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Analysis of Continuous Switching Systems: Theory and Examples*

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Abstract

This paper details work on ordinary differential equations that continuously switch among regimes of operation. In the first part, we develop some tools for analyzing such systems. We prove an extension of Bendixson's Theorem to the case of Lipschitz continuous vector fields. We also prove a lemma dealing with the robustness of differential equations with respect to perturbations that preserve a linear part, which we call the Linear Robustness Lemma. We then give some simple propositions that allow us to use this lemma in studying certain singular perturbation problems.

In the second part, the attention focuses on example systems and their analysis. We use the tools from the first part and develop some general insights. The example systems arise from a realistic aircraft control problem.

The extension of Bendixson's Theorem and the Linear Robustness Lemma have applicability beyond the systems discussed in this paper.

1 Introduction

1.1 Background and Motivation

This paper details work on ordinary differential equations (ODEs) that continuously switch among regimes of operation. We have in mind the following model as a prototypical example of a *switching system*:

$$\dot{x}(t) = F_i(x(t), t), \qquad x(0) = x_0 \tag{1}$$

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where $x(\cdot) \in R^n$ and $i = 1, \ldots, N$. Such systems are of "variable structure" or "multi-modal"; they are a special case of hybrid systems.[1] For instance, the particular i at any given time may be chosen by some "higher process," such as a controller, computer, or human operator. It may also be a function of time or state or both. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear time-varying equation. However, one might gain some leverage in the analysis of such systems by considering them to be amalgams of simpler systems. We add the assumptions that (1) each F_i is globally Lipschitz continuous and (2) the i's are picked in such a way that we have finite switches in finite time.

In this paper we study *continuous* switching systems. A *continuous switching system* is a switching system with the additional constraint that the switched subsystems agree at the switching time. More specifically, consider Equation (1) and suppose that at times t_j, $j = 1, 2, 3, \ldots$, there is a switch from $F_{k_{j-1}}$ to F_{k_j}. Then we require $F_{k_{j-1}}(x(t_j), t_j) = F_{k_j}(x(t_j), t_j)$. That is, we require that the vector field is continuous over time.

This constraint leads to a simpler class of systems to consider. At the same time, it is not overly restrictive since many switching systems naturally satisfy this constraint. Indeed they may even arise from the discontinuous logic present in hybrid systems. For example, we might have an aircraft where some surface, say the elevator, controls the motion. But this elevator is in turn a controlled surface, whose desired action is chosen by a digital computer that makes some logical decisions. Based on these decisions, the computer changes elevator inputs (say current to its motor) in an effectively discontinuous manner. However, the elevator angle and angular velocity do not change discontinuously. Thus, from the aircraft's point of view (namely, at the level of dynamics relevant to it), there are continuous switchings among regimes of elevator behavior. Therefore, continuous switching systems arise naturally from abstract hybrid systems acting on real objects. In this paper we develop tools that are useful in analyzing these situations.

Another problem arises in examples like the one we just introduced: the problem of unmodeled dynamics. Suppose the pilot, from some quiescent operating point, decides to invoke hard upward normal acceleration. The elevator starts swinging upward until it is swinging upward at maximum angular velocity (in an effort track the violent maneuver requested by the pilot). Then, some higher process makes a calculation and decides that continuing this command would result in an unsafe configuration (say attack angle beyond critical). It decides to begin braking the elevator motor immediately to avoid this situation. In this case, the *desired* angular velocity profile of the elevator (over the whole move) is most probably trapezoidal. However, the elevator is a dynamic element that can't track that desired profile exactly. We may want to know how taking these unmodeled dynamics into account affects our already modeled dynamics. We may also want to know how high our control gains should be to track within a certain error. We develop some tools that allow us to answer both these questions.

[1]Hybrid systems are those that inherently combine logical and continuous processes, *e.g.*, coupled finite automata and ODEs.

1.2 Overview

In the first part of the paper (Section 2), we develop general tools for analyzing continuous switching systems. For instance, we prove an extension of Bendixson's Theorem to the case of Lipschitz continuous vector fields. This gives us a tool for analyzing the existence of limit cycles of continuous switching systems. We also prove a lemma dealing with the continuity of differential equations with respect to perturbations that preserve a linear part. Colloquially, this lemma demonstrates the robustness of ODEs with a linear part. For purpose of discussion, we call it the *Linear Robustness Lemma.* This lemma is useful in easily deriving some of the common robustness results of nonlinear ODE theory (as given in, for instance, [2]). This lemma also becomes useful in studying singular perturbations if the fast dynamics are such that they maintain the corresponding algebraic equation to within a small deviation. We give some simple propositions that allow us to do this type of analysis.

The extension of Bendixson's Theorem and the Linear Robustness Lemma have uses beyond those explicitly espoused here and should be of general interest to systems theorists.

In the second part (Sections 3 and 4), we use the above tools to analyze some example continuous switching systems motivated by a realistic aircraft control problem. In Section 3, we present an example continuous switching control problem. This system is inspired from one used in the longitudinal control of modern aircraft such as the F-8 [15]. The control law uses a "logical" function (max) to pick between one of two stable controllers: the first a servo that tracks pilot inputs, the second a regulator about a fixed angle of attack. The desired effect of the total controller is to "track pilot inputs except when those inputs would cause the aircraft to exceed a maximum angle of attack." We analyze the stability of this hybrid system in the case where the pilot input is zero and the controllers are linear full-state feedback. We call this the max system. While the restriction to this case seems strong, one should note that the stability of such systems is typically verified only by extensive simulation [15]. In this paper, we use the tools discussed above to *prove* nontrivial statements about the controller's behavior. For example, we show that no limit cycles exist by applying our extension of Bendixson's Theorem. We also show that the family of linear full-state feedback max systems can be reduced to a simpler family via a change of basis and analysis of equilibria. Finally, we give a Lyapunov function that proves that all systems of this canonical form are globally asymptotically stable. The Lyapunov function itself has a logical component, and the proof that it diminishes along trajectories is split into logical cases.

In Section 4, we analyze a "simulation" of the max system. Specifically, we use a dynamic variable (output of a differential equation) instead of the output given by the max function directly. This corresponds to a dynamical smoothing or switching hysteresis, as we motivated above. By using our lemma on the robustness of linear ODEs, we conclude stability properties of this (singular perturbation) simulation from those of the original max system.

Finally, Section 5 presents a summary and conclusions. Appendix A collects the more tedious proofs. Appendix B treats the background, statement, and proof of our extension of Bendixson's Theorem.

1.3 Notation

Throughout this paper we use the notation of modern control theory [9, 14, 6, 3], ODEs [1, 5], and nonlinear systems analysis [17, 13].

2 Theory

In this section we summarize some general tools for the analysis of continuous switching systems. The first new tool we introduce is an extension of Bendixson's Theorem on the existence of limit cycles to the case of Lipschitz continuous vector fields. The second is a lemma we recently proved dealing with the robustness of ODE's with respect to perturbations that preserve a linear part. Finally, we prove some simple propositions that allow us to use the robustness lemma to analyze certain singular perturbation problems. Later, we apply each of these results to analyze our example problems.

2.1 Limit Cycles

Suppose we are interested in the existence of limit cycles of continuous switching systems in the plane. The traditional tool for such analysis is Bendixson's Theorem. But under our model, systems typically admit vector fields that are Lipschitz, with no other smoothness assumptions. Bendixson's Theorem, as it is traditionally stated (*e.g.*, [4, 17]), requires continuously differentiable vector fields and is thus not of use in general. Therefore, we offer an extension of Bendixson's Theorem to the more general case of Lipschitz continuous vector fields. Its proof is based on results in geometric measure theory (which are discussed in Appendix B).

Theorem 1 (Extension of Bendixson's Theorem) *Suppose D is a simply connected domain in R^2 and $f(x)$ is a Lipschitz continuous vector field on D such that the quantity $\nabla f(x)$ (the* divergence *of f, which exists almost everywhere) defined by*

$$\nabla f(x) = \frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2)$$

is not zero almost everywhere over any subregion of D and is of the same sign almost everywhere in D. Then D contains no closed trajectories of

$$\begin{aligned} \dot{x}_1(t) &= f_1[x_1(t), x_2(t)] \\ \dot{x}_2(t) &= f_2[x_1(t), x_2(t)] \end{aligned}$$

Proof Similar to that of Bendixson's Theorem [17, pp. 31–32] after using an extension of the divergence theorem known as the Gauss-Green-Federer Theorem [10, pp. 114–115]. (See Appendix B.) □

2.2 Robustness of ODEs

In this subsection, we summarize some results that show the robustness of solutions of ordinary differential equations with respect to perturbations of the vector field. First, we give and prove a basic lemma in ODE theory that demonstrates robustness of solutions to arbitrary perturbations. Then, we consider perturbations that preserve a linear part. This allows us to obtain more useful bounds. We call the result the Linear Robustness Lemma.

The proofs of both lemmas depend critically on the well-known Bellman-Gronwall inequality [3, p. 252], which is reprinted in Appendix A for convenience. The first is a basic lemma in ODE theory that was given without proof in [18]. It is useful as a comparison with our new result, Lemma 3. For completeness, we furnish a proof in Appendix A.[2]

Lemma 2 *Given*

$$\begin{aligned} \dot{x} &= F(x,t), \qquad x(0) = x_0 \\ \dot{y} &= G(y,t), \qquad y(0) = x_0 \end{aligned}$$

Suppose that F is globally Lipschitz continuous and "close to G," i.e.,

$$\begin{aligned} \|F(x,t) - F(y,t)\| &\le L\|x-y\|, \qquad \text{for all } x,y,t \\ \|F(x,t) - G(x,t)\| &\le \epsilon, \qquad \text{for all } x,t \end{aligned}$$

Then if $L \neq 0$

$$\|x(t) - y(t)\| \le \frac{\epsilon}{L}\left(e^{Lt} - 1\right), \qquad \text{for all } t \ge 0$$

If $L = 0$, then $\|x(t) - y(t)\| \le \epsilon t$.

Proof (See Appendix A.) □

The problem with this result is that (except in the trivial case) $L > 0$, so the bound diverges exponentially. Thus it is not useful in deducing stability of a nearby system, nor in examining robustness of a well-behaved model to perturbations in the vector field. There are some tools for this in the literature, under the heading "stability under persistent disturbances." For example, [11, p. 72] gives a local result. We are more interested in what one can say globally. Along these lines we consider perturbations that preserve a well-defined portion of the dynamics, a linear part. Here is our main result:

Lemma 3 (Linear Robustness Lemma) *Given*

$$\begin{aligned} \dot{x} &= Ax + F(x,t), \qquad x(0) = x_0 \\ \dot{y} &= Ay + G(y,t), \qquad y(0) = x_0 \end{aligned}$$

Suppose that F is globally Lipschitz continuous and "close to G," i.e.,

$$\begin{aligned} \|F(x,t) - F(y,t)\| &\le L\|x-y\|, \qquad \text{for all } x,y,t \\ \|F(x,t) - G(x,t)\| &\le \epsilon, \qquad \text{for all } x,t \end{aligned}$$

Then

$$\|x(t) - y(t)\| \le \frac{\epsilon c}{\eta + cL}\left(e^{(\eta + cL)t} - 1\right), \qquad \text{for all } t \ge 0$$

when

$$\|e^{At}\|_i \le c e^{\eta t} \tag{2}$$

where $\|\cdot\|_i$ is the induced norm associated with the norm $\|\cdot\|$ and $c \ge 1$, $\eta + cL \neq 0$, $\eta \neq 0$, and $L > 0$.

[2] Proofs that interrupt the flow of discussion or do not use novel techniques are relegated to Appendix A.

Proof (See Appendix A.) □

Corollary 4 *In some special cases not covered above we have:*

1. *If* $L = 0$ *but* $\eta \neq 0$, *then*

$$\|x(t) - y(t)\| \leq \frac{\epsilon c}{\eta}\left(e^{\eta t} - 1\right)$$

2. *If* $\eta = 0$ *and* $L = 0$, *then*

$$\|x(t) - y(t)\| \leq c\epsilon t$$

3. *If* $\eta = 0$ *but* $L > 0$, *then*

$$\|x(t) - y(t)\| \leq \frac{\epsilon}{L}\left(e^{cLt} - 1\right)$$

4. *If* $\eta \neq 0$ *and* $L > 0$ *but* $\eta + cL = 0$ *(this means* $\eta < 0$*), then*

$$\|x(t) - y(t)\| \leq \frac{\epsilon c}{(-\eta)}\left[cLt + e^{-cLT} - e^{\eta t}\right]$$

Proof (See Appendix A.) □

The similarity of Lemmas 2 and 3 is easy to see. Their proofs are also similar. The most important distinction arises when A is stable and η can be chosen negative. Indeed, if $\eta + cL < 0$, then we can guarantee nondivergence of the solutions.

The proof can easily be extended to the case where A is time-varying:

Corollary 5 *Lemma 3 and Corollary 4 hold when A is time varying, with Equation (2) replaced by*

$$\|\Phi(t, s)\|_i \leq ce^{\eta(t-s)}$$

where $\Phi(t, s)$ is the transition matrix of the time-varying linear matrix $A(t)$.

Proof Proof is the same as that for Lemma 3, replacing $e^{A(t-s)}$ by $\Phi(t, s)$. □

Then, the case $L = 0$ subsumes some of the global results of stability under persistent disturbances, *e.g.*, [2, p. 167].

2.3 Singular Perturbations

The standard singular perturbation model is [7]

$$\dot{x} = f(x, z, \epsilon, t), \qquad x(t_0) = x_0, \qquad x \in R^n \tag{3}$$

$$\epsilon\dot{z} = g(x, z, \epsilon, t), \qquad z(t_0) = z_0, \qquad z \in R^m \tag{4}$$

in which the derivatives of some of the states are multiplied by a small positive scalar ϵ. When we set $\epsilon = 0$, the state-space dimension reduces from $n + m$ to n and the second differential equation degenerates into an algebraic equation. Thus, Equation (3) represents a reduced-order model, with the resulting parameter perturbation being "singular." The reason for this terminology is seen when we divide both sides of Equation (4) by ϵ and let it approach zero.

We make use of the simpler model

$$\begin{aligned}\dot{x} &= f(x,z,t), & x(t_0) = x_0, \quad x \in R^n \\ \dot{z} &= \alpha^2[g(x,t) - z], & z(t_0) = z_0, \quad z \in R^m\end{aligned}$$

where we have $\epsilon = 1/\alpha^2$, α a nonzero real number. With this rewriting, one sees why Equation (4) is said to represent the "fast transients," or fast dynamics. The following lemma shows explicitly a certain case where the dynamics can be made so fast that the resulting "tracking error" between $u(t) \equiv g(x(t),t)$ and $z(t)$ is kept small.

Lemma 6 *Let*

$$\dot{z}(t) = \alpha^2(u(t) - z(t))$$

where u is a Lipschitz continuous (with constant L) function of time. Given any $\epsilon > 0$, if

$$|z(0) - u(0)| = \epsilon_0 < \epsilon$$

we can choose α large enough so that

$$|z(t) - u(t)| < \epsilon, \qquad t \geq 0$$

Proof (See Appendix A.) □

The result can be extended to higher dimensions as follows:

Lemma 7 *Let*

$$\dot{z}(t) = \alpha^2 A(u(t) - z(t))$$

where u and z are elements of R^n and $A \in R^{n \times n}$. Further, assume that A is positive definite and that each coordinate of u is a Lipschitz continuous (with constant L) function of time. Given any $\epsilon > 0$, if

$$\|z(0) - u(0)\| = \epsilon_0 < \epsilon$$

we can choose α large enough so that

$$\|z(t) - u(t)\| < \epsilon, \qquad t \geq 0$$

Proof Similar to the proof of Lemma 6. [Hint: consider the time derivative of $e^T e$, $e = (z - u)$, and use equivalence of norms on R^n.] □

These lemmas allow us to use the robustness lemmas of the previous section to analyze certain singular perturbation problems. The idea of the preceding lemmas is that the fast dynamics are such that they maintain the corresponding algebraic equation, $z(t) = u(t)$, to within a small deviation (cf. invariant manifolds [7, p. 18]).

3 Example 1: Max System

As an example of a switched system, we consider a problem combining logic in a continuous control system. Specifically, we start with the system

$$\Sigma: \quad \frac{d}{dt}\begin{bmatrix} q \\ \alpha \end{bmatrix} = \begin{bmatrix} -1 & -10 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} -1 \\ 0.1 \end{bmatrix}\delta$$

$$\begin{bmatrix} \alpha \\ n_z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -300 \end{bmatrix}\begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 30 \end{bmatrix}\delta$$

or, symbolically,

$$\begin{aligned}\dot{x} &= Ax + B\delta \\ \alpha &= C_1 x + D_1 \delta = C_1 x \\ n_z &= C_2 x + D_2 \delta\end{aligned}$$

These equations arise from the longitudinal dynamics of an aircraft (see Figure 1) with reasonable values chosen for the physical parameters. The variable θ is the pitch angle and α is the angle of attack. The input command δ is the angle of the elevator. The normal acceleration, n_z, is the output variable which we would like to track, *i.e.*, we assume that the pilot requests desired values of n_z with his control stick. As a constraint, the output variable α must have a value not much larger than α_{lim} (for the model to be valid and the plane to be controlled adequately). A successful controller would satisfy both of these objectives simultaneously to the extent that this is possible: we desire good tracking of the pilot's input without violating the constraint $\alpha \leq \alpha_{\text{lim}} + \epsilon$, for $\epsilon \geq 0$ some safety margin.

Now, suppose that two controllers, K_1 and K_2, have been designed to output δ_1 and δ_2 such that (1) Σ is regulated about $\alpha = \alpha_{\text{lim}}$ when $\delta = \delta_1$; and (2) Σ tracks command r—the pilot's desired n_z—when $\delta = \delta_2$, respectively. Finally, suppose that we add the following logical block: $\delta = \max(\delta_1, \delta_2)$. Control configurations much like this (see Figure 2) have been used to satisfy the objectives of our aircraft control problem [15].

To our knowledge, the stability of such systems has only been probed via extensive simulation [15]. In the remainder of this section, we examine the stability of certain cases of this control system. First, we limit ourselves to the case where both controllers are implemented with full-state feedback. We discuss the well-posedness of the system and show an example output run. Next, we consider the equilibrium points of the system and their stability in the case where the pilot's reference input (desired normal acceleration) is clamped to zero. More practically, we answer the question, What is the behavior of this control system if the pilot lets go of the control stick?

3.1 Preliminary Analysis of the Example System

First note that in our example system, the pair (A, B) is controllable. To make some headway, we restrict ourselves to the special case where the controllers K_1 and K_2 take the form of full-state feedback plus an offset term (for nonzero outputs):

$$\begin{aligned}\delta_1 &= -Fx + [C_1(-A + BF)^{-1}B]^{-1}\alpha_{\text{lim}} \\ \delta_2 &= -Gx + [(C_2 - D_2 G)(-A + BG)^{-1}B + D_2]^{-1} r\end{aligned}$$

For convenience, we let

$$\begin{aligned}k_1 &= [C_1(-A + BF)^{-1}B]^{-1} \\ k_2 &= [(C_2 - D_2 G)(-A + BG)^{-1}B + D_2]^{-1}\end{aligned}$$

Such constants generally need not exist. However, for our system we are keenly interested in the existence of k_1.[3] We have the following

[3]We assume k_2 exists since it does not affect our later analysis, which is in the case $r = 0$.

Fact 8 *The constant k_1 is guaranteed to exist for our system whenever F is chosen such that $(A - BF)$ is stable.*[4]

Proof (See Appendix A.) □

Thus, the resulting max control law exists. It is simply

$$\delta = \max(-Fx + k_1\alpha_{\text{lim}}, -Gx + k_2 r) \tag{5}$$

To get a feel for how the example system behaves, we have included some simulations. Figure 3 shows an example run of the just the tracking portion of the control system ($\alpha_{\text{lim}} = 0.6$, F chosen to place the closed-loop poles at -6 and -7). Part (a) shows normal acceleration tracks the desired trajectory well; (b) shows the α_{lim} constraint is violated to achieve this tracking.

Figure 4 shows the outputs when the full max control system is activated (with both $F = G$ chosen as F above). One easily sees that the controller acts as expected: it tracks the desired command well, except in that portion where tracking the command requires that the α_{lim} constraint be violated. In this portion, α is kept close to its constraint value (the maximum value of α in this simulation run was 0.6092).

3.2 Analysis for the Case $r \equiv 0$

The first thing we do is examine stability of Σ using the max control law in the case where $r \equiv 0$. The closed-loop system equations are then

$$\begin{aligned} \dot{x} &= Ax + B\max(-Fx + k_1\alpha_{\text{lim}}, -Gx) \\ &= (A - BG)x + B\max((G - F) + k_1\alpha_{\text{lim}}, 0) \end{aligned}$$

In our analysis below, we suppose that we have done a reasonable job in designing the feedback controls F and G. That is, we assume $(A - BF)$ and $(A - BG)$ are stable. This is possible because (A, B) controllable.

Now, recall that (A, B) controllable implies that $(A - BG, B)$ is controllable. Thus, it suffices to analyze the following equation, which we call the max system:

Definition 9 (Max System)

$$\Sigma_{\text{max}}: \qquad \dot{z} = Az + B\max(Fz + \gamma, 0)$$

where A and $A + BF$ are stable and (A, B) is controllable and $\gamma = k_1\alpha_{\text{lim}}$.

To fix ideas, let's look at simulation results. Figure 5 shows a max system trajectory with

$$A = \begin{bmatrix} -0.1 & 1 \\ -1 & -0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad F = \begin{bmatrix} -9 & 0 \end{bmatrix}, \qquad \gamma = -1$$

Figure 7[5] shows the trajectory resulting from the same initial condition for the system $\dot{x} = Ax$; Figure 8 for the system $\dot{x} = (A + BF)x$. Both component systems are stable.

[4] We say a matrix is stable when all its eigenvalues are strictly in the left-half plane.

[5] We intentionally skipped a figure to allow better figure placement for comparison.

To simplify analysis of the max system, we can make a change of basis ($x = Pz$), yielding

$$\begin{aligned}\dot{x} &= P\dot{z} \\ &= PAz + PB\max(Fz + \gamma, 0) \\ &= PAP^{-1}x + PB\max(FP^{-1}x + \gamma, 0)\end{aligned}$$

where P is any nonsingular matrix. In particular, P can be chosen so that the matrices PAP^{-1} and PB are in the so-called controller canonical form:

$$PAP^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \quad (6)$$

$$PB = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (7)$$

Note that $a_i > 0$ since PAP^{-1} is a stable matrix. Renaming matrices again, we have

Fact 10 *The max system $\Sigma_{\max}$ can be reduced to the system:*

$$\dot{x} = Ax + B\max(Fx + \gamma, 0)$$

where A and $A + BF$ are stable, (A, B) is controllable, and the matrices A and B are in controller canonical form.

We can do one more thing to simplify the max system just derived: expand the equations using the fact that A and B have controller canonical form. Doing this—and some equilibrium point analysis—we obtain

Remark 11 (Canonical Max System) *1. The max system can be reduced to the following canonical form:*

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -ax - by + \max(fx + gy + \gamma, 0)\end{aligned}$$

where a, b, $a - f$, and $b - g$ are greater than zero.

2. Further, without loss of generality, we may assume that $\gamma \leq 0$, in which case the only equilibrium point of this system is the origin.

Proof The first part is a straightforward calculation, with the inequalities on the constants arising from the assumed stability of A and $A + BF$.

Now, let's analyze the equilibrium points of the canonical max system. The relevant equations are $y = 0$ and $ax = \max(fx + \gamma, 0)$. This second one must be analyzed in two cases:

$$\begin{aligned} ax = 0, &\quad fx + \gamma \leq 0 \\ ax = fx + \gamma, &\quad fx + \gamma > 0\end{aligned}$$

Thus, $(0, 0)$ is an equilibrium point if and only if $\gamma \leq 0$; $(\gamma/(a - f), 0)$ is an equilibrium point if and only if

$$\begin{aligned}\frac{f}{a-f}\gamma + \gamma &> 0 \\ \frac{a}{a-f}\gamma &> 0 \\ \gamma &> 0\end{aligned}$$

where the last line follows from a and $a-f$ greater than zero. Therefore, the canonical max system has exactly one equilibrium point.

Finally, if $\gamma > 0$, changing coordinates to $z = x - \gamma/(a-f)$ yields $\dot{z} = y$ and

$$\begin{aligned}
\dot{y} &= -a\left(z + \frac{\gamma}{a-f}\right) - by + \max\left(f\left(z + \frac{\gamma}{a-f}\right) + gy + \gamma, 0\right) \\
&= -az - by - \frac{a\gamma}{a-f} + \max\left(fz + gy + \frac{a\gamma}{a-f}, 0\right) \\
&= -az - by + fz + gy + \max\left(0, -fz - gy - \frac{a\gamma}{a-f}\right) \\
&= -(a-f)z - (b-g)y + \max\left((-f)z + (-g)y + \left(-\frac{a\gamma}{a-f}\right), 0\right)
\end{aligned}$$

Now introducing new variables for the constants in parentheses, we obtain

$$\begin{aligned}
\dot{z} &= y \\
\dot{y} &= -\tilde{a}z - \tilde{b}y + \max(\tilde{f}z + \tilde{g}y + \tilde{\gamma}, 0)
\end{aligned}$$

It is easy to check that the new variables satisfy the inequalities of the canonical form. Further, we have $\tilde{\gamma} < 0$, and thus $(0,0)$ the only equilibrium. □

Next, note that this is equivalent to the second-order system:

$$\ddot{x} = -ax - b\dot{x} + \max(fx + g\dot{x} + \gamma, 0)$$

which we use below.

We have the following global results for the max system in the case where the reference input, r, is zero:

1. Limit cycles don't exist. Our max system consists of a logical (though Lipschitz continuous) switching between two stable linear systems, both of which admit negative divergence in their respective regions. Therefore, by Theorem 1, no limit cycles can exist.

2. The system is globally asymptotically stable. The proof is detailed below.

To prove global asymptotic stability, we first show

Remark 12 *The following is a Lyapunov function for the canonical max system:*

$$\begin{aligned}
V &= \frac{1}{2}\dot{x}^2 + \int_0^x [a\xi - \max(f\xi + \gamma, 0)]d\xi \\
&\equiv \frac{1}{2}\dot{x}^2 + \int_0^x c(\xi)d\xi
\end{aligned}$$

Proof The proof has two major parts: (i) V is a positive definite (p.d.) function, and (ii) $\dot{V} \leq 0$.

(i) To show that V is a p.d. function, it is enough to show that $xc(x) > 0$ when $x \neq 0$ and $c(0) = 0$. The second fact follows from $\gamma \leq 0$. Computing

$$\begin{aligned}
xc(x) &= ax^2 - x\max(fx + \gamma, 0) \\
&= \begin{cases} ax^2, & fx + \gamma \leq 0 \\ ax^2 - fx^2 - \gamma x, & fx + \gamma > 0 \end{cases}
\end{aligned}$$

That the desired condition holds in the first case follows immediately from $a > 0$. For the second case, we consider

1. $x > 0$:

$$ax^2 - fx^2 - \gamma x = (a-f)x^2 + (-\gamma)x > 0 + 0 = 0$$

2. $x < 0$:

$$ax^2 - fx^2 - \gamma x = ax^2 + (-x)(fx+\gamma) > 0 + 0 = 0$$

Thus V is a p.d. function.

(ii) Next, we wish to show that $\dot{V} \leq 0$. To that end, we compute

$$\begin{aligned} \dot{V} &= \dot{x}\ddot{x} + c(x)\dot{x} \\ &= \dot{x}[-ax - b\dot{x} + \max(fx + g\dot{x} + \gamma, 0)] + ax\dot{x} - \max(fx+\gamma,0)\dot{x} \\ &= -b\dot{x}^2 + \dot{x}\max(fx+g\dot{x}+\gamma,0) - \dot{x}\max(fx+\gamma,0) \end{aligned}$$

Now, there are four cases to be dealt with:

1. If $fx + g\dot{x} + \gamma \leq 0$ and $fx + \gamma \leq 0$, then $\dot{V} = -b\dot{x}^2 \leq 0$.
2. If $fx + g\dot{x} + \gamma > 0$ and $fx + \gamma > 0$, then $\dot{V} = -(b-g)\dot{x}^2 \leq 0$.
3. If $fx + g\dot{x} + \gamma \leq 0$ and $fx + \gamma > 0$, then

$$\dot{V} = -b\dot{x}^2 - \dot{x}(fx+\gamma)$$

If $\dot{x} \geq 0$, then $\dot{V} \leq 0$. If $\dot{x} < 0$, then, using $(b-g) > 0$, we obtain

$$\begin{aligned} b &> g \\ b\dot{x} &< g\dot{x} \\ fx+\gamma+b\dot{x} &< fx+\gamma+g\dot{x} \\ fx+\gamma+b\dot{x} &< 0 \\ -\dot{x}[fx+\gamma+b\dot{x}] &< 0 \\ \dot{V} &< 0 \end{aligned}$$

4. If $fx + g\dot{x} + \gamma > 0$ and $fx + \gamma \leq 0$, then

$$\dot{V} = -b\dot{x}^2 + \dot{x}(fx + \dot{x}g + \gamma)$$

If $\dot{x} \leq 0$, then $\dot{V} \leq 0$. If $\dot{x} > 0$,

$$\dot{V} = -(b-g)\dot{x}^2 + \dot{x}(fx+\gamma) \leq 0$$

□

Global asymptotic stability results from the facts that (1) the origin is the only invariant set for which $\dot{V} = 0$ and (2) $V(x) \to \infty$ as $\|x\| \to \infty$ [13].

4 Example 2: The Simulated Max System

In this section we analyze a variant of the max system introduced in Section 3. Specifically, recall that the max system can be reduced to the canonical form of Remark 11:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ax - by + \max(fx + gy + \gamma, 0) \end{aligned}$$

where a, b, $a - f$, and $b - g$ are greater than zero and $\gamma \leq 0$. It was shown in Section 3 that the only equilibrium point of this system is the origin, which is globally asymptotically stable.

In the *simulated* max system (i.e., using a differential equation for the max function) we have the following system of equations (cf. Definition 9):

Definition 13 (Simulated Max System)

$$\begin{aligned} \dot{x} &= Ax + B\delta \\ \dot{\delta} &= \alpha^2[\max(Fx + \gamma, 0) - \delta] \end{aligned}$$

where A and $A + BF$ are stable and (A, B) is controllable. Also, $\gamma = k_1 \alpha_{\text{lim}}$ and $\alpha \neq 0$.

This equation represents a smoothing of the max function's output; it provides a type of hysteresis that smooths transitions. Note that this equation represents a singular perturbation of the original max system. It can be used to model the fact that the elevator angle does not exactly track the desired control trajectory specified by the max function. To compare the max and simulated systems, consider Figure 6. This figure shows the simulation of the max system trajectory of Figure 5 with $\alpha^2 = 16$ and $\delta(0) = \max(Fx(0) + \gamma, 0)$. Note that, compared with the original max system, the switching is "delayed" and the trajectories are smoother, as expected.

By changing basis with the matrix $T = \text{blockdiag}\{P, 1\}$, where P is chosen so that the matrices PAP^{-1} and PB are in the so-called controller canonical form (see Equations (6) and (7)), we obtain

Definition 14 (Canonical Simulated Max System)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ax - by + \delta \\ \dot{\delta} &= \alpha^2[\max(fx + gy + \gamma, 0) - \delta] \end{aligned}$$

subject to initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad \delta(0) = \max(fx_0 + gy_0 + \gamma, 0)$$

where a, b, $a - f$, and $b - g$, are greater than zero; $\gamma \leq 0$; and $\alpha \neq 0$.

This is the system we will study in the remainder of this section. Note the added constraint on the initial condition of δ.[6]

Remark 15 *The only equilibrium point of this system is the origin, which is locally asymptotically stable (when $\gamma < 0$).*

Proof From the first two equations, we have the constraints $y = 0$ and $\delta = ax$. From the last one we obtain the following two cases:

1. $fx + \gamma \leq 0$: $-ax = 0$, which implies $x = \delta = 0$.

[6] The constraint on $\delta(0)$ is for convenience. It can be relaxed to be within ϵ of this value (where ϵ arises in our proofs below), with the same analytical results holding true. Specifically, Fact 16 still holds when Equation (8) is replaced by $|\delta(0) - \max(fx(0) + gy(0) + \gamma, 0)| < \epsilon$.

2. $fx + \gamma > 0$: $(-a + f)x + \gamma = 0$, which implies $x = \gamma/(a - f)$. However, this can't occur since

$$fx + \gamma = \frac{f\gamma}{(a-f)} + \gamma = \frac{a\gamma}{(a-f)} \leq 0$$

The origin is locally asymptotically stable because it is a linear system in some neighborhood of the origin (since $\gamma < 0$). □

This system is globally asymptotically stable when $f = g = 0$, because it reduces to a stable linear system in this case. For the special case where $\gamma = 0$, both component linear systems can be made stable by choosing α large enough. However, this in itself does not imply that the whole system is stable. We say more about this case at the end of the section.

The rest of this section explores the stability of the simulated max system by using Lemma 3.

4.1 Asymptotic Stability within Arbitrary Compact Sets

In this subsection we show that the simulated max system can be made asymptotically stable to the origin—within an arbitrary compact set containing the origin—by choosing the parameter α large enough. **Important note**: Since δ is subject to initial conditions depending on x and y (see Definition 13), this stability is with respect to arbitrary sets of initial conditions for x **and** y **only**. This subsection only considers the case $\gamma < 0$. In this case, the plane $fx + gy + \gamma = 0$ is a positive distance, call it d, away from the origin. Further, the three-dimensional linear system associated with the simulated max system about the origin with matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ -a & -b & 1 \\ 0 & 0 & -\alpha^2 \end{bmatrix}$$

is asymptotically stable, so there is some (open) ball around the origin (in R^3) of radius $\Delta_s \leq d$ such that once a trajectory of the simulated max system enters the ball of radius Δ_s, it tends toward the origin. Similarly, the max system is an asymptotically stable linear system near the origin, so there is some ball around the origin (in R^2) of radius $\Delta_m \leq d$ such that once a solution to the max system enters the ball of radius Δ_m, it tends towards the origin. For convenience, define $\Delta = \min(\Delta_m, \Delta_s)$.

Now, note that the max system and simulated max system can be written in the form required by Lemma 3 by choosing

$$A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

i.e.,

$$\begin{aligned} F(x, y, t) &= \max(fx + gy + \gamma, 0) \\ G(x, y, t) &= \delta \end{aligned}$$

where $\dot{\delta} = \alpha^2[\max(fx + gy + \gamma, 0) - \delta]$. An important fact is the following:

Fact 16 *Given*

$$\delta(0) = \max(fx(0) + gy(0) + \gamma, 0) \tag{8}$$

and $\epsilon > 0$, *we can choose* α *large enough so that*

$$|\delta(t) - \max(fx(t) + gy(t) + \gamma, 0)| < \epsilon, \qquad t \geq 0$$

Proof Since $\max(fx + gy + \gamma, 0)$ is Lipschitz continuous, we can apply Lemma 6 with $u(\cdot) \equiv \max(fx(\cdot) + gy(\cdot) + \gamma, 0)$, $x(\cdot) \equiv \delta(\cdot)$, and $\epsilon_0 = 0$. □

Below, let $\mu(t)$ and $\sigma(t)$ represent solutions to the max and simulated max systems, respectively. Next, consider the projection operator

$$\begin{aligned} \pi : R^3 &\rightarrow R^2 \\ \pi\left([x, y, \delta]^T\right) &= [x, y]^T \end{aligned}$$

Remark 17 *If* $\gamma < 0$, *the simulated max system can be made asymptotically stable to the origin within an arbitrary compact set containing it by choosing the parameter* α *large enough.*

Proof First, pick any compact set, Ω, containing the origin (of the max system). Next, we examine the trajectories of the max and simulated max systems from an arbitrary initial condition, $\mu_0 \in \Omega$. Recall that δ_0, and hence σ_0, is completely determined by μ_0. In particular, $\delta_0 = \max(fx_0 + gy_0 + \gamma, 0)$ and $\pi(\sigma_0) = \mu_0$.

Since the max system is globally asymptotically stable, there is a time, $T(\mu_0, \Delta)$, such that for $t > T$, we have $\|\mu(t)\| < \Delta/3$. Thus, we have $\max(fx(t)+gy(t)+\gamma, 0) = 0$ for all $t > T$. Now, according to Fact 16 we can pick α large enough so that

$$\epsilon < \min\left(\frac{\Delta}{3}, \frac{\Delta(\eta + cL)}{3c\left(e^{(\eta+cL)T} - 1\right)}\right) \tag{9}$$

At this point, we have, from Lemma 3,

$$\|\mu(T) - \pi(\sigma(T))\| \leq \frac{\epsilon c}{\eta + cL}\left(e^{(\eta+cL)T} - 1\right) \leq \frac{\Delta}{3}$$

Now, by construction we have $|\delta| < \Delta/3$. Thus, we have $\|\sigma(T)\| < \Delta$. From this point on, $\sigma(t)$ tends asymptotically toward the origin.

Finally, since Ω is compact, there is a finite time $\tau \geq T(\mu_0, \Delta)$ for all $\mu_0 \in \Omega$. Thus, we can pick ϵ (and then α) to achieve the desired inequality for all initial conditions. □

Note that if $\eta + cL < 0$, then ϵ—and hence α—can be chosen constant for all T. On the other hand, if $\eta + cL > 0$, restrictions on the magnitude of α may only guarantee asymptotic stability within some finite distance from the origin.

It is also important to realize that the same analysis holds for any other dynamic or nondynamic continuous variable used to approximate the max function, if it is such that it can be kept within ϵ of the max function for arbitrary ϵ. (Also recall Footnote 6.)

4.2 The Case $\gamma = 0$

For the special case $\gamma = 0$, the simulated max system represents a switching between the following component linear systems:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -a & -b & 1 \\ 0 & 0 & -\alpha^2 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -a & -b & 1 \\ f\alpha^2 & g\alpha^2 & -\alpha^2 \end{bmatrix}$$

Remark 18 *Both component linear systems can be made stable by choosing α large enough.*

Proof (See Appendix A.) □

Thus the component linear systems of the simulated max system with $\gamma = 0$ can be chosen so both are stable. However, this in itself does not imply that the whole system is stable.[7]

The comparison arguments of the previous subsection do not apply now since we cannot find a Δ like we did there. Thus, we can only use Lemma 3 to get bounds on the trajectories of the simulated max system. Note, however, that if $\eta + cL < 0$ then global asymptotic stability of the max system implies ultimate boundedness of the simulated max system.

One may be able to say more about specific instances of the simulated max system (*i.e.*, knowledge of the constants). For example, some cases may yield to our robustness of linear ODEs lemma by comparing the case $\gamma = 0$ with $\gamma = -\epsilon$. Alternatively, one could invoke robustness results in the literature, *e.g.*, [2, Theorem 6.1]. These tools can't be invoked in the general case because, roughly, the parameter α affects both η and L in conflicting fashion.

5 Summary and Conclusions

This paper detailed work on continuous switching systems. Although one might not guess from the presentation above, the theory in Section 2 was developed to analyze the example systems of Sections 3 and 4. Here, we set the record straight.

In Section 3, we presented an example hybrid control problem: the max system. This system was inspired from one used in the control of modern aircraft. The control law uses a logical function (max) to pick between one of two stable controllers: one a servo that tracks pilot inputs, the second a regulator about a fixed angle of attack. Typically, engineers resort to extensive simulation of even such simple systems because the analysis is too hard with their present toolbox. However, we analyzed the stability of this hybrid system in the case where the pilot input is zero and the controllers are linear full-state feedback. We showed that no limit cycles exist by proving and applying an extension of Bendixson's Theorem to the case of Lipschitz continuous vector fields;

[7] A counterexample can be constructed after one that appears in [16]: Use the asymptotically stable systems of Figures 7 (System I) and 8 (System II), activating System I in quadrants 4 and 2, System II in quadrants 3 and 1.

we also gave a Lyapunov function that proved all systems of this form are globally asymptotically stable. Interestingly, the Lyapunov equation used a logical switching.

In Section 4, we presented an analysis of a simulation of the max system. That is, we used a differential equation to obtain a "smooth" function instead of using the output given by the max function directly. By extending a result in the theory of continuity of solutions of ordinary differential equations, we proved stability properties of the simulation from those of the original max system.

Although the attention was focused by our examples and their analysis, we developed tools and insights along the way that should prove useful in tackling general hybrid systems problems.

A Assorted Proofs

A.1 Continuity Lemmas

The proofs of our continuity lemmas depend critically on the well-known Bellman-Gronwall inequality [3, p. 252]:

Lemma 19 (Bellman-Gronwall) *Let*

1. *f, g, k; $R_+ \to R$ and locally integrable;*
2. *$g \geq 0$, $k \geq 0$;*
3. *$g \in L_e^\infty$;*
4. *gk is locally integrable on R_+.*

Under these conditions, if $u : R_+ \to R$ satisfies

$$u(t) \leq f(t) + g(t) \int_0^t k(\tau)u(\tau)d\tau, \qquad \text{for all } t \in R_+$$

then

$$u(t) \leq f(t) + g(t) \int_0^t k(\tau)f(\tau) \left[\exp \int_\tau^t k(\sigma)g(\sigma)d\sigma\right] d\tau, \qquad \text{for all } t \in R_+$$

Proof [of Lemma 2] For any $t \geq 0$,

$$\begin{aligned} x(t) &= x_0 + \int_0^t F(x,\tau)d\tau \\ y(t) &= x_0 + \int_0^t G(y,\tau)d\tau \end{aligned}$$

Subtracting yields

$$\begin{aligned} x(t) - y(t) &= \int_0^t F(x,\tau)d\tau - \int_0^t G(y,\tau)d\tau \\ &= \int_0^t [F(x,\tau) - F(y,\tau) + F(y,\tau) - G(y,\tau)]d\tau \\ \|x(t) - y(t)\| &= \left\| \int_0^t [F(x,\tau) - F(y,\tau) + F(y,\tau) - G(y,\tau)]d\tau \right\| \\ &\leq \int_0^t \|F(x,\tau) - F(y,\tau)\|d\tau + \int_0^t \|F(y,\tau) - G(y,\tau)\|d\tau \\ &\leq L \int_0^t \|x(\tau) - y(\tau)\|d\tau + \epsilon t \end{aligned}$$

Using the Bellman-Gronwall Lemma, we obtain

$$\begin{aligned}\|x(t)-y(t)\| &\le \epsilon t+\int_0^t L\epsilon\tau\left[\exp\int_\tau^t Ld\sigma\right]d\tau \\ &= \epsilon\left\{t+L\int_0^t \tau e^{L(t-\tau)}d\tau\right\} \\ &= \epsilon\left\{t+Le^{Lt}\int_0^t \tau e^{-L\tau}d\tau\right\}\end{aligned}$$

If $L=0$

$$\|x(t)-y(t)\|=\epsilon t$$

and if $L>0$, we compute

$$\begin{aligned}\int_0^t \tau e^{-L\tau}d\tau &= \left[\frac{e^{-L\tau}}{L^2}(-L\tau-1)\right]_0^t \\ &= -\frac{e^{-Lt}}{L^2}(Lt+1)+\frac{1}{L^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\|x(t)-y(t)\| &\le \epsilon\left\{t-t-\frac{1}{L}+\frac{1}{L}e^{Lt}\right\} \\ &= \frac{\epsilon}{L}\left(e^{Lt}-1\right)\end{aligned}$$

□

Proof [of Lemma 3] For any $t\ge 0$,

$$\begin{aligned}x(t) &= e^{At}x_0+\int_0^t e^{A(t-\tau)}F(x,\tau)d\tau \\ y(t) &= e^{At}x_0+\int_0^t e^{A(t-\tau)}G(y,\tau)d\tau\end{aligned}$$

Subtracting yields

$$\begin{aligned}x(t)-y(t) &= \int_0^t e^{A(t-\tau)}[F(x,\tau)-F(y,\tau)+F(y,\tau)-G(y,\tau)]d\tau \\ \|x(t)-y(t)\| &\le \int_0^t \|e^{A(t-\tau)}\|_i\|F(x,\tau)-F(y,\tau)\|d\tau+\int_0^t \|e^{A(t-\tau)}\|_i\|F(y,\tau)-G(y,\tau)\|d\tau \\ &\le L\int_0^t \|e^{A(t-\tau)}\|_i\|x(\tau)-y(\tau)\|d\tau+\epsilon\int_0^t \|e^{A(t-\tau)}\|_i d\tau\end{aligned}$$

Now we are given that $\|e^{As}\|_i\le ce^{\eta s}$ so that

$$\begin{aligned}\int_0^t \|e^{A(t-\tau)}\|_i d\tau &\le ce^{\eta t}\int_0^t e^{-\eta\tau}d\tau \\ &\le \frac{c}{\eta}e^{\eta t}\left(1-e^{-\eta t}\right) \\ &\le \frac{c}{\eta}\left(e^{\eta t}-1\right) \qquad (10)\end{aligned}$$

Therefore,

$$\|x(t) - y(t)\| \leq cLe^{\eta t} \int_0^t e^{-\eta\tau} \|x(\tau) - y(\tau)\| d\tau + \frac{\epsilon c}{\eta} \left(e^{\eta t} - 1\right) \tag{11}$$

Using the Bellman-Gronwall Lemma, we obtain

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \frac{\epsilon c}{\eta}\left(e^{\eta t} - 1\right) + cLe^{\eta t} \int_0^t e^{-\eta\tau} \frac{\epsilon c}{\eta} \left(e^{\eta\tau} - 1\right) \left[\exp \int_\tau^t e^{-\eta\sigma} cLe^{\eta\sigma} d\sigma\right] d\tau \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + cLe^{\eta t} \int_0^t \left(1 - e^{-\eta\tau}\right) e^{cL(t-\tau)} d\tau\right] \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + cLe^{\eta t} e^{cLt} \int_0^t \left(e^{-cL\tau} - e^{-(\eta+cL)\tau}\right) d\tau\right] \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + cLe^{(\eta+cL)t}\left(\frac{1}{cL} - \frac{1}{cL}e^{-cLt} + \frac{1}{\eta + cL}e^{-(\eta+cL)t} - \frac{1}{\eta+cL}\right)\right] \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + e^{(\eta+cL)t} - e^{\eta t} + \frac{cL}{\eta+cL} - \frac{cL}{\eta+cL}e^{(\eta+cL)t}\right] \\
&= \frac{\epsilon c}{\eta}\left[\frac{cL}{\eta+cL} - 1\right]\left[1 - e^{(\eta+cL)t}\right] \\
&= \frac{\epsilon c}{\eta}\left[\frac{-\eta}{\eta+cL}\right]\left[1 - e^{(\eta+cL)t}\right] \\
&= \frac{\epsilon c}{\eta+cL}\left[e^{(\eta+cL)t} - 1\right]
\end{aligned} \tag{12}$$

□

Proof [of Corollary 4] Now we deal some special cases not covered above:

1. If $L = 0$ but $\eta \neq 0$, then Equation (11) gives

$$\|x(t) - y(t)\| \leq \frac{\epsilon c}{\eta}\left(e^{\eta t} - 1\right)$$

If $\eta = 0$ then Equation (10) is replaced by

$$\int_0^t \|e^{A(t-\tau)}\|_i d\tau \leq ct \tag{13}$$

2. So, if $\eta = 0$ and $L = 0$

$$\|x(t) - y(t)\| \leq c\epsilon t$$

3. If $\eta = 0$ and $L > 0$ then Equation (11) is replaced by

$$\|x(t) - y(t)\| \leq cL \int_0^t \|x(\tau) - y(\tau)\| d\tau + \epsilon ct$$

in which case the Bellman-Gronwall Lemma gives

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \epsilon ct + cL \int_0^t \epsilon c\tau \left[\exp \int_\tau^t cL\, d\sigma\right] d\tau \\
&\leq \epsilon ct + \epsilon c^2 L \int_0^t \tau e^{cL(t-\tau)} d\tau \\
&\leq \epsilon ct + \epsilon c^2 L e^{cLt} \int_0^t \tau e^{cL\tau} d\tau
\end{aligned}$$

Now repeating the calculation of Equation (10) with cL identified with L:

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \epsilon ct + \epsilon c^2 L e^{cLt}\left[-\frac{e^{-cLt}}{(cL)^2}(Lt+1) + \frac{1}{(cL)^2}\right] \\
&\leq \epsilon ct - \epsilon ct - \frac{\epsilon}{L} + \frac{\epsilon}{L}e^{cLt} \\
&\leq \frac{\epsilon}{L}\left(e^{cLt} - 1\right)
\end{aligned}$$

4. If $\eta \neq 0$ and $L > 0$ but $\eta + cL = 0$ (this means $\eta < 0$), then Equation (12) and the further computations simplify to

$$\begin{aligned}
\|x(t) - y(t)\| &= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + cL\int_0^t \left(e^{-cL\tau} - 1\right) d\tau\right] \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + cL\left(\frac{1}{cL} - \frac{1}{cL}e^{-cLt} - t\right)\right] \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - 1 + 1 - e^{-cLt} - cLt\right] \\
&= \frac{\epsilon c}{\eta}\left[e^{\eta t} - e^{-cLt} - cLt\right]
\end{aligned}$$

□

A.2 Singular Perturbation Lemmas

Proof [of Lemma 6] Let $e = z - u$. Then

$$\begin{aligned}
\frac{d^+e}{dt} &= \frac{d^+z}{dt} - \frac{d^+u}{dt} \\
&= \alpha^2(u - z) - \frac{d^+u}{dt} \\
&= -\alpha^2 e - \frac{d^+u}{dt}
\end{aligned}$$

where

$$\frac{d^+z}{dt} = \lim_{h\to 0^+}\frac{z(t+h) - z(t)}{h}$$

Now, since u is Lipschitz, we have

$$\left|\frac{d^+u}{dt}\right| \leq L$$

Thus, if we choose α such that $\alpha^2 > L/\epsilon$, then when $e \geq \epsilon$, we have

$$\frac{d^+e}{dt} \leq -\alpha^2\epsilon + L < 0$$

Similarly, when $e \leq -\epsilon$, we have

$$\frac{d^+e}{dt} \geq \alpha^2\epsilon - L > 0$$

Thus, the set $|e| < \epsilon$ is an invariant set. □

A.3 Max System

Proof [of Fact 8] We first need the following theorem [8, Theorem 3.10]:

Theorem 20 (Kwakernaak and Sivan) *Consider the time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t), \end{aligned}$$

where z and u have the same dimensions. Consider any asymptotically stable time-invariant control law

$$u(t) = -Fx(t) + u'(t).$$

Let $H(s)$ be the open-loop transfer matrix

$$H(s) = D(sI - A)^{-1}B,$$

and $H_c(s)$ the closed-loop transfer matrix

$$H_c(s) = D(sI - A + BF)^{-1}B.$$

Then $H_c(0)$ is nonsingular and the controlled variable $z(t)$ can under steady-state conditions be maintained at any constant value z_0 by choosing

$$u'(t) = H_c^{-1}(0)z_0$$

if and only if $H(s)$ has a nonzero numerator polynomial that has no zeroes at the origin.

For our system, we have

$$C_1(sI - A)^{-1}B = \frac{0.1(s-9)}{s^2 + 2s + 11}$$

which has a nonzero numerator with no zeroes at the origin. □

A.4 Simulated Max System

Proof [of Remark 18] A_1 is stable because it is upper block triangular with stable blocks. One can check that the characteristic polynomial of A_2 is

$$\lambda^3 + (b + \alpha^2)\lambda^2 + [a + \alpha^2(b - g)]\lambda + [\alpha^2(a - f)] \equiv \lambda^3 + \alpha'\lambda^2 + \beta'\lambda + \gamma'$$

The Routh test (to verify stable roots) reduces here to [12, p. 175]:

1. α', β', $\gamma' > 0$
2. $\beta' > \gamma'/\alpha'$

The first of these is verified by our conditions on a, b, f, g, and α. The second says we need

$$a + \alpha^2(b - g) > \frac{\alpha^2(a - f)}{b + \alpha^2}$$

which reduces to

$$\begin{aligned}\alpha^4(b-g)+\alpha^2[b(b-g)+f]+ab &> 0\\ \alpha^4+\alpha^2\left[b+\frac{f}{(b-g)}\right]+\frac{ab}{(b-g)} &> 0\end{aligned}$$

Since the last term on the left-hand is positive by our previous conditions (a, b, and $b-g$ greater than zero) it is sufficient for

$$\begin{aligned}\alpha^4+\alpha^2\left[b+\frac{f}{(b-g)}\right] &> 0\\ \alpha^2+b+\frac{f}{(b-g)} &> 0\end{aligned}$$

Again, since $b>0$, it is sufficient for

$$\begin{aligned}\alpha^2+\frac{f}{(b-g)} &> 0\\ \alpha^2 &> \frac{-f}{(b-g)}\end{aligned}$$

□

B Bendixson Extension

This appendix treats the background, statement, and proof of our extension of Bendixson's Theorem.

Bendixson's theorem gives conditions under which a region cannot contain a periodic solution (or limit cycle). It is usually stated as follows (statement and footnote adapted from [17, pp. 31-32]):

Theorem 21 (Bendixson's Theorem) *Suppose D is a simply connected*[8] *domain in R^2 such that the quantity $\nabla f(x)$ (the* divergence *of f) defined by*

$$\nabla f(x)=\frac{\partial f_1}{\partial x_1}(x_1,x_2)+\frac{\partial f_2}{\partial x_2}(x_1,x_2)$$

is not identically zero over any subregion of D and does not change sign in D. Then D contains no closed trajectories of

$$\begin{aligned}\dot{x}_1(t) &= f_1[x_1(t),x_2(t)]\\ \dot{x}_2(t) &= f_2[x_1(t),x_2(t)]\end{aligned}$$

The proof of Bendixson's Theorem depends, in a critical way, on Green's Theorem. The usual statement of Green's Theorem says [10] that a C^1 vector field $f(x)$ on a compact region A in R^n with C^1 boundary B satisfies

$$\int_B f(x)\cdot\mathbf{n}(A,x)d\sigma=\int_A \nabla f(x)d\mathcal{L}^n x$$

[8] A *connected* region can be thought of as a set that is in one piece, *i.e.*, one in which every two points in the set can be connected by a curve lying entirely within the set. A set is *simply connected* if (1) it is connected and (2) its boundary is connected.

where $\mathbf{n}(A, x)$ is the exterior unit normal to A at x, $d\sigma$ is the element of area on B, and $\mathcal{L}^n$ is the Lebesgue measure on R^n. It is possible, however, to treat more general regions and vector fields that are merely Lipschitz continuous. A general extension is the so-called Gauss-Green-Federer Theorem given in [10]. Even the statement of this theorem requires the development of a bit of the language of geometric measure theory. We state a relaxed version of this theorem that is still suitable for our purposes. In the final formula, ∇f exists almost everywhere because a Lipschitz continuous function is differentiable almost everywhere.

Theorem 22 (Relaxation of Gauss-Green-Federer) *Let A be a compact region of R^n with C^1 boundary B. Then for any Lipschitz vector field $f(x)$,*

$$\int_B f(x) \cdot \mathbf{n}(A, x) d\sigma = \int_A \nabla f(x) d\mathcal{L}^n x$$

Now we can prove our version of Bendixson's Theorem, which we repeat for convenience:

Theorem 1 (Extension of Bendixson's Theorem) *Suppose D is a simply connected domain in R^2 and $f(x)$ is a Lipschitz continuous vector field on D such that the quantity $\nabla f(x)$ (the* divergence *of f, which exists almost everywhere) defined by*

$$\nabla f(x) = \frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2)$$

is not zero almost everywhere over any subregion of D and is of the same sign almost everywhere in D. Then D contains no closed trajectories of

$$\dot{x}_1(t) = f_1[x_1(t), x_2(t)] \tag{14}$$

$$\dot{x}_2(t) = f_2[x_1(t), x_2(t)] \tag{15}$$

Proof The proof is similar to that of Bendixson's Theorem in [17, pp. 31–32].

Suppose, for contradiction, that J is a closed trajectory of Equations (14)–(15). Then at each point $x \in J$, the vector field $f(x)$ is tangent to J. Then $f(x) \cdot \mathbf{n}(S, x) = 0$ for all $x \in J$, where S is the area enclosed by J. But by Theorem 22

$$0 = \int_J f(x) \cdot \mathbf{n}(A, x) dl = \int_S \nabla f(x) d\mathcal{L}^2 x$$

Therefore, we must have either (i) $\nabla f(x)$ is zero almost everywhere, or (ii) the sets $\{x \in S | \nabla f(x) < 0\}$ and $\{x \in S | \nabla f(x) > 0\}$ both have positive measure. But if S is a subset of D, neither can happen. Hence, D contains no closed trajectories of Equations (14)–(15). □

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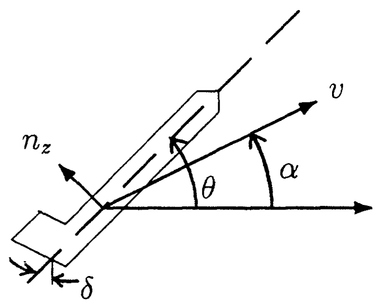

Figure 1: Longitudinal Aircraft View

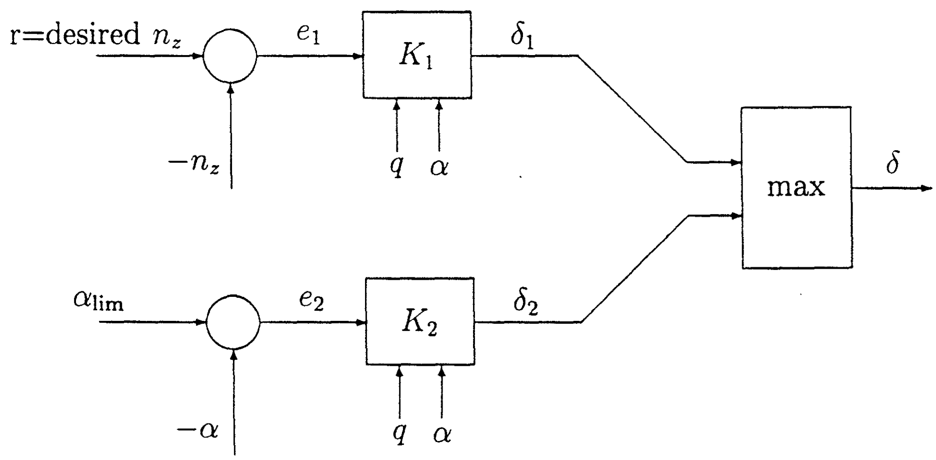

Figure 2: The Max Control System

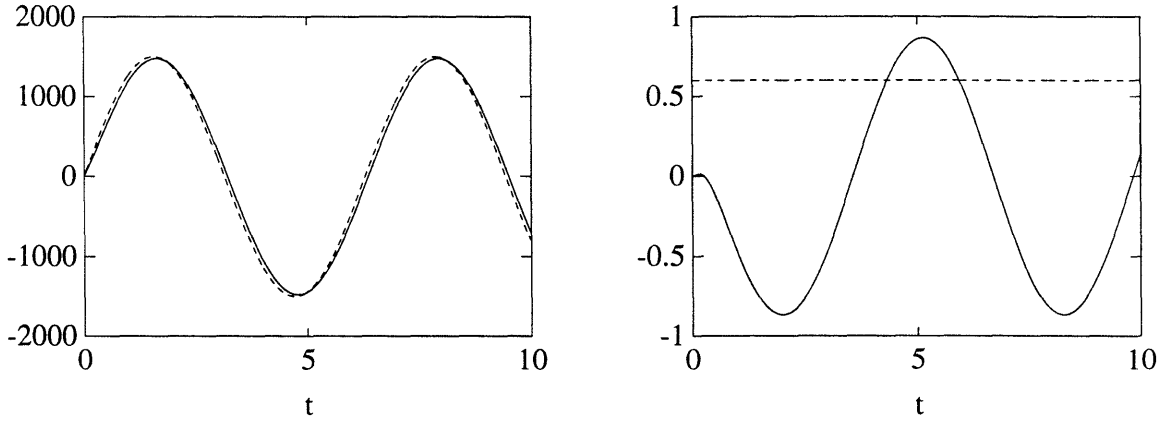

Figure 3: Outputs of the tracking controller: (a) normal acceleration, n_z (solid), and desired normal acceleration, r (dashed); (b) angle of attack, α (solid), and α's limit (dashed).

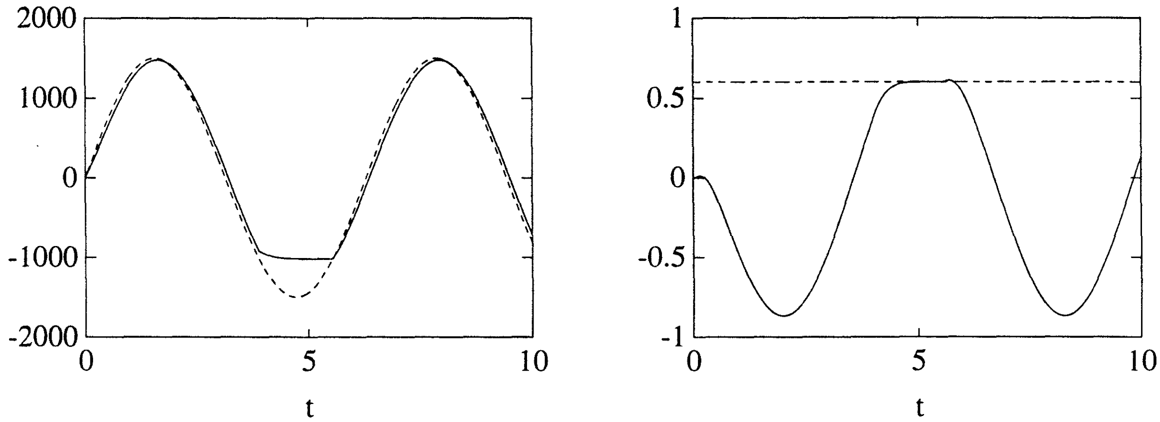

Figure 4: Outputs of the max controller: (a) normal acceleration, n_z (solid), and desired normal acceleration, r (dashed); (b) angle of attack, α (solid), and α's limit (dashed).

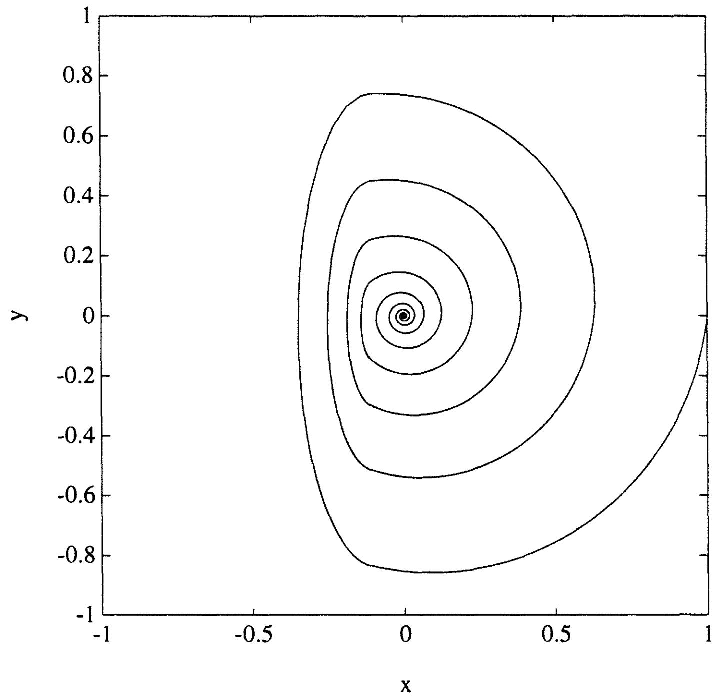

Figure 5: Max System Trajectory.

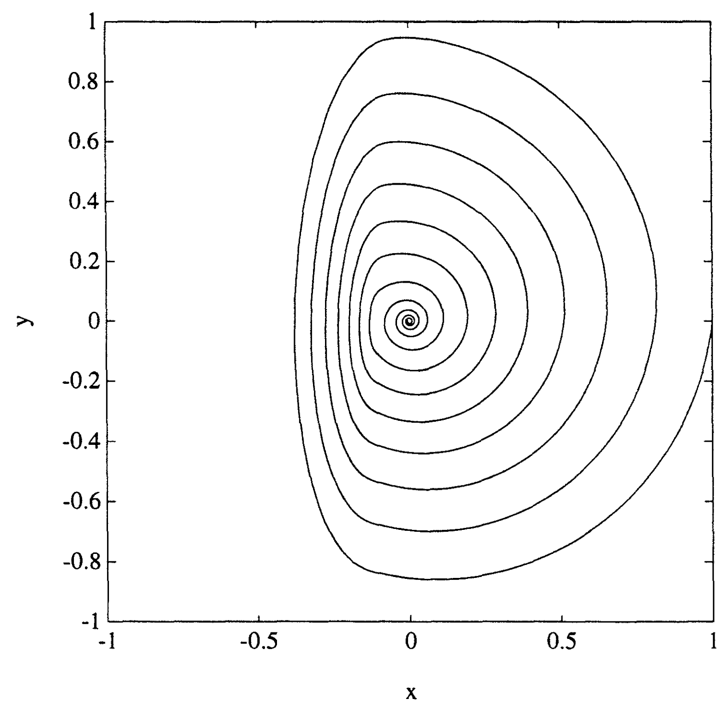

Figure 6: Simulated Max System Trajectory.

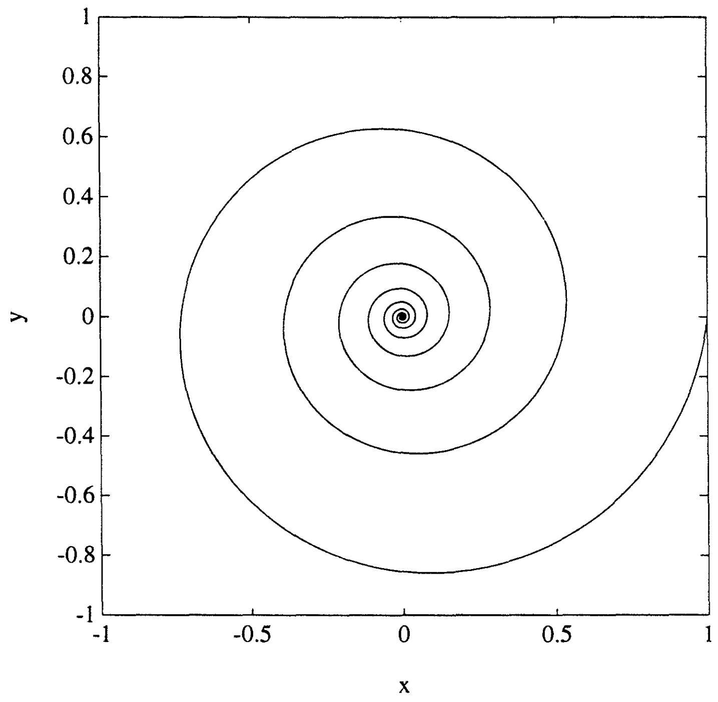

Figure 7: A System Trajectory.

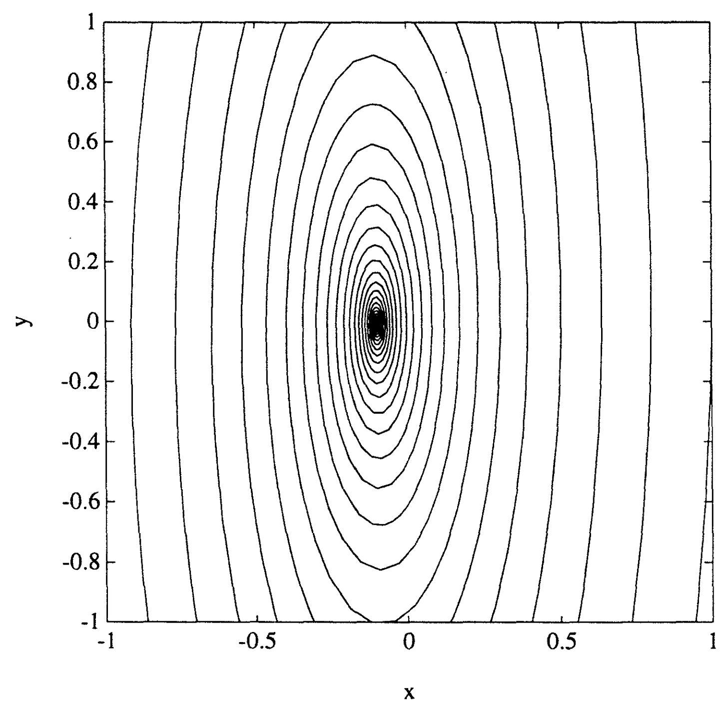

Figure 8: $A + BF$ System Trajectory.