March 1990

LIDS-P-1955

Optical CDMA via Temporal Codes

Murat Y. Azizoğlu
Laboratory for Information and Decision Systems, MIT

Jawad A. Salehi
Bell Communications Research

Ying Li
Laboratory for Information and Decision Systems, MIT

March 16, 1990

Abstract

We provide an analysis to the performance of Optical Orthogonal Codes(OOC) in an Optical Code Division Multiple Access(CDMA) network by considering the probability distribution of the interference patterns. We show that the actual performance is close to a previous estimate. We also consider a less structured temporal code in which the code words are allowed to overlap at two pulse positions. We obtain the bit error probability for this class of codes for two cases: with and without optical hard-limiting at the receivers. We show that this code may increase the number of users in the network considerably without a significant loss in the performance. We also provide a simple partial construction scheme for this code.

1 Introduction

Optical Code Division Multiple Access (CDMA) has been proposed as an alternative to frequency and time based multiple access methods for the future high speed fiber optic networks [1, 2, 3, 4, 5, 6]. A typical fiber-optic CDMA system is shown in Figure 1. In this paper, we study an optical CDMA network which is based on temporal codes. The users in this network are assigned distinct temporal code words with desired distinguishability properties. A code word consists of F chips, each of duration τ_c, with $T_b = F\tau_c$ being the duration of one information bit. K of these chips are occupied with short light pulses. The position of these K pulses in the frame constitutes the encoding, resulting in a temporal code. The users in the network send their data in on–off modulation format, so that for a "0" bit a user sends nothing and for a "1" bit it sends its temporal signature. The receiver checks for the occupancy of the K pulse positions that the desired transmitter is supposed to occupy. It compares the number of pulses in these positions to a threshold Th ($\leq K$). In the absence of thermal and shot noise, no errors can occur when the data bit is "1", and when it is "0" an error occurs if interferers result in at least Th pulses. Hard-limiting the incoming light power results in a performance improvement, since the possibility of an error due to interference heavily localized in a few pulse positions is eliminated [1]. We analyze the performance of systems both with and without hard-limiting at the receivers. For mathematical convenience, we assume that the users in the system are chip-synchronous, so that the pulses of different users are perfectly aligned although their bit frames may not be aligned. It is known that this assumption results in an upper bound in the performance [7].

For the users to cause minimal interference to each other, the temporal code must have small auto- and cross-correlations [1]. In particular, when the maximum cross-correlation and maximum off-peak auto-correlation is bounded by 1, we have the optimal code. In fact, such codes are designed in [1, 7], they are called Optical Orthogonal Codes (OOC) with $\lambda = 1$. Here λ refers to the maximum value of the auto- and cross-correlations. While these codes result in the optimal performance, they put a limitation on the number of users due to the limited number of distinct code words. For brevity, we refer to an Optical Orthogonal Code with $\lambda = 1$ as an OOC in this paper. For a code with a λ value other than 1 we explicitly specify this value.

The organization of the rest of this paper is as follows. In Section 2, we obtain the exact performance of the OOC in a CDMA network by obtaining the probability distribution of interference patterns. We show that the approximation given in [2] is very close to the actual performance. We observe the dependence of the system performance on various system parameters. In Section 3, we consider a temporal code that is less structured than OOC. This code may have up to 2 overlaps between code words. With this temporal code, the network can accommodate more users and the code words can have more pulses per frame. As a result, the network can be expanded with little loss in the performance. The details of the analysis are contained in the appendices. We also give a simple, partial construction scheme for a $\lambda = 2$ code in the appendix.

2 Exact Analysis of OOC

In this section, we develop an analytical framework that will be used in an exact analysis of the Optical Orthogonal Code CDMA system described in the previous section.

In OOC, two code words can not overlap at more than one pulse position. There are K^2 ways of pairing the K pulses of two users. Then, the probability that a pulse belonging to an interfering user overlaps with one of the pulses of the desired user is given by

$$q = \frac{K^2}{2F} \tag{1}$$

where the factor 1/2 accounts for the probability that the interferer is transmitting a "1". Therefore the number of users that interfere with the desired user has a Binomial distribution with parameters $M - 1$ and q, where M is the total number of users [1, 2]. If $\Pr(l)$ denotes the probability of l interfering users, we have

$$\Pr(l) = \binom{M-1}{l} q^l (1-q)^{M-1-l} \quad l = 0, 1, \ldots, M-1 \,. \tag{2}$$

The number of interfering users does not completely specify the pattern in which the interference occurs. Given that there are l interfering users, each interfering at exactly one pulse position, there is a variety of interference patterns. To describe these patterns we define a K dimensional interference vector $\vec{\alpha}(l)$ whose i'th element, $\alpha_i(l)$ represents the number of pulses that overlap with the i'th pulse of the desired user. Since every interfering user contributes one and only one pulse, this vector must satisfy

$$\sum_{i=1}^{K} \alpha_i(l) = l \quad \alpha_i(l) \in \{0, 1, \ldots, l\} \tag{3}$$

For a given l there is a set of vectors $\mathcal{F}_l$ that satisfy (3). In particular if a vector $\vec{\alpha}$ satisfies (3), then so do all of its permutations. The set $\mathcal{F}_l$ can be written as

$$\mathcal{F}_l = \left\{ \vec{\alpha} : \sum_{i=1}^{K} \alpha_i = l,\ \alpha_i = 0, 1, \ldots, l \right\}. \tag{4}$$

As an example, consider the case $l = 5$, $K = 3$. The set $\mathcal{F}_5$ consists of 5 distinct vectors, $500, 410, 320, 311, 221$, and all the permutations of these vectors, $050, 005, 401$, etc.

The bit error probability, P_E can be written as

$$P_E = \sum_{l=1}^{M-1} \Pr(l) \sum_{\vec{\alpha} \in \mathcal{F}_l} P(\vec{\alpha}; \mathcal{F}_l) P_E(\vec{\alpha}) \tag{5}$$

where $\Pr(l)$ is as given in (2), $P(\vec{\alpha}; \mathcal{F}_l)$ is the probability that $\vec{\alpha} \in \mathcal{F}_l$ is the interference pattern given l interfering users, and $P_E(\vec{\alpha})$ is the probability of error given the interference pattern $\vec{\alpha}$. We first calculate $P(\vec{\alpha}; \mathcal{F}_l)$. A vector $\vec{\alpha}$ in $\mathcal{F}_l$ is an interference pattern of l interfering users each of which contributing in exactly one pulse position. A user is equally likely to interfere at any one of the K pulse positions independent of all other users. Thus $\vec{\alpha}$ obeys a multinomial distribution given as

$$P(\vec{\alpha}; \mathcal{F}_l) = \frac{l!}{\prod_{i=1}^{K} \alpha_i!} \frac{1}{K^l}. \tag{6}$$

$P_E(\vec{\alpha})$ depends on whether or not hard-limiting is performed at the receiver. We first consider the case with no hard-limiting. In this case, the vector $\vec{\alpha}$ will result in an error if $\sum_{i=1}^{K} \alpha_i \geq Th$ and if the transmitted bit is a "0". The first of these events occur if and only if $l \geq Th$. Therefore, the probability of error in this case is simply

$$P_E = \frac{1}{2} \sum_{l=Th}^{M-1} \binom{M-1}{l} q^l (1-q)^{M-1-l} \tag{7}$$

which is the same result given in [2].

In the rest of this section, we consider the hard-limiting case, for which an approximation to the performance is given in [2]. When the incoming light is hard-limited, the receiver is not affected by the actual entries of $\vec{\alpha}$, but only by their being 0 or not. We first reformulate (5) in a slightly different form. To do this, we make the following observation. The receiver does not have any preference among the K pulse positions; it simply counts the occupied pulse positions. Therefore two interference patterns which are permutations of each other will result in the same error probability, i.e. $P_E(\vec{\alpha}) = P_E(\vec{\beta})$ if $\vec{\alpha}$ and $\vec{\beta}$ are permutations of each other. Then the number of vectors that must be considered in determining the bit error probability can be substantially reduced by taking only one vector as a representative of all its permutations. Without loss of generality we take this representative vector to be the one with nonincreasing components. For the example above, 410 is the representative of the set $\{410, 401, 140, 104, 041, 014\}$. Thus (5) can be rewritten as

$$P_E = \sum_{l=1}^{M-1} \Pr(l) \sum_{\vec{\alpha} \in \mathcal{G}_l} P(\vec{\alpha}; \mathcal{G}_l) P_E(\vec{\alpha}) \tag{8}$$

where $\mathcal{G}_l$ is the set of representative vectors defined as

$$\mathcal{G}_l = \left\{ \vec{\alpha} : \sum_{i=1}^{K} \alpha_i = l, \ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K \geq 0, \ \alpha_i \text{ integer} \right\}. \tag{9}$$

The probability of the vector $\vec{\alpha}$ is modified so as to account for the probabilities of its permutations, as

$$P(\vec{\alpha}; \mathcal{G}_l) = \sum_{\vec{\beta} \in \Pi(\vec{\alpha})} P(\vec{\beta}; \mathcal{F}_l) \tag{10}$$

where $\Pi(\vec{\alpha})$ denotes the set of all the permutations of $\vec{\alpha}$.

The representation in (8) has two important advantages over the one in (5). First, it is notationally simpler to express $P_E(\vec{\alpha})$ when the vector $\vec{\alpha}$ has ordered elements. More importantly, there are much fewer vectors in $\mathcal{G}_l$ than in $\mathcal{F}_l$ for every l. (In the example above $\mathcal{F}_5$ has 21 elements while $\mathcal{G}_5$ has only 5 elements.)

Equation (10) can be calculated explicitly as follows. We observe from (6) that a vector $\vec{\alpha}$ and any of its permutations are equally probable in $\mathcal{F}_l$. Using this fact in (10) we obtain

$$P(\vec{\alpha}; \mathcal{G}_l) = N(\vec{\alpha}) P(\vec{\alpha}; \mathcal{F}_l) \tag{11}$$

where $N(\vec{\alpha})$ is the number of distinct permutations of $\vec{\alpha}$. If all the elements in $\vec{\alpha}$ were distinct, then $\vec{\alpha}$ would have $K!$ permutations. On the other hand, if an element α_i is repeated $R(\alpha_i)$ times in $\vec{\alpha}$, then every distinct permutation has $R(\alpha_i)!$ copies. Therefore, the number of distinct permutations is given by

$$N(\vec{\alpha}) = \frac{K!}{\prod_i R(\alpha_i)!} \tag{12}$$

where the product is understood to be taken over i for which α_i are distinct. (In the example, $N(500) = 3!/1!2! = 3$.)

$P_E(\vec{\alpha})$ can be determined as follows. The i'th pulse position of the desired user is interfered by α_i pulses, for $i = 1, 2, \ldots, K$. An error will occur only when the transmitted bit is a "0" and at least Th pulse positions are interfered. With the ordering we described, this can be formulated as

$$P_E(\vec{\alpha}) = \frac{1}{2}\left[1 - \delta(\alpha_{Th})\right] \tag{13}$$

where $\delta(\cdot)$ is the discrete unit-impulse.

Using Equations (2), (6) and (11)-(13) in (8) we obtain

$$P_E = \frac{1}{2}\sum_{l=1}^{M-1}\binom{M-1}{l} q^l (1-q)^{M-1-l} \sum_{\vec{\alpha}\in\mathcal{G}_l} \frac{K!}{\prod_i R(\alpha_i)!} \frac{l!}{\prod_{i=1}^{K}\alpha_i!} \frac{1}{K^l}\left[1 - \delta(\alpha_{Th})\right] \tag{14}$$

where q is given by (1). Equation (14) can be calculated by listing the vectors in $\mathcal{G}_l$. Since $\alpha_{Th} \geq \alpha_K$ for every $1 \leq Th \leq K$, it is clear that P_E will be minimized by the choice of $Th = K$. This choice also enables a simplification in (14) as will be seen below.

For $Th = K$, the inner summation in (14) becomes the probability of an interference pattern whose smallest entry is nonzero. This probability can be expressed as

$$P_E(l) \triangleq \Pr\left(\min\{\alpha_i :\ 1 \leq i \leq K\} > 0 : \vec{\alpha} \in \mathcal{F}_l\right) = \sum_{\substack{\vec{\alpha}\in\mathcal{F}_l \\ \vec{\alpha}>0}} \frac{l!}{\prod_{i=1}^{K}\alpha_i!}\,\frac{1}{K^l}\,. \tag{15}$$

By Lemma 1, that we state and prove in the Appendix A, this becomes

$$P_E(l) = \sum_{i=0}^{K-1}(-1)^i\binom{K}{i}\left(1 - \frac{i}{K}\right)^l . \tag{16}$$

In Lemma 2 of the Appendix A we show that the error probability reduces to

$$P_E = \frac{1}{2}\sum_{i=0}^{K}(-1)^i\binom{K}{i}\left(1 - \frac{qi}{K}\right)^{M-1} . \tag{17}$$

It is also proven in Appendix A that if $M - 1 < K$, then $P_E = 0$, since there are not sufficiently many interferers.

Figure 2 shows the error probability as a function of K for $Th = K$, for different values of M. The figure also shows the approximate result given in [1]. The exact and the approximate results are indistinguishable for most of the parameter values. Therefore we conclude that the approximation is in perfect agreement with the exact result.

For lower threshold values, we calculate P_E directly from Equation (14). The results are shown in Figure 3. It can be observed that the error probability depends critically on the threshold and that it decreases monotonically as Th changes from 1 to K as expected.

3 A Less Structured Temporal Code

The Optical Orthogonal Code has the desirable property of minimal cross-correlation and off-peak autocorrelation [1]. As a result, the interference from the other users is minimized over all possible temporal codes. The drawback, however, is that the number of different code words, i.e., the number of users in the system, is limited. The number of users that can be accommodated in an OOC system is upper bounded by $(F-1)/K(K-1)$ [7]. This puts an upper bound on the number of pulses, K, that can be used for a given number of users. Therefore there is a tradeoff between the number of users and the error performance. Our goal in this section is to analyze the performance of a less structured code which allows a large increase in the number of users without a significant loss in the performance.

The temporal code we consider in this section is one in which the auto- and cross-correlations are bounded by 2. We refer to this code as the $\lambda = 2$ code. It is known that [7] the constraint on the number of users with this code is

$$M \leq \frac{(F-1)(F-2)}{K(K-1)(K-2)} \tag{18}$$

which is considerably higher than OOC. For typical parameters of $F = 1000$ and $K = 4$, we can have at most 83 users with OOC and 41500 users with a $\lambda = 2$ code.

In order to evaluate the performance of a $\lambda = 2$ code, we need to find the probability that two code words overlap at one pulse position and at two pulse positions. Consider two code words a^1 and a^2, each of length F and weight K. Let a_j^k be 1 if the j'th pulse position of code word a^k contains a pulse, and 0 otherwise, for $1 \leq j \leq F$ and $k = 1, 2$. Let I_l be the correlation between a^1 and a^2 with l shifts. Then

$$I_l = \sum_{j=1}^{F} a_j^1 a_{j\oplus l}^2 \tag{19}$$

where $\oplus$ denotes addition modulo F. With the assumption that the users are nonsynchronous in their bit frames, the shift l between a^1 and a^2 is an integer-valued random variable which is uniformly distributed between 0 and $F-1$. Then the expected value of I_l is

$$E(I_l) = \sum_{l=0}^{F-1} \frac{1}{F} \sum_{j=1}^{F} a_j^1 a_{j\oplus l}^2 = \frac{K^2}{F} \tag{20}$$

The average correlation does not depend on the structure of the code but only on its weight and length. For a $\lambda = 2$ code, I_l is a ternary random variable that takes on values 0, 1 and

2. If p_1 and p_2 are the probabilities that I_l is 1 and 2 respectively, then

$$E(I_l) = p_1 + 2p_2 = \frac{K^2}{F} \tag{21}$$

by the use of (20).

Equation (21) is valid for any code with $\lambda = 2$. The particular values of p_1 and p_2 depends on the particular code. For example an OOC is a $\lambda = 2$ code with $p_1 = K^2/F$ and $p_2 = 0$. On the other hand, it is shown in the Appendix B that a (F, K) code with $\lambda = 2$ can be obtained from a $(F/2, K/2)$ OOC by splitting each chip of the OOC into two chips and preserving the pulse positions. It is shown that for this code $p_1 = K^2/2F - K^4/4F^2$ and $p_2 = K^2/4F + K^4/8F^2$. Finally, for the $(4, 2)$ code 1010 and 0101, $p_1 = 0$ and $p_2 = 1/2$. From these examples, it is seen that p_1 and p_2 may take a range of values that satisfy (21). Furthermore, p_1 and p_2 need not be uniform among different code word pairs of the same code. This would be the case, for instance, when a subset of the code forms an OOC.

We first analyze the case without hard-limiting. In this case, an error will occur when the data bit is a "0" and the interference contributes a total of Th or more pulses in the K pulse positions of the desired user's code word. The number of interfering users that contribute 1 and 2 pulses admit a trinomial distribution with parameters $M - 1$, $q_1 \triangleq p_1/2$ and $q_2 \triangleq p_2/2$, with the factors 1/2 introduced due to equiprobable on-off data bits. Then we have

$$\Pr(l_1, l_2) = \frac{(M-1)!}{l_1! l_2! (M-1-l_1-l_2)!} q_1^{l_1} q_2^{l_2} (1 - q_1 - q_2)^{M-1-l_1-l_2} \tag{22}$$

with $l_1 + l_2 < M$. Given l_1 users interfering at 1 pulse position and l_2 users interfering at 2 pulse positions, the total number of interfering pulses is $l_1 + 2l_2$. Therefore the error probability is

$$P_E = \frac{1}{2} \sum_{\substack{l_1 + 2l_2 \geq Th \\ l_1 + l_2 < M}} \Pr(l_1, l_2) \tag{23}$$

which can be rewritten as

$$P_E = \frac{1}{2} - \frac{1}{2} \sum_{l_1=0}^{Th-1} \sum_{l_2=0}^{\lfloor (Th-1-l_1)/2 \rfloor} \frac{(M-1)!}{l_1! l_2! (M-1-l_1-l_2)!} q_1^{l_1} q_2^{l_2} (1 - q_1 - q_2)^{M-1-l_1-l_2} \,. \tag{24}$$

Since $q_1 + 2q_2 = K^2/2F$, the quantity $c \triangleq q_1/(K^2/2F)$ measures the extent to which the $\lambda = 2$ code deviates from an OOC in interference statistics. For $c = 1$, the code is an OOC. For $c = 0$, the code is one with only two overlaps, which will be considered in detail for the hard-limiting case. Figure 4 shows the error probability as a function of c for the optimum threshold ($Th = K$) with $F = 1000$. The error probability increases with decreasing c, as expected. However, the number of users that can be accommodated in the network increases when a $\lambda = 2$ code is employed. Therefore the goal in the code design must be to obtain a code with a desired number of distinct code words and the largest possible c.

We now concentrate on the case with hard-limiting. The variation of the interference statistics among different $\lambda = 2$ codes and even among different code word pairs of the same code makes it difficult to have a general performance analysis for $\lambda = 2$ codes with hard-limiting. Therefore we will analyze a worst case situation which will provide an upper bound to the performance. This is the case where every interference causes an overlap of two pulses, which corresponds to $c = 0$ in the discussion above. Then, it is seen from (21) that $p_1 = 0$ and $p_2 = K^2/2F$. This probability distribution also maximizes the variance of the correlation, I_l, subject to the fixed average constraint in (21). This can be seen as follows. The variance is given by $\sigma^2 = (p_1 + 4p_2) - (p_1 + 2p_2)^2$, which by (21) becomes $\sigma^2 = 2p_2 + K^2/F - K^4/F^2$. This is maximized by choosing p_2 as large as possible.

We only consider the $Th = K$ case, since this is the optimal threshold. Under these conditions, the error probability can be written as

$$P_E = \sum_{l=1}^{M-1} \binom{M-1}{l} p^l (1-p)^{M-1-l} P_E(l) \tag{25}$$

where $p = K^2/4F$, with the factor of 1/2 introduced due to equiprobable on-off data bits, and $P_E(l)$ is the probability of error conditioned on l interfering users. With $Th = K$, an error occurs only when the interference pattern $\vec{\alpha}(l)$ has all nonzero entries. The interference pattern now belongs to the set

$$\mathcal{A}_l = \left\{ \vec{\alpha} : \sum_{i=1}^{K} \alpha_i = 2l,\ \alpha_i = 0, 1, \ldots, l \right\} \tag{26}$$

since every interferer contributes two pulses to the interference at distinct pulse positions. Therefore $P_E(l)$ can be written as

$$P_E(l) = \frac{1}{2} \sum_{\substack{\vec{\alpha} \in \mathcal{A}_l \\ \vec{\alpha} > 0}} \Pr(\vec{\alpha}; \mathcal{A}_l) \tag{27}$$

where $\vec{\alpha} > 0$ implies that all entries of $\vec{\alpha}$ are strictly positive. It is shown in Appendix C that the exact calculation of $\Pr(\vec{\alpha}; \mathcal{A}_l)$ requires the determination of all nonnegative integer solutions of a system of linear equations with $\binom{K}{2}$ unknowns and K equations. This system has a unique solution for $K = 2$ and $K = 3$. The resulting error probability is

$$P_E = \frac{1}{2}[1 - (1-p)^{M-1}] = \frac{1}{2}[1 - (1 - 1/F)^{M-1}] \tag{28}$$

for $K = 2$; and

$$P_E = \frac{1}{2} - \frac{3}{2}\left(1 - \frac{3}{2F}\right)^{M-1} + \left(1 - \frac{9}{4F}\right)^{M-1} \tag{29}$$

for $K = 3$.

For $K \geq 4$, the exact calculation of P_E becomes infeasible since the solution of the linear system above is not unique. Therefore, we provide upper and lower bounds to the error probability as follows. We note that the set $\mathcal{A}_l$ is a subset of the set $\mathcal{F}_{2l}$ defined in the previous section. Since the vectors in $\mathcal{A}_l$ are those vectors in $\mathcal{F}_{2l}$ whose entries are bounded by l, the probability that a vector randomly chosen from $\mathcal{A}_l$ has all positive entries is no less than the corresponding probability of $\mathcal{F}_{2l}$. Therefore $P_E(l)$ of $\lambda = 2$ code, as given in (27), is lower bounded by $P_E(2l)$ of the OOC with the same K. Using Equation (16) in conjunction with (25) we obtain a lower bound to the error probability as

$$P_E \geq \frac{1}{2}\sum_{i=0}^{K}(-1)^i \binom{K}{i} \left[1 - \frac{pi}{K}\left(2 - \frac{i}{K}\right)\right]^{M-1} . \tag{30}$$

An alternative way to explain this lower bound is to consider a probability experiment in which l persons are asked to play a lottery game. Each player picks a random pair of numbers from integers $1, 2, \ldots, K$. The probability that every number from 1 to K is picked by some person is exactly $P_E(l)$ for the $\lambda = 2$ code. If the game is modified such that the players now pick each of the two numbers independently, then the probability that every number is picked by some person is decreased from its previous value due to the possibility of a person picking non-distinct numbers. The latter probability also corresponds to the case of $2l$ persons picking one number each, which in turn results in $P_E(2l)$ of OOC.

When K is even, a similar upper bound can be obtained by considering an OOC system with $K/2$ pulses per bit frame. Consider a $K/2$ dimensional vector $\vec{\beta}$ from the set $\mathcal{F}_l$ of this OOC system. The concatenation of $\vec{\beta}$ with itself, $\vec{\alpha} = (\vec{\beta}, \vec{\beta})$, is a K dimensional vector which is in the set $\mathcal{A}_l$. If the vector $\vec{\beta}$ causes an error in the OOC, then the vector $\vec{\alpha}$ causes an error in the $\lambda = 2$ code. Thus, the subset of $\mathcal{A}_l$ which consists of the vectors that cause an error contains those vectors which are concatenations as above. Therefore $P_E(l)$ of $\lambda = 2$ code is upper bounded by $P_E(l)$ of the OOC with $K/2$ pulses per frame. This results in the following upper bound on the error probability :

$$P_E \leq \frac{1}{2} \sum_{i=0}^{K/2} (-1)^i \binom{K/2}{i} \left(1 - \frac{2pi}{K}\right)^{M-1} . \tag{31}$$

This upper bound can also be explained with the previous lottery analogy. The game is now modified to be played such that every person picks a number from 1 to $K/2$ and chooses his second number to be the first number plus $K/2$. With this modification, the probability that every number is picked by some person is increased from the original game where the players could pick any pair of numbers. This results in the upper bound in (31).

Figure 5 shows the upper and lower bound as a function of K for different values of M. It can be seen that the bounds determine the error probability within two orders of magnitude. The lower bound is expected to be tight for large K, since the probability of a user putting both of its pulses in the same chip is $1/K$ in the lower bounding argument. Since the error probability that we seek is already an upper bound to the performance of the family of $\lambda = 2$ codes the lower bound may be used as a reliable performance measure. In fact, the lower corresponds to the exact error probability of a code with $p_2 = (K-1)p_1$. A comparison of Figure 2 and Figure 5 indicates that the performance of a $\lambda = 2$ code is a few orders of magnitude poorer than that of a OOC with the same values of M, K and F. However, for the same number of users and the same number of chips per bit, the $\lambda = 2$ code may have a much larger value of K, hence compensating for the loss in the performance. For instance, for 50 users with $F = 1000$, $K_{\max} = 5$ for OOC, while $K_{\max} = 28$ for the $\lambda = 2$ code. The resulting error probability is 10^{-4} for OOC while $K = 12$ results in an error probability of 10^{-5} for the $\lambda = 2$ code. Hence the $\lambda = 2$ code may even surpass the OOC in the performance.

4 Conclusions

We have used simple combinatorial tools to characterize the probabilistic behavior of the interference patterns in CDMA networks which employ temporal codes. This characterization resulted in exact expressions for the bit error probability for $\lambda = 1$ codes, both with and without hard-limiting. We also extended this analysis to temporal codes with $\lambda = 2$. While these codes result in a larger number of interfering pulses, they enable the network to have more users in comparison with $\lambda = 1$ codes. Our analysis shows that these temporal codes may provide a way to increase the number of users in the network without a significant degradation in the performance of the network. In fact, $\lambda = 2$ codes have the potential of improving upon the performance of $\lambda = 1$ codes since they allow a larger number of pulses per frame.

Appendix

A Derivation of Error Probability for $Th = K$

In this appendix, we derive the equations (16) and (17) which give the probability of error in a simple form for the $Th = K$ case. We start with a Lemma that will be used to obtain (16) from Equation(15).

Lemma 1

$$\sum_{\substack{\vec{\alpha}\in\mathcal{F}_l \\ \vec{\alpha}>0}} \frac{l!}{\prod_{i=1}^{K} \alpha_i!} = \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} (K-i)^l \tag{A.1}$$

Proof : We will use induction on K. Let $S(K,l)$ denote the left hand side of (A.1). Then $S(1,l) = l!/l! = 1$ and the claim is true for $K = 1$. Assume the claim is also true for K. Then,

$$\begin{aligned}
S(K+1,l) &= \sum_{\alpha_{K+1}=1}^{l-1} \frac{l!}{\alpha_{K+1}!(l-\alpha_{K+1})!} S(K, l-\alpha_{K+1}) \\
&= \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} \sum_{\alpha_{K+1}=1}^{l-1} \binom{l}{\alpha_{K+1}} (K-i)^{l-\alpha_{K+1}} \\
&= \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} \left[(K+1-i)^l - (K-i)^l - 1\right] \\
&= \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} (K+1-i)^l - \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} (K-i)^l - \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} .
\end{aligned}$$

We note that the last summation above is $(-1)^{K+1}$. This can be observed by noting that it is a binomial sum of $(1-1)^K$ with the last term missing. Incrementing the index of the second summation by 1 we obtain

$$\begin{aligned}
S(K+1,l) &= \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} (K+1-i)^l + \sum_{i=1}^{K} (-1)^i \binom{K}{i-1} (K+1-i)^l + (-1)^K \\
&= (K+1)^l + (-1)^K K + \sum_{i=1}^{K-1} (-1)^i \left[\binom{K}{i} + \binom{K}{i-1}\right] (K+1-i)^l + (-1)^K .
\end{aligned}$$

Now we use the identity $\binom{K}{i} + \binom{K}{i-1} = \binom{K+1}{i}$, and regroup terms to obtain

$$S(K+1,l) = \sum_{i=0}^{K}(-1)^i \binom{K+1}{i} (K+1-i)^l$$

which is the claim for $K+1$. □

Equation (16) follows directly from Lemma 1 simply by the introduction of the factor $1/K^l$.

In the following lemma we prove that the error probability is given by Equation (17).

Lemma 2 The probability of error for OOC with $Th = K$ is given by

$$P_E = \frac{1}{2}\sum_{i=0}^{K}(-1)^i \binom{K}{i} (1 - \frac{qi}{K})^{M-1} . \tag{A.2}$$

Proof : P_E is given by

$$P_E = \frac{1}{2}\sum_{l=1}^{M-1} \Pr(l) P_E(l)$$

where

$$\Pr(l) = \binom{M-1}{l} q^l (1-q)^{M-1-l}$$

and $P_E(l)$ is given by (16). Then

$$\begin{aligned}
P_E &= \frac{1}{2}\sum_{l=1}^{M-1} \binom{M-1}{l} q^l (1-q)^{M-1-l} \sum_{i=0}^{K-1}(-1)^i \binom{K}{i} \left(1 - \frac{i}{K}\right)^l \\
&= \frac{1}{2}\sum_{i=0}^{K-1}(-1)^i \binom{K}{i} \sum_{l=1}^{M-1} \binom{M-1}{l} \left(\frac{q(K-i)}{K}\right)^l (1-q)^{M-1-l} \\
&= \frac{1}{2}\sum_{i=0}^{K-1}(-1)^i \binom{K}{i} \left[\left(1 - \frac{qi}{K}\right)^{M-1} - (1-q)^{M-1}\right] \\
&= \frac{1}{2}\sum_{i=0}^{K-1}(-1)^i \binom{K}{i} \left(1 - \frac{qi}{K}\right)^{M-1} - \frac{1}{2}(1-q)^{M-1} \sum_{i=0}^{K-1}(-1)^i \binom{K}{i} \\
&= \frac{1}{2}\sum_{i=0}^{K}(-1)^i \binom{K}{i} \left(1 - \frac{qi}{K}\right)^{M-1}
\end{aligned}$$

where we used the property mentioned in the proof of Lemma 1 in eliminating the second summation in the next to last line. □

The following lemma and its corollary states the fact that if there are not sufficiently many interferers then an error can not occur.

Lemma 3 $P_E(l) = 0$ for $l < K$.

Proof : Let $S(K,l)$ be as in the proof of Lemma 1. Then $S(K,l) = K^l P_E(l)$. We will show that $S(K,l) = 0$ for $l < K$. From Lemma 1

$$S(K,l) = \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} (K-i)^l \,.$$

Letting $j = K - i$ we obtain

$$S(K,l) = (-1)^K \sum_{j=1}^{K} (-1)^j \binom{K}{j} j^l \,.$$

Let a be a real number and define the function $h(a)$ as

$$h(a) = \sum_{j=0}^{K} \binom{K}{j} a^j = (1+a)^K \,.$$

Then,

$$\begin{aligned}
[ah'(a)]_{a=-1} &= (-1)^K S(K,1) = 0 \\
[a^2 h''(a)]_{a=-1} &= (-1)^K [S(K,2) - S(K,1)] = 0 \\
[a^n h^{(n)}(a)]_{a=-1} &= (-1)^K \sum_{i=1}^{n} b_i S(K,i) = 0 \quad n < K
\end{aligned}$$

where b_i are the coefficient of the term x^i in the expansion of the polynomial $x(x-1)\cdots(x-i+1)$. From the above system of equations $S(K,n)$ can be recursively solved to be 0 for all $n < K$. □

An immediate result of this lemma is that $P_E = 0$ if $M - 1 < K$.

B Partial Construction of a $\lambda = 2$ Code

In this appendix, we give a method of constructing a $\lambda = 2$ code from a given $\lambda = 1$ code. The construction is not complete in the sense that more code words could be added to the code without changing the $\lambda = 2$ property. Our goal here is to demonstrate a nontrivial $\lambda = 2$ code and its statistical properties, rather than study the construction of $\lambda = 2$ codes in detail.

The construction of $\lambda = 1$ codes has been studied in detail in [7]. Let's consider one such code with length $F/2$ and weight $K/2$. For appropriately chosen F and K, one can obtain the maximum number of code words, i.e.,

$$M = \left\lfloor \frac{F/2 - 1}{\frac{K}{2}\left(\frac{K}{2} - 1\right)} \right\rfloor \tag{B.1}$$

code words following the design methods given in [7]. A $\lambda = 2$ code with length F and weight K can be obtained from this $\lambda = 1$ code by splitting each chip into two while preserving the pulses. For example, if a code word in $\lambda = 1$ code is 10100, then the corresponding code word in the new code is 1100110000. The new code will have its auto- and cross-correlations bounded by 2, since an overlap in the original code corresponds to two overlaps in the new code.

Consider a pair of code words in the original $\lambda = 1$ code and the corresponding pair of code words in the new $\lambda = 2$ code. Let I_l $(l = 0, 1, \ldots, F/2 - 1)$ be the cross-correlation of the two original code words with l shifts, and similarly let J_l $(l = 0, 1, \ldots, F - 1)$ be the cross-correlation of the corresponding code words in the new code. A cyclic shift of l chips in the original code corresponds to a cyclic shift of $2l$ chips in the new code. Therefore

$$J_{2l} = 2I_l \quad l = 0, 1, \ldots, F/2 - 1 \ . \tag{B.2}$$

On the other hand, odd number of shifts in the new code corresponds to half-integer shifts in the original code. When the new code is shifted $2l+1$ chips, the original code is in transition from l shifts to $l + 1$ shifts. There are four cases to consider. If $I_l = I_{l+1} = 0$, there are no pulse overlaps in the initial and final shifts and hence there can be no pulse overlaps in transition. Thus, $J_{2l+1} = 0$. If $I_l = 1$ and $I_{l+1} = 0$, there is one pulse overlap with l shift which disappears after one more shift. Then, in the midway, there is a half pulse overlap, i.e., $J_{2l+1} = 1$. The same is true when $I_l = 0$ and $I_{l+1} = 1$. Finally, when $I_l = I_{l+1} = 1$, there are one pulse overlaps at both initial and final shifts. In the midway, there will be two half-pulse overlaps, resulting in $J_{2l+1} = 2$. Therefore, in general the relation

$$J_{2l+1} = I_l + I_{l+1} \quad l = 0, 1, \ldots, F/2 - 1 \tag{B.3}$$

holds. Combining (B.2) and (B.3) one obtains

$$J_l = \begin{cases} 2I_{l/2} & l \text{ even,} \\ I_{(l-1)/2} + I_{(l+1)/2} & l \text{ odd .} \end{cases} \tag{B.4}$$

Equation (B.4) not only serves as a formal proof of the fact that the new code has $\lambda = 2$, but it can also be used to obtain the statistics of pulse overlaps. The probability of one overlap in the new code can be written as

$$\begin{aligned} p_1 &= \Pr(J_l = 1) \\ &= \frac{1}{2}\Pr(2I_{l/2} = 1 \mid l \text{ even}) + \frac{1}{2}\Pr(I_{(l-1)/2} + I_{(l+1)/2} = 1 \mid l \text{ odd}) \end{aligned}$$

where we used the fact that the shift l is uniformly distributed over the integers 0 through $F-1$. The first term above vanishes since I_k is either 0 or 1 for all k. Because of the same fact, the second term is the probability that either $I_{(l-1)/2}$ or $I_{(l+1)/2}$, but not both, is 1, which can be rewritten to yield

$$\begin{aligned} p_1 &= \frac{1}{2}\Pr(I_{(l-1)/2} \neq I_{(l+1)/2} \mid l \text{ odd}) \\ &= \frac{1}{2}\left[2\frac{(K/2)^2}{F/2}\left(1 - \frac{(K/2)^2}{F/2}\right)\right] \\ &= \frac{K^2}{2F} - \frac{K^4}{4F^2} \end{aligned} \tag{B.5}$$

where we used the symmetry and the previous result for the probability of overlap for a $\lambda = 1$ code (see (1) in Section 2).

The probability of two overlaps can be similarly found as:

$$\begin{aligned} p_2 &= \Pr(J_l = 2) \\ &= \frac{1}{2}\Pr(I_{l/2} = 1 \mid l \text{ even}) + \frac{1}{2}\Pr(I_{(l-1)/2} + I_{(l+1)/2} = 2 \mid l \text{ odd}) \\ &= \frac{1}{2}\frac{K^2}{2F} + \frac{1}{2}\left(\frac{K^2}{2F}\right)^2 \\ &= \frac{K^2}{4F} + \frac{K^4}{8F^2}\,. \end{aligned} \tag{B.6}$$

Note from (B.5) and (B.6) that $p_1 + 2p_2 = K^2/F$ as predicted by Equation (21) of Section 3.

Although this partial construction scheme does not increase the number of code words from the original $\lambda = 1$ code, this number is more than doubled from what could be obtained by a length F, weight K, $\lambda = 1$ code. This is because with the latter code the maximum number of code words is

$$M(\lambda = 1) = \frac{F-1}{K(K-1)} \tag{B.7}$$

while with the $\lambda = 2$ code we obtained, the number of code words is

$$M(\lambda = 2) = \frac{F/2 - 1}{\frac{K}{2}\left(\frac{K}{2} - 1\right)}\,. \tag{B.8}$$

The ratio of these two numbers is

$$\frac{M(\lambda = 2)}{M(\lambda = 1)} = 2\,\frac{F-2}{F-1}\frac{K-1}{K-2} \tag{B.9}$$

which is larger than 2 for all $K < F$. Therefore, this construction enables the number of users to be doubled for a given F and K. However, $\lambda = 2$ codes have a much larger potential in this aspect.

C Error Probability Calculation for $\lambda = 2$ Codes

In this appendix, we investigate the calculation of the error probability of the worst-case error probability for a $\lambda = 2$ code given by Equations (25)-(27). Essential to this calculation is the determination of $\Pr(\vec{\alpha}; \mathcal{A}_l)$, the probability that a particular vector $\vec{\alpha}$ in $\mathcal{A}_l$ is the interference pattern conditioned on l interfering users. We define n_{ij} $(1 \le i < j \le K)$ to be the number of users which interfere at pulse positions i and j. Then, $\vec{\alpha}$ is related to $\{n_{ij}\}$ via

$$\begin{aligned} \alpha_1 &= n_{12} + n_{13} + \ldots n_{1K} \\ \alpha_2 &= n_{12} + n_{23} + \ldots n_{2K} \\ &\vdots \\ \alpha_K &= n_{1K} + n_{2K} + \ldots n_{K-1,K} \end{aligned} \tag{C.1}$$

For a given $\vec{\alpha}$, this is a set of linear equations with $\binom{K}{2}$ unknowns and K equations. If this system has N_l nonnegative integer solutions $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_{N_l}$, where each vector $\vec{n}_k$ denotes a $\binom{K}{2}$ dimensional vector of $\{n_{ij}(k)\}$, then the desired probability is

$$\Pr(\vec{\alpha}; \mathcal{A}_l) = \sum_{k=1}^{N_l} \frac{l!}{\prod_{i=1}^{K-1} \prod_{j=i+1}^{K} (n_{ij}(k)!)} \binom{K}{2}^{-l} \tag{C.2}$$

It is difficult to obtain all the nonnegative integer solutions to the system in (C.1) and then to evaluate (C.2), when $\binom{K}{2} > K$. Therefore, the exact calculation of error probability is impractical for $K \ge 4$. We obtain this probability for $K = 2$ and $K = 3$ here.

For $K = 2$, the two equations in (C.1) are redundant with the solution $n_{12} = \alpha_1 = \alpha_2 = l$, i.e., with $\mathcal{A}_l$ having a single vector $\vec{\alpha} = (l, l)$. Then, (C.2) reduces to $\Pr(\vec{\alpha}; \mathcal{A}_l) = 1$ and the resulting error probability is, via (25) and (27),

$$P_E = \frac{1}{2}[1 - (1-p)^{M-1}] = \frac{1}{2}[1 - (1 - 1/F)^{M-1}] \tag{C.3}$$

which states that an error will occur whenever the data bit is "0" and at least one user interferes.

For $K = 3$, (C.1) has a single solution given by $n_{12} = l - \alpha_3$, $n_{13} = l - \alpha_2$, $n_{23} = l - \alpha_1$, which by the use of (C.2) results in

$$\Pr(\vec{\alpha}; \mathcal{A}_l) = \frac{l!}{(l-\alpha_1)!(l-\alpha_2)!(l-\alpha_3)!}(\frac{1}{3})^l . \tag{C.4}$$

Using (26), we obtain, after some manipulations,

$$\begin{aligned} P_E(l) &= \sum_{i=0}^{l}\sum_{j=0}^{l-i} \frac{l!}{i!j!(l-i-j)!}(\frac{1}{3})^l - 3(\frac{1}{3})^l \\ &= 1 - \left(\frac{1}{3}\right)^{l-1} \end{aligned}$$

which in conjunction with (25) results in

$$P_E = \frac{1}{2} - \frac{3}{2}\left(1 - \frac{3}{2F}\right)^{M-1} + \left(1 - \frac{9}{4F}\right)^{M-1} . \tag{C.5}$$

References

[1] J. A. Salehi, "Code division multiple-access techniques in optical fiber networks–Part I: Fundamental principles," *IEEE Trans. Commun.*, vol. 37, pp. 824-833, Aug. 1989.

[2] J. A. Salehi and C. A. Brackett, "Code division multiple-access techniques in optical fiber networks–Part II: Systems Performance Analysis," *IEEE Trans. Commun.*, vol. 37, pp. 834-842, Aug. 1989.

[3] J. Y. Hui, "Pattern code modulation and optical decoding–A novel code division multiplexing technique for multifiber networks," *IEEE J. Selected Areas Commun.*, vol. SAC-3, pp. 916-927, Nov. 1985.

[4] P. R. Prucnal, M. A. Santoro, and T. R. Fan, "Spread spectrum fiber-optic local area network using optical processing," *J. Lightwave Technol.*, vol. LT-4, pp. 547-554, May 1986.

[5] G. J. Foschini and G. Vanucci, "Using spread spectrum in a high capacity fiber-optic local area network," *J. Lightwave Technol.*, vol. LT-6, pp. 370-379, March 1988.

[6] J. A. Salehi, A. M. Weiner, and J. P. Heritage, "Coherent ultrashort light pulse code division multiple access communication systems," *J. Lightwave Technol.*, vol. LT-8, March 1990.

[7] F. R. K. Chung, J. A. Salehi, and V. K. Wei, "Optical orthogonal codes: Design, analysis, and applications," *IEEE Trans. Inform. Theory*, vol. 37, pp. 595-604, May 1989.

[8] M. Y. Azizoğlu, J. A. Salehi, and Y. Li, "Optical CDMA via temporal codes," submitted to *IEEE Trans. Commun.*, March 1990.

List of Figure Captions

Figure 1: A typical optical CDMA system.

Figure 2: Exact and approximate error probability for $F = 1000$, $M = 10, 30, 50$.

Figure 3: Dependence of error probability on threshold for $\lambda = 1$ code for $F = 1000$, $M = 30$.

Figure 4: Dependence of error probability on the statistics of the $\lambda = 2$ code without hard-limiting: a) $M = 10$, b) $M = 50$.

Figure 5: Upper and lower bounds to the worst-case error probability for $\lambda = 2$ code with hard-limiting for $F = 1000$, $M = 10, 50$.

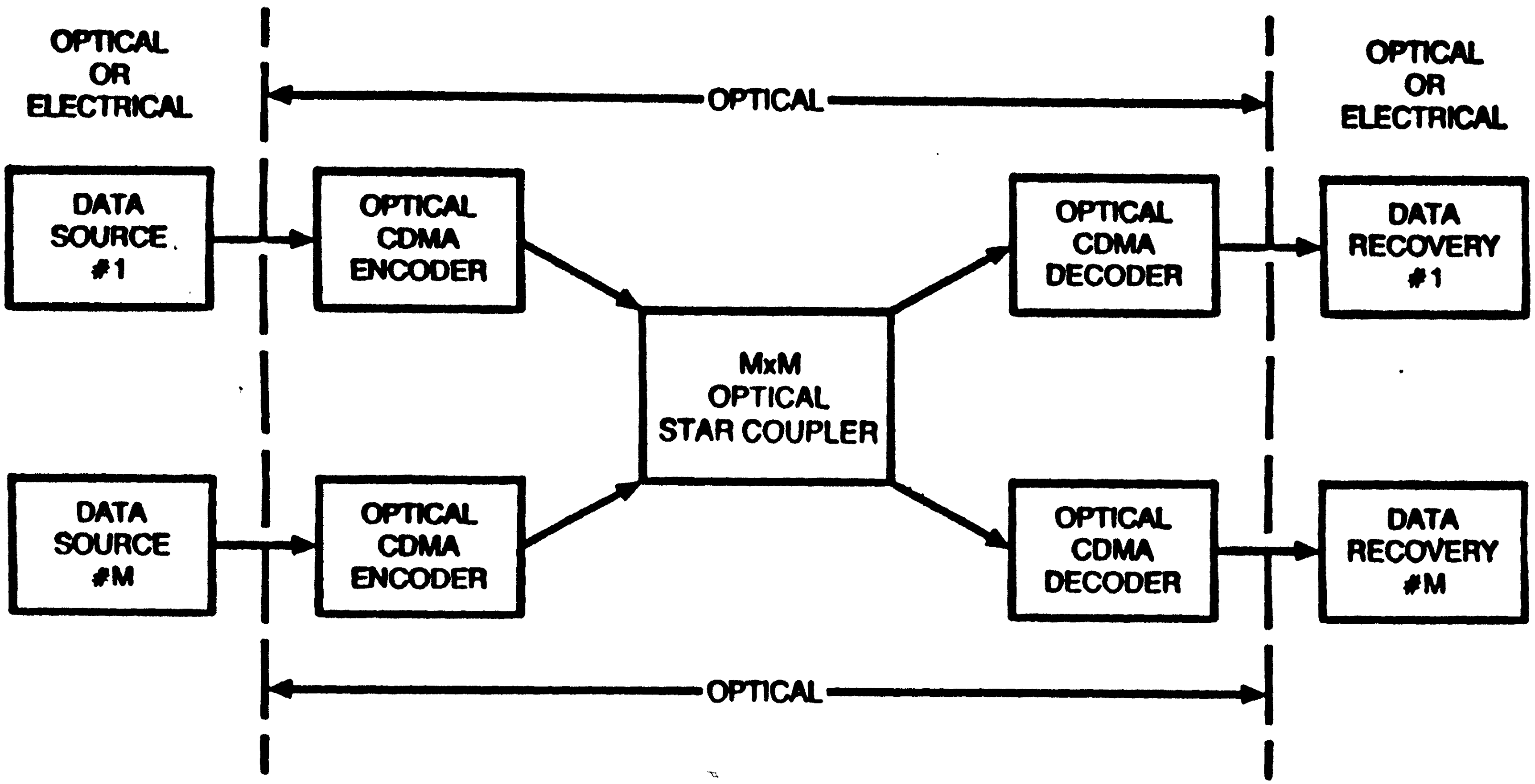
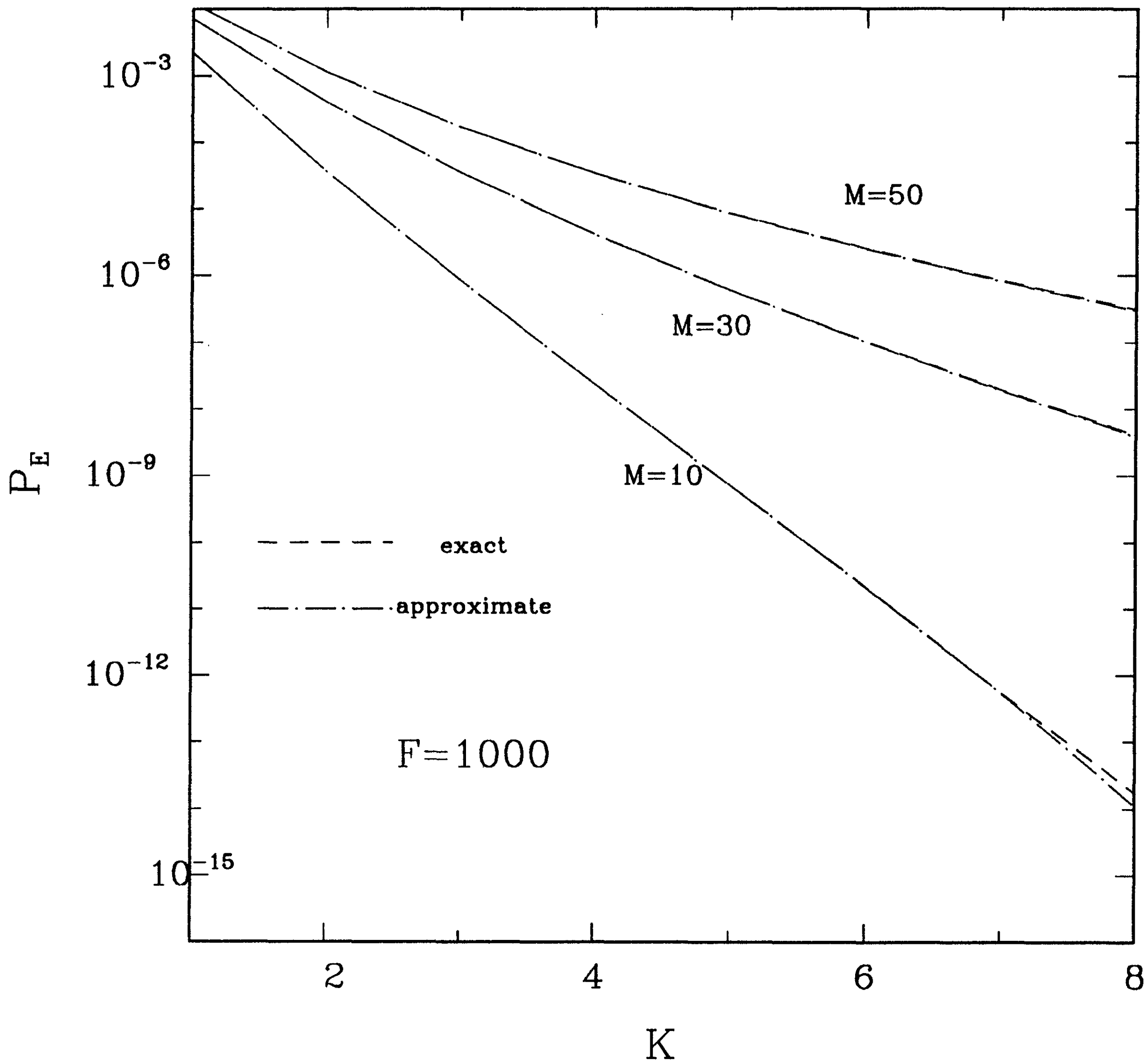

OPTICAL OR ELECTRICAL
OPTICAL
OPTICAL OR ELECTRICAL
DATA SOURCE #1
OPTICAL CDMA ENCODER
MxM OPTICAL STAR COUPLER
OPTICAL CDMA DECODER
DATA RECOVERY #1
DATA SOURCE #M
OPTICAL CDMA ENCODER
OPTICAL CDMA DECODER
DATA RECOVERY #M
OPTICAL

FIGURE 1



10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
P_E
M=50
M=30
M=10
exact
approximate
F=1000
2
4
6
8
K

Fig. 2

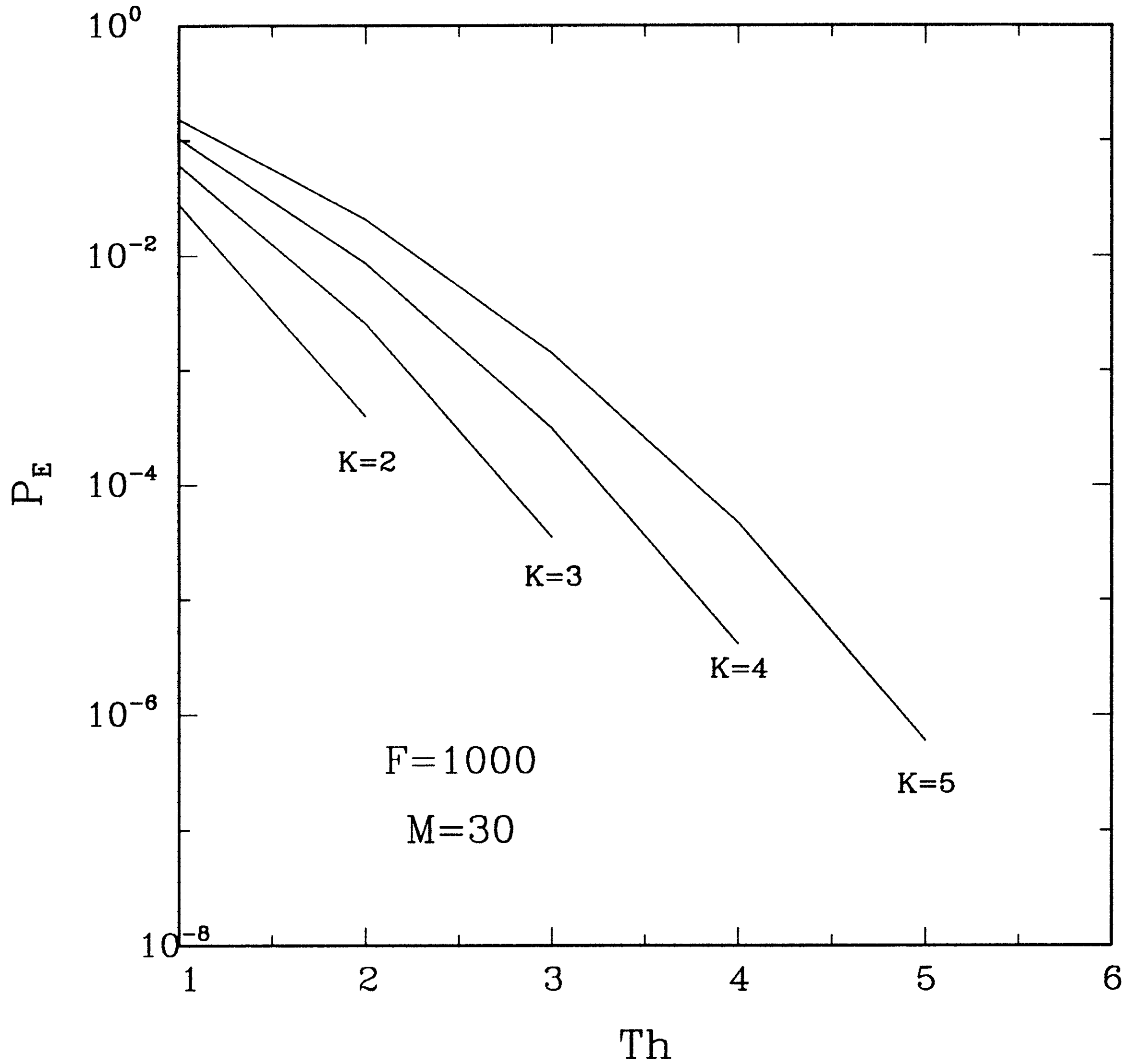


Fig. 3

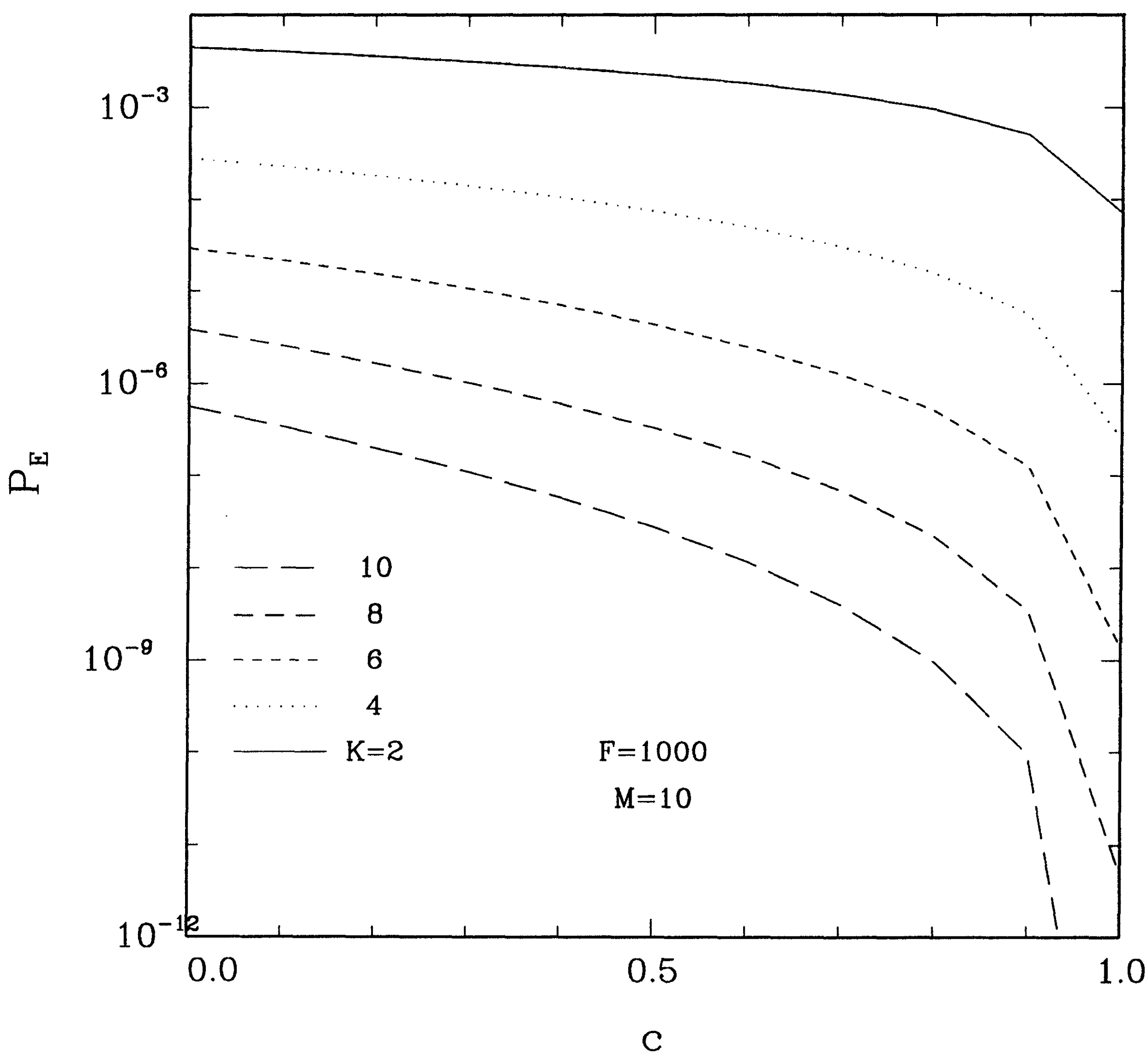


Fig. 4a

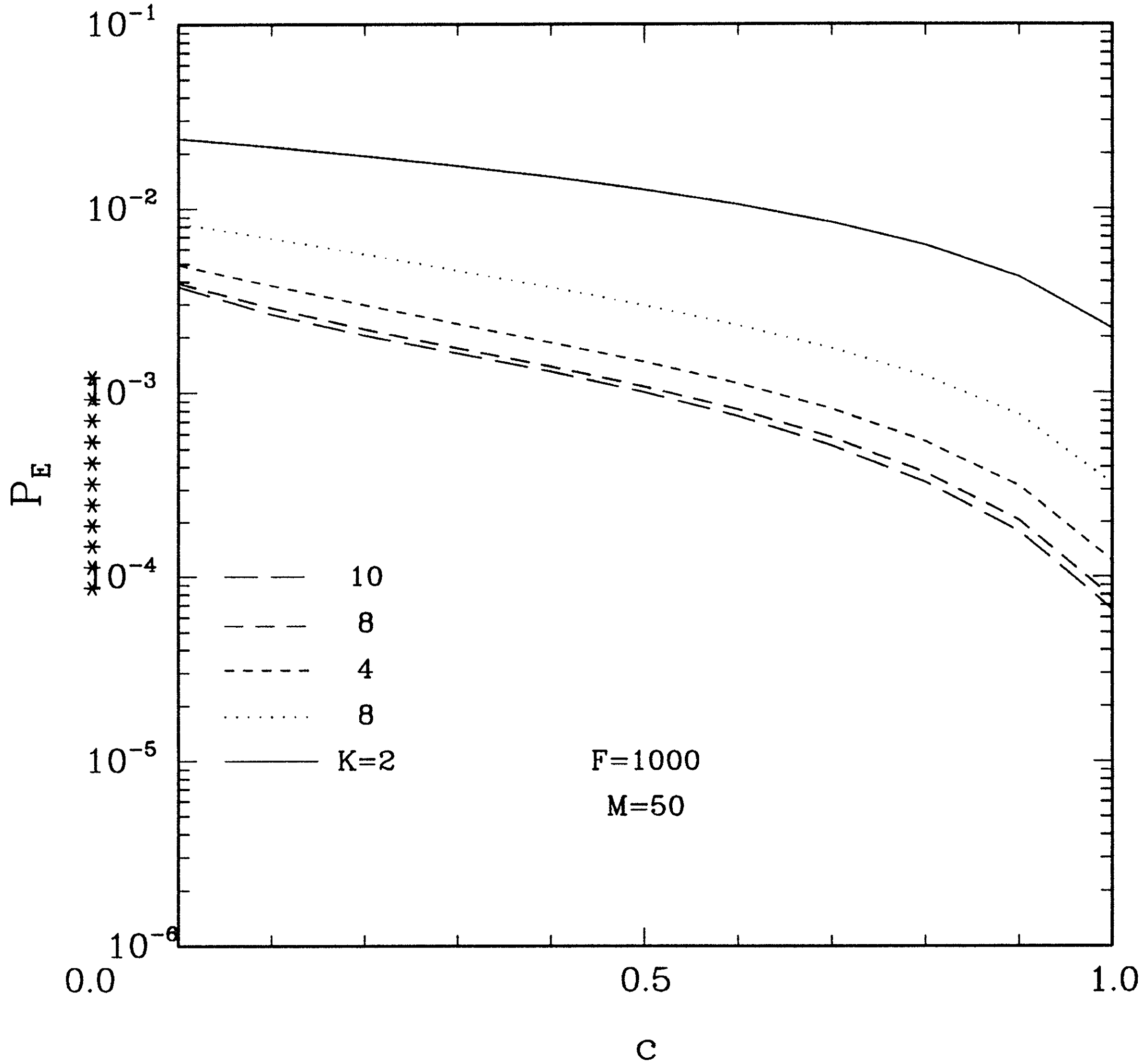


Fig. 4b

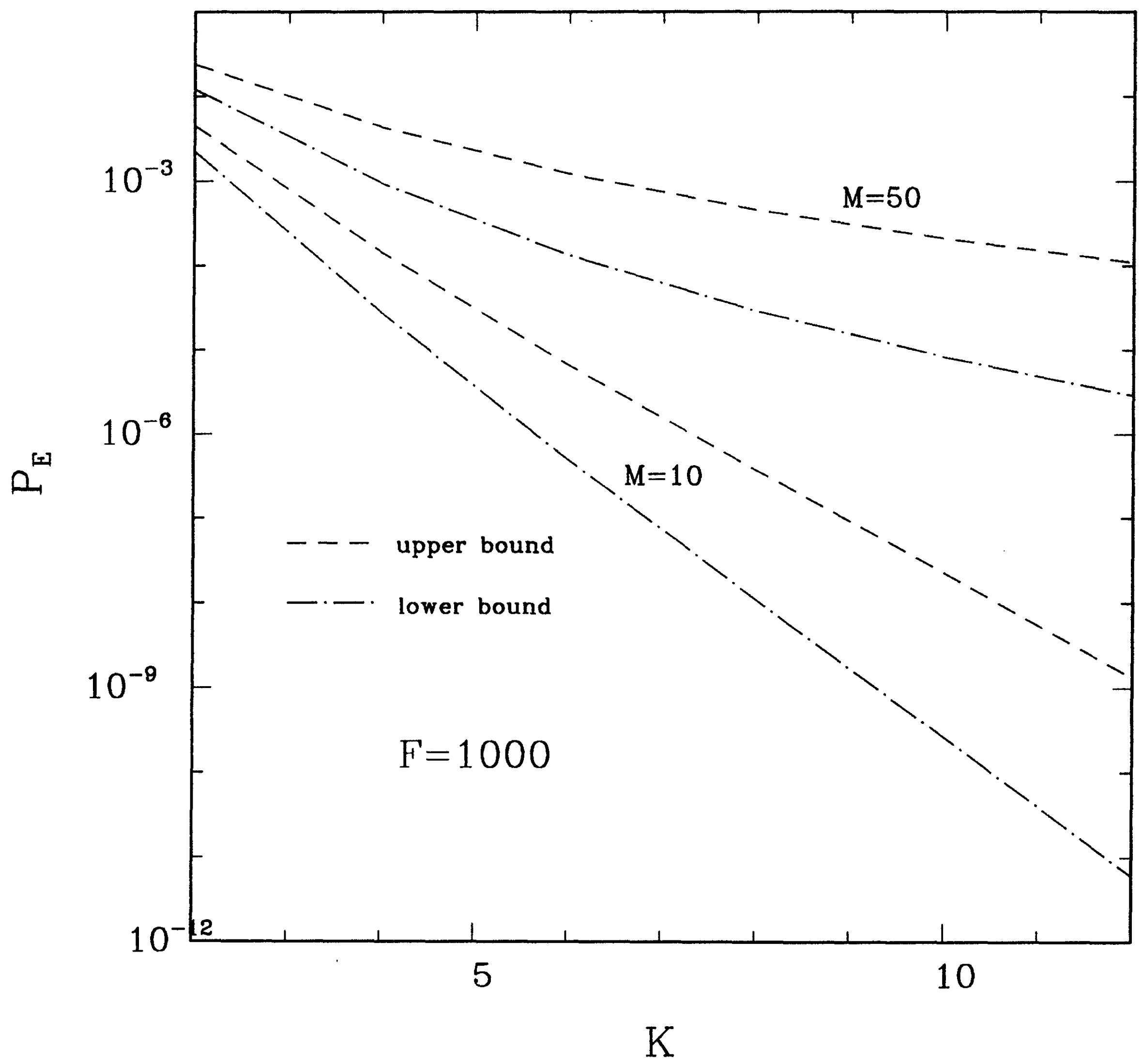

M=50
M=10
upper bound
lower bound
F=1000
10^{-3}
10^{-6}
10^{-9}
10^{-12}
5
10
K
P_E

Fig. 5