A Simple Bisection Algorithm for the $\mathcal{L}^2$ Induced Norm of a Sampled-Data System

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Abstract

An algorithm for computing the $\mathcal{L}^2$ induced norm of a sampled-data system is summarized. The computational details of this algorithm are spelled out, bringing together the recent advances in $\mathcal{H}^\infty$ theory on sampled-data systems with the discrete-time $\mathcal{H}^\infty$ problem. The algorithm is stated in an explicit and rigorous way, and implementation-oriented issues are considered.

1 Introduction

The theory of designing $\mathcal{H}^\infty$ optimal controllers[10] has undergone a significant advancement with the development of state-space solutions for the general problem[9]. This theory, however, does not handle the problem of designing digital controllers for continuous-time systems with continuous-time performance specifications, otherwise known as the sampled-data problem. This has been the target of vigorous research by many researchers [2, 3, 5, 6, 17, 18] for the $\mathcal{H}^\infty$ problem, and [1, 19, 22] for the ℓ^1 problem.

In [2], Bamieh and Pearson developed a theory for designing digital controllers for continuous-time systems that minimize the $\mathcal{L}^2$ induced norm of the closed loop system. Their solution was based on a lifting technique by which one converts the periodic closed loop operator to a time-invariant infinite-dimensional one. (Similar ideas appear in [3, 24].) It was shown that the solution of this problem is equivalent to the solution of a pure discrete-time problem.

In this report, the results in [2] are utilized to provide a simple bisection algorithm for computing the $\mathcal{L}^2$ induced norm for a hybrid sampled-data system. This gives a way

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to analyze closed loop systems, with a given controller. Methods for computing the $\mathcal{L}^2$ induced norm have been proposed in [5], in which only an approximate solution is given. Also, an alternate approach is given in [17, 18], however, the results should be supplemented in order to form it as a complete algorithm.

This report is primarily based on [2]. The basic objective is to summarize the steps involved in computing the $\mathcal{H}^\infty$ norm of a sampled-data system, and discuss some of the computational issues involved. In particular, this report brings several pieces of the theory together to present a rigorous algorithm that can be effectively used for computation.

2 Preliminaries

As usual, a signal is modeled as a function (continuous-time case) or a sequence (discrete-time case) and a system is modeled as a linear operator between such spaces.

The space of real numbers is denoted by $\Re$. Symbols $\mathcal{L}^2[0,\infty)$ and $\mathcal{L}^2[0,\tau)$ denote usual time-domain Lebesgue spaces, and $\mathcal{L}_e^2[0,\infty)$ denotes the extended version of $\mathcal{L}^2[0,\infty)$[8]. For simplicity sometimes we omit the dimension of spaces and, for example, write $\mathcal{L}^2[0,\infty)$ instead of $\mathcal{L}^2[0,\infty)^n$ when there is no fear of confusion.

When X is Banach space we define ℓ_X as $\{\{f_i\}|\ f_i \in X \ \text{ for } \ i = 0,1,\ldots\}$ and ℓ_X^2 as a set of all elements of ℓ_X that satisfy $\|\{f_i\}\|_{\ell_X^2} := \{\sum_{i=0}^{\infty}\{\|f_i\|_X\}^2\}^{1/2} < \infty$. Here ℓ_X^2 is actually a Banach space itself with the above norm $\|\{f_i\}\|_{\ell_X^2}$ [7, III.4.4]. In this report we use mostly $\ell_{\Re^n}$ and $\ell_{\mathcal{L}^2[0,\tau)^n}$.

The norms of Banach spaces X, Y induce a norm on a linear operator M from X to Y by $\sup_{x\in X,\|x\|_X=1}\|Mx\|_Y$. When X and Y are vector spaces over the same space, say Z, we simply write the norm as $\|M\|_Z$ regardless of their dimensions. Furthermore sometimes a norm is denoted without a symbol of the space when it is obvious from the context.

A sampled-data system considered in this report is shown in Figure 1. It is made of a linear continuous-time time-invariant (LCTI) generalized plant G, a linear discrete-time time-invariant (LDTI) controller C, an ideal sampler S_τ (with period τ) and a zero-order hold H_τ (with period τ).

Here we assume $G : \mathcal{L}_e^2[0,\infty)^w \oplus \mathcal{L}_e^2[0,\infty)^u \to \mathcal{L}_e^2[0,\infty)^z \oplus \mathcal{L}_e^2[0,\infty)^y$ has the form:

$$G : \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \mapsto \begin{bmatrix} z(t) \\ y(t) \end{bmatrix};$$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t), \end{aligned} \tag{1}$$

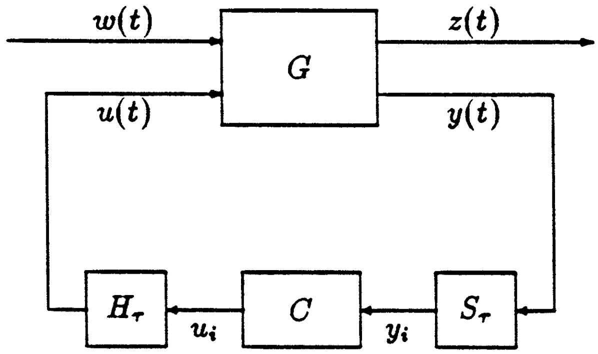

Figure 1: Sampled-data system

and $D_{11}, D_{12}, D_{21}, D_{22}$ are all equal to 0. (The notation $\mathcal{L}_e^2[0,\infty)^w, \mathcal{L}_e^2[0,\infty)^u, \dots$ stands for w being the dimension of the signal $w(t)$ and u the dimension of the signal $u(t)$, etc...) Also we assume $C : \ell_{\Re^y} \to \ell_{\Re^u}$ has the form:

$$
\begin{aligned}
&C : \{y_i\} \mapsto \{u_i\}; \\
x_{c,i+1} &= A_c x_{c,i} + B_c y_i, \\
u_i &= C_c x_{c,i} + D_c y_i.
\end{aligned}
\tag{2}
$$

The matrices $A, B_1, B_2, \dots, A_c, B_c, \dots$ are all real and constant-valued.

Remark: It is assumed that an anti-aliasing filter has been introduced before the sampler and is absorbed in G. This justifies the assumption that $D_{21}, D_{22} = 0$. Note that $y(t)$ is actually a continuous signal because of this and it guarantees validity of operation of S_τ on $y(t)$. See [2] for more detail. The rather strong assumptions, $D_{11}, D_{12} = 0$, are needed to get explicit formulas for the operator compositions in Proposition 3.

For notation convenience, we write the above as

$$
G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right]_c ,
$$

$$
C = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]_d .
$$

Moreover a sampler S_τ : (continuous elements of $\mathcal{L}_e^2[0,\infty)^y$) $\rightarrow \ell_{\Re^y}$ and a holder H_τ: $\ell_{\Re^u} \rightarrow \mathcal{L}_e^2[0,\infty)^u$ are defined as

$$
\begin{aligned}
(S_\tau y(t))_i &:= y(i\tau); && (3)\\
(H_\tau\{u_i\})(t) &:= u_i \ \text{ for } \ i\tau \le t < (i+1)\tau. && (4)
\end{aligned}
$$

Let $\mathcal{S}(P,K)$ denote a closed loop system made of a generalized plant P and a controller K, and let $\mathcal{F}(P,K)$ denote the mapping from the exogenous input to the regulated output of $\mathcal{S}(P,K)$. Then the system of Figure 1 can be described as $\mathcal{S}(G,H_\tau CS_\tau)$ and the mapping from $w(t)$ to $z(t)$ can be described as $\mathcal{F}(G,H_\tau CS_\tau)$.

Finally we define a stability of the sampled-data system. We say the system $\mathcal{S}(G,H_\tau CS_\tau)$ is *internally stable* if there exists positive constants $\alpha,\beta,\alpha_c,\beta_c$ such that with $w(t)=0$ and for every initial condition $(x(0),x_{c,0})$ we have

$$
\begin{aligned}
\|x(t)\| &\le \beta e^{-\alpha t}\|x(0)\| && \text{for all } \ t\ge 0,\\
\|x_{c,i}\| &\le \beta_c e^{-\alpha_c i}\|x_{c,0}\| && \text{for all } \ i\ge 0,
\end{aligned}
$$

where $\|.\|$ is the usual 2-norm for vectors. The next proposition follows from Proposition 3.7 in [18].

Proposition 1 *If the system $\mathcal{S}(G,H_\tau CS_\tau)$ is internally stable then $\|\mathcal{F}(G,H_\tau CS_\tau)\|$ is finite.*

The converse is not true in general. (It holds if some kind of controllability and observability conditions are satisfied[18].) So our goal is to find an algorithm that 1) tests internal stability, and 2) computes the norm of $\mathcal{S}(G,H_\tau CS_\tau)$ if stable.

3 The Lifting Technique

In this section we summarize the lifting technique of Bamieh and Pearson and state their result on the sampled-data system of Figure 1. See [2] for details.

First a lifting operator $W_\tau : \mathcal{L}_e^2[0,\infty) \rightarrow \ell_{\mathcal{L}^2[0,\tau)}$ is defined as

$$
W_\tau : f(t) \mapsto \{\hat{f}_i(t)\},
$$

$$
\hat{f}_i(t) := f(i\tau + t) \quad \text{for} \ \ 0 \le t < \tau.
$$

Lifting implies that the continuous-time signal is broken into a sequence of signals supported on an interval of length τ (see Figure 2). Clearly, the lifting operator is linear and bijective. The restriction of W_τ on $\mathcal{L}^2[0,\infty)$ ranges over $\ell^2_{\mathcal{L}^2[0,\tau)}$ and in fact it is an isomorphism (norm preserving).

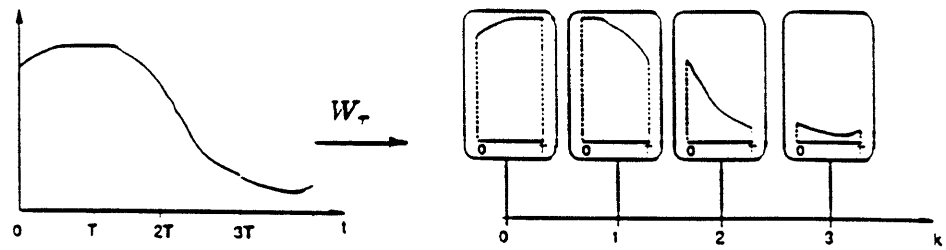

Figure 2: Lifting operator W_τ (from [1])

Next a lifting of a LCTI system $M : \mathcal{L}_e^2[0,\infty) \to \mathcal{L}_e^2[0,\infty)$ is defined as $W_\tau M W_\tau^{-1}$. From the isomorphic property, if $\|M\|_{\mathcal{L}^2[0,\infty)} < \infty$ then

$$\|M\|_{\mathcal{L}^2[0,\infty)} = \|W_\tau M W_\tau^{-1}\|_{\ell^2_{\mathcal{L}^2[0,\tau)}}.$$

Applying the above to the closed loop sampled-data system, we get $W_\tau \mathcal{F}(G, H_\tau C S_\tau) W_\tau^{-1}$. We can write this as $\mathcal{F}(\tilde{G}, C)$ and the system as $\mathcal{S}(\tilde{G}, C)$ by defining $\tilde{G}$ as

$$\tilde{G} = \begin{bmatrix} W_\tau \\ S_\tau \end{bmatrix} G \begin{bmatrix} W_\tau^{-1} & H_\tau \end{bmatrix}$$

(see Figure 3). The system $\tilde{G}$ has a 'state-space representation' as follows:

$$\begin{aligned} x((i+1)\tau) &= e^{A\tau}x(i\tau) + \int_0^\tau e^{A(\tau-s)}B_1\hat{w}_i(s)ds + \Psi(\tau)B_2u_i, \\ \hat{z}_i(t) &= C_1e^{At}x(i\tau) + \int_0^t C_1e^{A(t-s)}B_1\hat{w}_i(s)ds + C_1\Psi(t)B_2u_i, \\ y_i &= C_2x(i\tau), \end{aligned} \tag{5}$$

where $\hat{w}_i(t) = w(i\tau+t)$, $\hat{z}_i(t) = z(i\tau+t)$ and $\Psi(t) = \int_0^t e^{As}ds$. This is similar to a state-space representation of an ordinary LDTI system except that $\hat{w}_i(t), \hat{z}_i(t)$ are functions instead of constant vectors and operated on by linear operators instead of constant matrices. The similarities become clearer if we write the presentation as

$$\begin{aligned} x((i+1)\tau) &= \tilde{A}x(i\tau) + \tilde{B}_1\hat{w}_i(t) + \tilde{B}_2u_i, \\ \hat{z}_i(t) &= \tilde{C}_1x(i\tau) + \tilde{D}_{11}\hat{w}_i(t) + \tilde{D}_{12}u_i, \\ y_i &= \tilde{C}_2x(i\tau), \end{aligned} \tag{6}$$

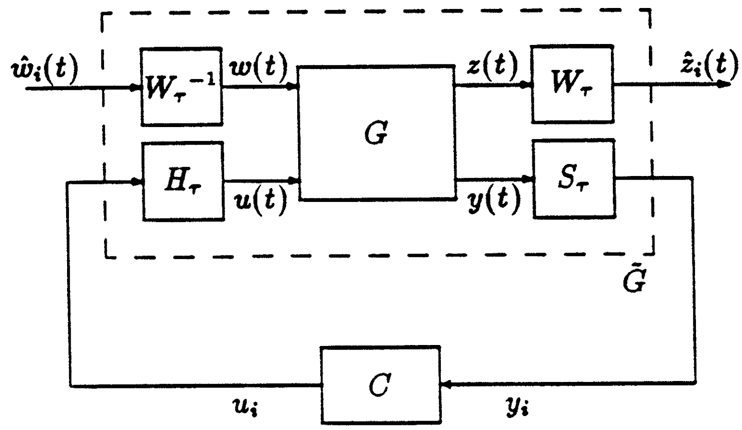

Figure 3: Lifted system $\tilde{G}$

where $\tilde{A} = e^{A\tau}, \tilde{B}_2 = \Psi(\tau)B_2$, and $\tilde{C}_2 = C_2$ are constant matrices, and $\tilde{B}_1, \tilde{C}_1, \tilde{D}_{11}$ and $\tilde{D}_{12}$ are linear operators. In our previous notation

$$\tilde{G} = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right]_l .$$

The system $\mathcal{S}(\tilde{G}, C)$ is internally stable if for any given initial state $(x(0), x_{c,0})$, the state $(x(i\tau), x_{c,i})$ goes to 0 exponentially fast as i goes to ∞, with $\hat{w}_i(t) = 0$ for $\forall i \geq 0, 0 \leq \forall t < \tau$. It is easy to see that internal stability of $\mathcal{S}(G, H_\tau C S_\tau)$ implies internal stability of $\mathcal{S}(\tilde{G}, C)$. In fact the converse is true[11]. This is summarized below.

Proposition 2 *The system $\mathcal{S}(G, H_\tau C S_\tau)$ is internally stable if and only if $\mathcal{S}(\tilde{G}, C)$ is.*

By this technique, we can regard the sampled-data system as a LDTI system, possibly infinite-dimensional. Bamieh and Pearson derive the next proposition on this. It enables us to relate the stability and the norm of a lifted system to ones of some particular LDTI system.

Proposition 3 (Theorem 4 and 6 in [2]) *Given a lifted system $\tilde{G}$ in the form of (6), where $\tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{C}_1, \tilde{C}_2, \tilde{D}_{11}, \tilde{D}_{12}$ are linear operators as:*

$$\begin{array}{ll} \tilde{A} : \Re^x \to \Re^x, & \\ \tilde{B}_1 : \mathcal{L}^2[0,\tau)^w \to \Re^x, & \tilde{B}_2 : \Re^u \to \Re^x, \\ \tilde{C}_1 : \Re^x \to \mathcal{L}^2[0,\tau)^z, & \tilde{C}_2 : \Re^x \to \Re^y, \\ \tilde{D}_{11} : \mathcal{L}^2[0,\tau)^w \to \mathcal{L}^2[0,\tau)^z, & \tilde{D}_{12} : \Re^u \to \mathcal{L}^2[0,\tau)^z. \end{array}$$

Assume $\|\tilde{D}_{11}\|_{\mathcal{L}^2[0,\tau)} < \gamma$ and form the matrices

$$\tilde{B}_1(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1}\tilde{B}_1^* \;=:\; T_B^* \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix} T_B,$$

$$\begin{bmatrix} \tilde{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix} (I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix} \;=:\; T_{CD}^* \begin{bmatrix} \Sigma_{cd} & 0 \\ 0 & 0 \end{bmatrix} T_{CD},$$

where Σ_b, Σ_{cd} are diagonal and nonsingular, and T_B, T_{CD} are nonsingular square matrices. Define a LDTI system

$$\acute{G} = \left[\begin{array}{c|cc} \acute{A} & \acute{B}_1 & \acute{B}_2 \\ \hline \acute{C}_1 & 0 & \acute{D}_{12} \\ \acute{C}_2 & 0 & 0 \end{array}\right]_d,$$

where

$$\acute{B}_1 := T_B^* \begin{bmatrix} \Sigma_b^{1/2} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \acute{C}_1 & \acute{D}_{12} \end{bmatrix} := \begin{bmatrix} \Sigma_{cd}^{1/2} & 0 \end{bmatrix} T_{CD},$$

$$\acute{A} := \tilde{A} + \tfrac{1}{\gamma^2}\tilde{B}_1\tilde{D}_{11}^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{C}_1,$$

$$\acute{B}_2 := \tfrac{1}{\gamma^2}\tilde{B}_1\tilde{D}_{11}^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{D}_{12} + \tilde{B}_2, \qquad \acute{C}_2 := \tilde{C}_2.$$

(Note that all the quantities defined above are constant matrices.) Then the following are equivalent.

- *$\mathcal{S}(\tilde{G}, C)$ is internally stable and $\|\mathcal{F}(\tilde{G}, C)\|_{\ell^2_{\mathcal{L}^2[0,\tau)}} < \gamma$.*
- *$\mathcal{S}(\acute{G}, C)$ is asymptotically stable and $\|\mathcal{F}(\acute{G}, C)\|_{\ell^2_{\Re}} < \gamma$.*

Particularly in the case of $\tilde{D}_{11} = 0$ we get:

- *$\mathcal{S}(\tilde{G}, C)$ is internally stable if and only if $\mathcal{S}(\acute{G}, C)$ is asymptotically stable.*
- *When stability holds, $\|\mathcal{F}(\tilde{G}, C)\|_{\ell^2_{\mathcal{L}^2[0,\tau)}} = \|\mathcal{F}(\acute{G}, C)\|_{\ell^2_{\Re}}$.*

When $\tilde{G}$ is defined as (5), it is rare that we have $\tilde{D}_{11} = 0$. However, in that case, all the operator compositions required can be computed via matrix computations. We have

$$\begin{aligned}
\acute{A} &= \tilde{A} + \tfrac{1}{\gamma^2}\tilde{B}_1\tilde{D}_{11}^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{C}_1 = \Gamma_{22}(\tau) - \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau)\Gamma_{12}(\tau), & (7)\\
\acute{B}_2 &= \tfrac{1}{\gamma^2}\tilde{B}_1\tilde{D}_{11}^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{D}_{12} + \tilde{B}_2 = [\Phi_{22}(\tau) - \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau)]B_2, & (8)\\
\acute{C}_2 &= \tilde{C}_2 = C_2, & (9)\\
\acute{B}_1\acute{B}_1^* &= \tilde{B}_1(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1}\tilde{B}_1^* = \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau), & (10)\\
\acute{C}_1^*\acute{C}_1 &= \tilde{C}_1^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{C}_1 = -\gamma^2\Gamma_{11}^{-1}(\tau)\Gamma_{12}(\tau), & (11)\\
\acute{C}_1^*\acute{D}_{12} &= \tilde{C}_1^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{D}_{12} = -\gamma^2\Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau)B_2, & (12)\\
\acute{D}_{12}^*\acute{D}_{12} &= \tilde{D}_{12}^*(I - \tfrac{1}{\gamma^2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\tilde{D}_{12} = \gamma^2 B_2^*[\Omega_{12}(\tau) - \Phi_{11}(\tau)\Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau)]B_2; & (13)
\end{aligned}$$

where $\Psi(t) = \int_0^t e^{As}ds$ as before, and

$$\begin{aligned}
\Gamma(t) = \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{bmatrix} &:= \exp\left\{\begin{bmatrix} -A^* & -\frac{1}{\gamma^2}C_1^*C_1 \\ B_1B_1^* & A \end{bmatrix} t\right\}, \qquad (14)\\
\Phi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix} &:= \int_0^t \Gamma(s)ds,\\
\Omega(t) = \begin{bmatrix} \Omega_{11}(t) & \Omega_{12}(t) \\ \Omega_{21}(t) & \Omega_{22}(t) \end{bmatrix} &:= \int_0^t \left(\int_0^s \Gamma(r)dr\right) ds
\end{aligned}$$

($\Phi(t), \Omega(t)$ are partitioned conformably with $\Gamma(t)$). Note that the integrations for Ψ, Φ, Ω can be calculated via matrix exponentials using the formula

$$\exp\left\{\begin{bmatrix} P & I \\ 0 & 0 \end{bmatrix} t\right\} = \begin{bmatrix} e^{Pt} & \int_0^t e^{Ps}ds \\ 0 & I \end{bmatrix},$$

which is true for any square matrix P. Thus, we can compute $\Psi(t)$ and $\Phi(t)$ using this formula once, and $\Omega(t)$ using it twice. The assumption $\|\tilde{D}_{11}\| < \gamma$ guarantees the existence of $\Gamma_{11}^{-1}(\tau)$. See [2] for all the details.

We show the explicit state-space representation of $\mathcal{F}(\acute{G}, C)$ for later use:

$$\mathcal{F}(\acute{G}, C) = \left[\begin{array}{cc|c} \acute{A} + \acute{B}_2D_c\acute{C}_2 & \acute{B}_2C_c & \acute{B}_1 \\ B_c\acute{C}_2 & A_c & 0 \\ \hline \acute{C}_1 + \acute{D}_{12}D_c\acute{C}_2 & \acute{D}_{12}C_c & 0 \end{array}\right]_d =: \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & 0 \end{array}\right]_d. \qquad (15)$$

Actually in the algorithm shown later for computing the norm, we need only $\mathcal{A}, \mathcal{B}\mathcal{B}^*, \mathcal{C}^*\mathcal{C}$ but not $\mathcal{B}, \mathcal{C}$ themselves. This means (7)–(13) are enough for the calculation and we do not have to evaluate $\acute{B}_1, \acute{C}_1, \acute{D}_{12}$ themselves.

We conclude this section by discussing how to compute $\|\tilde{D}_{11}\|$.

The first way is based on the discretization of $\tilde{D}_{11}$. Define a 'fast' sampler S_n : (continuous elements of $\mathcal{L}^2[0,\tau)^z$) $\rightarrow \Re^{nz}$ and a holder $H_n : \Re^{nw} \rightarrow \mathcal{L}^2[0,\tau)^w$ as

$$S_n: \quad (S_n\hat{z}(t))_k := \hat{z}(k\frac{\tau}{n}) \quad \text{for} \quad k = 0, \ldots, n-1; \quad \hat{z}(t) \in \mathcal{L}^2[0,\tau)^z;$$

$$H_n: \quad (H_n\{\hat{w}_k\})(t) := \hat{w}_{\lfloor \frac{nt}{\tau} \rfloor} \quad \text{for} \quad 0 \le t < \tau; \quad \{\hat{w}_k\} \in \Re^{nw}.$$

And then form a $nz \times nw$ matrix $S_n\tilde{D}_{11}H_n$ by defining its (k,l) block of size $z \times w$ by

$$(S_n\tilde{D}_{11}H_n)_{k,l} := \begin{cases} C_1 e^{A\frac{\tau}{n}(k-l-1)}\Psi(\tau/n)B_1 & \text{for } k-l \ge 1 \\ 0 & \text{for } k-l \le 0 \end{cases},$$

where w is a dimension of each $\hat{w}_k$, and z is a dimension of a signal $\hat{z}(t)$. Here we can show that the maximum singular value of $S_n\tilde{D}_{11}H_n$ goes to $\|\tilde{D}_{11}\|$ as n goes to infinity. Thus $\|\tilde{D}_{11}\|$ can be obtained through the maximum singular value of $S_n\tilde{D}_{11}H_n$ using a large enough n.

The second way is to examine the spectrum of $\tilde{D}_{11}^*\tilde{D}_{11}$. In [15] it is shown that the operator $(I - \frac{1}{\gamma^2}\tilde{D}_{11}^*\tilde{D}_{11})$ is invertible if and only if the matrix $\Gamma_{11}(\tau)$ in (14) is invertible (or equivalently $\Gamma_{22}(\tau)$ is invertible). Thus we can find the value of $\|\tilde{D}_{11}\|$ by drawing a graph of the minimum singular value of $\Gamma_{11}(\tau)$ over an appropriate range of γ and looking for the biggest γ which has the minimum singular value of $\Gamma_{11}(\tau)$ equal to 0.

4 ℓ^2 Induced Norm of a LDTI System

In this section we present a strong tool for computing the $\mathcal{H}^\infty$ norm of a LDTI system. This is the discrete counterpart of an analysis method for a LCTI system introduced in [4]. Since the norm of the sampled-data system is given by the norm of an equivalent LDTI system, this result will be of considerable importance.

Proposition 4 *Consider a LDTI system* $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_d$ *and write its z-transform as* $M(z) = C(zI - A)^{-1}B + D$. *Suppose A does not have any eigenvalues on the unit circle,* $\gamma > 0$ *is not a singular value of D, and* $\omega_0 \in [0, 2\pi)$. *Then* γ *is a singular value of* $M(e^{j\omega_0})$ *if and only if* $\det(P_1 - e^{j\omega_0}P_2) = 0$, *where*

$$P_1 = \begin{bmatrix} A + \frac{1}{\gamma^2}B(I - \frac{1}{\gamma^2}D^*D)^{-1}D^*C & 0 \\ \frac{1}{\gamma^2}C^*(I - \frac{1}{\gamma^2}DD^*)^{-1}C & I \end{bmatrix}, \qquad P_2 = \begin{bmatrix} I & B(I - \frac{1}{\gamma^2}D^*D)^{-1}B^* \\ 0 & A^* + \frac{1}{\gamma^2}C^*D(I - \frac{1}{\gamma^2}D^*D)^{-1}B^* \end{bmatrix}.$$

Proof (Almost the same as Theorem 1 in [4]): Write $R = (I - \frac{1}{\gamma^2}D^*D), S = (I - \frac{1}{\gamma^2}DD^*)$, respectively.

Let γ be a singular value of $M(e^{j\omega_0})$. Then we have a nonzero u, v such that

$$M(e^{j\omega_0})u = \gamma v, \qquad M^*(e^{j\omega_0})v = \gamma u,$$

that is

$$\begin{aligned} \{C(e^{j\omega_0}I - A)^{-1}B + D\}u &= \gamma v, \\ \{B^*(e^{-j\omega_0}I - A^*)^{-1}C^* + D^*\}v &= \gamma u. \end{aligned} \tag{16}$$

Define

$$\begin{aligned} r &= (A - e^{j\omega_0}I)^{-1}Bu, \\ s &= \tfrac{1}{\gamma}(I - e^{j\omega_0}A^*)^{-1}C^*v, \end{aligned} \tag{17}$$

then we get

$$\begin{aligned} -\tfrac{1}{\gamma}Cr + \tfrac{1}{\gamma}Du &= v, \\ e^{j\omega_0}B^*s + \tfrac{1}{\gamma}D^*v &= u. \end{aligned} \tag{18}$$

Now solving for u and v in terms of r and s

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\gamma}D & I \\ I & -\frac{1}{\gamma}D^* \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{\gamma}C & 0 \\ 0 & e^{j\omega_0}B^* \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \tag{19}$$

Here note that the assumption on γ guarantees the existence of the inverse and that $\begin{bmatrix} r \\ s \end{bmatrix} \neq 0$ holds from the above. From (17)

$$\begin{bmatrix} Bu \\ \frac{1}{\gamma}C^*v \end{bmatrix} = \begin{bmatrix} (A - e^{j\omega_0}I)r \\ (I - e^{j\omega_0}A^*)s \end{bmatrix}. \tag{20}$$

And from (19) and (20) we get

$$\begin{bmatrix} B & 0 \\ 0 & \frac{1}{\gamma}C^* \end{bmatrix} \begin{bmatrix} -\frac{1}{\gamma}D & I \\ I & -\frac{1}{\gamma}D^* \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{\gamma}C & 0 \\ 0 & e^{j\omega_0}B^* \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} (A - e^{j\omega_0}I)r \\ (I - e^{j\omega_0}A^*)s \end{bmatrix}.$$

It is straightforward to show

$$\begin{bmatrix} -\frac{1}{\gamma}D & I \\ I & -\frac{1}{\gamma}D^* \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\gamma}R^{-1}D^* & R^{-1} \\ S^{-1} & \frac{1}{\gamma}DR^{-1} \end{bmatrix}.$$

Substitute this to get

$$\left\{ \begin{bmatrix} A - e^{j\omega_0}I & 0 \\ 0 & I - e^{j\omega_0}A^* \end{bmatrix} \right.$$

$$-\begin{bmatrix} B & 0 \\ 0 & \frac{1}{\gamma}C^* \end{bmatrix}\begin{bmatrix} \frac{1}{\gamma}R^{-1}D^* & R^{-1} \\ S^{-1} & \frac{1}{\gamma}DR^{-1} \end{bmatrix}\begin{bmatrix} -\frac{1}{\gamma}C & 0 \\ 0 & e^{j\omega_0}B^* \end{bmatrix}\Bigg\}\begin{bmatrix} r \\ s \end{bmatrix} = 0,$$

then

$$\left\{\begin{bmatrix} A - e^{j\omega_0}I & 0 \\ 0 & I - e^{j\omega_0}A^* \end{bmatrix} + \begin{bmatrix} \frac{1}{\gamma^2}BR^{-1}D^*C & -e^{j\omega_0}BR^{-1}B^* \\ \frac{1}{\gamma^2}C^*S^{-1}C & -e^{j\omega_0}\frac{1}{\gamma^2}C^*DR^{-1}B^* \end{bmatrix}\right\}\begin{bmatrix} r \\ s \end{bmatrix} = 0,$$

$$\left\{\begin{bmatrix} A + \frac{1}{\gamma^2}BR^{-1}D^*C & 0 \\ \frac{1}{\gamma^2}C^*S^{-1}C & I \end{bmatrix} - e^{j\omega_0}\begin{bmatrix} I & BR^{-1}B^* \\ 0 & A^* + \frac{1}{\gamma^2}C^*DR^{-1}B^* \end{bmatrix}\right\}\begin{bmatrix} r \\ s \end{bmatrix} = 0. \quad (21)$$

To prove the converse suppose $\det(P_1 - e^{j\omega_0}P_2) = 0$ first. That means (21) holds for some $\begin{bmatrix} r \\ s \end{bmatrix} \neq 0$. Define u, v by (19), then $\begin{bmatrix} u \\ v \end{bmatrix} \neq 0$ follows. Here (19) and (21) are equivalent to (19) and (20), moreover to (17) and (18). Then (16) is derived. ∎

The above proposition provides us with a strong tool for obtaining the norm. From now on let $\bar{\sigma}(P)$ denote the maximum singular value of a matrix P.

Proposition 5 *With the notation of Proposition 4, let M be asymptotically stable (i.e. all of eigenvalues of A are in the open unit disk). Given $\gamma > 0$ such that γ is not a singular value of D and suppose there exists $\omega \in [0, 2\pi)$ such that $\gamma > \bar{\sigma}(M(e^{j\omega}))$. Then $\|M\|_{\ell^2_{\Re}} < \gamma$ if and only if a generalized eigenvalue problem $\det(P_1 - \lambda P_2) = 0$ has no solution λ on the unit circle.*

Proof: As is well known $\|M\|_{\ell^2_{\Re}} = \sup_{\omega\in[0,2\pi)} \bar{\sigma}(M(e^{j\omega}))$[10]. It is easy to see that the assumtion on γ implies that $\|M\|_{\ell^2_{\Re}} < \gamma$ if and only if there is no $\omega_0 \in [0, 2\pi)$ such that $M(e^{j\omega_0})$ has γ as its singular value. Now Proposition 4 is applicable and gives the desired result immediately. ∎

Proposition 5 is immediately applicable to a LDTI system $\mathcal{S}(\acute{\mathcal{G}}, C)$. In this case

$$P_1 = \begin{bmatrix} \mathcal{A} & 0 \\ \frac{1}{\gamma^2}C^*C & I \end{bmatrix}, \quad P_2 = \begin{bmatrix} I & \mathcal{B}\mathcal{B}^* \\ 0 & \mathcal{A}^* \end{bmatrix}, \quad (22)$$

with the notation of (15).

5 Algorithm

Next, an algorithm for computing $\|\mathcal{F}(G, H_\tau C S_\tau)\| = \|\mathcal{F}(\tilde{G}, C)\|$ is proposed.

Algorithm:

1 $\gamma_L := \|\tilde{D}_{11}\|$;
2 if $\mathcal{S}(\tilde{G}, C)$ is internally stable
3 then $\gamma_U :=$ (upper bound for $\|\mathcal{F}(\tilde{G}, C)\|$)
4 else output '$\mathcal{S}(G, H_\tau C S_\tau)$ is not internally stable' and exit;
5 while $\gamma_U - \gamma_L > \varepsilon\gamma_L$ do {
6 $\quad\gamma := (\gamma_L + \gamma_U)/2$;
7 $\quad$form $\mathcal{A}, \mathcal{B}\mathcal{B}^*, \mathcal{C}^*\mathcal{C}$ of $\mathcal{S}(\acute{G}, C)$ using current γ and (7)–(13);
8 $\quad$if $\mathcal{S}(\acute{G}, C)$ is asymptotically stable then {
9 $\quad\quad$if $\bar{\sigma}(\mathcal{F}(\acute{G}, C)(e^{j\omega})) < \gamma$ for some particular $\omega \in [0, 2\pi)$, say 0, then {
10 $\quad\quad\quad$form (P_1, P_2) via (22);
11 $\quad\quad\quad$if $\det(P_1 - \lambda P_2) = 0$ has no solution λ on the unit circle
12 $\quad\quad\quad$then $\gamma_U := \gamma$ /* $\|\mathcal{F}(G, H_\tau C S_\tau)\| < \gamma$ */
13 $\quad\quad\quad$else $\gamma_L := \gamma$; /* $\|\mathcal{F}(G, H_\tau C S_\tau)\| \geq \gamma$ */
$\quad\quad$}
14 $\quad\quad$else $\gamma_L := \gamma$; /* $\|\mathcal{F}(G, H_\tau C S_\tau)\| \geq \gamma$ */
$\quad$}
15 $\quad$else $\gamma_L := \gamma$; /* $\|\mathcal{F}(G, H_\tau C S_\tau)\| \geq \gamma$ */
}
16 output γ;
end.

Suppose for the time being that we can process lines 2–4 in some way. Then the validity of the algorithm can be shown as follows.

Proposition 6 *Consider the sampled-data system $\mathcal{S}(G, H_\tau C S_\tau)$ defined by (1)–(4) and assume $D_{11}, D_{12}, D_{21}, D_{22} = 0$. Then the above algorithm judges the internal stability of $\mathcal{S}(G, H_\tau C S_\tau)$ and gives its norm if stable.*

Proof: We will see afterwards lines 2–4 can be carried out.

Line 7 is valid because $\gamma > \|\tilde{D}_{11}\|$ there. To do line 8 check if the all eigenvalues of $\mathcal{A}$ are in the open unit disk.

In lines 11–13, Proposition 5 is applicable on $\mathcal{S}(\acute{G}, C)$ (all the assumptions are satisfied). Then in the case of line 12, $\|\mathcal{F}(\acute{G}, C)\| < \gamma$ is derived and hence $\|\mathcal{F}(\tilde{G}, C)\| < \gamma$ holds by Proposition 3 since $\mathcal{S}(\acute{G}, C)$ is asymptotically stable. In the case of line 13, $\|\mathcal{F}(\acute{G}, C)\| \geq \gamma$ and then $\|\mathcal{F}(\tilde{G}, C)\| \geq \gamma$ follows by the same proposition because $\mathcal{S}(\tilde{G}, C)$ is internally stable.

In line 14, note that $\|\mathcal{F}(\acute{G}, C)\| = \sup_{\omega\in[0,2\pi)} \bar{\sigma}(\mathcal{F}(\acute{G}, C)(j\omega))$ because of the asymptotic stability. Here $\bar{\sigma}(\mathcal{F}(\acute{G}, C)(j\omega)) \geq \gamma$ for some ω thus $\|\mathcal{F}(\acute{G}, C)\| \geq \gamma$ holds. Then by the same argument as before $\|\mathcal{F}(\tilde{G}, C)\| \geq \gamma$ follows.

In line 15, $\mathcal{F}(\acute{G},C)$ is not asymptotically stable. So we can use Proposition 3 in the same way again to get $\|\mathcal{F}(\tilde{G},C)\| \geq \gamma$.

This verifies the claim. ∎

Two points should be supplemented.

1) To check if $\bar{\sigma}(\mathcal{F}(\acute{G},C)(e^{j\omega})) < \gamma$ we note

$$\begin{aligned}|\bar{\sigma}(\mathcal{F}(\acute{G},C)(e^{j\omega}))|^2 &= |\bar{\sigma}(\mathcal{C}(e^{j\omega}I-\mathcal{A})^{-1}\mathcal{B})|^2\\ &= \bar{\lambda}(\mathcal{C}(e^{j\omega}I-\mathcal{A})^{-1}\mathcal{B}\mathcal{B}^*(e^{-j\omega}I-\mathcal{A}^*)^{-1}\mathcal{C}^*)\\ &= \bar{\lambda}((e^{j\omega}I-\mathcal{A})^{-1}\mathcal{B}\mathcal{B}^*(e^{-j\omega}I-\mathcal{A}^*)^{-1}\mathcal{C}^*\mathcal{C}),\end{aligned}$$

where $\bar{\lambda}(.)$ denotes the maximum eigenvalue. Hence this needs only $\mathcal{A}, \mathcal{B}\mathcal{B}^*, \mathcal{C}^*\mathcal{C}$ again.

2) We have to show how to process lines 2–4 of the algorithm.

Note that

$$\mathcal{F}(\tilde{G},C) = \tilde{D}_{11} + \mathcal{F}(\left[\begin{array}{c|cc}\tilde{A} & \tilde{B}_1 & \tilde{B}_2\\ \hline \tilde{C}_1 & 0 & \tilde{D}_{12}\\ \tilde{C}_2 & 0 & 0\end{array}\right]_l, C). \tag{23}$$

Then apply the procedure of Proposition 3 and transform the first variable of the second term. We write its result as

$$\mathcal{F}(\left[\begin{array}{c|cc}\acute{A}_0 & \acute{B}_{10} & \acute{B}_{20}\\ \hline \acute{C}_{10} & 0 & \acute{D}_{120}\\ \acute{C}_{20} & 0 & 0\end{array}\right]_d, C) = \left[\begin{array}{cc|c}\acute{A}_0+\acute{B}_{20}D_c\acute{C}_{20} & \acute{B}_{20}C_c & \acute{B}_{10}\\ B_c\acute{C}_{20} & A_c & 0\\ \hline \acute{C}_{10}+\acute{D}_{120}D_c\acute{C}_{20} & \acute{D}_{120}C_c & 0\end{array}\right]_d =: \left[\begin{array}{c|c}\mathcal{A}_0 & \mathcal{B}_0\\ \hline \mathcal{C}_0 & 0\end{array}\right]_d.$$

Here

$$\acute{A}_0 = \tilde{A} = e^{A\tau}, \tag{24}$$

$$\acute{B}_{20} = \tilde{B}_2 = \Psi(\tau)B_2, \tag{25}$$

$$\acute{C}_{20} = \tilde{C}_2 = C_2, \tag{26}$$

$$\acute{B}_{10}\acute{B}_{10}^* = \tilde{B}_1\tilde{B}_1^* = \int_0^\tau e^{A(\tau-s)}B_1B_1^*e^{A^*(\tau-s)}ds, \tag{27}$$

$$\acute{C}_{10}^*\acute{C}_{10} = \tilde{C}_1^*\tilde{C}_1 = \int_0^\tau e^{A^*s}C_1^*C_1e^{As}ds, \tag{28}$$

$$\acute{C}_{10}^*\acute{D}_{120} = \tilde{C}_1^*\tilde{D}_{12} = \int_0^\tau e^{A^*s}C_1^*C_1\Psi(s)B_2ds, \tag{29}$$

$$\acute{D}_{120}^*\acute{D}_{120} = \tilde{D}_{12}^*\tilde{D}_{12} = \int_0^\tau B_2^*\Psi^*(s)C_1^*C_1\Psi(s)B_2ds. \tag{30}$$

We cannot use the operator composition formulas (7)–(13) this time, however, this is the special case of Proposition 3 ($\tilde{D}_{11} = 0$). So by the eigenvalues of $\mathcal{A}_0$ we can judge internal stability of $\mathcal{S}(\tilde{G}, C)$ and it is equivalent to internal stability of $\mathcal{S}(G, H_\tau C S_\tau)$ by Proposition 2.

Now if internal stability holds then $\|\mathcal{F}(\tilde{G}, C)\|$ is finite by Proposition 1 and 2. Evaluate the norm in (23) and use the norm equality in the special case of Proposition 3 to get

$$
\begin{aligned}
\|\mathcal{F}(\tilde{G}, C)\| &\leq \|\tilde{D}_{11}\| + \|\mathcal{F}(\left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & 0 & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right]_l, C)\| \\
&= \|\tilde{D}_{11}\| + \|\mathcal{F}(\left[\begin{array}{c|cc} \acute{A}_0 & \acute{B}_{10} & \acute{B}_{20} \\ \hline \acute{C}_{10} & 0 & \acute{D}_{120} \\ \acute{C}_{20} & 0 & 0 \end{array}\right]_d, C)\| \\
&= \|\tilde{D}_{11}\| + \left\| \left[\begin{array}{c|c} \mathcal{A}_0 & \mathcal{B}_0 \\ \hline \mathcal{C}_0 & 0 \end{array}\right]_d \right\|. \qquad (31)
\end{aligned}
$$

Moreover we can bound the second term of (31) by the Hankel singular values[14] as

$$
\begin{aligned}
\left\| \left[\begin{array}{c|c} \mathcal{A}_0 & \mathcal{B}_0 \\ \hline \mathcal{C}_0 & 0 \end{array}\right]_d \right\| &\leq 2\sum_i \sigma_{Hi} \\
&\leq 2\sqrt{n\,\mathrm{tr}(W_c W_o)}. \qquad (32)
\end{aligned}
$$

Here σ_{Hi} are the Hankel singular values of the system $\left[\begin{array}{c|c} \mathcal{A}_0 & \mathcal{B}_0 \\ \hline \mathcal{C}_0 & 0 \end{array}\right]_d$, n is a dimension of $\mathcal{A}_0$, and W_c, W_o are the controllability and observability Grammians, respectively. It is known that W_c and W_o are the unique solutions of

$$
\begin{aligned}
\mathcal{A}_0 W_c \mathcal{A}_0^* + \mathcal{B}_0 \mathcal{B}_0^* &= W_c, \\
\mathcal{A}_0^* W_o \mathcal{A}_0 + \mathcal{C}_0^* \mathcal{C}_0 &= W_o,
\end{aligned} \qquad (33)
$$

respectively, when all the eigenvalues of $\mathcal{A}_0$ are in the open unit disk[13].

In short, to obtain the upper bound, first calculate $\mathcal{A}_0, \mathcal{B}_0\mathcal{B}_0^*, \mathcal{C}_0^*\mathcal{C}_0$ using (24)–(30), then find the unique solutions of (33), and use (31) and (32).

Here the integrations in (27)–(30) can be obtained via matrix exponentials again. Let us write

$$
\begin{aligned}
\begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ 0 & \Theta_{22}(t) \end{bmatrix} &:= \exp\left\{ \begin{bmatrix} -A & B_1 B_1^* \\ 0 & A^* \end{bmatrix} t \right\}, \\
\begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) & \Lambda_{13}(t) & \Lambda_{14}(t) \\ 0 & \Lambda_{22}(t) & \Lambda_{23}(t) & \Lambda_{24}(t) \\ 0 & 0 & \Lambda_{33}(t) & \Lambda_{34}(t) \\ 0 & 0 & 0 & \Lambda_{44}(t) \end{bmatrix} &:= \exp\left\{ \begin{bmatrix} -A^* & I & 0 & 0 \\ 0 & -A^* & C_1^* C_1 & 0 \\ 0 & 0 & A & B_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} t \right\},
\end{aligned}
$$

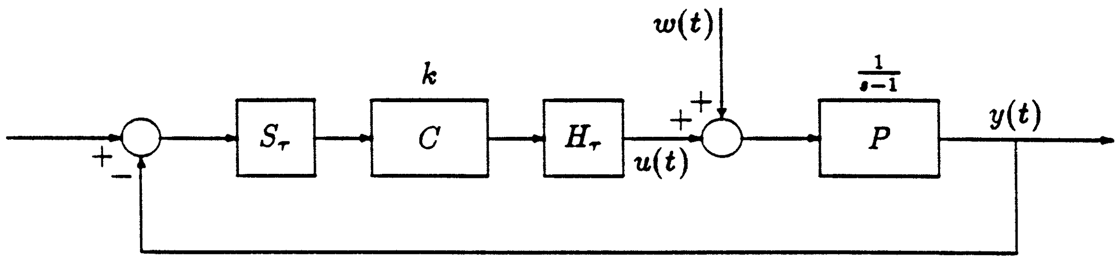

Figure 4: Example

where the matrices in the left hand sides are partitioned conformably with ones in the right hand sides. Then from [23] we have

$$
\begin{aligned}
(27) &= \Theta_{22}^*(\tau)\Theta_{12}(\tau), \\
(28) &= \Lambda_{33}^*(\tau)\Lambda_{23}(\tau), \\
(29) &= \Lambda_{33}^*(\tau)\Lambda_{24}(\tau), \\
(30) &= B_2^*\Lambda_{33}^*(\tau)\Lambda_{14}(\tau) + \Lambda_{14}^*(\tau)\Lambda_{33}(\tau)B_2.
\end{aligned}
$$

Now the desired problem is solved completely.

6 Example

In this section we show how the algorithm works using an easy example.

Suppose the system in Figure 4. Here we like to do a digital proportional feedback control on an unstable LCTI plant P ($\frac{1}{s-1}$ in Laplace transform) using a LDTI controller C (a multiplication of k). Our purpose is to evaluate $\mathcal{L}^2$ induced norm of the operator from a disturbance $w(t)$ to an output $y(t)$ in order to investigate the performance robustness of the system.

We can formalize the system to the form of Figure 1 by defining a generalized plant G as a operator from $\begin{bmatrix} w(t) \\ u(t) \end{bmatrix}$ to $\begin{bmatrix} y(t) \\ -y(t) \end{bmatrix}$, that is,

$$
G = \begin{bmatrix} P & P \\ -P & -P \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]_c .
$$

Then our aim is to compute $\|\mathcal{F}(G, H_\tau C S_\tau)\|$ $(= \|\mathcal{F}(\tilde{G}, C)\|)$. So we can put $A = B_1 = B_2 = C_1 = 1, C_2 = -1, A_c = B_c = C_c = 0$, and $D_c = k$.

From now on, we assume that the sampling period $\tau = 1$ and a controller gain $k = 1.873$.

First, we have to compute $\|\tilde{D}_{11}\|$. Using one of two methods proposed in Section 3 we have $\|\tilde{D}_{11}\| = 1.000$. So we put a lower bound $\gamma_L = 1.000$ (line 1).

Next, we form $\mathcal{A}_0, \mathcal{B}_0\mathcal{B}_0^*, \mathcal{C}_0^*\mathcal{C}_0$ through (24)–(30). The result is

$$\mathcal{A}_0 = \begin{bmatrix} -0.5001 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_0\mathcal{B}_0^* = \begin{bmatrix} 3.195 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C}_0^*\mathcal{C}_0 = \begin{bmatrix} 0.3235 & 0 \\ 0 & 0 \end{bmatrix}.$$

Because both of the two eigenvalues of $\mathcal{A}_0$ are in the unit disc, $\mathcal{S}(\tilde{G}, C)$ is (or equivalently $\mathcal{S}(G, H_\tau C S_\tau)$ is) internally stable (line 2).

Substituting the above matrices to (33), we have

$$W_c = \begin{bmatrix} 4.260 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_o = \begin{bmatrix} 0.4314 & 0 \\ 0 & 0 \end{bmatrix}.$$

(Actually we can obtain them through solving just scaler equations.) Now we have an upper bound for $\|\mathcal{F}(\tilde{G}, C)\|$:

$$\gamma_U = \|\tilde{D}_{11}\| + 2\sqrt{2\,\mathrm{tr}(W_c W_o)} = 4.834$$

(line 3).

Here we finish preparation and enter the bisection loop of the algorithm.

Put $\gamma = (\gamma_L + \gamma_U)/2 = 2.917$ and form $\mathcal{A}, \mathcal{B}\mathcal{B}^*, \mathcal{C}^*\mathcal{C}$ via (7)–(13) (line 6,7). After some calculation we have

$$\mathcal{A} = \begin{bmatrix} -0.4782 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}\mathcal{B}^* = \begin{bmatrix} 3.595 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C}^*\mathcal{C} = \begin{bmatrix} 0.3264 & 0 \\ 0 & 0 \end{bmatrix}$$

(line 9). Judging from the eigenvalues of $\mathcal{A}$, $\mathcal{S}(\acute{G}, C)$ is asymptotically stable. And we have

$$\begin{aligned} \bar{\sigma}(\mathcal{F}(\acute{G}, C)(e^{j0})) &= \left\{\bar{\lambda}((I - \mathcal{A})^{-1}\mathcal{B}\mathcal{B}^*(I - \mathcal{A}^*)^{-1}\mathcal{C}^*\mathcal{C})\right\}^{1/2} \\ &= 0.7328 \\ &< \gamma. \end{aligned}$$

Furthermore, λ, satisfying $\det(P_1 - \lambda P_2) = 0$, are -0.5921, -1.6890, 0, ∞, where P_1, P_2 are formed according to (22) (line 11). So we conclude $\|\mathcal{F}(G, H_\tau C S_\tau)\| < \gamma$, put $\gamma_U := \gamma$ (line 12), go back to the beginning of the loop, and proceed as before to compute the norm.

For example, we have $\|\mathcal{F}(G, H_\tau C S_\tau)\| = 2.110$ with four digits precision.

Figure 5 shows how $\|\mathcal{F}(G, H_\tau C S_\tau)\|$ changes depending on the sampling period τ, when k is chosen to be $(e^\tau + 0.5)/(e^\tau - 1)$ in order to fix $\mathcal{A}_0$ equal to $\begin{bmatrix} -0.5 & 0 \\ 0 & 0 \end{bmatrix}$. For comparison

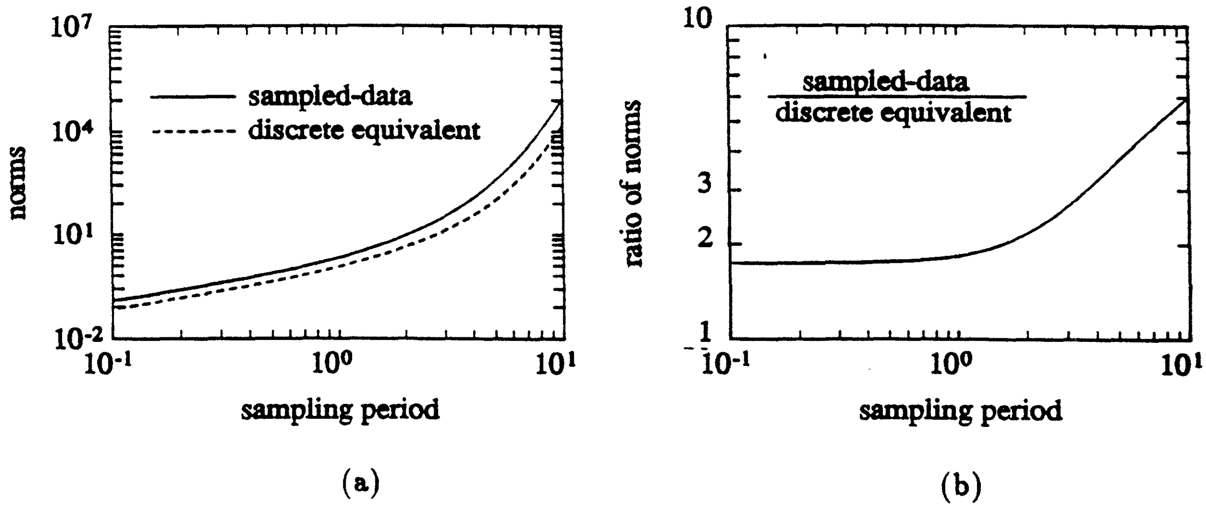

Figure 5: Dependency of the norms on a sampling period; (a) Norms of the sampled-data system and its zero-order hold discrete-time equivalent, (b) Ratio of the two norms

we also show the ℓ^2 induced norm of the operator from w_i to y_i in Figure 6. Note that this system can be dealt with as purely LDTI because P_d ($= S_\tau P H_\tau$) is LDTI. This P_d is called the 'zero-order hold discrete-time equivalent' of P and often used for a design and an analysis of a sampled-data system[12]. We can see in Figure 5(a) that a performance robustness of the sampled-data system gets worse as we use a longer sampling period, and in Figure 5(b) that the discrete-time equivalent does not give a good approximation when a sampling period is long.

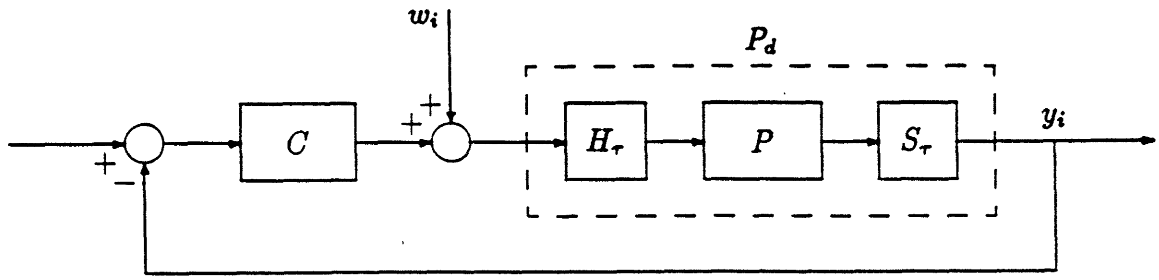

Figure 6: Approximation using a zero-order hold discrete-time equivalent

7 Conclusion

An algorithm to compute the $\mathcal{L}^2$ induced norm of a certain type of a sampled-data system is shown. This uses only matrix computations in its main part such as matrix exponentials and inverses, combined with a bisection algorithm.

It should be noted that when implementing this algorithm, numerical round-off errors resulting from the matrix computations can be quite critical (see [16]). For such computations, the recent self-validating numerical computation will be quite useful[20, 21].

References

[1] B. Bamieh, M. A. Dahleh and J. B. Pearson, 'Minimization of the $\mathcal{L}^\infty$-Induced Norm for Sampled-Data Systems,' *Report P-2072*, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 1991.

[2] B. Bamieh and J. B. Pearson, 'A General Framework for Linear Periodic Systems with Application to $\mathcal{H}^\infty$ Samepled-Data Control,' *IEEE Transactions on Automatic Control*, vol. 37 (1992), pp. 418–435.

[3] B. Bamieh, J. B. Pearson, B. A. Francis and A. Tannenbaum, 'A Lifting Technique for Linear Periodic Systems with Applications to Sampled-Data Control,' *Systems & Control Letters*, vol. 17 (1991), pp. 79–88.

[4] S. Boyd, V. Balakrishnan and P. Kabamba, 'A Bisection Method for Computing the $\mathcal{H}^\infty$ Norm of a Transfer Matrix and Related Problems,' *Mathematics of Control, Signals, and Systems*, vol. 2 (1989), pp. 207–219.

[5] T. Chen and B. A. Francis, 'On the $\mathcal{L}^2$-Induced Norm of a Sampled-Data System,' *Systems & Control Letters*, vol. 15 (1990), pp. 211–219.

[6] T. Chen and B. A. Francis, 'Input-Output Stability of Sampled-Data Systems,' *IEEE Transactions on Automatic Control*, vol. 36 (1991), pp. 50–58.

[7] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer-Verlag, New York, 1990.

[8] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[9] J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, 'State-Space Solutions to Standard $\mathcal{H}^2$ and $\mathcal{H}^\infty$ Control Problems,' *IEEE Transactions on Automatic Control*, vol. 34 (1989), pp. 831–847.

[10] B. A. Francis, *A Course in $\mathcal{H}^\infty$ Control Theory*, Springer-Verlag, New York, 1987.

[11] B. A. Francis and T. T. Georgiou, 'Stability Theory for Linear Time-Invariant Plants with Periodic Digital Controllers,' *IEEE Transactions on Automatic Control*, vol. 33 (1988), pp. 820–833.

[12] G. F. Franklin, J. D. Powell and M. L. Workman, *Digital Control of Dynamic Systems*, 2nd ed., Addison-Wesley Publishing, Reading, 1990.

[13] F. R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing, New York, 1959.

[14] K. Glover, 'All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and their $\mathcal{L}^\infty$-Error Bounds,' *International Journal of Control*, vol. 39 (1984), pp. 1115–1193.

[15] I. Gohberg and M. A. Kaashoek, 'Time Varying Linear System with Boundary Conditions and Integral Operators, I. The Transfer Operator and its Properties,' *Integral Equations and Operator Theory*, vol. 7 (1984), pp. 325–391.

[16] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd ed., The Johns Hopkins University Press, Baltimore, 1989.

[17] P. T. Kabamba and S. Hara, 'On Computing the Induced Norm of Sampled Data Systems,' *Proceeding of the 1990 American Control Conference*, San Diego, pp. 319–320.

[18] P. T. Kabamba and S. Hara, 'Worst Case Analysis and Design of Sampled Data Control Systems,' submitted to *IEEE Transactions on Automatic Control*.

[19] M. Khammash, 'Necessary and Sufficient Conditions for the Robustness of Time-Varying Systems with Applications to Sampled-Data Systems,' *Technical Report*, Electrical and Computer Engineering, Iowa State University, 1991.

[20] U. Kulisch and W. L. Miranker, *Computer Arithmetic in Theory and Practice*, Academic Press, New York, 1981.

[21] S. M. Rump, 'Solving Algebraic Problems with High Accuracy,' *A New Approach to Scientific Computation* (U. Kulisch and W. L. Miranker eds.), Academic Press, New York, 1983, pp. 51–120.

[22] N. Sivashankar and P. P. Khargonekar, 'Robust Stability and Performance Analysis of Sampled-Data Systems,' submitted to *IEEE Transactions on Automatic Control*, to appear.

[23] C. F. Van Loan, 'Computing Integrals Involving the Matrix Exponential,' *IEEE Transactions on Automatic Control*, vol. 23 (1978), pp. 395–404.

[24] Y. Yamamoto, 'New Approach to Sampled-Data Control Systems —A Function Space Method,' *Proceeding of the 29th Conference on Decision and Control*, Honolulu, 1990, pp. 1882–1887.