A^D (EA^D)^{-k} B & 1-N \leq k \leq 0 . \end{cases} \tag{2.17}$$

Given an extendible stationary TPBVDS over $[0,N]$ with weighting pattern $W_N(k)$, it is of interest to ask whether it is possible to extend this TPBVDS in a consistent way over intervals of increasing lengths, so that this progressive extension process gives rise to a unique extended weighting pattern $W(k)$ defined for all k. It is shown in [12] that for an interval of length $M > N$, if we select

$$\tilde{V}_i = V_i E^N (E^D)^M \ , \ \tilde{V}_f = V_f A^N (A^D)^M \ , \tag{2.18}$$

as the new boundary matrices over this larger interval, the TPBVDS $(C, \tilde{V}_i, \tilde{V}_f, E, A, B, M)$ is an extension of $(C, V_i, V_f, E, A, B, N)$ which is normalized, stationary and extendible. By using this extension procedure, we find that

$$W(k) = \begin{cases} C(V_i E^N) E^D (AE^D)^{k-1} B & k > 0 \\ -C[I - (V_i E^N)] A^D (EA^D)^{-k} B & k \leq 0 , \end{cases} \tag{2.19}$$

is the desired *extended weighting pattern.*

3. Internal Description of a Weighting Pattern

The matrix $V_i E^N$ specifies entirely the effect of the boundary conditions on the extended weighting pattern $W(k)$ given by (2.19). This motivates the introduction of the following concept.

Definition 3.1: Let $(C, V_i, V_f, E, A, B, N)$ be a stationary and extendible TPBVDS. Then, P is a *projection matrix* of this system if

$$O_s P R_s = O_s (E^N V_i) R_s \ . \tag{3.1}$$

The extended weighting pattern (2.19) can be expresssed in terms of P as

$$W(k) = \begin{cases} CPE^D (AE^D)^{k-1} B & k > 0 \\ -C(I-P) A^D (EA^D)^{-k} B & k \leq 0 . \end{cases} \tag{3.2}$$

Also, by using (2.9), (2.16), (3.1), and the fact that $\text{Im}(R_s)$ and $\text{Ker}(O_s)$ are E- and A- invariant, it is easy to check that a projection matrix P satisfies

$$O_s(PA - AP)R_s = O_s(PE - EP)R_s = 0 \tag{3.3a}$$

$$O_s(P - PEE^D)R_s = O_s[(I-P) - (I-P)AA^D]R_s = 0 \ . \tag{3.3b}$$

As is clear from Definition 3.1, one particular choice of projection matrix is $P = V_i E^N$. This choice is not unique in general. If P is a projection matrix, so is $P + Q$, where Q is any matrix such that $O_s QR_s$ equals zero.

The expression (3.2) for the extended weighting pattern $W(k)$ motivates the introduction of the following concept.

Definition 3.2: A 5-tuple (C,P,E,A,B) is said to be an *internal description* of the acausal weighting pattern $W(k)$ if it satisfies (3.2) and (3.3), and if $\{E,A\}$ is in standard form. Furthermore, (C,P,E,A,B) is *minimal* is it has the smallest dimension among all internal descriptions of $W(k)$.

Given an acausal weighting pattern $W(k)$, a possible procedure for constructing a minimal, extendible, stationary TPBVDS (C,V_i,V_f,E,A,B,N) which admits $W(k)$ as extended weighting pattern consists therefore in dividing the realization problem into two steps. First, find a minimal internal description (C,P,E,A,B) of $W(k)$. Next, given a finite interval $[0,N]$, find some appropriate boundary matrices V_i and V_f such that the corresponding TPBVDS is extendible and stationary, and such that P is a projection matrix associated to these matrices. The following result guarantees the validity of this two-step realization approach.

Theorem 3.1: Let (C,P,E,A,B) be an internal description of $W(k)$. Then, for any interval length N, there exists matrices V_i and V_f such that the TPBVDS (C,V_i,V_f,E,A,B,N) is normalized, extendible, stationary, and has for extended weighting pattern $W(k)$. P is a projection matrix of (C,V_i,V_f,E,A,B,N). Furthermore, this TPBVDS is minimal if and only if the internal description (C,P,E,A,B) of $W(k)$ is minimal.

Proof: Let

$$V_i = P(E^D)^N + \sigma X(\sigma E^N + A^N)^{-1} \tag{3.4a}$$

$$V_f = (I-P)(A^D)^N + X(\sigma E^N + A^N)^{-1} \ , \tag{3.4b}$$

where

$$X = I - PEE^D - (I-P)AA^D = (I-P)EE^D + PAA^D - EE^D AA^D \ , \tag{3.5}$$

and where σ is any scalar such that $\sigma E^N + A^N$ is invertible. The relations (3.4)-(3.5) specify a TPBVDS (C,V_i,V_f,E,A,B,N). By direct calculation, it is easy to check that V_i and V_f are normalized, and that this TPBVDS is stationary, extendible, and realizes $W(k)$. Noting that $O_s XR_s = 0$, we can also verify that P is a projection matrix of the TPBVDS. By construction, the minimality of the TPBVDS (C,V_i,V_f,E,A,B,N) is equivalent to that of the internal description (C,P,E,A,B). □

Given an internal description (C,P,E,A,B) of the weighting pattern $W(k)$, the following result, which was derived in [12], shows that it is possible to characterize the minimality of this internal description directly, without invoking minimality conditions for an associated TPBVDS.

Theorem 3.2: The internal description (C,P,E,A,B) of $W(k)$ is minimal if and only if

(a) $R_w = [R_s \;\; PR_s]$ has full row rank (3.6a)

(b) $O_w = \begin{bmatrix} O_s \\ O_s P \end{bmatrix}$ has full column rank (3.6b)

(c) $\mathrm{Ker}(O_s) \subset \mathrm{Im}(R_s)$. (3.6c)

By analogy with the weak reachability and observability matrices which were introduced in [9]-[10] to characterize the concepts of weak reachability and observability for a TPBVDS (C,V_i,V_f,E,A,B,N), the matrices R_w and O_w are called the *weak reachability* and *weak observability* matrices of the internal description (C,P,E,A,B).

It is also shown in [12] that Theorem **3.2** implies that two minimal internal descriptions of a weighting pattern can be related as follows.

Corollary: Consider two minimal internal descriptions (C_j,P_j,E_j,A_j,B_j), with $j=1,2$, of the same weighting pattern $W(k)$, which are in standard form for the same α and β. Then, there exists an invertible matrix T such that relations (2.14a), (2.14c)-(2.14d), and

$$O_s^1(P_1 - T^{-1}P_2T)R_s^1 = 0 \tag{3.7}$$

are satisfied.

The procedure that we develop here for constructing a minimal internal description (C,P,E,A,B) of $W(k)$ relies on the introduction of a new transform, the (s,t) transform, and on formulating the realization problem as a factorization problem for rational matrices in s and t.

4. (s,t)-Transform and Rational Matrix Factorization

One difficulty associated with the use of the z-transform for analyzing discrete-time descriptor systems, is that since the dynamics of such systems are singular, infinite frequencies cannot be handled in the same way as other frequencies [14]. This motivates the introduction of the transform

$$H(s,t) = \sum_{k=-\infty}^{\infty} H(k)t^{k-1}/s^k \ . \tag{4.1}$$

It can be expressed in terms of the standard z-transform $H(z)$ as

$$H(s,t) = H(s/t)/t \ . \tag{4.2}$$

From this observation, we see that when $H(s,t)$ exists, it is always stricly proper in (s,t) in the sense that

$$\lim_{c \to \infty} H(cs,ct) = 0 \ . \tag{4.3}$$

Note however that it is not necessarily strictly proper in s and t separately, so that the corresponding z-transform may not be proper.

In the following, we shall restrict our attention to the case when $H(z)$ and $H(s,t)$ are *rational*. Then, from (4.2), we see that the numerator and denominator polynomials of all entries of $H(s,t)$ are *homogeneous*. Furthermore, the relative degree in s and t of all entries of $H(s,t)$, i.e., the difference between the denominator and numerator degrees, is exactly one. Thus, the transformation (4.2) has the effect of transforming rational matrices $H(z)$, proper or not, into strictly proper homogeneous rational matrices in the two variables s and t with relative degree one.

4.1. Formulation of the Realization Problem

In the causal case, the z-transform plays an important role in the solution of the minimal realization problem. Specifically, given a causal weighting pattern $W(k)$, the minimal realization problem is equivalent to finding matrices (C,A,B) of minimal dimension such that the z-transform $W(z)$ admits the the factorization

$$W(z) = C(zI - A)^{-1}B \ . \tag{4.4}$$

For the case of acausal weighting patterns, the situation is more complex. If (C,P,E,A,B) is an internal description of the weighting pattern $W(k)$, and if $W_f(k)$ and $W_b(k)$ are the causal and anticausal parts of $W(k)$, the (s,t)-transforms of $W_f(k)$ and $W_b(k)$ can be expressed as

$$\begin{aligned} W_f(s,t) &= \sum_{k=1}^{\infty} CPE^D(AE^D)^{k-1}Bt^{k-1}/s^k \\ &= CPE^D(sI - tAE^D)^{-1}B = CP(sE - tA)^{-1}B \end{aligned} \tag{4.5a}$$

$$\begin{aligned} W_b(s,t) &= \sum_{k=-\infty}^{0} -C(I-P)A^D(EA^D)^kBt^{k-1}/s^k \\ &= C(I-P)A^D(sEA^D - tI)^{-1}B = C(I-P)(sE - tA)^{-1}B \ . \end{aligned} \tag{4.5a}$$

Note that $W_f(s,t)$ and $W_b(s,t)$ do not have in general the same regions of convergence. However, by analytic continuation, it is possible to extend their domains of definition to the whole plane, while using the same notation. This yields the factorizations

$$W_f(s,t) + W_b(s,t) = C(sE - tA)^{-1}B \tag{4.6}$$

$$[W_f(s,t) \ W_b(s,t)] = C(sE - tA)^{-1}[PB \ (I-P)B] \tag{4.7}$$

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} CP \\ C(I-P) \end{bmatrix}(sE - tA)^{-1}B \ . \tag{4.8}$$

Since the specification of an acausal weighting pattern $W(k)$ is equivalent to the specification of $W_f(s,t)$ and $W_b(s,t)$, we see from (4.6)-(4.8) that the construction of an internal description (C,P,E,A,B) of $W(k)$ involves the factorization of three homogeneous rational matrices in s and t, instead of a single rational matrix for causal systems.

4.2. Factorization of Homogeneous Rational Matrices in s and t

The above discussion motivates the following *minimal factorization problem*: given an homogeneous rational matrix function $H(s,t)$ of relative degree one, find matrices (K,D,F,G) of lowest possible dimension such that

$$H(s,t) = K(sD - tF)^{-1}G \ . \tag{4.9}$$

An important feature of this factorization problem, is that even if we impose the additional requirement that $\{D,F\}$ should be in standard form, i.e., that there exists α and β such that

$$\alpha D + \beta F = I \ , \tag{4.10}$$

the matrices (K,D,F,G) are not unique. To insure uniqueness, α and β must be chosen a priori. They can be chosen arbitrarily, as long as $H(\alpha,-\beta)$ is defined.

Theorem 4.1: A matrix function $H(s,t)$ admits a factorization of the form (4.9) if and only if it is homogeneous in s and t with relative degree one. Under these conditions, if (α,β) is a pair of scalars such that $H(\alpha,-\beta)$ exists, $H(s,t)$ admits a unique minimal factorization, up to a similarity transform, satisfying (4.9)-(4.10). The dimension r of this minimal realization, i.e., the size of D and F, is given by

$$r = d(H(\alpha z, 1 - \beta z)) \ , \tag{4.11}$$

where $d(.)$ denotes the usual McMillan degree, and where $H(\alpha z,1-\beta z)$ is a strictly proper rational matrix in z.

Proof: Necessity is obvious. To prove sufficiency, let α and β be such that $H(\alpha,-\beta)$ exists. Then, consider the rational matrix $H(\alpha z,1-\beta z)$. This matrix is strictly proper in z because

$$\lim_{z\to\infty} H(\alpha z, 1 - \beta z) = \lim_{z\to\infty} H(\alpha,-\beta)/z = 0 \ . \tag{4.12}$$

It can be realized as

$$H(\alpha z, 1 - \beta z) = K(zI - F)^{-1}G \ . \tag{4.13}$$

Now, assume that $\alpha \neq 0$ (otherwise, reverse the roles of D and F), and let

$$w = \alpha/(\alpha t + \beta s) \quad , \quad z = s/(\alpha t + \beta s) \ . \tag{4.14}$$

In this case

$$s = \alpha z / w \quad , \quad t = (1 - \beta z)/w \ , \tag{4.15}$$

which implies that

$$H(s,t) = wH(\alpha z, 1 - \beta z) = wK(zI - F)^{-1}G = K(sD - tF)^{-1}G \ , \tag{4.16}$$

where

$$D = (I - \beta F)/\alpha \ . \tag{4.17}$$

Since there is a one to one correspondence between the factorization (4.13) of $H(\alpha z, 1-\beta z)$ and the factorization (4.16)-(4.17) of $H(s,t)$, the dimension and uniqueness properties of these two factorizations are the same. This implies that minimal factorizations of $H(s,t)$ of the form (4.9)-(4.10) are related by a similarity transform, and have a dimension r equal to the McMillan degree of $H(\alpha z, 1-\beta z)$. □

Corollary: The factorization (4.9)-(4.10) is minimal if and only if (D,F,G) is strongly reachable and (K,D,F) is strongly observable. Furthermore, the dimension of a minimal factorization is equal to the rank of the Hankel matrix $O_s R_s$, where O_s and R_s are the strong observability and reachability matrices associated respectively to (K,D,F) and (D,F,G).

One unsatisfactory aspect of Theorem 4.1 is that the dimension r of a minimal factorization of $H(s,t)$ is characterized in terms of the McMillan degree of the 1-D rational matrix $H(\alpha z, 1-\beta z)$, and not directly in terms of $H(s,t)$. It turns out that it is possible to characterize r directly from $H(s,t)$ by extending the concept of McMillan degree as follows.

Definition 4.1: Given a homogeneous and strictly proper rational matrix $H(s,t)$ in s and t, the *McMillan degree* of $H(s,t)$ is defined as the degree of the least common multiple of the denominators of all minors of $H(s,t)$.

It was shown in [12] that:

Theorem 4.2: If $H(s,t)$ is factorizable, i.e., if it is homogeneous of relative degree one, the dimension of a minimal factorization of $H(s,t)$ is equal to its McMillan degree.

5. Minimal Realization

In Section 4, it was shown that the specification of an internal description (C,P,E,A,B) of a weighting pattern $W(k)$ yields the three rational matrix factorizations (4.6)-(4.8). This suggests that the construction of a minimal internal description of $W(k)$ can be formulated as a minimal factorization problem.

5.1. Dimension of a Minimal Realization

Theorem 5.1: The dimension n of a minimal internal description of $W(k)$ is given by

$$n = d([W_f(s,t) \ \ W_b(s,t)]) + d(\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}) - d(W_f(s,t) + W_b(s,t)) \ , \tag{5.1}$$

where $d(.)$ is the generalized McMillan degree introduced in Definition 4.1.

Proof: Let (C,P,E,A,B) be a minimal internal description of $W(k)$, and let ω, ρ and τ be defined as follows

$$\omega = d(C(sE - tA)^{-1}[PB \ (I-P)B]) = d([W_f(s,t) \ \ W_b(s,t)]) \tag{5.2a}$$

$$\rho = d(\begin{bmatrix} CP \\ C(I-P) \end{bmatrix}(sE - tA)^{-1}B) = d(\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}) \qquad (5.2b)$$

$$\tau = d(C(sE - tA)^{-1}B) = d(W_f(s,t) + W_b(s,t)) \; . \qquad (5.2c)$$

From the corollary of Theorem 4.1, it follows that ω, ρ and τ are the ranks of the Hankel matrices $O_s R_w$, $O_w R_s$ and $O_s R_s$, respectively. Then, from the minimality conditions (3.6a)-(3.6b), R_w and O_w have full rank, which implies that ω and ρ are the ranks of the strong observability and reachability matrices O_s and R_s, respectively. From condition (3.6c), we can also deduce that the rank of $O_s R_s$ equals the rank of O_s plus that of R_s minus n, so that

$$\tau = \omega + \rho - n \; , \qquad (5.3)$$

which implies (5.1). □

Example 5.1: Consider the weighting pattern

$$W(k) = \begin{cases} a^k & k \geq 1 \\ ba^k & k < 1 \end{cases}, \qquad (5.4)$$

where a and b are scalar parameters with $a < 1$. From Theorem 5.1, we find that the dimension of a minimal internal description of $W(k)$ is given by

$$n = d([\frac{a}{s - at} \; \frac{ab}{s - at}]) + d(\begin{bmatrix} \frac{a}{s - at} \\ \frac{-ab}{s - at} \end{bmatrix}) - d(\frac{(1-b)a}{s - at})$$

$$= \begin{cases} 1 + 1 - 1 = 1 & \text{for } b \neq 1 \\ 1 + 1 - 0 = 2 & \text{for } b \neq 1 \; . \end{cases} \qquad (5.5)$$

When $b \neq 1$, a minimal internal description of $W(k)$ is

$$C = a/(1-b) \quad P = 1/(1-b) \quad E = 1 \quad A = a \quad B = 1 \; , \qquad (5.6)$$

and for $b = 1$, we can select

$$C = [a \;\; a] \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E = I \quad A = aI \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (5.7)$$

Thus, $b = 1$ can be viewed as a singularity, in the sense that the dimension of a minimal internal description of W is 2 only when b is exactly equal to 1. □

5.2. Minimal Realization Procedure

One interesting aspect of Theorem 5.1 is that as an intermediate step in the evaluation of the dimension n of a minimal internal description of $W(k)$, we obtain ω and ρ, which are respectively the ranks of the strong observability and reachability matrices of a minimal internal description. This observation was used in [12] to obtain the following realization procedure.

Step 1: Construct the minimal factorizations

$$[W_f(s,t)\ W_b(s,t)] = \overline{C}(s\overline{E} - t\overline{A})^{-1}[\overline{B}_f\ \overline{B}_b] \tag{5.8}$$

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} \tilde{C}_f \\ \tilde{C}_b \end{bmatrix}(s\tilde{E} - t\tilde{A})^{-1}\tilde{B} \ , \tag{5.9}$$

where if α and β are such that $W_f(\alpha,-\beta)$ and $W_b(\alpha,-\beta)$ are defined, the pairs $\{\overline{E},\overline{A}\}$ and $\{\tilde{E},\tilde{A}\}$ satisfy the normalization condition (2.4) for the same α and β.

Step 2: Let

$$\overline{B} = \overline{B}_f + \overline{B}_b \qquad \tilde{C} = \tilde{C}_f + \tilde{C}_b \ . \tag{5.10}$$

From (5.8)-(5.9), we find

$$\begin{aligned} W(s,t) &= W_f(s,t) + W_b(s,t) \\ &= \overline{C}(s\overline{E} - t\overline{A})^{-1}\overline{B} = \tilde{C}(s\tilde{E} - t\tilde{A})^{-1}\tilde{B} \ , \end{aligned} \tag{5.11}$$

so that $(\overline{C},\overline{E},\overline{A},\overline{B})$ and $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ are two factorizations, in general non-minimal, of $W(s,t)$. The minimality of factorizations (5.8) and (5.9) implies that $(\overline{C},\overline{E},\overline{A},\overline{B})$ and $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ are respectively strongly observable and strongly reachable. By decomposing these two factorizations in strongly reachable/unreachable, and strongly observable/unobservable components, respectively, we obtain

$$\overline{C} = [\overline{C}_1\ \overline{C}_2] \quad \overline{E} = \begin{bmatrix} \overline{E}_1 & \overline{E}_2 \\ 0 & \overline{E}_4 \end{bmatrix} \quad \overline{A} = \begin{bmatrix} \overline{A}_1 & \overline{A}_2 \\ 0 & \overline{A}_4 \end{bmatrix} \quad \overline{B} = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} \tag{5.12}$$

and

$$\tilde{C} = [0\ \tilde{C}_2] \quad \tilde{E} = \begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 \\ 0 & \tilde{E}_4 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}. \tag{5.13}$$

Step 3: From (5.11), we find that

$$W(s,t) = \overline{C}_1(s\overline{E}_1 - T\overline{A}_1)^{-1}\overline{B}_1 = \tilde{C}_2(s\tilde{E}_4 - T\tilde{A}_4)^{-1}\tilde{B}_2 \ , \tag{5.14}$$

where the factorizations $(\overline{C}_1,\overline{E}_1,\overline{A}_1,\overline{B}_1)$ and $(\tilde{C}_2,\tilde{E}_4,\tilde{A}_4,\tilde{B}_2)$ are both strongly reachable and observable. This implies that they must be related by a similarity transformation, i.e., there exists a matrix T such that

$$\overline{C}_1 = \tilde{C}_2T^{-1} \quad \overline{E}_1 = T\tilde{E}_4T^{-1} \quad \overline{A}_1 = T\tilde{A}_4T^{-1} \quad \overline{B}_1 = T\tilde{B}_2 \ . \tag{5.15}$$

The matrix T is given by

$$T = \overline{M}_s\tilde{M}_s^T(\tilde{M}_s\tilde{M}_s^T)^{-1} \ , \tag{5.16}$$

where $\overline{M}_s$ and $\tilde{M}_s$ denote respectively the strong reachability matrices of $(\overline{E}_1,\overline{A}_1,\overline{B}_1)$ and $(\tilde{E}_4,\tilde{A}_4,\tilde{B}_2)$.

Step 4: The matrices C, E, A and B of a minimal internal description can be selected as

$$E = \begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 T^{-1} & * \\ 0 & \overline{E}_1 & \overline{E}_2 \\ 0 & 0 & \overline{E}_4 \end{bmatrix} \quad A = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 T^{-1} & * \\ 0 & \overline{A}_1 & \overline{A}_2 \\ 0 & 0 & \overline{A}_4 \end{bmatrix} \tag{5.17a}$$

$$C = [0 \; \overline{C}_1 \; \overline{C}_2] \quad B = \begin{bmatrix} \tilde{B}_1 \\ \overline{B}_1 \\ 0 \end{bmatrix}, \tag{5.17b}$$

where * indicates an arbitrary block entry. The role of the similarity transformation T is to guarantee that the component which is common to factorizations (5.8) and (5.9) is expressed in the same coordinate system. Note that (5.17) corresponds to a four part Kalman decomposition of (C,E,A,B) into strongly reachable/unreachable and observable/unobservable parts, where there is no unreachable and unobservable component, since the internal description that we construct must be *minimal.*

Step 5: The matrix P is obtained by solving

$$O_s VR_s = H_f \;, \tag{5.18a}$$

where H_f denotes the Hankel matrix associated to the causal part $W_f(k)$ of the weighting pattern, and setting

$$P = VE^{2n-1} \;. \tag{5.18b}$$

Example 5.2: Let

$$W(k) = \begin{cases} 0 & k = 1 \\ -1 & k \neq 1 \;. \end{cases} \tag{5.19}$$

Then

$$W_f(s,t) = -t/s(s-t) \qquad W_b(s,t) = 1/(s-t) \;, \tag{5.20}$$

and according to Theorem 5.1, the dimension of a minimal internal description of $W(k)$ is

$$n = 2 + 2 - 1 = 3 \tag{5.21}$$

Since $\omega = \rho = 2$, we can also conclude that the minimal internal description is neither strongly reachable nor strongly observable. To obtain a minimal description, the first step is to perform the minimal factorizations

$$[W_f \;\; W_b] = [\frac{-t}{s(s-t)} \;\; \frac{1}{s-t}] = [1 \; 1](sI - t\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \tag{5.22a}$$

$$\begin{bmatrix} W_f \\ W_b \end{bmatrix} = \begin{bmatrix} \dfrac{-t}{s(s-t)} \\ \dfrac{1}{s-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}(sI - t\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{5.22b}$$

which satisfy the normalization condition (2.4) with $\alpha = 1$, $\beta = 0$. This yields

$$\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \tilde{C} = [1\ 0]\ . \tag{5.23}$$

In this case, we can select $T = 1$, and

$$C = [0\ 1\ 1] \quad E = I \quad A = \begin{bmatrix} 1 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \tag{5.24}$$

where * denotes an arbitrary entry. P is obtained from (5.18), which yields

$$P = \begin{bmatrix} * & * & * \\ 0 & 1 & * \\ 1 & 0 & * \end{bmatrix}. \tag{5.25}$$

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