Irreducible Representations of Braid Groups via Quantized Enveloping Algebras

by

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Abstract

The q-deformations of enveloping algebras of classical Lie algebras, or quantized enveloping algebras $\mathbf{U}(\mathfrak{g})$, and their representations provide natural settings for the action, due to Lusztig, of the corresponding braid groups. Objects of particular interest are the zero weight spaces of $\mathbf{U}(\mathfrak{g})$-modules, since they are stable under the braid group action. We prove that for a certain class of simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules, the action of the braid group $\mathcal{B}_n$, of type A_n, on the zero weight space is irreducible. In particular, we show that there is a two parameter family of simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules for which the $\mathcal{B}_n$ action on the zero weight space is irreducible. Considering a special case, we show that for each $k \in \mathbb{N}$, there is an irreducible $\mathcal{B}_n$-module of dimension $\binom{n+k-1}{k}$.

The special case of type A_2 is studied in detail, and from it we deduce that $\mathcal{B}_2$ acts irreducibly on the zero weight space of all simple $\mathbf{U}(\mathfrak{sl}_3)$-modules. We also deduce some interesting results about the braid group action at roots of unity from this case.

Thesis Supervisor: George Lusztig
Title: Professor of Mathematics

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Being away from home for the first time and not being certain if research mathematics was what I wanted as a career combined to make the past three and a half years at MIT most difficult. I was often resigned to giving up and returning home without any idea of what I may do, and the existence of this thesis[1] is not a reflection of the persistence or dedication on my part, but rather a testimony to the most wonderful and generous people who encouraged and supported me through those years. I dedicate this thesis to them, and take this opportunity to express my sincere thanks.

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Irreducible Representations of Braid Groups via Quantized Enveloping Algebras

Oh Kang Kwon

Introduction

Let $\mathbf{U}$ be a quantized enveloping algebra corresponding to a Cartan matrix (a_{ij}), and let V be an irreducible representation of $\mathbf{U}$. In [**L1**] and [**L3**] Lusztig defined an action of the braid group, $\mathcal{B}$, corresponding to (a_{ij}) on $\mathbf{U}$ and on V, thereby providing natural settings for the action of the braid group. The action of $\mathcal{B}$ permutes the non-zero weight spaces of V – according to the action of the Weyl group on the weights – while keeping the zero weight space stable. The question arises whether the restricted action of the braid group on the zero weight space is irreducible, and the main aim of this thesis is to answer that question in the affirmative for a certain class of simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules. In particular, we prove that for each $n \geq 2$, there is a two parameter family of $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules for which the action of $\mathcal{B}_n$ on the zero weight space is irreducible. Looking at a special case, we deduce that for each $k \in \mathbf{N}$ there is an irreducible $\mathcal{B}_n$-module of dimension $\binom{n+k-1}{k}$.

In the case of type A_2, some computations are carried out to determine explicitly the matrices of the braid group generators with respect to Lusztig's canonical basis. From these we deduce that the braid group of type A_2 acts irreducibly on the zero weight space of every simple $\mathbf{U}(\mathfrak{sl}_3)$-module. We also deduce some interesting results about the braid groups of type A_n when v is specialized to a root of unity from these computations.

We include an algebraic proof that the action of the braid groups of type A_n, on the zero weight space of the tensor power $V^{\otimes n+1}$ of the standard representation of $\mathbf{U}$, reduces to a Hecke algebra action. And from this we obtain a proof of the irreducibility of the braid group action on the zero weight space of the simple constituents of $V^{\otimes n+1}$.

Having obtained a class of irreducible representations of braid groups of type A_n, a natural question arises: To what extent do they exhaust the set of all irreducible representations of the braid groups of type A_n? A small digression is made to determine this for the 2-dimensional representations in the case $n = 2$.

1. Preliminaries

We collect here some basic definitions and well-known results which we will need. The basic references for this chapter are [**L1**] and [**L3**].

1.1. Quantized Enveloping Algebras. We begin by giving a brief description of quantized enveloping algebras. Let $(a_{ij}), 1 \leq i,j \leq n$, be a Cartan matrix, and let $(d_1, \dots, d_n)$, where $d_i \in \{1,2,3\}$, be the vector with $d_1 + \cdots + d_n$ minimal such that the matrix $(d_i a_{ij})$ is symmetric.

Let v be an indeterminate, and for $n \in \mathbf{Z}$ and $d \in \mathbf{N}_{>0}$ define $[n]_d = (v^{dn} - v^{-dn})/(v^d - v^{-d})$. Note that $[-n]_d = -[n]_d$. Next, for $n \in \mathbf{Z}$ and $r \in \mathbf{N}$, define

$$[r]^!_d = \prod_{k=1}^{r} [k]_d, \quad \text{and} \quad \begin{bmatrix} n \\ r \end{bmatrix}_d = \frac{[n]_d[n-1]_d \cdots [n-r+1]_d}{[r]^!_d}.$$

The *quantized enveloping algebra* corresponding to (a_{ij}) (see [**L1**, Section 2]) is then the $\mathbf{C}(v)$-algebra $\mathbf{U}$ defined by the generators E_i, F_i, K_i, K_i^{-1} $(1 \leq i \leq n)$ and the relations

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_j = v^{d_ia_{ij}}E_jK_i, \qquad K_iF_j = v^{-d_ia_{ij}}F_jK_i,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}},$$
$$\sum_{p+q=1-a_{ij}} (-1)^q E_i^{(p)}E_jE_i^{(q)} = 0 \quad \text{if } i \neq j,$$
$$\sum_{p+q=1-a_{ij}} (-1)^q F_i^{(p)}F_jF_i^{(q)} = 0 \quad \text{if } i \neq j,$$

where $E_i^{(p)} = E_i^p/[p]_{d_i}^!$ and $F_i^{(p)} = F_i^p/[p]_{d_i}^!$. There is a Hopf algebra structure on $\mathbf{U}$ with comultiplication Δ, antipode S, and counit ϵ defined by

$$\begin{aligned} &\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, && \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, && \Delta K_i = K_i \otimes K_i,\\ &SE_i = -K_i^{-1}E_i, && SF_i = -F_iK_i, && SK_i = K_i^{-1},\\ &\epsilon E_i = 0, && \epsilon F_i = 0, && \epsilon K_i = 1. \end{aligned}$$

The iteration of Δ gives an algebra homomorphism $\Delta^{(n)}: \mathbf{U} \to \mathbf{U}^{\otimes n}$ which is well defined by the coassociativity of Δ.

Let $\mathbf{U}^-$, $\mathbf{U}^0$, and $\mathbf{U}^+$ be the $\mathbb{C}(v)$-subalgebras of $\mathbf{U}$ generated, respectively, by the F_i's, $K_i^{\pm 1}$'s, and E_i's. Then we have the triangular decomposition $\mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+ \cong \mathbf{U}$, where the isomorphism is given by $u^- \otimes u^0 \otimes u^+ \mapsto u^-u^0u^+$ (see [**L3**, 3.2]).

If $\mathfrak{g}$ is the Lie algebra corresponding to (a_{ij}), then we will often write $\mathbf{U}(\mathfrak{g})$ to denote the quantized enveloping algebra corresponding to (a_{ij}).

Example 1.1.1. $\mathbf{U}(\mathfrak{sl}_{n+1})$. In this case the Cartan matrix (a_{ij}) is given by $a_{ii} = 2$, $a_{ij} = -1$ ($|i-j| = 1$), and $a_{ij} = 0$ ($|i-j| > 1$), where $1 \leq i, j \leq n$. So the explicit relations for $\mathbf{U}(\mathfrak{sl}_{n+1})$ are

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iE_j = \begin{cases} v^2E_jK_i & \text{if } i = j,\\ v^{-1}E_jK_i & \text{if } |i-j| = 1,\\ E_jK_i & \text{if } |i-j| > 1, \end{cases} \qquad K_iF_j = \begin{cases} v^{-2}F_jK_i & \text{if } i = j,\\ vF_jK_i & \text{if } |i-j| = 1,\\ F_jK_i & \text{if } |i-j| > 1, \end{cases}$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v - v^{-1}},$$
$$E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \qquad \text{if } |i-j| > 1,$$
$$E_i^{(2)}E_j - E_iE_jE_i + E_jE_i^{(2)} = 0 \qquad \text{if } |i-j| = 1,$$
$$F_i^{(2)}F_j - F_iF_jF_i + F_jF_i^{(2)} = 0 \qquad \text{if } |i-j| = 1.$$

1.2. Representations of U. Let $\mathfrak{g}$ be the Lie algebra corresponding to a Cartan matrix (a_{ij}). Then the category of (type I) integrable highest weight $\mathbf{U}(\mathfrak{g})$-modules is isomorphic to the category of integrable highest weight $\mathfrak{g}$-modules. In particular, the simple highest weight $\mathbf{U}(\mathfrak{g})$-modules are parametrized by the dominant weights of $\mathfrak{g}$, and the dimensions of weight spaces of those modules are given by the dimensions of the weight spaces of the corresponding $\mathfrak{g}$-modules. In the case of type A_n, or equivalently where $\mathfrak{g} = \mathfrak{sl}_{n+1}$, a dominant weight is an n-tuple, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$, and the corresponding irreducible $\mathbf{U}(\mathfrak{sl}_{n+1})$-module $V(\lambda)$ is characterized by the following conditions:

(1) The highest weight space $V(\lambda)_\lambda$ is 1-dimensional, and
(2) If $0 \neq \xi \in V(\lambda)_\lambda$, then $E_i(\xi) = 0$ and $K_i(\xi) = v^{\lambda_i}\xi$ for $1 \leq i \leq n$.

1.3. Braid Group Action. Let (a_{ij}) be a Cartan matrix. For $k \in \mathbf{N}$ write $(\alpha\beta)_k = \alpha\beta\alpha\beta\ldots$, with k terms in the product, and for $a_{ij}a_{ji} = 0,1,2$, or 3, let $m_{ij} = 2,3,4$, or 6 respectively. Then the braid group $\mathcal{B}$ corresponding to (a_{ij}) is defined by

$$\mathcal{B} = \langle T_1, T_2, \ldots, T_n \mid (T_iT_j)_{m_{ij}} = (T_jT_i)_{m_{ij}} \rangle.$$

In the case of type A_n, the braid group has the following presentation

$$\mathcal{B} = \langle T_1, T_2, \ldots, T_n \mid T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \ (1 \leq i < n), \text{ and } T_iT_j = T_jT_i \ (|i-j| > 1) \rangle.$$

Let $\mathbf{U}$ be the quantized enveloping algebra corresponding to (a_{ij}), and let V be an integrable $\mathbf{U}$-module. Then there is an action of $\mathcal{B}$ on V, due to Lusztig, given by:

$$T_i(x) = \sum_{a,b,c\geq 0} (-1)^b v^{d_i(c^2-a^2-ac+ab-bc+a+c)} F_i^{(a)} E_i^{(b)} F_i^{(c)} K_i^{(a-c)}(x),$$

for $1 \leq i \leq n$ and $x \in V$. Note that there is only a finite number of non-zero terms in the sum, V being an integrable $\mathbf{U}$-module.

If x belongs to the p-weight space relative to K_i, so that $K_ix = v^{d_ip}x$, then it turns out that the only terms that contribute to the sum are those for which $a - b + c = p$. So for x in the p-weight space of K_i, we have

$$T_i(x) = \sum_{a-b+c=p} (-1)^b v^{d_i(c^2-a^2-ac+ab-bc+a+c)} F_i^{(a)} E_i^{(b)} F_i^{(c)} K_i^{(a-c)}(x).$$

In [**L3**, 5.2.1], Lusztig defines operators $T'_{i,1}$ by the formula

$$T'_{i,1}(x) = \sum_{a-b+c=p} (-1)^b v^{d_i(-ac+b)} F_i^{(a)} E_i^{(b)} F_i^{(c)}(x),$$

for x in the p-weight space of K_i. The operators T_i and $T'_{i,1}$ are related by the following:

Lemma 1.3.1. *We have* $T'_{i,1} = T_iK_i^{-1}$.

Proof: It suffices to check that for a,b,c satisfying $a-b+c=p$, we have $-ac+b = c^2 - a^2 - ac + ab - bc + a + c + (a-c)p - p$, and this is easily verified. □

This lemma allows us to use all the results pertaining to $T'_{i,1}$ found in [**L3**]. In particular, it implies that the T_i's satisfy the braid group relations (see [**L3**, Chapter 37]).

Consider the special case where $\mathbf{U} = \mathbf{C}(v)[F_i, K_i^{\pm 1}, E_i]$ is a copy of $\mathbf{U}(\mathfrak{sl}_2)$, and V is an irreducible $\mathbf{U}$-module of dimension $n+1$. If we denote by ξ the highest weight vector in V, then a basis (in fact Lusztig's canonical basis) of V is given by the set $\{\xi, F_i\xi, F_i^{(2)}\xi, \ldots, F_i^{(n)}\xi\}$, and we have the following explicit formula, due to Lusztig, for the action of T_i on this basis.

Proposition 1.3.2. *Let V be as above with highest weight vector ξ. Then*

$$T_i(F_i^{(j)}\xi) = (-1)^j v^{d_i(j+1)(n-j)} F_i^{(n-j)}\xi.$$

Proof: This follows from [**L3**, 5.2.2] and Lemma 1.3.1. □

We will make a frequent use of this result in the following sections, and often without explicit reference. Note that, when we specialize $v \mapsto \pm 1$, the braid group action factors through the finite group $(\mathbf{Z}/2)^2 \times S_2$. More generally it turns out that upto signs, the braid group action factors through the Weyl group of $\mathfrak{g}$ when v is specialized to ± 1. The question whether it factors through a finite group when $v \mapsto \zeta$, where $\zeta \neq \pm 1$ is a root of unity, is considered in section 4.6.

1.4. Gaussian Binomial Coefficients. We collect here some properties of the Gaussian binomial coefficients which will be useful later. Recall that for $n \in \mathbf{Z}$, $d \in \mathbf{N}_{>0}$, and $r \in \mathbf{N}$ they are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_d = \frac{[n]_d[n-1]_d \cdots [n-r+1]_d}{[r]_d^!}.$$

From this, it follows that for $n \in \mathbf{Z}$ and $r \in \mathbf{N}$, we have

$$\begin{bmatrix} n \\ r \end{bmatrix}_d = (-1)^r \begin{bmatrix} -n+r-1 \\ r \end{bmatrix}_d, \quad \text{and} \quad \begin{bmatrix} n \\ r \end{bmatrix}_d = 0 \text{ if } 0 \leq n < r.$$

Note that under the specialization $v \mapsto 1$ the Gaussian integer $[n]_d \mapsto n$, and more generally $\begin{bmatrix} n \\ r \end{bmatrix}_d \mapsto \binom{n}{r}$.

It is easily checked that the Gaussian binomial coefficients satisfy the following identity:

$$\prod_{j=0}^{n-1}(1+v^{2dj}x) = \sum_{j=0}^{n} v^{dj(n-1)} \begin{bmatrix} n \\ j \end{bmatrix}_d x^j,$$

where $n \in \mathbf{N}_{\geq 1}$.

Lemma 1.4.1. *For $r \in \mathbf{N}$ and $m, n \in \mathbf{Z}$, we have*

$$\begin{bmatrix} m+n \\ r \end{bmatrix}_d = \sum_{\substack{s,t \in \mathbf{N} \\ s+t=r}} v^{d(mt-ns)} \begin{bmatrix} m \\ s \end{bmatrix}_d \begin{bmatrix} n \\ t \end{bmatrix}_d.$$

Proof: See [**L3**, 1.3.1]. □

In the special case where $d = 1$ we omit the subscripts from the binomial coefficients so that $\begin{bmatrix} n \\ r \end{bmatrix} := \begin{bmatrix} n \\ r \end{bmatrix}_1$.

Now given a quantized enveloping algebra $\mathbf{U}$ corresponding to (a_{ij}), we define

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \frac{\prod_{s=0}^{t-1}(v^{d_i(c-s)}K_i - v^{-d_i(c-s)}K_i^{-1})}{\prod_{s=1}^{t}(v^{d_i s} - v^{-d_i s})},$$

for $c \in \mathbf{Z}$ and $t \in \mathbf{N}$. Then we have the following commutation formula for $E_i^{(p)}$ and $F_i^{(r)}$ due to Kostant [**K**] in the classical case and extended by Kac to the quantum case.

Lemma 1.4.2. *For $p, r \in \mathbf{N}$, we have*

$$E_i^{(p)}F_i^{(r)} = \sum_{0 \leq t \leq min(p,r)} F_i^{(r-t)} \begin{bmatrix} K_i; 2t-p-r \\ t \end{bmatrix} E_i^{(p-t)}.$$

Proof: See [**L4**, 4.3.1] □

2. Irreducibility of the Braid Group Action

We now restrict our attention to the case where (a_{ij}) is the Cartan matrix of type A_n, and denote by $\mathcal{B}_n$ the corresponding braid group. The proof of the irreducibility of the $\mathcal{B}_n$-action on the zero weight space of certain simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules is by induction, and in order to carry out the inductive process, we need to know, explicitly, the way in which the simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules decompose as a direct sum of $\mathbf{U}(\mathfrak{sl}_n)$-submodules. This is established, essentially, by the work of Gelfand and Tsetlin [**GT1**], and its q-analogue by Jimbo [**J**], in which they provide explicit combinatorial parametrization for bases of simple $\mathbf{U}(\mathfrak{gl}_{n+1})$-modules, along with the explicit formulae for the action of the Chevalley generators with respect to these bases. A Gelfand-Tsetlin basis consists of weight vectors compatible with the standard inclusions $\mathbf{U}(\mathfrak{gl}_i) \to \mathbf{U}(\mathfrak{gl}_{n+1})$, where $1 \le i \le n+1$. Trivial modification of their results then gives the corresponding formulae for $\mathbf{U}(\mathfrak{sl}_{n+1})$.

2.1. Gelfand-Tsetlin Bases for $\mathbf{U}(\mathfrak{sl}_{n+1})$. The treatment of the Gelfand-Tsetlin bases for the simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules given here will be that of Ueno, Takebayashi, and Shibukawa (see [**UTS**]) who prove Jimbo's q-analogue of the classical Gelfand-Tsetlin results using the lowering operator method. Their approach closely parallels the treatment for the classical case given by Zhelobenko in [**Z**]. The rather unnatural notation we adopt for the dominant weights of $\mathfrak{sl}_{n+1}$, which reflect their origins in the $\mathfrak{gl}_{n+1}$ case of Gelfand and Tsetlin, will be seen to be more than justified for their combinatorial usefulness in what follows.

Let $\Phi_{n+1} \subset \mathbf{N}^{n+1}$ be the set of $(n+1)$-tuples, $\mu_{n+1} = (\mu_{1,n+1}, \mu_{2,n+1}, \dots, \mu_{n+1,n+1})$, with the property $\mu_{i,n+1} \ge \mu_{i+1,n+1}$, $1 \le i \le n$. (Thus Φ is a subset of dominant weights for $\mathfrak{gl}_{n+1}$.) Then each $\mu_{n+1} \in \Phi_{n+1}$ can be seen to represent a dominant weight $\lambda(\mu_{n+1}) = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbf{N}^n$, of $\mathfrak{sl}_{n+1}$, where $\lambda_i = \mu_{i,n+1} - \mu_{i+1,n+1}$. Of course $\lambda(\mu_{n+1})$ and $\lambda(\mu'_{n+1})$ represent the same dominant weight if and only if $\mu_{i,n+1} = \mu'_{i,n+1} + \alpha$, for $1 \le i \le n+1$ and some $\alpha \in \mathbf{Z}$. With this in mind, we use the elements of Φ_{n+1} as the 'dominant weights' of $\mathfrak{sl}_{n+1}$, and write $V(\mu_{n+1})$ for the simple $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules they define.

Recall that $\mathbf{U} = \mathbf{U}(\mathfrak{sl}_{n+1})$ is a $\mathbf{C}(v)$-algebra with generators F_i, $K_i^{\pm 1}$, and E_i $(1 \le i \le n)$. There is a $\mathbf{C}(v)$-algebra involution * on $\mathbf{U}$ defined by

$$(K_i^{\pm 1})^* = K_i^{\pm 1}, \quad E_i^* = F_i, \quad \text{and } F_i^* = E_i.$$

For each simple $\mathbf{U}$-module $V(\mu_{n+1})$, we define its dual, $V(\mu_{n+1})^*$, as the irreducible ***right*** $\mathbf{U}$-module with the property that:

(1) The highest weight space $V(\mu_{n+1})^*_{\mu_{n+1}}$ is 1-dimensional, and
(2) If $0 \ne \eta \in V(\mu_{n+1})^*_{\mu_{n+1}}$, then $\eta F_i = 0$ and $\eta K_i = v^{\mu_{i,n+1} - \mu_{i+1,n+1}} \eta$.

If we fix highest weight vectors $\xi \in V(\mu_{n+1})$ and $\eta \in V(\mu_{n+1})^*$, then by the triangular decomposition of $\mathbf{U}$ (see section 1.1), there exists a natural pairing $V(\mu_{n+1})^* \otimes V(\mu_{n+1}) \to \mathbf{C}(v)$ given by $\eta x \otimes y\xi \mapsto \langle \eta | xy | \xi \rangle$. Here, $\langle \eta | xy | \xi \rangle$ has the following meaning: if in the decomposition $xy = u^- u^o u^+$, with $u^\pm \in U^\pm$ and $u^0 \in U^0$, $u^- \ne 1$ or $u^+ \ne 1$, then $\langle \eta | xy | \xi \rangle = 0$, and if $xy = u^0 \in U^0$, then $\langle \eta | xy | \xi \rangle$ is the eigenvalue of u^0 on η and ξ, which are necessarily the same by definition. This allows us to define an inner product $(\ ,\ )$ on $V(\mu_{n+1})$ by $(x\xi, y\xi) = \langle \eta | x^* y | \xi \rangle$. It turns out that the Gelfand-Tsetlin basis, described below, is orthonormal with respect to this inner product.

For $\mu_n = (\mu_{1,n}, \mu_{2,n}, \dots, \mu_{n,n}) \in \Phi_n$ write $\|\mu_n\| = \sum_{1 \le \alpha \le n} \mu_{\alpha,n}$ for the *norm* of μ_n, and if $\mu_{n+1} = (\mu_{1,n+1}, \mu_{2,n+1}, \dots, \mu_{n+1,n+1}) \in \Phi_{n+1}$, write $\mu_n \prec \mu_{n+1}$ if and only if $\mu_{i,n+1} \ge \mu_{i,n} \ge \mu_{i+1,n+1}$ $(1 \le i \le n)$. More generally, given $i < n$ and $\mu_i \in \Phi_i$, write $\mu_i \prec \mu_n$ if and only if there exists a sequence $(\mu_{i+1}, \mu_{i+2}, \dots, \mu_{n-1}) \in \Phi_{i+1} \times \Phi_{i+2} \times \dots \times \Phi_{n-1}$ such that $\mu_i \prec \mu_{i+1} \prec \dots \prec \mu_{n-1} \prec \mu_n$. A sequence of vectors $\mu = (\mu_{n+1}, \mu_n, \dots, \mu_1) \in \Phi_{n+1} \times \Phi_n \times \dots \times \Phi_1$ will be called a *Gelfand-Tsetlin scheme* if it satisfies the condition $\mu_i \prec \mu_{i+1}$ for $1 \le i \le n$.

For $1 \le j \le i \le n+1$, we define the *lowering operators* $D_{ij} \in U^0 \otimes U^-$ inductively as follows:

$$D_{i,i} = 1, \qquad D_{i,i-1} = F_i, \quad \text{and}$$
$$D_{i,j} = \langle K_{i,j} + i - j \rangle F_i D_{i-1,j} - \langle K_{i,j} + i - j - 1 \rangle D_{i-1,j} F_i \quad \text{if } j < i,$$

where $K_{i,j} = \prod_{j \le \alpha \le i-2} K_\alpha$, and $\langle K_{i,j} + k\rangle = (v^k K_{i,j} - v^{-k} K_{i,j}^{-1})/(v - v^{-1})$.

Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}) \in \mathbb{N}^{i-1}$, we set $D_i^\alpha = D_{i,1}^{\alpha_1} D_{i,2}^{\alpha_2} \cdots D_{i,i-1}^{\alpha_{i-1}}$, and for a given Gelfand-Tsetlin scheme $\mu = (\mu_{n+1}, \mu_n, \dots, \mu_1)$, we let $D^\mu = D_2^{\mu_2-\mu_1} D_3^{\mu_3-\mu_2} \cdots D_{n+1}^{\mu_{n+1}-\mu_n}$, where $\mu_i - \mu_{i-1} = (\mu_{1,i} - \mu_{1,i-1}, \mu_{2,i} - \mu_{2,i-1}, \dots, \mu_{i-1,i} - \mu_{i-i,i-1})$. Then we have following two results, first established by Gelfand and Tsetlin for the $\mathfrak{gl}_{n+1}$ case, and extended to the quantum case by Jimbo.

Theorem 2.1.1 (Branching Law for $\mathbf{U}(\mathfrak{sl}_i) \downarrow \mathbf{U}(\mathfrak{sl}_{i-1})$). *Let $\mathbf{U}_{i+1} = \mathbf{C}[F_j, K_j^{\pm 1}, E_j \mid 1 \le j \le i]$, for $0 \le i \le n$, be a chain of subalgebras in $\mathbf{U}$. Then for any $\mu_i \in \Phi_i$, $V(\mu_i)$ decomposes as $V(\mu_i) = \oplus_{\mu_{i-1} \prec \mu_i} V(\mu_{i-1})$, where $V(\mu_{i-1})$ with $\mu_{i-1} \in \Phi_{i-1}$ are irreducible $\mathbf{U}_{i-1}$-submodules.*

Proof: See [**UTS**, Theorem 6]. □

Starting with a simple $\mathbf{U}$-module $V(\mu_{n+1})$ and applying the above theorem successively we see that the Gelfand-Tsetlin schemes $\mu' = (\mu'_{n+1} \succ \mu'_n \succ \dots \succ \mu'_1)$, with $\mu'_{n+1} = \mu_{n+1}$, give rise to a parametrization for a basis of $V(\mu_{n+1})$, since the branching process stops at irreducible $\mathbf{U}_1$-modules which correspond to lines. The connection between the basis so obtained and the lowering operators described above is contained in following:

Theorem 2.1.2. *Let μ_{n+1} be a dominant weight for $\mathbf{U}$, and let ξ be the highest weight vector of $V(\mu_{n+1})$. Then the vectors $|\lambda\rangle = N_\lambda^{-1} D^\lambda \xi$, for each Gelfand-Tsetlin scheme λ with $\lambda_{n+1} = \mu_{n+1}$, forms an orthonormal basis under the inner product $(\, , \,)$, where*

$$N_\lambda^2 = \prod_{k=2}^{n+1} \left\{ \prod_{1 \le i < j \le k-1} \frac{[\lambda_{i,k} - \lambda_{j,k-1} + j - i]!}{[\lambda_{i,k-1} - \lambda_{j,k-1} + j - i]!} \prod_{1 \le i \le j \le k} \frac{[\lambda_{i,k} - \lambda_{j,k} + j - i - 1]!}{[\lambda_{i,k-1} - \lambda_{j,k} + j - i - 1]!} \right\},$$

and N_λ has the property that $N_\lambda\big|_{v=1} > 0$.

Proof: See [**UTS**, Proposition 7]. □

The basis $\{|\mu\rangle\}$ so obtained is called the Gelfand-Tsetlin basis. Note that the Gelfand-Tsetlin basis vectors $|\mu\rangle$ are compatible with natural inclusions $\mathbf{U}_i \to \mathbf{U}$ in the sense that $|\mu\rangle$ belongs to the irreducible $\mathbf{U}_i$-submodule of highest weight μ_i for all $2 \le i \le n+1$.

2.2. Some Consequences. We now turn to the action of $\mathbf{U}$ on the Gelfand-Tsetlin basis. The action of the Chevalley generators on the Gelfand-Tsetlin basis was first computed by Gelfand and Tsetlin [**GT1**] for $\mathfrak{gl}_{n+1}$, and by Jimbo [**J**] for $\mathbf{U}(\mathfrak{gl}_{n+1})$. However, in order to use their results we will need to extend our field $\mathbf{C}(v)$, since their formulae involve taking square roots. We will write $\mathfrak{C}$ to denote the algebraic closure of $\mathbf{C}(v)$, and take $\mathbf{U}$ to be an algebra over $\mathfrak{C}$ throughout this chapter. Then a trivial modification of Jimbo's results yields the following for the action of the Chevalley generators E_i, F_i, and $K_i^{\pm 1}$ of $\mathbf{U}$.

Theorem 2.2.1. *The generators E_i, F_i, and K_i act on the Gelfand-Tsetlin basis according to the following formulae:*

$$\begin{aligned} K_i(|\mu\rangle) &= v^{(\|\mu_i\| - \|\mu_{i-1}\|) - (\|\mu_{i+1}\| - \|\mu_i\|)} |\mu\rangle \\ E_i(|\mu\rangle) &= \sum_{1 \le \alpha \le i} a_{\alpha,i}(|\mu\rangle) \, |\mu + \epsilon_{\alpha,i}\rangle \\ F_i(|\mu\rangle) &= \sum_{1 \le \alpha \le i} a_{\alpha,i}(|\mu - \epsilon_{\alpha,i}\rangle) \, |\mu - \epsilon_{\alpha,i}\rangle, \end{aligned}$$

where $(\epsilon_{\alpha,i})_{s,t} = \delta_{s,\alpha}\delta_{t,i}$, and

$$a_{k,i}(|\mu\rangle) = \left\{ \frac{\prod_{1\le\alpha\le i+1} [(\mu_{\alpha,i+1}-\alpha)-(\mu_{k,i}-k)] \prod_{1\le\alpha\le i-1} [(\mu_{\alpha,i-1}-\alpha)-(\mu_{k,i}-k)-1]}{\prod_{\substack{1\le\alpha\le i\\ \alpha\ne k}} [(\mu_{\alpha,i}-\alpha)-(\mu_{k,i}-k)] \prod_{1\le\alpha\le i} [(\mu_{\alpha,i}-\alpha)-(\mu_{k,i}-k)-1]} \right\}^{\frac{1}{2}} .$$

Proof: The formulae for E_i and F_i are just those given by Jimbo, and the formulae for $K_i^{\pm 1}$ are obtained from Jimbo's formulae for $q^{\pm\epsilon_i}$ by using $K_i^{\pm 1} = (q^{\epsilon_i} q^{-\epsilon_{i+1}})^{\pm 1}$. □

We make the following simple observation which will be useful later.

Lemma 2.2.2. *For all $1 \le i \le n$ and $1 \le k \le i$, we have $a_{k,i}(|\mu\rangle)\big|_{v=1} \ge 0$. Moreover, $a_{k,i}(|\mu\rangle)\big|_{v=1} > 0$ if $\mu_{k,i} < \min(\mu_{k-1,i-1}, \mu_{k,i+1})$. In particular, this implies $a_{k,i}(|\mu\rangle) \ne 0$ if $\mu_{k,i} < \min(\mu_{k-1,i-1}, \mu_{k,i+1})$.*

Proof: The content of this lemma is that the quantity

$$a_{k,i}(|\mu\rangle)^2\Big|_{v=1} = \frac{\prod_{1\le\alpha\le i+1} \{(\mu_{\alpha,i+1}-\alpha)-(\mu_{k,i}-k)\} \prod_{1\le\alpha\le i-1} \{(\mu_{\alpha,i-1}-\alpha)-(\mu_{k,i}-k)-1\}}{\prod_{\substack{1\le\alpha\le i\\ \alpha\ne k}} \{(\mu_{\alpha,i}-\alpha)-(\mu_{k,i}-k)\} \prod_{1\le\alpha\le i} \{(\mu_{\alpha,i}-\alpha)-(\mu_{k,i}-k)-1\}}$$

is non-negative for $1 \le k \le i \le n$. It is clear that the first product in the denominator contains $i-k$ negative terms, corresponding to $k+1 \le \alpha \le i$, and that the second product contains $i-k+1$ negative terms corresponding to $k \le \alpha \le i$. The sign of the denominator is thus $(-1)^{2i-2k+1}$. In the numerator, since $\mu_{j,i+1} \ge \mu_{j,i} \ge \mu_{j+1,i+1}$ by definition of $\prec$, we have that $(\mu_{\alpha,i+1}-\alpha)-(\mu_{k,i}-k) < 0$ for $k+1 \le \alpha \le i+1$, and $(\mu_{\alpha,i-1}-\alpha)-(\mu_{k,i}-k)-1 < 0$ for $k \le \alpha \le i-1$. The remaining terms are non-negative, and so either the numerator is zero, or the sign of the numerator is $(-1)^{2i-2k+1}$. The first assertion follows immediately. Next $(\mu_{\alpha,i+1}-\alpha)-(\mu_{k,i}-k) > 0$ for $1 \le \alpha \le k-1$, and $(\mu_{k,i+1}-k)-(\mu_{k,i}-k) > 0$ if $\mu_{k,i+1} > \mu_{k,i}$. And similarly, $(\mu_{\alpha,i-1}-\alpha)-(\mu_{k,i}-k)-1 > 0$ for $1 \le \alpha \le k-2$, and $(\mu_{k-1,i-1}-k+1)-(\mu_{k,i}-k)-1 > 0$ if $\mu_{k-1,i-1} > \mu_{k,i}$. This proves the second assertion, and the final assertion is an immediate corollary. □

The above lemma has the following interesting corollaries.

Corollary 2.2.3. *The action of the Chevalley generators e_i and f_i of $\mathfrak{sl}_{n+1}$ have the following 'positivity' property:*

$$e_i^r(|\mu\rangle) = \sum_{\substack{\mu' \\ \mu'_j=\mu_j\ (j\ne i) \\ \mu_{i-1}\prec\mu'_i\prec\mu_{i+1} \\ \mu'_{ji}\ge\mu_{ji} \\ \|\mu'_i-\mu_i\|=r}} c_{\mu'}|\mu'\rangle, \qquad f_i^r(|\mu\rangle) = \sum_{\substack{\mu' \\ \mu'_j=\mu_j\ (j\ne i) \\ \mu_{i-1}\prec\mu'_i\prec\mu_{i+1} \\ \mu'_{ji}\le\mu_{ji} \\ \|\mu_i-\mu'_i\|=r}} d_{\mu'}|\mu'\rangle ,$$

where $c_{\mu'}, d_{\mu'} > 0$. In particular, for the generators E_i and F_i of $\mathbf{U}(\mathfrak{sl}_{n+1})$ we have

$$E_i^{(r)}(|\mu\rangle) = \sum_{\substack{\mu' \\ \mu'_j=\mu_j\ (j\ne i) \\ \mu_{i-1}\prec\mu'_i\prec\mu_{i+1} \\ \mu'_{ji}\ge\mu_{ji} \\ \|\mu'_i-\mu_i\|=r}} c_{\mu'}|\mu'\rangle, \qquad F_i^{(r)}(|\mu\rangle) = \sum_{\substack{\mu' \\ \mu'_j=\mu_j\ (j\ne i) \\ \mu_{i-1}\prec\mu'_i\prec\mu_{i+1} \\ \mu'_{ji}\le\mu_{ji} \\ \|\mu_i-\mu'_i\|=r}} d_{\mu'}|\mu'\rangle ,$$

where $c_{\mu'}, d_{\mu'} \neq 0$. □

Corollary 2.2.4. *Let* $m = m(|\mu\rangle, i) = \max\{k \in \mathbb{N} \mid E_i^k(|\mu\rangle) \neq 0\}$. *Then* $0 \neq E_i^{(m)}(|\mu\rangle) \in \langle\, |\mu'\rangle\,\rangle$, *where* $\mu'_j = \mu_j$ *for* $i \neq j$, *and* $\mu'_{j,i} = \min(\mu_{j,i+1}, \mu_{j-1,i-1})$ *for* $1 \leq j \leq i$. *Similarly let* $m' = m'(|\mu\rangle, i) = \max\{k \in \mathbb{N} \mid F_i^k(|\mu\rangle) \neq 0\}$. *Then* $F_i^{(m')}(|\mu\rangle) \in \langle\, |\mu'\rangle\,\rangle$, *where* $\mu'_j = \mu_j$ *for* $i \neq j$, *and* $\mu'_{j,i} = \max(\mu_{j+1,i+1}, \mu_{j,i-1})$ *for* $1 \leq j \leq i-1$. □

Example 2.2.5. Take $n = 2$, $\mu_3 = (3,2,1)$, and let

$$\xi = |(3,2,1),(3,2),(2)\rangle, \qquad \zeta_1 = |(3,2,1),(2,2),(2)\rangle,$$
$$\zeta_2 = |(3,2,1),(3,1),(2)\rangle, \qquad \eta = |(3,2,1),(2,1),(2)\rangle.$$

Then $W = \langle \xi, \zeta_1, \zeta_2, \eta \rangle_{\mathbb{C}}$ is a $\mathbb{C}[F_2, K_2^{\pm 1}, E_2]$-submodule of $V(\mu_3)$, and we have that the coefficients $a_{k,i}(|\mu\rangle)$ are given by

$$a_{12}(\xi) = 0, \qquad a_{22}(\xi) = 0,$$
$$a_{12}(\zeta_1) = \sqrt{\frac{[1][-1][-3][-1]}{[-1][-1][-2]}} = \sqrt{\frac{[3]}{[2]}}, \qquad a_{22}(\zeta_1) = 0,$$
$$a_{12}(\zeta_2) = 0, \qquad a_{22}(\zeta_2) = \sqrt{\frac{[3][1][-1][2]}{[3][2][-1]}} = 1,$$
$$a_{12}(\eta) = \sqrt{\frac{[1][-1][-3][-1]}{[-2][-1][-3]}} = \sqrt{\frac{1}{[2]}}, \qquad a_{22}(\eta) = \sqrt{\frac{[3][1][-1][2]}{[2][1][-1]}} = \sqrt{[3]}.$$

Note that the coefficients are non-negative under the specialization $v \mapsto 1$, and that the statements of corollaries 2.2.3 and 2.2.4 are easily seen to hold true in this case.

2.3. Decomposition of $V(\mu_{n+1})$ as a $\mathbf{U}(\mathfrak{sl}_n)$-module. By theorem 2.1.1, we have that $V(\mu_{n+1})$ decomposes as $V(\mu_{n+1}) = \oplus_{\mu_n \prec \mu_{n+1}} V(\mu_n)$ as a $\mathbf{U}_n$-module, where each summand is an irreducible $\mathbf{U}_n$-submodule. The decomposition is not multiplicity free in general since μ_n and μ'_n can give isomorphic $\mathbf{U}_n$-modules without their being equal. (Note that the decomposition is multiplicity free when regarded as a $\mathbf{U}(\mathfrak{gl}_n)$-module.) However, we will shortly see that this problem with multiplicities can be resolved once we restrict to zero weight spaces.

For $\mu_{n+1} \in \Phi_{n+1}$ and $k \in \mathbb{N}$ such that $\|\mu_{n+1}\| = (n+1)k$, let

$$\Sigma_i(\mu_{n+1}) = \{\mu_i \in \Phi_i \mid \mu_i \prec \mu_{n+1} \text{ and } \|\mu_i\| = ik\},$$

for $1 \leq i \leq n$, and define $\Sigma_{n+1}(\mu_{n+1}) = \{\mu_{n+1}\}$.
Let $V(\mu_{n+1})_0$ be the zero weight space of $V(\mu_{n+1})$. Then we have the following consequences of theorem 2.2.1.

Lemma 2.3.1. *If* $V(\mu_{n+1})_0 \neq 0$, *then there exists* $k \in \mathbb{N}$ *such that the basis of* $V(\mu_{n+1})_0$ *consists of* $|\mu\rangle$ *satisfying* $\|\mu_i\| = ik$ *for* $1 \leq i \leq n+1$.

Proof: Firstly note that since the vectors $|\mu\rangle$ are weight vectors, a basis for $V(\mu_{n+1})_0$ will consist of a subset of the Gelfand-Tsetlin basis for $V(\mu_{n+1})$ which are of weight zero. So let $|\mu\rangle \in V(\mu_{n+1})_0$, whence $K_i(|\mu\rangle) = v^{(\|\mu_i\| - \|\mu_{i+1}\|) - (\|\mu_{i-1}\| - \|\mu_i\|)}|\mu\rangle = |\mu\rangle$ for $1 \leq i \leq n$. Putting $i = 1$, we find $\|\mu_2\| = 2\|\mu_1\|$, and proceeding by induction we find $\|\mu_i\| = i\|\mu_1\|$. The result now follows since μ_{n+1}, and hence $\|\mu_{n+1}\|$, is fixed. □

Corollary 2.3.2. $V(\mu_{n+1})_0 \neq 0$ *if and only if* $(n+1) \mid \|\mu_{n+1}\|$. □

The problem of multiplicities can now be resolved.

Proposition 2.3.3. *We have $V(\mu_{n+1})_0 = \oplus_{\mu_n \in \Sigma_n(\mu_{n+1})} V(\mu_n)_0$. Furthermore, the decomposition is multiplicity free in the sense that for $\mu_n, \lambda_n \in \Sigma(\mu_{n+1})$, $V(\mu_n) \cong V(\lambda_n)$ as $\mathbf{U}_n$-modules if and only if $\mu_n = \lambda_n$.*

Proof: The first statement follows directly from the definition of $\Sigma_n(\mu_{n+1})$ and lemma 2.3.1. For the second statement, note that $V(\mu_n) \cong V(\lambda_n)$ as $\mathbf{U}_n$-modules if and only if $\mu_{i,n} = \lambda_{i,n} + \alpha$, for $1 \leq i \leq n$ and $\alpha \in \mathbf{Z}$. However, for $\mu_n, \lambda_n \in \Sigma_n(\mu_{n+1})$, this is possible if and only if $\mu_n = \lambda_n$, as we require $\|\mu_n\| = nk = \|\lambda_n\|$. This completes the proof. □

2.4. The Graph $\Gamma(\mu_{n+1})$. As mentioned earlier, our method for the proof of the irreducibility of the $\mathcal{B}_n$-action on the zero weight spaces is by induction. The approach we adopt is the following: we begin with $V(\mu_{n+1})_0$ and use proposition 2.3.3 to decompose it into $\mathcal{B}_{n-1}$-submodules. We then attach a graph to μ_{n+1}, or essentially to $V(\mu_{n+1})_0$, whose vertices correspond to the summands that appear in proposition 2.3.3, and relate the properties of the resulting graph to the irreducibilty of $V(\mu_{n+1})_0$ as a $\mathcal{B}_n$-module. It turns out that the connectedness of this graph, which is very easy to prove, has the consequence of establishing the irreducibility. We now define the graph, and prove its connectedness.

Let $\mu_{n+1} \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k$ for some $k \in \mathbf{N}$ (see corollary 2.3.2). We attach a graph, $\Gamma(\mu_{n+1})$, to μ_{n+1} as follows. The vertices of $\Gamma(\mu_{n+1})$ are the elements of $\Sigma_n(\mu_{n+1})$, and two vertices μ_n and μ'_n are joined by an edge if and only if $\Sigma_{n-1}(\mu_n) \cap \Sigma_{n-1}(\mu'_n) \neq \emptyset$.

Lemma 2.4.1. *$\Gamma(\mu_{n+1})$ is connected for $n \geq 1$.*

Proof: The case $n = 1$ is trivial since in that case $\Gamma(\mu_{n+1})$ consists of a single vertex. So assume $n \geq 2$. It is enough to show that vertices μ_n and μ'_n are joined by an edge where $\mu_{\alpha,n} = \mu'_{\alpha,n}$ for all $\alpha \notin \{i,j\}$, and $\mu_{i,n} = \mu'_{i,n} \pm 1$ and $\mu_{j,n} = \mu'_{j,n} \mp 1$. If $\mu'_n = (k,k,\ldots,k)$, then the condition $\mu_{1,n} \geq \mu_{2,n} \geq \cdots \geq \mu_{n,n}$ implies that $\mu_{1,n} = k+1$ and $\mu_{1,1} = k-1$. In this case, we have that $(k,k,\ldots,k) \in \Sigma_{n-1}(\mu_n) \cap \Sigma_{n-1}(\mu'_n)$. The case $\mu_n = (k,k,\ldots,k)$ is considered in a similar way. Thus we may assume that $\mu_n \neq (k,k,\ldots,k)$ and $\mu'_n \neq (k,k,\ldots,k)$, whence $\mu'_{1,n} > k > \mu'_{n,n}$ and $\mu_{1,n} > k > \mu_{n,n}$. Without loss of generality, we may assume $\mu_{i,n} = \mu'_{i,n} + 1$ and $\mu_{j,n} = \mu'_{j,n} - 1$, where $i < j$, and $\mu_{\alpha,n} = \mu'_{\alpha,n}$ for $\alpha \notin \{i,j\}$. (The case $i > j$ is obtained using the same method with μ_n and μ'_n interchanged.) Consider $S = \{\mu_{n-1} \in \Phi_{n-1} \mid \mu_{n-1} \prec \mu_n \text{ and } \mu_{n-1} \prec \mu'_n\}$. Then we need to show that there is a $\mu_{n-1} \in S$ with the property $\|\mu_{n-1}\| = (n-1)k$. Let μ_{n-1}^{min} and μ_{n-1}^{max} be elements of S with minimum and maximum norms respectively. Then we find

$$\|\mu_{n-1}^{max}\| = \begin{cases} nk - \mu'_{n,n} - 1 & \text{if } j < n, \\ nk - \mu'_{n,n} & \text{if } j = n, \end{cases} \qquad \|\mu_{n-1}^{min}\| = \begin{cases} nk - \mu'_{1,n} + 1 & \text{if } i > 1, \\ nk - \mu'_{1,n} & \text{if } i = 1. \end{cases}$$

In every case, we have that $\|\mu_{n-1}^{max}\| \geq nk - \mu'_{n,n} - 1$, and $\|\mu_{n-1}^{min}\| \leq nk - \mu'_{1,n} + 1$. Since $\mu'_{n,n} < k < \mu'_{1,n}$, we have that $\|\mu_{n-1}^{min}\| \leq nk - \mu'_{1,n} + 1 \leq (n-1)k$, and $\|\mu_{n-1}^{max}\| \geq nk - \mu'_{n,n} - 1 \geq (n-1)k$. And since there is a $\mu_{n-1} \in S$ of norm p, for every p between $\|\mu_{n-1}^{min}\|$ and $\|\mu_{n-1}^{max}\|$, we conclude that there exists a $\mu_{n-1} \in S$ such that $\|\mu_{n-1}\| = (n-1)k$, and this completes the proof. □

2.5. Main Lemmas. In this section we state and prove some results which establish the connection between $\Gamma(\mu_{n+1})$ and the $\mathcal{B}_n$-module $V(\mu_{n+1})$. But first we lay down some notation and groundwork.

Let $\mathbf{U}' = \mathbb{C}[F_n, K_n^{\pm 1}, E_n] \subset \mathbf{U}$ be a copy of $\mathbf{U}(\mathfrak{sl}_2)$ generated by F_n, $K_n^{\pm 1}$, and E_n and consider $V(\mu_{n+1})$ as a $\mathbf{U}'$-module. Then for each $\lambda = (\lambda_{n-1} \succ \lambda_{n-2} \succ \cdots \succ \lambda_1) \in \Phi_{n-1} \times \Phi_{n-2} \times \cdots \times \Phi_1$, we have, by theorem 2.2.1, that

$$W(\lambda) = \sum_{\substack{\mu' : \mu'_{n+1} = \mu_{n+1} \\ \lambda_{n-1} \prec \mu'_n \prec \mu_{n+1} \\ \mu'_i = \lambda_i \ (1 \leq i \leq n-1)}} \mathbb{C} \cdot |\mu'\rangle$$

is a $\mathbf{U}'$-submodule of $V(\mu_{n+1})$, since F_n, $K_n^{\pm 1}$, and E_n only affect the μ_n component of $|\mu\rangle$. It is clear that $V(\mu_{n+1}) = \oplus_\lambda W(\lambda)$. Fix a weight $\mu_{n+1} \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k$ for some $k \in \mathbf{N}$, and let

$$\Lambda = \{\lambda \in \Phi_{n-1} \times \Phi_{n-2} \times \cdots \times \Phi_1 \mid \lambda_i \in \Sigma_i(\lambda_{i+1}),\ 1 \le i \le n-1\}.$$

Lemma 2.5.1. *$W(\lambda) \cap V(\mu_{n+1})_0 \neq 0$ if and only if $\lambda \in \Lambda$, and $V(\mu_{n+1})_0 = \oplus_{\lambda \in \Lambda} W(\lambda)_0$.*

Proof: This is a consequence of lemma 2.3.1 and proposition 2.3.3. □

Now fix $\lambda \in \Lambda$, and consider $W(\lambda)$. As a $\mathbf{U}'$-module, $W(\lambda)$ decomposes as $W(\lambda) = \sum_\alpha W(\lambda, \alpha)$, where $W(\lambda,\alpha)$ is isomorphic to the irreducible $\mathbf{U}(\mathfrak{sl}_2)$-module of highest weight $2n_\alpha$. Let $0 \neq \xi_{\lambda,\alpha} \in W(\lambda,\alpha)$ be the highest weight vectors. Then in particular, we have $W(\lambda)_0 = \oplus_\alpha \mathbb{C} \cdot F_n^{(n_\alpha)} \xi_{\lambda,\alpha}$.

Lemma 2.5.2. *Let α_λ have the property that $n_{\alpha_\lambda} = \max(n_\alpha)$. Then $W(\lambda, \alpha_\lambda)$ occurs in $W(\lambda)$ with multiplicity one, and $\xi_{\lambda,\alpha_\lambda} \in \langle |\mu'\rangle \rangle$, where $\mu'_{n+1} = \mu_{n+1}$, $\mu'_{i,n} = \min(\mu_{i,n+1}, \lambda_{i-1,n-1})$ for $1 \le i \le n$, and $\mu'_j = \lambda_j$ for $1 \le j \le n-1$.*

Proof: Let $|\mu''\rangle \in W(\lambda)$. Then by definition we have $\mu''_{n+1} = \mu_{n+1}$, and $\mu''_i = \lambda_i$ for $1 \le i \le n-1$. In particular, μ''_i is fixed for all $i \neq n$. Now by theorem 2.2.1, $K_n(|\mu''\rangle) = v^{2\|\mu''_n\| - \|\mu''_{n+1}\| - \|\mu''_{n-1}\|}|\mu''\rangle$, and as μ''_{n-1} and μ''_{n+1} are fixed, the vectors in $W(\lambda)$ with the highest weight, respect to K_n, are $|\mu''\rangle$ with $\|\mu''_n\|$ maximal. However, there is a unique vector $|\mu'\rangle$ with maximal norm satisfying the condition $\lambda_{n-1} \prec \mu'_n \prec \mu_{n+1}$, and it is precisely the one given in the statement of this lemma. It is clear that this $|\mu'\rangle$ is a highest weight vector for an irreducible $\mathbf{U}'$-submodule of $W(\lambda)$, and the uniqueness of $|\mu'\rangle$ implies that $W(\lambda, \alpha_\lambda)$ occurs with multiplicity one. This completes the proof. □

The modules over the braid group $\mathcal{B}_n$ are, in general, not completely reducible. However for the situations that arise in this thesis, namely $V(\mu_{n+1})_0$, for certain $\mu_{n+1} \in \Phi_{n+1}$ (see theorem 2.6.1), and $\mathcal{B}_{n-1}$-submodules thereof, we will find that we can nevertheless decompose the spaces into a direct sum of irreducible $\mathcal{B}_{n-1}$-submodules, and this will play a crucial role in our inductive process. Our ability to obtain such a decomposition is facilitated by the following algebraic lemma, found for example in [**B**, §3 Proposition 9].

Lemma 2.5.3. *Let V be a module over an algebra A such that it permits a decomposition, $V = \oplus_{\omega \in \Omega} V_\omega$, into irreducible A-submodules. Then any A-submodule $W \subset V$ is a direct sum of its isotypic components. In particular, if the decomposition is multiplicity free then we have $W = \oplus_{\omega \in \Omega' \subset \Omega} V_\omega$.* □

We make an immediate use of this lemma.

Lemma 2.5.4. *Suppose that in the decomposition $V(\mu_{n+1})_0 = \oplus_{\mu_n \in \Sigma_n(\mu_{n+1})} V(\mu_n)_0$, each summand is irreducible as a $\mathcal{B}_{n-1}$-module and that the sum is mutiplicity free as a $\mathcal{B}_{n-1}$-module. Let $M \subset V(\mu_{n+1})_0$ be a $\mathcal{B}_n$-submodule and suppose that an element of the Gelfand-Zetlin basis $|\mu\rangle = (\mu_{n+1}, \mu_n, \dots, \mu_1) \in M$. Then $\bigoplus_{\substack{\lambda_n \in \Sigma_n(\mu_{n+1}) \\ \mu_{n-1} \prec \lambda_n \prec \mu_{n+1}}} V(\lambda_n)_0 \subset M$.*

Proof: Let $\lambda = (\mu_{n-1}, \mu_{n-2}, \dots, \mu_1)$. Then by lemma 2.3.1, we have $\lambda \in \Lambda$, and in the above notation, $|\mu\rangle \in W(\lambda)_0$. Hence, using lemma 2.5.2, $|\mu\rangle$ can be written as a sum $|\mu\rangle = cF_n^{(n_{\alpha_\lambda})} \xi_{\lambda,\alpha_\lambda} + \zeta$, where $\zeta \in \oplus_{\{\alpha | n_\alpha < n_\lambda\}} W(\lambda,\alpha)_0$. By corollary 2.2.4 and lemma 2.5.2, we have $E_n^{(n_{\alpha_\lambda})}|\mu\rangle \neq 0$, and since $E_n^{(n_{\alpha_\lambda})}\zeta = 0$ (as $n_\alpha < n_{\alpha_\lambda}$ for $\alpha \neq \alpha_\lambda$), this implies $c \neq 0$. Furthermore, since $F_n^{(n_{\alpha_\lambda})} \xi_{\lambda,\alpha_\lambda}$ is the unique T_n-eigenvector in $W(\lambda)_0$ (upto a non-zero multiple) with the 'maximal' eigenvalue $(-1)^{n_{\alpha_\lambda}} v^{n_{\alpha_\lambda}(n_{\alpha_\lambda}+1)}$ and M is T_n-invariant, we conclude $F_n^{(n_{\alpha_\lambda})} \xi_{\lambda,\alpha_\lambda} \in$

M. Now, by corollary 2.2.3 we may write $F_n^{(n_{\alpha_\lambda})}\xi_{\lambda,\alpha_\lambda} = \sum_{\mu_{n-1}\prec\lambda_n\in\Sigma_n(\mu_{n+1})} c_{\lambda_n} v_{\lambda_n}$, where $v_{\lambda_n} \in V(\lambda_n)_0$, and all $c_{\lambda_n} \neq 0$. Regarding M as a $\mathcal{B}_{n-1}$-module we then have, by lemma 2.5.3, that $v_{\lambda_n} \in M$ for all $\mu_{n-1} \prec \lambda_n \in \Sigma_n(\mu_{n+1})$, since by assumption the sum $\oplus_{\lambda_n\in\Sigma_n(\mu_{n+1})}V(\lambda_n)_0$ is multiplicity free as a $\mathcal{B}_{n-1}$-module. And as $V(\lambda_n)_0$ are irreducible $\mathcal{B}_{n-1}$-modules, again by assumption, we conclude $V(\lambda_n)_0 \subset M$ for all $\mu_{n-1} \prec \lambda_n \in \Sigma_n(\mu_{n+1})$. This completes the proof. □

An important connection between $\Gamma(\mu_{n+1})$ and the $\mathcal{B}_n$-module $V(\mu_{n+1})$ is given by the following:

Proposition 2.5.5. *Suppose that in the decomposition $V(\mu_{n+1})_0 = \oplus_{\mu_n\in\Sigma_n(\mu_{n+1})}V(\mu_n)_0$, each summand is irreducible as a $\mathcal{B}_{n-1}$-module, and that the sum is mutiplicity free as in lemma 2.5.4. Further, let M be a $\mathcal{B}_n$-submodule of $V(\mu_{n+1})_0$ such that $V(\mu_n)_0 \subset M$ for some $\mu_n \in \Sigma_n(\mu_{n+1})$. Then $V(\lambda_n)_0 \subset M$ for all λ_n in the path-component of μ_n in $\Gamma(\mu_{n+1})$.*

Proof: It is enough to consider $\lambda_n \in \Sigma_n(\mu_{n+1})$ which are adjacent to μ_n in the graph $\Gamma(\mu_{n+1})$. But then by the definition of the edges in $\Gamma(\mu_{n+1})$ there exists $\mu'_{n-1} \in \Sigma_{n-1}(\mu_n)\cap\Sigma_{n-1}(\lambda_n)$, and it is easy to check that we can find a sequence $(\mu'_{n-2} \succ \mu'_{n-3} \succ \cdots \succ \mu'_1) \in \Phi_{n-2}\times\Phi_{n-1}\times\cdots\times\Phi_1$ such that $|\mu'\rangle = (\mu_{n+1},\mu_n,\mu'_{n-1},\mu'_{n-2},\cdots,\mu'_1) \in V(\mu_{n+1})_0$. Now $|\mu'\rangle \in M$, and so by lemma 2.5.4 we have $V(\mu_n)_0 \oplus V(\lambda_n)_0 \subset \oplus_{\mu'_{n-1}\prec\lambda'_n\in\Sigma_n(\mu_{n+1})}V(\lambda'_n)_0 \subset M$. The result now follows.□

2.6. The Main Theorem. We can now state and prove our main result.

Theorem 2.6.1. *Let $\mu_{n+1} \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k$ for some $k \in \mathbb{N}$. Suppose that for all $2 \le i \le n$ and $\mu_{i+1},\mu'_{i+1} \in \Sigma_{i+1}(\mu_{n+1})$, we have $\Sigma_i(\mu_{i+1}) = \Sigma_i(\mu'_{i+1})$ if and only if $\mu_{i+1} = \mu'_{i+1}$. Then for all $1 \le i \le n$, we have $V(\mu_{i+1})_0$ is an irreducible $\mathcal{B}_i$-module. Further, for $\mu_{i+1},\mu'_{i+1} \in \Sigma_{i+1}(\mu_{n+1})$, we have $V(\mu_{i+1})_0 \cong V(\mu'_{i+1})_0$ as $\mathcal{B}_i$-modules if and only if $\mu_{i+1} = \mu'_{i+1}$.*

Proof: Fix μ_{n+1} satisfying the given conditions, and consider $\mu_{i+1} \in \Sigma_{i+1}(\mu_{n+1})$. We proceed by induction on i. If $i = 1$, we are in the case of $\mathbf{U}(\mathfrak{sl}_2)$-modules. Irreducibility is clear since the zero weight space is 1-dimensional. Next, two $\mathcal{B}_1$-modules $V(\mu_2)_0$ and $V(\mu'_2)_0$, with $\|\mu_2\| = \|\mu'_2\| = 2k$, are isomorphic if and only if $\mu_2 = \mu'_2$ since, by proposition 1.3.2, T_1 acts as scalars $(-1)^a v^{a(a+1)}$ and $(-1)^{a'} v^{a'(a'+1)}$ respectively on $V(\mu_2)_0$ and $V(\mu'_2)_0$, where $a = (\mu_{1,2} - \mu_{2,2})/2$ and $a' = (\mu'_{1,2} - \mu'_{2,2})/2$, and $a = a'$ if and only if $\mu_2 = \mu'_2$ as $\|\mu_2\| = \|\mu'_2\|$. Now suppose the result is true for i, where $1 \le i \le n-1$, and consider $\mu_{i+2} \in \Sigma_{i+2}(\mu_{n+1})$. Recall that $V(\mu_{i+2})_0$ decomposes as a direct sum, $V(\mu_{i+2})_0 = \oplus_{\mu_{i+1}\in\Sigma_{i+1}(\mu_{i+2})}V(\mu_{i+1})_0$, as a $\mathcal{B}_i$-module. By inductive hypothesis, each summand is an irreducible $\mathcal{B}_i$-module, and the decomposition is multiplicity free as a $\mathcal{B}_i$-module.

We first show that $V(\mu_{i+2})_0$ is irreducible under the $\mathcal{B}_{i+1}$-action. For this, let $0 \neq M \subset V(\mu_{i+2})_0$ be a $\mathcal{B}_{i+1}$-submodule. Then regarding M as a $\mathcal{B}_i$-module, we have by lemma 2.5.3 that $V(\mu_{i+1})_0 \subset M$ for some $\mu_{i+1} \in \Sigma_{i+1}(\mu_{i+2})$ (recall the inductive hypothesis). But then by proposition 2.5.5 we have that $V(\lambda_{i+1})_0 \subset M$ for all λ_{i+1} in the path-component of μ_{i+1} in $\Gamma(\mu_{i+2})$, and since $\Gamma(\mu_{i+2})$ is connected by lemma 2.4.1, we conclude that $V(\mu_{i+2})_0 = \oplus_{\mu_{i+1}\in\Sigma_{i+1}(\mu_{i+2})}V(\mu_{i+1})_0 \subset M$. This proves the irreducibility of $V(\mu_{i+2})_0$ under the $\mathcal{B}_{i+1}$-action.

Finally, suppose $\mu_{i+2},\mu'_{i+2} \in \Sigma_{i+2}(\mu_{n+1})$ such that $\mu_{i+2} \neq \mu'_{i+2}$. Then by assumption on μ_{n+1}, we have $\Sigma_{i+1}(\mu_{i+2}) \neq \Sigma_{i+1}(\mu'_{i+2})$. Without loss of generality we may assume there exists $\mu_{i+1} \in \Sigma_{i+1}(\mu_{i+2}) - \Sigma_{i+1}(\mu'_{i+2})$. But then $V(\mu_{i+2})_0 = \oplus_{\mu_{i+1}\in\Sigma_{i+1}(\mu_{i+2})}V(\mu_{i+1})_0 \ncong \oplus_{\mu_{i+1}\in\Sigma_{i+1}(\mu'_{i+2})}V(\mu_{i+1})_0 = V(\mu'_{i+2})_0$ as $\mathcal{B}_i$-modules, and thus as $\mathcal{B}_{i+1}$-modules. This completes the proof. □

We now give examples of μ_{n+1} which satisfy the conditions of the preceding theorem.

Corollary 2.6.2. *Every $\mu_3 \in \Phi_3$ such that $\|\mu_3\| = 3k$, for some $k \in \mathbf{N}$, has the property described in theorem 2.6.1. Thus the action of $\mathcal{B}_2$ on $V(\mu_3)_0$ is irreducible for every simple $\mathrm{U}(\mathfrak{sl}_3)$-module with nontrivial zero weight space.*

Proof: The conditions of theorem 2.6.1 place no further restrictions on $\mu_3 \in \Phi_3$, and the corollary follows. □

Lemma 2.6.3. *Let $n \geq 2$, and let $\mu_{n+1} = (a, k, k, \ldots, k, b, c)$, $\mu'_{n+1} = (a', k, k \ldots, k, b', c') \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k = \|\mu'_{n+1}\|$, and for $n = 2$, assume $max(b, b') \leq k$. Then $\Sigma_n(\mu_{n+1}) = \Sigma_n(\mu'_{n+1})$ if and only if $\mu_{n+1} = \mu'_{n+1}$.*

Proof: Suppose $\mu_{n+1} \neq \mu'_{n+1}$. We will construct a $\mu_n \in \Phi_n$ such that $\mu_n \in \Sigma_n(\mu_{n+1}) - \Sigma_n(\mu'_{n+1})$, or $\mu_n \in \Sigma_n(\mu'_{n+1}) - \Sigma_n(\mu_{n+1})$. Consider firstly the case when $b = b'$. Then since $a + b + c = 3k = a' + b' + c'$, we have $a + c = a' + c'$, or equivalently, $a - a' = c' - c$. By symmetry, we may assume without loss of generality that $a > a'$ and $c < c'$, as $\mu_{n+1} \neq \mu_{n+1}$ by assumption. Let S be the set of $\mu_n \in \Phi_n$ with the following property: $\mu_{n,n} = c$, and $\mu_{i,n} \in [\mu_{i+1,n+1}, \mu_{i,n+1}]$ for $1 \leq i \leq n-1$. Then we have $\mu_n \prec \mu_{n+1}$ but $\mu_n \not\prec \mu'_{n+1}$ for all $\mu_n \in S$. It remains to show that there exists a $\mu_n \in S$ such that $\|\mu_n\| = nk$. Let μ_n^{min} and μ_n^{max} be the elements in S with the minimum and maximum norms respectively. Then we find

$$\|\mu_n^{min}\| = (n+1)k - a \leq nk\,, \quad \text{and} \quad \|\mu_n^{max}\| = (n+1)k - b \geq nk.$$

Since there exists a μ_n in S of norm p, for every p betwen the minimum and maximum norms, we conclude that there exists a $\mu_n \in S$ with the property $\|\mu_n\| = nk$, and hence a $\mu_n \in \Sigma_n(\mu_{n+1}) - \Sigma_n(\mu'_{n+1})$. Consider now the case where $b \neq b'$. Without loss of generality, we may assume $b > b'$. Let S' be the set of $\mu_n \in \Phi_n$ with the following property: $\mu_{n,n} = b$, and $\mu_{i,n} \in [\mu_{i+1,n+1}, \mu_{i,n+1}]$ for $1 \leq i \leq n-1$. Then, as before, we have $\mu_n \prec \mu_{n+1}$ but $\mu_n \not\prec \mu'_{n+1}$ for all $\mu_n \in S'$, and we need to show that there exists a $\mu_n \in S'$ such that $\|\mu_n\| = nk$. Similar computation gives

$$\begin{aligned} \|\mu_n^{min}\| &= (n+1)k - a + b - c \\ &= (n+1)k - (a+b+c) + 2b \\ &= (n+1)k - 3k + 2b \\ &= nk - 2(k-b) \\ &\leq nk, \quad \text{since } b \leq k, \end{aligned} \qquad \begin{aligned} \|\mu_n^{max}\| &= (n+1)k - c \\ &\geq nk. \end{aligned}$$

It follows that there exists a $\mu_n \in \Sigma_n(\mu_{n+1}) - \Sigma_n(\mu'_{n+1})$, and this completes the proof. □

Corollary 2.6.4. *Let $n \geq 2$, and let $\mu_{n+1} = (a, b, k, k, \ldots, k, c)$, $\mu'_{n+1} = (a', b', k, k \ldots, k, c') \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k = \|\mu'_{n+1}\|$, and for $n = 2$, assume $min(b, b') \geq k$. Then $\Sigma_n(\mu_{n+1}) = \Sigma_n(\mu'_{n+1})$ if and only if $\mu_{n+1} = \mu'_{n+1}$.*

Proof: Similar argument to the preceding lemma applies. □

Lemma 2.6.5. *Let $n \geq 2$, and let $\mu_{n+1} = (a, k, k, \ldots, k, b, c) \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k$, and for $n = 2$, $b \leq k$. Then for $2 \leq i \leq n$, and $\mu_{i+1} \in \Phi_{i+1}$ such that $\|\mu_{i+1}\| = (i+1)k$, we have $\mu_{i+1} \in \Sigma_{i+1}(\mu_{n+1})$ if and only if $\mu_{i+1} = (\alpha, k, k, \ldots, k, \beta, \gamma)$, with $\alpha \leq a$, $\beta \geq b$, and $\gamma \geq c$, and for $i = 2$, $\beta \leq k$.*

Proof: This follows from reverse induction on i, and using the definition of $\Sigma_i(\mu_{n+1})$. □

A similar argument establishes the following for the weights that appear in corollary 2.6.4.

Corollary 2.6.6. *Let $n \geq 2$, and let $\mu_{n+1} = (a, b, k, k, \ldots, k, c) \in \Phi_{n+1}$ such that $\|\mu_{n+1}\| = (n+1)k$, and for $n = 2$, $b \geq k$. Then for $2 \leq i \leq n$ and $\mu_{i+1} \in \Phi_{i+1}$ such that $\|\mu_{i+1}\| = (i+1)k$, we have $\mu_{i+1} \in \Sigma_{i+1}(\mu_{n+1})$ if and only if $\mu_{i+1} = (\alpha, \beta, k, k, \ldots, k, \gamma)$, with $\alpha \leq a$, $\beta \leq b$, and $\gamma \geq c$, and for $i = 2$, $\beta \geq k$.* □

The lemmas 2.6.3 and 2.6.5, and corollaries 2.6.4 and 2.6.6 shows that for $n \geq 2$, the weights of the form $(a, k, k, \ldots, b, c)$ and $(a, b, k, k, \ldots, k, c)$, with $a + b + c = 3k$, satisfy the conditions of theorem 2.6.1. Thus we have the following:

Proposition 2.6.7. *Let $n \geq 2$. Then $V(\mu_{n+1})_0$ is an irreducible $\mathcal{B}_n$-module whenever $\mu_{n+1} \in \Phi_{n+1}$ is of the form $(a, k, k, \ldots, k, b, c)$, or $(a, b, k, k, \ldots, k, c)$, with $a + b + c = 3k$.* □

So we have a two parameter family of irreducible $\mathcal{B}_n$-modules for $n \geq 2$, where the parameters are $a - b$ and $b - c$ in the weights μ_{n+1} given in proposition 2.6.7.

Corollary 2.6.8. *Let $\mu_{n+1} = (2k, k, k, \ldots, k, 0)$. Then $V(\mu_{n+1})_0$ is an irreducible $\mathcal{B}_n$-module of dimension $\binom{n+k-1}{k}$. In particular there is an irreducible $\mathcal{B}_n$-module of arbitrarily large dimension.*

Proof: The irreducibility follows from proposition 2.6.7, and for the dimension note that, by lemma 2.6.5, we have a parametrization for the basis of $V(\mu_{n+1})_0$ given by the set

$$\{(a_n, a_{n-1}, \ldots, a_2) \mid 2k \geq a_n \geq a_{n-1} \geq \cdots \geq a_2 \geq k\}.$$

Since there is a bijection between this set and the set of monomials of degree $n - 1$ in $k + 1$ variables, we conclude that $\dim V(\mu_{n+1})_0 = \binom{n+k-1}{k}$. □

Remark 2.6.9. By specializing $v \mapsto \zeta$, where $\zeta \in \mathbf{C}^\times$ is not a root of unity, we obtain a class of simple $\mathcal{B}_n$-modules parametrized by a pair (μ_{n+1}, ζ), where $\mu_{n+1} \in \Phi_{n+1}$ satisfies the conditions of theorem 2.6.1.

Remark 2.6.10. It would be interesting to determine the necessary and sufficient conditions under which $V(\mu_{n+1})_0$ is an irreducible $\mathcal{B}_n$-module. It seems likely that the result is true for all μ_{n+1} such that $V(\mu_{n+1})_0 \neq 0$.

3. Other Weight Spaces

Until now our interest was restricted to the zero weight spaces of irreducible $\mathbf{U}(\mathfrak{sl}_{n+1})$-modules, since the zero weight space was the only weight space stable under the braid group action. Suppose we replace the braid group $\mathcal{B}_n$ by the subgroup $\mathcal{B}_n^{(2)}$, generated by T_i^2 ($1 \leq i \leq n$). Then *every* weight space is stable under the action of $\mathcal{B}_n^{(2)}$, and we may ask which of the weight spaces are irreducible under the $\mathcal{B}_n^{(2)}$-action. The purpose of this chapter is to give some (partial) results in answer to this question.

3.1. Irreducibility in the case $n = 2, 3$. Let $\mathcal{B}_n^{(2)} = \langle T_1^2, T_2^2, \ldots, T_n^2 \rangle \subset \mathcal{B}_n$. Then the arguments used in the previous chapter imply that for μ_{n+1} satisfying the conditions of theorem 2.6.1, $V(\mu_{n+1})_0$ is an irreducible $\mathcal{B}_n^{(2)}$-module and that for $\mu_{j+1}, \mu'_{j+1} \in \Sigma_{j+1}(\mu_{n+1})$, we have $V(\mu_{j+1})_0 \cong V(\mu'_{j+1})_0$, as $\mathcal{B}_j^{(2)}$-modules, if and only if $\mu_{j+1} = \mu'_{j+1}$.

For any $\nu \in \mathbf{Z}^n$ let $V(\mu_{n+1})_\nu$ denote the ν-weight space of $V(\mu_{n+1})$. Then we have that $T_i(V(\mu_{n+1})_\nu) \subset V(\mu_{n+1})_{s_i\nu}$, and so $T_i^2(V(\mu_{n+1})_\nu) \subset V(\mu_{n+1})_\nu$, where s_i is the generator of the Weyl group of $\mathfrak{sl}_{n+1}$ corresponding to T_i (that is, $s_i = (i, i+1) \in S_{n+1}$). Hence each weight space is stable under the action of $\mathcal{B}_n^{(2)}$, and we would like to determine if they are irreducible as $\mathcal{B}_n^{(2)}$-modules. The method of proof employed for the zero weight space case yields the following result.

Proposition 3.1.1. *Let $n = 1$ or 2, $\mu_{n+1} \in \Phi_{n+1}$, and let $\nu \in \mathbf{Z}^n$ such that $V(\mu_{n+1})_\nu \neq 0$. Then $V(\mu_{n+1})_\nu$ is an irreducible $\mathcal{B}_n^{(2)}$-module, and if $\|\mu_2\| = \|\mu_2'\|$, then $V(\mu_2)_\nu \cong V(\mu_2')_\nu$ if and only if $\mu_2 = \mu_2'$.*

Proof: Consider firstly the case $n = 1$. Then we are in the case of $\mathbf{U}(\mathfrak{sl}_2)$-modules. The irreducibility is clear since the weight spaces are 1-dimensional. Let $\mu_2, \mu_2' \in \Phi_2$ such that $\|\mu_2\| = \|\mu_2'\|$ but $\mu_2 \neq \mu_2'$. Then $a = \mu_{2,2} - \mu_{1,2} \neq \mu_{2,2}' - \mu_{1,2}' = a'$. Since $V(\mu_2)$ corresponds to the irreducible $\mathbf{U}(\mathfrak{sl}_2)$-module of highest weight a, and $V(\mu_2')$ corresponds to the $\mathbf{U}(\mathfrak{sl}_2)$-module of highest weight a', we have, in particular, that $V(\mu_2) = V(a) \not\cong V(a') = V(\mu_2')$. Without loss of generality assume $a \geq a'$. Then since we are assuming $V(a)$ and $V(a')$ have a weight space in common, $2 \mid a - a'$. So let $a = a' + 2r$ and suppose $\nu = a' - 2i$. Then the eigenvalues of T_1^2 are

$$\begin{aligned} \alpha &= (-1)^{a'} v^{(r+i+1)(a'+r-i)+(r+i)(a'+r-i+1)} \quad \text{and} \\ \alpha' &= (-1)^{a'} v^{(i+1)(a'-i)+i(a'-i+1)} \end{aligned}$$

respectively on $V(a)_\nu$ and $V(a')_\nu$. Since $\alpha' = \alpha$ if and only if $a = a'$, the result is proved for $n = 1$. The proof for the case $n = 2$ is identical to the proof used for the zero weight space case and is omitted. Note that we are able to apply the same arguments, as those used in the previous chapter, because $\mu_2 \neq \mu_2'$ implies $V(\mu_2)_\nu \not\cong V(\mu_2')_\nu$. □

3.2. Failure for $n \geq 3$. It was hinted in the previous section that for $n \geq 3$ the weight spaces $V(\mu_{n+1})_\nu$ are in general not irreducible as $\mathcal{B}_n^{(2)}$-modules. We now illustrate this with a simple example.

Proposition 3.2.1. *Let $\mu_4 = (3,2,1,0)$ and $\nu = (-2,2,0)$. Then $V(\mu_4)_\nu$ has a nontrivial $\mathcal{B}_3^{(2)}$-submodule.*

Proof: By theorem 2.2.1, we have that $V(\mu_4)_\nu$ has a basis consisting of vectors $(\mu_4, \mu_3, \mu_2, \mu_1)$ such that $\|\mu_3\| = 5$, $\|\mu_2\| = 4$, and $\|\mu_1\| = 1$, and $\mu_i \prec \mu_{i+1}$ for $1 \leq i \leq 3$. Such vectors are easily enumerated and they are

$$\begin{aligned} x &= |\,(3,2,1,0),(3,2,0),(3,1),(1)\rangle \\ y &= |\,(3,2,1,0),(3,1,1),(3,1),(1)\rangle \end{aligned}$$

It is clear that x and y are eigenvectors for T_1^2 and T_2^2, and moreover belong to the same T_1^2 and T_2^2-eigenspaces. Now consider

$$z = F_3 \mid (3,2,1,0),(3,2,1),(3,1),(1)\rangle = a_{3,3}(x)x + a_{2,3}(y)y\,.$$

Then z is an eigenvector for T_3^2 and, by above, also for T_1^2 and T_2^2. Hence $\langle z \rangle \subset V(\mu_4)_\nu$ is a $\mathcal{B}_3^{(2)}$-submodule of $V(\mu_4)_\nu$. □

Given $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbf{Z}^n$, we let $\nu_{\langle i \rangle} = (\nu_1, \nu_2, \ldots, \nu_i)$. Then the arguments of the previous chapter indicate the following.

Conjecture 3.2.2. *Let $\mu_{n+1} \in \Phi_{n+1}$ and $\nu \in \mathbf{Z}^n$. Assume that for all $i \in [1, n+1]$, all $\mu_i \prec \mu_{n+1}$, and all $\mu_{i-1}, \mu_{i-1}' \prec \mu_i$ such that $\|\mu_{i-1}\| = \|\mu_{i-1}'\|$, we have $V(\mu_{i-1})_{\nu_{\langle i-2 \rangle}} \cong V(\mu_{i-1}')_{\nu_{\langle i-2 \rangle}}$ as $\mathcal{B}_{i-2}^{(2)}$-modules if and only if $\mu_{i-1} = \mu_{i-1}'$. Then $V(\mu_{n+1})_\nu$ is an irreducible $\mathcal{B}_n^{(2)}$-module.* □

Remark 3.2.3. (1) It would be interesting to determine the precise conditions on the pair (μ_{n+1}, ν) for which $V(\mu_{n+1})_\nu$ is irreducible under the $\mathcal{B}_n^{(2)}$-action.

(2) Given $\mu_{n+1} \in \Phi_{n+1}$, and $\nu \in \mathbf{Z}^n$, the sum $\oplus_{w \in S_{n+1}} V(\mu_{n+1})_{w\nu}$ is stable under the $\mathcal{B}_n$-action. Another interesting question is: For what weights ν is $\oplus_{w \in S_{n+1}} V(\mu_{n+1})_{w\nu}$ an irreducible $\mathcal{B}_n$-module?

4. Explicit Computations in the Case $n = 2$

In the next few sections we examine the case of $\mathbf{U}(\mathfrak{sl}_3)$ in detail. In particular we give another proof of the irreducibility of the $\mathcal{B}_2$-action, on the zero weight space, by computing the matrix coefficients of T_1 and T_2 on the zero weight space explicitly. For this we use a different set of basis – namely Lusztig's canonical basis (see **[L1]** or **[L3]**) – for the zero weight space which facilitates simpler computation.

4.1. Lusztig's Canonical Basis for $\mathbf{U}(\mathfrak{sl}_3)$-modules. We begin by making a change in the notation. Write $\mathbf{U} = \mathbf{U}(\mathfrak{sl}_3)$, and for each pair $(n_1, n_2) \in \mathbb{N}^2$, denote by $L(n_1, n_2)$ the corresponding irreducible $\mathbf{U}$-module of highest weight (n_1, n_2). By symmetry we may assume, for our purposes, that $n_1 \leq n_2$.

An explicit formula for the canonical basis is not known for quantized enveloping algebras other than $\mathbf{U}(\mathfrak{sl}_2)$ and $\mathbf{U}(\mathfrak{sl}_3)$. In these two simplest cases the canonical basis for the 'minus part' (and the 'positive part') was computed by Lusztig (see **[L1]**). In the case of $\mathbf{U} = \mathbf{U}(\mathfrak{sl}_3)$, the canonical basis for $\mathbf{U}^-$ is given by the set $\{F_1^{(a)}F_2^{(b)}F_1^{(c)},\ F_2^{(a)}F_1^{(b)}F_2^{(c)} \mid a + c \leq b\}$.

Let $L = L(n_1, n_2)$ be as above, L_0 be the zero weight space of L, and let $\xi \in L$ be the highest weight vector. Then $L_0 \neq 0$ if and only if $3 \mid (n_2 - n_1)$, and if we let $N = (2n_1 + n_2)/3$, then the canonical basis of L_0 is given by the set $\{F_1^{(N-i)}F_2^{(2N-n_1)}F_1^{(i)}\xi \mid 0 \leq i \leq n_1\}$.

4.2. The Braid Group Action. In section 1.3, we gave a definition of the braid group action, due to Lusztig, in terms of an infinite sum. That definition is not convenient for the purposes of our computations in this section, and we use instead an alternative (but equivalent) definition of the braid group action defined by Lusztig in [**L3**, 5.2.4, 5.2.6, 37.1]. The action of $\mathcal{B} = \mathcal{B}_2$ on the generators of $\mathbf{U}$ are given by:

$$T_iE_j = -E_iE_j + v^{-1}E_jE_i, \qquad T_iE_i = -F_iK_i, \qquad T_iK_j = K_iK_j,$$
$$T_iF_j = -F_jF_i + vF_iF_j, \qquad T_iF_i = -K_i^{-1}E_i, \qquad T_iK_i = K_i^{-1},$$

where $i, j \in \{1, 2\}$ and $i \neq j$, and by proposition 1.3.2 the action of $\mathcal{B}$ on the highest weight vector $\xi \in L$ is given by

$$T_i(\xi) = v^{n_i}F_i^{(n_i)}\xi\,.$$

The action of $\mathcal{B}$ on $L(n_1, n_2)$ is then defined by

$$T_i(u\xi) = T_i(u)T_i(\xi),$$

where $u \in \mathbf{U}$. Using the above formulae, we can compute the action of $\mathcal{B}$ on the canonical basis of L_0.

Lemma 4.2.1. *The following formulae are valid:*

$$T_i(F_i^{(c)}\xi) = (-1)^c v^{(n_i-c)(c+1)}F_i^{(n_i-c)}\xi\,, \qquad T_i(F_j^{(c)}\xi) = v^{n_i+c}F_i^{(n_i+c)}F_j^{(c)}\xi\,,$$
$$T_i(F_i^{(b)}F_j^{(c)}\xi) = (-1)^b v^{(n_i-b+c)(b+1)}F_i^{(n_i-b+c)}F_j^{(c)}\xi\,,$$
$$T_i(F_j^{(b)}F_i^{(c)}\xi) = (-1)^{b+c}v^{(n_i-c)(c+1)}\sum_{0\leq j\leq b}\ \sum_{k+l=n_i-c}(-1)^j v^j \xi_{ijk}F_j^{(\alpha-j-k)}F_i^{(\alpha)}F_j^{(j-l)}\xi\,,$$
$$T_i(F_j^{(a)}F_i^{(b)}F_j^{(c)}\xi) = (-1)^{a+b}v^{\alpha(b-a+1)}\sum_{0\leq j\leq a}\ \sum_{k+l=\alpha}(-1)^j v^j \gamma_{jkl}F_1^{(a-j-k)}F_2^{(a+\alpha)}F_1^{(\alpha+c+j-l)}\xi\,,$$

where $\xi_{ijk} = \begin{bmatrix}\alpha - j\\ n_i - c\end{bmatrix}\begin{bmatrix}n_i - c\\ k\end{bmatrix}$, $\gamma_{jkl} = \begin{bmatrix}n_i - b + c - j\\ j\end{bmatrix}\begin{bmatrix}n_i - b + c\\ k\end{bmatrix}\begin{bmatrix}n_i - b + 2c + j - l\\ c\end{bmatrix}$, *and* $\alpha = n_i - b + c$.

Proof: These follow from the definition of the braid group action given above. The first can be obtained directly from propostion 1.3.2 by considering the irreducible $\mathbf{U}(\mathfrak{sl}_2)$-submodules with

ξ as the highest weight vector, and the next two can be obtained by considering the irreducible $U(\mathfrak{sl}_2)$-submodules with the highest weight vector $F_j^{(c)}\xi$. The remaining two formulae follow from simple computations which are omitted. □

4.3. Matrix Coefficients. We now compute the matrix coefficients $(T_2)_{ij}$ of the braid group generator on L_0 with respect to the canonical basis

$$\{F_1^{(N)}F_2^{(2N-n_1)}\xi,\ F_i^{(N-1)}F_2^{(2N-n_1)}F_1\xi,\dots,F_1^{(N-n_1)}F_2^{(2N-n_1)}F_1^{(n_1)}\xi\}.$$

Proposition 4.3.1. *The matrix coefficients $(T_2)_{ij}$ of T_2 on the canonical basis are given by*

$$(T_2)_{i,j} = (-1)^{n_1+n_2+i}v^{(N-n_1+i)(N-n_1+j+1)}\begin{bmatrix} i \\ j \end{bmatrix},$$

where $0 \le i,j \le n_1$.

Proof: Consider the last equation of lemma 4.2.1 and fix $i = j + k$, whence $l = \alpha - i + j$. Let C_i be the coefficient of $F_1^{(a-i)}F_2^{(a+\alpha)}F_1^{(c+i)}\xi$. Then we have

$$\begin{aligned}
C_i &= (-1)^{a+b}v^{\alpha(b-a+1)}\sum_{0\le j\le i}(-1)^j v^j \gamma_{j,i-j,\alpha-i+j}\\
&= (-1)^{a+b}v^{\alpha(b-a+1)}\sum_{0\le j\le i}(-1)^j v^j \begin{bmatrix} \alpha+j \\ \alpha \end{bmatrix}\begin{bmatrix} \alpha \\ i-j \end{bmatrix}\begin{bmatrix} c+i \\ c \end{bmatrix}\\
&= (-1)^{a+b}v^{\alpha(b-a+1)}\begin{bmatrix} c+i \\ c \end{bmatrix}\sum_{0\le j\le i} v^j \begin{bmatrix} -\alpha-1 \\ j \end{bmatrix}\begin{bmatrix} \alpha \\ i-j \end{bmatrix}\\
&= (-1)^{a+b}v^{\alpha(b-a+1)+i(\alpha+1)}\begin{bmatrix} c+i \\ c \end{bmatrix}\sum_{0\le j\le i} v^{(i-j)(-\alpha-1)-j\alpha}\begin{bmatrix} -\alpha-1 \\ j \end{bmatrix}\begin{bmatrix} \alpha \\ i-j \end{bmatrix}\\
&= (-1)^{a+b}v^{\alpha(b-a+1)+i(\alpha+1)}\begin{bmatrix} c+i \\ c \end{bmatrix}\begin{bmatrix} -1 \\ i \end{bmatrix} \qquad \text{by lemma 1.4.1}\\
&= (-1)^{a+b+i}v^{\alpha(b-a+1)+i(\alpha+1)}\begin{bmatrix} c+i \\ c \end{bmatrix}.
\end{aligned}$$

Hence we have that the coefficient of $F_1^{(N-j-k)}F_2^{(2N-n_1)}F_1^{(i)}\xi$ in $T_2(F_1^{(N-j)}F_2^{(2N-n_1)}F_1^{(j)}\xi)$ is

$$\begin{aligned}
(T_2)_{i,j} &= (-1)^{3N-n_1+i}v^{(n_1+n_2+j-2N)(N-n_1+j+1)+(i-j)(j+n_1+n_2-N+1)}\begin{bmatrix} i \\ j \end{bmatrix}\\
&= (-1)^{n_1+n_2+i}v^{(N-n_1+j)(N-n_1+j+1)+(i-j)(N-n_1+j+1)}\begin{bmatrix} i \\ j \end{bmatrix}\\
&= (-1)^{n_1+n_2+i}v^{(N-n_1+i)(N-n_1+j+1)}\begin{bmatrix} i \\ j \end{bmatrix}.
\end{aligned}$$

This completes the proof. □

Thus, if we let $r = N - n_1$, then the matrix for T_2 is the following:

$$(-1)^{n_1+n_2}\begin{pmatrix}
v^{r(r+1)} & 0 & \cdots & 0\\
-v^{(r+1)(r+1)} & -v^{(r+1)(r+2)} & \cdots & 0\\
v^{(r+2)(r+1)} & v^{(r+2)(r+2)}\begin{bmatrix} 2 \\ 1 \end{bmatrix} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
(-1)^{n_1}v^{(r+n_1)(r+1)} & (-1)^{n_1}v^{(r+n_1)(r+2)}\begin{bmatrix} n_1 \\ 1 \end{bmatrix} & \cdots & (-1)^{n_1}v^{(r+n_1)(r+n_1+1)}\begin{bmatrix} n_1 \\ n_1 \end{bmatrix}
\end{pmatrix}.$$

The computation required to directly determine the matrix coefficients of T_1 is more complicated, and so we adopt another approach. We begin by recalling some definitions and results from [**L2**]. Let $i \mapsto \hat{\imath}$ be the permutation of $\{1,2\}$ defined by $w_0 s_i w_0^{-1} = s_{\hat{\imath}}$, where w_0 is the longest element of W. It is clear that $\hat{1} = 2$ and $\hat{2} = 1$. (The definition of $\hat{\imath}$ was phrased in a way to indicate Lusztig's general treatment of what follows.) Then there is a $\mathbf{C}(v)$-algebra automorphism $\Lambda: \mathbf{U} \to \mathbf{U}$ such that $E_i \mapsto F_{\hat{\imath}}, F_i \mapsto E_{\hat{\imath}}, K_i \mapsto K_{\hat{\imath}}^{-1}$. It induces a $\mathbf{U}$-module automorphism $\Lambda: L(n_1, n_2) \to L(n_1, n_2)$ given by $\Lambda(\xi) = \eta$, where η is the element of the canonical basis in the lowest weight space. There is another $\mathbf{C}(v)$-algebra automorphism $P: \mathbf{U} \to \mathbf{U}$ defined by $P(E_i) = -E_i K_i, P(F_i) = -K_i^{-1} F_i, P(K_i) = K_i$, which induces a $\mathbf{U}$-module automorphism P of $L(n_1, n_2)$ given by $P(\xi) = \xi$.

Proposition 4.3.2 ([L2, 5.5]). *We have $T_{w_0} = P \circ \Lambda$ as automorphisms of $L(n_1, n_2)$.* □

By [**L2**, 5.9] we know that the action of T_{w_0} with respect to the canonical basis is given by a monomial matrix, and that on any given weight space, T_{w_0} is a scalar times a permutation matrix. We can explicitly determine this permutation matrix for the zero weight space.

Lemma 4.3.3. *Let $\dim(L_0) = n$. Then the matrix of T_{w_0} on L_0 is given by $\pm v^d \sigma$, where $d \in \mathbf{Z}$ and σ is the matrix with $\sigma_{i,j} = \delta_{i,n-j+1}$.*

Proof: The lowest weight vector of L is given by $\eta = F_1^{(n_2)} F_2^{(n_1+n_2)} F_1^{(n_1)} \xi$, and by proposition 4.3.2 we have $T_{w_0} = P \circ \Lambda$. So relative to the canonical basis of L we have, by repeated application of lemma 1.4.2,

$$\begin{aligned}
T_{w_0}(F_1^{(i)} F_2^{(2N-n_1)} F_1^{(N-i)} \xi) &= P(E_2^{(i)} E_1^{(2N-n_1)} E_2^{(N-i)} F_1^{(n_2)} F_2^{(n_1+n_2)} F_1^{(n_1)} \xi) \\
&= \pm v^d E_2^{(i)} E_1^{(2N-n_1)} F_1^{(n_2)} F_2^{(n_1+n_2-N+i)} \begin{bmatrix} K_2; N-i-n_1-n_2 \\ N-i \end{bmatrix} F_1^{(n_1)} \xi \\
&= \pm v^d E_2^{(i)} \sum_{0 \le k \le 2N-n_1} F_1^{(n_2-k)} \begin{bmatrix} K_1; 2k+n_1-n_2-2N \\ k \end{bmatrix} E_1^{(2N-n_1-k)} \\
&\quad \times F_2^{(n_1+n_2-N+i)} \begin{bmatrix} K_2; N-i-n_1-n_2 \\ N-i \end{bmatrix} F_1^{(n_1)} \xi \\
&= \pm v^d \sum_{0 \le k \le 2N-n_1} F_1^{(n_2-k)} \begin{bmatrix} K_1; 2k+n_1-n_2-2N+i \\ k \end{bmatrix} E_2^{(i)} F_2^{(n_1+n_2-N+i)} \\
&\quad \times \begin{bmatrix} K_2; 3N-i-2n_1-n_2-k \\ N-i \end{bmatrix} E_1^{(2N-n_1-k)} F_1^{(n_1)} \xi \\
&= \pm v^d \sum_{2N-2n_1 \le k \le 2N-n_1} F_1^{(n_2-k)} \begin{bmatrix} K_1; 2k+n_1-n_2-2N+i \\ k \end{bmatrix} F_2^{(n_1+n_2-N)} \\
&\quad \times \begin{bmatrix} K_2; N-n_1-n_2 \\ i \end{bmatrix} \sum_{0 \le u \le 2N-n_1-k} F_1^{(n_1-u)} \begin{bmatrix} K_1; 2u-2N+k \\ u \end{bmatrix} E_1^{(2N-n_1-k-u)} \xi \\
&= \pm v^d \sum_{2N-2n_1 \le k \le 2N-n_1} F_1^{(n_2-k)} F_2^{(2N-n_1)} F_1^{(2n_1-2N+k)} \begin{bmatrix} K_1; 4N-4n_1-n_2+i \\ k \end{bmatrix} \\
&\quad \times \begin{bmatrix} K_2; n_1-N-n_2+k \\ i \end{bmatrix} \begin{bmatrix} K_1; 2N-2n_1-k \\ 2N-n_1-k \end{bmatrix} \xi \\
&= \pm v^d \sum_{2N-2n_1 \le k \le 2N-n_1} F_1^{(n_2-k)} F_2^{(2N-n_1)} F_1^{(2n_1-2N+k)} \\
&\quad \times \begin{bmatrix} 4N-3n_1-n_2+i \\ k \end{bmatrix} \begin{bmatrix} n_1+k-N \\ i \end{bmatrix} \xi.
\end{aligned}$$

Since $2(N-n_1) \le k \le 2N-n_1$, $N-n_1 \le i \le N$, and $\begin{bmatrix} p \\ q \end{bmatrix} = 0$ for $0 \le p < q$, we have that the only nonzero term in the sum is the one for which $k = i + (n_2 - n_1)/3 = i + N - n_1$. Hence

$$T_{w_0}(F_1^{(i)} F_2^{(2N-n_1)} F_1^{(N-i)} \xi) = \pm v^d F_1^{(2N-n_1-i)} F_2^{(2N-n_1)} F_1^{(n_1-(N-i))} \xi ,$$

and the lemma is proved. □

The matrix coefficients of T_1 can now be computed.

Corollary 4.3.4. *We have* $T_1 = \sigma T_2 \sigma$.

Proof: From the above lemma $T_{w_0} = \pm v^d \sigma$, and so using $T_1T_2T_1 = T_{w_0} = T_2T_1T_2$, we have $T_1 = T_1(T_2T_1T_2)(T_1T_2T_1)^{-1} = (T_1T_2T_1)T_2(T_1T_2T_1)^{-1} = T_{w_0}T_2T_{w_0}^{-1} = \sigma T_2 \sigma$. □

4.4. Eigenvectors of T_i. In this section we calculate, explicitly, the eigenvectors of T_1 and T_2 on L_0. We begin with an identity which will play an important role in the determination of the eigenvectors.

Lemma 4.4.1. *For each pair of non-negative integers a and b, the following identity holds*

$$v^{-a(b+1)} \frac{[a+b]!}{[a+2b+1]!} = \sum_{k=0}^{a} (-1)^k v^{k(a+b)} \begin{bmatrix} a \\ k \end{bmatrix} \frac{[b+k]!}{[2b+k+1]!}.$$

Proof: The above identity is equivalent to

$$v^{-a(b+1)} \begin{bmatrix} a+b \\ b \end{bmatrix} = \sum_{k=0}^{a} v^{k(a+b)} \frac{1}{[k]![a-k]!} \prod_{l=1}^{k} [b-l] \prod_{l=k+1}^{a} [2b+l+1].$$

But

$$rhs = \sum_{k=0}^{a} (-1)^k v^{k(a+b)} \begin{bmatrix} b+k \\ k \end{bmatrix} \begin{bmatrix} a+2b+1 \\ a-k \end{bmatrix} = \sum_{k=0}^{a} v^{k(a+b)} \begin{bmatrix} -b-1 \\ k \end{bmatrix} \begin{bmatrix} a+2b+1 \\ a-k \end{bmatrix}$$

$$= v^{-a(b+1)} \sum_{k=0}^{a} (-1)^k v^{k(a+b)} \begin{bmatrix} -b-1 \\ k \end{bmatrix} \begin{bmatrix} a+2b+1 \\ a-k \end{bmatrix} = v^{-a(b+1)} \begin{bmatrix} a+b \\ b \end{bmatrix},$$

where the last equality follows by lemma 1.4.1, and the lemma is proved. □

Since the matrix for T_2 was lower triangular, by considering the diagonal entries, we have from the previous section that the eigenvalues, λ_i, of T_2 are given by

$$\lambda_i = (-1)^{n_1+n_2+i} v^{(N-i)(N-i+1)} .$$

Proposition 4.4.2. *The eigenvector $f_i = (f_i^0, f_i^1 \dots, f_i^{n_1})^t$ of T_2 corresponding to the eigenvalue λ_i is given by*

$$f_i^j = (-1)^{i+j-n_1} \begin{bmatrix} j \\ n_1 - i \end{bmatrix} \frac{[N-n_1+j]!}{[N-i]!} \frac{[2N-i+1]!}{[2N-n_1-i+j+1]!},$$

where $0 \le i \le n_1$.

Proof: We need to show that $\lambda_i f_i^j = \sum_{0 \le k \le i+j-n_1} (T_2)_{j,n_1-i+k} f_i^{n_1-i+k}$. On substituting the values of $(T_2)_{i,j}$ and f_i^j and simplifying we obtain

$$rhs = (-1)^{n_1+n_2+j} v^{(N-n_1+j)(N-n_1+i+1)} \begin{bmatrix} j \\ n_1 - i \end{bmatrix} \frac{[2N-i+1]!}{[N-i]!}$$

$$\sum_{0 \le k \le i+j-n_1} (-1)^k v^{k(N-n_1+j)} \begin{bmatrix} i+j-n_1 \\ k \end{bmatrix} \frac{[N-i+k]!}{[2N-2i+k+1]!}$$

After a similar substitution for the *lhs* and cancelling, the equation that we need to verify becomes

$$v^{(n_1-i-j)(N-i+1)}\frac{[N-n_1+j]!}{[2N-n_1+j+1]!}=\sum_{k=0}^{i+j-n_1}(-1)^k v^{k(N-n_1+j)}\begin{bmatrix} i+j-n_1 \\ k \end{bmatrix}\frac{[N-i+k]!}{[2N-2i+k+1]!}.$$

But letting $a = i + j - n_1$ and $b = N - i$, this is precisely the equality proved in lemma 4.4.1 above. □

Since $T_1 = \sigma T_2 \sigma$, the eigenvectors of T_1 are given as follows.

Corollary 4.4.3. *The eigenvectors e_i of T_1 are given by $e_i = \sigma f_i$.* □

4.5. Irreducibility of the Braid Group Action. Given $f/g \in \mathbf{C}(v)$, with $f, g \in \mathbf{C}[v]$, define $\deg(f/g)$ by $\deg(f/g) = \deg(f) - \deg(g)$, where $\deg(f)$, for $f \in \mathbf{C}[v]$, is the usual polynomial degree. Then we note from the previous sections the following:

(1) In the matrix T_1, we have $\deg((T_1)_{i,j}) > \deg((T_1)_{i+k,j+l})$.
(2) In the matrix T_2, we have $\deg((T_2)_{i,j}) < \deg((T_2)_{i+k,j+l})$.
(3) For the T_1-eigenvectors e_i, we have $\deg(e_i^j) < \deg(e_i^{j+k})$.
(4) For the T_2-eigenvectors f_i, we have $\deg(f_i^j) > \deg(f_i^{j+k})$.

Proposition 4.5.1. *$\mathcal{B}$ acts irreducibly on L_0.*

Proof: Let $M \subset L_0$ be a $\mathcal{B}$-submodule. Then since T_1 has distinct eigenvalues on L_0 (see above), M contains a T_1-eigenvector e_i for some i. By above computations, we have $e_i^0 \neq 0$, and so writing e_i as a linear combination of the f_i's we see that f_{n_1} appears with a nonzero coefficient. It follows that $f_{n_1} \in M$.
Let $d_i = \deg((T_1)_{i,i})$ and $\delta_i = \deg(f_{n_1}^i)$. Then $\deg((T_1^i f_{n_1})^j) = \delta_j + i d_j$, and by an argument analogous to that used to prove the nonvanishing of the van der Monde determinant, we have $T_1^i f_{n_1}$, for $0 \le i \le n_1$, are linearly independent and belong to M. Since $\dim(L_0) = n_1 + 1$, we have $M = L_0$, and the proposition is proved. □

4.6. Some Questions at Roots of Unity. It is known that for $v = \pm 1$ the braid group action factors through the (finite) group $(\mathbf{Z}/2)^r \times W$ for some $r \in \mathbf{N}$, where W is the corresponding Weyl group. In this section we show that for $v = \zeta$, a root of unity, the braid group action factors through a finite group if and only if $\zeta = \pm 1$.

From the explicit description of the action of T_1 and T_2 we have the following result.

Lemma 4.6.1. *Let ζ be a root of unity.*

(1) *If $\zeta^{4l} = 1$ then*

$$T_1 = \begin{pmatrix} \zeta^l & 1 \\ 0 & \zeta^l \end{pmatrix}, \qquad T_2 = \begin{pmatrix} \zeta^l & 0 \\ 1 & \zeta^l \end{pmatrix}$$

is a representation of $\mathcal{B}$.

(2) *If $\zeta^{2l+1} = 1$ then*

$$T_1 = \begin{pmatrix} \zeta & [2]_\zeta \zeta^{l+1} & 1 \\ 0 & -1 & -\zeta^{l+1} \\ 0 & 0 & \zeta \end{pmatrix}, \qquad T_2 = \begin{pmatrix} \zeta & 0 & 0 \\ -\zeta^{l+1} & -1 & 0 \\ 1 & [2]_\zeta \zeta^{l+1} & \zeta \end{pmatrix}$$

is a representation of $\mathcal{B}$.

(3) *If $\zeta^{2(2l+1)} = 1$ then*

$$T_1 = \begin{pmatrix} \zeta^2 & [2]_{\zeta^2} \zeta^{2(l+1)} & 1 \\ 0 & -1 & -\zeta^{2(l+1)} \\ 0 & 0 & \zeta^2 \end{pmatrix} \qquad T_2 = \begin{pmatrix} \zeta^2 & 0 & 0 \\ -\zeta^{2(l+1)} & -1 & 0 \\ 1 & [2]_{\zeta^2} \zeta^{2(l+1)} & \zeta^2 \end{pmatrix}$$

is a representation of $\mathcal{B}$.

Proof: We just need to check the braid relations which, in this case, are very simple. □

Lemma 4.6.2. *For the representations of $\mathcal{B}$ given in lemma 4.6.1, the following holds.*

(1) *For the case $\zeta^{4l} = 1$, we have $(T_1^n)_{12} = n\zeta^{(n-1)l}$.*
(2) *For the case $\zeta^{2l+1} = 1$, we have $(T_1^n)_{13} = \{n\zeta^n(1-\zeta^2) + (\zeta^n - (-1)^n)(1+\zeta^2)\}/(1+\zeta)^2$.*
(3) *For the case $\zeta^{2(2l+1)} = 1$, we have $(T_1^n)_{13} = \{n\zeta^{2n}(1-\zeta^4) + (\zeta^{2n} - (-1)^n)(1+\zeta^4)\}/(1+\zeta^2)^2$.*

Proof: (1) The result is clearly true for $n = 1$, and it is clear that $(T_1^n)_{11} = \zeta^{nl} = (T_1^n)_{22}$. So assuming $(T_1^n)_{12} = n\zeta^{(n-1)l}$, we have that $(T_1^{n+1})_{12} = (T_1)_{11}(T_1^n)_{12} + (T_1)_{12}(T_1^n)_{22} = n\zeta^{nl} + \zeta^{nl} = (n+1)\zeta^{nl}$, and the result follows by induction.

(2) We again use induction. A similar, albeit a little messier, computation gives $(T_1^n)_{11} = \zeta^n$ and $(T_1^n)_{12} = (-1)^n\{1 + (-1)^{n+1}\zeta^n\}/(1+\zeta)$, and the result follows from this.
(3) This case can be obtained from the previous one by replacing ζ with ζ^2 everywhere in the proof. □

We are now ready to prove the result alluded to at the beginning of this section. We can prove the following stronger statement.

Proposition 4.6.3. *Let $\zeta \neq \pm 1$ be a root of unity. Then there exists a finite dimensional representation of $\mathcal{B}$ in which the generators T_i have infinite order.*

Proof: (1) Suppose $\zeta^{4l} = 1$. Then in the above representation of $\mathcal{B}$, we have, by the preceding lemma, that $(T_1^n)_{1,2} = n\zeta^{(n-1)l} \neq 0$ for all n. Hence T_1 has infinite order as required.

(2) Suppose $\zeta^{2l+1} = 1$. Then in the above representation of $\mathcal{B}$, we have, by the preceding lemma, that $(T_1^n)_{1,3} = \{n\zeta^n(1-\zeta^2) + (\zeta^n - (-1)^n)(1+\zeta^2)\}/(1+\zeta)^2 \neq 0$ for n large since $1 - \zeta^2 \neq 0$. Hence T_1 has infinite order.
(3) Suppose $\zeta^{2(2l+1)} = 1$. Then the same argument as the previous case with ζ replaced by ζ^2 shows that T_1 again has infinite order.

This completes the proof. □

Corollary 4.6.4. *Let $\zeta^l = 1$. Then the braid group action at $v = \zeta$ factors through a finite group if and only if $\zeta = \pm 1$.*

Proof: This follows immediately from the above proposition since the T_i would have finite order if the action of $\mathcal{B}$ factored through a finite group. □

Remark 4.6.5. The representations of $\mathcal{B}$ given above are obtained by choosing appropriate values of n_1 and n_2 in the explicit formulae for the matrices of T_i computed in the previous sections, and then specializing $v \mapsto \zeta$. Before the specialization, the T_i's were semisimple as they had distinct eigenvalues. However, they do not remain semisimple after the specialization, since otherwise they would have finite order, contrary to proposition 4.6.4.

4.7. A digression. In the previous sections, we have found many simple $\mathcal{B}$-modules, and even found explicit matrices for the generators of the braid group on these modules. It would be interesting to determine to what extent the simple $\mathcal{B}$-modules that arise as the zero weight spaces of simple **U**-modules exhaust the set of all simple $\mathcal{B}$-modules. It is clear that such modules exhaust all 1-dimensional $\mathcal{B}$-modules. In this section we consider this question for the 2-dimensional $\mathcal{B}$-modules, and leave the remaining cases for future investigation.

Proposition 4.7.1. *The 2-dimensional irreducible $\mathcal{B}$-modules over $\mathbb{C}$ occur in two types. The action of the braid group generators on these modules are given by,*

$$(i) \quad T_1 = \begin{pmatrix} a & \pm\sqrt{-ac} \\ 0 & c \end{pmatrix}, \qquad T_2 = \begin{pmatrix} c & 0 \\ \pm\sqrt{-ac} & a \end{pmatrix},$$

$$(ii) \quad T_1 = \begin{pmatrix} a & \pm\sqrt{ac - a^2 - c^2} \\ 0 & c \end{pmatrix}, \qquad T_2 = \begin{pmatrix} a & 0 \\ \pm\sqrt{ac - a^2 - c^2} & c \end{pmatrix},$$

where $a, c \in \mathbb{C}^\times$.

Proof: Let $V = \mathbb{C}^2$ and suppose $\mathcal{B}$ acts irreducibly on V. Then without loss of generality we may choose $\{v, w\}$ as a basis of V, where v is an eigenvector of T_1 and w is an eigenvector of T_2. This allows us to assume T_1 to be upper triangular and T_2 to be lower triangular. Note that the irreducibilty implies v and w are independent. Since $T_1T_2T_1 = T_2T_1T_2$, we have $T_2 = (T_1T_2)T_1(T_1T_2)^{-1}$ and that T_1 and T_2 are conjugate. Hence they share the same set of eigenvalues. Suitable rescaling of v and w allows us to write

$$T_1 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad T_2 = \begin{pmatrix} c & 0 \\ b & a \end{pmatrix},$$

if T_1 and T_2 have their eigenvalues appearing along the diagonal in the opposite order, or

$$T_1 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad T_2 = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

if the eigenvalues occur in the same order. In both cases, the braid relation and the requirement of irreducibilty force b to have the form given in the statement of the proposition. □

Hence the 2-dimensional $\mathcal{B}$-modules are parametrized, essentially, by $S_2 \times (\mathbb{C}^\times \times \mathbb{C}^\times)$. The 2-dimensional $\mathcal{B}$-modules that arise as the zero weight spaces of $\mathbf{U}$-modules have

$$T_1 = \pm \begin{pmatrix} v^{r(r+1)} & 0 \\ -v^{(r+1)(r+1)} & -v^{(r+1)(r+2)} \end{pmatrix}, \qquad T_2 = \pm \begin{pmatrix} -v^{(r+1)(r+2)} & -v^{(r+1)(r+1)} \\ 0 & v^{r(r+1)} \end{pmatrix},$$

and these belong to the type (i) representations given above. They consist of type (i) representations

$$T_1 = \begin{pmatrix} a & \pm\sqrt{-ac} \\ 0 & c \end{pmatrix}, \qquad T_2 = \begin{pmatrix} c & 0 \\ \pm\sqrt{-ac} & a \end{pmatrix}$$

which have $a^{r+2} + (-1)^{r+1}c^r = 0$ for some $r \in \mathbb{N}$. Hence the simple 2-dimensional $\mathcal{B}$-modules that arise as the zero weight spaces of $\mathbf{U}$-modules account for

$$\cup_{r\in\mathbb{N}}\{(x_1, x_2) \in \mathbb{C}^\times \times \mathbb{C}^\times \mid x_1^{r+2} + (-1)^{r+1}x_2^r = 0\} \subset \mathbb{C}^\times \times \mathbb{C}^\times$$

of the type (i) representations of $\mathcal{B}$.

5. Braid Group Action as Hecke Algebra Action

Let $V = \mathbb{C}(v)^{n+1}$ be the module affording the standard representation of $\mathbf{U} = \mathbf{U}(\mathfrak{sl}_{n+1})$ and let $\mathcal{L} = V^{\otimes n+1}$. In this section, we restrict our attention to the irreducible $\mathbf{U}$-modules appearing in $\mathcal{L}_0$, and show that the action of $\mathcal{B} = \mathcal{B}_n$ on $\mathcal{L}_0$ reduces to the Hecke algebra action. In particular, this will lead to a proof of the irreducibility of the $\mathcal{B}$-action on the the zero weight spaces of those modules.

Let $\{e_1, e_2, \ldots, e_{n+1}\}$ be the standard basis of V. Then the zero weight space, $\mathcal{L}_0$, of $\mathcal{L}$ has the basis $\{e_{\pi 1} \otimes e_{\pi 2} \otimes \cdots \otimes e_{\pi(n+1)} \mid \pi \in S_{n+1}\}$, and E_i, F_i, $K_i^{\pm 1}$ act on $\mathcal{L}$ according to:

$$\begin{aligned} E_i &= \sum_{1 \le k \le n+1} K_i \otimes \cdots \otimes K_i \otimes E_i \otimes 1 \otimes \cdots \otimes 1\,, \\ F_i &= \sum_{1 \le k \le n+1} 1 \otimes \cdots \otimes 1 \otimes F_i \otimes K_i^{-1} \otimes \cdots \otimes K_i^{-1}\,, \quad \text{and} \\ K_i^{\pm 1} &= K_i^{\pm 1} \otimes \cdots \otimes K_i^{\pm 1}\,, \end{aligned}$$

where there are $n+1$ terms in the product, and in the first two formulae E_i and F_i appear in the k-th position (see the definition of comultiplication Δ in Section 1.1). Recall from Section 1.3 that we have the following definition of the braid group generator T_i,

$$T_i = \sum_{a,b,c \ge 0} (-1)^b v^{c^2 - a^2 - ac + ab - bc + a + c} F_i^{(a)} E_i^{(b)} F_i^{(c)} K_i^{(a-c)}.$$

Note that we can disregard the $K_i^{(a-c)}$ terms on the zero weight space.

Now the standard action of $\mathbf{U}$ on V is given by,

$$E_i(e_j) = \delta_{i+1,j} e_i\,, \qquad F_i(e_j) = \delta_{i,j} e_{i+1}\,, \qquad K_i^{\pm 1}(e_j) = \delta_{i,j} v^{\pm 1} e_i + \delta_{i+1,j} v^{\mp 1} e_{i+1}\,.$$

So on the zero weight space with the above basis, the non-zero operators appearing in the expression for T_i are those for which $c \le 1$, $b \le c+1$, $a + c \le b + 1$, since $E_i^{(2)}$ and $F_i^{(2)}$ annihilate the zero weight space. Now the triples (a, b, c) satisfying the above condition can be enumerated. They are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(2,1,0)$, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(0,2,1)$, $(1,2,1)$, and $(2,2,1)$. Since the T_i preserve the zero weight space, we require $b = a + c$, and thus the triples that contribute non-zero terms in the expression for T_i are $(0,0,0)$, $(1,1,0)$, $(0,1,1)$, and $(1,2,1)$. It is immediate from above that in studying the action of T_i on the zero weight space, we can restrict to the case of $\mathbf{U}(\mathfrak{sl}_2)$ action on $V_2 \otimes V_2$, where V_2 is the standard $\mathbf{U}(\mathfrak{sl}_2)$-module. We drop the subscripts from T_i, E_i, F_i, and K_i. We then have the following formulae easily deduced from above.

$$\begin{aligned} &E(e_1 \otimes e_2) = v e_1 \otimes e_1\,, \quad &&E(e_2 \otimes e_1) = e_1 \otimes e_1\,, \quad &&E(e_2 \otimes e_2) = e_1 \otimes e_2 + v^{-1} e_2 \otimes e_1\,, \\ &F(e_1 \otimes e_2) = v e_2 \otimes e_2\,, \quad &&F(e_2 \otimes e_1) = e_2 \otimes e_2\,, \quad &&F(e_1 \otimes e_1) = e_1 \otimes e_2 + v^{-1} e_2 \otimes e_1\,. \end{aligned}$$

Using these we obtain the following for the action of T.

$$T(e_1 \otimes e_2) = (1 - v^2) e_1 \otimes e_2 - v e_2 \otimes e_1\,, \qquad T(e_2 \otimes e_1) = -v e_1 \otimes e_2\,.$$

And thus we have,

$$\begin{aligned} T^2(e_1 \otimes e_2) &= (1 - v^2) T(e_1 \otimes e_2) + v^2 e_1 \otimes e_2 \\ T^2(e_2 \otimes e_1) &= (1 - v^2) T(e_2 \otimes e_1) + v^2 e_2 \otimes e_1. \end{aligned}$$

This demonstrates the required quadratic relation $T^2 = (1 - v^2)T + v^2 1$, and so we have:

Proposition 5.1. *Let L be a simple constituent of $V^{\otimes n+1}$ with $L_0 \neq 0$. Then the action of $\mathcal{B}$ on L_0 is a Hecke algebra action.* □

Corollary 5.2. *With L as above, L_0 is an irreducible $\mathcal{B}$-module.*

Proof: Setting $v = 1$ we obtain the action of the symmetric group on the tensor power of the standard $\mathfrak{sl}_{n+1}$-module. By a classical result on double centralizers (see **[W]**) we have that the action of S_{n+1} on $(V^{\otimes n+1})_0\big|_{v=1}$ corresponds to the S_{n+1}-action on its group algebra, and that $L_0\big|_{v=1}$'s are precisely the irreducible constituents of this regular representation of S_{n+1}. Hence S_{n+1} acts irreducibly on $L_0\big|_{v=1}$, and it follows that $\mathcal{B}_n$ acts irreducibly on L_0. □

Concluding Remarks

Let (a_{ij}) be a Cartan matrix, $\mathcal{B}$ the corresponding braid group, $\mathfrak{g}$ the corresponding Lie algebra, and $\mathbf{U} = \mathbf{U}(\mathfrak{g})$. Then we may once again study the $\mathcal{B}$-action on the zero weight space of the simple $\mathbf{U}$-modules. For types B_n and D_n, Gelfand and Tsetlin gave a parametrization for the bases of simple $\mathfrak{g}$-modules in [**GT2**], and they may prove to be useful in the investigation of the $\mathcal{B}$-action. However unlike in the case of type A_n, the basis vectors are not weight vectors, and the explicit formulae for the action of the Chevalley generators have not been worked out. It would be nice to find a general argument that will prove the irreducibility of the $\mathcal{B}$-action on the zero weight space of the simple $\mathbf{U}$-modules, for all types, if the action is indeed irreducible.

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