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TORSION OF A THIN RECTANGULAR BEAM WITH AXIAL PRESTRESS AND ENDS CONSTRAINED FROM WARPING

by

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ABSTRACT

This paper documents the analysis of a thin rectangular shaft with axial prestress, twisted by couples applied at ends constrained from warping. Four equations were derived:

1. Torsional stiffness, K
2. Axial stress, σ_{zz}
3. Shear stress, σ_{xz}
4. Shear stress, σ_{yz}

A finite element analysis was used to verify the equations derived.

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SECTION 1

INTRODUCTION

The technology of fabricating miniaturized integrated electronic circuits, from silicon wafers, has led to the development of a new generation of applications referred to as "micro-machining", or the application of silicon fabrication technology to the building of moving mechanical devices. Micro-machining techniques have revolutionized the ability of designers to miniaturize moving mechanical systems for use in compact electro-mechanical sensors and instruments.

Because the mechanical mechanisms within these instruments are fabricated from silicon, conventional machine components such as bearings, gears, linkages, etc. are impractical. Freedom of motion is obtained through flexible joints which elastically deform to give the desired displacements.

One such flexible joint, which provides a rotational degree of freedom, is a thin rectangular shaft which twists axially about its cross sectional center of gravity. This flexure design is of use in a variety of micro-mechanical devices, and will be the focus of study in this paper.

The torsional spring constant or "stiffness" of the flexure pivot is of primary importance in the design of the micro-mechanical instrumentation because this constant is often directly related to key performance factors, and the overall dynamic bevavior of the instruments.

The classical St. Venant solution, found in most texts on elasticity, relating the applied moment to angular twist of a thin rectangular beam, does not account for two additional effects unique to the flexure under consideration.

First, because of the typical design of these instruments, warping is prevented at both ends of the flexure. Warping, or the axial distortion of the cross section, occurs whenever the cross section of the shaft is non-circular. The prevention of this warping at the ends of the flexure alters the result of the classical St. Venant solution where it is assumed the ends are free. (See Part II, Background Theory, Section B: St. Venant Solution.)

Second, an axial pre-stress is initially present in the flexure. This pre-stress is a consequence of the fabrication technique employed.

The objective of this analysis is to develop the equations necessary to calculate the torsional stiffness and resultant stresses of a thin rectangular beam with an initial pre-stress and ends constrained from warping.

This paper is divided into six sections: 1. Introduction; 2. Background Theory; 3. Analysis; 4. Conclusions; 5. References; 6. Appendix.

SECTION 2

BACKGROUND THEORY

A. Flexure Description

Figure 1 is a diagram of the flexure pivot. A rectangular cartesian coordinate system is centered at the fixed end of the flexure, where the flexure attaches to a rigid base. The y-axis is parallel to the height of the pivot, b; the x-axis is parallel to the thickness dimension, c; and the z-axis is parallel to the lengthwise dimension, ℓ. In all cases, calculations will be done in metric units.

A couple of magnitude, T, and a constant axial pre-stress of magnitude, σ_o, act on the cross-sectional surface, S1, at $z=\ell$, where the flexure connects to the moving part of the device. Warping is prevented at both ends of the flexure on surfaces S1 and S2.

B. St. Venant Solution

The St. Venant solution is the classical formulation of the torsion problem for a thin rectangular beam twisted by couples applied at ends free to warp. No pre-stress is applied. The solution is derived from the fundamental torsion equations:

$$\nabla^2\psi = -2G\theta \qquad (2-1)$$

and

$$Msv = \iint \psi \, dA \qquad (2-2)$$

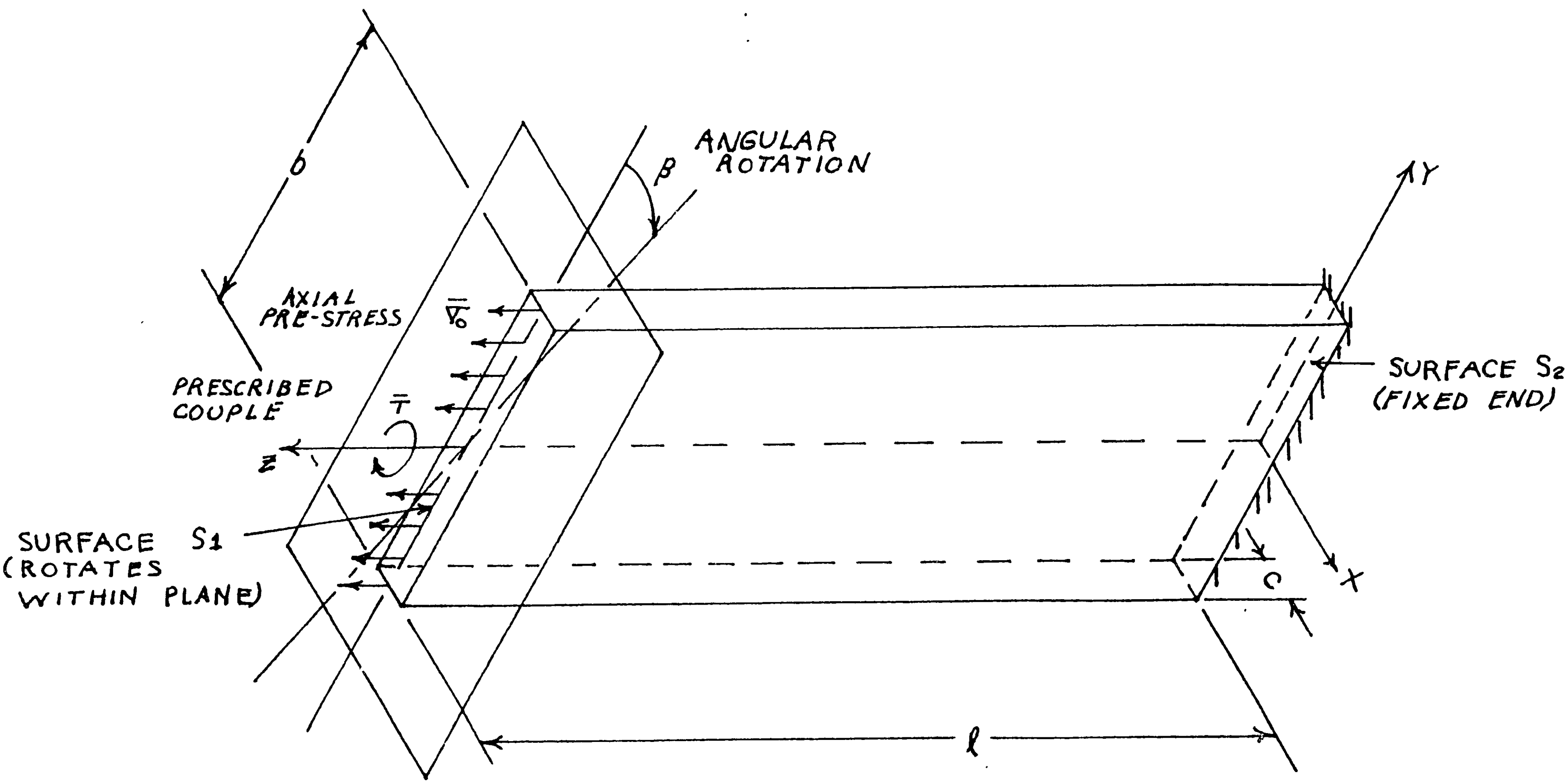

Figure 1. Flexure pivot.

where

∇^2 = LaPlacian in x & y

ψ = stress function

G = torsional modulus

θ = twist/unit length

Msv = applied moment.

The sv subscript, here and in what follows, refers to the St. Venant solution.

The shear stresses acting on the face of the cross section are related to the stress function by the following definitions:

$$\sigma_{zx} = \frac{\partial \psi}{\partial y} \tag{2-3}$$

$$\sigma_{zy} = -\frac{\partial \psi}{\partial x} \tag{2-4}$$

where

σ_{zx} = shear stress acting on z face in x direction

σ_{zy} = shear stress acting on z face in y direction.

The LaPlacian in (2-1) is first nondimensionalized such that for large values of the flexure height, b, the $\frac{\partial^2 \psi}{\partial y^2}$ term can be neglected. The nondimensional equation is:

$$\frac{\partial^2 \psi}{\partial \eta^2} + \left(\frac{c}{b}\right)^2 \frac{\partial^2 \psi}{\partial \xi^2} = -\frac{G\theta}{2} c^2 \qquad \left(\frac{c}{b}\right)^2 < 1.0$$

where

η = 2x/c

ξ = 2y/b

This simplification allows us to directly integrate the remaining terms in (2-1) with respect to η. Using the boundary condition $\psi(\pm c/2) = 0$, a parabolic stress function dependent only on x results:

$$\psi = \frac{G\theta c^2}{4} [1 - (2x/c)^2] \quad (2-5)$$

where

c = flexure thickness.

The "membrane analogy"[1] requires that the stress function, ψ, be zero along the boundary of the shaft. Equation (2-5) only approximately satisfies this requirement because ψ is not zero in the vincinity of y = ±b/2. The error, however, is small.

The shear stresses are then calculated using equations (2-3) and (2-4). The following results are obtained.

$$\sigma_{zx} = 0$$

$$\sigma_{zy} = 2G\theta x.$$

Note that σ_{zx} has a small non-zero near the upper and lower edges of the flexure, at y = ±b/2, but this stress is small, and is neglected. Note also that σ_{zy} has a maximum value at the sides of the flexure at x = ±c/2.

The moment equation is found by substituting (2-5) into (2-2) and integrating over the area of the flexure cross section. The following equation results:

$$Msv = 1/3bC^3G\theta \quad (2-6)$$

Equation (2-6) is the usual approximate St. Venant solution for the specialized case of torsion of a thin rectangular beam with free ends and no pre-stress.[2] Equation (2-6) can be further modified if θ, the twist/unit length, is redefined as β/ℓ, where ℓ equals the length of the flexure. β represents an angular rotation in radians. This modification is permissible because no intermediate restraints, moments, or stresses are applied to the flexure. In the absence of these loads, and within the realm of the linear displacement assumptions, θ, the twist per unit length, is assumed not to vary with length; therefore, if the twist/unit length, θ, is integrated over the length of the flexure, the product of $\theta \cdot \ell$ equals β, in radians.

The St. Venant solution in final form is:

$$Msv = \frac{1/3bC^3G}{\ell}\beta \qquad (2\text{-}7)$$

where

Msv = applied moment

b = height of flexure

c = thickness of flexure

ℓ = length of flexure

β = angular rotation (radians)

G = torsional modulus.

C. Reissner's Principle

A variational method referred to as "Reissner's Principle" will be used in this analysis.[3] In general, variational methods are powerful tools in solving complex problems in elasticity, and serve as the underlying principles behind most computer finite element programs.

Reissner's Principle is a unique variational statement because the stresses as well as the displacements are varied independently. These variations in stress and displacement produce both the stress/displacement relations and the equilibrium relations as Euler equations. Thus, any assumed set of stress and displacement conditions will satisfy both the equilibrium requirements, and a variationally consistent set of constitutive relations.

Perhaps the greatest advantage of Reissner's Principle is that any number of arbitrary displacement and stress boundary condition assumptions can be made at the start of an analysis, with the guarantee that the allowable variations of the remaining unknown stresses and displacements will produce at least the same number of independent simultaneous differential equations describing the result. The resulting equations will appear in mixed form, with both stresses and displacements, thus giving better insight into the nature of the problem.

Reissner's equation written in full is as follows:

$$\delta\left\{\iiint F\,dv - \iint_{S1}(\bar{P}_x u + \bar{P}_y v + \bar{P}_z w)\,dS - \iint_{S2}\left[P_x(u-\bar{u}) + P_y(v-\bar{v}) + P_z(w-\bar{w})\right]dS\right\} = 0 \qquad (2\text{-}8)$$

where

$$F = \sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \sigma_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \sigma_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z} - W(\sigma_{xx},\ldots,\sigma_{yz});$$

$$W(\sigma_{xx},\ldots,\sigma_{yz}) = 1/2E(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \nu/E(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz}) + 1/2G(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2).$$

$\sigma_{xx},\ldots,\sigma_{yz}$ and u,v,w are the stresses and displacements.

The terms inside the two double surface integrals are quantities similar to the work of externally applied stresses, or "prescribed surface tractions", $\bar{P}$, acting on surfaces S1; and externally imposed displacement conditions, or "prescribed displacements", $\bar{U}$, acting on surfaces S2.

Prescribed displacements and stresses are not subject to variation, and are indicated by an overhead bar sign. In vector notation, these quantities are written as:

$$\bar{\underset{\sim}{P}} = \bar{P}_x \hat{i} + \bar{P}_y \hat{j} + \bar{P}_z \hat{k}$$

$$\bar{\underset{\sim}{U}} = \bar{u} \hat{i} + \bar{v} \hat{j} + \bar{w} \hat{k}.$$

Application of the variational method yields the precribed surface tractions in terms of the internal stresses by the following relations:

$$\bar{P}_x = \sigma_{xx}\cos(n,x) + \sigma_{xy}\cos(n,y) + \sigma_{xz}\cos(n,z)$$

$$\bar{P}_y = \sigma_{xy}\cos(n,x) + \sigma_{yy}\cos(n,y) + \sigma_{yz}\cos(n,z)$$

$$\bar{P}_z = \sigma_{xz}\cos(n,x) + \sigma_{yz}\cos(n,y) + \sigma_{zz}\cos(n,z)$$

Reissner's equation in its present form does not account for the non-linear strain terms resulting from large rotations. It is these non-linear terms which, when multiplied by the axial pre-stress, contribute to the torsional moment. Therefore, the task is to modify Reissner's equation to include the axial pre-stress with the appropriate non-linear strain terms.

SECTION 3

ANALYSIS

A. Strain Displacement Relations

The complete strain displacement relations including non-linear terms may be written as

$$\varepsilon_{ij} = \frac{\frac{1}{2}[\bar{r},_i \cdot \bar{u},_j + \bar{u},_i \cdot \bar{r},_j + \bar{u},_i \cdot \bar{u},_j]}{|\bar{r},_i| \quad |\bar{r},_j|} \tag{3-1}$$

where, in a rectangular x,y,z coordinate system,

$$\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \tag{3-2}$$

$$\bar{U} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} \tag{3-3}$$

and subscripts, i,j = x,y,z.

$\bar{r}$ is the position vector of an arbitrary particle in the unde formed body, and $\bar{U}$ is the displacement vector of that particle as it moves uniformly from the undeformed state to the deformed state. The commas in expression (3-1) indicate partial differentiation with respect to the particular coordinate index subscript, i or j.

Equation (3-1) is derived from the fundamental definition of the strain tensor.[4]

$$dS^2 - ds^2 = 2 \sum_{ij}^{3} \varepsilon_{ij} \left|\bar{r},_i\right| dx_i \left|\bar{r},_j\right| dx_j \qquad (3\text{-}4)$$

where, $dS^2 - ds^2$ represents the change in the square of the length of a differential line segment, as it translates and rotates from the undeformed state, to the deformed state. Equation (3-4) has been slightly generalized in that a right-handed 1,2,3 coordinate system is used in the summation. Details of the derivation of (3-1) from (3-4) are contained in Appendix A.

Equation (3-1) written in full is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]$$

$$\varepsilon_{yz} = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z}\right]$$

$$\varepsilon_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\cdot\frac{\partial w}{\partial x}\right]$$

These relations can be used to modify Reissner's variational statement to include the non-linear effects of large displacements and rotations, and the effect of the axial pre-stress.

For this analysis, the rotation of the flexure is small, two degrees nominal twist, and sufficient accuracy is obtained by neglecting the non-linear portions of the strains associated with the normal and shearing stresses σ_{xx}, ..., σ_{yz}. The nonlinear strain term, ε_{zz}, multiplied by the pre-stress σ_o, in the u and v directions, however, is not negligible.

σ_o is included as an additional constant prescribed stress added to σ_{zz}. σ_o is not subject to variation, $\delta\sigma_o = 0$, and does not vary with length, $\frac{\partial\sigma_o}{\partial z} = 0$.

The complete non-linear term associated with σ_o and σ_{zz} is:

$$(\sigma_o + \sigma_{zz})\left[\frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]\right]$$

We restrict our attention to cases where σ_o is large compared to σ_{zz}; i.e., $(\sigma_o/\sigma_{zz}) \approx 0(10)$; therefore, the nonlinear terms multiplied by σ_{zz} can by neglected in comparison with σ_o. Neglecting the nonlinear products of the σ_{zz} term gives:

$$(\sigma_o + \sigma_{zz})\frac{\partial w}{\partial z} + \frac{1}{2}\sigma_o\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \qquad (3\text{-}5)$$

We anticipate $\partial u/\partial z$ and to $\partial v/\partial z$ be on the order of the rotation at the cross-section at any z, i.e., 2 degrees max, small with respect to 1.0 but not small with respect to the elastic shear strains. $\partial w/\partial z$ is allowed to be of the same order of magnitude as $\partial u/\partial z$ and $\partial v/\partial z$ (we shall see this is true near the ends) but is also small with respect to 1.0. We thus neglect $(\partial w/\partial z)^2$ with respect to $\partial w/\partial z$ but retain $(\partial u/\partial z)^2$ and $(\partial v/\partial z)^2$.

$$\sigma_o\left(\frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2\right) \approx \sigma_o\left(\frac{\partial w}{\partial z}\right)$$

so that,

$$(\sigma_o + \sigma_{zz})\frac{\partial w}{\partial z} + \frac{1}{2}\sigma_o[(\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2] \tag{3-5}$$

will replace the $\sigma_{zz}(\partial w/\partial z)$ term in Reissner's equation (2-8), and will account for any changes in the stiffness of the pivot caused by axial pre-stress.

It is further assumed that the surfaces on the sides of the flexure, and on the top and bottom, are free from stress; thus, $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$. Surface S1, in Figure 1, will now represent a cross-sectional surface, where surface tractions are prescribed, and surface S2, where displacements are prescribed.

Rewriting (2-8) under these assumptions

$$\delta\{\iiint [(\sigma_o + \sigma_{zz})\frac{\partial w}{\partial z} + \frac{1}{2}\sigma_o[(\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2] + \sigma_{xz}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$$

$$+ \sigma_{yz}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) - \sigma_{zz}{}^2/2E - (\sigma_{xz}{}^2 + \sigma_{yz}{}^2)/2G]dV$$

$$- \iint_{S_1}(\bar{P}_x u + \bar{P}_y v + \bar{P}_z w)dxdy - \iint_{S_2}[P_x(u - \bar{u})$$

$$+ P_y(v - \bar{v}) + P_z(w - \bar{w})]dxdy\} = 0 \tag{3-6}$$

Applying the variational method to (3-6), with σ_{ij}, $P_x, \ldots,$ $u, \ldots$ subject to variation, yields the following:

(i) The equilibrium relations:

$$\text{x-direction:} \quad \frac{\partial \sigma_{xz}}{\partial z} + \sigma_o \frac{\partial^2 u}{\partial z^2} = 0$$

$$\text{y-direction:} \quad \frac{\partial \sigma_{yz}}{\partial z} + \sigma_o \frac{\partial^2 v}{\partial z^2} = 0$$

$$\text{z-direction:} \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

(ii) The boundary conditions on S1:

$$\overline{P}_x = \sigma_{xz} + \sigma_o \frac{\partial u}{\partial z}$$

$$\overline{P}_y = \sigma_{yz} + \sigma_o \frac{\partial v}{\partial z}$$

$$\overline{P}_z = \sigma_{zz} + \sigma_o$$

(iii) The boundary conditions on S2:

$$u = \overline{u}$$

$$v = \overline{v}$$

$$w = \overline{w}$$

(iv) The constituitive relations:

$$\sigma_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\sigma_{zz} = E\,\frac{\partial w}{\partial z}$$

These equilibrium equations can be derived from Flugge[5] by using the same displacement and stress assumptions, and by replacing σ_{zz} in Flugge's equations by $(\sigma_o + \sigma_{zz})$ with the restriction $\partial\sigma_o/\partial z = 0$.

The boundary terms relating external surface tractions to the interior stresses can also be derived from Flugge by considering the equilibrium condition of an external pseudostress vector with the interior pseudostresses. The pseudostress is a nonlinear definition of stress where the stress components are related to the original area before deformation.

From the above boundary conditions on S1, we define a prescribed torque, $\overline{T}$, as:

$$\overline{T} = \iint_{S_1} (\overline{P}_x y - \overline{P}_y x)dxdy$$

or

$$\overline{T} = \iint_{S_1} \left(\sigma_{xz} y - \sigma_{yz} x + \sigma_o \frac{\partial u}{\partial z} y - \sigma_o \frac{\partial v}{\partial z} x\right)dxdy$$

B. <u>Further Displacement Assmuptions and Derivation of Torsion Equations</u>

Further displacement assumptions are[3]:

$$u = \beta(z)y$$

$$v = -\beta(z)x$$

$$w = \alpha(z) \cdot \Phi(x,y)$$

where

$\beta(z)$ = angle of twist in radians

$\Phi(x,y)$ = warping function

$\alpha(z)$ = twist per unit length.

The boundary conditions are:

on S2 $u = \overline{u}$

$v = \overline{v}$

$w = \overline{w}$

on S1 $T = \overline{T}$ (prescribed torque)

$P_z = \overline{P}_z$ (prescribed axial stress)

Note that the above displacement assumptions are similar to the assumptions used in the conventional displacement approach of the St. Venant torsion theory.

$$u = \theta yz$$

$$v = -\theta xz$$

$$w = \theta \cdot \Phi(x,y)$$

where

θ = uniform twist per unit length.

The important difference is that $\beta(z)$ and $\alpha(z)$ vary with the length of the flexure because of the axial restraint of the no warping condition at the ends.

Introducing the displacement assumptions and boundary conditions into (3-6), and defining the torque as: (Refer to Page 18.)

$$T(z) = \iint\left(\sigma_{xz}y - \sigma_{yz}x + \sigma_o\left(x^2 + y^2\right)\beta'(z)\right)dxdy$$

gives

$$\delta\Big\{\iiint\Big[\left(\sigma_o + \sigma_{zz}\right)\frac{\partial\alpha}{\partial z}\Phi + \frac{1}{2}\sigma_o\left(x^2 + y^2\right)\left(\frac{\partial\beta}{\partial z}\right)^2 + \sigma_{xz}\left(\frac{\partial\beta}{\partial z}y + \Phi,_x\alpha\right)$$

$$+ \sigma_{yz}\left(\Phi,_y\alpha - x\frac{\partial\beta}{\partial z}\right) - \sigma_{zz}{}^2/2E - (\sigma_{xz}{}^2 + \sigma_{yz}{}^2/2G\Big]dV = \overline{T}\beta_{s_1}$$

$$+ \alpha\iint_{s_1}\overline{P}_z\phi dxdy + T(\overline{\beta} - \beta)_{s_2} + (\overline{\alpha} - \alpha)\iint_{s_2}P_z\phi dxdy\Big\}$$

Varying the stresses and displacements:

$$\iiint\Big[\left(\sigma_o + \sigma_{zz}\right)\phi\left(\frac{\partial}{\partial z}(\delta\alpha)\right) + \sigma_o\left(x^2 + y^2\right)\beta'\left(\frac{\partial}{\partial z}(\delta\beta)\right)$$

$$+ \sigma_{xz}\left(y\left(\frac{\partial}{\partial z}(\delta\beta)\right) + \Phi,_x\delta\alpha\right) + \sigma_{yz}\left(\Phi,_y\delta\alpha - x\left(\frac{\partial}{\partial z}(\delta\beta)\right)\right) + \delta\sigma_{zz}(\Phi\alpha')$$

$$+ \delta\sigma_{xz}\left(y\beta' + \Phi,_x\alpha\right) + \delta\sigma_{yz}\left(\Phi,_y\alpha - x\beta'\right) - \delta\sigma_{zz}\left(\sigma_{zz}/E\right)$$

$$- \left(\sigma_{xz}\delta\sigma_{xz} + \sigma_{yz}\delta\sigma_{yz}\right)/G\Big]dV = \overline{T}\delta\beta_{s_1} + \delta\alpha\iint_{s_1}\overline{P}_z\phi dxdy$$

$$+ \delta T(\overline{\beta} - \beta)_{s_2} + (\overline{\alpha} - \alpha)\iint_{s_2}\delta P_z\phi dxdy - T\delta\beta_{s_2} - \delta\alpha\iint_{s_2}P_z\phi dxdy$$

collecting terms with $\delta\beta = \delta\alpha = 0$ on S2:

$$\iiint\Big[(\sigma_o + \sigma_{zz})\phi\,\frac{\partial}{\partial z}(\delta\alpha) + (\sigma_{xz}\Phi,_x + \sigma_{yz}\Phi,_y)\delta\alpha$$

$$+ \left[\sigma_o(x^2 + y^2)\beta' + \sigma_{xz}y - \sigma_{yz}x\right]\frac{\partial}{\partial z}(\delta\beta)$$

$$+ (\Phi,_y\alpha - x\beta' - \sigma_{yz}/G)\delta\sigma_{yz} + (y\beta' + \Phi,_x\alpha - \sigma_{xz}/G)\delta\sigma_{xz}$$

$$+ (\Phi\alpha' - \sigma_{zz}/E)\delta\sigma_{zz}\Big]dV = \overline{T}\delta\beta_{s_1} + \delta\alpha\iint_{s_1}\overline{P}_z\phi dxdy + \delta T(\overline{\beta} - \beta)_{s_2}$$

$$+ (\overline{\alpha} - \alpha)\iint_{s_2}\delta P_z\phi dxdy$$

and integrating by parts, gives:

$$\iiint[(\sigma_{xz}\Phi,_x + \sigma_{yz}\Phi,_y - \frac{\partial\sigma_{zz}}{\partial z}\Phi)\delta\alpha - \frac{\partial}{\partial z}[\sigma_o(x^2 + y^2)\beta' + \sigma_{xz}y$$

$$- \sigma_{yz}x]\delta\beta + (\Phi,_y\alpha - x\beta' - \sigma_{yz}/G)\delta\sigma_{yz}$$

$$+ (y\beta' + \Phi,_x\alpha - \sigma_{xz}/G)\delta\sigma_{xz} + (\Phi\alpha' - \sigma_{zz}/E)\delta\sigma_{zz}]dV$$

$$+ [\iint[\sigma_o(x^2 + y^2)\beta' + \sigma_{xz}y - \sigma_{yz}x]dxdy - \overline{T}]\delta\beta_{s_1}$$

$$+ [\iint[(\sigma_{zz} + \sigma_o) - \overline{P}_z]\phi dxdy]_{s_1}\delta\alpha + \delta T(\beta - \overline{\beta})_{s_2}$$

$$+ (\alpha - \overline{\alpha})\iint_{s_2}\delta P_z\phi dxdy = 0$$

The independent variational variables are then equated to zero.

In the interior:

$$\delta\alpha: \quad \iint(\sigma_{xz}\Phi,_x + \sigma_{yz}\Phi,_y - \frac{\partial\sigma_{zz}}{\partial z}\phi)dxdy = 0 \qquad (3\text{-}7)$$

$$\delta\beta: \quad \frac{\partial}{\partial z}[\iint[\sigma_o(x^2 + y^2)\beta' + \sigma_{xz}y - \sigma_{yz}x]dxdy] = 0 \qquad (3\text{-}8)$$

$$\delta\sigma_{xz}: \quad \sigma_{xz} = G(y\beta' + \Phi,_x\alpha) \qquad (3\text{-}9)$$

$$\delta\sigma_{yz}: \quad \sigma_{yz} = G(\Phi,_y\alpha - x\beta') \qquad (3\text{-}10)$$

$$\delta\sigma_{zz}: \quad \sigma_{zz} = E\phi\alpha' \qquad (3\text{-}11)$$

On the boundary on S_1:

$$\delta\beta: \quad \overline{T} = \iint[\sigma_o(x^2 + y^2)\beta' + \sigma_{xz}y - \sigma_{yz}x]dxdy \tag{3-12}$$

$$\delta\alpha: \quad \iint\overline{P}_z\phi dxdy = \iint[\sigma_{zz} + \sigma_o]\phi dxdy$$

On the boundary on S_2:

$$\delta T: \quad \beta = \overline{\beta}$$

$$\delta P_z: \quad \alpha = \overline{\alpha}$$

giving five first order differential equations, and five unknowns α, β, σ_{xz}, σ_{yz}, σ_{zz}.

Equation (3-8) is a statement of equilibrium, $\partial T/\partial z = 0$ meaning the torque does not vary along the length of the flexure. Equation (3-12) is the boundary term corresponding to the torque. Equations (3-9) through (3-11) are the constitutive relations. The double integrals over the cross section in equations (3-7) and (3-8) have been retained because of the dependence on x & y through the warping function.

In what follows, we set the boundary conditions on $z = 0$, on the S2 surface, to be $\overline{\beta} = \overline{\alpha} = 0$. At the other end, $z = \ell$, we set the mixed condition $T = \overline{T}$, and $w = 0$ (hence $\alpha(\ell) = 0$).

Substituting (3-9) through (3-11) into (3-7) & (3-12), and defining:

$$D^* = \iint(x\Phi,_y - y\Phi,_x)dxdy$$

$$I_p = \iint(x^2 + y^2)dxdy$$

$$\Gamma = \iint \Phi^2 dxdy$$

$$D = \iint \left(\Phi,_x^2 + \Phi,_y^2\right) dxdy$$

gives two differential equations:

$$\overline{T} = G\left(I_p\beta' - D^*\alpha\right) + \sigma_o I_p \beta' \tag{3-13}$$

$$G(D\alpha - D^*\beta') - E\Gamma\alpha'' = 0 \tag{3-14}$$

Solving (3-13) for α,

$$\alpha = \frac{(\sigma_o + G)I_p}{GD^*}\beta' - \frac{\overline{T}}{GD^*}$$

Then substituting into (3-14), and using the relation, $\partial T/\partial z = 0$, gives:

$$G\left[\frac{(\sigma_o + G)I_p}{G} - \frac{D^{*2}}{D}\right]\beta' - E\Gamma\left[\frac{I_p(\sigma_o + G)}{DG}\right]\beta''' = \overline{T} \tag{3-15}$$

with boundary conditions:

$$\beta(0) = 0$$

$$(\alpha(0) = 0): \qquad \beta'(0) = \frac{\overline{T}}{(\sigma_o + G)I_p}$$

$$(\alpha(\ell) = 0): \qquad \beta'(\ell) = \frac{\overline{T}}{(\sigma_o + G)I_p}$$

Although we anticipate that σ_o is small in comparison to G, one must be careful in neglecting σ_o in comparison to G in the above terms. For example, even for small values of σ_o, $\sigma_o/G << 1.0$, σ_o cannot be neglected in the bracketed term multiplied by β' in (3-15), since this factor represents the difference between two numbers of relatively the same magnitude. That is, for a thin rectangular cross-section we find, with $\phi = -xy$, that $I_p = D$, and $D \approx D^*$; hence,

$$\left[\frac{(\sigma_o + G)I_p}{G} - \frac{D^{*2}}{D}\right] \approx \left[\frac{\sigma_o}{G} D - \frac{D^2 - D^{*2}}{D}\right]$$

where, $(\sigma_o D)/G$ is of the same order as $(D^2 - D^{*2})/D$, and therefore, cannot be neglected.

Alternatively, we can pose the problem in terms of $\alpha(z)$. Solving (3-13) for β',

$$\beta' = \frac{\overline{T}}{(\sigma_o + G)I_p} + \frac{GD^*}{(\sigma_o + G)I_p}\alpha$$

and substituting into (3-14) gives:

$$G\left[\frac{(\sigma_o + G)I_p D}{GD^*} - D^*\right]\alpha - E\Gamma\left[\frac{I_p(\sigma_o + G)}{D^*G}\right]\alpha'' = \overline{T} \qquad (3\text{-}16)$$

with boundary conditions:

$$\alpha(0) = 0$$

$$\alpha(\ell) = 0$$

C. <u>General Solution of the Torsion Equations</u>

(1) Equation (3-15)

Define:

$$a = \frac{G^2}{(\sigma_o + G)E\Gamma}\left[\frac{DI_p - D^{*2}}{I_p} + \frac{\sigma_o}{G} D\right]$$

$$d = \frac{\overline{T}GD}{(\sigma_o + G)E\Gamma I_p}$$

$$d/a = \frac{\overline{T}}{G\left[\frac{(\sigma_o + G)I_p}{G} - \frac{D^{*2}}{D}\right]}$$

(These terms are expanded in Appendix B.)

and Equation (3-15) simplifies to:

$$\beta''' - a\beta' = -d$$

The general solution is:

$$\beta(z) = C_1 + C_2 \sinh\sqrt{(a)}\, z + C_3 \cosh\sqrt{(a)}\, z + (d/a)z \qquad (3\text{-}17)$$

The constants of integration are determined from the boundary conditions associated with (3-15) as follows:

$$C_3 = \left[\frac{\overline{T}}{(\sigma_o + G)I_p} - \frac{d}{a}\right] \cdot \left[\frac{1 - \cosh\sqrt{(a)}\ell}{\sqrt{(a)}\sinh\sqrt{(a)}\ell}\right]$$

$$C2 = \frac{1}{\sqrt{(a)}} \cdot \left[\frac{\overline{T}}{(\sigma_o + G)I_p} - \frac{d}{a}\right]$$

$$C_1 = -C_3$$

Inserting these constants into (3-17) gives:

$$\beta(z) = \left[\frac{\bar{T}}{(\sigma_o + G)I_p} - \frac{d}{a}\right] \left[\frac{1 - \cosh\sqrt{(a)}\,\ell}{\sqrt{(a)}\,\sinh\sqrt{(a)}\,\ell}\right](\cosh\sqrt{(a)}\,z - 1)$$

$$+ \frac{1}{\sqrt{(a)}} \left[\frac{\bar{T}}{(\sigma_o + G)I_p} - \frac{d}{a}\right] \sinh\sqrt{(a)}\,z + (d/a)z \qquad (3\text{-}18)$$

(2) Equation (3-16)

Define:

$$a = \frac{G^2}{(\sigma_o + G)E\Gamma} \left[\frac{DI_p - D^{*2}}{I_p} + \frac{\sigma_o}{G} D\right]$$

$$d' = \frac{\bar{T}GD^*}{(\sigma_o + G)E\Gamma I_p}$$

$$d'/a = \frac{\bar{T}}{G\left[\frac{\sigma_o I_p D}{D^*G} + \frac{DI_p - D^{*2}}{D^*}\right]}$$

and (3-16) simplifies to:

$$\alpha'' - a\alpha = -d'/a$$

The general solution is:

$$\alpha(z) = C_1 \sinh\sqrt{(a)}\,z + C_2 \cosh\sqrt{(a)}\,z + d'/a$$

The constants are evaluated from the boundary conditions associated with (3-16)

$$C_1 = \frac{d'}{a} \cdot \frac{(\cosh\sqrt{(a)}\,\ell - 1)}{\sinh\sqrt{(a)}\,\ell}$$

$$C_2 = -\frac{d'}{a}$$

Inserting these constants into the general solution gives:

$$\alpha(z) = \frac{d'}{a} \cdot \frac{(\cosh\sqrt{(a)}\,\ell - 1)}{\sinh\sqrt{(a)}\,\ell} \sinh\sqrt{(a)}\, z - \frac{d'}{a} \cosh\sqrt{(a)}\, z + \frac{d'}{a} \qquad (3\text{-}19)$$

D. Solution of Torsion Equations for Thin Rectangular Cross-section

(1) Stiffness Equation

The generalized equation for the torsional stiffness is

$$T = K\beta(\ell)$$

where

T = applied torque

K = stiffness

$\beta(\ell)$ = angular rotation (radians).

To calculate $\beta(\ell)$, Equation (3-18) will be calculated at $z = \ell$ and specialized for the case of a thin flexure with rectangular cross-section.

$$\beta(\ell) = \frac{1}{\sqrt{(a)}}\left[\frac{T}{(\sigma_o + G)I_p} - \frac{d}{a}\right]\left[-\frac{\left(1 - \cosh\sqrt{(a)}\,\ell\right)^2}{\sinh\sqrt{(a)}\,\ell} + \sinh\sqrt{(a)}\,\ell\right] + (d/a)\ell$$

From the calculation in Appendix B, $\sqrt{(a)}\,\ell = 92.7$; therefore, the cosh and sinh terms in the square brackets are expanded, and approximated as:

$$\cosh\sqrt{(a)}\,\ell \approx \sinh\sqrt{(a)}\,\ell$$

and

$$1/(\sinh\sqrt{(a)}\,\ell) \approx 0$$

The result gives:

$$\beta(\ell) = \frac{2}{\sqrt{(a)}}\left[\frac{T}{(\sigma_o + G)I_p} - \frac{d}{a}\right] + (d/a)\ell \qquad (3\text{-}20)$$

This equation is further expanded as:

$$\beta(\ell) = \frac{2}{\sqrt{(a)}}\left[\frac{T}{(\sigma_o + G)I_p} - \frac{T}{G\left[\frac{(\sigma_o + G)I_p}{G} - \frac{D^{*2}}{D}\right]}\right] + \frac{T\ell}{G\left[\frac{(\sigma_o + G)I_p}{G} - \frac{D^{*2}}{D}\right]}$$

Manipulating this equation and using the results of Appendix B gives the following reductions:

$$\beta(\ell) = \frac{2T}{\sqrt{(a)}\,(\sigma_o + G)} \left[\frac{-1}{1/3bc^3[1 + 1/4(b/c)^2\sigma_o/G]}\right]$$

$$+ \frac{T\ell}{1/3bc^3G[1 + 1/4(b/c)^2\sigma_o/G]}$$

$$\beta(\ell) = \frac{T}{1/3bc^3G[1 + 1/4(b/c)^2\sigma_o/G]}$$

$$\cdot \left[\ell - b\frac{\sqrt{E}}{[12(\sigma_o + G)(1 + 1/4(b/c)^2\sigma_o/G)]^{1/2}}\right] \qquad (3\text{-}21)$$

Abbreviating (3-21) as follows

$$\mu = 1 + 1/4(b/c)^2\sigma_o/G$$

$$L_c = b\frac{\sqrt{E}}{[12(\sigma_o + G)(1 + 1/4(b/c)^2\sigma_o/G)]^{1/2}}$$

gives

$$\beta(\ell) = \frac{T}{1/3bc^3G\mu}(\ell - L_c) \qquad (3\text{-}22)$$

The stiffness is

$$K = \frac{1/3bc^3G\mu}{(\ell - L_c)}$$

Equation (3-22) is the final form of the linear torsion equation relating the applied torque to angular twist of a thin rectangular beam, with an initial pre-stress, and ends constrained from warping. If L_c = 0 in (3-22), the equation reduces to, in expanded form

$$\beta(\ell) = \frac{T\ell}{1/3bc^3G[1 + 1/4(b/c)^2\sigma_o/G]}$$

This no warping equation matches the result of Biot[6], where the torsion equation was derived for a thin rectangular strip under initial axial pre-stress with free ends. Thus, the L_c term subtacted from the length of the flexure represents the increase in torsional stiffness caused by the restraint of warping. Because L_c has units of length, and is subtracted from ℓ in (3-22), the term can be thought of as a length correction factor where the stiffness is increased by a reduction in length.

Note that if the ends of the flexure are free and the pre-stress zero, the equation reduces to the classical St. Venant torsion equation (2-7) derived in Section A of the Background Theory.

(2) Stresses

(a) Axial Stress, σ_{zz}.

Equation (3-11) combined with equation (3-19), differentiated with respect to z, will give the axial stress. From (3-19)

$$\alpha'(z) = \frac{d'}{a}\sqrt{(a)}\left[\frac{(\cosh\sqrt{(a)}\,\ell - 1)}{\sinh\sqrt{(a)}\,\ell}\cosh\sqrt{(a)}\,z - \sinh\sqrt{(a)}\,z\right] \quad (3\text{-}23)$$

Expanding (3-23) and using the results of Appendix B gives:

$$\alpha'(z) = \frac{T}{G\left[\frac{\sigma_o I_p D}{D^*G} + \frac{DI_p - D^{*2}}{D^*}\right]} \cdot \left[\frac{G^2}{(\sigma_o + G)E\Gamma}\left[\frac{DI_p - D^{*2}}{I_p} + \frac{\sigma_o}{G}D\right]\right]^{1/2}$$

$$\cdot \left[\frac{(\cosh\sqrt{a}\,\ell - 1)}{\sinh\sqrt{a}\,\ell}\cosh\sqrt{a}\,z - \sinh\sqrt{a}\,z\right]$$

$$\alpha'(z) = \frac{T}{1/3bc^3 G[1 + 1/4(b/c)^2 \sigma_o/G]}$$

$$\cdot \left[\frac{G^2}{(\sigma_o + G)E} \frac{48}{b^2} (1 + 1/4(b/c)^2 \sigma_o/G]^{1/2} \qquad (3\text{-}24)$$

$$\cdot \left[\frac{(\cosh\sqrt{a}\,\ell - 1)}{\sinh\sqrt{a}\,\ell} \cosh\sqrt{a}\,z - \sinh\sqrt{a}\,z\right]$$

Solving (3-21), for T and inserting into (3-24) gives

$$\alpha'(z) = \frac{G}{Eb^2}$$

$$\cdot \left[\frac{48b\sqrt{E}\quad [1 + 1/4(b/c)^2 \sigma_o/G]}{\ell[48(\sigma_o + G)(1 + 1/4(b/c)^2 \sigma_o/G)]^{1/2} - 2b\sqrt{E}}\right]$$

$$\cdot \left[\frac{(\cosh\sqrt{a}\,\ell - 1)}{\sinh\sqrt{a}\,\ell} \cosh\sqrt{a}\,z - \sinh\sqrt{a}\,z\right]\beta(\ell) \qquad (3\text{-}25)$$

Combining with (3-11) gives the axial stress as:

$$\sigma_{zz} = \frac{-xy}{b} G \cdot \left[\frac{24\sqrt{E}[1 + 1/4(b/c)^2 \sigma_o/G]}{\ell[12(\sigma_o + G)[1 + 1/4(b/c)^2 \sigma_o/G])]^{1/2} - b\sqrt{E}}\right]$$

$$\cdot \left[\frac{\cosh\sqrt{a}\,\ell - 1}{\sinh\sqrt{a}\,\ell} \cosh\sqrt{a}\,z - \sinh\sqrt{a}\,z\right]\beta(\ell) \qquad (3\text{-}26)$$

In deriving (3-26), the warping function $\phi = -xy$ was used. This warping function closely approximates the actual St. Venant warping function for a thin rectangular beam[7] and gives, as we shall see,

$$\sigma_{xz} \approx 0$$

and

$$\sigma_{yz} = -Gx(\alpha + \beta')$$

away from the ends.

Abbreviating (3-26) as follows:

$$\mu = 1 + 1/4(b/c)^2 \sigma_o/G$$

gives

$$\sigma_{zz} = \frac{-xy}{b} G \cdot \left[\frac{24\sqrt{(E)}\ \mu}{\ell\left[12(\sigma_o + G)\mu\right]^{1/2} - b\sqrt{E}}\right]$$

$$\cdot \left[\frac{(\cosh\sqrt{(a)}\ \ell - 1}{\sinh\sqrt{(a)}\ \ell} \cosh\sqrt{(a)}\ z - \sinh\sqrt{(a)}\ z\right]\beta(\ell)$$

where

$$\sqrt{(a)} = \frac{G}{b}\left[\frac{48\mu}{(\sigma_o + G)E}\right]^{1/2} \qquad (3\text{-}27)$$

The bracketed sinh and cosh terms will now be approximated near the ends of the flexure and in the interior regions.

At $z \approx 0$ the following approximations are made:

$$\frac{\cosh\sqrt{(a)}\ \ell}{\sinh\sqrt{(a)}\ \ell} \approx 1, \text{ and } \frac{-\cosh\sqrt{(a)}\ z}{\sinh\sqrt{(a)}\ \ell} \approx 0$$

thus, the bracketed sinh and cosh terms reduce to:

$$(\cosh\sqrt{(a)}\, z - \sinh\sqrt{(a)}\, z) = e^{-\sqrt{(a)}z}$$

Rewriting (3-27) gives:

$$\sigma_{zz} = \frac{-xy}{b} G \cdot \left[\frac{24\sqrt{(E)}\ \mu\beta(\ell)}{\ell\left[12(\sigma_o + G)\mu\right]^{1/2} - b\sqrt{E}}\right] e^{-\sqrt{(a)}z} \qquad (3\text{-}28)$$

At $z \approx \ell$, $\sinh\sqrt{(a)}\, z \approx \cosh\sqrt{(a)}z$, therefore, the bracketed sinh and cosh terms in (3-27) are approximated as:

$$-\frac{(\cosh\sqrt{(a)}\, z}{\sinh\sqrt{(a)}\, \ell} \approx -e^{\sqrt{(a)}(z-\ell)}$$

Rewriting (3-27) gives:

$$\sigma_{zz} = \frac{xy}{b} G \cdot \left[\frac{24\sqrt{(E)}\ \mu\beta(\ell)}{\ell\left[12(\sigma_o + G)\mu\right]^{1/2} - b\sqrt{E}}\right] e^{\sqrt{(a)}(z-\ell)} .$$

At $z \approx \ell/2$, the interior portion, the exponential terms in (3-28) and (3-29) become negligibly small; thus,

$$\sigma_{zz} \approx 0$$

in the interior.

The maximum axial stress due to warping occurs at both ends, $z = \ell$, and $z = 0$, at $x = \pm c/2$ and $y = \pm b/2$. Using $x = c/2$ and $y = b/2$, at $z = \ell$, (3-29) gives

$$\sigma_{zzmax} = cG\left[\frac{6\sqrt{(E)}\ \mu\beta(\ell)}{\ell\left[12(\sigma_o + G)\mu\right]^{1/2} - b\sqrt{E}}\right]$$

To find the total axial stress the pre-stress should be added to equations (3-28) and (3-29).

$$\sigma_{zztotal} = \sigma_{zz} + \sigma_o$$

The stress distribution, caused by warping at $z = \ell$, is shown in Figure 2. Note that because the warping function is an odd function, the integral of the axial stress over the cross-sectional area is zero. Thus, the axial stress produced by the no warping constraint, at the ends of the flexure, is self-equilibriated, and rapidly diminishes to zero moving away from the ends.

Figure 2 also shows the correct direction of the rotation β, and torque, T. The direction of rotation is found by considering the definition of rotation in the z direction.

$$\Omega_{zz} = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Substituting the displacement assumptions for u and v gives:

$$\Omega_{zz} = -\beta$$

(b) Shear Stress, σ_{xz}

Equation (3-9) will be combined with Equation (3-18), $\beta(z)$, differentiated with respect to z; and Equation (3-19), $\alpha(z)$.

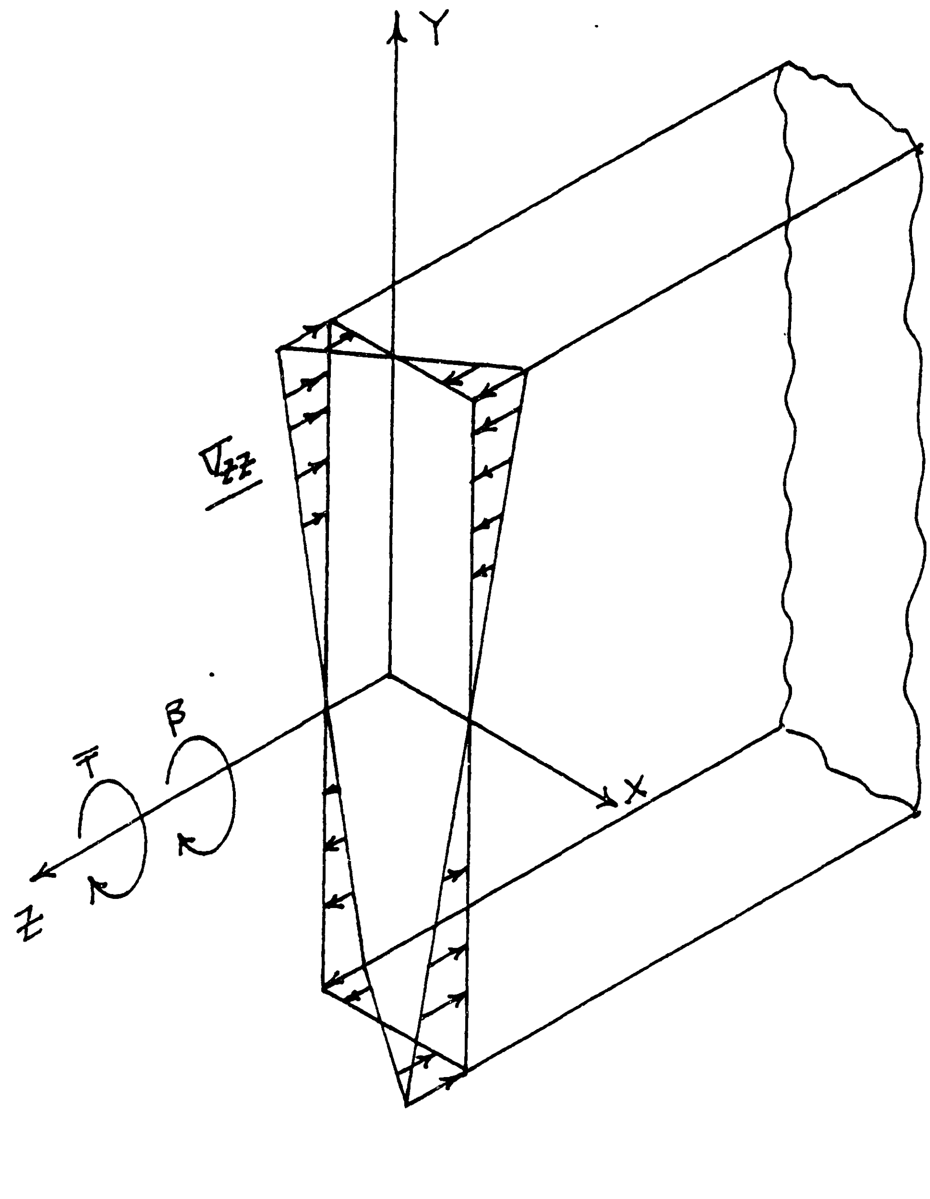

Figure 2. Stress distribution, z = ℓ.

Differentiating (3-18) gives

$$\beta'(z) = \left[\frac{T}{(\sigma_o + G)I_p} - \frac{d}{a}\right]\left[\frac{(1 - \cosh\sqrt{(a)}\,\ell)}{\sinh\sqrt{(a)}\,\ell}\sinh\sqrt{(a)}\,z + \cosh\sqrt{(a)}\,z\right] + (d/a) \qquad (3\text{-}30)$$

At z ≈ 0, the following approximations are made:

$$-\frac{\cosh\sqrt{(a)}\,\ell}{\sinh\sqrt{(a)}\,\ell} \approx -1, \text{ and } \frac{\sinh\sqrt{(a)}\,z}{\sinh\sqrt{(a)}\,\ell} \approx 0$$

thus,

$$\beta'(z) = \left[\frac{T}{(\sigma_o + G)I_p} - \frac{d}{a}\right](\cosh\sqrt{(a)}z - \sinh\sqrt{(a)}z) + d/a$$

or,

$$\beta'(z) = \left[\frac{T}{(\sigma_o + G)I_p} - \frac{d}{a}\right]e^{-\sqrt{(a)}z} + d/a \qquad (3\text{-}31)$$

and, using the same approximation with the sign reversed, (3-19) becomes

$$\alpha(z) = \frac{d'}{a}(\sinh\sqrt{(a)}\,z - \cosh\sqrt{(a)}\,z + 1)$$

or,

$$\alpha(z) = (d'/a)(-e^{-\sqrt{(a)}z} + 1) \qquad (3\text{-}32)$$

Combining Equations (3-31) and (3-32), and noting from Appendix B, d/a = d'/a, the following expression for σ_{xz} results.

$$\sigma_{xz} = Gy \frac{T}{(\sigma_o + G)I_p} e^{-\sqrt{(a)}z} \qquad (3\text{-}33)$$

where

$$\sqrt{(a)} = \frac{G}{b}\left[\frac{48\ \mu}{(\sigma_o + G)E}\right]^{1/2}$$

Inserting (3-22) into (3-33) gives σ_{xz} as a function of $\beta(\ell)$.

$$\sigma_{xz} = G^2 y \frac{4(c/b)^2 \mu e^{-\sqrt{(a)}z}}{(\sigma_o + G)(\ell - L_c)} \beta(\ell) \qquad (3\text{-}33)$$

Equation (3-33) is the equation for the shear stress in the x-direction near the end $z \approx 0$.

The maximum shear stress σ_{xzmax} occurs at the top and bottom edges of the flexure, $y = \pm b/2$, and at the end of the flexure. Using $y = b/2$, and $z = 0$, (3-33) becomes:

$$\sigma_{xzmax} = G^2 \frac{2(c^2/b)\mu}{(\sigma_o + G)(\ell - L_c)} \beta(\ell) \qquad (3\text{-}34)$$

Returning to Equation (3-30) and approximating the bracketed cosh and sinh terms at $z \approx \ell$ gives

$$\left[\frac{(1 - \cosh\sqrt{(a)}\ \ell)}{\sinh\sqrt{(a)}\ \ell} \sinh\sqrt{(a)}\ z + \cosh\sqrt{(a)}\ z\right]$$

$$\approx \frac{\sinh\sqrt{(a)}\ z}{\sinh\sqrt{(a)}\ \ell} = e^{\sqrt{(a)}(z-\ell)}$$

Proceeding as before the following expression for σ_{xz} at $z \approx \ell$ is derived.

$$\sigma_{xz} = G^2 y \frac{4(c/b)^2 \mu e^{\sqrt{(a)}(z-\ell)}}{(\sigma_o + G)(\ell - L_c)} \beta(\ell) \qquad (3\text{-}35)$$

Again σ_{xz} approaches a maximum value at $z = \ell$, and $y = \pm b/2$. Inserting $y = b/2$ and $z = \ell$ into (3-35) gives (3-34).

(c) Shear Stress, σ_{yz}.

Equation (3-30), $\beta'(z)$, and (3-19), $\alpha(z)$, will be combined with (3-10) to derive the equation for σ_{yz}. Proceeding as before two equations, one for each end, result.

For $z \approx 0$:

$$\sigma_{yz} = -Gx\left[\left[\frac{T}{(\sigma_o + G)I_p} - \frac{2d}{a}\right]e^{-\sqrt{(a)}z} + \frac{2d}{a}\right] \qquad (3\text{-}36)$$

where,

$$d/a = \frac{T}{1/3bc^3 G\left[1 + 1/4(b/c)^2 \sigma_o/G\right]}$$

or, in abbreviated form,

$$d/a = \frac{T}{1/3bc^3 G\mu}$$

Soving (3-22) for T and substituting into (3-36) gives:

$$\sigma_{yz} = \frac{-Gx}{(\ell - L_c)}\left[\left[\frac{4(c/b)^2 \mu G}{(\sigma_o + G)} - 2\right]e^{-\sqrt{(a)}z} + 2\right]\beta(\ell) \qquad (3\text{-}37)$$

where,

$$d/a = \frac{\beta(\ell)}{(\ell - L_c)}$$

Equation (3-37) is the equation for the shear stress in the y direction at $z \approx 0$.

The maximum stress, σ_{yzmax}, occurs at $z = 0$, and $x = \pm c/2$. Using $x = c/2$ and $z = 0$, (3-37) becomes:

$$\sigma_{yzmax} = \frac{-2G^2(c^3/b^2)\mu}{(\ell - L_c)(\sigma_o + G)} \beta(\ell) \qquad (3\text{-}38)$$

For $z \approx \ell$, the sinh and cosh terms are approximated as before in deriving (3-35), and (3-22) is used to substitute for T.

$$\sigma_{yz} = \frac{-Gx}{(\ell - L_c)} \left[\left[\frac{4(c/b)^2\mu G}{(\sigma_o + G)} - 2\right]e^{\sqrt{(a)}(z-\ell)} + 2\right]\beta(\ell) \qquad (3\text{-}39)$$

Again, σ_{yz} has a maximum value at $x = \pm c/2$ and $z = \ell$. Inserting these values into (3-39) gives (3-38).

The exponential terms in Equations (3-33), (3-35), (3-37), and (3-39) approach zero rapidly moving away from the ends. Therefore, within the interior of the flexure,

$$\sigma_{xz} = 0$$

$$\sigma_{yz} = -2Gx \frac{\beta(\ell)}{(\ell - L_c)}$$

This result closely matches the form of the equations derived for the shear stresses derived in Section B, St. Venant Solution, of the Background Theory. In particular, the shear stresses are identical if the ends of the flexure are free to warp, $L_C = 0$; and $\beta(\ell)/\ell$ is redefined as θ, twist per unit length.

E. Calculations

(1) Stress Distribution at $z \approx 0$.

Table 1 shows the change in axial and shear stesses from $z = 0$ to $z = 4$ μm.

The stresses, particularly lthe σ_{zz} stress, change most between $z = 0$ and $z = 2$ μm. At $z = 4$ μm, the σ_{zz} and σ_{xz} stresses are negligibly small, and the σ_{yz} stress has increased to a maximum indicating the transition from the zone affected by the warping constraint, to the interior region where the predominating stress distribution corresponds in form to the St. Venant stresses derived in the Background Theory.

The stress distribution at $z \approx \ell$ is identical to the values in Table 1 with the exception the σ_{zz} stress is opposite in sign.

Table 1. Stress at $z \approx 0$.

$\sigma_o = 700\ \mu N/\mu m^2$

2 Degree Rotation

z (μm)	$\sigma_{zz_{MAX}}\ \frac{\mu N}{(\mu m)^2}$	$\sigma_{xy_{MAX}}\ \frac{\mu N}{(\mu m)^2}$	$\sigma_{yz_{MAX}}\ \frac{\mu N}{(\mu m)^2}$
0.0	-61.1	2.31	-0.031
0.25	-36.5	1.38	-1.81
0.5	-21.8	0.824	-2.87
0.75	-13.0	0.492	-3.51
1.0	-7.78	0.294	-3.89
1.25	-4.61	0.175	-4.11
1.5	-2.77	0.105	-4.25
2.0	-0.989	0.037	-4.38
2.5	-0.353	0.013	-4.42
3.0	-0.125	0.004	-4.44
3.5	-0.045	0.002	-4.45
4.0	-0.016	0.000	-4.45

The stress distribution at $z \approx \ell$ is identical to the above values with the sign reversed for σ_{zz}.

Appendix C contains the equations, in reduced form, used to calculate the stresses in Table 1.

(2) Finite Element Comparison

A computer finite element analysis was employed to verify the stiffness and stress equations derived in this paper. The results of the analysis are tabulated in Tables 2 through 4. Two pivots, Pivot 1 and Pivot 2, of different dimensions were analyzed (see Table 5). Note that in Table 5 the elastic constants are different than those previously used.

Table 2 compares the calculated stiffness values versus the asterisked finite element values for zero pre-stress. Overall, the numbers correlated well. Two additional columns, one for each pivot headed K_{SV}, are included which show the value of the St. Venant stiffness calculated from Eq. (2-7) in the Background Theory.

Table 2. Stiffness.

$\sigma_o = 0$

2 Degree Rotation

c(μm)	$K_{(PIVOT\ 1)}$ $\frac{\mu n \cdot \mu m}{rad}$	K_{SV} $\frac{\mu n \cdot \mu m}{rad}$	$K_{(PIVOT\ 2)}$ $\frac{\mu n \cdot \mu m}{rad}$	K_{SV} $\frac{\mu n \cdot \mu m}{rad}$
0.11	7.15 *7.18	6.46	7.49 *7.43	6.74
0.22	57.2 57.02	51.71	59.9 *57.44	53.9

The calculated value for Pivot 2, of 0.22 μm thickness, differs by 4% from the finite element value. In general, the data appears to

Table 3. Stiffness.

$\sigma_o = 1346\ \mu n/(\mu m)^2$

2 Degree Rotation

c	K_{PIVOT2} $\frac{\mu n \cdot \mu m}{rad}$	K_{SV} $\frac{\mu n \cdot \mu m}{rad}$
0.11	31.86 *31.74	6.74

Table 4. Principal stress.

$\sigma_o = 0$

2 Degree Rotation

c(μm)	z(μm)	$\sigma_{p(PIVOT\ 1)}$ $\frac{\mu n}{\mu m^2}$	St. Vennant Shear Stress σ_{zyMAX}	$\sigma_{p(PIVOT\ 2)}$ $\frac{\mu n}{\mu m^2}$	St. Venant Shear Stress σ_{zyMAX}
0.11	0	19.43 *[illegible]2.7		61.0 *32.1	
	1	12.6 *12.7		25.3 *25.3	
	2	9.43 *10.0		22.2 *22.0	
	3	8.1 *8.21	6.23	21.7 22.0	19.43
0.22	0	38.8 *22.9		122 *65.8	
	1	25.3 *24.0		50.7 *48.4	
	2	18.85 *20.0		44.4 *40.0	
	3	16.1 *16.0	12.43	43.5 *40.0	38.86

Table 5. Pivot dimensions.

	PIVOT 1	PIVOT 2
b μm c μm ℓ μm	9 0.11, 0.22 44.11	3 0.11, 0.22 14.11
G $\frac{\mu n}{(\mu m)^2}$	7.14×10^4	7.14×10^4
E $\frac{\mu n}{(\mu m)^2}$	1.9×10^5	1.9×10^5

suggest greater error with increasing thickness and decreasing width. It is unclear, however, whether the source of error lies within the finite element analysis or the equations derived in this paper.

Consider the approximations of I_p, D, and D*. For Pivot 2, with c = 0.22 μm, and b = 3 μm, the error in neglecting c^2 in comparison with b^2 is 0.5%. This approximation introduces only a slight error, and is an unlikely source of significant error in the stiffness calculation.

Increasing the accuracy of the warping function is of no consequence. From Sokolnikoff[7], the second order approximation of the warping function is

$$\phi = xy - \frac{8c^2}{\pi^3} \frac{\sinh[(\pi/c)y]}{\cosh[(\pi/2)(b/c)]} \sin[(\pi/c)x]$$

Introducing this approximation does not alter the previous results becuase the additional terms subtracted from xy, in the equation above, integrate to zero in Γ, D, and D*.

Table 3 compares the calculated value of stiffness versus the finite element value for Pivot 2, c = 0.11 μm thick, under an initial axial pre-stress of 1346 $\mu N/\mu m^2$. Again, the calculated value correlates closely with the finite element value. A nonlinear analysis option was employed with the computer software in obtaining the finite element stiffness.

Table 4 compares the maximum principal stress of the finite element program, identified by the asterisks, to the calculated values. Again, the pre-stress is zero. Two additional columns, one for each pivot headed σ_{yzmax}, are included which show the maximum St. Venant shear stress calcualted from the St. Venant shear stress formula presented on Page 8 of the Background Theory.

The large difference in stress, at z = 0, is attributed to the inaccuracy of the finite element program. Originally, the objective of the finite element analysis was to determine the stiffness of the flexure. Because stiffness is related to displacement, sufficient accuracy was obtained by modeling the pivot with a course mesh of elements. Stress, however, depends of the derivative of displacement; thus, small errors in the stiffness calculations will be magnified into large errors in stress calculations. Generally, to compensate for this increase in error a dense mesh is used. Thus, for accurate predictions of stress a dense mesh should have been employed at the ends of the flexure.

Because the stresses in the flexure pivot change abruptly over the short distance between z = 0 and z = 1 μm, and because a course mesh was used, large differences in the calculated values versus finite values result.

Note, however, that the error of the finite element method diminishes rapidly with increasing length, and at z = 3 μm the two values roughly match.

In summary, the variational method provides a closed form solution, where equations describe the result, giving insight into the nature of the problem. The finite element method, however, yields only a number given a particular input, and therefore, requires some interpretation of the results. In particular, unless one is aware of how nonlinear and localized effects, such as axial pre-stress and restraint of warping, can alter the solution, the finite element method may give erroneous results. In general, it is wise to use both methods together; the variational method giving the closed form solution, and the finite element method providing a useful check.

SECTION 4

CONCLUSIONS

A set of equations were derived which calculate the torsional stiffness and resultant stresses of a thin rectangular shaft under initial axial pre-stress with ends constrained from warping.

The analysis was linear, assuming small rotations and displacements, except for the additional nonlinear displacement term added to Reissner's equation to account for the significant effect of the axial pre-stress.

The equations were then specialized and approximated for the case of a thin rectangular shaft where c^2, the thickness squared, is negligible in comparison with b^2, the width squared.

The equations were then verified with a finite element analysis. All calculated values of stiffness correlated well with the finite element values. The values of stress correlated well only within the interior region of the flexure. At the ends, large differences in the normal stress were noted becuase of the abrupt change in the predominant axial stress, and the inability of the finite element program to accurately calcualte such a large change given the course mesh model of the pivot.

SECTION 5

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SECTION 6

APPENDIX A

Figure 3 shows the change in an arbitrary differential line segment as it translates and rotates from the undeformed configuration, denoted by $\overline{pq}$, to the deformed configuration, $\overline{PQ}$. The coordinate system is a rectangular 1,2,3 system.

Consider the square of the change in length of this segment as:

$$\bar{PQ}^2 - \bar{pq}^2 = dS^2 - ds^2$$

where

$$dS = |d\bar{R}|$$

and

$$ds = |d\bar{r}|$$

Thus

$$dS^2 - ds^2 = d\bar{R} \cdot d\bar{R} - d\bar{r} \cdot d\bar{r} \tag{6a-1}$$

Equation (6a-1) expresses the change in the square of the length of the differential line segment as the dot product of the line segment vector, with itself, in the undeformed configuration subtracted from the dot product of the vector in the deformed state.

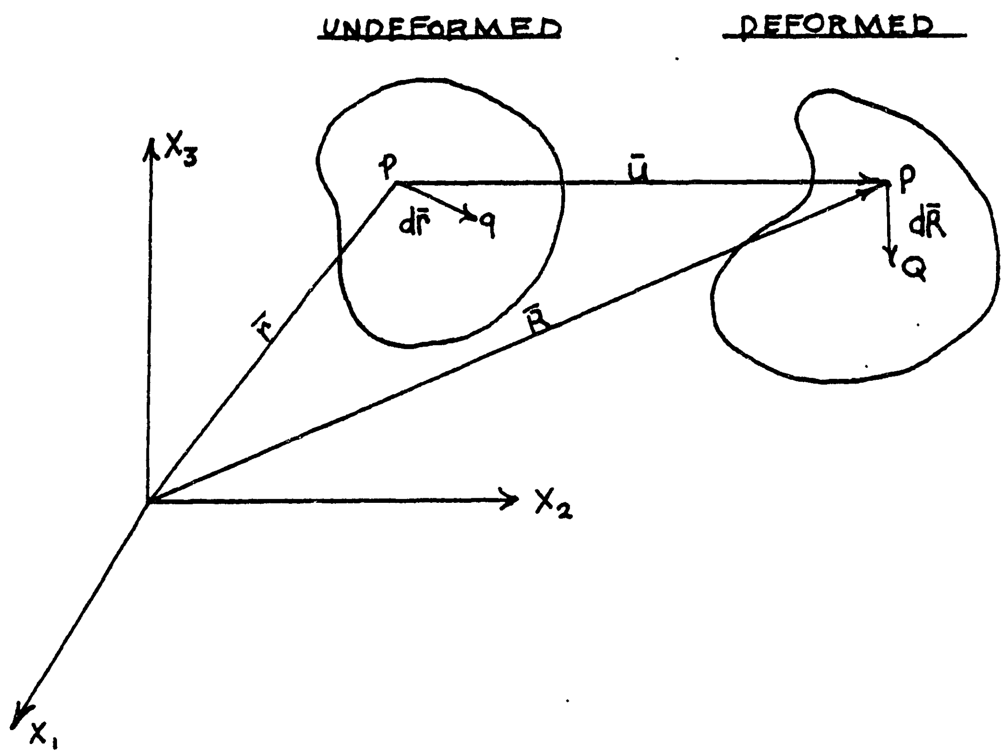


Figure 3. Strain displacement relation.

$$d\bar{R} = \frac{\partial \bar{R}}{\partial x_1} dx_1 + \frac{\partial \bar{R}}{\partial x_2} dx_2 + \frac{\partial \bar{R}}{\partial x_3} dx_3$$

and

$$d\bar{r} = \frac{\partial \bar{r}}{\partial x_1} dx_1 + \frac{\partial \bar{r}}{\partial x_2} dx_2 + \frac{\partial \bar{r}}{\partial x_3} dx_3$$

or, in shorthand notation,

$$d\bar{R} = \sum_{i}^{3} \bar{R},_i dx_i \tag{6a-2}$$

$$d\bar{r} = \sum_{i}^{3} \bar{r},_i dx_i \tag{6a-3}$$

The commas indicate partial differentiation with respect to the coordinate index i.

Using (6a-2) and (6a-3), (6a-1) can be written as:

$$dS^2 - ds^2 = \sum_{ij}^{3} (\bar{R},_i \cdot \bar{R},_j - \bar{r},_i \cdot \bar{r},_j)dx_i dx_j \tag{6a-4}$$

where the summation is carried out over all nine combinations of i and j. Use of the different subscript j ensures that the dot product is carried out fully over all possible terms of the line segment vectors maintaining complete generality.

From Fig. 3:

$$\bar{R} = \bar{r} + \bar{u} \tag{6a-5}$$

Substituting (6a-5) into (6a-4) gives:

$$dS^2 - ds^2 = \sum_{ij}^{3} (\bar{r},_i \cdot \bar{u},_j + \bar{r},_j \cdot \bar{u},_i + \bar{u},_i \cdot \bar{u}_j) dx_i dx_j \qquad (6a\text{-}5)$$

Now define strain as:

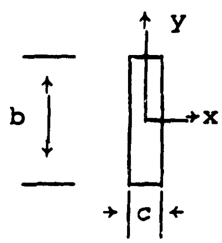
$$dS^2 - ds^2 = 2\sum_{ij}^{3} \varepsilon_{ij} \left|\bar{r},_i\right| dx_i \left|\bar{r},j\right| dx_j \qquad (6a\text{-}7)$$

Inserting (6a-7) into (6a-6) gives:

$$\varepsilon_{ij} = \frac{\frac{1}{2}[\bar{r},_i \cdot \bar{u},_j + \bar{u},_i \cdot \bar{r},_j + \bar{u},_i \cdot \bar{u},_j]}{\left|\bar{r},_i\right| \quad \left|\bar{r},_j\right|} \qquad (6a\text{-}8)$$

Equation (6a-8) is the generalized strain displacement relation including non-linear terms. Note that although the analysis was carried out assuming a rectangular 1,2,3 coordinate system, any arbitrary orthogonal curvilinear system may have been used.

APPENDIX B



Flexure X-sec

	Dimensions (m)
b	8.00 E-6
c	0.11 E-6
ℓ	45.00 E-6

<u>Mechanical Properties of Silicon</u>

G: Torsional Modulus (n/m^2) = 5.1 E10

E: Young's Modulus (n/m^2) = 1.7 E11

For all cases σ_o = 7.0 E8 (n/m^2)

(1) $$D^* = \iint (x\Phi,_y - y\Phi,_x)dxdy$$

$$D^* = \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} (y^2 - x^2)dxdy$$

$$D^* = 1/12bc(b^2 - c^2)$$

(2) $$I_p = \iint (x^2 + y^2)dxdy$$

$$I_p = \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} (x^2 + y^2)dxdy$$

$$I_p = 1/12bc(b^2 + c^2)$$

(3) $$\Gamma = \iint \Phi^2 \, dxdy$$

$$\Gamma = \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} (x^2y^2)dxdy$$

$$\Gamma = 1/144\ b^3c^3$$

$$D = \iint (\Phi,_x^2 + \Phi,_y^2)dxdy$$

$$D = \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} (x^2 + y^2)dxdy$$

$$D = 1/12bc(b^2 + c^2)$$

NOTE: $D = I_p$

From equations (3-18) and (3-19):

$$a = \frac{G^2}{(\sigma_o + G)E\Gamma}\left[\frac{DI_p - D^{*2}}{I_p} + \frac{\sigma_o}{G} D\right]$$

$$a = \frac{G^2}{(\sigma_o + G)E1/144b^3c^3}\left[\frac{4/144\ b^4c^4}{1/12\ b^3c} + \frac{\sigma_o}{G}\ 1/12b^3c\right]$$

Note: $I_p \approx 1/12b^3c$ for $c << b$

$DI_p - D^{*2} = 4/144\ b^4c^4$ exactly.

$$a = \left[\frac{G^2}{(\sigma_o + G)E}\frac{48}{b^2}\ (1 + 1/4(b/c)^2\sigma_o/G)\right]$$

$$a = \frac{48(5.1\ E10)^2 \cdot [1+0.25(8/.11)^2(7.0\ E8/5.1\ E10)]}{(7.0\ E8 + 5.1\ E10)(1.7\ E11)(8.0\ E{-6})^2}$$

$$a = 4.249\ E12\ /m^2$$

$$\sqrt{(a)} \cdot \ell = (4.249\ E12)^{1/2}\ (45.0\ E{-6})$$

$$\sqrt{(a)} \cdot \ell = 92.7$$

From equations (3-15) and (3-16)

$$d/a = \frac{T}{G\left[\frac{(\sigma_o + G)I_p}{G} - \frac{D^{*2}}{D}\right]}$$

$$d/a = \frac{T}{G\left[\frac{\sigma_o I_p}{G} - \frac{DI_p - D^{*2}}{D}\right]}$$

$$d/a = \frac{T}{G\left[\frac{4/144\ b^4c^4}{1/12\ b^3c} + \frac{\sigma_o}{G}\ 1/12b^3c\right]}$$

where $DI_p - D^{*2} = 4/144\ b^4c^4$ exactly

and $D \approx 1/12\ b^3c$ for $c << b$

$$d/a = \frac{T}{1/3bc^3G[1 + 1/4(b/c)^2\sigma_o/G]}$$

Note that for Equation (3-19) $d'/a = d/a$

APPENDIX C

SAMPLE CALCULATIONS

Parameters

$x = c/2$

$y = b/2$

$\sigma_o = 700\ \mu n/\mu m^2$

$\beta(\ell) = 2\ \text{deg} = \pi/90\ \text{rad}$

$G = 5.1 \times 10^4\ \mu n/\mu m^2$

$E = 1.7 \times 10^5\ \mu n/\mu m^2$

$b = 8\ \mu m$

$c = 0.11\ \mu m$

$\ell = 45\ \mu m$

Conversion Factor

$$10^6 \frac{\mu N}{\mu m^2} = \frac{1N}{m^2}$$

1. $$\sigma_{zz} = -cG\left[\frac{6\sqrt{E}\ \mu\ \beta(\ell)}{\ell\left[12(\sigma_o + G)\mu\right]^{1/2} - b\sqrt{E}}\right]e^{-\sqrt{a}\ z}$$

$$\mu = 1 + \frac{1}{4}\left(\frac{b}{c}\right)^2 \frac{\sigma_o}{G}$$

$$\mu = 1 + \frac{1}{4}\left(\frac{8}{0.11}\right)^2 \frac{7 \times 10^8}{5.1 \times 10^{10}}$$

$$\mu = 19.15$$

From Appendix B

$$a = 4.249 \times 10^{12}/m^2$$

$$\sqrt{a} = 2.062 \times 10^6/\text{m}$$

$$\sqrt{a} = 2.062/\mu\text{m}$$

$$\sigma_{zz} = -(0.11)(5.1 \times 10^4)\left[\frac{6\sqrt{1.7 \times 10^5}\ (19.15)(\pi/90)}{45\left[12(700 + 5.1 \times 10^4)19.15\right]^{1/2} - 8\sqrt{1.7 \times 10^5}}\right] e^{-2.062\ z}$$

$$\sigma_{zz} = -61.1\ e^{-2.062\ z}\ \mu\text{N}/\mu\text{m}^2$$

2. $$\sigma_{xz} = G^2\ \frac{2(c^2/b)\mu\beta(\ell)}{(\sigma_o + G)(\ell - L_c)}\ e^{-\sqrt{a}\ z}$$

$$L_c = b\ \frac{\sqrt{E}}{\left[12(\sigma_o + G)\mu\right]^{1/2}}$$

$$L_c = 8\left[\frac{1.7 \times 10^5}{12(700 + 5.1 \times 10^4)(19.15)}\right]^{1/2}$$

$$L_c = 0.957$$

$$\sigma_{xz} = \frac{(5.1 \times 10^4)^2\ 2\ \frac{(0.11)^2}{8}\ (19.15)(\frac{\pi}{90})}{(700 + 5.1 \times 10^4)(45 - 0.957)}\ e^{-2.062\ z}$$

$$\sigma_{xz} = 2.31\ e^{-2.062\ z}$$

3. $$\sigma_{yz} = \frac{-GC}{(\ell - L_c)}\left[\left(\frac{2(c/b)^2\mu G}{(\sigma_o + G)} - 1\right)e^{-\sqrt{a}\ z} + 1\right]\beta(\ell)$$

$$\sigma_{yz} = \frac{(-5.1 \times 10^4)(0.11)(\pi/90)}{45 - 0.957}\left[\left(\frac{2(\frac{0.11}{8})^2(19.15)(5.1 \times 10^4)}{(700 + 5.1 \times 10^4)} - 1\right)e^{-\sqrt{a}\ z} + 1\right]$$

$$\sigma_{yz} = -4.45\left[-0.993\ e^{-2.062\ z} + 1\right]$$