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PRESENCE OF MAGNITUDE BOUNDED REAL
UNCERTAINTY¹

by

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STABILITY AND PERFORMANCE ANALYSIS IN THE PRESENCE OF MAGNITUDE BOUNDED REAL UNCERTAINTY

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Abstract

This paper is concerned with stability and performance Riccati equation based analysis methods for perturbed feedback systems. The elements of the state space representation of the systems are assumed to be linearly perturbed with real, magnitude bounded uncertainties. Performance is defined as the value of the H_∞ norm of the transfer function of interest. An analysis method based on simultaneous satisfaction of a set of Riccati inequalities is studied as well as a method based on a single, "overall" Riccati equation. Special attention is paid to the conservatism of these methods. It is shown when the perturbation is treated as real, the result can be at least as conservative as if the perturbation were unstructured, frequency dependent uncertainty. This is further used to show that existing synthesis methods for H_∞ norm minimization may be used for obtaining a compensator that makes the closed loop system satisfy the single Riccati equation criterion.

1. Introduction

Real parameter perturbations are very important in stability and performance analysis and synthesis. In this paper we discuss Riccati equation analysis methods for the stability and performance robustness in the presence of real uncertainty in the state space representation of the closed loop system. The first method is based on the existence of the common solution for a set of Riccati type inequalities. It is obtained by extending Horisberger and Belanger's (1976) result on Lyapunov stability with respect to

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perturbation in the "A" matrix only to the case where the perturbation enter all parts of the state space representation of the system. Furthermore, a performance specification is introduced as the bound on the H_∞ norm of the certain transfer function that should remain satisfied for the perturbed system. This is done without constructing an auxiliary passive system as it was done in (Boyd and Yang,1988).

In order to reduce the numerical complexity of the above method, a more conservative but numerically simpler analysis method based on solving a single Riccati equation is discussed. The results of Bernstein and Haddad (1988) for the perturbation in "A" matrix and those of Yeh et al (1989) based on existence of a solution to the single Riccati equation, are extended to all the elements of the closed loop system state space representation. This is done by using the Peterson-Hollot bounding function (Peterson and Hollot,1986), originally introduced for the perturbations in the "A" matrix only. Conservatism of the single Riccati equation method is studied in detail. It is shown that in cases where the perturbation is confined to the "A" matrix as in (Bernstein and Haddad,1988) and (Madiwale et al,1988), the resulting condition based on the existence of a solution to the overall Riccati equation is equal to the "Small Gain Theorem" condition (Zames,1965) for the perturbed system. This means that the structure of the perturbation is neglected completely and that the real uncertainty is treated as frequency dependent. When the perturbation is present in all the elements of the state space representation, it is shown that the single Riccati equation condition is equal to the H_∞ norm bound on an auxiliary system. The latter is obtained by augmenting additional inputs and outputs to the nominal system. This is further exploited to show that any synthesis method that satisfies the H_∞ norm condition for the augmented system will also satisfy the overall Riccati equation criterion.

Since the present paper discusses and extends some existing methods, the outline of different methods for stability and performance robustness with respect to real uncertainty

is now given.

1.1 Outline of Existing Methods

The existing approaches for analysis are classified according to the representation of the system. Hence, there are polynomial approaches among which the best known result is Kharitonov's Theorem (Kharitonov, 1978). It establishes the conditions on the coefficients of the characteristic polynomial, which are assumed to be independently varying, so that it remains Hurwitz.

Besides Kharitonov's result, the other group of methods for checking the stability of a perturbed characteristic polynomial are those based on the "zero exclusion criterion." They exploit the fact that the roots of polynomials are continuous functions of their coefficients. The assumptions are that the nominal system is stable and that the coefficients of the characteristic polynomial depend on the independent, magnitude bounded, parameters. Then, stability is guaranteed if the polynomial has no roots on the imaginary axis for any value of the varying coefficients. Based on the above criterion, de Gaston and Safonov (1988) have defined a nonconservative measure for the largest stability hyperbox in the parameter space. They have used the properties of the image of the hyperbox vertices mapped into the complex plane through the characteristic polynomial of the perturbed system. A more general result was presented by Saeki (1986), where the uncertain parameters could be frequency dependent.

Another group of approaches deal with the presence of real parameter perturbation in the state space representation of a system. They are based on the concept of Lyapunov stability. The results, except in special cases, give only sufficient conditions and are conservative as shown in (Bhattacharyya, 1987).

A very detailed survey of the literature on parametric uncertainty can be found in (Bhattacharyya, 1987) and in (Siljak, 1989). We will mention only those results that deal with real parameter perturbation in state space models. Special attention will be given to the

results on robust performance of perturbed systems when the performance is associated with the infinity norm of a particular transfer function.

A sufficient condition for robust stability in the presence of real parameter perturbations in the state space was originally presented in (Chang and Peng,1972) and (Patel,Toda, and Sridhar,1977). The "A" matrix of the perturbed system was given by its nominal value A_0 and the associated error matrix $E(t)$ that could be time varying. The corresponding Lyapunov equation for the nominal system is

$$A_0' P + P A_0 + 2Q = 0 \quad (1.1)$$

where $Q > 0$ and A_0 is stable. Then, it was shown that the system remains stable if the maximum singular value of the error matrix E is strictly smaller than the ratio between the smallest singular value of Q and the largest singular value of P . This ratio represents the upper bound on the "two" norm of the admissible perturbation matrix E and it is clearly dependent on the choice of the matrix Q . It was shown by Patel and Toda (1980) that the choice for Q in (1.1) that maximizes above defined ratio is the identity matrix. Furthermore, the structure of the perturbation matrix E was taken into account by checking the largest magnitude of its elements against the scaled ratio defined above.

The structural information was further taken in consideration by Yedavalli (1985a,b) and Yedavalli and Liang (1986). It was shown (Yedavalli and Liang,1986) that this approach depends on the state space representation of the system. An algorithm for improving the stability bounds based on the state space similarity transformation with diagonal scaling matrices was introduced.

The stability hypersphere in the parameter space of a system given by its state space representation was treated by Keel et al (1988). An assumption was made that the uncertain parameters enter the "A" matrix of the system linearly. Sufficient conditions in the form of upper bounds on the two norm of the perturbation vector were derived in order to guarantee stability of the perturbed system. A "robustification" algorithm was introduced where the

stability radius was gradually increased by changing the original controller.

Horisberger and Belanger (1976) posed the stability problem, in the presence of magnitude bounded real uncertainty entering the "A" matrix of the closed loop system linearly, as the existence of a matrix $P=P' \geq 0$ that simultaneously satisfies a set of Lyapunov inequalities. This approach was further exploited by Boyd and Yang (1988).

A Riccati equation synthesis approach for linear control systems was developed by Peterson (1987a) and Peterson and Hollot (1986) for magnitude bounded time varying or time invariant parameter perturbation. It requires that the uncertain parameters enter the state space representation of the system linearly and that certain "matching" conditions be satisfied. Then, a solution to the appropriate Riccati equation in (Peterson,1987a) exists and the stability of the closed loop system is guaranteed for the given range of perturbation. This holds since it can be shown that the latter is an upper bound for all Riccati equations corresponding to any point in the perturbation space. Hence, it is said that it contains a Peterson-Hollot bounding function.

It turns out that the same Riccati equation guarantees the bound on the H_∞ norm of the obtained system as was observed in (Peterson,1987b). Combining these two results led to the disturbance attenuation problem in the presence of real parameter perturbation. For stability analysis only, the different bounding functions are discussed in (Bernstein and Haddad,1988). Synthesis approaches based on Bernstein and Haddad's (1989) " H_2/H_∞ " design methodology , were introduced in (Madiwale et al,1988) and (Yeh et al,1989).

2. Stability and Performance in the Presence of Magnitude Bounded Real Uncertainty

Let the closed loop system $M(s, \mathbf{q})$ be defined as follows :

$$M(s, \mathbf{q}) = \mathbf{C}(\mathbf{q}) [s\mathbf{I} - \mathbf{A}(\mathbf{q})]^{-1} \mathbf{B}(\mathbf{q}) \quad \mathbf{A}(\mathbf{q}) \in \mathbb{R}^{n \times n} \quad (2.1)$$

where \mathbf{q} is the vector of the real parameter perturbations which belong to the set Ω ,

$$\Omega = \{ \mathbf{q} \text{ s.t. } \mathbf{q} = [q_1, \dots, q_m]' \in \mathbb{R}^m \mid |q_i| \leq r_i, \quad r_i \in \mathbb{R}_+ \} \quad (2.2)$$

Therefore, the uncertainty belongs to a "hyperbox" Ω in parameter space. In the case of $m = 2$, the "hyperbox" is depicted in Figure 1.

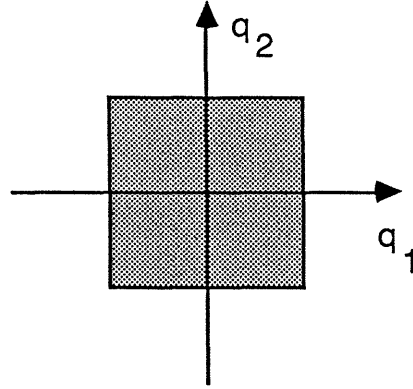


Figure 1. Parameter Box in Two-Dimensional Parameter Space

Furthermore, let us assume that the uncertain parameters enter the elements of the state space representation of the system linearly, i.e.

$$\mathbf{A}(\mathbf{q}) = \mathbf{A}_o + \sum_1^m q_i \mathbf{E}_i = \mathbf{A}_o + \Delta \mathbf{A} \quad (2.3)$$

$$\mathbf{B}(\mathbf{q}) = \mathbf{B}_o + \sum_1^m q_i \mathbf{F}_i = \mathbf{B}_o + \Delta \mathbf{B} \quad (2.4)$$

$$C(\mathbf{q}) = C_o + \sum_1^m q_i G_i = C_o + \Delta C \quad (2.5)$$

where $q_i \in \Omega$ is defined in (2.2) and $[A_o, B_o, C_o, 0]$ is the state space representation of the nominal system $M(s, \mathbf{q}=0)$. The matrices $E_i, F_i,$ and G_i define the structure of uncertainty with respect to each parameter perturbation q_i . They can always be scaled in such a way that $r_i = 1$ for $i = 1, 2, \dots, m$.

The nominal system, obtained for $\mathbf{q} = 0$, is assumed to be stable and with the infinity norm $\|M(s, \mathbf{q}=0)\|_\infty < \gamma$ where $\|M(s, \mathbf{q}=0)\|_\infty := \sup_{\omega} \overline{\sigma}(M(j\omega, \mathbf{q}=0))$. This is satisfied if the following holds

Lemma 2.1

Let $M(s, \mathbf{q}=0) = C_o [sI - A_o]^{-1} B_o$, (A_o, B_o) stabilizable and (C_o, A_o) detectable. Then, the necessary and sufficient condition for $M(s, \mathbf{q}=0) \in RH_\infty$ and $\|M(s, \mathbf{q}=0)\|_\infty < \gamma$ is the existence of $P=P'$ s.t.

- i) $P \geq 0$
 - ii) $A_o'P + PA_o + (1/\gamma)^2 P B_o B_o' P + C_o' C_o = 0$
 - iii) $[A_o + (1/\gamma)^2 B_o B_o' P]$ is stable
- (2.6)

Proof : It is presented in (Doyle et al, 1988).



The first two conditions i) and ii) are sufficient for $\|M(s, \mathbf{q}=0)\|_\infty \leq \gamma$. The third condition is needed for the strict inequality i.e. $\|M(s, \mathbf{q}=0)\|_\infty < \gamma$.

Corollary 2.1.1

Let $M(s, q=0) = C_0[sI - A_0]^{-1}B_0$. Then, a sufficient condition for $M(s, q=0) \in RH_\infty$ and $\|M(s, q=0)\|_\infty < \gamma$ is the existence of $P=P' > 0$ s.t.

$$A_0'P + PA_0 + \left(\frac{1+\varepsilon}{\gamma}\right)^2 PB_0B_0'P + C_0'C_0 < 0 \quad (2.7)$$

where ε is an arbitrarily small positive number.

Proof :

The strict inequality in (2.7) implies that $\exists Q > 0$ s.t.

$$A_0'P + PA_0 + \left(\frac{1+\varepsilon}{\gamma}\right)^2 PB_0B_0'P + C_0'C_0 + Q = 0 \quad (2.8)$$

This guarantees that the system $M_1 = [C_0 \ Q^{1/2}] [sI - A_0]^{-1} [(1+\varepsilon)B_0]$ has the following properties :

i) $(A_0, [C_0 \ Q^{1/2}])$ is observable since $(A_0, Q^{1/2})$ is. The latter is true because $Q^{1/2}$ is a square matrix with full rank.

ii) A_0 is a stable matrix. This follows from the Lyapunov stability criterion since $P > 0$ and (2.8) can be rewritten as the following Lyapunov equation :

$$A_0'P + PA_0 + Q_1 = 0 \quad (2.9)$$

where

$$Q_1 = \left(\frac{1+\varepsilon}{\gamma}\right)^2 PB_0B_0'P + C_0'C_0 + Q > 0 \quad (2.10)$$

iii) $\|M_1(j\omega)\|_\infty \leq \gamma$. This result follows from (2.8) and Lemma 2.1 since A_0 is stable. On the other hand, it implies the following :

$$\|M(j\omega)\|_\infty < |1 + \varepsilon| \|M(j\omega)\|_\infty \leq \|M_1(j\omega)\|_\infty \leq \gamma \quad ; \quad \varepsilon > 0 \quad (2.11)$$

This completes the proof. ■

Our goal is to derive conditions under which the system remains stable and with norm

smaller than γ for all $\mathbf{q} \in \Omega$. Once this is done, their conservatism will be studied and the connections with already existing results established.

According to Lemma 2.1 and its corollary, a sufficient condition for stability and norm boundedness of the system $M(\mathbf{q})$ in the hyperbox is given in the following lemma.

Lemma 2.2

Let $M(\mathbf{q})$ be defined as in (2.1) : (2.5). If $\exists P=P' \in \mathbb{R}^{n \times n}$ s.t. $P > 0, \epsilon > 0$, and

$$R(\mathbf{q}) := A(\mathbf{q})'P + PA(\mathbf{q}) + \left(\frac{1+\epsilon}{\gamma}\right)^2 PB(\mathbf{q})B(\mathbf{q})'P + C(\mathbf{q})'C(\mathbf{q}) < 0 \quad \forall \mathbf{q} \in \Omega \quad (2.12)$$

then $\|M(\mathbf{q})\|_{\infty} < \gamma \quad \forall \mathbf{q} \in \Omega$.

Proof :

This statement is the extension of the result in Corollary 2.1.1 to all the points in the hyperbox Ω with a common matrix $P=P'$.



The condition in (2.12) represents an infinite dimensional problem because it requires a common $P=P'$ for all the points in the hyperbox Ω which is an uncountable set. To make this approach useful we have to try to reduce its dimensionality, i.e. to make it finite dimensional. We now show that the existence of a matrix $P=P'$ s.t. $R(\mathbf{q})$ is convex with respect to $\mathbf{q} \in \Omega$ and negative definite at all the vertices of the hyperbox is sufficient to achieve condition (2.12) for all $\mathbf{q} \in \Omega$. This can be seen as an extension of the result for Lyapunov functions presented in (Horisberger and Belanger,1976).

Lemma 2.3

Let the function $R(\mathbf{q})$ defined in (2.12) be convex with respect to $\mathbf{q} \in \Omega$ for every

$P=P'>0$. Then, $R(\mathbf{q}) < 0 \quad \forall \mathbf{q} \in \Omega$ if $\exists P=P'>0$ s.t. $R(\mathbf{q}_j) < 0$, $j = 1, 2, \dots, 2^m$, where \mathbf{q}_j corresponds to the "j'th" vertex of the hyperbox Ω .

Proof:

The hyperbox Ω is a convex set in parameter space. The function $R(\mathbf{q})$ is convex over the same set. Therefore, for any $\mathbf{q} \in \Omega$ we have

$$R(\mathbf{q}) \leq \sum_1^{2^m} \alpha_j R(\mathbf{q}_j); \quad \sum_1^{2^m} \alpha_j = 1; \quad \alpha_j \geq 0 \quad (2.13)$$

Hence, $R(\mathbf{q}) < 0$ for all $\mathbf{q} \in \Omega$ since $R(\mathbf{q}_j) < 0$. This completes the proof. ■

The previous lemma requires that $R(\mathbf{q})$ be a convex function for every $P=P'>0$. This will hold if $R(\mathbf{q})$ is linearly dependent on every parameter perturbation q_i . The overall expression for $R(\mathbf{q})$, with the perturbation defined as in (2.3) - (2.5), has the following form :

$$\begin{aligned} R(\mathbf{q}) = & R_o + \sum_1^m q_i [(E_i'P + PE_i) + \left(\frac{1+\varepsilon}{\gamma}\right)^2 P(F_i B_o' + B_o F_i')P + (G_i' C_o + C_o' G_i)] + \\ & + \left(\frac{1+\varepsilon}{\gamma}\right)^2 P \left(\sum_1^m q_i F_i\right) \left(\sum_1^m q_i F_i'\right)P + \left(\sum_1^m q_i G_i\right) \left(\sum_1^m q_i G_i\right) \end{aligned} \quad (2.14)$$

where $R_o := A_o'P + PA_o + \left(\frac{1+\varepsilon}{\gamma}\right)^2 PB_o B_o'P + C_o' C_o$. The last two elements in the above expression are the only ones where the parameter perturbation appears in nonlinear form. If they could be bounded from above with a function linear in q_i , the resulting $R(\mathbf{q})$ would be linear and, therefore, convex in \mathbf{q} . A possible linear bounds are be presented as follows :

$$\Delta C' \Delta C = \left(\sum_1^m q_i G_i' \right) \left(\sum_1^m q_i G_i \right) \leq \left(\sum_1^m \bar{\sigma}(G_i) \right)^2 I = \alpha_C^2 I \quad (2.15)$$

where $|q_i| \leq 1$. Furthermore,

$$\alpha_B^2 I \geq \Delta B \Delta B' \quad (2.16)$$

can be obtained in an analogous way. There are possibly different methods that can be used for bounding quadratic perturbation terms. An alternative method based on properties of positive matrices was used in (Yeh et al,1989) where all the quadratic and linear terms in ΔB and ΔC were bounded from above.

The function $R(\mathbf{q})$ can now be bounded from above with some $R_1(\mathbf{q})$ defined as :

$$\begin{aligned} R_1(\mathbf{q}) = R_o + \sum_1^m q_i [(E_i' P + P E_i) + \left(\frac{1+\varepsilon}{\gamma} \right)^2 P (F_i B_o' + B_o F_i') P + (G_i' C_o + C_o' G_i)] + \\ + \left(\frac{1+\varepsilon}{\gamma} \right)^2 \alpha_B^2 P P + \alpha_C^2 I = R_o + \sum_1^m q_i \Delta R_i + \left(\frac{1+\varepsilon}{\gamma} \right)^2 \alpha_B^2 P P + \alpha_C^2 I \geq R(\mathbf{q}) \quad (2.17) \end{aligned}$$

This function is linear in \mathbf{q} and, hence, convex for every $P=P'$. Therefore, we have managed to construct a function that bounds $R(\mathbf{q})$ from above and is convex in \mathbf{q} . By using the result of Lemma 2.3, we can formulate the following.

Corollary 2.3.1

Let the system be described as in Lemma 2.3. If $\exists P=P'>0$ s.t. $R_1(\mathbf{q},P) < 0$ at every vertex of the hyperbox in parameter space, then the perturbed system $M(\mathbf{q})$ is stable and $\|M(\mathbf{q})\|_\infty < \gamma$ for all $\mathbf{q} \in \Omega$.

Proof:

Stability of the system in the hyperbox is guaranteed from a Lyapunov type argument. For every $\mathbf{q} \in \Omega$ the $R_1(\mathbf{q},P)$ can be rewritten as the Lyapunov equation

$$A(\mathbf{q})' P + P A(\mathbf{q}) + Q_1(\mathbf{q}) = 0 \quad P, Q_1(\mathbf{q}) > 0 \quad (2.18)$$

as was done in (2.9). This implies stability of $A(q)$.

It is important to notice that, by having the strict inequalities in $P > 0$ and $R_1(q, P) < 0$, there was no need to require stabilizability and detectability of $(A(q), B(q))$ and $(C(q), A(q))$ for every $q \in \Omega$ in order to check stability of $A(q)$.

The above conditions also guarantee that $\|M(s, q)\|_\infty < (1 + \varepsilon) \|M(s, q)\|_\infty \leq \gamma \quad \forall q \in \Omega$. This completes the proof. ■

The previous statement establishes the condition for the norm boundedness of the system in the parameter box in terms of the existence of a symmetric P that simultaneously satisfies inequalities at vertices. This represents an important result because it reduces an infinite dimensional problem to a finite dimensional one.

Let $R_1(q)$ at the vertex "j" be given as :

$$R_1(q_j) = A'(q_j)P + PA(q_j) + \left(\frac{1+\varepsilon}{\gamma}\right)^2 PB_oB_o'P + C_o'C_o + \left(\frac{1+\varepsilon}{\gamma}\right)^2 PW(q_j)P + Y(q_j) + \left(\frac{1+\varepsilon}{\gamma}\right)^2 \alpha_B^2 P P + \alpha_C^2 I \quad (2.19)$$

where $W(q) = \sum_1^m q_i (F_i B_o' + B_o F_i')$ and $Y(q) = \sum_1^m q_i (G_i' C_o + C_o' G_i)$.

The expression in (2.19) can be rewritten in the form of an ordinary Riccati equation as :

$$R_1(q_j) = A'(q_j)P + PA(q_j) + \left(\frac{1+\varepsilon}{\gamma}\right)^2 P\tilde{B}(q_j)\tilde{B}(q_j)'P + \tilde{C}(q_j)'\tilde{C}(q_j) \quad (2.20)$$

where $\tilde{B}(q_j)$ and $\tilde{C}(q_j)$ are augmented input and output matrices. They are defined as follows:

$$\tilde{B}(q_j) = [B_o \quad W(q_j)^{0.5} \quad \alpha_B I] \quad \text{and} \quad \tilde{C}(q_j) = \begin{bmatrix} C_o \\ Y(q_j)^{0.5} \\ \alpha_C I \end{bmatrix} \quad (2.21)$$

The condition defined in Corollary 2.3.1 requires a single matrix $P=P'$ which simultaneously satisfies 2^m Riccati inequalities. However, there are some situations where the requirement can be further simplified.

Remark

It is easy to see that, if the matrices ΔR_i as defined in (2.17) are definite in sign for the given P , then there is a single Riccati equation corresponding to one of the vertices whose solution would simultaneously satisfy the remaining inequalities.

A possible way to come up with a matrix P that satisfies inequalities $R_1(q) < 0$, $j=1, \dots, 2^m$ for a fixed magnitude of perturbation, is to formulate the following minimization problem

$$\min_{P=P'} \lambda_{\max} [\text{diag} \{ R_1(q_j), -P \}] \quad j = 1, 2, \dots, 2^m \quad (2.22)$$

where λ_{\max} is the largest eigenvalue. This approach was originally presented in (Horisberger and Belanger, 1976) for the Lyapunov inequalities. If λ_{\max} is negative, then the system will remain stable and with infinity norm smaller than γ .

The set of all symmetric matrices P is a convex set. Furthermore, it can be scaled, without loss of generality, to the set of all symmetric matrices P whose entries have bounded magnitude, i.e. each $|P_{kl}| \leq 1$, $k=1, \dots, n$, $l=1, \dots, n$. This is possible since for fixed j and with $P>0$, the inequality $R_1(q_j, A, \tilde{B}, \tilde{C}, P) < 0$ holds if and only if $R_1(q_j, A, \beta^{1/2} \tilde{B}, \beta^{-1/2} \tilde{C}, \beta^{-1/2} P) < 0$ for all scalar $\beta \in \mathbb{R}_+$.

Unfortunately, the function (2.22) is not convex in P as it was originally obtained in (Horisberger and Belanger,1976) by using Lyapunov inequalities. We now transform the original problem into a convex one with respect to matrix P . This is shown in the following lemma.

Lemma 2.4

A matrix $P = P' > 0$ satisfies

$$A'P + PA + \left(\frac{1+\epsilon}{\gamma}\right)^2 P\tilde{B}\tilde{B}'P + \tilde{C}'\tilde{C} < 0$$

if the following holds for all $\alpha \in \mathbb{R}_+$

$$T = \begin{bmatrix} A'P + PA + \alpha^{-1} \tilde{C}'\tilde{C} & \left(\frac{1+\epsilon}{\gamma}\right) P\tilde{B} \\ \tilde{B}'P \left(\frac{1+\epsilon}{\gamma}\right) & -\alpha^{-1} I \end{bmatrix} < 0 \quad (2.23)$$

Proof :

The matrix T can be rewritten as :

$$\begin{bmatrix} A'P + PA + \alpha \left(\frac{1+\epsilon}{\gamma}\right)^2 P\tilde{B}\tilde{B}'P + \alpha^{-1} \tilde{C}'\tilde{C} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\alpha \left(\frac{1+\epsilon}{\gamma}\right)^2 P\tilde{B}\tilde{B}'P & \left(\frac{1+\epsilon}{\gamma}\right) P\tilde{B} \\ \tilde{B}'P \left(\frac{1+\epsilon}{\gamma}\right) & -\alpha^{-1} I \end{bmatrix} < 0 \quad (2.24)$$

Let the first matrix be T_1 and the second T_2 , s.t. $T = T_1 + T_2$. It is easy to show that T_2 is always negative semidefinite whenever $P > 0$.

The sufficiency is proved by assuming $T < 0$. At the same time $T_2 \leq 0$ with the null space defined as $[x' \quad P\tilde{B}'x']'$ where $x \in \mathbb{R}^n$. Therefore, for every nonzero vector in the null space of T_2 and, therefore, for all $x \in \mathbb{R}^n / x \neq 0$ we require

$$x' [A'P + PA + \alpha \left(\frac{1+\epsilon}{\gamma}\right)^2 P\tilde{B}\tilde{B}'P + \alpha^{-1} \tilde{C}'\tilde{C}] x < 0. \quad \forall x \in \mathbb{R}^n / x \neq 0 \quad (2.25)$$

Hence, the Riccati expression in T_1 is negative definite if $T < 0$. This completes the proof. ■

It is easy to check that if $P \geq 0$ and the Riccati expression in (2.25) is required to be negative semidefinite, then $T \leq 0$ becomes both a necessary and sufficient condition. By applying Lemma 2.4, the convexity in P is achieved without constructing an auxiliary passive system as it was done in (Boyd and Yang, 1988).

Remark

The matrix T being negative definite implies the following :

$$[x' \ u'] T [x' \ u']' < 0 \quad \text{for every } x \in \mathbb{R}^n, \quad u \in \mathbb{R}^s \quad (2.26)$$

The expression in (2.26) is equivalent to:

$$2x'P \left[Ax + \left(\frac{1+\epsilon}{\gamma}\right) \tilde{B}u \right] < u'u - y'y \quad (2.27)$$

where $\frac{dx}{dt} = Ax + \left(\frac{1+\epsilon}{\gamma}\right) \tilde{B}u$; $y = \tilde{C}x$.

If the former inequality is integrated from $t_0 = 0$ to $t = \tau$ with $x(t_0) = 0$ and $u(t_0) = y(t_0) = 0$, we have :

$$0 \leq V(x) = x' P x < \int_0^{\tau} u'u dt - \int_0^{\tau} y'y dt \quad (2.28)$$

where $V(x)$ is a Quadratic Lyapunov function associated with the system $M_a = [A, \left(\frac{1+\epsilon}{\gamma}\right) \tilde{B}, \tilde{C}, 0]$. This system has its infinity norm strictly smaller than one since, at any point in time, the energy at its output is smaller than the energy at the input. Therefore, $\epsilon=0$ can be used in this case.



According to the previous lemma, the minimization of the maximal eigenvalue of the matrix $T(q_j)$, corresponding to a vertex "j", implies minimization of the associated Riccati inequality. Therefore, the minimization process in (2.22) is transformed into the following:

$$\min_{P=P'} \lambda_{\max}[\text{diag} \{ T(q_j), -P \}] \quad j = 1, 2, \dots, 2^m \quad (2.29)$$

The minimization in (2.29) is done by searching for individual entries of the matrix P . If its rank is equal to n , then there are $n(1+n)/2$ different elements that we are looking for. For example, a system of the fifth order and 4 uncertain parameters will give rise to $j=16$ inequalities in (2.29) while the number of unknown elements in $P=P'$ will be 15. This illustrates the complexity of this type of approach even for relatively low order systems with few uncertain parameters. A very good and detailed survey of different methods for solving this problem can be found in (Boyd and Yang,1988).

In order to simplify the above presented approach based on simultaneously satisfying 2^m Riccati type inequalities, we will derive a single, "overall" Riccati equation. The positive definite solution $P=P'$ of the latter, if it exists, will simultaneously satisfy all the inequalities at the vertices and, therefore, it will guarantee stability and norm boundedness of the perturbed system.

2.1 Stability and Performance Analysis Based on the Single Riccati Equation

A condition for the stability and norm boundedness of the perturbed system was said to be guaranteed if there is a common $P=P'>0$ s.t. $R_1(q,P)$, defined as in (2.17), which is negative definite for all $q \in \Omega$. The perturbation dependent terms in $R_1(q,P)$ are given in the following form :

$$\sum_1^m q_i [(E_i' P + P E_i) + \left(\frac{1+\epsilon}{\gamma} \right)^2 P (F_i B_o' + B_o F_i') P + (G_i' C_o + C_o' G_i)] \quad (2.30)$$

The idea is to bound all three perturbation terms in (2.30) from the above s.t.

$$\begin{aligned} \Pi_A(P) &\geq \sum_1^m q_i (E_i' P + P E_i) && \forall q \in \Omega \\ \Pi_B(P) &\geq \sum_1^m q_i P (F_i B_o' + B_o F_i') P && \forall q \in \Omega \\ \Pi_C &\geq \sum_1^m q_i (G_i' C_o + C_o' G_i) && \forall q \in \Omega \end{aligned} \quad (2.31)$$

After substituting these bounding functions in $R_1(q,P)$, we construct $R_2(q,P)$ which bounds $R(q,P)$ from above for all $\forall q \in \Omega$ as follows:

$$\begin{aligned} R(q,P) &\leq R_1(q,P) \leq R_2(q,P) = \\ &= R_o + \Pi_A(P) + \left(\frac{1+\epsilon}{\gamma} \right)^2 \Pi_B(P) + \Pi_C + \alpha_B^2 P P \left(\frac{1+\epsilon}{\gamma} \right)^2 + \alpha_C^2 I = 0 \end{aligned} \quad (2.32)$$

We will now construct a bounding function for the perturbation in A matrix, i.e. Π_A . Let $\Pi_1(P)$ be the auxiliary bounding function defined as :

$$\Pi_1(P) = \sum_1^m | (E_i' P + P E_i) | \quad (2.33)$$

meaning that $2x' | P E_i | x \geq 0$ for every $x \in R^n$. This represents the natural choice for $\Pi_A(P)$, but the corresponding equation is difficult to solve. Hence, the previous form of

$\Pi_A(P)$, i.e. $\Pi_1(P)$, has to be modified in order to obtain a more convenient one.

The matrices E_i are, or can be expressed as the sum of rank one matrices. Because of the simplicity, we assume that they are of rank one. This implies that they can be represented as the product of two vectors, i.e. $E_i = b_i c_i'$. Therefore, we have:

$$x' \Pi_1(P) x = 2 \sum_1^m | (x' P b_i) (c_i' x) | \quad (2.34)$$

where the expressions in parentheses are scalar. This representation is not unique. By using the well known inequality for scalars x, y and $z_i \neq 0$, $2 | xy | \leq x^2 / z_i^2 + y^2 z_i^2$, we have :

$$\Pi_A(P) = (1+\epsilon) \sum_1^m (1/z_i^2) (P b_i b_i' P) + \sum_1^m z_i^2 (c_i' c_i) \geq \Pi_1(P) \quad (2.35)$$

If $z_i = 1$ for all $i = 1: m$, the following holds with $\epsilon > 0$:

$$\Pi_A(P) = (1+\epsilon)^2 P B_1 B_1' P + C_1' C_1 \quad (2.36)$$

This bounding function, with $\epsilon=0$, is introduced by Peterson and Hollot (1986) for the perturbation in the "A" matrix only and it is frequently used by different authors. It is obvious that the construction of a bounding function is not unique. Different bounding functions are presented in (Bernstein and Haddad, 1988).

Following the same procedure, and assuming without loss in generality that the matrices $(F_i B_o)$ and $(G_i' C_o)$ are of rank one, we can construct the following bounding functions :

$$\Pi_B = P (B_2 B_2' + B_3 B_3') P \geq \sum_1^m q_i P (F_i B_o' + B_o F_i') P ; \quad \forall q \in \Omega \quad (2.37)$$

$$\Pi_C = C_2' C_2 + C_3' C_3 \geq \sum_1^m q_i (G_i' C_o + C_o' G_i) ; \quad \forall q \in \Omega \quad (2.38)$$

The overall single Riccati equation is, therefore, defined as :

$$\begin{aligned}
R_2(q,P) &= R_o + \Pi_A(P) + \left(\frac{1+\varepsilon}{\gamma}\right)^2 \Pi_B(P) + \Pi_C + \alpha_B^2 P P \left(\frac{1+\varepsilon}{\gamma}\right)^2 + \alpha_C^2 I \\
&= A_o'P + PA_o + \left(\frac{1+\varepsilon}{\gamma}\right)^2 P(B_oB_o' + \gamma^2 B_1 B_1' + B_2 B_2' + B_3 B_3' + \alpha_B^2 I) P + \\
&\quad + (C_o' C_o + C_1' C_1 + C_2' C_2 + C_3' C_3 + \alpha_C^2 I) = 0
\end{aligned}
\tag{2.39}$$

The existence of $P=P'>0$ that satisfies (2.39) is the condition for norm boundedness of the augmented system whose input matrix is $(1+\varepsilon)/\gamma [B_o \ \gamma B_1 \ B_2 \ B_3 \ \alpha_B I]$ and the output matrix is $[C_o' \ C_1' \ C_2' \ C_3' \ \alpha_C I]'$. This at the same time guarantees that the perturbed system in (2.1) will be stable and norm bounded for all $q \in \Omega$.

3. Conservatism of the Riccati Equation Based Methods

In the previous section two different methods for checking stability and infinity norm boundedness of the linearly perturbed, nominally stable system were presented. The first method is based on the existence of a matrix $P=P'$ that simultaneously satisfies Riccati inequalities at all vertices of the hyperbox in parameter space. The conservatism of this method is based on :

- i) the requirement for a common $P=P'>0$ for all points within the hyperbox
- ii) linearization of the quadratic terms in the Riccati expression in order to make it convex in q for a given P .

If the perturbation were confined to the "A" matrix only, the second step would be

unnecessary. This case was treated in (Horisberger and Belenger,1976) for checking stability of the perturbed system only. Lyapunov equations were used instead Riccati equations since norm boundedness was not considered. A way of constructing a common $P=P'$ matrix was introduced as in (2.22). Convexity in P was required and it was guaranteed since Lyapunov equations were used.

The second analysis method is based on construction of a single Riccati equation that bounds from above all Riccati inequalities corresponding to the hyperbox in the parameter space. It was constructed by defining certain bounding functions. The Peterson-Hollot bounding function (Peterson and Hollot,1986) originally derived for bounding the perturbation in the "A" matrix only, was also used herein. Furthermore, using the same approach, we derived the analogous bounding functions Π_B and Π_C for perturbations in the "B" and "C" matrices respectively. Introduction of these bonding functions made the resulting criterion given in the form of the overall Riccati equation easy to check. On the other hand, this has further increased the conservatism of the previous methodology based on checking of the vertices of the hyperbox in parameter space. Therefore, there is a clear tradeoff between the numerical complexity and conservatism of the introduced methods for checking stability.

We will now study the conservatism of the single Riccati equation approach in more detail. Let the perturbations be confined to the "A" matrix only. The overall Riccati equation in this case will have the following form:

$$A_0'P + PA_0 + [(1+\varepsilon)/\gamma]^2 P [B_0 B_0' + \gamma^2 B_1 B_1'] P + C_0' C_0 + C_1' C_1 = 0 \quad (2.40)$$

implying $\|M(s,P)\|_\infty < 1$ with $P \geq 0$ where $M(s,P) = [A_0, (\gamma^{-1} B_0 \ B_1), (C_0' \ C_1)']', 0]$ is depicted in Figure 2.

Under the assumption of E_i , $i = 1, \dots, 2^m$, being rank one matrices and B_1 and C_1 derived as in the Peterson-Hollot bounding function, we have $Q = I \mathbf{q}$, $\|Q\| \leq 1$. Therefore, the perturbation matrix Q is norm bounded and it has a block diagonal structure. Since the existence of a $P=P' \geq 0$ that satisfies (2.40) guarantees $\|M(s,P)\|_\infty < 1$, this is the "Small

Gain Theorem (SGT)" (Zames,1965) condition for stability of the same system with respect to all perturbation matrices $\Delta(s)$ where $\begin{bmatrix} d & d1 \end{bmatrix} = \Delta(s) \begin{bmatrix} e & e1 \end{bmatrix}$ and $\|\Delta\|_{\infty} \leq 1$. Furthermore, this holds also for the case where $\Delta(s) = \text{diag} \{ \Delta p(s), Q \}$ where $\|\Delta p\|_{\infty} \leq 1$ corresponds to the performance requirement.

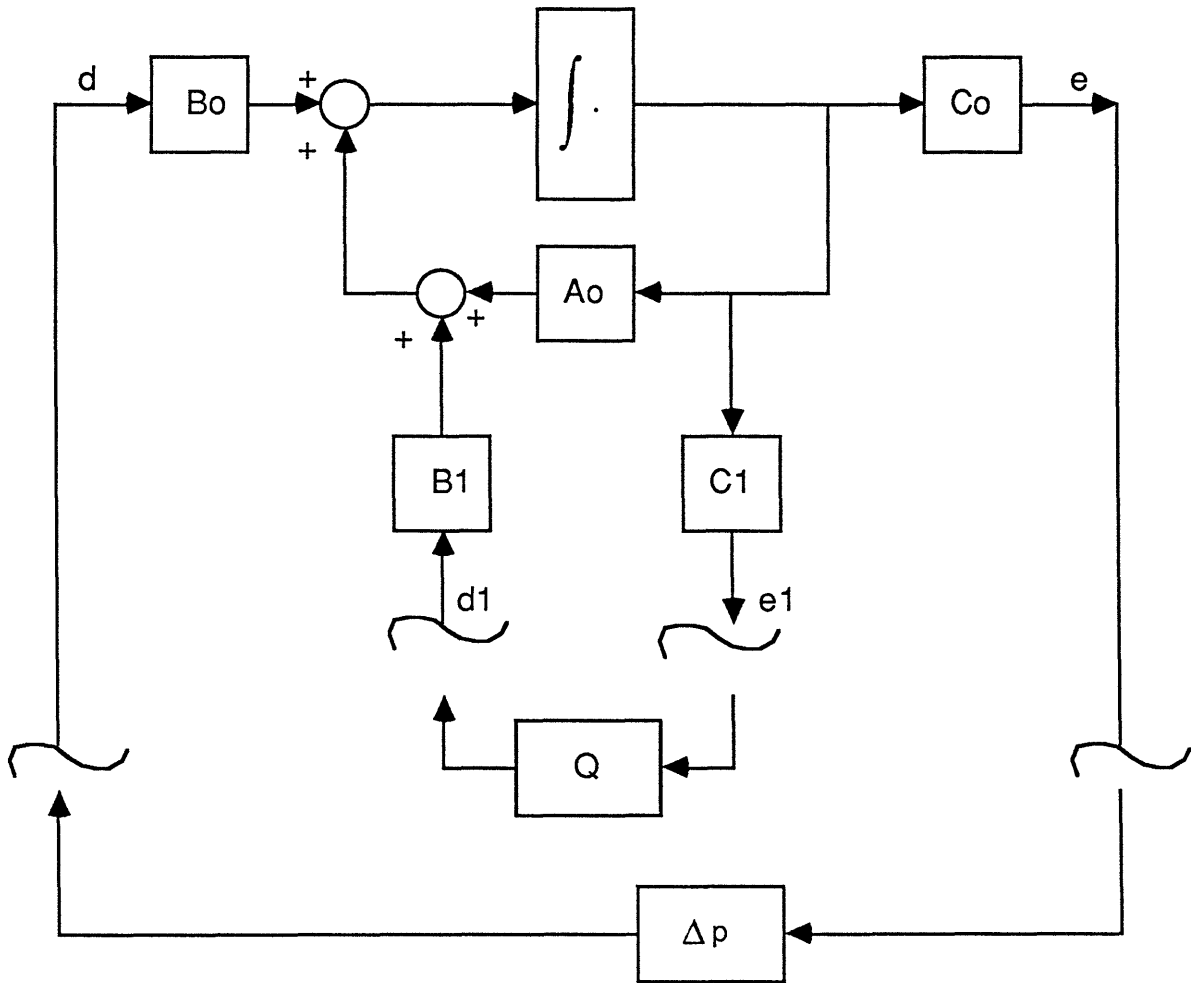


Figure 2. Perturbation in the "A" Matrix Only

This is a conservative condition since the real parameter perturbation is treated as frequency dependent and, furthermore, the block diagonal structure of the overall

perturbation and of Q itself is neglected. The conservatism may be reduced by treating this problem as a " μ " problem (Doyle,1982) with $(m+1)$ blocks and also by providing different B_1 and C_1 matrices in the Peterson-Hollot bounding function.

In the case of perturbation entering the "B" and "C" matrices of the state space representation, the condition defined in (2.39) is even more conservative due to the linear upper bounding of quadratic terms. The resulting Riccati equation (2.39) can be viewed as the H_∞ norm condition for some auxiliary system with the augmented inputs and outputs that do not necessarily correspond to the location of uncertainty in the original system.

Understanding that the stability and norm boundedness criterion based on the overall Riccati equation is essentially either the "SGT" condition for the perturbed system or a H_∞ norm condition on the augmented system in (2.39), has very important implication on the possible synthesis methods. Consequently, it is clear that existing H_∞ minimization methods such as the one of Doyle et al (1988) will provide a controller, if it exists, that guarantees bound of the H_∞ norm of the closed loop system corresponding to equation in (2.39). This will, on the other hand, guarantee the existence of a matrix P that satisfies the same equation and, therefore, the overall Riccati equation criterion.

The other possible approach for finding a compensator that satisfies the Riccati equation criterion for the closed loop system is the Bernstein and Haddad's " H_2/H_∞ " design methodology presented in (Madiwale et al,1988). The perturbation was originally restricted to be in the "A" matrix of the closed loop system only while the " H_2 " cost was not considered. By linearizing the quadratic terms in case of perturbations being present in "C" matrix of the closed loop system, Yeh et al (1989) have loosened the restriction on the location of real uncertainty in the system state space description. The applied methodology in general results in the system of coupled Riccati and Lyapunov equations without providing any guarantees about the existence of their solution. Only when the certain weights in the both " H_2 " and " H_∞ " cost functions are chosen to be the same, the obtained equations decouple and the result is identical to the one in (Doyle et al,1989) as was shown

in (Mustafa,1989).

In the case of full state feedback, Khargonekar et al (1987) have shown that quadratic stabilizability of a class of systems perturbed with the time varying real uncertainty is achieved iff certain "SGT" condition holds. They have introduced a control law that stabilizes the closed loop system which is essentially the solution of the full state " H_∞ " minimization problem. In order to accommodate uncertainty in the "B" matrix and guarantee necessity, they have required a special matching condition on the matrices $\Delta A(t)$ and $\Delta B(t)$ and no structure in the perturbation matrix $Q(t)$, $\| Q(t) \| \leq 1$, introduced in Figure 2. In the case treated in the present paper where real uncertainty enters the state space representation linearly, the perturbation matrix will always be diagonal, i.e. $Q = \text{diag}\{q_1, \dots, q_m\}$. Therefore, the "SGT" condition for stability will be only sufficient even if the perturbation is time varying.

4. Conclusions

Two different analysis methods based on Riccati type condition for stability and infinity norm boundedness of a system perturbed with real uncertainty were discussed. The perturbation was assumed to be magnitude bounded and to enter linearly all the elements of the state space description of the system. By linearizing the quadratic "perturbation" terms in the Riccati expression, a condition based on simultaneous satisfaction of a set of Riccati inequalities was established. In order to achieve convexity in the matrix $P=P'$, an equivalent condition was derived and the construction of P was then posed as an optimization problem.

An alternative analysis approach based on the existence of a solution to the "overall" Riccati equation was introduced. The latter bounds from above Riccati inequalities

corresponding to all the points of the hyperbox in parameter space. Perturbation in all elements of the state space realization were bounded with Peterson-Hollot like bounding functions.

Furthermore, the conservatism of the methodologies introduced and their connections to previous results have been discussed. Special attention was given to the comparison between the Small Gain Theorem and the method based on the single Riccati equation. It is shown that the latter is at least conservative as if real perturbation were treated as complex, unstructured uncertainty. This was further used to show that usual H_∞ synthesis may be used to design compensator that will guarantee overall Riccati equation criterion for the resulting closed loop system.

References

- Bernstein D.S. and W.M. Haddad (1988). Robust Stability and Performance Analysis for State Space Systems via Quadratic Lyapunov Bounds, submitted for publication.
- Bernstein D.S. and W.M. Haddad (1989). LQG Control with an H_∞ Performance Bound : A Riccati Equation Approach, *IEEE Trans. Auto. Cont.*, vol. 34, No. 3, pp. 293-305.
- Bhattacharyya S.P. (1987). *Robust Stabilization Against Structured Perturbation*, Springer-Verlag, Berlin.
- Boyd S. and Q. Yang (1988). Structured and Simultaneous Lyapunov Functions for System Stability Problems, *Techn. Report No. L-104-88-1*, Stanford University.
- Chang S.S.L. and T.K.C. Peng (1972). Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters, *IEEE Trans. Auto. Cont.*, vol. AC-17, pp. 474-483.
- De Gaston R.R.E. and M.G. Safonov (1988). Exact Calculation of the Multiloop Stability Margin, *IEEE Trans. Auto. Cont.*, vol. 33, No. 2, pp. 156-171.

- Doyle J.C. (1982). Analysis of Feedback Systems with Structured Uncertainties, *IEE Proceedings*, vol.129, Part D, No. 6, pp.242-250.
- Doyle J.C., K. Glover, P. Khargonekar, and B. Francis (1988). State Space Solutions to Standard H_2 and H_∞ Control Problems, *Proc. Amer. Cont. Conf.*, pp. 1691-1700, Atlanta, GA.
- Horisberger H.P. and P.R. Belanger (1976). Regulators for Linear, Time Invariant Plants with Uncertain Parameters, *IEEE Trans. Auto. Cont.*, AC-21, pp. 705-708.
- Kharitonov V.L. (1978). Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations, *Differential. Uravnen.*, vol. 14, No. 14, pp. 2086-2088.
- Khargonekar P.P., I.R. Petersen, and K. Zhou (1990). Robust Stabilization of Uncertain Systems and H_∞ Optimal Control, *IEEE Trans. Auto. Cont.*, vol. 35, No. 3, pp. 256-361.
- Kheel L.H., S.P. Bhattacharyya, and J.W. Howze (1988). Robust Control with Structured Perturbations, *IEEE Trans. Auto. Cont.*, vol. 33, No. 1, pp. 68-78.
- Madiwale A.N., D.S. Bernstein, and W.M. Haddad (1988). Robust H_∞ Control Design for Systems with Structured Parameter Uncertainty, *Proc. 27th. IEEE Conf. Dec. Cont.*, pp. 965-972, Austin, TX.
- Mustafa D. (1989). Relations between maximum-entropy/ H_∞ control and combined H_∞ /LQG control, *Syst. & Cont. Letters*, vol. 12, pp. 193-203.
- Patel R.V., M. Toda, and R. Sridhar (1979). Robustness of Linear Quadratic State Feedback Design in Presence of System Uncertainty, *IEEE Trans. Auto. Cont.*, vol. AC-22, pp. 945-947.
- Patel R.V. and M. Toda (1980). Quantitative Measures of Robustness for Multivariable Systems, *Proc. Joint Cont. Conf.*, TP8A-9, San Francisco, CA.

- Petersen I.R. and C.V. Hollot (1986). A Riccati Equation Approach to the Stabilization of Uncertain Linear Systems, *Automatica*, vol. 22, No. 4, pp. 397-411.
- Petersen I.R. (1987a). A Stabilization Algorithm for a Class of Uncertain Linear Systems, *Syst. & Cont. Letters*, vol. 8, pp. 351-357.
- Petersen I.R. (1987b). Disturbance Attenuation and H_∞ Optimization : A Design Method Based on the Algebraic Riccati Equation, *IEEE Trans. Auto. Cont.*, vol. AC-32, No. 5, pp. 427-429.
- Saeki M. (1986). A Method of Robust Stability Analysis with Highly Structured Uncertainties, *IEEE Trans. Auto. Cont.*, vol. AC-31, No. 10, pp. 935-940.
- Siljak D.D. (1989). Parameter Space Methods for Robust Control Design : A Guided Tour, *IEEE Trans. Auto. Cont.*, vol. 34, No. 7, pp. 674-689.
- Yedavalli R.K. (1985a). Improved Measures of Stability Robustness for Linear State Space Models, *IEEE Trans. Auto. Cont.*, AC-30 pp. 577-579.
- Yedavalli R.K. (1985b). Perturbation Bounds for Robust Stability In LNear State Space Models, *Int. J. Cont.*, vol. 42, pp. 1507-1517.
- Yedavalli R.K. and Z.Liang (1986). Reduced Conservatism in Stability Robustness Bounds by State Transformation, *IEEE Trans. Auto. Cont.*, AC-31, pp. 863-866.
- Yeh H., S.S. Banda, A.C. Bartlett, and S.A. Heise (1989). Robust Design of Multivariable Feedback Systems with Real Parameter Uncertainty and Unmodelled Dynamics, *Proc. Amer. Cont. Conf.*, Pittsburgh, PA.
- Zames G. (1965). Nonlinear Time-Varying Feedback Systems - Condition for L_∞ Boundedness Derived Using Conic Operators on Exponentially Weighted Spaces, *Proc. 3rd. Allerton Conf.*