VIBRATIONAL STATES OF ODD NUCLEI
BY THE QUASI-PARTICLE METHOD

by

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ABSTRACT

The properties of the low lying energy levels of an odd mass nucleus in the vibrational region are investigated using the quasi-particle method. A non-pairing ("long-range") type of interaction is assumed to act between the nucleons. We expand this interaction in multipoles and keep only the quadrupole term to carry out diagonalizations in two different approaches: (i) the quasi-Boson plus quasi-particle scheme, which imagines that the lowest levels of the adjacent even-even nucleus are well described as quasi-Boson states to which an odd quasi-particle may be coupled, and (ii) the three quasi-particle scheme which treats directly the odd mass nucleus as an assemblage of quasi-particles. The wavefunctions generated are used to study electric transition rates and inelastic alpha scattering cross sections. Because of readily available and fairly complete experimental data Cu^{63} was chosen as a test case for these two methods. Scheme (ii) gives many more low lying states than scheme (i); however, in both schemes a "quartet" of spins, $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$, appears with strong electric transitions and large inelastic alpha scattering cross sections that also obey the well known parity phase rules for excitation by alpha-particles. Such states are identified with the "collective" levels observed experimentally for Cu^{63}. It is found that scheme (i) is far superior to (ii) as regards agreement between calculation and experiment for negative parity states. In the case of positive parity levels, however, scheme (i) turns out to be unsuitable and scheme (ii) is used. In this case eight positive parity levels of a collective nature are produced. Comparison with observed energies and excitation cross sections in alpha particle scattering is made in all cases where such experimental data is available. Sum rules for electric quadrupole transitions are also derived and are compared with experiment.

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INTRODUCTION

The properties of a nucleus consisting of a tightly bound group of nucleons (the "core") plus an odd "valence" nucleon have been considered by many workers. The possible excitations are those of the "core", the single nucleon, and their coupled combination. In most cases a specific model has been used to describe the modes of core excitation with the odd particle serving as a perturbation. Of particular interest in the present investigation will be the effect of such an odd nucleon on the vibrational excitations of a "core" possessing spherical symmetry.

It is well known that[1)] a weak coupling of a particle of spin j to a pure even-even nucleus excited "core" state of spin I produces $2j+1$ (for $j < I$) or $2I+1$ (for $I < j$) states. These states are predicted to be at energies such that a "center of gravity" rule is obeyed. De Shalit[2)] has used this idea to describe the systematics of excited states in several nuclei. Like Reference 1), no specific "core" excitation mechanism is assumed. For a nucleon coupled to the 2^+ first excited state of an even-even nucleus, all levels obtained in the weak coupling should have equal B(E2)'s to the ground state. In addition these transitions can have no M1 component.

To explain the fact that such levels do indeed possess in many cases rather large B(M1)'s, Braunstein and De Shalit[3] allowed for a modicum of configuration mixing. This was also able to justify the observation that the j excited state to j ground state B(E2) was weaker than that from the other excited states to the ground state. Despite the above mentioned merits, the weak coupling scheme does not appear to provide a systematic description of the excited states of odd mass nuclei. In fact, it is very hard to understand the spin ordering in any simple way. For example, a naive prediction of spins would place the level of highest spin lowest; the next level would have smaller spin, and finally the highest energy level would be that of smallest spin. This is seen by considering the odd nucleon as "caught" in the attractive potential of the "core", so that the higher the total angular momentum, the more likely is the particle to be moving with rather than away from this attraction. This is not in accord with observation, e.g., Tl^{203} has a $1/2^+$ ground state followed by $3/2^+$ and $5/2^+$ states at 279 and 401 kev respectively.

Coulomb excitation[4] has long been known as a mechanism for producing excited states of a collective nature. Inelastic alpha scattering experiments with odd mass targets can yield anomolously large cross sections that are also indicative of collective excitation.[5-7] Because of the energy separation of these levels the idea of weak coupling seems questionable.

In particular, recent experiments on Cu^{63} have shown a disagreement with the weak coupling scheme, so it appears a general investigation of the properties of an odd mass nucleus in the vibrational region is indicated.

The weak coupling of the odd proton to the 2^+ state of Ni^{62} predicts a quartet of low lying levels of spins, $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$. Of these only the $1/2^-$, $5/2^-$, and $7/2^-$ have been found by Gove[8] to have measurable B(E2)'s to the ground state. The $3/2^-$ level should be present if the idea of Reference 1) is correct. Even an admixture of the $3/2^-$ single particle in the excited state is inadequate, for then too high an M1 transition is predicted. Bayman and Silverberg[9] have shown that such a decrease in $3/2^-$ to $3/2^-$ B(E2) can only be produced in strong coupling.

An inelastic alpha scattering experiment by Meriwether, et al,[10] has shown that the low lying negative parity levels of Cu^{63} have differential cross sections that are out of phase with the elastic. Again the $3/2^-$ excited state is not discernable. Five positive parity levels are found near 3 mev, and all have large inelastic differential cross sections in phase with the elastic. Weak coupling to the Ni^{62} 3.5 mev 3^- state predicts only 4 levels. Thus, while the obeying of the phase rule and the magnitude of the cross sections indicate "collective" behavior, the origin of this "core" type of excitation must still be explained.

Bouten and Van Leuven[11)] have obtained reasonable agreement with the energy levels and spins of Cu^{63} by assuming a coupling of the $2p_{3/2}$-, $1f_{5/2}$-, and $2p_{1/2}$- single protons to quadrupole surface vibrations of up to two phonons. They have, however, not indicated where the other energy levels predicted from their diagonalizations are. Indeed, if there were many low lying levels, then some criterion must be set for selecting the "collective" ones. Only by fitting "core" parameters and single proton energies to give the best agreement with experiment are they able to obtain results. The B(E2)'s that they predict do not agree well with experiment.

Approaches to the problem of the structure of the low lying levels of an odd nucleus in the vibrational region that do not involve arbitrary parameters, and that might be considered more intrinsic to the odd mass nucleus are considered in the present investigation. Because of the rather complete data one now has for Cu^{63}, this nucleus will serve as a test case for the approaches we use. In the next few pages the stages leading to the basic ideas of this investigation will be described in order to provide the proper context.

Recent advances in the many body problem of nuclear physics have been made by several workers (e.g. Belyaev[12)]) through application of the work of Bardeen, et al[13)] in superconductivity. In Reference 13) the energy gap between the normal and superconducting states in a metal is described as a result of a pairing interaction with the help of a Bogoliubov-Valatin[14)] transformation, i.e., conversion of real electron annihilation and creation operators to those of quasi-electrons. In a similar way, the large energy gap between the ground and first excited states of most even-even nuclei has been explained on the basis of a pairing force with such a transformation applied to the nucleon annihilation and creation operators.

For nuclei that are several nucleons beyond or before closed shells, however, the first 2^+ state appears at an energy of roughly half the gap. A residual interaction of a quadrupole nature in addition to the pairing correlation has been successful in explaining the energy and large B(E2) of this low lying 2^+ state. Such a force was first suggested by Elliot[15)] and applied by Arvieu and Veneroni[16)] to the quasi-particle representation. Kisslinger and Sorenson[17)] showed that such an interaction when treated as a perturbation to existing quasi-particle levels, could not sufficiently lower the 2^+ state or yield a large enough B(E2). Using collective coordinates and the adiabatic approximation, however, they were able to obtain the frequency pertinent to such a quadrupole deformation. The systematics of single closed shell even-even nuclei were then explained by fitting pairing and long-range force strengths to the available data.

The method of the random phase approximation as first shown by Baranger[18] was also able to give the correct energy and B(E2) for the 2^+ first excited state. Such a method assumes that the quasi-Fermions behave "collectively" like a Boson provided there are not too many Fermions, for then the Pauli principle becomes of overwhelming importance. The Boson approximation while lacking the physical picture implicit in a core vibration, is much more suitable in showing from what terms "collective" behavior arises.

In a more recent paper Kisslinger and Sorenson[19] demonstrated the equivalence of the adiabatic and Boson approximations for energies at about half the gap. Only the latter is successful, however, in the limit of a very weak quadrupole interaction. The Boson approximation simplifies calculations in that one need never know the wavefunctions for the various states. This comes about through application of the Boson commutation rules, and the use of quasi-Boson creation and annihilation operators such that, e.g., the ground and first excited states have respectively zero and one quasi-Boson.

In this same paper[19] the introduction of a neutron-proton component of the long-range interaction has made possible an extension of these ideas to all even-even nuclei where the quasi-particle picture is applicable (i.e., not just nuclei of a single closed shell nature).

Kisslinger and Sorenson in their first paper[17] only dealt with one quasi-particle states when discussing odd mass nuclei. The first attempt at coupling the phonons of vibration with single quasi-particles through the long-range interaction was made by Sorenson.[20] He justified the use of the adiabatic approximation in certain cases to describe the phonons, but generally concluded that the Boson approximation is the more suitable approach. For the first time a diagonalization was necessary, in that the coupling of odd quasi-particle to phonon removed the energy degeneracy. In general the energies and spin assignments he gets for such single closed shell odd mass nuclei as Pb, Sn, and Ni are not in very good agreement with experiment.

In the second Kisslinger and Sorenson paper,[19] wave-functions are derived for the states of odd mass nuclei using the Boson approximation from the outset. Subsequent papers by Sorenson[21,22] have met with moderate success in fitting experimental B(E2)'s in the Ni to Pb region, and attributing l-forbidden M1 transitions to quadrupole coupling. Yoshida[23] has used similar wavefunctions as obtained from both quadrupole and octupole long-range interactions.

The energies, spins, B(E2)'s, B(E3)'s, inelastic alpha scattering cross sections etc. that he obtains only agree in certain cases with experiment. In both that and the previously mentioned papers, the phonon plus quasi-particle coupling schemes can't be depended on to give reasonable answers for an arbitrary odd mass nucleus.

The inadequacy would seem to lie in the evaluation of matrix elements. The second paper of Kisslinger and Sorenson[19)] and the single papers of Sorenson[20,21,22)] utilize the zero to one and two to one phonon matrix elements as obtained by Choudhury.[24)] Yoshida uses those calculated by Ford and Levinson.[25)] However, these matrix elements are based on specific assumptions regarding the weak coupling of nucleons to surface vibrations. The "collective" assumptions regarding the use of surface tension and mass parameters that one has avoided in the Boson approximation are once again brought out, i.e., the quasi-Boson scheme is literally abandoned when treating odd mass nuclei.

The aim of the present investigation will be to use the correct matrix elements, i.e., to apply the properties of quasi-Bosons to odd mass nuclei. At no time will there be a reversion to the ideas of an incompressible fluid, and most importantly there will be no arbitrary parameters or assumptions relating to a "surface". It will be shown that without these assumptions only states of large B(E2) (and hence"collective") have form factors, as determined by inelastic alpha scattering, that seem to be peaked at the "surface" of the nucleus. Thus "surface" vibrations will be the logical.physical interpretation rather than the actual cause of the collective properties of the nuclei under consideration.

In Chapter I there will be a brief review of the quasi-particle technique for diagonalizing the pairing plus shell model Hamiltonian. The long-range (i.e., non-pairing) interaction will then be introduced to include all multipole orders, and the Boson approximation method for its diagonalization will be summarized. As a starting point for the present investigation, the appropriate wavefunctions for odd mass nuclei will be introduced which include up to one quasi-Boson. In order to learn just how good the Boson-approximation is, an alternate procedure will use wavefunctions of up to three quasi-particles. For reasons to be discussed in Chapter I, only couplings involving:

(1) a quasi-Boson made up of quasi-neutrons (protons) plus an odd quasi-proton (neutron), and

(2) two quasi-neutrons (protons) plus an odd quasi-proton (neutron) will be investigated.

Chapter II will present the general expressions for the matrix elements in both of these schemes. In particular certain elements will be seen to vanish because of parity restrictions implicit in the multipole order of the long-range interaction. The two different types of matrix elements will be compared and the actual terms dropped in the Boson approximation will be shown to occur naturally in the two quasi-neutron scheme. In addition, interaction graphs will be presented to help visualize differences in the two schemes.

In Chapter III the expressions for reduced electric transition probabilities will be derived in terms of the wavefunctions that result from the diagonalizations of Chapter II. Also expressions for the inelastic scattering cross sections of alpha particles will be obtained in both the Born and distorted wave Born Approximations. The former will employ a delta function for the alpha-nucleon interaction, whereas, a Gaussian will be used as the interaction in the latter case. Kisslinger[26)] used the Boson approximation to explain some of the experimental features of the differential cross sections in alpha and electron scattering from Ni^{58}. This will, however, be the first application of the quasi-Boson scheme to odd mass nuclei where the wavefunctions are indeed of true quasi-Boson origin. The two quasi-neutrons plus quasi-proton scheme will also be used throughout this chapter.

The specific application to Cu^{63} to obtain level schemes, electric transition probabilities, and inelastic alpha scattering cross sections will occupy Chapter IV. The relative merits of the alternate schemes will become manifest as one compares predictions with the previously mentioned experimental results of Gove,[8] and Meriwether, et al.[10]

Chapter V summarizes and discusses the main conclusions of this work and points out possible extensions.

CHAPTER I

This presents an investigation of the theoretical properties of the vibrational states in odd mass nuclei, with particular reference to those nuclei adjacent to spherical even-even nuclei having one major closed shell.

The following assumptions are made:

1. To lowest order nucleons move independently in a shell model potential.

2. The first correction to this independent motion is a pairing interaction between nucleons.

3. Only pairing correlations between nucleons of like charge are considered important. This assumption may, however, be questionable.[19)]

4. The shell model plus pairing Hamiltonian may be approximately diagonalized by employing the method of quasi-particles.[27)]

5. The "collective" properties of even-even nuclei may

be described by the addition of a long-range interaction between nucleons.

6. The approximate diagonalization of the entire Hamiltonian (shell model plus pairing plus long-range) may be obtained for even-even nuclei by means of the so-called Boson approximation[18] (also known as the random phase approximation and the Sawada[28] approximation).

The present work will seek to determine the worth of the Boson approximation when applied to even-odd nuclei. This will involve a coupling of the odd particle to the states of the adjacent even-even nucleus. In addition, there will be an investigation more intrinsic to the odd mass nucleus, i.e., the approximate diagonalization of the entire Hamiltonian without making the Boson approximation. This will not be a simple coupling of the odd particle to the even-even nucleus. A further description of the methods to be employed will be delayed until Section D) of this Chapter.

A very brief explanation of the basic ideas behind the quasi-particle technique and the Boson approximation is necessary. This will give the proper perspective to what follows in the present investigation, and serve to introduce the equations that will be referred to quite often. The notation follows very closely that of Bayman[27] whose paper may be consulted for further details.

This Chapter will consist of four main sections, the fourth of which, Section D, has been previously alluded to. Section A) will pertain to the shell model plus pairing Hamiltonian, and its approximate diagonalization by use of the quasi-particle scheme. Section B) will introduce the long range interaction, and Section C) will summarize the Boson approximation in even-even nuclei. The original work for even-odd nuclei will commence in Section D).

SECTION A. The Shell Model Plus Pairing Hamiltonian and the Quasi-Particle Transformation.

1. General Description:

The energy gap occuring in even-even nuclei between the ground and first excited states may be explained by considering a pairing force acting only between nucleons in the same (ℓ, j) shell, coupling pairs of these nucleons to zero total angular momentum. The approximate diagonalization of the shell model plus pairing Hamiltonian may be accomplished by transforming from the set of real annihilation and creation operators to a set of operators that annihilate and create fictitious (or quasi-) particles. This diagonalization is at the sacrifice of the constancy of the number of real particles, and results in a smearing out of the Fermi surface, i.e., single real particle states will have certain probabilities of being occupied. Excited nuclear states are obtained by creating quasi-particles into the quasi-particle vacuum, and the number of such particles is the analog of the term seniority (or number of unpaired particles). In no case may the number of quasi-particles exceed the number of real particles.

2. Notation Conventions:

Before giving a summary of the pertinent equations, a few conventions in notation to be used throughout this paper shall be established for simplicity.

a) The letters N and P when used as subscripts, superscripts, or arguments denote neutron and proton respectively. The neutron or proton referred to may be real or quasi, depending on the context.

b) The letter j with appropriate subscripts shall be used to represent the angular momentum of a single particle (real or quasi). For example, j_{a_N} is the angular momentum of a neutron in the single neutron state of energy, ϵ_{a_N} , or of a quasi-neutron in the single quasi-neutron state of energy, E_{a_N} .

c) The z component of j will always be referred to by the letter, m , with the same subscripts as j . For example, m_{a_N} , is the z component of j_{a_N} .

d) The orbital angular momentum of a single particle will always be written as ℓ with appropriate subscripts. Hence, ℓ_{a_N} is the orbital angular momentum associated with the total angular momentum, j_{a_N} .

e) The single particle radial quantum number will be written as n with the desired subscripts. This means that the radial quantum number associated with j_{a_N} is n_{a_N} .

f) Whenever j is referred to, then ℓ and n are also implied. Thus, j_{a_N} means the angular momentum of a (real or quasi) neutron that is also characterized by ℓ_{a_N} and n_{a_N} . In particular, $\sum_{j_{a_N}}$ is the same as $\sum_{j_{a_N}, \ell_{a_N}, n_{a_N}}$
However, m_{a_N} will be explicitly written out when a particular z component of j_{a_N} is desired.

g) When j_{a_N} is used explicitly as e.g., $(-1)^{1/2 + j_{a_N}}$, then, of course, only the particular value of j_{a_N} is used (n_{a_N} and ℓ_{a_N} being irrelevant).

h) The only occasion for the explicit writing out of ℓ_{a_N} will be in terms like $(-1)^{\ell_{a_N}}$.

i) The only occurrence of n_{a_N} will be for matrix elements such as $\langle n_{a_N} | f(r) | n_{b_N} \rangle$, where $f(r)$ is some function of neutron (real or quasi) coordinates.

j) The creation operator for a neutron in state, j_{a_N} , with z component, m_{a_N} , will be written as $b^{+j_a}_{m_a}(N)$. This eliminates having to write the subscript N twice. However, in all other parts of an expression j_{a_N} and m_{a_N} will both be subscripted, e.g., $(-1)^{j_{a_N}+m_{a_N}} b^{+j_a}_{m_a}(N)$. By logical extension, a similar notation is used for quasi-proton annihilators, proton creators, etc.

3. Summary of Important Equations:

Invoking these rules for notation, the shell model Hamiltonian is given in the real nucleon occupation number representation by

$$H_{\substack{shell\\model}} = \sum_{\eta j_{a_\eta} m_{a_\eta}} \epsilon_{j_{a_\eta}} b^{+j_a}_{m_a}(\eta)\, b^{j_a}_{m_a}(\eta) \qquad (I\ 1)$$

and the pairing interaction in this same representation is

$$H_{pair.} = -\frac{G_\eta}{4} \sum_{\substack{\eta j_{1_\eta} m_{1_\eta}\\ j_{2_\eta} m_{2_\eta}}} (-1)^{l_{1_\eta}+j_{1_\eta}+m_{1_\eta}+l_{2_\eta}+j_{2_\eta}+m_{2_\eta}} b^{+j_1}_{m_1}(\eta)\, b^{+j_1}_{-m_1}(\eta) \times b^{j_2}_{-m_2}(\eta)\, b^{j_2}_{m_2}(\eta) \qquad (I\ 2)$$

The index η may be N or P, and $G_\eta > 0$ gives an attractive pairing force of strength measured by G_η. The creation and annihilation operators of nucleons are respectively $b^+(\eta)$ and $b(\eta)$. In order to insure the correct transformation properties under rotations, the annihilator $b(\eta)$ must be replaced by $c(\eta)$ via

$$c^{j_a}_{m_a}(\eta) = (-1)^{j_{a_\eta}-m_{a_\eta}} b^{j_a}_{-m_a}(\eta) \qquad (I\ 3)$$

The quasi-particle creators and annihilators are $\beta^+(\eta)$ and $\beta(\eta)$ with $\gamma(\eta)$ (by analogy to (I3)) replacing the latter via

$$\gamma^{j_a}_{m_a}(\eta) = (-1)^{j_{a_\eta}-m_{a_\eta}} \beta^{j_a}_{-m_a}(\eta) \qquad (I\ 4)$$

If $u_{j_{a_\eta}}$ and $v_{j_{a_\eta}}$ are transformation coefficients satisfying the normalization condition

$$u_{j_{a_\eta}}^2 + v_{j_{a_\eta}}^2 = 1$$

(I5)

then $\beta^{+j_a}_{m_a}(\eta)$ and $\gamma^{j_a}_{m_a}(\eta)$ are given by

$$\beta^{+j_a}_{m_a}(\eta) = -(-1)^{l_{a_\eta}} v_{j_{a_\eta}} c^{j_a}_{m_a}(\eta) + u_{j_{a_\eta}} b^{+j_a}_{m_a}(\eta) \quad \text{(I6a)}$$

and

$$\gamma^{j_a}_{m_a}(\eta) = u_{j_{a_\eta}} c^{j_a}_{m_a}(\eta) + (-1)^{l_{a_\eta}} v_{j_{a_\eta}} b^{j_a}_{m_a}(\eta) \quad \text{(I6b)}$$

The reverse transformations are

$$b^{+j_1}_{m_1}(\eta) = u_{j_{1\eta}} \beta^{+j_1}_{m_1}(\eta) + (-1)^{l_1} v_{j_{1\eta}} \gamma^{j_1}_{m}(\eta) \quad \text{(I6c)}$$

and

$$c^{j_1}_{m_1}(\eta) = -(-1)^{l_{1\eta}} v_{j_{1\eta}} \beta^{+j_1}_{m_1}(\eta) + u_{j_{1\eta}} \gamma^{j_1}_{m_1}(\eta) \quad \text{(I6d)}$$

In the quasi-particle occupation number representation (I1) plus (I2) eventually becomes

$$H_{\substack{\text{Shell}\\ \text{Model}}} + H_{\text{Pair.}} = H_{oo} + \sum_{\eta j_{a_\eta} m_{a_\eta}} (-)^{j_{a_\eta}+m_{a_\eta}} E_{j_{a_\eta}} \beta^{+j_a}_{m_a}(\eta) \gamma^{j_a}_{-m_a}(\eta) \quad \text{(I7)}$$

where H_{oo} is the approximate quasi-particle vacuum energy. The quasi-particle energy, $E_{j_{a_\eta}}$, is given in terms of the reduced single nucleon energy, $\tilde{\epsilon}_{j_{a_\eta}}$, the Fermi energy, δ_η, and the semi-gap, Δ_η, by

$$E_{j_{a_\eta}} = \sqrt{(\tilde{\epsilon}_{j_{a_\eta}} - \delta_\eta)^2 + \Delta_\eta^2} \quad \text{(I8)}$$

The definition of Δ_η is

$$\Delta_\eta \equiv G_\eta \sum_{j_{a_\eta}} (j_{a_\eta} + \tfrac{1}{2}) u_{j_{a_\eta}} v_{j_{a_\eta}} \qquad (I9)$$

and that of $\tilde{\epsilon}_{j_{a_\eta}}$ is

$$\tilde{\epsilon}_{j_{a_\eta}} \equiv \epsilon_{j_{a_\eta}} - G_\eta v^2_{j_{a_\eta}} \qquad (I10)$$

The $u_{j_{a_\eta}}$ and $v_{j_{a_\eta}}$ coefficients may also be expressed in terms of $\tilde{\epsilon}_{j_{a_\eta}}$, λ_η, and Δ_η by

$$u^2_{j_{a_\eta}} = \frac{1}{2}\left[1 + \frac{\tilde{\epsilon}_{j_{a_\eta}} - \lambda_\eta}{\sqrt{(\tilde{\epsilon}_{j_{a_\eta}} - \lambda_\eta)^2 + \Delta^2_\eta}}\right] \qquad (I11)$$

$$v^2_{j_{a_\eta}} = \frac{1}{2}\left[1 - \frac{\tilde{\epsilon}_{j_{a_\eta}} - \lambda_\eta}{\sqrt{(\tilde{\epsilon}_{j_{a_\eta}} - \lambda_\eta)^2 + \Delta^2_\eta}}\right] \qquad (I12)$$

and

$$2 u_{j_{a_\eta}} v_{j_{a_\eta}} = \frac{\Delta_\eta}{\sqrt{(\tilde{\epsilon}_{j_{a_\eta}} - \lambda_\eta)^2 + \Delta^2_\eta}} \qquad (I13)$$

The quasi-particle vacuum, $|\tilde{0}_\eta\rangle$, is defined by

$$\gamma^{j_a}_{m_a}(\eta)|\tilde{0}_\eta\rangle = 0 \qquad (I14)$$

and is a mixture of states with $0, 2, 4, \ldots 2\Omega_\eta$ real particles. The symbol, Ω_η, stands for the total number of pairs and is given by

$$\Omega_\eta = \frac{1}{2} \sum_{j_{a_\eta}} (2j_{a_\eta} + 1) \qquad (I15)$$

A single particle state, j_{a_η}, m_{a_η}, has a probability of being occupied in $|\tilde{0}_\eta\rangle$ given by

$$\frac{\langle \tilde{o}_\eta | b^{+j_a}_{m_a}(\eta) b^{j_a}_{m_a}(\eta) | \tilde{o}_\eta \rangle}{\langle \tilde{o}_\eta | \tilde{o}_\eta \rangle} = v^2_{j_{a_\eta}}$$

(I16)

and the average number of particles, $\bar{N}'_\eta$, in state $|\tilde{o}_\eta\rangle$ is then

$$\bar{N}'_\eta = \sum_{j_{a_\eta}} (2 j_{a_\eta} + 1) v^2_{j_{a_\eta}} \tag{I17}$$

By the appropriate choice of δ_η,

$$\bar{N}'_\eta = N'_\eta \tag{I18}$$

where N'_η is the actual number of particles.

4. Coupling Definitions and Commutation Relations:

The vector coupling of the nucleon annihilation and creation operators to give an integer total angular momentum, I, with z component, M, are defined by

$$\left[b^{+j_1}(\eta) b^{+j_2}(\eta) \right]^I_M \equiv \sum_{m_{1_\eta} m_{2_\eta}} \langle j_{1_\eta} m_{1_\eta} j_{2_\eta} m_{2_\eta} | I M \rangle b^{+j_1}_{m_1}(\eta) b^{+j_2}_{m_2}(\eta)$$

(I19a)

$$\left[c^{j_1}(\eta) c^{j_2}(\eta) \right]^I_M \equiv \sum_{m_{1_\eta} m_{2_\eta}} \langle j_{1_\eta} m_{1_\eta} j_{2_\eta} m_{2_\eta} | I M \rangle c^{j_1}_{m_1}(\eta) c^{j_2}_{m_2}(\eta)$$

(I19b)

and

$$\left[b^{+j_1}(\eta) c^{j_2}(\eta) \right]^I_M \equiv \sum_{m_{1_\eta} m_{2_\eta}} \langle j_{1_\eta} m_{1_\eta} j_{2_\eta} m_{2_\eta} | I M \rangle b^{+j_1}_{m_1}(\eta) c^{j_2}_{m_2}(\eta)$$

(19c)

where $\langle j_{1\eta}\, m_{1\eta}\, j_{2\eta}\, m_{2\eta} | I M \rangle$ is a Clebsch-Gordon coefficient.

For identical nucleons, the commutation rules are just those for Fermions, i.e.,

$$\left[c^{j_1}_{m_1}(\eta)\ ,\ c^{j_2}_{m_2}(\eta) \right]_+ = 0 \tag{I20a}$$

$$\left[b^{+j_1}_{m_1}(\eta), b^{+j_2}_{m_2}(\eta) \right]_+ = 0 \tag{I20b}$$

(Throughout this paper, a Kronecker delta involving angular momenta implies that the angular momenta have the same parity, e.g., $\pi_{j_1} = \pi_{j_2}$ is implicit in $\delta_{j_1 j_2}$)
and

$$\left[b^{+j_1}_{m_1}(\eta), c^{j_2}_{m_2}(\eta) \right]_+ = (-1)^{j_{2\eta} - m_{2\eta}} \left[b^{+j_1}_{m_1}(\eta), b^{j_2}_{-m_2}(\eta) \right]_+$$

$$= \delta_{j_{1\eta} j_{2\eta}}\, \delta_{m_{1\eta}, -m_{2\eta}} (-1)^{j_{1\eta} + m_{1\eta}} \tag{I20c}$$

where $[\ \ ]_+$ means the anti-commutator. For $b^+(\eta)$ and $c(\eta)$ coupled to an angular momentum (I,M) one then finds

$$\left[b^{+j_1}(\eta)\, c^{j_2}(\eta) \right]^I_M = -(-1)^{j_{1\eta} + j_{2\eta} + I} \left[c^{j_2}(\eta)\, b^{+j_1}(\eta) \right]^I_M$$

$$- \delta_{j_{1\eta} j_{2\eta}}\, \delta_{M0}\, \delta_{I0} \sqrt{2 j_{1\eta} + 1} \tag{I20d}$$

Since protons and neutrons will be considered as non-identical particles, one must supplement equations (I20) with the commutation rules

$$\left[c^{j_1}_{m_1}(N), c^{j_2}_{m_2}(P) \right] = 0$$

(I21a)

$$\left[b^{+j_1}_{m_1}(N), b^{+j_2}_{m_2}(P) \right] = 0$$

(I21b)

$$\left[b^{+j_1}_{m_1}(\eta), c^{j_2}_{m_2}(\eta') \right] = 0$$

(I21c)

and

$$\left[b^{+j_1}(\eta)\, c^{j_2}(\eta') \right]^I_M = (-1)^{j_{1\eta}+j_{2\eta'}+I} \left[c^{j_2}(\eta)\, b^{+j_1}(\eta') \right]^I_M$$

(I21d)

with η =N and η' =P, or vice versa, in the last two equations. The factor $(-1)^{j_{1N}+j_{2P}+I}$ in (I21d) comes from the interchange of $j_{1\eta}$ and $j_{2\eta'}$ in the Clebsch-Gordon coefficient.

By replacing $b^+(\eta)$, $b(\eta)$, and $c(\eta)$ by $\beta^+(\eta)$,$\beta(\eta)$, and $\gamma(\eta)$ respectively, sets of equations analogous to (I19a,b,c,d), (I20a,b,c,d), and (I21a,b,c,d) exist for quasi-particles. These equations involving the coupling and commutation rules for quasi-particle annihilation and creation operators shall be referred to as (I19'a,b,c,d), (I20'a,b,c,d), and (I21'a,b,c,d).

where $\langle j_{1\eta} m_{1\eta} j_{2\eta} m_{2\eta} | IM \rangle$ is a Clebsch-Gordon coefficient.

For identical nucleons, the commutation rules are just those for Fermions, i.e.,

$$\left[c^{j_1}_{m_1}(\eta) , c^{j_2}_{m_2}(\eta) \right]_+ = 0 \tag{I20a}$$

$$\left[b^{+j_1}_{m_1}(\eta), b^{+j_2}_{m_2}(\eta) \right]_+ = 0 \tag{I20b}$$

(Throughout this paper, a Kronecker delta involving angular momenta implies that the angular momenta have the same parity, e.g., $\pi_{j_1}=\pi_{j_2}$ is implicit in $\delta_{j_1 j_2}$)

and

$$\left[b^{+j_1}_{m_1}(\eta), c^{j_2}_{m_2}(\eta) \right]_+ = (-1)^{j_{2\eta}-m_{2\eta}} \left[b^{+j_1}_{m_1}(\eta), b^{j_2}_{-m_2}(\eta) \right]_+$$

$$= \delta_{j_{1\eta} j_{2\eta}} \delta_{m_{1\eta}, -m_{2\eta}} (-1)^{j_{1\eta}+m_{1\eta}} \tag{I20c}$$

where $[\ ]_+$ means the anti-commutator. For $b^+(\eta)$ and $c(\eta)$ coupled to an angular momentum (I,M) one then finds

$$\left[b^{+j_1}(\eta) c^{j_2}(\eta) \right]^I_M = -(-1)^{j_{1\eta}+j_{2\eta}+I} \left[c^{j_2}(\eta) b^{+j_1}(\eta) \right]^I_M$$

$$- \delta_{j_{1\eta} j_{2\eta}} \delta_{M0} \delta_{I0} \sqrt{2j_{1\eta}+1} \tag{I20d}$$

SECTION B. The Long-Range Interaction.

Consider the long-range Hamiltonian

$$\mathcal{H}_{L.R.} \equiv H_{N,P} + \mathcal{H}^{NN} + \mathcal{H}^{PP} \tag{I22a}$$

where one defines

$$H_{N,P} \equiv \frac{1}{2}\left(\mathcal{H}^{NP} + \mathcal{H}^{PN}\right) \tag{I22b}$$

in order to insure Hermiticity. The symbol, $H_{N,P}$, stands for the total long-range interaction between neutron and proton, while $\mathcal{H}^{NN}$ and $\mathcal{H}^{PP}$ are neutron-neutron and proton-proton interactions respectively. The long-range interaction between two particles, i_η and $j_{\eta'}$, is chosen to be

$$\mathcal{H}^{\eta\eta'} = -\sum_k \frac{4\pi}{2k+1} F^k \sum_{ij} r_{i_\eta}^k r_{j_{\eta'}}^k \sum_{m=-k}^{k} Y_{km}(\theta_{i_\eta}, \phi_{i_\eta}) \times Y_{km}^*(\theta_{j_{\eta'}}, \phi_{j_{\eta'}}) \tag{I23}$$

The minus sign indicates an attractive force, and F^k, the interaction strength, is defined to be greater than zero. Particles i and j are indicated by the indices (i, j), and the type of particle (proton or neutron) is described by η and η' for i and j respectively. The multipole order is k, and Y_{km} is a spherical harmonic. The neutron-neutron, proton-proton, and neutron-proton interaction strengths are all assumed equal. The interaction (I23) may then be written for a particular multipole order k as

$$\mathcal{H}^{\eta\eta'}_{(k)} = -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \left[V^{k}(\eta)\, V^{k}(\eta') \right]^{0}_{0}$$

(I24)

with the notation

$$\left[V^{k}(\eta)\, V^{k}(\eta') \right]^{0}_{0} = \sum_{m=-k}^{k} \langle k\, m\, k\, -m | 0\, 0 \rangle\, V^{k}_{m}(\eta)\, V^{k}_{-m}(\eta')$$

(I25)

and

$$V^{k}_{m}(\eta) = \sum_{i} r^{k}_{i_\eta}\, Y_{km}(\Omega_{i_\eta}) = \sum_{i} v^{k}_{m}(\eta_i)$$

(I26)

In the occupation number representation (I26) becomes

$$V^{k}_{m}(\eta) = \sum_{\substack{j_{1\eta}\, m_{1\eta} \\ j_{2\eta}\, m_{2\eta}}} \langle \psi^{j_{1\eta}}_{m_{1\eta}} | v^{k}(\eta) | \psi^{j_{2\eta}}_{m_{2\eta}} \rangle\, b^{+j_1}_{m_1}(\eta)\, b^{j_2}_{m_2}(\eta)$$

(I27)

The symbol, η , has been included to distinguish between neutrons and protons, and $\psi^{j_{1\eta}}_{m_{1\eta}}$ is the single particle wave-function. Using the Wigner-Eckart theorem[29)] and equations (I3) and (I19c), equation (I27) becomes

$$V^{k}_{m}(\eta) = \sum_{j_{1\eta}\, j_{2\eta}} q^{k}(j_{1\eta}\, j_{2\eta}) \left[b^{+j_1}(\eta)\, c^{j_2}(\eta) \right]^{k}_{m}$$

(I28)

where $q^{k}(j_{1\eta}\, j_{2\eta})$ is related to the reduced single particle matrix element and is defined by

$$q^{k}(j_{1\eta}\, j_{2\eta}) \equiv -\frac{1}{\sqrt{2k+1}} \langle j_{1\eta} \| r^{k}(\eta) \| j_{2\eta} \rangle$$

(I29)

Interchanging $j_{1\eta}$ and $j_{2\eta}$ in (I29) reveals that

$$q^{k}(j_{2\eta}\, j_{1\eta}) = (-1)^{j_{2\eta}-j_{1\eta}}\, q^{k}(j_{1\eta}\, j_{2\eta}) \qquad \text{(I30)}$$

Equation (I24) may then be written as

$$\mathcal{H}^{\eta\eta'}_{(k)} = -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1\eta}\, j_{2\eta} \\ j_{3\eta'}\, j_{4\eta'}}} q^{k}(j_{1\eta}\, j_{2\eta})\, q^{k}(j_{3\eta'}\, j_{4\eta'})$$

$$\times \left\{ \left[b^{+j_1}(\eta)\, c^{j_2}(\eta) \right]^{k} \left[b^{+j_3}(\eta')\, c^{j_4}(\eta') \right]^{k} \right\}^{0}_{0}$$

(I31)

where the indicated coupling of the annihilation and creation operators is a logical extension of the notation of equations (I19). Applying the quasi-particle transformation (I6c,d) to (I31) yields sixteen terms (prior to any simplification). These terms are defined to make up three components of $\mathcal{H}^{\eta\eta'}(k)$, i.e.,

$$\mathcal{H}^{\eta\eta'}(k) \equiv \mathcal{H}^{k}_{\eta\eta'\,(40)} + \mathcal{H}^{k}_{\eta\eta'\,(22)} + \mathcal{H}^{k}_{\eta\eta'\,(31)} \qquad \text{(I32)}$$

with

$$\mathcal{H}^{k}_{\substack{\eta\eta' \\ (40)}} \equiv -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1\eta} j_{2\eta} \\ j_{3\eta'} j_{4\eta'}}} q^{k}(j_{1\eta} j_{2\eta})\, q^{k}(j_{3\eta'}, j_{4\eta'})$$

$$\times \left[h^{k}_{\beta^{+}_{1\eta} \beta^{+}_{2\eta} \beta^{+}_{3\eta'} \beta^{+}_{4\eta'}} + h^{k}_{\gamma_{1\eta} \gamma_{2\eta} \gamma_{3\eta'} \gamma_{4\eta'}} \right] \tag{I33a}$$

$$\mathcal{H}^{k}_{\substack{\eta\eta' \\ (22)}} = -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1\eta} j_{2\eta} \\ j_{3\eta'} j_{4\eta'}}} q^{k}(j_{1\eta} j_{2\eta})\, q^{k}(j_{3\eta'}, j_{4\eta'})$$

$$\times \Big[h^{k}_{\beta^{+}_{1\eta} \beta^{+}_{2\eta} \gamma_{3\eta'} \gamma_{4\eta'}} + h^{k}_{\gamma_{1\eta} \gamma_{2\eta} \beta^{+}_{3\eta'} \beta^{+}_{4\eta'}}$$

$$+ h^{k}_{\beta^{+}_{1\eta} \gamma_{2\eta} \beta^{+}_{3\eta'} \gamma_{4\eta'}} + h^{k}_{\beta^{+}_{1\eta} \gamma_{2\eta} \gamma_{3\eta'} \beta^{+}_{4\eta'}}$$

$$+ h^{k}_{\gamma_{1\eta} \beta^{+}_{2\eta} \beta^{+}_{3\eta'} \gamma_{4\eta'}} + h^{k}_{\gamma_{1\eta} \beta^{+}_{2\eta} \gamma_{3\eta'} \beta^{+}_{4\eta'}} \Big] \tag{I33b}$$

and

$$\mathcal{H}^{k}_{\substack{\eta\eta' \\ (31)}} = -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1\eta} j_{2\eta} \\ j_{3\eta'} j_{4\eta'}}} q^{k}(j_{1\eta} j_{2\eta})\, q^{k}(j_{3\eta'}, j_{4\eta'})$$

$$\times \Big[h^{k}_{\beta^+_{1\eta}\,\beta^+_{2\eta}\,\beta^+_{3\eta'}\,\gamma_{4\eta'}} + h^{k}_{\beta^+_{1\eta}\,\beta^+_{2\eta}\,\gamma_{3\eta'}\,\beta^+_{4\eta'}}$$

$$+ h^{k}_{\gamma_{1\eta}\,\gamma_{2\eta}\,\beta^+_{3\eta'}\,\gamma_{4\eta'}} + h^{k}_{\gamma_{1\eta}\,\gamma_{2\eta}\,\gamma_{3\eta'}\,\beta^+_{4\eta'}}$$

$$+ h^{k}_{\beta^+_{1\eta}\,\gamma_{2\eta}\,\beta^+_{3\eta'}\,\beta^+_{4\eta'}} + h^{k}_{\beta^+_{1\eta}\,\gamma_{2\eta}\,\gamma_{3\eta'}\,\gamma_{4\eta'}}$$

$$+ h^{k}_{\gamma_{1\eta}\,\beta^+_{2\eta}\,\beta^+_{3\eta'}\,\beta^+_{4\eta'}} + h^{k}_{\gamma_{1\eta}\,\beta^+_{2\eta}\,\gamma_{3\eta'}\,\gamma_{4\eta'}} \Big] \qquad \text{(I33c)}$$

The sixteen individual terms are given explicitly by

$$h^{k}_{\beta^+_{1\eta}\,\beta^+_{2\eta}\,\beta^+_{3\eta'}\,\beta^+_{4\eta'}} = (-1)^{l_{2\eta}+l_{4\eta'}}\, u_{j_{1\eta}}\, v_{j_{2\eta}}\, u_{j_{3\eta'}}\, v_{j_{4\eta'}} \left\{ \left[\beta^{+j_1}(\eta)\,\beta^{+j_2}(\eta)\right]^{k} \left[\beta^{+j_3}(\eta')\,\beta^{+j_4}(\eta')\right]^{k} \right\}^{0}_{0} \qquad \text{(I34a)}$$

$$h^{k}_{\gamma_{1\eta}\,\gamma_{2\eta}\,\gamma_{3\eta'}\,\gamma_{4\eta'}} = (-1)^{l_{1\eta}+l_{3\eta'}}\, v_{j_{1\eta}}\, u_{j_{2\eta}}\, v_{j_{3\eta'}}\, u_{j_{4\eta'}} \left\{ \left[\gamma^{j_1}(\eta)\,\gamma^{j_2}(\eta)\right]^{k} \left[\gamma^{j_3}(\eta')\,\gamma^{j_4}(\eta')\right]^{k} \right\}^{0}_{0} \qquad \text{(I34b)}$$

$$h^{k}_{\beta^+_{1\eta}\,\beta^+_{2\eta}\,\gamma_{3\eta'}\,\gamma_{4\eta'}} = -(-1)^{l_{2\eta}+l_{3\eta'}}\, u_{j_{1\eta}}\, v_{j_{2\eta}}\, v_{j_{3\eta'}}\, u_{j_{4\eta'}} \left\{ \left[\beta^{+j_1}(\eta)\,\beta^{+j_2}(\eta)\right]^{k} \left[\gamma^{j_3}(\eta')\,\gamma^{j_4}(\eta')\right]^{k} \right\}^{0}_{0} \qquad \text{(I34c)}$$

$$h^{k}_{\gamma_{1_3}\gamma_{2_3}\beta^+_{3_{3'}}\beta^+_{4_{3'}}} = -(-1)^{l_{1_3}+l_{4_{3'}}} v_{j_{1_3}} w_{j_{2_3}} w_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\gamma^{j_1}_{(3)}\gamma^{j_2}_{(3)}\right]^k \left[\beta^{+j_3}_{(3')}\beta^{+j_4}_{(3')}\right]^k \right\}^0_0$$

(I34d)

$$h^{k}_{\beta^+_{1_3}\gamma_{2_3}\beta^+_{3_{3'}}\gamma_{4_{3'}}} = w_{j_{1_3}} w_{j_{2_3}} w_{j_{3_{3'}}} w_{j_{4_{3'}}} \left\{ \left[\beta^{+j_1}_{(3)}\gamma^{j_2}_{(3)}\right]^k \left[\beta^{+j_3}_{(3')}\gamma^{j_4}_{(3')}\right]^k \right\}^0_0$$

(I34e)

$$h^{k}_{\beta^+_{1_3}\gamma_{2_3}\gamma_{3_{3'}}\beta^+_{4_{3'}}} = -(-1)^{l_{3_{3'}}+l_{4_{3'}}} w_{j_{1_3}} w_{j_{2_3}} v_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\beta^{+j_1}_{(3)}\gamma^{j_2}_{(3)}\right]^k \left[\gamma^{j_3}_{(3')}\beta^{+j_4}_{(3')}\right]^k \right\}^0_0$$

(I34f)

$$h^{k}_{\gamma_{1_3}\beta^+_{2_3}\beta^+_{3_{3'}}\gamma_{4_{3'}}} = -(-1)^{l_{1_3}+l_{2_3}} v_{j_{1_3}} v_{j_{2_3}} w_{j_{3_{3'}}} w_{j_{4_{3'}}} \left\{ \left[\gamma^{j_1}_{(3)}\beta^{+j_2}_{(3)}\right]^k \left[\beta^{+j_3}_{(3')}\gamma^{j_4}_{(3')}\right]^k \right\}^0_0$$

(I34g)

$$h^{k}_{\gamma_{1_3}\beta^+_{2_3}\gamma_{3_{3'}}\beta^+_{4_{3'}}} = (-1)^{l_{1_3}+l_{2_3}+l_{3_{3'}}+l_{4_{3'}}} v_{j_{1_3}} v_{j_{2_3}} v_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\gamma^{j_1}_{(3)}\beta^{+j_2}_{(3)}\right]^k \left[\gamma^{j_3}_{(3')}\beta^{+j_4}_{(3')}\right]^k \right\}^0_0$$

(I34h)

$$h^{k}_{\beta^+_{1_3}\beta^+_{2_3}\beta^+_{3_{3'}}\gamma_{4_{3'}}} = -(-1)^{l_{2_3}} w_{j_{1_3}} v_{j_{2_3}} w_{j_{3_{3'}}} w_{j_{4_{3'}}} \left\{ \left[\beta^{+j_1}_{(3)}\beta^{+j_2}_{(3)}\right]^k \left[\beta^{+j_3}_{(3')}\gamma^{j_4}_{(3')}\right]^k \right\}^0_0$$

(I34i)

$$h^{k}_{\beta^+_{1_3}\beta^+_{2_3}\gamma_{3_{3'}}\beta^+_{4_{3'}}} = (-1)^{l_{2_3}+l_{3_{3'}}+l_{4_{3'}}} w_{j_{1_3}} v_{j_{2_3}} v_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\beta^{+j_1}_{(3)} \beta^{+j_2}_{(3)}\right]^{k} \left[\gamma^{j_3}_{(3')} \beta^{+j_4}_{(3')}\right]^{k} \right\}^{0}_{0}$$

(I34j)

$$h^{k}_{\gamma_{1_3}\gamma_{2_3}\beta^+_{3_{3'}}\gamma_{4_{3'}}} = (-1)^{l_{1_3}} v_{j_{1_3}} w_{j_{2_3}} w_{j_{3_{3'}}} w_{j_{4_{3'}}} \left\{ \left[\gamma^{j_1}_{(3)} \gamma^{j_2}_{(3)}\right]^{k} \left[\beta^{+j_3}_{(3')} \gamma^{j_4}_{(3')}\right]^{k} \right\}^{0}_{0}$$

(I34k)

$$h^{k}_{\gamma_{1_3}\gamma_{2_3}\gamma_{3_{3'}}\beta^+_{4_{3'}}} = (-1)^{l_{1_3}+l_{3_{3'}}+l_{4_{3'}}} v_{j_{1_3}} w_{j_{2_3}} v_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\gamma^{j_1}_{(3)} \gamma^{j_2}_{(3)}\right]^{k} \left[\gamma^{j_3}_{(3')} \beta^{+j_4}_{(3')}\right]^{k} \right\}^{0}_{0}$$

(I34l)

$$h^{k}_{\beta^+_{1_3}\gamma_{2_3}\beta^+_{3_{3'}}\beta^+_{4_{3'}}} = -(-1)^{l_{4_{3'}}} w_{j_{1_3}} w_{j_{2_3}} w_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\beta^{+j_1}_{(3)} \gamma^{j_2}_{(3)}\right]^{k} \left[\beta^{+j_3}_{(3')} \beta^{+j_4}_{(3')}\right]^{k} \right\}^{0}_{0}$$

(I34m)

$$h^{k}_{\beta^+_{1_3}\gamma_{2_3}\gamma_{3_{3'}}\gamma_{4_{3'}}} = (-1)^{l_{3_{3'}}} w_{j_{1_3}} w_{j_{2_3}} v_{j_{3_{3'}}} w_{j_{4_{3'}}} \left\{ \left[\beta^{+j_1}_{(3)} \gamma^{j_2}_{(3)}\right]^{k} \left[\gamma^{j_3}_{(3')} \gamma^{j_4}_{(3')}\right]^{k} \right\}^{0}_{0}$$

(I34n)

$$h^{k}_{\gamma_{1_3}\beta^+_{2_3}\beta^+_{3_{3'}}\beta^+_{4_{3'}}} = (-1)^{l_{1_3}+l_{2_3}+l_{4_{3'}}} v_{j_{1_3}} v_{j_{2_3}} w_{j_{3_{3'}}} v_{j_{4_{3'}}} \left\{ \left[\gamma^{j_1}_{(3)} \beta^{+j_2}_{(3)}\right]^{k} \left[\beta^{+j_3}_{(3')} \beta^{+j_4}_{(3')}\right]^{k} \right\}^{0}_{0}$$

(I34o)

$$h^{k}_{\gamma_{1_\eta}\beta^{+}_{2_\eta}\gamma_{3_{\eta'}}\gamma_{4_{\eta'}}} = -(-1)^{l_{1_\eta}+l_{2_\eta}+l_{3_{\eta'}}} N_{j_{1_\eta}} N_{j_{2_\eta}} N_{j_{3_{\eta'}}} N_{j_{4_{\eta'}}} \left\{ \left[\gamma^{j_1}_{(\eta)}\beta^{+j_2}_{(\eta)}\right]^{k} \left[\gamma^{j_3}_{(\eta')}\gamma^{j_4}_{(\eta')}\right]^{k} \right\}^{0}_{0}$$

(I34p)

The parentheses around the subscripts in (I32) indicates that not all annihilation operators are written to the right of all creation operators.

Finally, to obtain $\mathcal{H}^{k}_{L.R.}$ ((I22) with fixed k) one uses equations (I32), (I33), and (I34) with the appropriate symbols η , and η' , i.e.,

$$\mathcal{H}^{k}_{L.R.} = \frac{1}{2}\left[\mathcal{H}^{\eta=P,\,\eta'=N}_{(k)} + \mathcal{H}^{\eta=N,\,\eta'=P}_{(k)}\right] + \mathcal{H}^{\eta=N,\,\eta'=N}_{(k)} + \mathcal{H}^{\eta=P,\,\eta'=P}_{(k)}$$

(I35)

<u>SECTION C</u>. <u>Even-Even Nuclei with One Major Closed Shell and The Boson Approximation</u>.

The single particle (shell model) plus pairing interaction diagonalization predicts that the first excited states of an even-even nucleus are those due to the creation of two quasi-particles of the same type (i.e., both quasi-protons or both quasi-neutrons). The two quasi-particle energy, $E_{j_{1_\eta} j_{2_\eta}}$, is given by

$$E_{j_{1_\eta} j_{2_\eta}} = E_{j_{1_\eta}} + E_{j_{2_\eta}} \qquad \text{(I36)}$$

so that each two quasi-particle state is separated from the ground state by an energy $\geq 2\Delta_\eta$, with Δ_η given by equation

(I9). Any particular two quasi-particle energy (I36) is degenerate in total angular momentum, I, (in addition to the (2I + 1) degeneracy in M, the z component of I), and has the wavefunction, $|(\widetilde{j_{1_n} j_{2_n}}) I M\rangle$, with

$$|(\widetilde{j_{1_n} j_{2_n}}) I M\rangle \equiv \frac{1}{\sqrt{1+\delta_{j_{1_n} j_{2_n}}}} \left[\beta^{+j_1}_{(n)} \beta^{+j_2}(n)\right]^{I}_{M} |\tilde{0}_n\rangle \tag{I37}$$

The square root in the denominator is for normalization (c.f. Appendix A1), and the $\sim$ over $(j_{1_n} j_{2_n})$ indicates quasi-particles.

To obtain vibrational levels, one seeks a set of annihilation and creation operators that obey Boson commutation rules. First define

$$B^{+k}_{q}(j_{1_n} j_{2_n}) \equiv \frac{1}{\sqrt{1+\delta_{j_{1_n} j_{2_n}}}} \left[\beta^{+j_1}_{(n)} \beta^{+j_2}_{(n)}\right]^{k}_{q} \tag{I38a}$$

and

$$C^{k}_{q}(j_{1_n} j_{2_n}) \equiv (-1)^{k-q} B^{+k}_{-q}(j_{1_n} j_{2_n}) =$$

$$-\frac{1}{\sqrt{1+\delta_{j_{1_n} j_{2_n}}}} \left[\gamma^{j_1}_{(n)} \gamma^{j_2}_{(n)}\right]^{k}_{q} \tag{I38b}$$

The indices j_{1_n} and j_{2_n} may be permuted by

$$B^{+k}_{q}(j_{1_n} j_{2_n}) = -(-1)^{j_{1_n}+j_{2_n}-k} B^{+k}_{q}(j_{2_n}, j_{1_n}) \tag{I39a}$$

and

$$C^{k}_{q}(j_{1\eta}\, j_{2\eta}) = -(-1)^{j_{1\eta}+j_{2\eta}-k}\, C^{k}_{q}(j_{2\eta}\, j_{1\eta})$$

(I39b)

From the anti-commutation rules of the $\beta^{+j}_{m}(\eta)$ (I20'b) and the $\gamma^{\ell j}_{m}$ (I20'a) one has

$$\left[B^{+R}_{\rho}(j_{1\eta}\, j_{2\eta}),\, B^{+S}_{\sigma}(j_{3\eta}\, j_{4\eta}) \right] = 0 \qquad \text{(I40a)}$$

and

$$\left[C^{R}_{\rho}(j_{1\eta}\, j_{2\eta}),\, C^{S}_{\sigma}(j_{3\eta}\, j_{4\eta}) \right] = \left[B^{R}_{\rho}(j_{1\eta}\, j_{2\eta}),\, B^{S}_{\sigma}(j_{3\eta}\, j_{4\eta}) \right] = 0$$

(I40b)

The commutator of $B^{R}_{\rho}(j_{1\eta}\, j_{2\eta})$ and $B^{+S}_{\sigma}(j_{3\eta}\, j_{4\eta})$ is given by

$$\left[B^{R}_{\rho}(j_{1\eta}\, j_{2\eta}),\, B^{+S}_{\sigma}(j_{3\eta}\, j_{4\eta}) \right] = \frac{\delta_{\rho\sigma}\delta_{RS}}{1+\delta_{j_{1\eta} j_{2\eta}}} \Big[\delta_{j_{2\eta} j_{4\eta}} \delta_{j_{1\eta} j_{3\eta}}$$

$$- \delta_{j_{1\eta} j_{4\eta}} \delta_{j_{2\eta} j_{3\eta}} (-1)^{j_{1\eta}+j_{2\eta}+R} \Big] - \frac{(-1)^{R+\rho+j_{1\eta}+j_{4\eta}} \sqrt{(2R+1)(2S+1)}}{\sqrt{(1+\delta_{j_{1\eta} j_{2\eta}})(1+\delta_{j_{3\eta} j_{4\eta}})}}$$

$$\times \sum_{I} \langle R\, -\rho\; S\, \sigma \,|\, I\, M_I \rangle \left(\qquad\qquad \right)$$

(I41a)

with $\left(\quad\right)$ of (I41a) given by

$$\left(\quad\right) = \delta_{j_{2_\eta} j_{3_\eta}} (-1)^{j_{1_\eta}+j_{4_\eta}} \begin{Bmatrix} j_{1_\eta} & j_{4_\eta} & I \\ S & R & j_{2_\eta} \end{Bmatrix} \left[\beta^{+j_4}_{(\eta)} \gamma^{j_1}{}_{(\eta)} \right]^{I}_{M_I}$$

$$+ \delta_{j_{2_\eta} j_{4_\eta}} (-1)^{j_{1_\eta}+j_{3_\eta}+S} \begin{Bmatrix} j_{1_\eta} & j_{3_\eta} & I \\ S & R & j_{2_\eta} \end{Bmatrix} \left[\beta^{+j_3}_{(\eta)} \gamma^{j_1}_{(\eta)} \right]^{I}_{M_I}$$

$$+ \delta_{j_{1_\eta} j_{3_\eta}} (-1)^{j_{2_\eta}+j_{4_\eta}+R} \begin{Bmatrix} j_{2_\eta} & j_{4_\eta} & I \\ S & R & j_{1_\eta} \end{Bmatrix} \left[\beta^{+j_4}_{(\eta)} \gamma^{j_2}_{(\eta)} \right]^{I}_{M_I}$$

$$+ \delta_{j_{1_\eta} j_{4_\eta}} (-1)^{j_{2_\eta}+j_{3_\eta}+R+S} \begin{Bmatrix} j_{2_\eta} & j_{3_\eta} & I \\ S & R & j_{1_\eta} \end{Bmatrix} \left[\beta^{+j_3}_{(\eta)} \gamma^{j_2}_{(\eta)} \right]^{I}_{M_I} \quad \text{(I41b)}$$

If the second term of (I41a) is neglected, then, $B^{R}_{\rho}(j_{1_\eta} j_{2_\eta})$ and $B^{+S}_{6}(j_{3_\eta} j_{4_\eta})$ may be considered as Boson annihilation and creation operators respectively. The first term of (I41a) states that $B^{+R}_{\rho}(j_{1_\eta} j_{2_\eta})$ and $B^{+S}_{6}(j_{3_\eta} j_{4_\eta})$ pertain to independent oscillators unless $\rho = 6$, $R = S$, and either $j_{1_\eta} = j_{3_\eta}$, $j_{2_\eta} = j_{4_\eta}$ or $j_{1_\eta} = j_{4_\eta}$, $j_{2_\eta} = j_{3_\eta}$.

The Boson approximation is valid if there are "far fewer" available quasi-Fermions than there are quasi-Fermion states. This may be seen very easily. If the number of quasi-Fermion states outnumbers the number of quasi-Fermions, then any particular quasi-Fermion state is more probably empty than occupied. Consequently, the annihilators in (I41b) are more likely to give zero when operating on excited states.

Specializing to even-even nuclei with one major closed shell, one need consider only quasi-neutrons or only quasi-protons. The shell model plus pairing Hamiltonian may then be written for independent Bosons as

$$H_{\substack{shell\\Model}} + H_{Pair.} \simeq (H_{oo})_{\eta} + \sum_{\substack{j_{1\eta} \geq j_{2\eta}\\ S,\sigma}} (E_{j_{1\eta}} + E_{j_{2\eta}})\, B_{\sigma}^{+S}(j_{1\eta}\, j_{2\eta})$$

$$\times (-1)^{S+\sigma}\, C_{-\sigma}^{S}(j_{1\eta}\, j_{2\eta}) \qquad \text{(I42)}$$

The "approximately equals" sign, $\simeq$, alludes to the Boson approximation, and the associated vacuum is still $|\tilde{0}_{\eta}\rangle$. The symbols, S and σ , are for the total angular momentum and its z component.

Now one introduces the long-range Hamiltonian for multipole order k (c.f. equations (I32), (I33), and (I34) with $\eta = \eta'$) as an interaction between Bosons. Terms $\mathcal{H}^{k}_{\eta\eta\,(31)}$ (I33c) and part of $\mathcal{H}^{k}_{\eta\eta\,(22)}$ (I33b) are dropped because these terms are really distributed over oscillators of many multipole orders. Then, from equations (I32), (I33), and (I35) $(\mathcal{H}^{k}_{L.R.})_{even-even}$ is given by

$$(\mathcal{H}^{k}_{L.R.})_{even-even} = -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1\eta} j_{2\eta}\\ j_{3\eta} j_{4\eta}}} q^{k}(j_{1\eta} j_{2\eta})\, q^{k}(j_{3\eta} j_{4\eta})$$

$$\times \Big[\big(h_{\beta^{+}_{1\eta} \beta^{+}_{2\eta} \beta^{+}_{3\eta} \beta^{+}_{4\eta}} + h_{\gamma_{1\eta} \gamma_{2\eta} \gamma_{3\eta} \gamma_{4\eta}} \big) + \big(h_{\beta^{+}_{1\eta} \beta^{+}_{2\eta} \gamma_{3\eta} \gamma_{4\eta}} + h_{\gamma_{1\eta} \gamma_{2\eta} \beta^{+}_{3\eta} \beta^{+}_{4\eta}} \big) \Big] \qquad \text{(I43)}$$

This may be rewritten using (I34) and (I38) as

$$(\mathcal{H}^{k}_{L.R.})_{\text{even-even}} \simeq -\frac{(-1)^{k}4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1\eta}\geq j_{2\eta} \\ j_{3\eta}\geq j_{4\eta}}} \bar{q}^{k}(j_{1\eta} j_{2\eta})\, \bar{q}^{k}(j_{3\eta} j_{4\eta})$$

$$\times \left\{ \left[B^{+k}(j_{1\eta} j_{2\eta}) + (-1)^{k} C^{k}(j_{1\eta} j_{2\eta}) \right] \left[B^{+k}(j_{3\eta} j_{4\eta}) + (-1)^{k} C^{k}(j_{3\eta} j_{4\eta}) \right] \right\}^{0}_{0}$$

(I44)

with

$$\bar{q}^{k}(j_{1\eta} j_{2\eta}) \equiv (-1)^{l_{2\eta}} \frac{q^{k}(j_{1\eta} j_{2\eta})}{\sqrt{1+\delta_{j_{1\eta} j_{2\eta}}}} \left[u_{j_{1\eta}} v_{j_{2\eta}} + u_{j_{2\eta}} v_{j_{1\eta}} \right]$$

(I45)

where $q^{k}(j_{1\eta} j_{2\eta})$ is given by (I29). The total Hamiltonian (considering only the long-range multipole of order k) is then

$$(H_{k})_{\text{even-even}} = H_{\substack{\text{Shell} \\ \text{Model}}} + H_{\text{Pair}} + (\mathcal{H}^{k}_{L.R.})_{\text{even-even}}$$

(I46)

All energies are measured relative to the vacuum energy, $(H_{00})_{\eta}$, in (I42), and $(\mathcal{H}^{k}_{L.R.})_{\text{even-even}}$ is given by (I44).

To diagonalize (I46) a quasi-Boson creation operator, $\Gamma^{k}_{m}(\eta)$, is defined using $B^{+k}_{m}(j_{1\eta} j_{2\eta})$ and $C^{k}_{m}(j_{1\eta} j_{2\eta})$

from (I38) to be

$$\Gamma^{k}_{m}(\eta) \equiv \sum_{j_{1\eta} \geq j_{2\eta}} \left[r(j_{1\eta} j_{2\eta}) B^{+k}_{m}(j_{1\eta} j_{2\eta}) + s(j_{1\eta} j_{2\eta}) C^{k}_{m}(j_{1\eta} j_{2\eta}) \right]$$

(I47a)

The quasi-Boson annihilation operator, Γ^{+k}_{m}, would then be

$$\Gamma^{+k}_{m}(\eta) = (-1)^{k+m} \sum_{j_{1\eta} \geq j_{2\eta}} \left[r(j_{1\eta} j_{2\eta}) C^{k}_{-m}(j_{1\eta} j_{2\eta}) + s(j_{1\eta} j_{2\eta}) B^{+k}_{-m}(j_{1\eta} j_{2\eta}) \right]$$

(I47b)

The $r(j_{1\eta} j_{2\eta})$ and $s(j_{1\eta} j_{2\eta})$ are a set of numbers and are obtained by requiring

$$\left[H^{k}_{even-even} , \Gamma^{k}_{m}(\eta) \right] = w^{k} \Gamma^{k}_{m}(\eta)$$

(I48)

where w^{k} is the energy of the oscillator of order k.
This leads to

$$r(i_\eta j_\eta) = \frac{-(-1)^{k}}{w^{k} - (E_{i_\eta} + E_{j_\eta})} \frac{8\pi F^{k}}{(2k+1)} \bar{q}^{k}(i_\eta j_\eta) \sum_{\mu_\eta \geq \nu_\eta} \bar{q}^{k}(\mu_\eta \nu_\eta) \left[(-1)^{k} r(\mu_\eta \nu_\eta) - s(\mu_\eta \nu_\eta) \right]$$

(I49a)

and

$$s(i_\eta j_\eta) = \frac{-1}{w^{k} + (E_{i_\eta} + E_{j_\eta})} \frac{8\pi F^{k}}{(2k+1)} \bar{q}^{k}(i_\eta j_\eta) \sum_{\mu_\eta \geq \nu_\eta} \bar{q}^{k}(\mu_\eta \nu_\eta) \left[(-1)^{k} r(\mu_\eta \nu_\eta) - s(\mu_\eta \nu_\eta) \right]$$

(I49b)

with $\bar{q}^{k}(ij)$ given by (I45).

The equations (I49) may be simultaneously solved to give

$$\sum_{i_\eta \geq j_\eta} \frac{\bar{q}^{2(k)}(i_\eta j_\eta)(E_{i_\eta}+E_{j_\eta})}{(E_{i_\eta}+E_{j_\eta})^2-(\omega^k)^2} = \frac{2k+1}{16\pi F^k}$$

(I50)

A plot of the left hand side of (I50) vs. ω^k with F^k as a parameter appears in Reference 27), and the lowest frequency for a fixed F^k is less than the possible two quasi-particle energies. A fairly complete discussion is also given in Reference 27). The operator, $\Gamma_m^k(\eta)$, thus creates a quasi-Boson into the vacuum, $|\tilde{\hat{0}}_\eta\rangle$. The quasi-Boson vacuum, $|\tilde{\hat{0}}_\eta\rangle$, is defined via

$$\Gamma_m^{+k}(\eta)\,|\tilde{\hat{0}}_\eta\rangle = 0 \quad \text{for all } k, m$$

(I51)

and is different from $|\hat{0}_\eta\rangle$, the quasi-particle vacuum. In fact, $|\hat{0}_\eta\rangle$ has components with 0,4,8.....quasi-particles.[18)]

SECTION D. Even-Odd Nuclei - The Appropriate Wavefunctions.

The essential foundations and equations having been set up in the previous three Sections, the actual work of this investigation will now begin.

1. Single Quasi-Particles:

The simplest interpretation of the lowest states of even-odd nuclei is that there are a group of seniority one levels, i.e., non-interacting one quasi-particle states. If $E_{j_{1\eta}}$ is a one quasi-particle energy and $E_{j_{2\eta}}$ is another, then to lowest order in $\frac{\epsilon_{j_{1\eta}}+\epsilon_{j_{2\eta}}-2\delta}{\Delta}$ [30] there follows

$$E_{j_{1\eta}} - E_{j_{2\eta}} = \left(\epsilon_{j_{2\eta}} - \epsilon_{j_{1\eta}}\right) \frac{\epsilon_{j_{2\eta}}+\epsilon_{j_{1\eta}}-2\delta}{2\Delta} \quad ;$$

$$\frac{\epsilon_{j_{2\eta}}+\epsilon_{j_{1\eta}}-2\delta}{2\Delta} \ll 1 \qquad \text{(I52)}$$

This energy difference is much too small to explain the spectrum of most odd nuclei.

2. Quasi-Boson Plus Quasi-Proton:

In order to establish a convention regarding the quasi-Boson makeup, this entire paper will assume that the quasi-Boson is of quasi-neutron origin.

The first approach will be to couple the odd quasi-proton to the vibrational states of the adjacent even-even nucleus. If this even-even nucleus has a closed major proton shell, the coupling will then simply be with the quasi-Boson described in the preceding paragraph. The long-range Hamiltonian (I35) will then be diagonalized with respect to single quasi-particle and coupled states. For simplicity, only one Boson states are used in the coupling. The departure from previous calculations[20-23] will be the use of actual Boson approximation wavefunctions in calculated matrix elements. This will eliminate the use of the empirical restoring force and inertial parameters, $(C)_w$ and $(B)_w$, respectively.

The quasi-Boson creation operator of angular momentum k (I47a) consisting of quasi-neutrons may be written as $\Gamma^{k}_{(N)}$ (the N denoting quasi-neutrons). Now couple this quasi-Boson to a quasi-proton of angular momentum, j_{cp}. The resulting state of total angular momentum, J, and z component, M, may be written as

$$\left|\left(\tilde{\hat{k}}_N ; \hat{\tilde{j}}_{cp}\right) JM\right\rangle \equiv \left[\Gamma^{k}_{(N)} \beta^{+j_c}_{(P)}\right]^J_M \left|\tilde{\hat{0}}_N ; \tilde{0}_P\right\rangle \tag{I53}$$

where $|\tilde{\hat{0}}_N;\tilde{0}_P\rangle$ represents the quasi-Boson vacuum, $|\tilde{\hat{0}}_N\rangle$, and the quasi-proton vacuum, $|\tilde{0}_P\rangle$. The notation, $|(\tilde{\hat{k}}_N ; \hat{\tilde{j}}_{cp})JM\rangle$, is self-evident. The set of wavefunctions (I53) is orthonormal (See Appendix A5), i.e.,

$$\left\langle \tilde{\hat{0}}_N ; \tilde{0}_P \right| \left[\Gamma^{k'}_{(N)} \beta^{+j'_c}_{(P)}\right]^{+J'}_{M'} \left[\Gamma^{k}_{(N)} \beta^{+j_c}_{(P)}\right]^J_M \left|\tilde{\hat{0}}_N ; \tilde{0}_P\right\rangle$$

$$= \delta_{kk'} \delta_{j_{cp} j'_{cp}} \delta_{JJ'} \delta_{MM'} \tag{I54}$$

The single quasi-proton state, $\beta_M^{+J}(P)|\tilde{\tilde{0}}_N ; \tilde{0}_P\rangle$, is orthogonal to all the states of (I53). This is seen from the fact that there is no operator to annihilate the created quasi-Boson, i.e.,

$$\langle \tilde{\tilde{0}}_N ; \tilde{0}_P | \beta_M^{J}(P) \left[\Gamma^{k}(N)\, \beta^{+j_c}(P) \right]_M^J | \tilde{\tilde{0}}_N ; \tilde{0}_P \rangle = 0 \tag{I55}$$

From (I20c) the single quasi-proton states are orthonormal.

$$\langle \tilde{\tilde{0}}_N ; \tilde{0}_P | \beta_{M'}^{J'}(P)\, \beta_M^{+J}(P) | \tilde{\tilde{0}}_N ; \tilde{0}_P \rangle = \delta_{JJ'}\, \delta_{MM'} \tag{I56}$$

Hence, the state of total angular momentum, J, and z component, M, may be written as

$$|JM\rangle_{0,k} = a_{0J}^{J}\, \beta_M^{+J}(P) | \tilde{\tilde{0}}_N ; \tilde{0}_P \rangle + \sum_{j_{cp}} a_{k j_{cp}}^{J} \left[\Gamma^{k}(N)\, \beta^{+j_c}(P) \right]_M^J \times | \tilde{\tilde{0}}_N ; \tilde{0}_P \rangle \tag{I57}$$

The notation, $|JM\rangle_{0,k}$, indicates the presence of a zero Boson term and a one Boson term. The letter k is just the angular momentum of the one Boson state in the adjacent even-even nucleus. The symbols, a_{0J}^{J} and $a_{k j_{cp}}^{J}$, are the coefficients of the zero and one phonon states respectively.

Orthonormality requires from (I54), (I55), and (I56) that

$$ {}_{0,k'}\langle J'M' | JM \rangle_{0,k} = \delta_{kk'} \delta_{JJ'} \delta_{MM'} \tag{I58} $$

This means that

$$ | a^{J}_{0J} |^2 + \sum_{j_{cp}} | a^{J}_{k\, j_{cp}} |^2 = 1 \tag{I59} $$

In addition, the $\Gamma^{k}_{m}(N)$ form a complete set of orthogonal states in the subspace, $| \tilde{\tilde{0}}_N \rangle$, hence

$$ \langle \tilde{\tilde{0}}_N | \Gamma^{+k'}_{m'}(N)\, \Gamma^{k}_{m}(N) | \tilde{\tilde{0}}_N \rangle = \delta_{kk'} \delta_{mm'} \tag{I60} $$

This requires (c.f. Appendix A4) that

$$ \sum_{j_{1N} \geq j_{2N}} \left[r^2(j_{1N} j_{2N}) - s^2(j_{1N} j_{2N}) \right]^2 = 1 \tag{I61} $$

The symbols, $r(j_{1N} j_{2N})$ and $s(j_{1N} j_{2N})$, have been defined previously in equations (I49a) and (I49b) respectively.

The procedure will be to take for a particular k the experimental value of w^k (defined in (I48)). Equations (I49a), (I49b), and (I61) will then be solved for the force strength, F^k , and the sets of coefficients, $r(j_{1N} j_{2N})$ and $s(j_{1N} j_{2N})$. The values of the quasi-particle transformation coefficients,

$(w_{j_{a_\eta}}, N_{j_{a_\eta}})$, may be obtained by using (I11), (I12), and the tables of Kisslinger and Sorenson.[17)] They list the single particle energies, $\epsilon_{j_{a_\eta}}$, and pairing parameters, G_η, λ_η, and Δ_η. One may also obtain the quadrupole force strength from Reference 17). The long-range Hamiltonian (I35) will be diagonalized with respect to the set of states (I57). This will yield not only the eigenvalues but via (I57) and (I59) the sets of coefficients, a_{0J}^{J} and $a_{k\, j_{c_p}}^{J}$. Electric transition probabilities and inelastic scattering cross-sections can then be calculated.

3. Quasi-Boson Plus Quasi-Neutron :

The coupling of a quasi-neutron of angular momentum, j_{c_N}, (created by $\beta^{+j_c}(N)$) with a quasi-Boson of angular momentum k (created by $\Gamma^{k}(N)$) is much more complicated. First of all, the quasi-Boson vacuum, $|\tilde{\tilde{o}}_N\rangle$, and the quasi-neutron vacuum are not separable as are $|\tilde{\tilde{o}}_N\rangle$ and $|\hat{o}_p\rangle$ in (I57). Secondly, the coupling of $\beta^{+j_c}(N)$ with each component quasi-neutron particle and hole of $\Gamma^{k}(N)$ must be considered; this implies a worsening of the validity of the Boson Approximation. In order to pursue this coupling in detail, one must have detailed knowledge of the quasi-Boson vacuum, $|\tilde{\tilde{o}}_N\rangle$. Now a prime advantage of the Boson approximation was to preclude knowing specifically the makeup of $|\tilde{\hat{o}}_N\rangle$. Clearly the coupling of a quasi-neutron to a quasi-Boson (consisting of quasi-neutrons) is against the very spirit of the Boson approximation. Such an approximation conceals the Fermion nature of the constituent quasi-neutrons; whereas, the odd quasi-neutron enhances this nature. An alternate procedure (Sections D4.,5.) will shed some light on this situation.

4. Three Quasi-Particles (two quasi-neutrons and one quasi-proton):

Consider an even-odd nucleus with one proton outside of a major closed shell. The long-range Hamiltonian (I35) should ideally be diagonalized relative to all states of one quasi-proton, one quasi-proton plus two quasi - neutrons,.......one quasi-proton plus η_N quasi-neutrons, where η_N is even and equal to the number of neutrons. States containing more than one quasi-proton and states containing more than η_N quasi-neutrons are spurious. Only states of one quasi-proton, and one quasi-proton plus two quasi-neutrons shall be considered in the present investigation. The algebraic complexity involved in the treatment of states with components of one quasi-proton plus more than two quasi-neutrons is the reason for this limitation. While more than two quasi-neutrons would appear to be at too high an energy to mix with lower configurations, the possibility should not be overlooked. An indication of the importance of considering more than two quasi-neutrons would be the extent of disagreement between the method of this section and experiment.

Consider a state of two quasi-neutrons of angular momenta, j_{a_N} and j_{b_N}, respectively and a quasi-proton of angular momentum, j_{cp}. If the three quasi-particles couple to total angular momentum, J with z component, M, then this state may be represented by

$$|JM\rangle_{\substack{2\ quasi\text{-}N \\ plus \\ 1\ quasi\text{-}p}} = \sum_{\substack{j_{a_N} \geq j_{b_N} \\ J_0\ j_{cp}}} \lambda^{J}_{(j_{a_N} j_{b_N}) J_0 j_{cp}} \left(\left[\beta^{+j_a}_{(N)} \beta^{+j_b}(N) \right]^{J_0} \beta^{+j_c}_{(p)} \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_p\rangle \tag{I62}$$

The set, $\lambda^{J}_{(j_{a_N} j_{b_N}) J_0 j_{cp}}$, are expansion coefficients. The two quasi-neutrons couple to intermediate angular momentum, J_0, and the quasi-proton couples with J_0 to produce J, M. The quasi-neutron vacuum, $|\tilde{0}_N\rangle$, and the quasi-proton vacuum, $|\tilde{0}_p\rangle$, are separable. An alternate representation of $|JM\rangle$ is

$$|JM\rangle_{\substack{2\ quasi\text{-}N \\ plus \\ 1\ quasi\text{-}p}} = \sum_{\substack{j_{a_N} \geq j_{b_N} \\ J_0\ j_{cp}}} \lambda'^{JM}_{(j_{a_N} j_{cp}) J_0 j_{b_N}} \left(\left[\beta^{+j_a}_{(N)} \beta^{+j_c}_{(p)} \right]^{J_0} \times \beta^{+j_b}(N) \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_p\rangle \tag{I63}$$

Equations (I62) and (I63) are not independent. This is shown mathematically in Appendix C, and is not unexpected in that the Pauli principle is obeyed in both (I62) and (I63). Clearly, the way one couples three particles (two of which are identical) to a total angular momentum, J, M, is independent of the intermediate coupling state provided the two particles remain identical. Another way of looking at this is that the total number of states before coupling must equal the number after coupling. Since the former is dependent on the Pauli principle, the only stipulation after coupling is that this principle be still obeyed.

Equation (I62) is reminiscent of the expression for a quasi-Boson coupled to a quasi-proton (I53). For this reason (I62) is considered now as the form of the three quasi-particles contribution to the state, $|JM\rangle$. The single quasi-proton contribution to the state, $|JM\rangle$, is

$$|JM\rangle_{1\,quasi-p} = a_{0J}^{J}\, \beta^{+J}_{M}(p)\, |\tilde{0}_N ; \tilde{0}_p\rangle \tag{I64}$$

The orthogonality of the set, $\left(\left[\beta^{+j_a}(N)\,\beta^{+j_b}(N)\right]^{J_0} \beta^{+j_c}(p)\right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_p\rangle$ is shown in Appendix A2. This is represented as follows

$$\langle \tilde{0}_p ; \tilde{0}_N | \left(\left[\beta^{+j_a'}(N)\,\beta^{+j_b'}(N)\right]^{J_0'} \beta^{+j_c'}(p)\right)^{+J'}_{M'}$$

$$\times \left(\left[\beta^{+j_a}_{(N)}\,\beta^{+j_b}_{(N)}\right]^{J_0} \beta^{+j_c}_{(p)}\right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_p\rangle = \delta_{J'J}\,\delta_{M'M}\,\delta_{j_{cp}'j_{cp}}\,\delta_{J_0'J_0}$$

$$\times \left(\delta_{j_{a_N}'j_{a_N}}\,\delta_{j_{b_N}'j_{b_N}} - (-1)^{j_{a_N}+j_{b_N}+J_0}\,\delta_{j_{a_N}j_{b_N}'}\,\delta_{j_{b_N}j_{a_N}'}\right) \tag{I65}$$

From Appendix A2 the normalization factor for the states, $\left(\left[\beta^{+j_a}_{(N)}\beta^{+j_b}_{(N)}\right]^{J_0}\beta^{+j_c}_{(P)}\right)|\tilde{0}_N;\tilde{0}_P\rangle$, is $\frac{1}{\sqrt{1+\delta_{j_{a_N}j_{b_N}}}}$. Hence,

$$\langle\tilde{0}_P;\tilde{0}_N|\left(\left[\beta^{+j_a}(N)\beta^{+j_b}(N)\right]^{J_0}\beta^{+j_c}(P)\right)^{+J}_M\left(\left[\beta^{+j_a}(N)\beta^{+j_b}(N)\right]^{J_0}\beta^{+j_c}_{(P)}\right)^J_M$$
$$\times|\tilde{0}_N;\tilde{0}_P\rangle = 1+\delta_{j_{a_N}j_{b_N}} \qquad (I66)$$

Obviously, the set, $|JM\rangle_{1\,quasi\text{-}P}$ (I64) and the set, $|JM\rangle_{2\,quasi\text{-}N\ plus\ 1\,quasi\text{-}P}$ (I62) are orthogonal.

The total wavefunction of state $|JM\rangle$ is then

$$|JM\rangle = \left[a^J_{0J}\beta^{+J}_M(P) + \sum_{\substack{j_{a_N}\geq j_{b_N}\\ J_0,\,j_{c_P}}} c^J_{(j_{a_N}j_{b_N})J_0 j_{c_P}}\frac{1}{\sqrt{1+\delta_{j_{a_N}j_{b_N}}}}\left(\left[\beta^{+j_a}_{(N)}\beta^{+j_b}_{(N)}\right]^{J_0}\right.\right.$$
$$\left.\left.\times\beta^{+j_c}_{(P)}\right)^J_M\right]|\tilde{0}_N;\tilde{0}_P\rangle \qquad (I67)$$

Note the similarity between (I67) and the corresponding quasi-Boson equation (I57). The set (I67) is orthonormal, and the set of coefficients, $c^J_{(j_{a_N}j_{b_N})J_0 j_{c_P}}$, with the single quasi-proton coefficient, a^J_{0J}, satisfy the relation

$$|a^J_{0J}|^2 + \sum_{\substack{j_{a_N}\geq j_{b_N}\\ J_0,\,j_{c_P}}}\left|c^J_{(j_{a_N}j_{b_N})J_0 j_{c_P}}\right|^2 = 1 \qquad (I68)$$

The procedure will be to diagonalize the long-range Hamiltonian (I35) with respect to the set of states (I67). The necessary pairing parameters, single particle energies and force strength (if quadrupole) may be obtained from Kisslinger and Sorenson.[17] As a consequence of the diagonalization, the set, $c^{J}_{(j_{a_N} j_{b_N})J_0 j_{c_P}}$, as well as a^{J}_{0J}, will be determined. Electric transition probabilities and scattering cross-sections may then be determined.

5. Three Quasi-Neutrons:

For three quasi-neutrons of angular momenta, $j_{a_N}, j_{b_N}, j_{c_N}$, the normalization equation analogous to (I66) is, from Appendix A3, given by

$$\langle \tilde{0}_N | \left([\beta^{+j_a}{}_{(N)} \beta^{+j_b}{}_{(N)}]^{J_0} \beta^{+j_c}{}_{(N)} \right)^{+J}_{M} \left([\beta^{+j_a}{}_{(N)} \beta^{+j_b}{}_{(N)}]^{J_0} \beta^{+j_c}{}_{(N)} \right)^{J}_{M} | \tilde{0}_N \rangle$$

$$= 1 + \delta_{j_{a_N} j_{b_N}} + (2J_0+1) \Big[\begin{Bmatrix} J & j_{c_N} & J_0 \\ j_{a_N} & j_{c_N} & J_0 \end{Bmatrix} \delta_{j_{b_N} j_{c_N}} + \begin{Bmatrix} J & j_{c_N} & J_0 \\ j_{b_N} & j_{c_N} & J_0 \end{Bmatrix} \delta_{j_{a_N} j_{c_N}}$$

$$+ 2 \begin{Bmatrix} J & j_{a_N} & J_0 \\ j_{a_N} & j_{a_N} & J_0 \end{Bmatrix} \delta_{j_{a_N} j_{b_N}} \delta_{j_{b_N} j_{c_N}} \delta_{j_{a_N} j_{c_N}} \Big] \tag{I69}$$

An attempt to obtain an orthogonality relation analogous to (I65) yields from Appendix A3

$$\langle \tilde{0}_N | \left([\beta^{+j_a'}_{(N)} \beta^{+j_b'}_{(N)}]^{J_0'} \beta^{+j_c'}_{(N)} \right)^{J'}_{M'} \left([\beta^{+j_a}{}_{(N)} \beta^{+j_b}{}_{(N)}]^{J_0} \beta^{+j_c}_{(N)} \right)^{J}_{M} | \tilde{0}_N \rangle$$

$$= \delta_{J'J} \delta_{M'M} \delta_{j_{c_P}' j_{c_P}} \delta_{J_0' J_0} \left(\delta_{j_{b_N}' j_{b_N}} \delta_{j_{a_N}' j_{a_N}} - (-1)^{j_{a_N}+j_{b_N}+J_0} \delta_{j_{a_N}' j_{b_N}} \delta_{j_{b_N}' j_{a_N}} \right)$$

$$- \delta_{JJ'} \delta_{M'M} \sqrt{(2J_0+1)(2J_0'+1)} \Bigg[(-1)^{j_{b_N}+J_0} \begin{Bmatrix} J & j_{c_N} & J_0 \\ j_{a_N} & j_{b_N} & J_0' \end{Bmatrix}$$

$$\times \left((-1)^{j_{c_N}+J_0'} \delta_{j_{b_N} j_{c_N}'} \delta_{j_{c_N} j_{b_N}'} \delta_{j_{a_N} j_{a_N}'} + (-1)^{j_{a_N}} \delta_{j_{b_N} j_{c_N}'} \delta_{j_{a_N} j_{b_N}'} \delta_{j_{c_N} j_{a_N}'} \right)$$

$$+ \begin{Bmatrix} J & j_{c_N} & J_0 \\ j_{b_N} & j_{a_N} & J_0' \end{Bmatrix} \left((-1)^{j_{b_N}+j_{c_N}+J_0'} \delta_{j_{a_N} j_{c_N}'} \delta_{j_{c_N} j_{b_N}'} \delta_{j_{b_N} j_{a_N}'} \right.$$

$$\left. - \delta_{j_{a_N} j_{c_N}'} \delta_{j_{b_N} j_{b_N}'} \delta_{j_{c_N} j_{a_N}'} \right) \Bigg] \qquad \text{(I73)}$$

The first term is recognizable as the analog of the right hand side of (I65). The second term, however, states that for $J_0 \neq J_0'$, the two different states of three quasi-particles are not orthogonal. The mathematical complications that result from trying to obtain an orthonormal set from linear combinations of states, $\left([\beta^{+j_{a(N)}} \beta^{+j_{b(N)}}]^{J_0} \beta^{+j_{c(N)}} \right)^J_M$, thwart pursuing this method.

Considerable information may, however, still be obtained about the three quasi-particles. Consider the diagonalization of the long-range force for two quasi-neutrons plus one quasi-proton (Section D4). Certain matrix elements will be independent of the odd quasi-proton's presence. These will be the terms that will give information about the validity of the Boson approximation of even-even nuclei, and hence if the coupling of the odd quasi-neutron to the other two quasi-neutrons is small, then, the results from the odd quasi-proton case will be approximately applicable. In other words, aside from the Pauli principle the coupling of an odd neutron to two other neutrons is like the coupling of an odd proton to the two neutrons. Just how well these arguments apply, will be determined in the ensuing chapters.

CHAPTER II

THE MATRIX ELEMENTS OF THE LONG-RANGE INTERACTION

In this chapter the long-range Hamiltonian (I35) will be diagonalized. This will be done specifically for the case of nuclei having one proton and any number of neutrons outside of a major closed shell.

In all of the following work, the validity of the shell model and quasi-particle representation is assumed; hence, the eigenvalues of the single particle plus pairing Hamiltonian (I7) are considered as already known.

Section A) assumes that the long range interaction between neutrons has already been diagonalized. This gives rise to the quasi-Boson representation. The neutron-proton long-range interaction matrix elements will be obtained with respect to states of one quasi-proton, and one quasi-proton plus one quasi-Boson.

Section B) includes the matrix elements of the long-range interaction between neutrons as well as between neutron and proton. The set of eigenstates will include states of one quasi-proton and two quasi-neutrons plus one quasi-proton.

Section C) points out some important differences between even and odd multipolarity long-range interactions. Parity arguments will show that for odd multipolarity many neutron-proton matrix elements vanish.

Section D) serves to compare the expressions of sections A) and B). In particular, terms dropped in the Boson approximation will be seen to occur naturally in the work of Section B.

Section E) is concerned with graphology. In this way the inherent properties of the long-range Hamiltonian and its matrix elements can be illustrated quite clearly.

Section A. Quasi-Boson Plus Quasi-Proton.

The Hamiltonian for the single particle plus pairing and long-range interactions between neutrons (I46) has already been diagonalized in the Boson-approximation.[18] The eigenvalues for a particular quasi-Boson order k are the energies, ω^{k}, $2\omega^{k}$, ..., with the energy ω^{k} defined in (I48). All energies are relative to the ground state, $|\tilde{\tilde{0}}_{N}\rangle$ (I51). The set ω^{k} represents one phonon vibrations, there being a single phonon for each mode k. In this scheme the presence of quasi-neutrons in the ground state is indicative of zero-point oscillations of an even-even nucleus. The creation of quasi-Bosons produces "collective" states in these nuclei.

Now consider the addition of a single quasi-proton to an even-even nucleus. This, of course, means that one has an even-odd nucleus. Suppose one attempts to explain the spectroscoptic properties of this even-odd nucleus by still retaining the Boson-approximation. If the odd proton in this even-odd nucleus is the only proton outside of a major closed shell, then the set of eigenstates (considering only one phonon contributions) is given by (I57). For a state of total angular momentum, J, with z component, M, which contains the contribution of a quasi-Boson of order, k, this was just

$$|JM\rangle_{o,k} = a^{J}_{oJ}\, \beta^{+J}_{M}(p)\, |\tilde{\tilde{0}}_N ; \tilde{0}_p\rangle + \sum_{j_{cp}} a^{J}_{k j_{cp}} \left[\Gamma^{k}_{(N)} \beta^{+j_c}_{(p)} \right]^{J}_{M} \times |\tilde{\tilde{0}}_N ; \hat{0}_p\rangle \qquad \text{(I57)}$$

where the quasi-Boson is made up of quasi-neutrons.

Since there is only one proton outside of a major closed shell, there is no proton-proton long-range interaction. The k'th multipole of the neutron-proton long-range interaction is given by (I32) and (I22b) as

$$H^{k}_{N,P} = \frac{1}{2}\left[\left(\mathcal{H}^{k}_{\substack{NP\\(40)}} + \mathcal{H}^{k}_{\substack{NP\\(22)}} + \mathcal{H}^{k}_{\substack{NP\\(31)}} \right) + \left(\mathcal{H}^{k}_{\substack{PN\\(40)}} + \mathcal{H}^{k}_{\substack{PN\\(22)}} + \mathcal{H}^{k}_{\substack{PN\\(31)}} \right) \right] \qquad \text{(II1)}$$

where (as mentioned in Chapter I) the parenthesis indicate that not all annihilation operators have been written to the right of all creation operators. The terms, $\mathcal{H}^{k}_{\substack{NP\\(40)}}$, $\mathcal{H}^{k}_{\substack{NP\\(22)}}$, $\mathcal{H}^{k}_{\substack{NP\\(31)}}$, $\mathcal{H}^{k}_{\substack{PN\\(40)}}$, $\mathcal{H}^{k}_{\substack{PN\\(22)}}$, and $\mathcal{H}^{k}_{\substack{PN\\(31)}}$ are given in (I33) for the appropriate η and η'. The terms, $\mathcal{H}^{k}_{\substack{NP\\(40)}}$ and $\mathcal{H}^{k}_{\substack{PN\\(40)}}$, plus parts of $\mathcal{H}^{k}_{\substack{NP\\(31)}}$, $\mathcal{H}^{k}_{\substack{PN\\(31)}}$, $\mathcal{H}^{k}_{\substack{NP\\(22)}}$, and $\mathcal{H}^{k}_{\substack{PN\\(22)}}$ involve the annihilation or creation of two quasi-protons.

These terms all give vanishing matrix elements with respect to the set of states (I57) since each state has only one quasi-proton.

From the equations (I33) the parts of $\mathcal{H}^{k}_{\eta\eta'}$ $(\eta=N, \eta'=P \text{ or } \eta=P, \eta'=N)$ (31) that will give non zero matrix elements are those involving

$$h^{k}_{\beta^+_{1\eta}\beta^+_{2\eta}\beta^+_{3\eta'}\gamma_{4\eta'}} \quad \text{(I34i)}, \quad h^{k}_{\beta^+_{1\eta}\beta^+_{2\eta}\gamma_{3\eta'}\beta^+_{4\eta'}} \quad \text{(I34j)}$$

$$h^{k}_{\gamma_{1\eta}\gamma_{2\eta}\beta^+_{3\eta'}\gamma_{4\eta'}} \quad \text{(I34k), and } h^{k}_{\gamma_{1\eta}\gamma_{2\eta}\gamma_{3\eta'}\beta^+_{4\eta'}} \quad \text{(I34l)}$$

Use of the commutation rule for an annihilator coupled to a creator (I20'd), and the definitions of $B^{+k}_{m}(j_{1\eta}j_{2\eta})$ and $C^{k}_{m}(j_{1\eta}j_{2\eta})$ (I38) enables one to combine these terms. If in addition one uses the results of Appendix C (Introduction) then

$$\underset{(31)}{H^{k}_{N,P}} \equiv \frac{1}{2}\Big[\underset{(31)}{\mathcal{H}^{k}_{NP}} + \underset{(31)}{\mathcal{H}^{k}_{PN}}\Big] = (-1)^{k}\frac{4\pi}{\sqrt{2k+1}}F^{k}\sum_{\substack{j_{1N}j_{2N}\\ j_{3P}j_{4P}}}\sqrt{1+\delta_{j_{1N}j_{2N}}}\, g^{k}(j_{1N}j_{2N})$$

$$\times\, g^{k}(j_{3P}j_{4P})(-1)^{l_{1N}} N_{1N}\, w_{2N} \cos(\tau_{3P}+(-1)^{k}\tau_{4P})\Big\{\big[C(j_{1N}j_{2N}) + (-1)^{k}B^{+}(j_{1N}j_{2N})\big]^{k}\big[\beta^{+j_3}_{(P)}\,\gamma^{j_4}_{(P)}\big]^{k}\Big\}^{0}_{0} \quad \text{(II1')}$$

Use has also been made of the conservation of parity conditions

$$(-1)^{l_{1\eta}+l_{2\eta}+k} = (-1)^{l_{3\eta'}+l_{4\eta'}+k} = 1 \quad \text{(II2)}$$

The following definitions are useful:

$$\cos(\vartheta_{a_\eta} \pm \vartheta_{b_\eta}) \equiv u_{ja_\eta} u_{jb_\eta} \mp v_{ja_\eta} v_{jb_\eta} \tag{II3a}$$

and

$$\sin(\vartheta_{a_\eta} \pm \vartheta_{b_\eta}) \equiv v_{ja_\eta} u_{ja_\eta} \pm v_{jb_\eta} u_{ja_\eta} \tag{II3b}$$

with

$$\sin \vartheta_{a_\eta} \equiv v_{ja_\eta}$$
$$\sin \vartheta_{b_\eta} \equiv v_{jb_\eta} \tag{II3c}$$

and

$$\cos \vartheta_{a_\eta} \equiv u_{ja_\eta}$$
$$\cos \vartheta_{b_\eta} \equiv u_{jb_\eta} \tag{II3d}$$

For ease of notation, the subscript, j, shall henceforth be dropped when dealing with the (u, v) coefficients, e.g., $u_{ja_\eta} \equiv u_{a_\eta}$ and $v_{ja_\eta} \equiv v_{a_\eta}$. Equation (II3a) has been used in (II 1').

For one proton outside of a major closed shell, a quasi-proton and a proton are the same, i.e., $u_{a_p} = 1$, and $v_{a_p} = 0$ (c.f.(I6)). Thus (II 1') may be simplified further. From (II3a), the expression $\cos(\vartheta_{3p} + (-1)^k \vartheta_{4p})$ is unity and (II 1') becomes finally

$$H^k_{\nu,p} = (-1)^k \frac{4\pi}{\sqrt{2k+1}} F^k \sum_{\substack{j_{1\nu} j_{2\nu} \\ j_{3p} j_{4p}}} \sqrt{1+\delta_{j_{1\nu} j_{2\nu}}}\, q^k(j_{1\nu} j_{2\nu})\, q^k(j_{3p} j_{4p}) \tag{31}$$

$$\times (-1)^{l_{1N}} v_{1N} w_{2N} \left\{ \left[c(j_{1N} j_{2N}) + (-1)^{k} B^{+}(j_{1N} j_{2N}) \right]^{k} \times \left[\beta^{+ j_3(p)} \gamma^{j_4(p)} \right]^{k} \right\}^{0}_{0}$$

(II4)

From the equations (I34) the only terms from $\mathcal{H}^{k}_{\eta\eta'\,(22)}$ (with $\eta = N$, $\eta' = P$ or vice versa) that will give non zero matrix elements are those involving

$h^{k}_{\beta^{+}_{1\eta} \gamma_{2\eta} \beta^{+}_{3\eta'} \gamma_{4\eta'}}$ (I34e), $h^{k}_{\beta^{+}_{1\eta} \gamma_{2\eta} \gamma_{3\eta'} \beta^{+}_{4\eta'}}$ (I34f),

$h^{k}_{\gamma_{1\eta} \beta^{+}_{2\eta} \beta^{+}_{3\eta'} \gamma_{4\eta'}}$ (I34g), and

$h^{k}_{\gamma_{1\eta} \beta^{+}_{2\eta} \gamma_{3\eta'} \beta^{+}_{4\eta'}}$ (I34h).

One now uses the commutation rule (I20'd), as well as (II3a), $\cos(\alpha_{3p} + \alpha_{4p}) = 1$, and the results of Appendix C (Introduction) to give for one proton outside of a major closed shell

$$H^{k}_{N,P\,(22)} \equiv \frac{1}{2}\left[\mathcal{H}^{k}_{NP\,(22)} + \mathcal{H}^{k}_{PN\,(22)} \right] = -(-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} q^{k}(j_{1N} j_{2N})$$

$$\times q^{k}(j_{3P} j_{4P}) \cos(\alpha_{1N} + \alpha_{2N}) \left\{ \left[\beta^{+ j_1(N)} \gamma^{j_2(N)} \right]^{k} \left[\beta^{+ j_3}(P) \gamma^{j_4}(P) \right]^{k} \right\}_{0}^{0}$$

(II5)

Thus, the long-range neutron-proton Hamiltonian (II 1) becomes for order k

$$H^{k}_{N,P} = H^{k}_{\substack{N,P\\(31)}} + H^{k}_{\substack{N,P\\(22)}} \tag{II6}$$

with $H^{k}_{\substack{N,P\\(31)}}$ and $H^{k}_{\substack{N,P\\(22)}}$ given by (II4) and (II5) respectively.

In order to calculate the one quasi-proton to one quasi-proton plus two quasi neutrons matrix elements, certain commutation rules are necessary. Since $\langle \tilde{\tilde{0}}_N | \Gamma^{k}_{m}(N) = 0$ where $\Gamma^{k}_{m}(N)$ is the quasi-Boson creation operator, one may write

$$\langle \tilde{\tilde{0}}_N | B^{+k'}_{m'}(j'_{1N} j'_{2N}) \Gamma^{k}_{m}(N) | \tilde{\tilde{0}}_N \rangle$$

$$= \langle \tilde{\tilde{0}}_N | \left[B^{+k'}_{m'}(j'_{1N} j'_{2N}), \Gamma^{k}_{m}(N) \right] | \tilde{\tilde{0}}_N \rangle \tag{II7}$$

From (I47a), and the Boson commutation rule for the operators $B^{+k'}_{m'}(j'_{1N} j'_{2N})$ and $B^{+k}_{m}(j_{1N} j_{2N})$, (II7) becomes

$$\sum_{j_{1N} \geq j_{2N}} s(j_{1N} j_{2N}) \langle \tilde{\tilde{0}}_N | \left[B^{+k'}_{m'}(j'_{1N} j'_{2N}), C^{k}_{m}(j_{1N} j_{2N}) \right] | \tilde{\tilde{0}}_N \rangle \tag{II8}$$

Using the Boson commutation rule (I41), and the normalization condition, $\langle \tilde{\tilde{0}}_N | \tilde{\tilde{0}}_N \rangle = 1$, one gets

$$\sum_{j_{1N} \geq j_{2N}} s(j_{1N} j_{2N}) \left[\frac{-(-1)^{k+m} \delta_{m,-m'} \delta_{kk'}}{1 + \delta_{j_{1N} j_{2N}}} \left(\delta_{j_{1N} j'_{1N}} \delta_{j_{2N} j'_{2N}} \right.\right.$$

$$-(-1)^{j_{1N}+j_{2N}+k}\delta_{j_{1N}j'_{2N}}\delta_{j_{2N}j'_{1N}}\Big)\Big] = -(-1)^{k+m}\delta_{m,-m'}\delta_{kk'}s(j'_{1N}j'_{2N})$$

(II9)

Hence,

$$\langle \tilde{\tilde{0}}_N | B^{+k'}_{m'}(j'_{1N}j'_{2N})\Gamma^k_m(N)|\tilde{\tilde{0}}_N\rangle = -(-1)^{k+m}\delta_{m,-m'}\delta_{kk'}s(j'_{1N}j'_{2N})$$

(II 10)

Similarly,

$$\langle \tilde{\tilde{0}}_N | C^{k'}_{m'}(j'_{1N}j'_{2N})\Gamma^k_m(N)|\tilde{\tilde{0}}_N\rangle = (-1)^{k+m}\delta_{m,-m'}\delta_{kk'}r(j'_{1N}j'_{2N})$$

(II 11)

The matrix elements of $H^k_{N,P\,(31)}$ are evaluated in Appendix Cla, with Hermiticity substantiated by the Introduction to Appendix C. The result is

$$\langle(\tilde{\tilde{0}}_N;\tilde{j}_{cp})j_{cp}m_{cp}|H^k_{N,P\,(31)}|(\tilde{\tilde{k}}_N;\tilde{j}'_{cp})JM\rangle$$

$$=\langle(\tilde{\tilde{k}}_N;\tilde{j}'_{cp})JM|H^k_{NP\,(31)}|(\tilde{\tilde{0}}_N;\tilde{j}_{cp})j_{cp}m_{cp}\rangle$$

$$=-4\pi F^k q^k(j'_{cp}J)\delta_{j_{cp}J}\delta_{m_{cp}M}\sum_{j_{1N}\geq j_{2N}}\frac{1}{\sqrt{1+\delta_{j_{1N}j_{2N}}}}q^k(j_{1N}j_{2N})(-1)^{l_{1N}}$$

$$\times\sin(\chi_{1N}+\chi_{2N})\left[r(j_{1N}j_{2N})-(-1)^k s(j_{1N}j_{2N})\right]$$

(II 12)

where the notation follows (II3b), and the factor, $\delta_{Jj_{cp}}\delta_{Mm_{cp}}$, simply provides the statement that $H^{k}_{N,P\,(31)}$ is a scalar.

Of particular interest in (II 12) is the separability. There is a part involving only the quasi-proton and the total angular momentum, $\mathcal{J}$. The factors after the summation sign involve only quasi-neutrons. One would expect something like this if the properties of the even nucleus are to be retained in the coupling to the odd proton.

The matrix elements of $H^{k}_{N,P\,(22)}$ (II5) should be quite small. This follows from $H^{k}_{N,P\,(22)}$ consisting of single quasi-particle scattering terms. These terms can easily be shown to be distributed over many oscillator modes rather than just the k'th mode.[27] This would greatly diminish the size of the terms. In fact, the analogous terms in the neutron-neutron interaction are neglected in the quasi-Boson description for an even-even nucleus.

The presence of single quasi-neutron operators precludes the use of $|\tilde{\tilde{0}}_N\rangle$, the quasi-Boson vacuum, with $H^{k}_{N,P\,(22)}$. The correct procedure is to obtain an explicit expression for $|\tilde{\tilde{0}}_N\rangle$ in terms of 0,4,8.......quasi-neutron states. But from the remarks of the preceding paragraph, the contribution from $H^{k}_{N,P\,(22)}$ would hardly seem worth this effort. Hence, the matrix elements of $H^{k}_{N,P\,(22)}$ will be obtained by approximating $|\tilde{\tilde{0}}_N\rangle$ by the quasi-neutron vacuum, $|\tilde{0}_N\rangle$. This means that $\Gamma^{k}_{m}(N)$ (I47a) is approximated by $\sum_{\gamma_{1N}\geq\gamma_{2N}} \lambda(\gamma_{1N}\gamma_{2N}) \times B^{+k}_{m}(\gamma_{1N}\gamma_{2N})$. The result (from Appendix C1b) for one proton outside of a major closed shell is given for order k

by

$$\langle (\tilde{\hat{k}}_N ; \hat{j}_{cp}) JM | H^{k}_{\substack{N,P\\(22)}} | (\tilde{\hat{k}}_N ; \tilde{j}_{cp}') J'M' \rangle$$

$$= \langle (\tilde{\hat{k}}_N ; \tilde{j}_{cp}') J'M' | H^{k}_{\substack{N,P\\(22)}} | (\tilde{\hat{k}}_N ; \hat{j}_{cp}) JM \rangle$$

$$\simeq (2k+1) 4\pi F^k g^k (j_{cp} j_{cp}') (-1)^{j_{cp}+J} \begin{Bmatrix} k & k & k \\ j_{cp} & j_{cp}' & J \end{Bmatrix} \delta_{JJ'} \delta_{MM'}$$

$$\times \sum_{j_{a_N} j_{b_N} j_{c_N}} g^k (j_{a_N} j_{b_N}) \cos(\tau_{a_N} + \tau_{b_N}) \sqrt{(1+\delta_{j_{b_N} j_{c_N}})(1+\delta_{j_{a_N} j_{c_N}})}$$

$$\times \lambda (j_{b_N} j_{c_N}) \lambda (j_{c_N} j_{a_N}) \begin{Bmatrix} k & k & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix}$$

(II 13)

The Hermiticity stated in (II 13) is also shown in Appendix C1b. One notes that (II 13), like (II 12), is separable into quasi-proton and quasi-neutron parts, but in (II 13) each part is weighted by six-j symbols. Again $\delta_{JJ'} \delta_{MM'}$ just states the scalar nature of $H^{k}_{\substack{N,P\\(22)}}$.

Section B. Two Quasi-Neutrons Plus One Quasi-Proton

The assumption is made that only the single particle Hamiltonian and the pairing interaction have been diagonalized. This then leaves the entire long-range Hamiltonian (I35) to be diagonalized. Since the present investigation deals with one proton outside of a major closed shell, there is no proton-proton interaction. Then $\mathcal{H}^{k}_{L.R.}$ is given (c.f.(II6)) by

$$\mathcal{H}^{k}_{L.R.} = H^{k}_{\underset{(31)}{N,P}} + H^{k}_{\underset{(22)}{N,P}} + \mathcal{H}^{NN}_{(k)} \qquad \text{(II 14)}$$

The eigen function for a state of total angular momentum, J, and z component, M, is given by

$$|JM\rangle = \Big[a^{J}_{oJ}\, \beta^{+J}_{M}(P) + \sum_{(j_{a_N} \geq j_{b_N}) J_0 j_{cP}} C_{(j_{a_N} j_{b_N}) J_0 j_{cP}} \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \Big(\big[\beta^{+j_a}_{(N)} \beta^{+j_b}_{(N)} \big]^{J_0} \times \beta^{+j_c}_{(P)} \Big)^{J}_{M} \Big] |\hat{0}_N ; \hat{0}_P\rangle \qquad \text{(I67)}$$

Since all components of the state, $|JM\rangle$, involve one quasi-proton, the same reasoning applied in the preceding section holds. This means that the only pertinent neutron-proton interaction terms (c.f. (I32)) are $H^{k}_{\underset{(31)}{N,P}}$ and $H^{k}_{\underset{(22)}{N,P}}$ given by (II4) and (II5) respectively. The best form for $H^{k}_{\underset{(31)}{N,P}}$ is, however, that which emphasizes the single-quasi neutron nature. The operators, β^{+} and C, in (II4) are not assumed to obey Boson commutation rules in the present situation. Using the definitions of β^{+} and C (I38), equation (II4) becomes for force order k and one proton outside of a major closed shell

$$H^{k}_{\underset{(31)}{N,P}} = (-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1_N} j_{2_N} \\ j_{3P} j_{4P}}} q^{k}(j_{1_N} j_{2_N})\, q^{k}(j_{3P} j_{4P}) (-1)^{l_{1_N}} v_{1_N} u_{2_N} \times \Big\{ \Big(-\big[\gamma^{j_1}_{(N)} \gamma^{j_2}_{(N)} \big] + (-1)^{k} \big[\beta^{+j_1}_{(N)} \beta^{+j_2}_{(N)} \big] \Big)^{k} \big[\beta^{+j_3}{}_{(P)}\, \gamma^{j_4}{}_{(P)} \big]^{k} \Big\}_{0}^{0} \qquad \text{(II 15a)}$$

where

$$\left(-\left[\gamma^{j_1}_{(N)}\gamma^{j_2}_{(N)}\right]+(-1)^{k}\left[\beta^{+j_1}(N)\,\beta^{+j_2}(N)\right]^{k}\right.$$

$$=-\left[\gamma^{j_1}(N)\,\gamma^{j_2}(N)\right]^{k}+(-1)^{k}\left[\beta^{+j_1}(N)\,\beta^{+j_2}(N)\right]^{k}$$

(II 15b)

The long-range interaction between neutrons is just (I32) with $\eta=\eta'=N$, i.e.

$$\mathcal{H}^{NN}(k)=\mathcal{H}^{k}_{\substack{NN\\(40)}}+\mathcal{H}^{k}_{\substack{NN\\(22)}}+\mathcal{H}^{k}_{\substack{NN\\(31)}}$$

(II 16)

and of these terms only $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ can give non-zero matrix elements. The vanishing of the matrix elements of $\mathcal{H}^{k}_{\substack{NN\\(40)}}$ is due to the existence of only zero or two quasi-neutrons in the component states of the basis considered. Since $\mathcal{H}^{k}_{\substack{NN\\(40)}}$ itself contains four quasi-neutron operators, all of which are annihilators or all of which are creators, the matrix elements must be zero.

At first glance there would seem to be a possibility of matrix elements of $\mathcal{H}^{k}_{\substack{NN\\(31)}}$ not vanishing. The elements would be between states of one quasi-proton and two quasi-neutrons plus one quasi-proton. Since $\mathcal{H}^{k}_{\substack{NN\\(31)}}$ does not involve quasi-proton operators, such operators will come only from the states themselves. In particular $\langle\tilde{0}_p|\gamma^{J}_{-M}(P)\,\beta^{+j_c}_{m_c}(P)|\tilde{0}_p\rangle$ would occur in the matrix element, $\langle JM|\mathcal{H}^{k}_{\substack{NN\\(31)}}|JM\rangle$.

Using the commutation rule (I20'c)

$$\langle\tilde{0}_p|\gamma^{J}_{-M}(P)\,\beta^{+j_c}_{m_c}(P)|\tilde{0}_p\rangle=\delta_{Jj_{cp}}\,\delta_{Mm_{cp}}(-1)^{j_{cp}+m_{cp}}$$

(II 17)

From (I67) one sees that $M = m_{cp}$ requires that the z component of J_0 be identically zero, where J_0 is the angular momentum of the two quasi-neutron state. The intermediate angular momentum, J_0, itself must then be zero, i.e., the two quasi-neutrons must be paired to zero. If such is the case, then the excited state, $|JM\rangle$, is spurious, for the quasi particle picture requires this to be the ground state. Thus, in order to avoid the introduction of spurious states the matrix elements of $\mathcal{H}^{k}_{NN(31)}$ must vanish.

The $\mathcal{H}^{k}_{NN(22)}$ terms (I34) may be divided into two classes. One of these consists of coupled annihilation operators and coupled creation operators (I34c and d), and the other consists of annihilation operators coupled to creation operators (I4e,f,g,h).

In the equations (I34) j_{1_N}, j_{2_N}, j_{3_N}, and j_{4_N} are dummy indices representing single quasi-neutron angular momenta. One may then, e.g., interchange j_{1_N} and j_{2_N} in (I34 d), and j_{3_N} and j_{4_N} in (I34c). By also using the interchange rules for g^k (I30) and Clebsch-Gordon coefficients (A1.4), plus the parity conservation rule (II2), equations (I34c,d) combine to give

$$\mathcal{H}^{k}_{NN(22)}(\text{class } 1) = \frac{4\pi F^k}{\sqrt{2k+1}} \sum_{\substack{j_{1_N} j_{2_N} \\ j_{3_N} j_{4_N}}} g^k(j_{1_N} j_{2_N})\, g^k(j_{3_N} j_{4_N})$$

$$\times\, u_{1_N} v_{2_N} u_{3_N} v_{4_N} (-1)^{l_{1_N}+l_{3_N}} \Big\{ \left[\beta^{+j_1}(N)\, \beta^{+j_2}(N)\right]^k \left[\gamma^{j_3}(N)\, \gamma^{j_4}(N)\right]^k$$

$$+ \left[\gamma^{j_1}(N)\, \gamma^{j_2}(N)\right]^k \left[\beta^{+j_3}(N)\, \beta^{+j_4}(N)\right]^k \Big\}^0_0 \qquad \text{(II 18)}$$

The notation (Class 1) in the left hand side of (II 18) will distinguish (II 18) from the terms consisting of annihilators coupled to creators. The term $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 2), may also be reduced to a simple form. This is done by interchanging j_{1_N} and j_{3_N} with j_{2_N} and j_{4_N} respectively in equations (I46e,f,g,h) in a way similar to that used in obtaining $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 1) (II 18). If one also uses the commutation rule for a coupled annihilator and creator (I21'd), the result is

$$\mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class } 2) = -(-1)^{k}\frac{4\pi}{\sqrt{2k+1}}F^{k}\sum_{\substack{j_{1_N}j_{2_N}\\j_{3_N}j_{4_N}}} g^{k}(j_{1_N}j_{2_N})\,g^{k}(j_{3_N}j_{4_N})$$

$$\times\cos(\tau_{1_N}+\tau_{2_N})\cos(\tau_{3_N}+\tau_{4_N})\left\{\left[\beta^{+j_1(N)}\gamma^{j_2(N)}\right]^{k}\left[\beta^{+j_3}_{(N)}\gamma^{j_4(N)}\right]^{k}\right\}^{0}_{0} \qquad \text{(II 19)}$$

Hence, the total long-range Hamiltonian for multipole order k (II 14) may be written as

$$\mathcal{H}^{k}_{L.R.} = H^{k}_{\substack{N,P\\(31)}} + H^{k}_{\substack{N,P\\(2,2)}} + \mathcal{H}^{NN}(k) = H^{k}_{\substack{N,P\\(31)}} + H^{k}_{\substack{N,P\\(22)}}$$

$$+\ \mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class } 1) + \mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class } 2) \qquad \text{(II20)}$$

with $H^{k}_{\substack{N,P\\(31)}}$ given by (II 15), $H^{k}_{\substack{N,P\\(22)}}$ by (II5), $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 1) by (II 18), and $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 2) by (II 19).

The matrix elements of (II 20) are calculated (Appendix C2) with respect to the set of states (I67). The Hermiticity of $\mathcal{H}^{k}_{NN\,(22)}$ (Classes 1 and 2) and $H^{k}_{N,\rho\,(22)}$ are also checked in Appendix C2 while that of $H^{k}_{N,\rho\,(31)}$ is verified in the Introduction to Appendix C. In summary the matrix elements of (II20) are as follows;

$$\langle [(\tilde{j}_{a_N}\, \tilde{j}_{b_N}) J_0\, \tilde{j}_{c_\rho}] JM \,|\, H^{k}_{N,\rho\,(22)} \,|\, [(\tilde{j}'_{a_N}\, \tilde{j}'_{b_N}) J'_0\, \tilde{j}'_{c_\rho}] J'M' \rangle$$

$$= \langle [(\tilde{j}'_{a_N}\, \tilde{j}'_{b_N}) J'_0\, \tilde{j}'_{c_\rho}] J'M' \,|\, H^{k}_{N,\rho\,(22)} \,|\, [(\tilde{j}_{a_N}\, \tilde{j}_{b_N}) J_0\, \tilde{j}_{c_\rho}] JM \rangle$$

$$= (-1)^{k} (\delta_{JJ'}\delta_{MM'})\, 4\pi F^{k} g^{k}(j_{c_\rho} j'_{c_\rho}) (-1)^{j_{c_\rho}+J} \begin{Bmatrix} J'_0 & J_0 & k \\ j_{c_\rho} & j'_{c_\rho} & J \end{Bmatrix}$$

$$\times \sqrt{(2J_0+1)(2J'_0+1)}\; \frac{1}{\sqrt{(1+\delta_{j_{a_N} j_{b_N}})(1+\delta_{j'_{a_N} j'_{b_N}})}}$$

$$\times \Big[-g^{k}(j_{a_N} j'_{a_N}) \cos(\alpha_{a_N}+\alpha'_{a_N}) (-1)^{j_{a_N}+j_{b_N}} \begin{Bmatrix} J'_0 & J_0 & k \\ j_{a_N} & j'_{a_N} & j_{b_N} \end{Bmatrix} \delta_{j_{b_N} j'_{b_N}}$$

$$+ g^{k}(j_{b_N} j'_{a_N}) \cos(\alpha_{b_N}+\alpha'_{a_N}) (-1)^{J_0} \begin{Bmatrix} J'_0 & J_0 & k \\ j_{b_N} & j'_{a_N} & j_{a_N} \end{Bmatrix} \delta_{j_{a_N} j'_{b_N}}$$

$$- q^{k}(j_{a_N}\, j'_{b_N}) \cos(\tau_{a_N} + \tau'_{b_N}) (-1)^{j_{a_N} + j'_{b_N} + J'_0} \begin{Bmatrix} J'_0 & J_0 & k \\ j_{a_N} & j'_{b_N} & j_{b_N} \end{Bmatrix} \delta_{j_{b_N} j'_{a_N}}$$

$$- q^{k}(j_{b_N}\, j'_{b_N}) \cos(\tau_{b_N} + \tau'_{b_N}) (-1)^{j_{a_N} + j'_{b_N} + J_0 + J'_0}$$

$$\times \begin{Bmatrix} J'_0 & J_0 & k \\ j_{b_N} & j'_{b_N} & j_{a_N} \end{Bmatrix} \delta_{j_{a_N} j'_{a_N}} \Bigg]$$

(II 21)

$$\langle \tilde{O}_N ; \widetilde{j_{cp}\, m_{cp}} \,|\, \underset{(31)}{H^{k}_{N,P}} \,|\, [(\tilde{j}'_{a_N}\, \tilde{j}'_{b_N}) J'_0\, \tilde{j}'_{cp}]\, JM \rangle$$

$$= \langle [(\tilde{j}'_{a_N}\, \hat{j}'_{b_N}) J'_0\, \tilde{j}'_{cp}]\, JM \,|\, \underset{(31)}{H^{k}_{N,P}} \,|\, \hat{O}_N ; \widetilde{j_{cp}\, m_{cp}} \rangle$$

$$= \frac{-4\pi F^{k}}{\sqrt{(2k+1)(2J+1)}}\, q^{k}(j'_{cp}\, J)\, q^{k}(j'_{a_N}\, j'_{b_N}) (-1)^{l'_{a_N}}$$

$$\times \sin(\tau'_{a_N} + \tau'_{b_N}) \frac{1}{\sqrt{1 + \delta_{j'_{a_N} j'_{b_N}}}}\, \delta_{J j_{cp}}\, \delta_{M m_{cp}}\, \delta_{k J'_0}$$

(II 22)

$$\langle [(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 j_{c_P}] JM \,|\, \mathcal{H}^{k}_{\substack{NN\\(22)}} (\text{class } 1) \,|\, [(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J'_0 \tilde{j}'_{c_P}] J'M' \rangle$$

$$= \langle [(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J'_0 \tilde{j}'_{c_P}] J'M' \,|\, \mathcal{H}^{k}_{\substack{NN\\(22)}} (\text{class} 1) \,|\, [(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{c_P}] J M \rangle$$

$$= (\delta_{JJ'} \delta_{MM'}) \left\{ \frac{-8\pi}{2k+1} F^{k} q^{k} (j_{a_N} j_{b_N}) q^{k} (j'_{a_N} j'_{b_N}) \right.$$

$$\times (-1)^{l_{b_N} + l'_{b_N}} \sin(x_{a_N} + x_{b_N}) \sin(x'_{a_N} + x'_{b_N}) \delta_{J_0 J'_0} \delta_{J_0 k} \delta_{j_{c_P} j'_{c_P}}$$

$$- 4\pi F^{k} (1 + \delta_{j_{a_N} j_{b_N}}) \delta_{J_0 J'_0} \delta_{j_{c_P} j'_{c_P}} \delta_{j_{a_N} j'_{a_N}} \delta_{j_{b_N} j'_{b_N}} \sum_{j_{1_N} \geq j_{2_N}} \left[q^{k} (j_{1_N} j_{2_N}) \right]^2$$

$$\times \sin^2 (x_{1_N} + x_{2_N}) \frac{1}{1 + \delta_{j_{1_N} j_{2_N}}}$$

$$+ 4\pi F^{k} \delta_{J_0 J'_0} \delta_{j_{c_P} j'_{c_P}} (1 + \delta_{j_{a_N} j_{b_N}}) \delta_{j_{a_N} j'_{a_N}} \delta_{j_{b_N} j'_{b_N}}$$

$$\times \sum_{j_{1_N}} \left(\frac{\left[q^k(j_{1_N} j_{b_N}) \right]^2}{2 j_{b_N} + 1} \sin^2(\alpha_{1_N} + \alpha_{b_N}) \right.$$

$$\left. + \frac{\left[q^k(j_{1_N} j_{a_N}) \right]^2}{2 j_{a_N} + 1} \sin^2(\alpha_{1_N} + \alpha_{a_N}) \right) \Bigg\} \frac{1}{\sqrt{(1+\delta_{j_{a_N} j_{b_N}})(1+\delta_{j'_{a_N} j'_{b_N}})}}$$

(II 23)

and,

$$\left\langle \left[(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{c_P} \right] JM \right| \mathcal{H}^{k}_{\substack{NN \\ (22)}} (\text{class } 2) \left| \left[(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J'_0 \tilde{j}'_{c_P} \right] J'M' \right\rangle$$

$$= \left\langle \left[(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J'_0 \tilde{j}'_{c_P} \right] J'M' \right| \mathcal{H}^{k}_{\substack{NN \\ (22)}} (\text{class } 2) \left| \left[(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{c_P} \right] JM \right\rangle$$

$$= \Big\{ -4 \pi F^k (1 + \delta_{j_{a_N} j_{b_N}}) \delta_{j_{a_N} j'_{a_N}} \delta_{j_{b_N} j'_{b_N}} \delta_{j_{c_P} j'_{c_P}} \delta_{J_0 J'_0}$$

$$\times \sum_{j_{1_N}} \left(\frac{\left[q^k(j_{1_N} j_{b_N}) \right]^2}{2 j_{b_N} + 1} \cos^2(\alpha_{1_N} + \alpha_{b_N}) \right.$$

$$+ \frac{\left[q^{k}(j_{1\nu}\, j_{a\nu})\right]^{2}}{2 j_{a\nu}+1} \cos^{2}(\alpha_{1\nu}+\alpha_{a\nu})\Big)\Big)$$

$$-8\pi F^{k}\, \delta_{j_{cp} j'_{cp}}\, \delta_{J_0 J_0'}\, (-1)^{j_{b\nu}+j'_{a\nu}} \Big[(-1)^{J_0}\, q^{k}(j_{a\nu}\, j'_{a\nu})\, q^{k}(j_{b\nu}\, j'_{b\nu})$$

$$\times \cos(\alpha_{a\nu}+\alpha'_{a\nu})\, \cos(\alpha_{b\nu}+\alpha'_{b\nu}) \begin{Bmatrix} j_{b\nu} & j_{a\nu} & J_0 \\ j'_{a\nu} & j'_{b\nu} & k \end{Bmatrix}$$

$$+ q^{k}(j_{b\nu}\, j'_{a\nu})\, q^{k}(j_{a\nu}\, j'_{b\nu})\, \cos(\alpha_{b\nu}+\alpha'_{a\nu})$$

$$\times \cos(\alpha_{a\nu}+\alpha'_{b\nu}) \begin{Bmatrix} j_{a\nu} & j_{b\nu} & J_0 \\ j'_{a\nu} & j'_{b\nu} & k \end{Bmatrix} \Big] \Big\}$$

$$\times \frac{1}{\sqrt{(1+\delta_{j_{a\nu} j_{b\nu}})(1+\delta_{j'_{a\nu} j'_{b\nu}})}}\, \delta_{JJ'}\, \delta_{MM'}$$

(II 24)

The notation, $\left| \left[(\tilde{j}_{a_N} \tilde{j}_{b_N})J_0 \quad \tilde{j}_{c_P} \right] JM \right\rangle$, refers to a particular component of the state, $|JM\rangle$, viz; that of two quasi-neutrons, j_{a_N} and j_{b_N}, coupled to J_0 which then couples with the quasi-proton, j_{c_P}, to give J, M. It should be noted that the normalization factor, $\frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}}$, is included in this notation. The state, $|\tilde{0}_N \; ; \tilde{j}_{c_P} \tilde{m}_{c_P}\rangle$ contains no quasi-neutrons and one quasi-proton of angular momentum, j_{c_P}, with z projection, m_{c_P}. Kronecker deltas, $\delta_{JJ'} \delta_{MM'}$ in (II21), (II23), and (II24), and $\delta_{J j_{c_P}} \delta_{M m_{c_P}}$ of (II22), emphasize that $\mathcal{H}^{k}_{L.R.}$ (II20) is a scalar. More will be said about these matrix elements in the following Sections.

Section C. Parity Restrictions on Matrix Elements of the Long-Range Interaction.

From (I23) the long-range interaction for the k'th multipole is seen to be proportional to $\sum_{m=-k}^{k} Y_{km}(\Omega_i) Y^*_{km}(\Omega_j)$. Where Y_k is the spherical harmonic and i and j refer to the i'th and j'th particles respectively. Since $Y^*_{km} = (-1)^m Y_{k,-m}$ one may write[29]

$$\sum_{m=-k}^{k} Y_{km}(\Omega_i) Y^*_{km}(\Omega_j) = Y_k(\Omega_i) \cdot Y_k(\Omega_j) \qquad \text{(II25)}$$

The matrix element of (II25) between states, $|(j_i j_j) I M_I\rangle$, is given by [29]

$$\langle (j_i' j_j') I' M_I' | Y_k(i) \cdot Y_k(j) | (j_i j_j) I M_I \rangle = (-1)^{j_i + j_j' + I} \delta_{II'} \delta_{M_I M_I'} \begin{Bmatrix} I & j_j' & j_i' \\ k & j_i & j_j \end{Bmatrix} \langle j_i' \| Y_k(i) \| j_i \rangle \langle j_j' \| Y_k(j) \| j_j \rangle \qquad \text{(II26)}$$

where j_i and j_j are the angular momenta of particles i and j respectively. These angular momenta couple to total angular momentum, I , with z component, M_I . The double barred matrix elements are the usual reduced matrix elements of a spherical tensor. Conservation of parity then requires that for k even, $\pi_{j_i'} = \pi_{j_i}$ and $\pi_{j_j'} = \pi_{j_j}$. Whereas for k odd, one must have $\pi_{j_i'} \neq \pi_{j_i}$ and $\pi_{j_j'} \neq \pi_{j_j}$.

Conservation of parity introduces still another restriction when one calculates the matrix elements of the long-range interaction between states like $|(J_{0_N} j_{cp}) JM\rangle$. The symbols, J_0 and j_{cp} , are integer and odd half integer respectively, and couple to give a total angular momentum, J , with z component M . To make this discussion more appropriate to the matrix elements of the preceding sections, let j_{cp} represent the odd quasi-proton. Also let $T^k(N) \cdot U^k(P)$ be a term in the long-range interaction between a neutron and a proton in the quasi-particle occupation number representation. Then analogous to (II26) one has

$$\langle (\tilde{J}_{0_N}' \tilde{j}_{cp}') J'M' | T^k(N) \cdot U^k(P) | (\tilde{J}_{0_N} \tilde{j}_{cp}) JM \rangle$$
$$= (-1)^{J_0 + j_{cp}' + J} \delta_{JJ'} \delta_{MM'} \begin{Bmatrix} J & j_{cp}' & J_0' \\ k & J_0 & j_{cp} \end{Bmatrix} \langle \tilde{J}_{0_N}' \| T^k(N) \| \tilde{J}_{0_N} \rangle$$
$$\times \langle \tilde{j}_{cp}' \| U^k(P) \| \tilde{j}_{cp} \rangle \qquad \text{(II27)}$$

and,

$$\langle (\tilde{\tilde{k}}_N' \tilde{j}_{cp}') J'M' | T^k(N) \cdot U^k(P) | (\tilde{k}_N \tilde{j}_{cp}) JM \rangle$$
$$= (-1)^{k + j_{cp}' + J} \delta_{JJ'} \delta_{MM'} \begin{Bmatrix} J & j_{cp}' & k' \\ k & k & j_{cp} \end{Bmatrix} \langle \tilde{\tilde{k}}_N' \| T^k(N) \| \tilde{k}_N \rangle$$
$$\times \langle \tilde{j}_{cp}' \| U^k(P) \| \tilde{j}_{cp} \rangle \qquad \text{(II28)}$$

The symbol, j_{cp} , in (II27) and (II28) refers to the odd quasi-proton. In (II27), J_0 , represents the angular momentum of two quasi-neutrons coupled together. The angular momenta of the individual quasi-neutrons are irrelevant to the present discussion and are not indicated in (II27).

1. Two Quasi-Neutrons Plus Quasi-Proton:

The implications of (II27) in the matrix elements of $H^{k}_{N,P\,(22)}$ (II21) are

for k even

$$\pi_{J_0} = \pi_{J_0'} \quad , \quad \pi_{j_{cp}} = \pi_{j_{cp}'} \qquad \text{(II29a)}$$

for k odd

$$\pi_{J_0} \neq \pi_{J_0'} \quad , \quad \pi_{j_{cp}} \neq \pi_{j_{cp}'} \qquad \text{(II29b)}$$

For the matrix elements of $H^{k}_{N,P\,(31)}$ (II22) equation (II27) requires that

for k even

$$\pi_{j_{cp}'} = \pi_J = \pi_{j_{cp}} \qquad \text{(II30a)}$$

for k odd

$$\pi_{j_{cp}'} \neq \pi_J \quad , \quad \pi_{j_{cp}'} \neq \pi_{j_{cp}} \qquad \text{(II30b)}$$

2. Quasi-Boson Plus Quasi-Proton:

There are two possible values of k' in (II28), i.e., $k' = k$ or $k' = 0$. The former indicates matrix elements of $H^{k}_{N,P\,(22)}$ (II 13) and the latter matrix elements of $\underset{(31)}{H}{}^{k}_{N,p}$(II 12).

The restrictions on the matrix elements of $\underset{(31)}{H}{}^{k}_{N,p}$ (II 12) due to (II28) are

for k even $\quad \pi_{j'_{cp}} = \pi_J = \pi_{j_{cp}}$

and (II31a)

for k odd $\quad \pi'_{j_{cp}} \neq \pi_J \;, \; \pi'_{j_{cp}} \neq \pi_{j_{cp}}$ (II31b)

The restrictions on the matrix elements of $\underset{(22)}{H}{}^{k}_{N,p}$ (II 13) due to (II28) are

for k even $\quad \pi_{j_{cp}} = \pi_{j'_{cp}}$

and (II32a)

for k odd

$$\pi_{j_{cp}} \neq \pi_{j'_{cp}} \qquad \text{(II32b)}$$

Equation (II32b) implies that the matrix elements of $\underset{(31)}{H}{}^{k}_{N,p}$ (II 13) must vanish for k odd. This is seen from the fact that $\underset{(22)}{H}{}^{k}_{N,p}$ can only have matrix elements between states of the same total angular momentum.

Section D. Comparison and Discussion of the Matrix Elements of Sections A) and B).

1. The Matrix Elements of $\underset{(31)}{H}{}^{k}_{N,p}$:

When $\underset{(31)}{H}{}^{k}_{N,p}$ is written in the form of (II4) its matrix elements are given by (II 12) in the quasi-Boson plus quasi-proton picture. For $\underset{(31)}{H}{}^{k}_{N,p}$ written in the form of (II 15) the matrix elements are given by (II22) in the two quasi-neutrons plus quasi-proton scheme. Comparing (II 12) and (II 22) one sees that they are quite similar, the only difference being

$$\sum_{j_{1_N} \geq j_{2_N}} \frac{1}{\sqrt{1+\delta_{j_{1_N} j_{2_N}}}} q^k (j_{1_N} j_{2_N}) (-1)^{l_{1_N}} \sin(\tau_{1_N} + \tau_{2_N}) \left[r(j_{1_N} j_{2_N}) - (-1)^k s(j_{1_N} j_{2_N}) \right]$$

quasi-Boson plus quasi-proton

$$\longrightarrow q^k (j_{a_N}' j_{b_N}') (-1)^{l_{a_N}'} \sin(\tau_{a_N}' + \tau_{b_N}') \qquad \text{(II33)}$$

two quasi-neutrons plus quasi proton

This is what one would expect since the use of the quasi-Boson scheme involves a "weighted" sum over the relative contributions of the possible two quasi-neutron states. The "weighting" is represented by the sets, $r(j_{1_N} j_{2_N})$ and $s(j_{1_N} j_{2_N})$, (I49) which are the coefficients in the definition of the quasi-Boson creation operator (I47a). From equation (I49) with $\omega^k < E_{j_{1_N}} + E_{j_{2_N}}$, one sees that each $r(j_{1_N} j_{2_N})$ and the corresponding $(-1)^k s(j_{1_N} j_{2_N})$ always have opposite signs. This means that $r(j_{1_N} j_{2_N})$ and $s(j_{1_N} j_{2_N})$ add coherently in (II33). The "collective" character inherent in the Boson approximation is thus further emphasized. Both the right and left hand sides of (II33) are related to the electric transition probability in the adjacent even-even nucleus. (c.f.Chapter III).

One sees that for arbitrary k the degeneracy in total angular momentum, J , is removed in both schemes by

$H^{k}_{N,P\,(31)}$ (c.f. equations (II 12) and (II 22)). The separability of (II 22) into quasi-neutron and quasi-proton parts indicates the possibility of "collectiveness" arising from the quasi-neutrons plus quasi-proton technique. One may make this statement since the quasi-Boson plus quasi-proton scheme has the same property of separability (II 12).

The matrices to be diagonalized in the two quasi-neutrons plus quasi-proton method will be much larger than the quasi-Boson plus quasi-proton matrices. This is due to the diagonalization that has already been performed in the quasi-Boson scheme, i.e., the sets, $\lambda(j_{1N}\, j_{2N})$ and $\lambda(j_{1N}\, j_{2N})$, are already supposed known from the adjacent even-even nucleus. In the two quasi-neutrons plus quasi-proton scheme the "weighting" of the $q^{k}(j'_{a_N}\, j'_{b_N}) \sin(\phi'_{a_N} + \phi'_{b_N})$ terms (II 33) will occur as a consequence of the entire diagonalization.

2. The Matrix Elements of $H^{k}_{N,P\,(22)}$

The term, $H^{k}_{N,P\,(22)}$, (II 5) has matrix elements in the quasi-Boson plus quasi-proton picture given by (II 13). The matrix elements for the case of two quasi-neutrons plus one quasi-proton are given by (II 21). The effect of $H^{k}_{N,P\,(22)}$ is restricted in the quasi-Boson plus quasi-proton scheme to even k (II 13) because of the parity requirement (II 32b). In the two quasi-neutrons plus quasi-proton scheme, however, the matrix elements of $H^{k}_{N,P\,(22)}$ need not vanish (II 21).

This is due to the fact that the angular momentum of the two quasi-neutron state need not equal the multipole order, L. For even L the basic difference is that the quasi-Boson plus quasi-proton scheme involves a weighted sum over single quasi-neutron states; whereas, the other scheme does not. Both equations are separable into quasi-neutron and quasi-proton parts, and each part is proportional to a six-j symbol. As previously mentioned, in both cases the elements depend on the total angular momentum, J ; consequently, states of different J will not be degenerate.

Also of interest are the dependencies of (II 13) and (II 21) on $\cos(\phi_{a_\nu}+\phi_{b_\nu})$ types of terms rather than $\sin(\phi_{a_\nu}+\phi_{b_\nu})$ as in (II 12) and (II 22). For quasi-particles near the Fermi energy, λ , μ and ν are approximately equal (I 11, 12), and from (II 3a) $\cos(\phi_{a_\nu}+\phi_{b_\nu})$ approaches zero. In this same region, however, the $\sin(\phi_{a_\nu}+\phi_{b_\nu})$ term would approach a peak since $\sin^2(\phi_{a_\nu}+\phi_{b_\nu})+\cos^2(\phi_{a_\nu}+\phi_{b_\nu})=1$.

In (II 21) one sees that J_0 and J_0' , the angular momenta to which two different two quasi-neutron configurations couple, need not equal L or each other. This implies that there are many off diagonal non-vanishing matrix elements in the two quasi-neutrons plus quasi-proton scheme. It is such elements that will make the matrices of sizes even larger than would be expected by counting the number of available states for each of two quasi-neutrons.

3. The Matrix Elements of $\mathcal{H}^{k}_{\underset{(22)}{NN}}$ (Class 1):

The term, $\mathcal{H}^{k}_{\underset{(22)}{NN}}$ (Class 1), is given by (II 18), and its matrix elements by (II23). These elements are not dependent on the value of the total angular momentum, $\mathcal{J}$. The presence of the odd quasi-proton is manifested only in the $\delta_{j_{cp} j'_{cp}}$ requirement. This just means that the quasi-proton does not influence the quasi-neutrons, the only restriction being that matrix elements must be between states containing the same quasi-proton.

There are no corresponding matrix elements in the quasi-Boson plus quasi-proton scheme. As mentioned before, this is because the neutron-neutron long-range interaction is already diagonalized in the Boson approximation.

Equation (II 18) is seen to consist of sums over $\left[B^{+k}(j_{1_N} j_{2_N}) C^{k}(j_{3_N} j_{4_N})\right]^{0}_{0}$ and $\left[C^{k}(j_{1_N} j_{2_N}) B^{+k}(j_{3_N} j_{4_N})\right]^{0}_{0}$ type terms. The latter may be rewritten in terms of $\left[B^{+k}(j_{3_N} j_{4_N}) C^{k}(j_{1_N} j_{2_N})\right]^{0}_{0}$ by using the commutation rule (I41). This will introduce the $\frac{1}{1+\delta_{j_{1_N} j_{2_N}}}\left[\delta_{j_{2_N} j_{4_N}} \delta_{j_{1_N} j_{3_N}} - \delta_{j_{1_N} j_{4_N}} \delta_{j_{2_N} j_{3_N}} (-1)^{j_{1_N}+j_{2_N}+k}\right]$ term and also single quasi-neutron scattering types of terms. These scattering terms are dropped in the Boson approximation but are kept in the present instance. The $\mathcal{H}^{k}_{\underset{(22)}{NN}}$ (Class 1) expression (II 18) thus contains terms proportional to sums over $2\left[B^{k}(j_{1_N} j_{2_N}) C^{k}(j_{3_N} j_{4_N})\right]^{0}_{0}$ and $\frac{1}{1+\delta_{j_{1_N} j_{2_N}}}\left[\delta_{j_{2_N} j_{4_N}} \delta_{j_{1_N} j_{3_N}} - \delta_{j_{1_N} j_{4_N}} \delta_{j_{2_N} j_{3_N}} (-1)^{j_{1_N}+j_{2_N}+k}\right]$, and also sums due to the single quasi-neutron scattering terms. This is shown in detail in Appendix C2, and leads to identification of the specific terms in (II23) as follows :

Contribution due to $\sum_{j_{1N} j_{2N}} 2\left[B^{+k}(j_{1N} j_{2N})\, C^{k}(j_{3N} j_{4N})\right]_0^0$

$$= \frac{-8\pi}{2k+1}\, F^{k}\, q^{k}(j_{aN} j_{bN})\, q^{k}(j_{a'N} j_{b'N})\,(-1)^{l_{bN}+l_{b'N}} \sin(\vartheta_{aN}+\vartheta_{bN})$$

$$\times \sin(\vartheta_{a'N}+\vartheta_{b'N})\, \delta_{J_0 J_0'}\, \delta_{J_0 k}\, \delta_{j_{cp} j_{cp}'}\, \delta_{JJ'}\, \delta_{MM'} \qquad \text{(II34a)}$$

Contribution due to $\sum_{j_{1N} j_{2N}} \frac{1}{1+\delta_{j_{1N} j_{2N}}}\left[\delta_{j_{2N} j_{4N}}\delta_{j_{1N} j_{3N}} - \delta_{j_{1N} j_{4N}}\delta_{j_{2N} j_{3N}}(-1)^{j_{1N}+j_{2N}+k}\right]$

$$= -4\pi F^{k}\, \delta_{JJ'}\, \delta_{MM'}\, \delta_{J_0 J_0'}\, \delta_{j_{cp} j_{cp}'}\,(1+\delta_{j_{aN} j_{bN}})\, \delta_{j_{aN} j_{a'N}}\, \delta_{j_{bN} j_{b'N}}$$

$$\times \sum_{j_{1N} \geq j_{2N}} \left[q^{k}(j_{1N} j_{2N})\right]^2 \sin^2(\vartheta_{1N}+\vartheta_{2N})\, \frac{1}{1+\delta_{j_{1N} j_{2N}}} \qquad \text{(II34b)}$$

Contribution due to single quasi-neutron scattering terms (neglected in the Boson approximation) =

$$+4\pi F^{k}\, \delta_{JJ'}\, \delta_{MM'}\, \delta_{J_0 J_0'}\, \delta_{j_{cp} j_{cp}'}\,(1+\delta_{j_{aN} j_{bN}})\, \delta_{j_{aN} j_{a'N}}\, \delta_{j_{bN} j_{b'N}}$$

$$\times \sum_{j_{1N}} \left(\frac{\left[q^{k}(j_{1N} j_{bN})\right]^2}{2j_{bN}+1} \sin^2(\vartheta_{1N}+\vartheta_{bN}) + \frac{\left[q^{k}(j_{1N} j_{aN})\right]^2}{2j_{aN}+1} \sin^2(\vartheta_{1N}+\vartheta_{aN}) \right) \qquad \text{(II34c)}$$

Equation (II34c) is of opposite sign from (II34a) and (II34b). If one associates (II34a) and (II34b) with the matrix elements involved in the Boson approximation diagonalization, then (II34c) would be expected to raise the eigenvalues. There is obviously more to the Boson approximation diagonalization than is evident in (II34a) and (II34b). Since only up to two quasi-neutrons have been used there has been no contribution from $\mathcal{H}^{k}_{NN(40)}$ (I43a), whereas the Hamiltonian (I56) diagonalized in the Boson approximation does include $\mathcal{H}^{k}_{NN(40)}$.

Equation (II34c) is seen to involve couplings between the angular momenta of the quasi-neutrons comprising the state considered (i.e., j_a and j_b) and other possible quasi-neutron angular momenta. There is a sum over such couplings; hence, (II34c) might not be negligible. The test of the ideas of these past few paragraphs will be a specific numerical example presented later in this paper. Before one can analyze the effect of an odd quasi-neutron on the Boson approximation, the validity in the presence of two quasi-neutrons themselves should be considered.

4. The Matrix Elements of $\mathcal{H}^{k}_{NN(22)}$ (Class 2):

The term, $\mathcal{H}^{k}_{NN(22)}$ (Class 2), is given by (II 19), and its matrix elements by (II24). This Hamiltonian term is neglected in the quasi-Boson picture (c.f. (I43)), and one would expect it to be quite small. The form of

(II 19) is similar to that of $H^{k}_{N,P\,(22)}$ (II5); however, the matrix elements of $\mathcal{H}^{k}_{NN\,(22)}$ (Class 2) are incapable of removing the degeneracy in the total angular momentum, $\mathcal{J}$. This is to be expected since the total angular momentum, $\mathcal{J}$, arises from the presence of the odd proton anyway.

Equation (II24) consists of two parts, one of which is dependent on six-j symbols. This term involving six-j symbols does not involve a sum over quasi-neutron states; hence, it is limited in magnitude. The other term involves a sum over single quasi-neutron states, but is proportional to $\cos^2(\phi_{1N}+\phi_{kN})$ types of factors. By reasoning similar to that mentioned in Section D2) this term should also be almost negligible. Indeed, the dependence on the square of the cosine supports this idea.

Section E. Graphology.

The differences between the quasi-Boson plus quasi-proton and two quasi-neutrons plus quasi-proton diagonalizations are vividly illustrated by using interaction graphs. In addition the nature of each long-range Hamiltonian term becomes recognizable in terms of real protons and neutrons. Even though the number of real particles is not a constant in the quasi-particle picture, the graphs are a convenient way of gaining insight into the nature of the interaction.

The complete long-range Hamiltonian including terms whose matrix elements vanish is (c.f. (II 14), (II 16), and

(II 20))

$$\mathcal{H}^{k}_{L.R.} = \mathcal{H}^{k}_{NN\,(40)} + \mathcal{H}^{k}_{NN\,(22)}(\text{class 1}) + \mathcal{H}^{k}_{NN\,(22)}(\text{class 2}) + \mathcal{H}^{k}_{NN\,(31)} + H^{k}_{N,P\,(22)} + H^{k}_{N,P\,(31)} \qquad \text{(II35)}$$

The first index in the subscript is the number of quasi-particle creation or annihilation operators. The second number is the number of quasi-particle annihilation or creation operators. In other words, the second index is the number of creators if the first index is the number of annihilators and vice versa. The meanings of (Class 1) and (Class 2) will become apparent from the different graphs used to represent them.

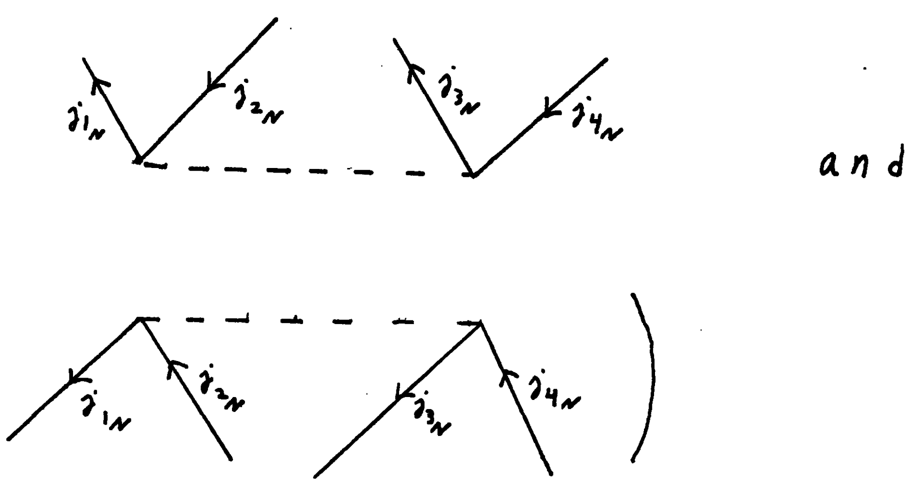
From (I29), $q^{k}(j_a\, j_b)$ is related to the reduced matrix element of the k'th multipole term of the long-range Hamiltonian expansion. The states with respect to which the matrix elements are taken are those of single quasi-particles. The quantum numbers of these states are not only those of real particles but also those of real holes, i.e., a quasi-particle is part real particle and part real hole. The (u, v) trans-formation coefficients (I6) indicate how much of the quasi-particle is particle and how much is hole. All long-range Hamiltonian terms involve terms like $q^{k}(j_{a_\eta}\, j_{b_\eta})\, q^{k}(j_{c_{\eta'}}\, j_{d_{\eta'}})$ multiplied by four (u, v) coefficients. If u_{a_η} appears then j_{a_η} refers to a real particle, and if v_{a_η} occurs

$\dot{j}_{a_\eta}$ pertains to a real hole. Similar associations hold for $\dot{j}_{b_\eta}$, $\dot{j}_{c_\eta}$, and $\dot{j}_{d_\eta}$. The question of whether there is annihilation or creation is arbitrarily resolved as follows.[31] In $q^k(j_{a_\eta} j_{b_\eta})$ if the first index, $\dot{j}_{a_\eta}$, refers to a real particle then the real particle is pictured as being created. If the second index, $\dot{j}_{b_\eta}$, refers to a real particle then the real particle is annihilated. Conversely the first index, $\dot{j}_{a_\eta}$, implies real hole annihilation and the second index, $\dot{j}_{b_\eta}$, pertains to real hole creation.

1. Graphs for the Long-Range Hamiltonian:

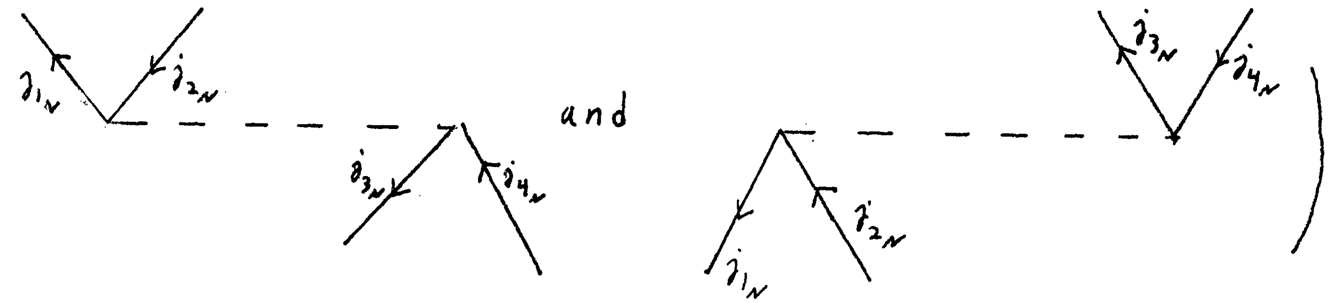
The non-simplified $\mathcal{H}_{L.R.}$ (I34) best shows the origin of the graphs. First the neutron-neutron interaction shall be considered. With the horizontal arrow in the following graphs meaning "may be pictured as" one has

$$\underset{(40)}{\mathcal{H}^k_{NN}} \xrightarrow{(I34\,a,b)} \sum_{\substack{j_{1_N} j_{2_N} \\ j_{3_N} j_{4_N}}} \Big(\text{Terms like}$$



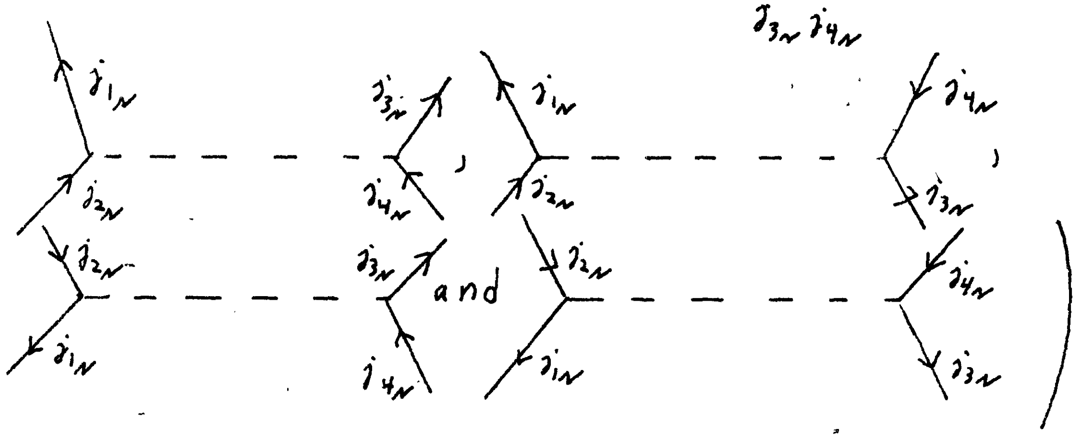
(II 36 a)

$$\mathcal{H}^{h}_{NN\,(22)} \text{ (class 1) } \xrightarrow[(\mathrm{I}\,34\,c,d)]{} \sum_{\substack{j_{1N}\, j_{2N}\\ j_{3N}\, j_{4N}}} \Big(\text{Terms like}$$



(II 36b)

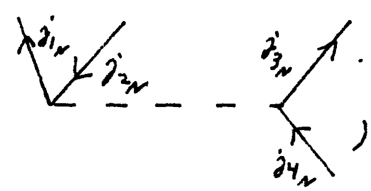
$$\mathcal{H}^{h}_{NN\,(22)} \text{ (class 2) } \xrightarrow[(\mathrm{I}\,34\,e,f,g,h)]{} \sum_{\substack{j_{1N}\, j_{2N}\\ j_{3N}\, j_{4N}}} \Big(\text{Terms like}$$

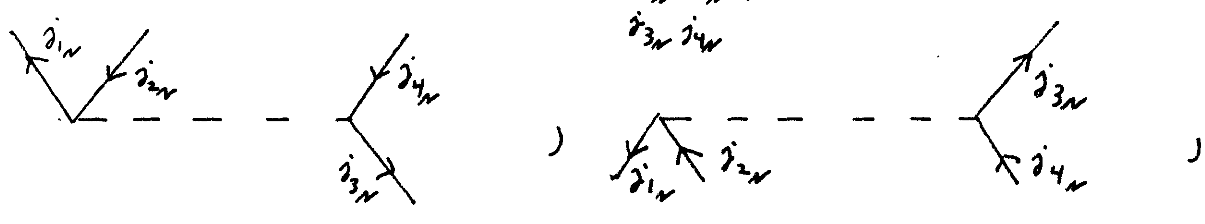


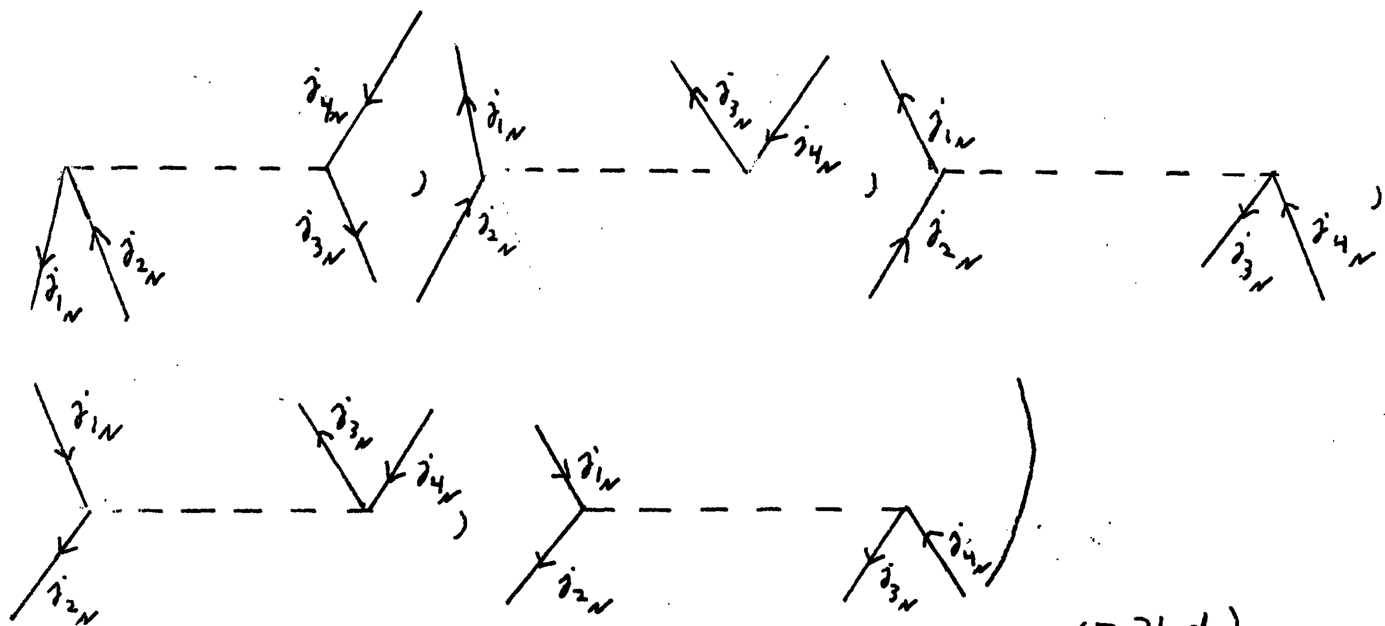
(II 36c)

and

$$\mathcal{H}^{h}_{NN\,(31)} \xrightarrow[(\mathrm{I}\,34\,i,j,k,l,m,n,o,p)]{} \sum_{\substack{j_{1N}\, j_{2N}\\ j_{3N}\, j_{4N}}} \Big(\text{Terms like}$$







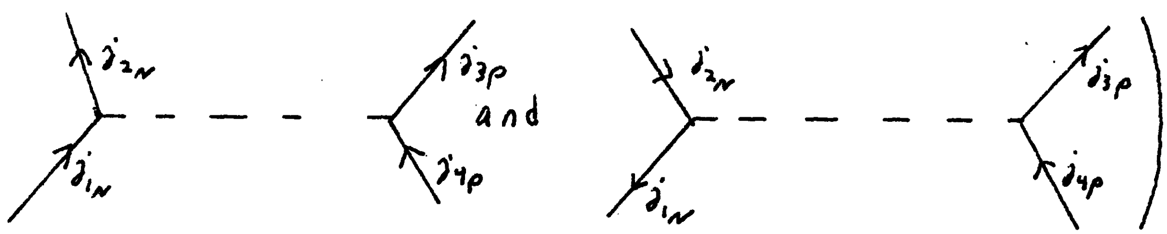
(II 36 d)

One sees easily from (II36b) and (II36c) the difference between Class 1 and Class 2. The former involves the creation and annihilation of particle-hole pairs. This may be looked at as a direct interaction. The latter represents the scattering of particles from one state to another, and the similar scattering of holes. This may be looked at as an exchange interaction and should be weaker than the direct one.

The terms, $\mathcal{H}^{k}_{NN\,(40)}$ and $\mathcal{H}^{k}_{NN\,(22)}$ (Class 1), are diagonalized in the even-even nucleus Boson approximation and $\mathcal{H}^{k}_{NN\,(22)}$ (Class 2) is neglected. In the two quasi-neutron diagonalization $\mathcal{H}^{k}_{NN\,(40)}$ gives zero for its matrix elements and $\mathcal{H}^{k}_{NN\,(22)}$ (Class 1) and $\mathcal{H}^{k}_{NN\,(22)}$ (Class 2) both contribute. The term, $\mathcal{H}^{k}_{NN\,(31)}$, gives zero between non-spurious states of an even number of quasi-neutrons (c.f. discussion following (II 16)).

When one considers the neutron-proton interaction for a proton outside of a major closed shell, the proton and quasi-proton are the same. This means that $u_{j_p}=1$ and $v_{j_p}=0$. From (I34) the only non-vanishing Hamiltonian terms are parts of $H^{k}_{N,P\,(22)}$ and $H^{k}_{N,P\,(31)}$, with the former being of (Class 2) rather than (Class 1) nature. The graphs are as follows

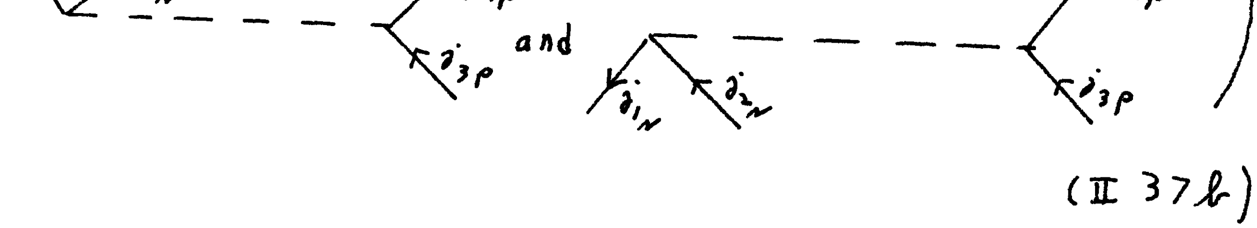
$$H^{k}_{N,P\,(22)} \xrightarrow[(I\,34\,l,g)]{} \sum_{\substack{j_{1N}\, j_{2N} \\ j_{3P}\, j_{4P}}} \Big(\text{Terms like}$$



(II 37a)

and

$$H^{k}_{N,P\,(31)} \xrightarrow[(I\,34\,i,k)]{} \sum_{\substack{j_{1N}\, j_{2N} \\ j_{3P}\, j_{4P}}} \Big(\text{Terms like}$$



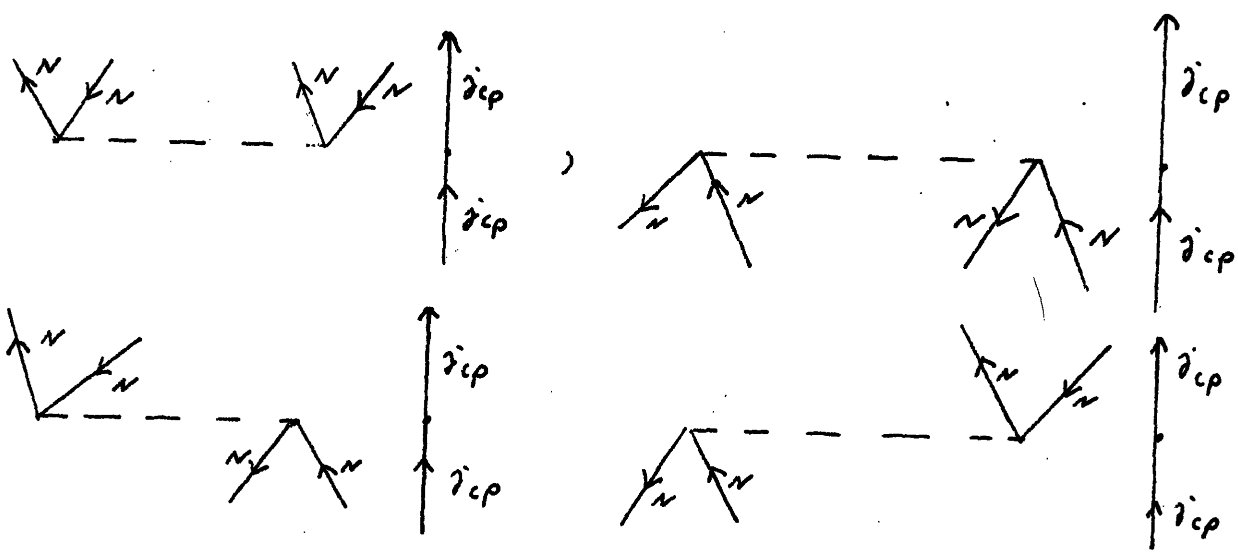
(II 37b)

2. Graphs for the Matrix Elements of the Long-Range Hamiltonian:

The resultant matrix elements of $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 1) (II23) may be replaced by graphs similar to (II36a,b). It is very significant to see that $\mathcal{H}^{k}_{\substack{NN\\(40)}}$ types of terms appear, for this agrees rather nicely with the similarity between the matrix elements in the two schemes. These terms are due to the Kronecker deltas that arise from the Fermion annihilator -creator commutation rule (I20'c).

$$\langle [(\tilde{\hat{j}}_{a_N} \tilde{\hat{j}}_{b_N}) J_0 \hat{j}_{c_P}] JM | \mathcal{H}^{k}_{\substack{NN\\(22)}} (\text{Class 1}) | [(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J_0' \tilde{j}'_{c_P}] JM \rangle$$

$\longrightarrow$ sums over terms like



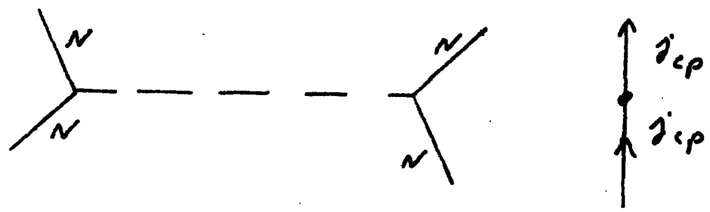
(II 38)

In the above and following graphs the actual neutron quantum numbers are omitted (just an N is indicated) for simplicity.

The matrix elements of $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 2) (II24) may be replaced by graphs like (II36c), i.e.,

$$\langle [(\tilde{j}_{a_N}\,\tilde{j}_{b_N})J_0\,\tilde{j}_{cp}]JM \,|\, \mathcal{H}^{k}_{\substack{NN\\(22)}} \,(\text{class } 2)\,|\,[(\tilde{j}'_{a_N}\,\tilde{j}'_{b_N})J_0'\,\tilde{j}'_{cp}]JM\rangle$$

$\longrightarrow$ sum over terms like



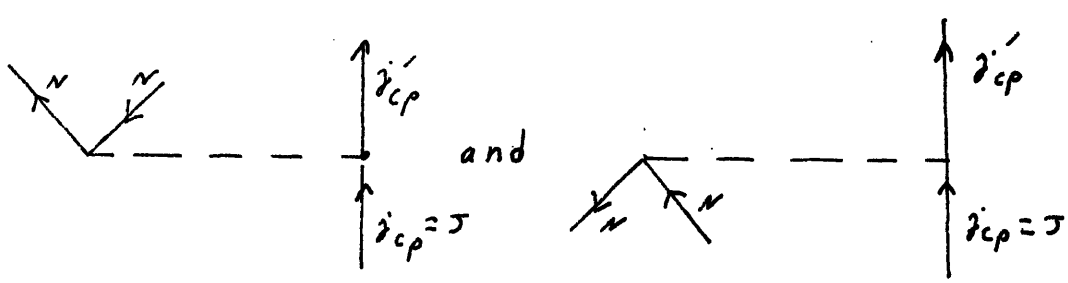
(II 39)

To avoid repetition the neutron arrows have been omitted. In both (II38) and (II39) the odd proton is seen to remain in the same state.

The matrix elements of $H^{k}_{\substack{N,p\\(31)}}$ (II 12) in the quasi-Boson plus quasi-proton scheme may be written as a sum over terms like (II37b), i.e.,

$$\langle (\tilde{\tilde{k}}_N \,;\, \tilde{j}'_{cp})JM \,|\, H^{k}_{\substack{N,p\\(31)}} \,|\, (\tilde{\tilde{0}}_N \,;\, \tilde{j}_{cp})\, j_{cp}\, m_{cp}\rangle$$

$\longrightarrow$ sum over terms like

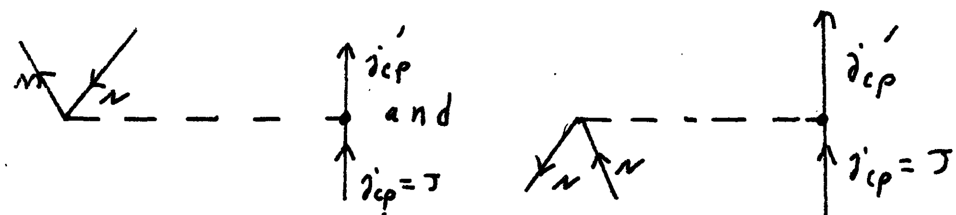


(II 40)

In the two quasi-neutrons plus quasi-proton method the matrix elements of $H^{k}_{N,p\,(31)}$ (II22) do not involve a sum, but may also be written for even k like (II37b). Hence

$$\left\langle \left[(\tilde{j}_{a_N'}\ \tilde{j}_{b_N'}) J_0'\ \tilde{j}_{cp}' \right] JM \right| H^{k}_{N,p\,(31)} \left| \tilde{0}_N ;\ \tilde{j}_{cp}\ \tilde{m}_{cp} \right\rangle$$

$\longrightarrow$

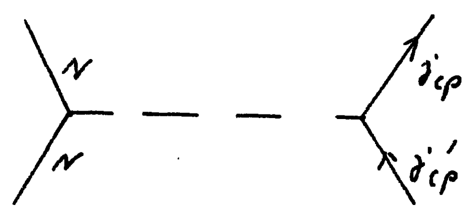


(II41)

The matrix elements of $H^{k}_{N,p\,(22)}$ (II13) in the quasi-Boson plus quasi-proton scheme give a sum over terms like (II37a). The neutron arrows are omitted for ease. Hence

$$\left\langle (\tilde{\tilde{k}}_N ;\ \tilde{j}_{cp}) JM \right| H^{k}_{N,p\,(22)} \left| (\tilde{\tilde{k}}_N ;\ \tilde{j}_{cp}') JM \right\rangle$$

$\longrightarrow$ sum over terms like

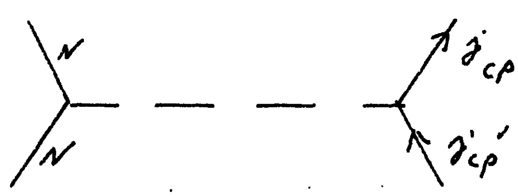


(II 42)

Similarly in the two quasi-neutrons plus quasi-proton scheme one gets for the matrix elements of $H^{k}_{N,p\,(22)}$ (II21) terms like (II37a), only without a sum, i.e.,

$$\left\langle \left[(\tilde{j}_{a_N}\ \tilde{j}_{b_N}) J_0\ \tilde{j}_{cp} \right] JM \right| H^{k}_{N,p\,(22)} \left| \left[(\tilde{j}_{a_N'}\ \tilde{j}_{b_N'}) J_0'\ \tilde{j}_{cp'} \right] JM \right\rangle$$

$\longrightarrow$ Terms like



(II 43)

One notes that $H^{k}_{N,\rho\,(31)}$ is of a direct neutron interaction type; whereas, $H^{k}_{N,\rho\,(22)}$ is of a neutron scattering type.

CHAPTER III

The diagonalizations described in the preceding Chapter yield the wave functions that describe the nuclear levels of our model. These wave functions will be used in this Chapter to calculate: (1) the electric transitions and (2) the inelastic alpha particle scattering cross sections. Only by such calculations will one be able to determine which states are "collective." Of particular importance will be the probing of the structure of such levels. Indeed, the techniques used in Chapter II do not guarantee a priori that coherent vibrations will be present.

We shall see that only certain states have prominent electric transition probabilities. These same levels will also have large inelastic alpha particle scattering cross sections. Before discussing these results, the relationships necessary for the calculations will be presented. In this way the role of the various components of the wave functions we have generated will become evident. Coherent states will be those whose components contribute additively to the resultant transition rates and scattering cross sections. These expressions will also show that essentially the same terms that speed up the former also increase the latter, i.e., there is a correspondence between enhanced electric transitions and enhanced scattering cross sections.

Section A) will deal with electric multipole transitions. This will be done for both the quasi-Boson plus quasi-proton and two quasi-neutrons plus quasi-proton schemes.

Section B) will treat the inelastic scattering of alpha particles in the Born Approximation. Of interest here will be the relative sizes of the different cross sections for inelastic alpha particle scattering from the various levels.

For comparison Section C) will treat the inelastic scattering of alpha particles in the Distorted Wave Born Approximation. The shape of differential cross section versus angle curves for each level will be dependent on the detailed nuclear structure. Hence, these shapes, in addition to magnitudes, may be directly compared to experiment. This and the preceding section will deal with scattering in both the quasi-Boson plus quasi-proton and two quasi-neutrons plus quasi-proton schemes.

Section A. Electric Multipole Transitions.

The reduced transition probability for an electric multipole of order k , $B(Ek)$, is given by [32]

$$\underset{J_i \to J_f}{B(Ek)} = \frac{1}{2J_i+1} \left| \langle J_f \| \frac{1}{e} \sum_{n,\eta} e_\eta \, r_{n_\eta}^{k} \, Y_k(\theta_{n_\eta}, \phi_{n_\eta}) \| J_i \rangle \right|^2 \qquad \text{(III 1)}$$

The initial state has a total angular momentum, J_i , and the final state a total angular momentum, J_f . Transitions will be considered as going from excited states to the ground state. This means that J_i refers to an excited state, and J_f pertains to the ground state. This distinction is very important because in the literature $B(E\lambda)$ often refers to an excitation from the ground state. These will be referred to as $B(E\lambda)_{J_f \to J_i}$, and in coulomb excitation experiments this is the quantity that is measured. With J_f still referring to the ground state spin and J_i to the excited state spin equation (III 1) and $B(E\lambda)_{J_f \to J_i}$ are related by

$$B(E\lambda)_{J_i \to J_f}\,(2J_i+1) = B(E\lambda)_{J_f \to J_i}\,(2J_f+1) \qquad \text{(III2)}$$

In equation (III 1) the symbol, m , is the particle index, while η stands for n or p i.e., neutron or proton. The symbol, Y_λ , is the spherical harmonic, while e_η is the charge of the η'th nucleon, and e is just the unit of positive charge. The notation, r^λ , implies the λ'th power of the radial coordinate, r .

When dealing with the wave functions of the states involved, a different form for $B(E\lambda)_{J_i \to J_f}$ (III 1) simplifies calculations. The reduced matrix component of the electric transition operator of multipole order λ may be written as

$$\langle J_f \| \frac{1}{e} \sum_{m,\eta} e_\eta r_{m_\eta}^{k} Y_k (\theta_{m_\eta} \phi_{m_\eta}) \| J_i \rangle$$

$$= \sqrt{2J_f+1} \; (-1)^{k+J_i-J_f} \left(\frac{1}{\langle k\, q\, J_i M_i | J_f \; q+M_i \rangle} \right)$$

$$\times \langle J_f \; M_i+q | \frac{1}{e} \sum_{m,\eta} e_\eta r_{m_\eta}^{2} Y_{kq} (\theta_{m_\eta} \phi_{m_\eta}) | J_i M_i \rangle$$

(III3)

One may replace (III 1) by

$$\underset{J_i \to J_f}{B(Ek)} = \frac{(2J_f+1) \left| \langle J_f M_i+q | \frac{1}{e} \sum_{m,\eta} e_\eta r_{m_\eta}^{k} Y_{kq} (\theta_{m_\eta} \phi_{m_\eta}) | J_i M_i \rangle \right|^2}{(2J_i+1) \left| \langle k\, q \; J_i M_i | J_f \; q+M_i \rangle \right|^2}$$

(III4)

The units of both (III 1) and (III3) are $(\text{length})^{2k}$. Expanding over the index, η , gives

$$\frac{1}{e} \sum_{\eta,m} e_\eta r_{m_\eta}^{k} Y_{kq} (\theta_{m_\eta} \phi_{m_\eta}) =$$

$$\sum_{m} r_{m_p}^{k} Y_{kq} (\theta_{m_p}, \phi_{m_p}) + \frac{e_N}{e} \sum_{m} r_{m_N}^{k} Y_{kq} (\theta_{m_N}, \phi_{m_N})$$

(III5)

The symbol, e_N , stands for the effective neutron change. The problem of calculating the charge involved in a transition involves a detailed description of the polarizing effect of N on the core. By using an effective neutron charge, this calculation is assumed to have already been carried out.

The proton term, $\sum_{m} r_{m_p}^{k} Y_{kq} (\theta_{m_p}, \phi_{m_p})$, of (III5) may be written in the proton occupation number representation as

$$\sum_{\substack{j_{a_p} m_{a_p} \\ j_{b_p} m_{b_p}}} \langle j_{a_p}\ m_{a_p} | r_p^k\ Y_{kq}(\theta_p \phi_p) | j_{b_p} m_{b_p} \rangle\ b_{m_a}^{+j_a}(p)\ b_{m_b}^{j_b}(p)$$

(III6)

One now transforms to the quasi-proton representation for one proton outside of a major closed shell (u=1,v-0), and introduces $q^k(j_{a_p} j_{b_p})$ (I29) so that (III6) becomes

$$\sum_{j_{a_p} j_{b_p}} q^k(j_{a_p} j_{b_p}) \sum_{m_{a_p} m_{b_p}} \langle j_{a_p}\ m_{a_p}\ j_{b_p}\ m_{b_p} | k q \rangle\ \beta_{m_a}^{+j_a}(p)\ \gamma_{m_b}^{j_b}(p)$$

$$= \sum_{j_{a_p} j_{b_p}} q^k(j_{a_p} j_{b_p}) \left[\beta^{+j_a(p)}\ \gamma^{j_b(p)} \right]_q^k$$

(III7)

In a similar way the neutron term, $\frac{e_N}{e} \sum_m r_{m_N}^k Y_{kq}(\theta_{m_N} \phi_{m_N})$, of (III5) may be written in the neutron occupation number representation as

$$\frac{e_N}{e} \sum_{\substack{j_{a_N} m_{a_N} \\ j_{b_N} m_{b_N}}} \langle j_{a_N}\ m_{a_N} | r_N^k\ Y_{kq}(\theta_N \phi_N) | j_{b_N}\ m_{b_N} \rangle\ b_{m_a}^{+j_a}(N)\ b_{m_b}^{j_b}(N)$$

(III8)

and in the quasi-neutron occupation number representation as

$$\frac{e_N}{e} \sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N}) \sum_{\substack{m_{a_N} \\ m_{b_N}}} \langle j_{a_N}\ m_{a_N}\ j_{b_N}\ m_{b_N} | k q \rangle \left[(-1)^{l_{b_N}}\ u_{a_N} v_{b_N} \right.$$

$$\times \left((-1)^{k} \gamma^{j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) - \beta^{+j_a}_{m_a}(N)\, \beta^{+j_b}_{m_b}(N) \right)$$

$$+ \cos(\psi_{a_N} + \psi_{b_N})\, \beta^{+j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) \Big] \qquad \text{(III9)}$$

The form (III9) is easily obtained by applying the quasi-particle transformation (I6) and the definition of $q^{k}(j_{a_N} j_{b_N})$ (I29) to (III8).

1. Quasi-Boson Plus Quasi-Proton:

Equation (III9) may be written in a form more appropriate to quasi-Boson calculations as

$$\frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^{k}(j_{a_N} j_{b_N}) \Big\{ -(-1)^{\ell_{b_N}} u_{a_N} v_{b_N} \left[(-1)^{k} C^{k}_{q}(j_{a_N} j_{b_N}) + B^{+k}_{q}(j_{a_N} j_{b_N}) \right] \sqrt{1+\delta_{j_{a_N} j_{b_N}}} + \cos(\psi_{a_N} + \psi_{b_N})\, \beta^{+j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) \Big\}$$

(III 10)

where $C^{k}_{q}(j_{a_N} j_{b_N})$ and $B^{+k}_{q}(j_{a_N} j_{b_N})$ are defined in (I38). The state, $|J_i M_i\rangle$, is written as $|J_i M_i\rangle_{0,k}$ in the quasi-Boson plus quasi-proton scheme and is given by

$$|J_i M_i\rangle_{0,k} = a^{J_i}_{0 J_i}\, \beta^{+J_i}_{M_i}(P)\, |\hat{\tilde{0}}_N ; \tilde{0}_P\rangle + \sum_{j_{cP}} a^{J_i}_{k j_{cP}} \left[\Gamma^{k}(N)\, \beta^{+j_c}(P) \right]^{J_i}_{M_i} \times |\hat{\tilde{0}}_N ; \tilde{0}_P\rangle \qquad \text{(I57)}$$

The coefficients, $a^{J_i}_{0 J_i}$ and $a^{J_i}_{k j_{cP}}$, are determined from the pertinent diagonalizations of Chapter II.

From (I57) and (I4) one has

$$\langle_{0,k} J_f \; M_i+q| = (-1)^{-J_f-M_i-q} \, a^{J_f}_{0J_f} \langle \hat{\tilde{0}}_N ; \tilde{0}_P | \gamma^{J_f}_{-M_i-q}(\rho)$$

$$+ \sum_{j'_{cp}} a^{J_f}_{k j'_{cp}} \langle \hat{\tilde{0}}_N ; \hat{0}_P | \left[\Gamma^{k}(N)\, \beta^{+j'_c}(\rho)\right]^{+J_f}_{M_i+q}$$

(III 11)

with $a^{J_f}_{0J_f}$ and $a^{J_f}_{kj'_{cp}}$, determined in the same way as $a^{J_i}_{0J_i}$ and $a^{J_i}_{kj_{cp}}$ of (I57).

The matrix element, $\langle J_f \; M_i+q | \frac{1}{e} \sum_{m,\eta} e_\eta r^k_{m_\eta} Y_{kq}(\theta_{m_\eta}\phi_{m_\eta}) | J_i M_i\rangle$, from (III4) becomes via (III5) through (III 10), (I57), and (III 11)

$$\langle_{0,k} J_f \; M_i+q | \sum_{j_{ap} j_{bp}} q^k(j_{ap} j_{bp}) \left[\beta^{+j_a}_{(\rho)} \gamma^{j_b}_{(\rho)}\right]^k_q$$

$$+ \frac{e_N}{e} \sum_{\substack{j_{a_N}\\ j_{b_N}}} q^k(j_{a_N} j_{b_N}) \Big\{ -(-1)^{l_{b_N}} u_{a_N} v_{b_N} \left[(-1)^k C^k_q(j_{a_N} j_{b_N})\right.$$

$$\left. + B^{+k}_q(j_{a_N} j_{b_N})\right] \sqrt{1+\delta_{j_{a_N} j_{b_N}}} + \cos(\tau_{a_N}+\tau_{b_N})\, \beta^{+j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) \Big\}$$

$$\times | J_i M_i\rangle_{0,k}$$

(III 12)

Equation (III 12) is evaluated in Appendix D1) and when the result is inserted into (III4) we obtain

$$\underset{J_i \to J_f}{B(Ek)} = |A_P + B_P + C_N + D_N|^2 \times \left(\frac{2J_f+1}{2J_i+1}\right)$$

(III 13)

where

$$A_p \equiv (-1)^k a^{J_i}_{0J_i} \, a^{J_f}_{0J_f} \, q^k(J_f \, J_i)(-1)^{J_i+J_f}\sqrt{\frac{2k+1}{2J_f+1}}$$

(III 13a)

$$B_p \equiv (-1)^k \sqrt{(2J_i+1)(2k+1)} \sum_{j'_{cp} j_{cp}} a^{J_f}_{k j'_{cp}} \, a^{J_i}_{k j_{cp}}$$

$$\times \; q^k(j'_{cp} \, j_{cp}) \, (-1)^{j_{cp}-J_i} \begin{Bmatrix} k & J_i & J_f \\ k & j'_{cp} & j_{cp} \end{Bmatrix}$$

(III 13b)

$$C_\nu \equiv \frac{l_\nu}{l}(-1)^k \left[-a^{J_f}_{k J_i} \, a^{J_i}_{0J_i} + (-1)^{J_i+J_f}\left(\frac{2J_i+1}{2J_f+1}\right)^{1/2} a^{J_f}_{0J_f} \, a^{J_i}_{kJ_f} \right]$$

$$\times \sum_{j_{a\nu} \geq j_{b\nu}} q^k(j_{a\nu} \, j_{b\nu}) \frac{1}{\sqrt{1+\delta_{j_{a\nu} j_{b\nu}}}} (-1)^{l_{a\nu}} \sin(\tau_{a\nu}+\tau_{b\nu}) \left[r(j_{a\nu} \, j_{b\nu}) - (-1)^k r(j_{a\nu} \, j_{b\nu}) \right]$$

(III 13c)

and with $r(j_{a\nu} \, j_{c\nu})$ and $r(j_{b\nu} \, j_{c\nu})$ given by (I59)

$$D_\nu \equiv \frac{l_\nu}{l}(-1)^k (2k+1)^{3/2}\sqrt{(2J_i+1)} \sum_{j_{cp}} a^{J_f}_{k j_{cp}} \, a^{J_i}_{k j_{cp}} \begin{Bmatrix} k & k & k \\ J_i & J_f & j_{cp} \end{Bmatrix}$$

$$(-1)^{j_{cp}-J_f} \sum_{j_{a\nu} j_{b\nu} j_{c\nu}} q^k(j_{a\nu} \, j_{b\nu}) \sqrt{1+\delta_{j_{a\nu} j_{c\nu}}} \sqrt{1+\delta_{j_{b\nu} j_{c\nu}}}$$

$$r(j_{a\nu} \, j_{c\nu}) \, r(j_{b\nu} \, j_{c\nu}) \cos(\tau_{a\nu}+\tau_{b\nu}) \begin{Bmatrix} k & k & k \\ j_{a\nu} & j_{b\nu} & j_{c\nu} \end{Bmatrix} (-1)^{j_{a\nu}+j_{c\nu}}$$

(III 13d)

The term, A_p , (III 13a) is the contribution due to the single quasi-proton parts of $|J_i M_i\rangle_{0,k}$ (I57) and ${}_{0,k}\langle J_f M_i+q|$ (III 11). If the transition were indeed pure single quasi-proton then $a_{0J_i}^{J_i} = a_{0J_f}^{J_f} = 1$ with all other coefficients identically zero. Then $B(Ek)_{J_i\to J_f}$ would be given from (III 13) by

$$B(Ek)_{\substack{s.p.\\ J_i\to J_f}} = \frac{2k+1}{2J_i+1}\left[q^k(J_f J_i)\right]^2 \tag{III 14}$$

This is readily verified from De Shalit and Talmi[32] with the definition of q^k (I29). One notes that the origin of A_p is the proton multipole term (III7).

The term, B_p , (III 13b) is the contribution of the one quasi-Boson plus one quasi-proton parts of both the excited and ground states, and is expected to be quite small. The proton multipole term (III7) is again the term describing the transition.

As given by (III 13c), C_N is the term that one expects to be the prime contributor to the resultant $B(Ek)_{J_i\to J_f}$. This is readily seen from the fact that the $B(Ek)$ of an even nucleus is (Appendix D1c)

$$B(Ek)_{k\to 0} = \left\{(-1)^k \frac{e_N}{e} \sum_{j_{a_N}\geq j_{b_N}} q^k(j_{a_N} j_{b_N}) \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} (-1)^{l_{a_N}} \sin(\tau_{a_N}+\tau_{b_N}) \times \left[(-1)^k r(j_{a_N} j_{b_N}) - s(j_{a_N} j_{b_N})\right]\right\}^2 \tag{III 15}$$

Thus, $B_{k\to 0}(Ek)$ occurs naturally in (III 13c) with k being the angular momentum of the one quasi-Boson state of the adjacent even-even nucleus. The interpretation of the coefficient, $\left[-a^{J_f}_{kJ_i}\, a^{J_i}_{0J_i} + (-1)^{J_i+J_f}\left(\frac{2J_i+1}{2J_f+1}\right)^{1/2} a^{J_f}_{0J_f}\, a^{J_i}_{kJ_f}\right]$, in (III 13c) is clear. The second term is what one usually calls the "core" to ground state transition. A single proton of spin, J_f, coupled to a "core" of spin, k, has a transition to the single proton (spin J_f) part of the ground state. In the most naive case $a^{J_f}_{0J_f} = a^{J_i}_{kJ_f} = 1$ and $a^{J_f}_{kJ_i} = a^{J_i}_{0J_i} = 0$. The first term, $-a^{J_f}_{kJ_i}\, a^{J_i}_{0J_i}$, relates to the single quasi-proton (J_i) part of the excited state and the one quasi-Boson plus quasi-proton (J_i) part of the ground state. One would expect this contribution to be rather small.

The origin of C_N is, of course, the neutron multipole term (III 10). But comparison to (II 12) shows that the sum in the expression for $B_{k\to 0}(Ek)$ (III 15) is exactly the one that occurs in the matrix element of $H^{k}_{N,P}$ (II 12). This results from the nature of the neutron (3 1) electric transition operator, $\frac{e_N}{e}\sum_m r^k_{m_N} Y_{kq}(\theta_{m_N}\phi_{m_N})$, in (III 5). This operator sans $\frac{e_N}{e}$ also occurs with respect to the long-range Hamiltonian in equation (I26). Incidentally, if the "core" part of an excited state interacts strongly with the single particle part of the ground state and this interaction is via the k 'th multipole of the long-range Hamiltonian; then, a relatively large $B_{J_i\to J_f}(Ek)$ is expected.

Lastly, $\mathcal{D}_N$ (III 13d) is due to the single quasi-neutron term, $\beta^{+j_a}_{m_a}(N)\,\gamma^{j_b}_{m_b}(N)$, of equation (III 10). Hence, a rigorous quasi-Boson calculation like that performed to get C_N is not possible here. The approximation (Clb.3) of using $|\hat{\tilde{0}}_N\rangle$ in place of $|\hat{\tilde{\tilde{0}}}_N\rangle$ is made. The fact that this scattering type of a term is neglected in the Boson Approximation for an even-even nucleus means that its contribution must be small. In essence, $\mathcal{D}_N$ represents the transition from the one phonon part of the excited state to the one phonon part of the ground state. It differs from B_P (III 13b) in that the former is due to the neutron multipole term (III 10) while the latter is due to the proton multipole term (III 7).

The components of excited and ground states that contribute to the k'thelectric multipole transition are restricted by parity conservation. Thus

In (III 13a) for k even $\quad \pi_{J_i} = \pi_{J_f}$

for k odd $\quad \pi_{J_i} \neq \pi_{J_f} \qquad$ (III 16)

In (III 13b) for k even $\quad \pi_{j'_{cp}} = \pi_{j_{cp}}$

for k odd $\quad \pi_{j'_{cp}} \neq \pi_{j_{cp}} \qquad$ (III 17)

Also (III 13d) = 0 for k odd (III 18)

Equations (III16) and (III 17) result from the transition being due to the odd proton. Equation (III 13c) has no restriction since the transition is due to the quasi-Boson itself. In other words, there is a transition from a one quasi-Boson to a zero quasi-Boson level with the quasi-proton remaining in its same state. Equation (III 18) arises from the requirement in (III 13d) that the odd quasi-proton in the initial and final states remains the same.

2. Two Quasi-Neutrons Plus Quasi-Proton:

The proton multipole term is given by (III7) and the best form for the neutron multipole contribution is (III9). The state, $|J_i M_i\rangle$, is

$$|J_i M_i\rangle = \Big[a^{J_i}_{0 J_i}\, \beta^{+\,J_i}_{M_i}(P) + \sum_{\substack{j_{1_N} \geq j_{2_N} \\ J_0,\, j_{c_P}}} c^{J_i}_{(j_{1_N} j_{2_N}) J_0\, j_{c_P}} \frac{1}{\sqrt{1+\delta_{j_{1_N} j_{2_N}}}} \times \Big(\big[\beta^{+j_1}_{(N)} \beta^{+j_2}_{(N)} \big]^{J_0} \beta^{+j_c}(P) \Big)^{J_i}_{M_i} \Big] |\tilde{0}_N ; \tilde{0}_P\rangle$$

(I67)

and, hence

$$\langle J_f\, M_i+q| = a^{J_f}_{0 J_f} (-1)^{-J_f - M_i - q} \langle \tilde{0}_N ; \tilde{0}_P | \gamma^{J_f}_{-M_i - q}(P) + \sum_{\substack{j'_{1_N} \geq j'_{2_N} \\ J'_0,\, j'_{c_P}}} c^{J_f}_{(j'_{1_N} j'_{2_N}) J'_0\, j'_{c_P}} \frac{1}{\sqrt{1+\delta_{j'_{1_N} j'_{2_N}}}} \langle \tilde{0}_N ; \tilde{0}_P | \Big(\big[\beta^{+j'_1}_{(N)} \beta^{+j'_2}_{(N)} \big]^{J'_0} \beta^{+j'_c}_{(P)} \Big)^{J_f\,\dagger}_{M_i+q}$$

(III 19)

The coefficients, $a^{J_i}_{0 J_i}$, $c^{J_i}_{(j_{1N} j_{2N}) J_0 j_{cp}}$, $a^{J_f}_{0 J_f}$, and $c^{J_f}_{(j'_{1N} j'_{2N}) J'_0 j'_{cp}}$ are determined from the appropriate diagonalizations of Chapter II.

Using (III5) through (III9), (I67), and (III 19), one may write

$$\langle J_f\ M_i + q \,|\, \frac{1}{\ell} \sum_{n,\eta} \ell_\eta\, r^k_{n_\eta} Y_{kq}(\theta_{n_\eta} \phi_{n_\eta}) \,|\, J_i M_i \rangle$$

$$= \langle J_f\ M_i + q \,|\, \sum_{j_{ap} j_{bp}} g^k(j_{ap}\, j_{bp}) \left[\beta^{+\, j_{a}}(p)\, \gamma^{j_{b}}(p) \right]^k_q$$

$$+ \frac{\ell_N}{\ell} \sum_{\substack{j_{a_N} \\ j_{b_N}}} g^k(j_{a_N}\, j_{b_N}) \sum_{\substack{m_{a_N} \\ m_{b_N}}} \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | kq \rangle$$

$$\times \Big[(-1)^{\ell_{b_N}} u_{a_N} v_{b_N} \left((-1)^k \gamma^{j_a}_{m_a}(N) \gamma^{j_b}_{m_b}(N) - \beta^{+ j_a}_{m_a}(N)\, \beta^{+ j_b}_{m_b}(N) \right)$$

$$+ \cos(\theta_{a_N} + \theta_{b_N})\, \beta^{+ j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) \Big] \,|\, J_i M_i \rangle$$

(III20)

Equation (III20) is evaluated in Appendix D2) and when placed in (III4) yields

$$B(Ek)_{J_i \to J_f} = \left| A_p + B'_p + C'_N + D'_N \right|^2 \times \left(\frac{2J_f + 1}{2J_i + 1} \right)$$

(III21)

Where A_p is given by (III 13a), and

$$B_p' \equiv \sqrt{(2J_i+1)(2k+1)} \sum_{\substack{j_{1N} \geq j_{2N} \\ J_0\, j_{cp}\, j_{cp}'}} c^{J_i}_{(j_{1N} j_{2N}) J_0 j_{cp}} \; c^{J_f}_{(j_{1N} j_{2N}) J_0 j_{cp}'}$$

$$\times\, q^k(j_{cp}' j_{cp}) (-1)^{j_{cp} - J_i + J_0} \begin{Bmatrix} k & J_i & J_f \\ J_0 & j_{cp}' & j_{cp} \end{Bmatrix}$$

(III22a)

$$C_N' \equiv \frac{l_N}{l} (-1)^k \sum_{j_{a_N} \geq j_{b_N}} \Big[-a_{0 J_i} \, c^{J_f}_{(j_{a_N} j_{b_N}) k J_i} + (-1)^{J_i + J_f}$$

$$\times \left(\frac{2J_i+1}{2J_f+1}\right)^{1/2} a_{0 J_f} \, c^{J_i}_{(j_{a_N} j_{b_N}) k J_f} \Big] \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \, q^k(j_{a_N} j_{b_N})$$

$$\times (-1)^{l_{a_N}} \sin(\tau_{j_{a_N}} + \tau_{j_{b_N}})$$

and (III22b)

$$D_N' \equiv \frac{l_N}{l} (-1)^k \sqrt{(2k+1)(2J_0+1)(2J_0'+1)(2J_i+1)}$$

$$\times \sum_{\substack{j_{a_N} j_{b_N} j_{c_N} \\ J_0\, J_0'\, j_{cp}}} c^{J_f}_{(j_{a_N} j_{c_N}) J_0' j_{cp}} \; c^{J_i}_{(j_{b_N} j_{c_N}) J_0 j_{cp}} (-1)^{j_{cp} - J_f} \begin{Bmatrix} k & J_0' & J_0 \\ j_{cp} & J_i & J_f \end{Bmatrix}$$

$$\times\, q^k(j_{a_N} j_{b_N}) \sqrt{(1+\delta_{j_{a_N} j_{c_N}})(1+\delta_{j_{b_N} j_{c_N}})} \; \cos(\tau_{a_N} + \tau_{b_N})$$

$$\times \begin{Bmatrix} J_0 & J_0' & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix} (-1)^{j_{c_N} + j_{a_N} + J_0 + J_0'}$$

(III22c)

One notes the strong similarity between (III22) and the quasi-Boson plus quasi-proton results (III 13). The single quasi-proton contribution (III 13a) is the same in both cases as it should be. The term, B_p', (III22a) is the contribution of the two quasi-neutrons plus quasi-proton parts of both excited and ground states. The proton multipole term (III7) produces this term just as it does B_p (III 13b), and one expects this contribution to be small.

The neutron multipole term (III9) is the origin of both C_N' (III22b) and D_N' (III22c). The former is analogous to C_N (III 13c) and the latter to D_N (III 13d). The important term, since it relates to the "core", is C_N'. One notes the similarity between the matrix elements of $H^{k}_{N,P\,(31)}$ (II22) and the expression for C_N' (III22b). This is again due to the similarity between the transition operator and the long-range interaction. The part of (II22) concerned with pairs of quasi-neutron states is $q^{k}(j_{a_N}' j_{b_N}')(-1)^{l_{a_N}'} \sin(\theta_{a_N}' + \theta_{b_N}') \frac{1}{\sqrt{1+\delta_{j_a j_b}}}$ and C_N' just sums over the possible (j_{a_N}', j_{b_N}') combinations. The weightings of each couple, (j_{a_N}', j_{b_N}'), are the pertinent coefficients from the excited and groud state wave functions. The long-range Hamiltonian term, $H^{k}_{N,P\,(31)}$, is quite important in removing the energy degeneracy in excited states of different spin in an odd mass nucleus. Thus C_N or C_N' is expected to be prominent in determining the electric transition probabilities of these levels.

The $a_{0\,J_f}^{J_f}\, c^{J_i}_{(j_{a_N} j_{b_N})\,k\,J_f}$ term of C_N' (III22b) describes the transition from a two quasi-neutrons plus quasi-proton part of the excited state to the single quasi-proton part of the ground state. Since one sums over such weighted contributions, the technique is somewhat analogous to the quasi-Boson method. In the quasi-Boson method the very concept of the phonon itself is based on these sums having been already performed.

The $a_{0J_i}^{J_i} \, r_{(j_{a_\nu} j_{b_\nu})\lambda J_i}^{J_f}$ term of C_ν' (III22b) represents the effect of the transition from the single quasi-proton part of the excited state to the ground state. Here the pertinent part of the ground state is the two quasi-neutrons plus quasi-proton component. This contribution like the corresponding term of C_ν (III 13c) is expected to be small.

The term, D_ν' , (III22c) may be calculated exactly within the confines of the two quasi-neutrons plus quasi-proton basis. This may be done since single quasi-neutron operators occur and the vacuum is just $|\tilde{0}_\nu\rangle$, the quasi-neutron vacuum. Like D_ν (III 13d) this term will probably be almost negligible.

Parity conservation imposes certain restrictions as to which components contribute to the λ 'th electric multipole transition. Hence

$$\text{In (III22a)} \quad \text{for } \lambda \text{ even} \quad \pi_{j_{cp}} = \pi_{j_{cp}'}$$
$$\text{for } \lambda \text{ odd} \quad \pi_{j_{cp}} \neq \pi_{j_{cp}'} \qquad \text{(III23)}$$

$$\text{In (III22c)} \quad \text{for } \lambda \text{ even} \quad \pi_{J_0} = \pi_{J_0'}$$
$$\text{for } \lambda \text{ odd} \quad \pi_{J_0} \neq \pi_{J_0'} \qquad \text{(III24)}$$

Equations (III23) and (III24) result respectively from the quasi-proton and quasi-neutrons making the transition.

There is no restriction on (III22b) since the pair of quasi-neutrons make the transition, and they are coupled to h. In other words the initial state has two quasi-neutrons coupled to h, and the final state is the quasi-neutron vacuum.

Section B. Inelastic Alpha Scattering in the Born Approximation.

The Born Approximation will be used to determine the relative cross sections for the inelastic scattering of alpha particles from different nuclear levels. In this way "collective" levels will be evident from the prominence of the differential cross sections for their excitation.

Alpha particles are used for simplicity since they are of spin zero. The nucleon-projectile interaction will be assumed to be spin independent and of zero range. The differential cross section for inelastic scattering may be written as

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4} \frac{k'}{k} \frac{1}{2J_f+1} \sum_{M_i M_f} |m|^2 \tag{III25}$$

The symbols, k and k', stand for the wave numbers of the incident and scattered alpha particles respectively, while μ is the nucleon reduced mass and J_f is the ground state spin of the target nucleus. The z component of J_f is M_f and M_i is the z component of J_i, the excited state spin. The reason for the use of the notation, J_i and J_f, is to tie in with the results of the previous section. The target nucleus is elevated from its ground state, J_f, to an excited state, J_i. Hence, the process is the opposite of an electric multipole transition which de-excites the nucleus from state, J_i, leaving it in the ground state, J_f.

$\mathcal{M}$ is the transition matrix element given in the Boson approximation by

$$\mathcal{M} = \int e^{-i\vec{k}'\cdot\vec{r}_\alpha}\, \Phi^*_{J_i M_i} \sum_{n,\eta} V(|\vec{r}_{n_\eta} - \vec{r}_\alpha|)\, e^{i\vec{k}\cdot\vec{r}_\alpha} \times \Phi_{J_f M_f}\, d\vec{r}_\alpha\, d\tau \qquad \text{(III26)}$$

The vector, $\vec{r}_\alpha$, is the alpha particle coordinate, while $\Phi_{J_f M_f}$ is the wave function for the target nucleus in its ground state, and $\Phi_{J_i M_i}$ is the target nucleus excited state wave function. The interaction is $\sum_{n,\eta} V_0(|\vec{r}_{n_\eta} - \vec{r}_\alpha|)$, where $\vec{r}_{n_\eta}$ is the nucleon coordinate. The symbol, η, again is N or P (neutron or proton), and n is the particle index. The differential volume element for the nucleus is $d\tau$.

If one assumes a zero range interaction, i.e.,

$$\sum_{n,\eta} V_0(|\vec{r}_{n_\eta} - \vec{r}_\alpha|) = V_0 \sum_{n,\eta} \delta(\vec{r}_{n_\eta} - \vec{r}_\alpha) \qquad \text{(III27)}$$

one gets for (III26)

$$\mathcal{M} = V_0 \int \Phi^*_{J_i M_i} \sum_{n,\eta} e^{i\vec{p}\cdot\vec{r}_{n_\eta}}\, \Phi_{J_f M_f}\, d\tau \qquad \text{(III28)}$$

with

$$\vec{p} = \vec{k} - \vec{k}'$$

(III29)

Now the expansion of the plane wave $e^{i\vec{p}\cdot\vec{r}_{n\eta}}$ is given by

$$e^{i\vec{p}\cdot\vec{r}_{n\eta}} = 4\pi \sum_{\ell=0}^{\infty} (i)^{\ell} j_\ell(p r_{n\eta}) \sum_{m=-\ell}^{\ell} Y_m^{*\ell}(\theta_p \phi_p) Y_m^{\ell}(\theta_{n\eta}, \phi_{n\eta})$$

(III30)

The integral (III28) is independent of θ_p, ϕ_p so that one can evaluate it at any value of θ_p. If $\theta_p = 0$, then

$$Y_m^{*\ell}(\theta_p = 0, \phi_p) = \delta_{m0} \sqrt{\frac{2\ell+1}{4\pi}}$$

(III31)

and $\mathcal{M}$ becomes

$$\mathcal{M} = V_0 \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\, i^{\ell} \int \Phi^*_{J_i M_i} \sum_{n,\eta} j_\ell(p r_{n\eta}) Y_0^{\ell}(\theta_{n\eta} \phi_{n\eta})\, \Phi_{J_f M_f}\, d\vec{\tau}$$

$$= V_0 \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\, i^{\ell} \langle J_i M_i | \sum_{n,\eta} j_\ell(p r_{n\eta}) Y_0^{\ell}(\theta_{n\eta} \phi_{n\eta}) | J_f M_f \rangle$$

(III32)

One should not confuse the imaginary i and the excited state subscript, i, in the label, J_i, M_i. The orthogonality of the spherical harmonics gives from (III32)

$$|\mathcal{M}|^2 = 4\pi V_0^2 \sum_{\ell=0}^{\infty} (2\ell+1) \left| \langle J_i M_i | \sum_{n,\eta} j_\ell(p r_{n\eta}) Y_0^{\ell}(\theta_{n\eta} \phi_{n\eta}) | J_f M_f \rangle \right|^2$$

(III33)

Introducing the reduced matrix element via Edmonds[29]

$$\langle J_i M_i | \sum_{m,3} j_l(p\,r_{m_3})\, Y_0^l(\theta_{m_3}\phi_{m_3}) | J_f M_f \rangle$$

$$= \frac{(-1)^{J_f - M_f}}{\sqrt{2l+1}} \langle J_i M_i\; J_f - M_f | l\, 0 \rangle \langle J_i \| \sum_{m,3} j_l(p\,r_{m_3})\, Y^l(\theta_{m_3}\phi_{m_3}) \| J_f \rangle$$

(III34)

and thus

$$\sum_{M_i M_f} |\mathcal{M}|^2 = 4\pi V_0^2 \sum_{l=0}^{\infty} \left| \langle J_i \| \sum_{m,3} j_l(p\,r_{m_3})\, Y^l(\theta_{m_3}\phi_{m_3}) \| J_f \rangle \right|^2$$

$$\times \sum_{M_i M_f} \left[(-1)^{J_f - M_f} \right]^2 \left| \langle J_i M_i\; J_f - M_f | l\, 0 \rangle \right|^2$$

$$= 4\pi V_0^2 \sum_{l=0}^{\infty} \left| \langle J_i \| \sum_{m,3} j_l(p\,r_{m_3})\, Y^l(\theta_{m_3}\phi_{m_3}) \| J_f \rangle \right|^2$$

(III35)

Finally from (III25) and (III35) the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{\pi \hbar^4} \frac{k'}{k} \frac{V_0^2}{2J_f+1} \left| \langle J_i \| \sum_{m,3} j_l(p\,r_{m_3})\, Y^l(\theta_{m_3}\phi_{m_3}) \| J_f \rangle \right|^2$$

$$\equiv \sum_l \left(\frac{d\sigma}{d\Omega} \right)_l$$

(III36)

where $\left(\frac{d\sigma}{d\Omega}\right)_l$ is the differential cross section for multipole order l .

The similarity between $\sum_{m,3} j_l(p\,r_{m_3})\, Y^l(\theta_{m_3}\phi_{m_3})$ (III36) and the electric multipole operator $\frac{1}{\ell} \sum_{n,\eta} \ell_\eta r_{n_\eta}^{k} Y_k(\theta_{n_\eta}\phi_{n_\eta})$ (III 1) is obvious. The only difference is that the former has $j_\ell(\rho r_{n_\eta})$ and the latter has $\frac{\ell_\eta}{\ell} r_{n_\eta}^{k} Y_k(\theta_{n_\eta},\phi_{n_\eta})$. The expressions derived for $\underset{J_i \to J_f}{B(Ek)}$ may, therefore, simply be applied. Hence, if we define

$$\underset{J_i \to J_f}{B(\lambda k)} \equiv \frac{1}{2J_i+1} \left| \langle J_f \| \sum_{n,\eta} j_k(\rho r_{n_\eta}) Y_k(\theta_{n_\eta} \phi_{n_\eta}) \| J_i \rangle \right|^2$$

(III37)

then from analogy to (III 1) and (III2)

$$\underset{J_f \to J_i}{B(\lambda k)} = \frac{2J_i+1}{2J_f+1} \underset{J_i \to J_f}{B(\lambda k)}$$

(III38)

The (λk) notation implies excitation of electric multipole order, k , due to the incident alpha. From (III36) one may also write

$$\underset{J_i \to J_f}{B(\lambda k)} = \frac{2J_f+1}{2J_i+1} \frac{\pi \hbar^4}{\mu^2} \frac{K}{K'} \frac{1}{V_0^2} \left(\frac{d\sigma}{d\Omega}\right)_k$$

(III39)

and insertion into (III38) gives

$$\left(\frac{d\sigma}{d\Omega}\right)_k = V_0^2 \frac{\mu^2}{\pi \hbar^4} \frac{K'}{K} \underset{J_f \to J_i}{B(\lambda k)}$$

(III40)

Thus using (III38), (III40), and the analogy between

$B(\lambda k)_{J_i \to J_f}$ and $B(Ek)_{J_i \to J_f}$ the differential cross sections may be simply inferred.

Define $q^{k}_{(\lambda)}(j_{1_3} j_{2_3})$ by analogy to $q^{k}(j_{1_3} j_{2_3})$ (I29) as

$$q^{k}_{(\lambda)}(j_{1_3} j_{2_3}) \equiv -\frac{1}{\sqrt{2k+1}} \langle j_{1_3} \| U^{k}(3) \| j_{2_3} \rangle \tag{III41}$$

where

$$U^{k}_{m_k}(3) \equiv j_k(\kappa r_3)\, Y_{km}(\theta_3, \phi_3) \tag{III42}$$

1. Quasi-Boson Plus Quasi-Proton:

From (III38), (III40), and (III13) one has

$$\left(\frac{d\sigma}{d\Omega}\right)_k \equiv \left| A_p^{(\lambda)} + B_p^{(\lambda)} + C_N^{(\lambda)} + D_N^{(\lambda)} \right|^2 \times V_0^2 \frac{\mu^2}{\pi \hbar^4} \frac{K'}{K} \tag{III43}$$

In (III43) $A_p^{(\lambda)}$, $B_p^{(\lambda)}$, $C_N^{(\lambda)}$, and $D_N^{(\lambda)}$ are given by A_p (III 13a), B_p (III 13b), C_N (III 13c), and D_N (III 13d) respectively provided $q^{k}_{(\lambda)}$ (III41) is used in place of q^{k} (I29).

2. Two Quasi-Neutrons Plus Quasi-Proton

Now use (III38), (III40), and (III 21) to give

$$\left(\frac{d\sigma}{d\Omega}\right)_k = \left| A_p^{(\lambda)} + B_p^{\prime(\lambda)} + C_N^{\prime(\lambda)} + D_N^{\prime(\lambda)} \right|^2 \times \frac{V_0^2 \mu^2}{\pi \hbar^4} \frac{K'}{K} \tag{III44}$$

Here $A_p^{(\lambda)}$ is the same as in the Quasi-Boson plus quasi-proton case. The terms, $B_p^{\prime(\lambda)}$, $C_N^{\prime(\lambda)}$, and $D_N^{\prime(\lambda)}$ are given by B_p' (III22a), C_N' (III22b), and D_N' (III22c) respectively provided $q^k_{(\lambda)}$ (III41) is used in place of q^k (I29).

Section C. Inelastic Alpha Scattering in the Distorted Wave Born Approximation (DWBA).

The DWBA has been adequately described in the literature.[33-35] We shall summarize this method briefly, however, in order to establish the proper context for the present work. In essence, the elastic scattering is the dominant effect and other direct processes are treated as perturbations. For the present purposes the direct process of interest is the inelastic scattering of alpha particles. The relative motion of the target and alpha particle is described in terms of distorted waves. These waves include the elastic scattering as determined from a (supposedly known) optical model potential. The transition matrix responsible for the inelastic scattering is then treated to first order only and is responsible for the inelastic scattering. The shapes of the differential cross section versus angle curves turn out to be severely dependent on these distortion effects. Of particular interest will be the differences between inelastic scattering from "collective" and "non-collective" states, and the reasons behind these differences.

Consider a nucleus left in an excited state of angular momentum, J_{ex}, M_{ex}, as a result of inelastic alpha scattering. If the nucleus was originally in the ground state of angular momentum, $J_{g.s.}$, $M_{g.s.}$, then the transition amplitude from Bassel, et al,[35] is

$$T_{ex, g.s.} = \int d\vec{r}_\alpha \, \chi_{ex}^{(-)*} (\vec{K}', \vec{r}_\alpha) \langle J_{ex} M_{ex} | V(\vec{r}_\alpha, \vec{\xi}) | J_{g.s.} M_{g.s.} \rangle \times \chi_{g.s.}^{(+)} (\vec{K}, \vec{r}_\alpha) \quad \text{(III45)}$$

Here the subscript, ex , refers to the excited or final state, and the subscript, $g.s.$, pertains to the initial or ground state. The symbol, K , is for the initial wave number of the alpha while K' stands for the final wave number. The vector, $\vec{r}_\alpha$, represents the positi n of the alpha-target reduced mass in the center of mass system , and $\vec{\xi}$ represents internal coordiantes of the target. The wave functions, $\chi_{g.s.}^{(+)}(\vec{K}, \vec{r}_\alpha)$ and $\chi_{ex}^{(-)}(\vec{K}', \vec{r}_\alpha)$, are distorted waves given by

$$\chi_{g.s.}^{(+)} (\vec{K}, \vec{r}_\alpha) = e^{i\vec{K}\cdot\vec{r}_\alpha} + f(\theta_K) \frac{e^{iKr_\alpha}}{r_\alpha} \quad \text{(III46a)}$$

and

$$\chi_{ex.}^{(-)} (\vec{K}', \vec{r}_\alpha) = e^{i\vec{K}'\cdot\vec{r}_\alpha} + f^*(\pi-\theta_{K'}) \frac{e^{-iK'r_\alpha}}{r_\alpha} \quad \text{(III46b)}$$

where θ_K and $\theta_{K'}$ are the respective angles of elastic scattering, and $\chi_{g.s}^{(+)}(\vec{K}, \vec{r}_\alpha)$ and $\chi_{ex.}^{(-)}(\vec{K}', \vec{r}_\alpha)$ are the wave functions which are solutions of the Schroedinger equations

$$\left[\nabla^2 + k^2 - \frac{2M}{\hbar^2} V(r_\alpha) - \frac{2M}{\hbar^2} V_c(r_\alpha)\right] \chi_{g.s.}^{(+)} (\vec{K}, \vec{r}_\alpha) = 0 \quad \text{(III46c)}$$

and

$$\left[\nabla^2 + K'^2 - \frac{2\mathcal{M}}{\hbar^2} U(r_\lambda) - \frac{2\mathcal{M}}{\hbar^2} V_c(r_\lambda)\right] \chi^{(-)}_{\ell\mathcal{M}}(\vec{K}', \vec{r}_\lambda) = 0 \tag{III46d}$$

respectively. The symbol, $\mathcal{M}$, stands for the reduced mass of the alpha-target system, while $U(r_\lambda)$ is the optical potential and $V_c(r_\lambda)$ is the coulomb potential. A Woods-Saxon form is used for $U(r_\lambda)$ in both the real and imaginary parts. Also each part will use the same distance parameters. Hence,

$$U(r_\lambda) = -\frac{1}{e^{x_\lambda}+1}\left[V_R + i\, V_{IM}\right] \tag{III47a}$$

with

$$x_\lambda = \frac{r_\lambda - R}{a} \tag{III47b}$$

and

$$R = r_0\, A^{1/3} \tag{III47c}$$

In (III47), V_R and V_{IM} are the amplitudes of the real and imaginary parts respectively. The parameters, r_0 and a , refer to the smallest nuclear radius and diffuseness parameter respectively, while A is the mass number of the target.

The coulomb potential, $V_c(r_\lambda)$, is taken to be that due to a uniformly charged sphere of radius R_c . Hence,

$$V_c(r_\lambda) = \frac{Z_\lambda Z_A e^2}{r_\lambda}\ , \quad r_\lambda \geq R_c \tag{III48a}$$

$$V_c(r_\lambda) = \frac{Z_\lambda Z_A e^2}{2R_c}\left(3 - \frac{r_\lambda^2}{R_c^2}\right), \quad r_\lambda \leq R_c \tag{III48b}$$

with

$$R_c = r_c \, A^{1/3} \quad ; \quad z_\alpha = 2$$
(III48c)

where r_c is a parameter, and e is the proton charge. The symbol, z_A, stands for the atomic number of the target, and $z_\alpha = 2$ is the atomic number of the alpha particle.

In (III45) the alpha-target interaction, $V(\vec{r}_\alpha, \xi)$, will be taken as

$$V(\vec{r}_\alpha, \vec{\xi}) = \sum_{m,\eta} V(|\overrightarrow{r_\alpha - r_{m_\eta}}|)$$
(III49)

where $\vec{r}_{m_\eta}$ has the same connotation as in the previous two Sections. Each $V(|\overrightarrow{r_\alpha - r_{m_\eta}}|)$ may be expanded in terms of Legendre Polynomials, $P_\ell(\cos\theta_{\alpha m_\eta})$, as

$$V(|\overrightarrow{r_\alpha - r_{m_\eta}}|) = \sum_\ell V_\ell(r_\alpha, r_{m_\eta}) \, P_\ell(\cos\theta_{\alpha m_\eta})$$
(III50)

The angle, $\theta_{\alpha m_\eta}$, is between the alpha-target reduced mass and the m_η'th nucleon. Equation (III50) has been written out for future comparison to an expansion by Swiatecki.[36] A more useful expansion is in terms of spherical harmonics, i.e.,

$$V(|\vec{r}_\alpha - \vec{r}_{m_\eta}|) = \sum_\ell V_\ell(r_\alpha, r_{m_\eta}) \sum_m \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta_{m_\eta} \phi_{m_\eta}) Y^*_{\ell m}(\theta_\alpha \phi_\alpha)$$
(III51)

(Bassel, et al,[35] leave out the $\frac{4\pi}{2\ell+1}$ in their expansion analogous to (III51).) The symbol, $Y_{\ell m}$, is the spherical harmonic, while θ_{m_η}, ϕ_{m_η} are the angles for the m_η'th nucleon, and θ_α, ϕ_α are the polar angles of $\vec{r}_\alpha$.

Define V_ℓ and $A_\ell F_\ell$ via

$$V_\ell \equiv \sum_{\eta, n} V_\ell(r_\lambda, r_{n_\eta}) Y_\ell(\theta_{n_\eta} \phi_{n_\eta})$$

(III52)

and

$$A_\ell F_\ell(r_\lambda) \equiv \frac{4\pi}{2\ell+1} \frac{1}{\sqrt{2J_{ex}+1}} \langle J_{ex} \| V_\ell \| J_{g.s.} \rangle$$

(III53)

Bassel, et al,[35] use a slightly different definition of the reduced matrix element. As a result of this and their previously mentioned omission of $\frac{4\pi}{2\ell+1}$, their equation analogous to (III53) does not contain $\frac{4\pi}{2\ell+1} \frac{1}{\sqrt{2J_{ex}+1}}$. The symbol, F_ℓ, stands for the form factor and A_ℓ is just interpreted as a "strength factor". One notes that F_ℓ is only dependent on r_λ. This comes about because $\| J_{g.s.}\rangle$ and $\langle J_{ex} \|$ are nuclear states and the nucleon radial coordinates, r_{n_η}, are integrated over in calculating (III53). Only $Y_\ell(\theta_\eta \phi_\eta)$ of (III52) really utilizes the reduced matrix element properties in $\langle J_{ex} \| V_\ell \| J_{g.s.}\rangle$ (III53).

Now expand $\chi^{(+)}_{g.s.}(\vec{K}, \vec{r}_\lambda)$ in terms of partial waves as

$$\chi^{(+)}_{g.s.}(\vec{K}, \vec{r}_\lambda) = \frac{4\pi}{K r_\lambda} \sum_L i^L \chi_L(K r_\lambda) \sum_M Y_{LM}(\theta_\lambda, \phi_\lambda) Y^*_{LM}(\theta_K, \phi_K)$$

(III54a)

and

$$\chi^{(-)*}_{ex}(\vec{K}', \vec{r}_\lambda) = \chi^{(+)}_{ex}(-\vec{K}', \vec{r}_\lambda)$$

(III54b)

In (III54a) θ_K, ϕ_K are the polar angles of K, and similarly $\theta_{K'}, \phi_{K'}$ would be the polar angles of K' upon expansion of (III54b). The $\vec{K}$ axis is chosen as the z direction and the y axis is along $\vec{K} \times \vec{K}'$. This means that $\theta_{K'}$ is the angle between $\vec{K}$ and $\vec{K}'$. For ease, let

$$\theta_{K'} \equiv \theta \qquad \text{(III55)}$$

Finally (III45) becomes

$$T_{ex,\,g.s.} = \left[\frac{(4\pi)^{1/2}}{(K')^2} \sum_{\ell} A_\ell \langle J_{g.s.} M_{g.s.}\, \ell\, M_{ex}-M_{g.s.} | J_{ex} M_{ex} \rangle \right.$$
$$\times (2\ell+1)^{1/2}\, \beta^{\ell, M_{ex}-M_{g.s.}}(\theta) \qquad \text{(III56)}$$

where

$$\beta^{\ell, M_{ex}-M_{g.s.}}(\theta) = (-1)^{M_{ex}-M_{g.s.}}\, \beta^{\ell, M_{g.s.}-M_{ex}}(\theta)$$
$$= \sum_{LL'} \Gamma_{L'L}^{\ell, M_{ex}-M_{g.s.}}\, P_{L'}^{M_{ex}-M_{g.s.}}(\theta)\, f_{L'L}^{\ell}\ , \quad M_{ex}-M_{g.s.} \geq 0$$

(III57)

with $P_{L'}^{M_{ex}-M_{g.s.}}(\theta)$ being the associated Legendre polynormial, and $f_{L'L}^{\ell}$ the redial integral defined by

$$f_{L'L}^{\ell} \equiv \frac{K'}{K} \int \chi_{L'}(K' r_2)\, F_\ell(r_2)\, \chi_L(K r_2)\, dr_2$$

(III58)

The $\Gamma_{L'L}^{\ell, M_{ex}-M_{g.s.}}$ are complicated coefficients defined by Bassel, et al.[35)] The differential cross section for inelastic scattering is then

$$\frac{d\sigma}{d\Omega} = \left(\frac{M}{2\pi\hbar^2} \right)^2 \left(\frac{K'}{K} \right) \sum_{ave} |T_{ex,\,g.s.}|^2$$

$$= \frac{2J_{ex}+1}{2J_{g.s}+1} \sum_{\ell} |A_\ell|^2 \sigma_\ell(\theta)$$

(III59)

where M_{ex} has been summed over and $M_{g.s.}$ has been averaged over. In (III59) $T_{ex,g.s.}$ is given by (III56) and $\sigma_\ell(\theta)$ by

$$\sigma_\ell(\theta) = \frac{\mu^2}{(k')^3 k \pi \hbar^4} \sum_{M_{ex}-M_{g.s.}} \left| \beta^{\ell, M_{ex}-M_{g.s.}}(\theta) \right|^2$$

(III60)

with $\beta^{\ell, M_{ex}-M_{g.s}}(\theta)$ given by (III57).

Each level to be considered is due to only one multipole order, ℓ. In (III60) $M_{ex}-M_{g.s.}$ is the z component of ℓ, and the angular momenta, J_{ex} and $J_{g.s}$, do not appear in (III60) other than to insure that

$$\vec{J}_{g.s} + \vec{\ell} = \vec{J}_{ex}$$

(III61)

In (III57) one sees that $\beta^{\ell, M_{ex}-M_{g.s}}(\theta)$, for a given ℓ, is dependent only on $\Gamma^{\ell, M_{ex}-M_{g.s.}}_{L'L}$, $P^{M_{ex}-M_{g.s.}}_{L'}(\theta)$, and $f^{\ell}_{L'L}$. The first two quantities are independent of the interaction that produced the excited state. Only in $f^{\ell}_{L'L}$ (III58) does an allusion to the interaction appear. This comes from $F_\ell(\Lambda_\lambda)$, the nuclear form factor, which is defined in (III53).

From (III59) it is apparent that $|A_\ell|^2$ is merely the "weight" or "strength" of the ℓ'th differential cross section, $\sigma_\ell(\theta)$. Thus, if the particular multipole order used is $\ell = k$ (i.e., only $\ell = k$ contributes significantly)

$$|A_\ell|^2 = 1 \qquad \ell = k \tag{III62a}$$

$$|A_\ell|^2 = 0 \qquad \ell \neq k \tag{III62b}$$

so that from (III59)

$$\frac{d\sigma}{d\Omega} = \frac{2J_{ex}+1}{2J_{g.s.}+1}\,\sigma_k(\theta) \tag{III63}$$

Combining (III52), (III53), and (III62) gives

$$F_k(r_\gamma) = \frac{4\pi}{(2k+1)\sqrt{2J_{ex}+1}} \langle J_{ex} \| \sum_{n,\gamma} V_k(r_\gamma, r_{n_\gamma}) Y_k(\theta_{n_\gamma}\phi_{n_\gamma}) \| \times J_{g.s}\rangle = \langle J_{ex} \| \frac{4\pi}{(2k+1)\sqrt{2J_{ex}+1}} \sum_{\gamma,n} V_k(r_\gamma, r_{n_\gamma}) Y_k(\theta_{n_\gamma}\phi_{n_\gamma}) \| J_{g.s}\rangle \tag{III64}$$

since $\frac{4\pi}{(2k+1)\sqrt{2J_{ex}+1}}$ is just a constant.

Equation (III64) is the quantity that is similar to the electric transition amplitude discussed in Section A). In order to clarify the comparison let us change the notation of the present section slightly by reading the symbols, $J_{g.s.}$ and J_{ex}, as J_f and J_i respectively. Then (III63) is rewritten as

$$\frac{d\sigma}{d\Omega} = \frac{2J_i+1}{2J_f+1}\,\sigma_k(\theta) \tag{III65}$$

and the expression (III64) becomes a matrix element of order, k , between the excited state of spin, J_i , and the ground state of spin, J_f , i.e.,

$$F_k(r_\lambda) = \langle J_i \| \frac{4\pi}{(2k+1)\sqrt{2J_i+1}} \sum_{n,\eta} V_k(r_n, r_{m_\eta}) Y_k(\theta_{m_\eta} \phi_{m_\eta}) \| J_f \rangle$$

(III66)

Now from (III2)

$$\underset{J_f \to J_i}{B(Ek)} = \frac{2J_i+1}{2J_f+1} \underset{J_i \to J_f}{B(Ek)}$$

(III67)

By analogy to (III 1) one also has

$$\underset{J_f \to J_i}{B(Ek)} = \frac{1}{2J_f+1} \left| \langle J_i \| \frac{1}{e} \sum_{n,\eta} e_\eta r_{m_\eta}^k Y_k(\theta_{m_\eta} \phi_{m_\eta}) \| J_f \rangle \right|^2$$

(III68)

Putting (III 1) in (III67) gives

$$\underset{J_f \to J_i}{B(Ek)} = \frac{1}{2J_f+1} \left| \langle J_f \| \frac{1}{e} \sum_{n,\eta} e_\eta r_{m_\eta}^k Y_k(\theta_{m_\eta}, \phi_{m_\eta}) \| J_i \rangle \right|^2$$

(III69)

So that comparing (III69) and (III68) gives

$$\langle J_f \| \frac{1}{e} \sum_{n,\eta} e_\eta r_{m_\eta}^k Y_k(\theta_{m_\eta}, \phi_{m_\eta}) \| J_i \rangle$$

$$= (-1)^\delta \langle J_i \| \frac{1}{e} \sum_{n,\eta} e_\eta r_{m_\eta}^k Y_k(\theta_{m_\eta}, \phi_{m_\eta}) \| J_f \rangle$$

(III70)

where $(-1)^\delta$ is the phase. Now multiply (III70) by $(-1)^{k+J_f-J_i}$ $\times \frac{1}{\sqrt{2J_f+1}}$ yielding

$$\langle J_f \| \frac{1}{\ell}(-1)^{k+J_f-J_i} \frac{1}{\sqrt{2J_f+1}} \sum_{n,\eta} \ell_\eta \lambda_{n_\eta}^k Y_k(\theta_{n_\eta}\phi_{n_\eta}) \| J_i \rangle$$

$$= \langle J_i \| \frac{1}{\ell}(-1)^{\delta+k+J_f-J_i} \frac{1}{\sqrt{2J_f+1}} \sum_{n,\eta} \ell_\eta \lambda_{n_\eta}^k Y_k(\theta_{n_\eta}\phi_{n_\eta}) \| J_f \rangle \qquad \text{(III71)}$$

It is this form of the reduced matrix element that is specifically calculated in Appendix D).

Thus, one may compare (III66) and (III71). Equation (III66) has $\frac{4\pi}{(2k+1)\sqrt{2J_i+1}}$ in place of $\frac{1}{\ell}(-1)^{\delta+k+J_f-J_i}\frac{1}{\sqrt{2J_f+1}}$ of (III71), and $V_k(\lambda_\lambda, \lambda_{n_\eta})$ in place of $\ell_\eta \lambda_{n_\eta}^k$.

Equation (III71) is evaluated in (Dlf.1) and (D2d.1). One may simply employ these results with the above mentioned changes to obtain $F_k(\lambda_\lambda)$ (III66). This means that

$$F_k(\lambda_\lambda) = \frac{\ell \times 4\pi\sqrt{2J_f+1}}{(2k+1)\sqrt{2J_i+1}}(-1)^{\delta+k+J_i-J_f} \times \text{(III71)} \qquad \text{(III72)}$$

provided $V_k(\lambda_\lambda, \lambda_{n_\eta})$ replaces $\ell_\eta \lambda_{n_\eta}^k$ in (III71)

1. Quasi-Boson Plus Quasi-Proton :

From (Dlf.1) and (III71)

$$\langle J_i \| \frac{1}{\ell}(-1)^{\delta+k+J_f-J_i} \frac{1}{\sqrt{2J_f+1}} \sum_{n,\eta} \ell_\eta \lambda_{n_\eta}^k Y_k(\theta_{n_\eta}\phi_{n_\eta}) \| J_f \rangle$$

$$= (-1)^{\delta} \left[A_P + B_P + C_N + D_N \right]$$

(III73)

where A_P, B_P, C_N, and D_N are given by (III 13a), (III 13b), (III 13c), and (III 13d) respectively.

Spin-orbit interactions are neglected, and the assumption is made that both neutrons and protons have the same interaction with the alpha particles. Each $V(|\vec{r}_\lambda - \vec{r}_{m_\eta}|)$ (III49) is taken as a Gaussian

$$V(|\vec{r}_\lambda - \vec{r}_{m_\eta}|) = V_0 \, e^{-\frac{(|\vec{r}_\lambda - \vec{r}_{m_\eta}|)^2}{b^2}}$$

(III74)

where V_0 is the strength and b is the range parameter. The terms, $V_k(r_\lambda, r_{m_\eta})$, (III50) (with k used as an index in place of ℓ) are then the coefficients in the expansion of the Gaussian (III74). Swiatecki,[36] has obtained the expansion of a Gaussian, but his coefficients, $V_k'(r_\lambda, r_{m_\eta})$, are defined slightly different. Namely

$$V_k'(r_\lambda, r_{m_\eta}) = \frac{1}{2k+1} V_k(r_\lambda, r_{m_\eta})$$

(III75)

Each $V_k(r_\lambda, r_{m_\eta})$ will contain the constant factor, V_0, so that one may define

$$v_k(r_\lambda, r_{m_\eta}) = \frac{1}{V_0} V_k(r_\lambda, r_{m_\eta})$$

(III76a)

$$v_k'(r_\gamma, r_{m_\eta}) = \frac{1}{V_0} V_k'(r_\gamma, r_{m_\eta})$$

(III76b)

with

$$v_k'(r_\gamma, r_{m_\eta}) = \frac{1}{2k+1} v_k(r_\gamma, r_{m_\eta})$$
(III76c)

Now recall the definition of $q^k(j_{1\eta} j_{2\eta})$ (I29, I26) as

$$q^k(j_{1\eta} j_{2\eta}) = -\frac{1}{\sqrt{2k+1}} \langle j_1 \| r_\eta^k Y_k \| j_2 \rangle$$

The use of (III72) will introduce the quantity $q^k_{(OWBA)_\gamma}(j_{1\eta} j_{2\eta})$ defined by

$$q^k_{(OWBA)_\gamma}(j_{1\eta} j_{2\eta}) \equiv -\frac{1}{\sqrt{2k+1}} \langle j_1 \| V_k(r_\gamma, r_\eta) Y_k \| j_2 \rangle$$

(III77)

which may be written using (III75) and (III76b) as

$$q^k_{(OWBA)_\gamma}(j_{1\eta} j_{2\eta}) = \frac{-V_0(2k+1)}{\sqrt{2k+1}} \langle j_1 \| v_k'(r_\gamma, r_\eta) Y_k \| j_2 \rangle$$
(III78)

Since $j_{1\eta}$ and $j_{2\eta}$ have been shorthand notations for the sets of single particle quantum numbers, $n_{1\eta}, l_{1\eta}, j_{1\eta}$ and $n_{2\eta}, l_{2\eta}, j_{2\eta}$, respectively, (III78) is more specifically

$$q^k_{(OWBA)_\gamma}(j_{1\eta} j_{2\eta}) = \frac{-V_0(2k+1)}{\sqrt{2k+1}} \langle n_{1\eta} l_{1\eta} | v_k'(r_\gamma, r_\eta) | n_{2\eta} l_{2\eta} \rangle$$
$$\times \langle j_{1\eta} \| Y_k \| j_{2\eta} \rangle$$
(III79)

The matrix element, $\langle n_{1\eta} l_{1\eta} | v_k'(r_\lambda, r_\eta) | n_{2\eta} l_{2\eta} \rangle$, is just a radial integral over r_η, the single quasi-particle (neutron or proton) coordinate. The reason that r_{m_η} has been replaced by r_η in (III77), (III78), and (III79) is that (III71) and hence (III72) are evaluated using the quasi-particle occupation number representation. In this representation only a typical single quasi-particle (proton or neutron) matrix element will appear.

Finally (III72), (III73), (III 13a,b,c,d), and (III79) give for $F_k(r_\lambda)$

$$F_k(r_\lambda) = -4\pi V_0 (-1)^{k+J_i-J_f} \frac{\sqrt{2J_f+1}}{\sqrt{(2k+1)(2J_i+1)}} \Big[A_p^{(DWBA)}(r_\lambda) + B_p^{(DWBA)}(r_\lambda) + C_N^{(DWBA)}(r_\lambda) + D_N^{(DWBA)}(r_\lambda) \Big] \quad \text{(III80)}$$

with

$$A_p^{(DWBA)}(r_\lambda) \equiv (-1)^k a_{0J_i}^{J_i} a_{0J_f}^{J_f} (-1)^{J_i+J_f} \sqrt{\frac{2k+1}{2J_f+1}} \times \langle J_f \| Y_k \| J_i \rangle \langle n_{J_f} l_{J_f} | v_k'(r_\lambda, r_p) | n_{J_i} l_{J_i} \rangle \quad \text{(III81a)}$$

$$B_p^{(DWBA)}(r_\lambda) \equiv (-1)^k \sqrt{(2J_i+1)(2k+1)} \sum_{j_{cp}' j_{cp}} a_{k j_{cp}'}^{J_f} a_{k j_{cp}}^{J_i}$$

$$\times (-1)^{j_{cp}-J_i} \begin{Bmatrix} k & J_i & J_f \\ k & j_{cp}' & j_{cp} \end{Bmatrix} \langle j_{cp}' \| Y_k \| j_{cp} \rangle$$

$$\times \langle n_{cp'} \, l_{cp'} | N_k'(r_2, r_p) | n_{cp} \, l_{cp} \rangle$$

(III81b)

$$C_N^{(DWBA)}(r_2) \equiv (-1)^k \left[-a_{kJ_i}^{J_f} a_{0J_i}^{J_i} + (-1)^{J_i+J_f} \left(\frac{2J_i+1}{2J_f+1} \right)^{1/2} a_{0J_f}^{J_f} a_{kJ_f}^{J_i} \right]$$

$$\times \sum_{j_{a_N} \geq j_{b_N}} \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} (-1)^{l_{a_N}} \sin(\sigma_{a_N} + \sigma_{b_N}) \left[r(j_{a_N} j_{b_N}) - (-1)^k s(j_{a_N} j_{b_N}) \right]$$

$$\times \langle j_{a_N} \| Y_k \| j_{b_N} \rangle \langle n_{a_N} l_{a_N} | N_k'(r_2, r_N) | n_{b_N} l_{b_N} \rangle$$

(III81c)

and

$$D_N^{(DWBA)}(r_2) \equiv (-1)^k (2k+1)^{3/2} \sqrt{2J_i+1} \sum_{j_{cp}} a_{kj_{cp}}^{J_f}$$

$$\times a_{kj_{cp}}^{J_i} \begin{Bmatrix} k & k & k \\ J_i & J_f & j_{cp} \end{Bmatrix} (-1)^{j_{cp}-J_f} \sum_{j_{a_N} j_{b_N} j_{c_N}} \sqrt{(1+\delta_{j_{a_N} j_{c_N}})(1+\delta_{j_{b_N} j_{c_N}})}$$

$$\times r(j_{a_N} j_{c_N}) r(j_{b_N} j_{c_N}) \cos(\sigma_{a_N} + \sigma_{b_N}) \begin{Bmatrix} k & k & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix}$$

$$\times (-1)^{j_{a_N}+j_{c_N}} \langle j_{a_N} \| Y_k \| j_{b_N} \rangle \langle n_{a_N} l_{a_N} | N_k'(r_2, r_N) | n_{b_N} l_{b_N} \rangle$$

(III81d)

One notes the similarity between (III80) and the Born approximation expression (III43).

2. Two Quasi-Neutrons Plus Quasi-Proton:

Utilizing the same alpha-nucleon interaction as in Section C1.), equations (III72) through (III79) also apply in the present case. Analogous to (III80) one may write

$$F_k(\Lambda_\lambda) = -4\pi V_0 (-1)^{k+J_i-J_f} \frac{\sqrt{2J_f+1}}{\sqrt{(2k+1)(2J_i+1)}} \left[A_p^{(OWBA)}(\Lambda_\lambda) + B_p'^{(OWBA)}(\Lambda_\lambda) + C_N'^{(OWBA)}(\Lambda_\lambda) + D_N'^{(OWBA)}(\Lambda_\lambda) \right]$$

(III82)

The single quasi-proton term, $A_p^{(OWBA)}(\Lambda_\lambda)$, is the same as (III81a). From equations (III22) the other quantities of (III82) are defined as

$$B_p'^{(OWBA)}(\Lambda_\lambda) \equiv \sqrt{(2J_i+1)(2k+1)} \sum_{\substack{j_{1N} \geq j_{2N} \\ J_0\, j_{cp}\, j_{cp}'}} c^{J_i}_{(j_{1N}j_{2N})J_0 j_{cp}}\, c^{J_f}_{j_{1N}j_{2N}(J_0)j_{c}}$$

$$\times (-1)^{j_{cp}-J_i+J_0} \begin{Bmatrix} k & J_i & J_f \\ J_0 & j_{cp}' & j_{cp} \end{Bmatrix} \langle j_{cp}' \| Y_k \| j_{cp} \rangle \langle n_{cp}' l_{cp}' | N_k'(\Lambda_\lambda, \Lambda_p) | n_{cp} l_c$$

(III83a)

$$C_N'^{(OWBA)}(\Lambda_\lambda) \equiv (-1)^k \sum_{j_{aN} \geq j_{bN}} \left[-a_{0J_i}^{J_i}\, c^{J_f}_{(j_{aN}j_{bN})kJ_i} + (-1)^{J_i+J_f} \left(\frac{2J_i+1}{2J_f+1}\right)^{1/2} a_{0J_f}^{J_f}\, c^{J_i}_{(j_{aN}j_{bN})kJ_f} \right] \frac{1}{\sqrt{1+\delta_{j_{aN}j_{bN}}}} (-1)^{l_{aN}}$$

$$\times \sin(\tau_{a_\nu} + \tau_{b_\nu}) \langle j_{a_\nu} \| Y_k \| j_{b_\nu} \rangle \langle n_{a_\nu} l_{a_\nu} | v_k'(r_\lambda, r_\nu) | n_{b_\nu} l_{b_\nu} \rangle$$

(III83b)

and

$$D'^{(DWBA)}_{(r_\lambda)} \equiv (-1)^k \sqrt{(2k+1)(2J_0+1)(2J_0'+1)(2J_i+1)}$$

$$\times \sum_{\substack{j_{a_\nu} j_{b_\nu} j_{c_\nu} \\ J_0 J_0' j_{cp}}} c^{J_f}_{(j_{a_\nu} j_{c_\nu}) J_0' j_{cp}} \, c^{J_i}_{(j_{b_\nu} j_{c_\nu}) J_0 j_{cp}} (-1)^{j_{cp} - J_f} \begin{Bmatrix} k & J_0' & J_0 \\ j_{cp} & J_i & J_f \end{Bmatrix}$$

$$\times \sqrt{(1+\delta_{j_{a_\nu} j_{c_\nu}})(1+\delta_{j_{b_\nu} j_{c_\nu}})} \cos(\tau_{a_\nu} + \tau_{b_\nu}) \begin{Bmatrix} J_0 & J_0' & k \\ j_{a_\nu} & j_{b_\nu} & j_{c_\nu} \end{Bmatrix}$$

$$\times (-1)^{j_{c_\nu} + j_{a_\nu} + J_0 + J_0'} \langle j_{a_\nu} \| Y_k \| j_{b_\nu} \rangle \langle n_{a_\nu} l_{a_\nu} | v_k'(r_\lambda, r_\nu) | n_{b_\nu} l_{b_\nu} \rangle$$

(III83c)

The result (III82) is quite similar to the Born Approximation result (III44).

CHAPTER IV

A TEST CASE - Cu^{63}

In this Chapter for purposes of illustration the expressions obtained in the preceding chapters will be applied to a particular nucleus, Cu^{63}. We have chosen the example because:

1. It possesses one proton outside of a major closed shell ($Z = 29$), and an even number of neutrons ($N = 34$).

2. There is an ample amount of experimental information available with which to compare. In particular, energies of the positive and negative parity levels, B(E2)'s , and both elastic and inelastic alpha particle scattering differential cross sections have been studied.[8,10]

3. There exists a number of quite significant disagreements between experiment and the simple "core" plus odd proton theory for this nucleus. Hence, the merits of the methods described so far will face stern challenges. For example, the ground state of Cu^{63} is $3/2^-$. Naively, excited levels of spins, $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$, should be expected by the weak coupling of the odd proton to the first 2^+ level of Ni^{62}, and all four negative parity levels should

retain the "collective" features of the 2^+ state. However, the $3/2^-$ excited state is not found in coulomb excitation experiments.[8] Furthermore, five positive parity states are found in the 3 mev region of excitation. Coupling of the odd proton of spin $3/2^-$ to the 3^- "collective" state of Ni^{62} gives only four levels.

The structure of both the positive and negative parity states will be considered in detail later in this Chapter from both points of view (i.e., quasi-Boson plus quasi-particle and three quasi-particles) and do provide insight into the possible reasons for these discrepancies.

4. The protons and neutrons are in the same shell region, i.e., there is one proton beyond the Z=28 major shell closure, and also six neutrons beyond the corresponding N=28. This means that the single proton and single neutron energies will probably not differ greatly. The latter are tabulated in Kisslinger and Sorenson.[17]

5. Other necessary parameters are available from the same paper. In particular, the Ni^{62} quasi-neutron transformation coefficients, and the quadrupole force strength may be so obtained.

6. The energy positions of the first 2^+ and 3^- states of Ni^{62} are well known.

7. Even though Ni^{63} is not stable, the number and energies of levels obtained experimentally in $Ni^{62}(d,p)Ni^{63*}$[37] reactions will be of interest. This can be compared, e.g.,

to $Cu^{63}(p,p')Cu^{63*}$ [38] results. In this way differences between core plus odd proton and core plus odd neutron predicted by theory may be checked.

The outline of the present chapter is as follows:

In Section A) we will obtain the energies of low lying negative and positive parity states along with their B(E2)'s or B(E3)'s to the ground state.

Section B) will consider inelastic alpha particle scattering in the Born approximation.

Section C) will utilize the Distorted Wave Born Approximation.

All of these sections will contain results from both schemes, i.e., quasi-Boson plus quasi-proton and two quasi-neutrons plus quasi-proton.

Before entering into the actual calculations we discuss the values of the parameters pertinent to the work that follows. If one assumes that the $1f_{7/2}$ shell is about 3 mev below the $2p_{3/2}$, then the $1f_{7/2}$ proton states do not contribute to the smearing of the Fermi surface to any appreciable extent, i.e., $u_{7/2}=0$, $v_{7/2}=1$. The appropriate proton levels are then just those between 28 and 50 nucleons. These states are also assumed to be applicable ones for neutrons. The pertinent states are the $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, and $1g_{9/2}$. The $2p_{3/2}$ single particle energy is zero, and the energies of the other three states are taken from

Kisslinger and Sorenson[17] for neutrons as 0.78 mev for the $1f_{5/2}$ state, 1.56 mev for the $2p_{1/2}$ state, and 4.52 mev for the $1g_{9/2}$ state. The quasi-proton will be assumed to have the same energies as a quasi-neutron would have in these states.

The quasi-neutron transformation coefficients, (u_ν, v_ν), are obtained from equations (I 11,I12) and the values from Kisslinger and Sorenson[17] of the Fermi energy, λ , the half-gap, Δ , and the pairing strength, G. The quasi-neutron energy, E_N, is given by (I8).

These parameters are tabulated in Table Ia. In addition, $E_{a_\nu} + E_{b_\nu}$, $\sin(\gamma_{a_\nu} + \gamma_{b_\nu})$ (II3b), $\cos(\gamma_{a_\nu} + \gamma_{b_\nu})$ (II3a), and $q^{k}(j_{a_\nu} j_{b_\nu})$ (I29) such that $\vec{j}_{a_\nu} + \vec{j}_{b_\nu} = \vec{k} = \vec{2}$ or $\vec{3}$ are listed in Table Ib. In determining $q^{k}(j_{a_\nu} j_{b_\nu})$, the reduced matrix components for Y_k are obtained from Edmonds,[29] and the radial matrix is gotten by using harmonic oscillator radial wavefunctions.[40] The units of $q^{k}(j_{a_\nu} j_{b_\nu})$ are $(4\pi)^{-1/2} \left(\frac{\hbar}{m\omega_{H.O.}}\right)^{k/2}$, where m is the nucleon mass and $\omega_{H.O.}$ is the harmonic oscillator frequency.

Section A. Energy Spectrum, B(E2)'s, and B(E3)'s.

1. Quasi-Boson Plus Quasi-Proton:

a) Quadrupole Interaction -

For $k=2$ the possible total angular momenta for negative parity excited states are $1/2^-$, $3/2^-$, $5/2^-$, $7/2^-$, and $9/2^-$. This follows from the coupling of the one quasi-Boson angular momentum, 2^+, to the single quasi-proton $1/2^-$, $3/2^-$, and $5/2^-$, spins. The only possible spin $9/2^-$ excited state is that due to the coupling of the 2^+ to the $5/2^-$ single quasi-proton spin. This state will have no B(E2) to the $3/2^-$ ground state of Cu^{63}.

The pertinent matrix elements depend on the expansion coefficients, $r(j_{1N} j_{2N})$ and $s(j_{1N} j_{2N}$. These are simply obtained from the sets of $r(j_{1N} j_{2N})$ and $s(j_{1N} j_{2N})$ equations (I49), the normalization equation (A5.5), and the energy, $w^{(2)}$, of the 2^+ state in Ni^{62} known to be

$$w^{(2)} = 1.17\ mev \qquad \text{(IV1)}$$

The values of $r(j_{1N} j_{2N})$ and $s(j_{1N} j_{2N})$ are listed in Table II. From Kisslinger and Sorenson[17)] the quadrupole (k =2) long-range interaction strength is given by

$$F^{(2)} \left(\frac{\hbar}{m\omega_{H.O.}}\right)^2 = 0.0114\ mev \qquad \text{(IV2)}$$

As shown in Chapter II the only contributing long-range Hamiltonian terms are $H^{(2)}_{N,P\,(31)}$ (II4) and $H^{(2)}_{N,P\,(22)}$ (II5). Their matrix elements are given by (II 12) and (II 13) respectively. With the help of Table I these elements have been calculated, and are listed in Table III. The arbitrary phase for $r(j_{1N} j_{2N})$ and $s(j_{1N} j_{2N})$ (TableII) comes about from the normalization equation (A5.5). This arbitrariness carries over into the matrix elements of $H^{k}_{N,P\,(31)}$ (Table III), and the "collective" contributing term to B(E2), C_N(III 13c). Henceforth, the bottom sign (Table II) will be used.

The sizes of the matrices to be diagonalized are $J=1/2^-(3x3)$, $J=3/2^-(4x4)$, $J=5/2^-(4x4)$, $J=7/2^-(2x2)$, and $J=9/2^-(1x1)$. Now let us write down an eigenfunction in the quasi-Boson plus quasi-proton scheme for a state, (J,M) as:

$$|JM\rangle_{0,2^+} = \Big(a^J_{0J}\, \beta^{+J}_M(P) + a^J_{2^+,1/2^-}\left[\Gamma^{2^+}_{(N)}\, \beta^{+1/2^-}_{(P)}\right]^J_M + a^J_{2^+,3/2^-}\left[\Gamma^{2^+}_{(N)}\, \beta^{+3/2^-}_{(P)}\right]^J_M + a^J_{2^+,5/2^-}\left[\Gamma^{2^+}_{(N)}\, \beta^{+5/2^-}_{(P)}\right]^J_M \Big) |\tilde{\hat{0}}_N ; \hat{0}_P\rangle \qquad \text{(IV3)}$$

with

$$a^J_{0J} = 0 \quad \text{for } J = 7/2^-, 9/2^- \qquad \text{(IV4a)}$$

$$a^J_{2^+,1/2^-} = 0 \quad \text{for } J = 1/2^-, 7/2^-, 9/2^- \qquad \text{(IV4b)}$$

and

$$a^J_{2^+,3/2^-} = 0 \quad \text{for } J = 9/2^- \qquad \text{(IV4c)}$$

Once this diagonalization has been performed, the B(E2)'s may be calculated via (III 13).

The eigenvalues and eigenfunctions are presented in Table IV, and the B(E2)'s using an effective neutron charge of 1 are listed in Table V in units of 10^{-52}sq.cm. In this latter table, $B(E2)\downarrow \equiv B(E2)_{J_i \to J_f}$ (III 1), i.e., the B(E2)'s for de-excitation are tabulated, and the nomenclature follows the interpretation in Chapter III of the terms of equations (III 13). The column labeled "single proton" contains A_p (III 13a). That labeled "one quasi-Boson exc. st. to one quasi-Boson gd. st. (due to the proton)", contains B_p (III 13b) and is so named since the transition is due to the quasi-proton multipole operator (III7) between the one quasi-Boson parts of both excited and ground states. The label "one quasi-Boson exc. st. to one quasi-Boson gd. st. (due to neutrons)", refers to D_N (III 13d) and is due to the quasi-neutron multipole operator (III 10) between the same components as mentioned above for B_p. "zero quasi-Boson exc. st. to one quasi-Boson gd. st. (due to neutrons)" refers to $C_N^{(1)}$ (D1d.2). This is the contribution due to the neutron multipole operator (III 10), and is between the zero quasi-Boson and one quasi-Boson parts of the excited and ground states respectively. The "one quasi-Boson exc. st. to zero quasi-Boson gd. st. (due to neutrons)" column is $C_N^{(2)}$ (D1d.5) and is the usual contribution from the "core" part of the excited state to the single quasi-proton part of the ground state.

The last column is the ratio of B(E2) to the pure single quasi-proton transition (III 14) from a $3/2^-$ excited state to the $3/2^-$ ground state. This particular pure single quasi-proton transition is arbitrarily chosen, and all such ratios in this investigation will be relative to this same transition. The energies, spins, and B(E2)'s of all levels are illustrated in Figure 1a.

Table V (or Figure 1a) indicates that there are only five levels of reasonably large B(E2)'s, viz., the 0.797

mev $1/2^-$, 1.40 mev $7/2^-$, 1.48 mev $5/2^-$, 1.54 mev $3/2^-$, and the 0.740 mev $5/2^-$. However, the B(E2) of the 1.54 mev $3/2^-$, and the 0.740 mev $5/2^-$ are only 58% and 35% respectively of the next lowest B(E2), that of the 1.48 mev $5/2^-$. These five states are shown in the level scheme of Figure 1b which also contains the predictions of the weak coupling model of De Shalit,[2)] as well as the experimental levels obtained in coulomb excitation.[8)] The weak coupling levels were normalized by setting the $1/2^-$ energy equal to the corresponding quasi-Boson plus quasi-proton prediction. In addition, Figure 1b includes the B(E2) of the 2^+ level of Ni^{62} as obtained from (D1c.7). This is compared with the values of Kisslinger and Sorenson[17)] and experiment.[8)] The agreement of the Cu^{63} quasi-Boson plus quasi-proton levels and the Ni^{62} level with the corresponding experimental values is quite good.

It is of interest to know the sensitivity of the predicted energies and B(E2)'s to the value of $F^{(2)}$. Now, the quasi-Boson frequency, $\omega^{(2)}$, is a function of $F^{(2)}$ (I50). Thus a change of $F^{(2)}$ requires a corresponding change in the energy used for the 2^+ state of Ni^{62}. However, for moderate force strengths the change in $\omega^{(2)}$ is not very much (c.f. Bayman[27)] p. 47a). Assuming a constant $\omega^{(2)}$, the dependence on $F^{(2)}$ of the energies and B(E2)'s is given in Figure 2. In this curve, the level of zero energy is the pure $3/2^-$ single particle state. This means that the actual energy of a level is the ordinate minus the energy of the Cu^{63} ground state.

b) Octupole Interaction -

For k=3, the matrix elements of $H^{k}_{N,P\,(31)}$ (II 12) exist only for J=9/2$^+$, since that is the only available single quasi-proton of positive parity. The matrix elements of $H^{k}_{N,P\,(22)}$ (II 13) vanish because of the parity requirement (II32b). Thus, one cannot obtain a set of positive parity levels of different spin in this scheme.

2. Two Quasi-Neutrons Plus Quasi-Proton (Quadrupole Interaction and Negative Parity Levels):

a) The Two Quasi-Neutrons Coupled to Arbitrary Angular Momentum -

The only excited states having B(E2)'s to the ground state have spins of 1/2$^-$, 3/2$^-$, 5/2$^-$, and 7/2$^-$. One is primarily interested in the B(E2)'s as indications of collective behavior. This consideration plus the difficulties involved in setting up extremely large matrices suggests we limit ourselves to only the above mentioned spins.

The sizes of the matrices to be diagonalized are J=1/2$^-$ (25x25), J=3/2$^-$ (42x42), J=5/2$^-$ (47x47), and J=7/2$^-$ (42x42). The bases with respect to which the matrices are diagonalized are listed in Table VI. Parity conservation requires that the matrix elements vanish between states in which the odd quasi-proton and/or the coupled quasi-neutron pair are in different parity states. This means that each matrix in Table VI can be broken up into two sub-matrices. One will involve two quasi-neutrons coupled to an even parity angular momentum, J_0, and then J_0 coupled to an odd parity quasi-proton. The other will have J_0 of odd parity coupled to a $9/2^+$ single quasi-proton. This latter sub-matrix will involve two $9/2^+$ quasi-particles, and consequently energies between 7 and 8 mev along the diagonal. Since these energies are much greater than those with which this paper is concerned, and the aforementioned parity restrictions (III23 and III24) forbid B(E2)'s from such levels to the ground state, these levels will no longer be considered. This leaves matrices of sizes $J=1/2^-$(19x19), $J=3/2^-$(32x32), $J=5/2^-$(35x35), and $J=7/2^-$(30x30). These are diagonalized using the same interaction strength as for the quasi-Boson plus quasi-proton scheme (IV2) and the parameters of Table I.

The matrix elements of $H^{k}_{N,p\,(22)}$ (II21) are of order 10^{-2} to 10^{-3} mev. The specific values are so small and numerous that these matrix elements are not tabulated. The major contributor to removing the degeneracy in total angular momentum, J, is $H^{k}_{N,p\,(31)}$ whose matrix elements are given by (II22). These elements are of the order of 10^{-1} mev and are listed in Table VII.

The matrix elements of $\mathcal{H}^{k}_{NN\,(22)}$ (Class 1) (II23 or II34) are listed in Tables VIII and IX. Table VIII contains the contribution due to $\sum_{j_{1N} j_{2N}} 2\left[B^{+2}(j_{1N} j_{2N}) C^{2}(j_{3N} j_{4N})\right]^{0}_{0}$ (II34a). In Table IX are the contributions of $\frac{1}{1+\delta_{j_{1N} j_{2N}}}\left[\delta_{j_{2N} j_{4N}}\delta_{j_{1N} j_{3N}} - (-1)^{j_{1N}+j_{2N}} \times \delta_{j_{1N} j_{4N}}\delta_{j_{2N} j_{3N}}\right]$ (III34b) and the single quasi-neutron scattering terms dropped in the Boson approximation (II34c). The matrix elements of $\mathcal{H}^{k}_{NN\,(22)}$ (Class 2) (II24) are listed in Table X.

We next calculate the B(E2)'s. There is found to be only one level of large B(E2) for each of the total angular momenta, $1/2^-$, $3/2^-$, $5/2^-$, $7/2^-$, and the energies of these "collective" levels are 2.26, 2.08, 2.05, and 2.13 mev respectively. The corresponding ratios of B(E2)$\downarrow$ to pure single proton are, in the same order, 1.70, 2.15, 2.15, and 1.96. Since there are so many levels for each spin, only the results for one typical angular momentum are tabulated. The $5/2^-$ spin is arbitrarily chosen and constitutes Table XI. In this Table the third, fourth, and fifth columns correspond to A_p (III 13a), B_p' (III22a), and C_N' (III22b) respectively. The term, D_N' (III22c), is neglected in the entire proceedings. The justification for this is that its calculation is quite unwieldy and a typical value is about a hundreth of B_p' which is itself small.

One notes in Table XI that there are 21 levels below the 2.05 mev state. This is quite different from the quasi-Boson plus quasi-proton results (c.f. Table V) in both number of levels and energy of the level of high B(E2). In the quasi-Boson plus quasi-proton scheme the level of high

B(E2) was at 1.48 mev, 0.57 mev below the present value. The energy discrepancy is actually greater than that, i.e., the "center of gravity" of the $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$ states of prominent B(E2) is 2.13 mev in the present case. This is in contrast to the 1.17 mev 2^+ state of Ni^{62} which served as the origin for the states in the quasi-Boson plus quasi-proton scheme. The 1 mev energy difference is attributable to the terms dropped in the Boson approximation (c.f. the last column of Table IX) and will be discussed further in Chapter V.

The dependence on the quadrupole force strength appears in Figure 3. Here the changes in the average energy of the quartet of high B(E2) levels, the B(E2)'s themselves, and the nature of the Cu^{63} ground state are indicated. The outstanding feature is the loss of the entire "effect" for interaction strengths greater than about 0.0148 mev. Above this strength there ceases to be any levels of high B(E2), and the ground state is no longer of spin $3/2^-$.

b) The Two Quasi-Neutrons Coupled to Angular Momentum 2^+ Only:

The question arises as to how important the assumption of an arbitrary two quasi-neutron angular momentum is to the results just described. For one thing, the use of only a 2^+ intermediate state would enable a more direct comparison to the quasi-Boson plus quasi-proton scheme, and secondly, the matrices and subsequent calculations involved would be considerably simplified.

By using only a 2^+ two quasi-neutron state, the matrices for spins, $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$, are reduced to sizes (13x13), (19x19), (19x19), and (12x12) respectively. The bases for these diagonalizations are just those states in Table VI for which $\mathcal{J}_0 = 2^+$. Similarly the matrix elements (exclusive of those of $H^{(2)}_{N,\rho\,(22)}$) are the $\mathcal{J}_0 = \mathcal{J}_0' = 2^+$ terms from Tables VII through X.

Using the same force strength (IV2) and the parameters of Table I, the eigenvalues and B(E2)'s are presented for $J=1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$ in Tables XIIa,b,c, and d respectively. The quantities listed are the same as in Table XI. Again one sees that there is only one level of large B(E2) for each of the four spins. The "quartet" of levels occurs at approximately the same energy as that predicted from arbitrary $\mathcal{J}_0$. For $J=5/2^-$ there are now only 9 levels below the quartet member compared to the 21 for arbitrary $\mathcal{J}_0$. All of the $J=1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$ levels of Tables XII are indicated in Figure 1c, with only the spins of the "quartet" explicitly written. The experimental levels are again presented for comparison. Also included are the positive parity states (c.f. Section A4).

Figure 4 depicts the dependence on interaction strength of the same quantities that were considered in Figure 3. The close similarity between the two curves clearly suggests the primary importance of the 2^+ intermediate state. In addition, the quasi-proton energies were changed by as much as 75% and altered the energies and B(E2)'s by only about 5%. The feature of a "quartet" of large B(E2) still remained.

3. Two Quasi-Neutrons Plus Quasi-Proton (Octupole Interaction and Positive Parity Levels):

For an octupole, (k=3) interaction, the matrix elements of $H^{k}_{N,P\,(31)}$ (II 22) exist only for J=9/2$^+$. For all other levels the only way to remove the degeneracy in total angular momentum, J, is through $H^{k}_{N,P\,(22)}$ (II21), and one can expect this term to be small in general.

By analogy to the preceding section, one need only use the 3^- two quasi-neutron intermediate state. The only way to obtain this state is by coupling the $9/2^+$ quasi-neutron to either the $3/2^-$ or $5/2^-$ quasi-neutron. The interaction strength may be obtained from the quasi-Boson results in Ni^{62}, even though by parity conservation the quasi-Boson plus quasi-proton method is not applicable to Cu^{63} (c.f. Section A1.b of this chapter). The value of $\omega^{(3)}$ is 3.50 mev, the energy of the 3^- level in Ni^{62}. Using the equations for the expansion coefficients, $\lambda(j_{1N} j_{2N})$ and $\Delta(j_{1N} j_{2N})$ (I49), and the normalization equation (A5.5), the interaction strength is found to be

$$F^{(3)} \left(\frac{\hbar}{m\omega_{H.O.}}\right)^3 = 0.00501 \text{ mev} \qquad \text{(IV5)}$$

which is about a factor of 2 less than the quadrupole strength(IV2).

The use of (IV5) and the parameters of Table I enables one to calculate the matrix elements of $\mathcal{H}^{(3)}_{NN\,(22)}$ (Class 1) (II23) and $\mathcal{H}^{(3)}_{NN\,(22)}$ (Class 2) (II24). These are listed in Table XIIIa. In this table the diagonal elements are also added to the appropriate two quasi-neutron energies (c.f. Table I). One sees that the resulting diagonal elements are only of the order of 0.3 to 0.4.mev. The addition of the odd quasi-proton and the subsequent degeneracy removal via $H^{k}_{N,P\,(22)}$ (II21) (or $H^{k}_{N,P\,(31)}$ (II 22) for $J=9/2^{+}$) could hardly be expected to produce levels even as high as 2 mev. Since experimental positive parity states of Cu^{63} [10)] are in the 3 mev region, the use of the octupole interaction is not adequate to produce such states.

4. Two Quasi-Neutrons Plus Quasi-Proton (Quadrupole Interaction and Positive Parity Levels):

The use of the quadrupole rather than octupole interaction turns out to produce levels in the correct energy region. Again the assumption is made that only the 3^{-} two quasi-neutron state need be considered. From the quadrupole interaction strength (IV2) and the parameters of Table I the matrix elements of $\mathcal{H}^{(2)}_{NN\,(22)}$ (Class 1) (II23) and $\mathcal{H}^{(2)}_{NN\,(22)}$ (Class 2) (II24) are calculated and tabulated in Table XIIIb. The sums of diagonal elements and the appropriate two quasi-neutron energies are also listed and are of the order of 3 mev.

Parity conservation requirements cause the matrix elements of $H^{(2)}_{N,P(31)}$ to vanish (II29a). Hence only $H^{(2)}_{N,P(22)}$ removes the degeneracy in total angular momentum, J, via (II21). The possible values of J are $1/2^+$, $3/2^+$, $5/2^+$, $7/2^+$, $9/2^+$, and $11/2^+$, and these involve 2x2, 4x4, 6x6, 6x6, 4x4, and 2x2 matrices respectively. There will also be a pure single quasi-proton level of spin $9/2^+$.

Despite the fact that these levels have been obtained by using a quadrupole interaction, one must bear in mind that the transition to the ground state is still octupole. The B(E3)'s are particularly easy to calculate. The ground state is a mixture of $3/2^-$ single quasi-proton and 2^+ coupled to quasi-proton. The excited states have only components due to 3^- coupled to quasi-proton. Thus the contributions of A_p (III 13a), B_p' (II22a), and $C_N'^{(1)}$ (D2b.4) are all zero. This comes about as follows: both A_p and $C_N'^{(1)}$ require a single quasi-proton part of the excited state, and B_p' requires both excited and ground states to have the same two quasi-neutron angular momentum. The contribution of D_N' (III22c) is expected to be so small that it is neglected. This leaves only $C_N'^{(2)}$ (D2b.9), i.e., the transition from 3^- plus $3/2^-$ to 0^+ plus $3/2^-$, where $3/2^-$ refers to the single quasi-proton, and 3^- to the two quasi-neutrons. This is, in essence, a "core" transition.

The eigenvalues, expansion coefficients from the eigenfunctions, i.e., $K^{J}_{(j_{a_\nu} j_{b_\nu})3^- j_{c_p}}$, and B(E3)'s in units of 10^{-78} cm^6 are listed in Table XIV. The B(E3)'s are obtained from (III21) assuming an effective neutron charge of 1, and compared to the B(E3) due to a pure single quasi-proton transition (III 14), viz., $9/2^+$ to $3/2^-$. Nine levels are found to have relatively large B(E3)'s, but one of these is just the $9/2^+$ pure single quasi-proton state at 4.61 mev. The other eight consist of two each of spins, $3/2^+$, $5/2^+$, $7/2^+$, and $9/2^+$. All of the levels of Table XIV are indicated in Figure 1c, with only the spins of the nine levels of large B(E3) explicitly written. The positive parity experimental levels[10)] are also presented for comparison. The fact that there are more than four such states near 3 mev agrees nicely with our results. The four levels predicted above 3 mev have B(E3)'s larger than the other four levels by a factor of about five, and are lower in energy than the experimentally observed ones. An investigation of the dependence of the nine energy levels of large B(E3) on interaction strength appears in Figure 5.

Section B. Inelastic Alpha Scattering in the Born Approximation.

The expressions for the cross sections have been given in Chapter III, Section B. Evaluation may be accomplished by using the eigenfunctions obtained in the preceding section, and the parameters of Table I. One needs, however, to relate the momentum transfer, $\vec{p}$ (III29), to the energy of the incident alpha particle and the pertinent Cu^{63} excited state. The relation is [41)]

$$p^2 = 1.91 \times 10^{25} \left[E_{\alpha'} + E_\alpha + 2\sqrt{E_{\alpha'} E_\alpha} \cos\theta \right] cm^{-2} \qquad \text{(IV6)}$$

where θ is the angle of scattering in the laboratory system, and E_α and $E_{\alpha'}$ are respectively the incident and scattered alpha energies in the laboratory system. The latter energy is given by

$$E_{\alpha'} = \left[0.0597\sqrt{E_\alpha} \cos\theta + \sqrt{E_\alpha (0.880 + 0.00356 \cos^2\theta) - 0.941W} \right. \qquad \text{(IV7)}$$

with W representing the energy of the excited state of Cu^{63}.

1. Negative Parity States:

a) Quasi-Boson Plus Quasi-Proton Scheme -

The differential cross section is given by (III43), but only relative values shall be calculated, i.e., no attempt will be made to obtain the alpha-nucleon interaction strength. The curves for the relative differential cross sections vs. angle are presented for $J=1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$ in Figures 6a,b,c, and d respectively. An incident alpha energy, E_α, of 50 mev is used.

The important feature of these curves is the occurrence at the same angle (~15°) of a peak for the five levels found to have large B(E2) (c.f. Figure 1). Of these five the 1.54 mev $3/2^-$ and 0.740 mev $5/2^-$ have their peaks smaller than the other three by about the same ratio as in the case of the B(E2)'s. All other levels have cross sections that are dwarfed by comparison to these five.

b) Two Quasi-Neutrons Plus Quasi-Proton Scheme -

The results obtained in Section A) indicate that only the 2^+ two quasi-neutron state need be considered. The differential cross section is given by (III44), and again only relative magnitudes of the cross sections are calculated. Using 50 mev incident alpha particles, the differential cross sections for certain levels of spins, $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$, are plotted in Figures 7a,b,c, and d respectively. In particular, the curves for levels found earlier to have prominent B(E2)'s (c.f. Tables XII a,b,c,d) are drawn. For comparison two other typical levels of each spin are also plotted, one at a nearby energy and one in the region of 1.20 mev.

Only the large B(E2) "quartet" members have large cross sections. The dominant feature is the peaking for all "quartet" members at the same angle ($\sim 15°$). This is also the angle at which high peaks occurred in the quasi-Boson plus quasi-proton scheme. The other two levels in each Figure are typical of the remaining states of each spin, and are seen to have peaks that are at most about 1/200 of those of the "quartet".

2. Positive Parity States:

Again using the results of Section A only the 3^- two quasi-neutron state is considered. The relative differential cross sections for 50 mev alphas may be determined from (III44). The levels investigated are those arising from the quadrupole interaction (c.f. SectionA4).

The nine levels previously found to have large B(E3)'s (c.f. Table XIV) are the only ones that have prominent peaks in the differential cross sections. One of these, the $9/2^+$ single particle level, is at too high an energy to be of interest. The other eight levels have the differential cross sections for their excitation plotted in Figures 8a,b,c, and d for spins, $3/2^+$, $5/2^+$, $7/2^+$, and $9/2^+$, respectively. For each spin, in addition to the pair having large B(E3), the cross section curve for a typical level is drawn. Such levels have cross section peaks that are at most about 1/200 of the other eight states. Again the large peaks occur at the same angle ($\sim 15°$), and one cannot directly distinguish the levels of 2^+ and 3^- origin. As mentioned in Chapter III a detailed description of the differential cross sections can only be meaningfully made by employing a more detailed study of the alpha-nucleon interaction. This will become evident in the next section.

Section C. Inelastic Alpha Scattering in the Distorted Wave Born Approximation.

1. Introduction:

This method has been described in Chapter III, Section C). In particular, one expects the differential cross sections for inelastic alpha scattering to reflect the "multipole origin" of the level concerned. Since the quasi-Boson plus quasi-proton scheme does not work for positive parity levels (c.f. Section Alb), the two quasi-neutrons plus quasi-proton method must be employed in order to obtain a comparison between states of opposite parity. This means that the "multipole origin" is not the interaction order (this is always quadrupole), but the angular momentum to which the two quasi-neutrons are coupled. This angular momentum is the multipole order of the electric transition to the ground state provided one uses the results of Section A, i.e., negative and positive parity states may be considered as due only to 2^+ and 3^- two quasi-neutron states respectively.

The total eigenfunctions that have been used thus far are based upon the use of harmonic oscillator single particle radial eigenfunctions. However, one would like to have a better approximation to the correct single particle eigenfunction when using the DWBA. As a compromise, the expansion coefficients (I67) and (III 19) are those used right along, but the single particle radial matrix elements (III79) are calculated more exactly. In particular, these single particle radial wavefunctions are obtained from the program ABACUS[42] using a Woods-Saxon well with additional spin-orbit interaction. The entire well is then

$$V = \frac{-V_0}{1+e^{\frac{r-R_0}{a}}} - V_{s.o.}\ \frac{1}{r}\left|\frac{d}{dr}\left(1+e^{\frac{r-R_0}{a}}\right)^{-1}\right|\left(\frac{\hbar}{m_\pi c}\right)^2 \vec{l}\cdot\vec{\sigma} \tag{IV8}$$

The spin-orbit well depth, $V_{s.o.}$, is fixed at 10 mev, R_o is given by $1.25A^{1/3}$ with A=63, and a is chosen as 0.5 Fermis. Given the binding energies of the states, and an initial well depth, V_o, ABACUS searches on V_o to obtain a matching of the external and internal logarithmic derivatives at the boundary. The binding energies are obtained from Cohen et al,[43] and are -9.52, -8.74, -7.96, and -5.00 mev for the $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, and $1g_{9/2}$ states respectively. Choosing an initial value of 45.0 mev for V_o, the finally used values of V_o in mev were 47.7, 48.9, 49.2, and 46.2 for these states in the order abovementioned.

The optical model and coulomb interaction parameters (III47 and III48) were obtained by the best fit to the experimental elastic alpha scattering from the ground state of Cu^{63}.[10] The theoretical results were obtained by using 40 partial waves. Figure 9 contains the comparison of the optical model prediction and experiment. The parameters used are

$$U_R = 42.5 \text{ mev} \quad \text{(IV9a)}$$

$$U_{IM} = 18.5 \text{ mev} \quad \text{(IV9b)}$$

$$\lambda_0 = 1.585 \text{ Fermis} \quad \text{(IV9c)}$$

$$\lambda_c = 1.585 \text{ Fermis} \quad \text{(IV9d)}$$

and

$$a = 0.5 \text{ Fermis} \quad \text{(IV9e)}$$

These values will be used for both entrance and exit channels in the DWBA inelastic alpha scattering calculations. The DWBA results are obtained by using the program, JULIE, a modified version of SALLY,[44] that allows for the introduction of an arbitrary form factor (III82).

2. Results:

The differential cross sections for inelastic scattering of 50 mev alphas from various excited states of Cu^{63} were determined. Figure 10 contains several features, one of which is a comparison of the experimental results for the 0.961 mev $5/2^-$ state and the theoretical prediction for the 2.05 mev $5/2^-$ "quartet" member obtained in the two quasi-neutrons (coupled to 2^+ only) plus quasi-proton scheme. The latter differential cross section was normalized to the experimental curve by equating the cross sections arbitrarily at 37°. The magnitudes of the differential cross sections for all other levels were also changed accordingly. The agreement between the $5/2^-$ experimental and theoretical curves is excellent.

Also shown in Figure 10 are the differential cross sections for two of the "collective" states of positive parity (arbitrarily the $5/2^+$ states are plotted). These states were obtained by the two quasi-neutrons (coupled to 3^- only) plus quasi-proton scheme. They are out of phase with the $5/2^-$ curve and in phase with the elastic scattering curve of Figure 9; thus, the parity phase rule[35),45)] for the strongly oscillatory parts of the angular distributions is obeyed. This agrees with the results of Meriwether.[10)] A typical cross section for a "non-collective" state of neg. parity (arbitrarily the $1/2^-$ state at 1.18 mev) is included in Figure 10 for comparison. The peaks of this curve are considerably smaller in magnitude than those of the "collective" states, but the parity phase rule still appears to be obeyed.

In Figure 11 the differential cross section curves for all of the negative parity "quartet" members are shown. Again the "non-collective" 1.18 mev $1/2^-$ curve is included for comparison. The "quartet" consists of the 2.25 mev $1/2^-$, 2.08 mev $3/2^-$, 2.05 mev $5/2^-$, and 2.06 mev $7/2^-$ states as obtained earlier in the two quasi-neutrons (coupled to 2^+ only) plus quasi-proton scheme. The four curves oscillate in phase with each other and have comparable magnitudes. The angles of maxima and the corresponding cross sections agree nicely with experiment (c.f. Table XVa) except for the predicted value of the $3/2^-$ cross section; it is too large by a factor of about 8.

The differential cross sections for the eight positive parity "collective" states are pictured in Figure 12. These states were obtained earlier using a quadrupole interaction in the two quasi-neutrons (coupled to 3^- only) plus quasi-proton scheme. There is an apparent break up into two groups of four. The four curves of largest cross section represent the higher energy levels and are due primarily to the $(9/2^+, 3/2^-)3^-$ intermediate state, while the other four curves are due mainly to the $(9/2^+, 5/2^-)3^-$ intermediate state (c.f. Table XIV). All eight curves oscillate in phase with each other. This agrees nicely with the experimental results (c.f. Table XVb). The pure $9/2^+$ single quasi-proton state at 4.61 mev has a differential cross section curve of magnitude comparable with these eight and oscillates in phase with them. (This is not included in any of the Figures.)

The absolute values of the form factors (III66 and III82) for the negative parity "quartet" members and some typical negative parity states are plotted in Figure 13 in arbitrary units. All states are based on the two quasi-neutrons (coupled to 2^+ only) plus quasi-proton scheme. The shapes for the "quartet" members are the same, and all four levels have form factors of comparable magnitude. Of the curves shown in Figure 13 for typical levels, only the 1.973 mev $7/2^-$ and 1.974 mev $5/2^-$ states have form factors similar to the "quartet". This would mean that the shape of their differential cross sections would resemble those of the "quartet". The 1.18 mev $1/2^-$ state has a form factor that tapers off sooner than these aforementioned levels, but even it produces a differential cross section curve (Figure 10) that is in phase with the "quartet" members. The 1.19 mev $5/2^-$ level as shown in Figure 13 has an oscillating form factor; hence, the 1.18 mev $1/2^-$ level produces an "intermediate" form factor with shape somewhere between the "collective" kind and the oscillatory kind (represented by the 1.19 mev $5/2^-$ level). Of course, all of the "non-collective" states have form factors that have peaks much smaller in magnitude than the "collective" states.

In Figure 14 the form factor for a typical "quartet" member in the quasi-Boson plus quasi-proton scheme (III80) is displayed (Arbitrarily the 1.48 mev $5/2^-$ state is presented). Comparison to the curve for the 2.05 mev $5/2^-$ "quartet" member in the two quasi-neutrons (coupled to 2^+ only) plus quasi-proton scheme shows that both will produce similar differential cross section curves. Also included in Figure 14 are the form factors for two "collective" and two typical positive parity states. These states are obtained from the two quasi-neutrons (coupled to 3^- only) plus quasi-proton scheme. The magnitudes of the "collective" form factors (arbitrarily the 2.77 mev and 3.19 mev $5/2^+$ states are represented) are more than 20 times greater than the magnitudes of the form factors for typical positive parity levels. The curves for the 3.47 mev $3/2^+$ and 3.53 mev $5/2^+$ states are arbitrarily displayed. The shapes of these typical form factors imply that they also would obey the phase rule.

CHAPTER V

CONCLUSIONS

In this chapter we review the main conclusions and implications of the calculations that we have carried out for the test case of Cu^{63}. For convenience Sections A), B), and C) will summarize the results that have been collected so far. In particular Section A) will refer to the energy levels, Section B) to the B(Ek)'s, and Section C) to the inelastic alpha scattering cross sections.

In Section D) sum rules involving the energies and B(Ek)'s will be presented and discussed in connection with available experimental data.[8,10] (The derivations of these are to be found in Appendix E).

A discussion of the contents of Sections A), B), and C) will comprise Section E). The differences between the two coupling schemes are summarized in Table XVI.

Section A. Energy Levels.

1. Quasi-Boson Plus Quasi-Proton Scheme:

   a) Negative Parity States -

   Using a quadrupole interaction, there are only 7 levels predicted below 2 mev in Cu^{63} (Figures la,b, and Table V).

b) Positive Parity States -

Because of parity restrictions (II32b) this scheme is unsuitable.

2. Two Quasi-Neutrons Plus Quasi-Proton Scheme:

a) Negative Parity States (arbitrary intermediate angular momentum) -

Using a quadrupole interaction, <u>85</u> levels are predicted below 2.3 mev. (The $5/2^-$ levels are typical of those of spins $1/2^-$, $3/2^-$, $7/2^-$, and are listed in Table XI.)

b) Negative Parity States (2^+ intermediate angular momentum only) -

There are <u>42</u> levels predicted below 2.3 mev on the basis of a quadrupole interaction. (Figure lc and Tables XIIa,b,c,d)

c) Positive Parity States (3^- intermediate angular momentum only) -

Again using a quadrupole interaction, <u>25</u> levels are predicted between about 2.5 and 5 mev (c.f. Figure lc and Tables XIV, XV).

<u>Section B</u>. <u>B(Ek) (k=2 or 3)</u>.

1. Quasi-Boson Plus Quasi- Proton Scheme Applied to Negative Parity States:

Only <u>4</u> (and possibly 5) levels have large B(E2)'s to the ground state (Figure 1a and Table V). The four levels are between about 0.7 and 1.6 mev and consist of one each of spins, $1/2^-$, $3/2^-$, $5/2^-$, $7/2^-$, with the B(E2) of the $3/2^-$ level being only 58% of the next smallest. The results are compared with experiment[8] in Figure 1b.

2. Two Quasi-Neutrons Plus Quasi-Proton Scheme:

   a) Negative Parity States (arbitrary intermediate angular momentum) -

   Of the 85 levels mentioned in Section A2a) only 4 have large B(E2)'s to the ground state, there being one each of spins, $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$. (The $5/2^-$ levels are typical and are listed in Table XI.) The four levels have approximately equal B(E2)'s and all are in the 2 mev range.

   b) Negative Parity States (2^+ intermediate angular momentum only) -

   Of the 42 levels referred to in Section A2b) only 4 (one each of spins $1/2^-$, $3/2^-$, $5/2^-$, and $7/2^-$) have strong B(E2)'s to the ground state. The 4 levels are in the 2 mev region, and the $3/2^-$ state has a B(E2) about 20% less than that of the next smallest. (Figure 1c and Tables XIIa,b,c,d).

   c) Positive Parity States (3^- intermediate angular momentum only) -

   Of the 25 levels mentioned in Section A2c), there are 8 (two distinct $3/2^+$, $5/2^+$, $7/2^+$, $9/2^+$ "quartets") that have strong B(E3)'s. Of these 8, the four of lowest energy have the smaller B(E3)'s (Figure 1c and Table XIV).

Section C. Inelastic Alpha Scattering Cross Sections.

1. Quasi-Boson Plus Quasi-Proton Scheme Applied to Negative Parity States:

Only the levels of large B(E2) have prominent cross sections. Figures 6a,b,c,d show that the $3/2^-$ level turns out to have the smallest cross section, being about 40% below that of the $5/2^-$, which is the next smallest. The peak for the $1/2^-$ curve (Figure 6a) is arbitrarily set equal to the $1/2^-$ cross section obtained at 26° by Meriwether, et al,[10)] (the maxima in Figures 6b,c,d, and Figures 7a,b,c,d being adjusted accordingly), and the four quasi-Boson plus quasi-proton peaks are compared with the experimental diffraction maxima in Table XV.

2. Two Quasi-Neutrons Plus Quasi-Proton Scheme:

a) Negative Parity States (2^+ intermediate angular momentum only) -

Again only the states of strong B(E2) have large cross sections, and the $3/2^-$ cross section is comparable to those of the $1/2^-$, $5/2^-$, and $7/2^-$ states (Figures 7a,b,c,d and 11). Comparison to the experimental results of Reference 10) is made in Figure 10, and Table XV.

b) Positive Parity States (3^- intermediate angular momentum only) -

The levels of strong B(E3) are the only ones that have large cross sections, and again there are two distinct $3/2^+$, $5/2^+$, $7/2^+$, $9/2^+$ "quartets"(c.f. Figure 12). Comparison to the experimental results[10)] is made in Table XV, and the parity phase rules are illustrated via Figures 9 and 10.

Section D. Sum Rules.

In Appendix E), the following sum rules involving the excited state, J_i , and ground state, J_f , are derived in the quasi-Boson plus quasi-proton scheme.

$$\sum_{J_i,\lambda_{J_i}} B(E\ell)_{J_i,\lambda_{J_i}\to J_f} \simeq |a_{0J_f}^{J_f}|^2 \sum_{J_i} \left[B(E\ell)_{s.p.\; J_i\to J_f} + B(E\ell)_{\ell\to 0^+}\right] \quad \text{(V1)}$$

and

$$\sum_{J_i,\lambda_{J_i}} (2J_i+1) E_{\lambda_{J_i}} B(E\ell)_{J_i,\lambda_{J_i}\to J_f} \simeq |a_{0J_f}^{J_f}|^2 \sum_{J_i,\lambda_{J_i}} (2J_i+1) E_{\lambda_{J_i}} \Big[|a_{0J_i}^{J_i}(\lambda_{J_i})|^2$$
$$\times B(E\ell)_{s.p.\; J_i\to J_f} + |a_{\ell J_f}^{J_i}(\lambda_{J_i})|^2 B(E\ell)_{\ell\to 0^+} \Big] \quad \text{(V2)}$$

The symbol, λ_{J_i} , indicates the particular member of the state of total angular momentum, J_i , that is considered, and $E_{\lambda_{J_i}}$ is the energy of that state. The left hand sides of (V1) and (V2) are exact sums, while the right hand side is not. (The neglected terms involve, e.g., "core" to "core" transitions and are shown in Appendix E) to be quite small.)

The sum rules (V1 and V2) involve sums over <u>all</u> states obtained via the appropriate diagonalizations. The ground state single particle coefficient, $a_{0J_f}^{J_f}$, is, of course, nearly unity (e.g., $|a_{0J_f}^{J_f}|$ =0.937 for Cu^{63}). One notes also that provision for the single particle transition, $B(E\lambda)_{S.P.\ J_i \to J_f}$, occurs naturally.

If one makes the following approximations:

(i) $\sum_{\alpha J_i} = 1$ and $E_{\alpha J_i} = E_{J_i}$ (there is only one level of each spin)

(ii) $B(E\lambda)_{S.P.\ J_i \to J_f} = 0$ (there is only a "core" transition)

(iii) $|a_{0\,J_f}^{J_f}|^2 = 1$ (pure single particle ground state)

(iv) $|a_{0\,J_i}^{J_i}|^2 = 0$

(v) $|a_{\lambda\,J_f}^{J_i}|^2 = 1$ (pure "core" excited state)

(vi) $E_{\alpha J_i} = E_{J_i} = \epsilon_{J_f} + \hbar\omega + \Delta E_\lambda(J_i)$

(ϵ_{J_f} is the single particle energy, $\hbar\omega$ that of the "core", and $\Delta E_\lambda(J_i)$ the shift of the level of spin, J_i, due to the λ'th multipole of the "core" - particle interaction).

(vii) $\sum_{J_i}(2J_i+1)\,\Delta E_\lambda(J_i) = 0$ ("center of gravity" rule)

then the weak coupling sum rule of De Shalit[2] follows from (V1), i.e.,

$$B(E\lambda)_{J_i \to J_f} = B(E\lambda)_{\lambda \to 0^+} \qquad \text{(V3)}$$

while (V2) becomes

$$\sum_{J_i}(2J_i+1)E_i\, B(E\lambda)_{J_i \to J_f} = (\hbar\omega + \epsilon_{J_f})\, B(E\lambda)_{\lambda \to 0^+} \sum_{J_i}(2J_i+1) \qquad \text{(V4)}$$

Let us now apply the sum rule (V1) to the results of the 2^+ quasi-Boson plus quasi-proton scheme for Cu^{63}. The $B(E2)_{J_i, \alpha_{J_i} \to 3/2^-}$ values and $|a^{3/2^- g.s.}_{0\, 3/2^-}|^2$ are obtained from Tables V and IV respectively, $B(E2)_{2^+ \to 0}$ from Figure 1b, and $B(E2)_{s.p.\, J_i \to 3/2^- g.s.}$ is given by (III 14). The result is that the dropped terms on the right hand side of (V1) are only about .7% of the total left hand side, i.e., to within .7% (V1) is an equality. (In a similar way the dropped terms on the right hand side of (V2) may be shown to be negligible.) It is also interesting to note that the 4 "collective" levels account for 92% of the left hand side of (V1) and $B(E2)_{2^+ \to 0^+}$ accounts for about 89% of the right hand side.

Next, the sum rule (V2) will be applied to the experimental results of Gove.[8] Assuming that there are only a quartet of levels, then his $B(E2)_{J_i \to J_f}$ values and $B(E2)_{2^+ \to 0}$ (Figure 1b) indicate that $B(E2)_{3/2^- \to 3/2^-}$ is only about 28% of the average B(E2) of the $1/2^-$, $5/2^-$, and $7/2^-$ states (in obtaining this result, the calculated $|a^{3/2^- g.s.}_{0\, 3/2^- g.s.}|^2$ and $B(E2)_{s.p.\, J_i \to 3/2^-}$ values were used). If again use is made of the calculated wave-functions (Table IV), then the sum rule (V2) predicts that $B(E2)_{3/2^- \to 3/2^-}$ increases as $E_{3/2^-}$ increases, e.g., at $E_{3/2^-} \sim 3.2$ mev, $B(E2)_{3/2^- \to 3/2^-}$ vanishes, and even at as high an energy as $E_{3/2^-} \sim 5.2$ mev, $B(E2)_{3/2^- \to 3/2^-}$ is only 2 single particle units (less than

1/4 of the average B(E2) of the $1/2^-$, $5/2^-$, and $7/2^-$ levels). The fact that Gove[8] does not observe the $3/2^-$ level agrees nicely with our results. In addition, the fact that Meriwether, et al,[10] observe a smaller (by about a factor of 7) inelastic alpha scattering cross section for the $3/2^-$ level than for the "quartet" level of next smallest cross section (Table XV) also fits in with our predictions.

Section E. Discussion.

1. Quasi-Boson Plus Quasi-Proton Scheme:

From Figure la, the only low lying levels predicted in the quasi-Boson plus quasi-proton scheme are those of large B(E2). This fits well with the experimental results of Gove[8] (c.f. Figure lb). Hence, the quadrupole coupling of a quasi-proton to the 2^+ quasi-Boson gives a quite reasonable description of the low energy spectrum of Cu^{63}.

In this scheme the levels of given J are split apart mainly through the interaction term, $H^{(2)}_{N,P\,(31)}$ (II4), except for $J=7/2^-$ (Table IIIa). This makes possible the maintenance of "collective" properties for some levels even in strong coupling despite the dispersing of the levels. It is, thus, quite necessary to take mixing into account.

The dependence on force strength, as shown in Figure II, illustrates that the $3/2^-$, higher $5/2^-$, and $7/2^-$ states consist mainly of the (2^+, $3/2^-$) admixture. This is also evident from the components of the eigenfunctions as listed in Table IV. By contrast, the $1/2^-$ level is due about equally to the $1/2^-$ single particle and $(2^+, 3/2^-)$ components. It is this $(2^+, 3/2^-)$ admixture that relates back to the idea of a 2^+ "core" coupled to the $3/2^-$ ground state. The presence of a fifth level (of lower B(E2) than the others) is due to the $5/2^-$ single particle state.

From Table V, the one quasi-Boson to zero quasi-Boson (i.e. "core") contribution to B(E2) is the most important as a rule. The other contributions are not always insignificant, however, and in some cases are larger than that of the "core", e.g., the single proton part of the B(E2) of the 2.77 mev $1/2^-$ state, and the one quasi-Boson to one quasi-Boson (due to the proton) contribution to the B(E2) of the 3.04 mev $5/2^-$ state. The effect of coherence is quite evident from this Table, e.g., the 0.797 mev $1/2^-$ rather than the 1.78 mev $1/2^-$ level is "collective" for this very reason. All contributions have the same sign in the former case, whereas the "core" contribution in the latter is opposite in sign to that of all the other components that determine the resultant B(E2). The 1.78 mev $1/2^-$ level is thus labeled as "non-collective" despite the fact that its "core" contribution is comparable to that of the "collective" $1/2^-$ state at 0.797 mev.

One may also learn something about the one quasi-

Boson 2^+ state itself. The large values of $\left(\frac{\lambda}{\Delta}\right)^2 (j_{1N}\, j_{2N})$ in Table II indicate that the $B^{+2}(N)$ (I38a) terms in the definition of $\Gamma^{2(N)}(N)$ (I47a) are much more important than the $C^{(2)}(N)$ (I38b) terms. This substantiates the approximation $\Gamma^2_m(N) \simeq \sum_{j_{1N} \geq j_{2N}} \lambda(j_{1N}\, j_{2N})\, B^{+2}_m(j_{1N}\, j_{2N})$ made in Chapter II in computing the matrix elements of the interaction term, $H^{k}_{N,P\,(22)}$ (II 13). In this same Table it is seen that the ($j_{1N} = j_{2N} = 5/2^-$) state is weighted at least 1.5 times stronger than any other state in contributing to $\Gamma^2(N)$. This is reasonable in that the $5/2^-$ quasi-neutron energy, $E_{5/2^-}$, is the smallest, so that $2E_{5/2^-}$ is the two quasi-neutron energy closest to the 1.17 mev $\omega^{(2)}$.

2. Comparison of the Two Quasi-Neutrons Plus Quasi-Proton and Quasi-Boson Plus Quasi-Proton Schemes (negative parity levels only):

The comparison is best made when only the 2^+ two quasi-neutron configuration is considered. That the 2^+ intermediate state is actually the dominant one may be readily seen as follows: the matrix elements of $H^{(2)}_{N,P\,(31)}$ require that the single quasi-proton couple only to the 2^+ two quasi-neutron state, and it is these terms that are the prime removers of the degeneracy in the total angular momentum, J. The $H^{(2)}_{N,P\,(22)}$ terms are so small by comparison that they really need not have been considered from the start. In essence, dropping the contributions of two quasi-neutrons coupled to an intermediate angular momentum other than 2^+ amounts to discarding $H^{(2)}_{N,P\,(22)}$. These matrix elements are calculated via (II21) and are so copious, yet small (of the order of 10^{-2} mev), that they are not tabulated. The matrix elements of $H^{(2)}_{N,P\,(31)}$ are calculated from (II22) and comprise Table VII. The difference between $H^{(2)}_{N,P\,(31)}$ and $H^{(2)}_{N,P\,(22)}$ is best respresented in the graphs of Chapter II, Section E).

The analogies between the matrix elements of the two quasi-neutrons plus quasi-proton scheme and the quasi-Boson plus quasi-proton scheme have been described earlier in Chapter II. The results of the two schemes as applied to Cu^{63} have many similarities. In particular, both schemes predict the existence of a "collective quartet" of levels. These levels possess in both schemes large B(E2)'s and also have large excitation cross sections for alpha particle scattering. There are, however, some differences between the "quartets" of the two schemes. In the two quasi-neutrons plus quasi-proton scheme the 4 levels are above 2 mev and all are within about 0.2 mev of each other (Figure 1c), whereas in the quasi-Boson plus quasi-proton approach, the energy spread is from .794 to 1.54 mev.

One can try to understand this difference in a qualitative way. The quasi-Boson plus quasi-proton scheme is based upon knowing the energy of the one quasi-Boson state. To this one simply couples the odd quasi-proton and then diagonalizes the interaction with respect to a basis of single quasi-protons and a 2^+ quasi-Boson. The quasi-Boson is the result of coherent contributions from all pertinent two quasi-neutron states. This establishes the inherent collective nature. By creating the quasi-Boson into a vacuum that has provisions for containing not only 0, but 4,8,12.......quasi-neutrons, one is in effect including the contributions of states of more than two quasi-particles, i.e., the quasi-Boson and its associated vacuum are a complete diagonalization of the long-range Hamiltonian with respect to a basis consisting of all allowable numbers of quasi-neutrons.

If one were to perform a two quasi-neutron diagonalization of the long-range Hamiltonian in Ni^{62}, one would get too high an energy for the 2^+ state. The extra "collectiveness" and subsequent lowering of the energy comes about only by including more quasi-neutrons. In coupling an odd quasi-proton to the possible two quasi-neutron states, one is essentially first solving the Ni^{62} problem. This is manifested in the interaction term, $\mathcal{H}^{(2)}_{NN\,(22)}$ (Class 1) (II 18). The $\mathcal{H}^{(2)}_{NN\,(40)}$ term (c.f. (II 16)) gives vanishing matrix elements, but this vanishing is due to the use of an incomplete basis, and really amounts to dropping $\mathcal{H}^{(2)}_{NN\,(40)}$. The separation of the long-range Hamiltonian into neutron-neutron and neutron-proton parts thus leads to the diagonalization of the latter, subject to the erroneous basis obtained by diagonalization of the former.

The last column of Table IX contains the values of matrix elements that result because one considers only Fermi commutation rules for the individual quasi-neutrons, rather than Boson commutation rules for their aggregate. These energies are all positive and raise the reference level to which one refers when coupling the odd quasi-proton. In fact, the energies are of the order of 1 mev, just the order of magnitude by which the levels are off in the two quasi-neutrons plus quasi proton scheme. The association of the "quartet" from one scheme with that of the other thus becomes more reasonable.

It is important that only the energies are affected. The B(E2)'s of the "quartet" in the two quasi-neutrons plus quasi-proton scheme, while somewhat smaller than those of the "collective" levels in the quasi-Boson plus quasi-proton scheme, are still the only prominent ones. However, the fact that one is dealing with such large matrices, yet still has only four levels that may be called "collective" is interesting and perhaps somewhat surprising, and indicates the importance of coherence among the different configurations that make up a level. Indeed, the very illustration of the origin of "collectiveness" via the coherent superposition of the eigenfunction components serves to illustrate what is glossed over in the quasi-Boson scheme. The existence, e.g., of a $5/2^-$ level at 2.04 mev as well as at 2.05 mev with only the latter being "collective" (c.f. Table XI) is an illustration of the coherence effect. Hence, a state of many components (due to an abundance of configuration mixing) may not be readily identifiable as collective from its eigenfunction, and even a large "core" B(E2) may be annulled by the sum of other contributions.

The many clusters of 4 levels (two quasi-neutrons plus quasi-proton scheme, Fig. 1c) come from the removal by $H^{(2)}_{N,P}$ (22) of the degeneracy in total angular momentum, J, for each three quasi-particle state. The effect of $H^{(2)}_{N,P}$ (31) is essentially to shift the clusters via the interaction of levels of the same J. The closeness (within about.2 mev) of the "collective" levels is due to the fact that while there are more off diagonal elements of $H^{(2)}_{N,P}$ (31) than in the quasi-Boson plus quasi-proton scheme, such elements are smaller than those in the latter scheme. Comparison of Tables IIIa and VII shows that typical matrix elements of $H^{(2)}_{N,P}$ (31) in the two quasi-neutrons plus quasi-proton scheme are about 1/10 the size of typical diagonal elements. Whereas in the quasi-Boson plus quasi-proton scheme the factor is about 1/3. It is not unexpected, then, that the spread of the "collective" levels in the quasi-Boson plus quasi-proton scheme (about .75 mev) is larger.

It is interesting to see just how "collective" the $3/2^-$ excited state is in the two quasi-neutrons plus quasi-proton scheme. Comparing Tables XIIa,b,c,d indicates that the $3/2^-$ "quartet" member has a B(E2) of 1.99 single particle units compared to 3.08, 2.43, and 2.56 for the $1/2^-$, $5/2^-$, and $7/2^-$ collective states respectively. Thus, while the $3/2^-$ state is not retarded as much as in the quasi-Boson plus quasi-proton scheme, the effect still remains. However, the cross sections for excitation by alpha particles (Table XV) do not indicate that the $3/2^-$ state has a reduction in its "collectiveness". This would imply that the electric quadrupole transition matrix elements are less sensitive to the structure of the nuclear eigenfunctions than are the scattering matrix elements. This is reasonable, since the wave numbers involved in the alpha scattering are many times larger than those in the electromagnetic decay, and hence take a more detailed look at the structure of the nucleus.

Let us next consider the existence of the critical force strength in Figures 3 and 4. Above this strength "collective" levels cease to appear and the ground state ceases to be of spin $3/2^-$. This could be due to the fact that a large interaction strength, and hence a strong attraction amongst Fermions, enhances the Boson description of Ni^{62}. The manifestation of this would be a lowering of the Ni^{62} 2^+ quasi-Boson energy. In that event the two quasi-neutrons plus quasi-proton scheme would become entirely inadequate, i.e., not only eigenvalues but also eigenfunctions would start to deviate strongly. This means that higher quasi-particle configurations would need to be included in order to recapture the "collective" essence of the Boson approximation.

If the interaction strength were to be increased even more, however, one would expect the entire quasi-Boson description itself to fail. For as the ground and 2^+ states get closer, the average number of quasi-particles in the former increases, and this enhances the Fermion nature of all available particles, i.e., the "Fermions begin to get in each others way".[27)]

3. Two Quasi-Neutrons (Coupled to 3^- only) Plus Quasi-Proton Scheme:

The positive parity states of Cu^{63} can only be obtained in this scheme. Removal of the degeneracy in total angular momentum, J, is brought about in a quadrupole interaction by means of $H^{(2)}_{N,P}$ (II21). The other degeneracy removing term, $^{(22)}H^{(2)}_{N,P}{}_{(31)}$ (II22), cannot contribute because of the requirement that the single quasi-proton and that coupled to the 3^- state be of the same parity.

The use of only the 3^- intermediate state is done by analogy to the 2^+ intermediate state's being the most important contributor to the negative parity states. One would certainly expect the 3^- two quasi-neutron state to be the most significant because of the B(E3) that such a state would produce, i.e., a "core" type of transition (and hence a connection to the usual idea of a "collective" contribution) can only be realized in a 3^- to 0^+ transition involving the quasi-neutrons.

It is interesting to note that experimentally there are more than 4 levels of large cross section which are in phase with each other and the elastic scattering.[10) This is just what our model gives (c.f. Figures 9 and 12).

While the 3^- two quasi-neutron angular momentum does agree with that of the octupole vibration state of Ni^{62}, the $(9/2^+, 3/2^-)3^-$ and $(9/2^+, 5/2^-)3^-$ states while in the right energy region are not at the same energy as either $\omega^{(3)}$ or each other. This would seem to imply collective properties of Cu^{63} which are intrinsic to it, and not directly connected with the known 3^- energy state of Ni^{62}.

There exists a "collective quartet" of states associated with each of the ways of getting a 3^- two quasi-neutron state, i.e., $(9/2^+, 3/2^-)3^-$ and $(9/2^+, 5/2^-)3^-$. From Table XIIIb, one sees that the two different 3^- states are shifted by approximately equal amounts (-2.44 mev for the former and -2.57 mev for the latter) due to $\mathcal{H}^{(2)}_{\mathcal{N}\mathcal{N}\,(22)}$ (Classes (1 and 2)). so that their energy separation remains due primarily to the difference in energy (0.28 mev) between the $3/2^-$ and $5/2^-$ quasi-neutrons. The interaction term, $H^{(2)}_{\mathcal{N},P\,(22)}$ (II21), breaks each 3^- state up into the various components of total angular momentum.

The matrices that must be diagonalized in the 3^- situation are much smaller than those of the 2^+ case. This is directly due to the fact that all but one of the single particle states available in the pertinent section of the shell model are of the same parity. In the 28-50 particle region, the $3/2^-$, $5/2^-$, $1/2^-$, and $9/2^+$ single particle states are present. The 3^- two quasi-neutron states can only be obtained provided one of the two quasi-neutrons is the $9/2^+$, and this immediately imposes a restriction that was not present in the 2^+ two quasi-neutron states.

Because of the small number of components of each eigenfunction, one may readily discover which component is the prime contributor to "collectivism". From Table XIV, the couplings $(9/2^+, 3/2^-)3^-, 3/2^-$ and $(9/2^+, 5/2^-)3^-, 3/2^-$ are clearly the most important. This is reasonable in that a B(E3) from the excited states to the $3/2^-$ ground state is of pure "core" nature when the odd quasi-proton remains in the same state. Each of the "non-collective" states in the present case is characterized by negligible $(9/2^+, 3/2^-)3^-, 3/2^-$ and $(9/2^+, 5/2^-)3^-, 3/2^-$ components.

Let us also compare the B(E3)'s, inelastic alpha scattering cross sections, and form factors for the two "quartets". From Table XIV the "quartet" of primarily $(9/2^+, 3/2^-)3^-$ origin have B(E3)'s that are a factor of 4 or 5 greater than those of the "quartet" of $(9/2^+, 5/2^-)3^-$ origin. A similar situation holds for the Born approximation curves of Figure 8. In the DWBA curves of Figure 12 the factor is seen to be about 8. The reason for this is primarily the difference between the reduced matrix elements, $\mathcal{J}^{(3)}(9/2^+, 3/2^-)$ and $\mathcal{J}^{(3)}(9/2^+, 5/2^-)$ (Table Ib). The former is larger by a factor of 3 because of the reduced matrix element of the spherical harmonic, i.e., in the one case one has $\langle g_{9/2} \| Y_3 \| p_{3/2} \rangle$ (no spin flip), and in the other $\langle g_{9/2} \| Y_3 \| f_{5/2} \rangle$ (spin flip) with the latter about a factor of 3 smaller than the former.

As is evidenced by Figure 10, the two "quartets" both obey the parity phase rule. The form factors (Figure 14) for the "quartet" of 2^+ origin are all peaked at a smaller radius than the levels of 3^- origin. This is due to the overwhelming importance of the $9/2^+$ single particle wavefunction in the latter case, i.e., the radial part of the $1g_{9/2^+}$ wavefunction is peaked at a larger radius than that of any of the other single particle wavefunctions. Also evident from Figure 14 is the peaking at different radii of the form factors for the members of the two different 3^- "quartets". In particular, the $(9/2^+, 3/2^-)3^-$ "quartet" yields peaks at a larger radius than the $(9/2^+, 5/2^-)3^-$. This is again primarily due to the 1g radial wavefunction which is weighted much more strongly in the former than latter case. Opposing this effect somewhat is the latter's 1f radial wavefunction which peaks at a larger radius than the first peak of the 2p radial wavefunction.

The wide range of form factor curves for the 2^+ state (Figure 13) does not occur for the 3^- state (Figure 14) because of the smaller amount of configuration mixing in the eigenfunctions of 3^- origin, and the omnipresence of the $1g_{9/2^+}$ single particle state.

Possible Consequences and Extensions.

1. Nuclear "Surface":

The DWBA calculations show that one may indeed picture the collective excitations as surface phenomena insofar as the form factor curves (Figures 13 and 14) show that all of the states of large cross section have peaks near the "surface" of the nucleus. For all of these "collective" states the dominant contributions to the cross sections come from partial waves of order near $L_0 = 20 \simeq KR_0$, where K is the wave number of incoming alpha (at 50 mev) and $R_0 = 1.585x(63)^{1/3}$ Fermis. Because of the overlap of such incident and outgoing waves with these states, one obtains the characteristics of the inelastic alpha scattering differential cross sections for collective states.

2. Differences Between "Core" Plus Proton and "Core" Plus Neutron:

The difference in spectra of two odd mass nuclei, the one of "core" plus proton and the other of "core" plus neutron origin, may possibly be understood via the relative degree of validity of the Boson approximation in the two cases.

Consider, e.g., the case of Cu^{63} vs. Ni^{63}. As pointed out many times, the low lying negative parity states of

Cu^{63} are fairly well described in the Boson approximation. Experimentally via $Cu^{63}(p,p)Cu^{63*}$ [38] only 6 negative parity levels are found below 2 mev, whereas $Ni^{62}(d,p)Ni^{63*}$ predicts 12.[37] Although the experiments are of a different nature, it might well be that the larger number of low lying levels in Ni^{63} indicates that the Boson approximation is not good in that case. This would be reasonable in that as a "core" plus odd neutron situation, Ni^{63} would have an extra available quasi-neutron as compared to the case of Cu^{63} with "core" plus odd proton. The matrix elements of the long-range interaction in the two quasi-neutrons (coupled to 2^+ only) plus quasi-neutron or quasi-proton schemes would only differ in details of coupling. This would come about from the non-commutability amongst the three quasi-neutron annihilation and creation operators. The previous results for the case of Cu^{63} in this scheme would then be applicable to Ni^{63}, i.e., there would be many low lying states and the "collective" states would be too high. Conceivably for Cu^{63} the "collective" states are lowered due to $\mathcal{H}_{NN}^{(2)}$ (40) (i.e., the consideration of wavefunction components of more than two quasi-neutrons), and this would serve as a verification of the applicability of the quasi-Boson scheme. For Ni^{63}, however, if the Boson approximation is not good, then higher numbers of quasi-neutrons would not lower the "collective" states. From Tables XIIa,b,c,d there are about 40 levels (for all spins) predicted below 2 mev, but Figure 1c illustrates that many of these levels are nearly degenerate. Thus, the 12 measured levels in Ni^{63} [37] below 2 mev as opposed to 6 in Cu^{63} [38] seem to support the preceding interpretation.

For an indication of the validity of the Boson approximation in the presence of an odd particle, one calculates

$$\langle \tilde{o}_P, \tilde{o}_N | (-1)^{-j_{5_N}-m_{5_N}} \gamma^{j_5}_{-m_5}(\eta) \left[C^2(j_{1_N} j_{2_N}), B^{+2}(j_{3_N} j_{4_N}) \right]^0_0 \times \beta^{+j_5}_{m_5}(\eta) | \tilde{o}_P, \tilde{o}_N \rangle \qquad \text{(V1)}$$

where η = N or P. From (I41) there are single quasi-neutron scattering terms that are dropped in the Boson approximation. Such terms still contribute nothing to (V1) if η = p because of the rules for the commutation of operators representing different types of Fermions (I21'). For η = N, however, the scattering terms may indeed couple with the odd quasi-particle. Such couplings are proportional to $\frac{1}{2j_{5_N}+1}$ which is to be expected from the Pauli principle, i.e., for small j_{5_N} there are less available m_{5_N} states for quasi-neutrons, and consequently more likelihood of their Fermion nature being important. Provided that j_{5_N} is an angular momentum present in a scattering term, the element (V3) can change value by anywhere from 17% ($j_{5_N} = 5/2^-$) to 50% ($j_{5_N} = 1/2^-$) which are non-negligible amounts.

3. Dependence on Shell Model Region:

The results of this calculation would seem to indicate that the systematics of an odd mass nucleus can be rather dependent on its relative place in the periodic table in the following senses:

(i) The pertinent single particle wavefunctions determine the degree of overlap involved in single particle matrix elements.

(ii) These same wavefunctions also determine the relative signs of such matrix elements, and consequently the amount of coherence.

(iii) The possible values of the total angular momentum, J, of an energy level, and the ways that one may couple to get each J are determined by the available single particle spins and parities.

(iv) Implicit in (iii) are the sizes of the matrices to be diagonalized, and hence the number of energy levels produced and the degree of configuration mixing involved in their eigenfunctions.

(v) The (u, v) coefficients associated with the single quasi-particle states are important in determining the relative magnitudes of matrix elements, i.e., these coefficients help determine which long-range Hamiltonian terms are important.

4. Nuclei Away From Closed Shells:

A calculation for the problem of a non-deformed odd mass nucleus away from closed shells would present greater difficulty. The solution would depend on knowing the origin of the 2^+ state in the adjacent even-even nucleus (i.e., whether that state is due to quasi-neutrons or quasi-protons). In addition, the makeup of the 3^- intermediate states in the odd mass nucleus itself must be known.

5. Summary:

In summary, the general conclusion reached in this investigation is that when applicable, the quasi-Boson plus quasi-particle scheme is much better than the three quasi-particle approach in describing the low lying states of odd mass nuclei.

The two quasi-neutrons (coupled to 2^+) plus quasi-proton scheme is unable to give the correct energies of "collective" levels (these are too high and too close, c.f. Figure 1c). Even though the B(E2)'s come out enhanced, and the inelastic alpha cross sections are large (and have the right shape vs. angle), one gets larger B(E2)'s and cross sections in the quasi-Boson plus quasi-proton scheme (and the same cross section shape vs. angle). The indication is that to obtain the optimum "collectivism", to lower the energies of the "collective" levels, and to disperse them, the quasi-Boson plus quasi-proton scheme is necessary.

In order to obtain comparable results in a scheme just using an assemblage of interacting quasi-particles one would have to consider configurations of more than three quasi-particles. However, for the case of the 3^- intermediate angular momentum state involved in making up the positive parity levels, it is unlikely that higher numbers of quasi-particles will produce a much better answer. This follows from the fact that there are no large terms analogous to the ones that raise the levels in the case of a 2^+ intermediate angular momentum (c.f. Tables XIIIb and IX), i.e., for the 2^+ case these terms are the ones that must be dropped to make the Boson approximation. The reason why these terms (II34c) are small for the 3^- case is that the number of ways of forming odd parity intermediate states is limited by the number of single particle states of the correct parity available to the quasi-neutrons in the region of mass number around A=63.

APPENDIX A

Orthonormality

The following expressions are written for the case of quasi-neutrons as the identical particles. The odd particle is considered as a quasi-proton. The results are also valid for quasi-protons as the identical particles and an odd quasi-neutron. Simply make the pertinent changes in wording and notation.

1. Two Quasi-Neutrons:

Let j_{a_N} and j_{b_N} be the angular momenta of the two quasi-neutrons, J_0 the total angular momentum to which they are coupled, and M_0 the z component of J_0. The two quasi-neutron state is written as

$$A_{(j_{a_N} j_{b_N})J_0} \left[\beta^{+\,j_a}(N)\,\beta^{+\,j_b}(N)\right]^{J_0}_{M_0} |\tilde{0}_N\rangle \tag{A1.1}$$

where $A_{(j_{a_N} j_{b_N})J_0}$ is the normalization, $|\tilde{0}_N\rangle$ the quasi-neutron vacuum, and $\beta^+(N)$ the creation operator of a quasi-neutron. To show that the set (A1.1) is orthogonal, compute

$$\langle \tilde{0}_N | [\beta^{+j_a'}(N)\, \beta^{+j_b'}(N)]^{+J_0'}_{M_0'} [\beta^{+j_a}(N)\beta^{+j_b}(N)]^{J_0}_{M_0} |\tilde{0}_N\rangle \tag{A1.2}$$

with j_{a_N}', j_{b_N}', J_0', and M_0' representing another set of quantum numbers. Equation (A1.2) may be written using (I20'c) and (I4) as

$$\sum_{m_{a_N}' m_{b_N}'} \sum_{m_{a_N} m_{b_N}} \langle j_{a_N}' m_{a_N}' j_{b_N}' m_{b_N}' | J_0' M_0' \rangle \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | J_0 M_0 \rangle$$

$$\times (-1)^{-j_{a_N}' - m_{a_N}' - j_{b_N}' - m_{b_N}'} \underbrace{\langle \tilde{0}_N | \gamma^{j_b'}_{-m_b'}(N)\, \gamma^{j_a'}_{-m_a'}(N)\, \beta^{+j_a}_{m_a}(N)\, \beta^{+j_b}_{m_b}(N) | \tilde{0}_N \rangle} \tag{A1.3}$$

The idea is to get all $\gamma^{j'}_{m'}(N)$ operators to the right of all $\beta^{+j}_{m}(N)$ operators and then utilize (I14). For ease in notation define terms like $\delta_{a_N' a_N}$ and $\delta_{a_N', -a_N}$ via

$$\delta_{a_N' a_N} \equiv \delta_{j_{a_N}' j_{a_N}} \delta_{m_{a_N}' m_{a_N}} \tag{A1.4a}$$

$$\delta_{a_N', -a_N} \equiv \delta_{j_{a_N}' j_{a_N}} \delta_{m_{a_N}', -m_{a_N}} \tag{A1.4b}$$

Continuous use of (I20'c) yields

$$\underbrace{\qquad\qquad} = \langle \tilde{0}_N | \gamma^{j_b'}_{-m_b'}(N)\, \beta^{+j_b}_{m_b}(N) | \tilde{0}_N \rangle \delta_{a_N a_N'} (-1)^{j_{a_N} + m_{a_N}}$$

$$- \langle \tilde{0}_N | \gamma^{j_b'}_{-m_b'}(N)\, \beta^{+j_a}_{m_a}(N) | \tilde{0}_N \rangle \delta_{b_N a_N'} (-1)^{j_{b_N} + m_{b_N}}$$

$$= (-1)^{j_{b_\nu}+m_{b_\nu}+j_{a_\nu}+m_{a_\nu}} \left(\delta_{b_\nu b_\nu'}\delta_{a_\nu a_\nu'} - \delta_{a_\nu b_\nu'}\delta_{b_\nu a_\nu'}\right)$$

$$\equiv f_{ab}+g_{ab} \qquad \text{(A1.5)}$$

with $f_{ab} \equiv (-1)^{j_{b_\nu}+m_{b_\nu}+j_{a_\nu}+m_{a_\nu}} \delta_{b_\nu b_\nu'}\delta_{a_\nu a_\nu'}$ (A1.6a)

and $g_{ab} \equiv -(-1)^{j_{b_\nu}+m_{b_\nu}+j_{a_\nu}+m_{a_\nu}} \delta_{a_\nu b_\nu'}\delta_{b_\nu a_\nu'}$ (A1.6b)

Insertion of (A1.6) into (A1.3) gives

$$\sum_{m_{a_\nu}' m_{b_\nu}'} \sum_{m_{a_\nu} m_{b_\nu}} \langle j_{a_\nu}' m_{a_\nu}' j_{b_\nu}' m_{b_\nu}' | J_0' M_0' \rangle \langle j_{a_\nu} m_{a_\nu} j_{b_\nu} m_{b_\nu} | J_0 M_0 \rangle$$

$$\times (-1)^{-j_{a_\nu}'-m_{a_\nu}'-j_{b_\nu}'-m_{b_\nu}'} (f_{ab}+g_{ab})$$

$$= \sum_{m_{a_\nu} m_{b_\nu}} \langle j_{a_\nu} m_{a_\nu} j_{b_\nu} m_{b_\nu} | J_0' M_0' \rangle \langle j_{a_\nu} m_{a_\nu} j_{b_\nu} m_{b_\nu} | J_0 M_0 \rangle$$

$$\times \left(\delta_{j_{b_\nu} j_{b_\nu}'} \delta_{j_{a_\nu} j_{a_\nu}'} - (-1)^{j_{a_\nu}+j_{b_\nu}+J_0'} \delta_{j_{a_\nu} j_{b_\nu}'} \delta_{j_{b_\nu} j_{a_\nu}'}\right)$$

(A1.7)

where use has been made of

$$\langle j_{a_\nu}' m_{a_\nu}' j_{b_\nu}' m_{b_\nu}' | J_0' M_0' \rangle = (-1)^{j_{a_\nu}'+j_{b_\nu}'+J_0'}$$

$$\times \langle j_{b_\nu}' m_{b_\nu}' j_{a_\nu}' m_{a_\nu}' | J_0' M_0' \rangle \qquad \text{(A1.8)}$$

with J_0' an integer.

The orthonormality of Clebsch-Gordon coefficients requires that (A1.7) becomes

$$\delta_{J_0 J_0'}\delta_{M_0 M_0'}(\delta_{j_{a_N} j_{a_N}'}\delta_{j_{b_N} j_{b_N}'} - (-1)^{j_{a_N}+j_{b_N}+J_0}\delta_{j_{a_N} j_{b_N}'}\delta_{j_{b_N} j_{a_N}'}$$

Thus, from (A1.2) we have the orthogonality relation

$$\langle \tilde{0}_N | [\beta^{+j_a'}(N)\beta^{+j_b'}(N)]^{+J_0'}_{M_0'} [\beta^{+j_a}(N)\beta^{+j_b}(N)]^{J_0}_{M_0} | \tilde{0} \rangle$$

$$= \delta_{J_0 J_0'}\delta_{M_0 M_0'}(\delta_{j_{a_N} j_{a_N}'}\delta_{j_{b_N} j_{b_N}'} - (-1)^{j_{a_N}+j_{b_N}+J_0}$$

$$\times \delta_{j_{a_N} j_{b_N}'}\delta_{j_{b_N} j_{a_N}'}) \qquad \text{(A1.9)}$$

To obtain the normalization constant, $A_{(j_{a_N} j_{b_N})J_0}$, let $j_{a_N}' = j_{a_N}$, $j_{b_N}' = j_{b_N}$, $J_0' = J_0$, and $M_0' = M_0$ in (A1.9). The right hand side of (A1.9) then becomes $1 - (-1)^{j_{a_N}+j_{b_N}+J_0}\delta_{j_{a_N} j_{b_N}}$. If $j_{a_N} = j_{b_N}$, then J_0 is even, and the right hand side of (A1.9) is just $1 + 1$.

Hence

$$|A_{(j_{a_N} j_{b_N})J_0}|^2 \langle \tilde{0} | [\beta^{+j_a}(N)\beta^{+j_b}(N)]^{J^+}_{M_0}$$

$$\times [\beta^{+j_a}(N)\beta^{+j_b}(N)]^{J_0}_{M_0} | \tilde{0}_N \rangle = |A_{(j_{a_N} j_{b_N})J_0}|^2 (1+\delta_{j_{a_N} j_{b_N}}) \qquad \text{(A1.10)}$$

For the right hand side of (A1.10) to be equa. .o 1,

$$A_{(j_{a_N} j_{b_N})J_0} = \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \qquad \text{(A1.11)}$$

From (A1.1), (A1.9), and (A1.11) the set of orthonormal states of two quasi-neutrons is then

$$\frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \left[\beta^{+j_a}(N)\, \beta^{+j_b}(N)\right]^{J_0}_{M_0} |\tilde{0}_N\rangle \tag{A1.12}$$

2. Two Quasi-Neutrons Plus One Quasi-Proton:

Consider a quasi-proton of angular momentum, j_{c_p}, coupled to the two quasi-neutron state, $(j_{a_N} j_{b_N})J_0$, of Appendix A1 to give a total angular momentum, J, and z component, M. The orthonormal set of states is just

$$\frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \left(\left[\beta^{+j_a}(N)\, \beta^{+j_b}(N)\right]^{J_0} \beta^{+j_c}(P)\right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_P\rangle \tag{A2.1}$$

This may be readily shown. Quasi-proton and quasi-neutron operators (annihilation or creation) commute; thus, equation (A2.1) may be written as

$$\sum_{M_0 m_{c_p}} \langle J_0 M_0 j_{c_p} m_{c_p} | J M\rangle \left(\frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \left[\beta^{+j_a}(N)\, \beta^{+j_b}(N)\right]^{J_0}_{M_0} |\tilde{0}_N\rangle\right) \times \left(\beta^{+j_c}_{m_c}(P)\, |\tilde{0}_P\rangle\right) \tag{A2.2}$$

where $\frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} \left[\beta^{+j_a}(N)\, \beta^{+j_b}(N)\right]^{J_0}_{M_0} |\tilde{0}_N\rangle$ is itself an orthonormal set of two quasi-neutron states (A1.12). With the notation of (A1.4)

$$\langle \tilde{0}_P | \beta^{j_c'}_{m_c'}(P)\, \beta^{+j_c}_{m_c}(P) | \tilde{0}_P\rangle = \delta_{c_p c_p'} \tag{A2.3}$$

Then, using (A2.2), (A1.9), (A2.3), and the orthonormality of Clebsch-Gordon coefficients, the orthogonality of the set (A2.1) is verified via

$$\frac{1}{\sqrt{1+\delta_{j_{a_N}' j_{b_N}'}}}\;\frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}}\;\langle \tilde{0}_P ; \tilde{0}_N | \left([\beta^{+}{}_{j_a'(N)}\, \beta^{+}{}_{j_b'(N)}]^{J_0}\, \beta^{+}{}_{j_c'(P)} \right)^{+J'}_{M'}$$

$$\times \left([\beta^{+}{}_{j_a(N)}\, \beta^{+}{}_{j_b(N)}]^{J_0}\, \beta^{+}{}_{j_c(P)} \right)^{J}_{M} | \tilde{0}_N ; \tilde{0}_P \rangle = \delta_{JJ'}\,\delta_{MM'}\,\frac{1}{\sqrt{(1+\delta_{j_{a_N}' j_{b_N}'}}}$$

$$\times \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}}\;\delta_{J_0 J_0'} \left(\delta_{j_{a_N} j_{a_N}'}\,\delta_{j_{b_N} j_{b_N}'} - (-1)^{j_{a_N}+j_{b_N}+J_0}\,\delta_{j_{a_N} j_{b_N}'} \right.$$

$$\left. \times\, \delta_{j_{b_N} j_{a_N}'} \right) \delta_{j_{c_P} j_{c_P}'} \qquad \text{(A2.4)}$$

For $J_0 = J_0'$, $j_{a_N} = j_{a_N}'$, $j_{b_N} = j_{b_N}'$, $j_{c_P} = j_{c_P}'$, $J = J'$, and M=M', the right hand side of (A2.4) becomes

$$\frac{1}{1+\delta_{j_{a_N} j_{b_N}}} \times \left(1+\delta_{j_{a_N} j_{b_N}}\right) = 1 \qquad \text{(A2.5)}$$

verifying the normality of the set (A2.1).

3. Three Quasi-Neutrons:

Consider the three quasi-neutron state

$$\left([\beta^{+}{}_{j_a}\, \beta^{+}{}_{j_b}]^{J_0}\, \beta^{+}{}_{j_c} \right)^{J}_{M} | \tilde{0}_N \rangle \qquad \text{(A3.1)}$$

Since there are only quasi-neutrons, the N notation is omitted for simplicity. The three quasi-neutrons have angular momenta, j_a, j_b, and j_c. Two of these, j_a and j_b, couple to give J_0, which is in turn coupled to j_c to yield a total angular momentum, J, of z component, M. To see if this is an orthogonal set calculate

$$\langle \tilde{0} | \left([\beta^{+j_a'} \beta^{+j_b'}]^{J_0'} \beta^{+j_c'} \right)^{+J'}_{M'} \left([\beta^{+j_a} \beta^{+j_b}]^{J_0} \beta^{+j_c} \right)^{J}_{M} | \tilde{0} \rangle$$

$$= \sum_{\substack{M_0 m_c \\ m_a m_b}} \langle J_0 M_0 j_c m_c | J M \rangle \langle j_a m_a j_b m_b | J_0 M_0 \rangle \sum_{\substack{M_0' m_c' \\ m_a' m_b'}} \langle J_0' M_0' j_c' m_c' | J' M' \rangle$$

$$\times \langle j_a' m_a' j_b' m_b' | J_0' M_0' \rangle (-1)^{-j_c' - m_c' - j_b' - m_b' - j_a' - m_a'}$$

$$\times \left(\quad \right) \qquad \text{(A3.2)}$$

with use made of (I4), and with $\left(\quad \right)$ defined via

$$\left(\quad \right) \equiv \langle \tilde{0} | \gamma^{j_c'}_{-m_c'} \gamma^{j_b'}_{-m_b'} \gamma^{j_a'}_{-m_a'} \beta^{+j_a}_{m_a} \beta^{+j_b}_{m_b} \times \beta^{+j_c}_{m_c} | \tilde{0} \rangle \qquad \text{(A3.3)}$$

If continual use is made of (I20'c) and (I 14), then in the notation of A1.4

$$\left(\quad \right) = (-1)^{j_a + m_a + j_b + m_b + j_c + m_c} \Big[(\delta_{cc'} \delta_{bb'} - \delta_{bc'} \delta_{cb'}) \delta_{aa'} - (\delta_{cc'} \delta_{ab'} - \delta_{ac'} \delta_{cb'}) \delta_{ba'} + (\delta_{bc'} \delta_{ab'} - \delta_{ac'} \delta_{bb'}) \delta_{ca'} \Big] \qquad \text{(A3.4)}$$

The right hand side of (A3.2) becomes

$$\sum_{\substack{M_0 m_c \\ m_a m_b}} \langle J_0 M_0 j_c m_c | J M \rangle \langle j_a m_a j_b m_b | J_0 M_0 \rangle \sum_{\substack{M_0' m_c' \\ m_a' m_b'}} \langle j_a' m_a' j_b' m_b' | J_0' M_0' \rangle$$

$$\times \langle J_0' M_0' j_c' m_c' | J' M' \rangle \times ((1)+(2)+(3)+(4)+(5)+(6)) \qquad (A3.5)$$

with

$$(1) = \delta_{cc'} \delta_{bb'} \delta_{aa'} \qquad (A3.6a)$$

$$(2) = -\delta_{bc'} \delta_{cb'} \delta_{aa'} \qquad (A3.6b)$$

$$(3) = -\delta_{cc'} \delta_{ab'} \delta_{ba'} \qquad (A3.6c)$$

$$(4) = \delta_{ac'} \delta_{cb'} \delta_{ba'} \qquad (A3.6d)$$

$$(5) = \delta_{bc'} \delta_{ab'} \delta_{ca'} \qquad (A3.6e)$$

$$(6) = -\delta_{ac'} \delta_{bb'} \delta_{ca'} \qquad (A3.6f)$$

Terms (1) and (3) involve no coupling of j_a or j_b to j_c, and are just the non-normalized analogs of (A2.4), i.e.,

$$\delta_{J_0 J_0'} \delta_{j_c j_c'} \left(\delta_{j_a j_a'} \delta_{j_b j_b'} - (-1)^{j_a + j_b + J_0} \times \delta_{j_a j_b'} \delta_{j_b j_a'} \right) \qquad (A3.7)$$

The remaining terms are more complicated and involve six-j symbols. The evaluation of one typical term will suffice to indicate how all of these terms are computed. Equations (A3.6b) and (A3.5) give for term (2)

$$\sum_{\substack{M_0 m_c \\ M_0'}} \sum_{m_a m_b} \langle J_0 M_0 j_c m_c | J M \rangle \langle j_a m_a j_b m_b | J_0 M_0 \rangle$$

$$\times \langle J_0' M_0' j_b m_b | J' M' \rangle \langle j_a m_a j_c m_c | J_0' M_0' \rangle \qquad (A3.8)$$

$$\times \delta_{j_b j_c'} \delta_{j_c j_b'} \delta_{j_a j_a'}$$

Since M_0, m_c, m_a, m_b, and M_0' are not all independent, (A3.8) may be written as

$$\sum_{m_b m_c} \langle J_0\, M-m_c\, j_c\, m_c | J M\rangle \langle j_a\, M-m_c-m_b\, j_b\, m_b | J_0\, M-m_c\rangle$$

$$\times \langle J_0'\, M-m_b\, j_b\, m_b | J' M'\rangle \langle j_a\, M-m_c-m_b\, j_c\, m_c | J_0'\, M-m_b\rangle$$

$$\times \delta_{j_b j_c'}\, \delta_{j_c j_b'}\, \delta_{j_a j_a'} \qquad \text{(A3.9)}$$

The calculation is simplified if three-j symbols are used. The three-j symbol is defined as [29]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-1/2}\langle j_1 m_1\, j_2 m_2 | j_3\, -m_3\rangle$$

$$m_1+m_2+m_3=0$$

$$\text{(A3.10)}$$

An even permutation of the columns of a 3-j symbol doesn't change its value, but an odd permutation does. From Reference 29)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

$$\text{(A3.11a)}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \qquad \text{(A3.11b)}$$

The use of (A3.10), (A3.11), and a simplification in the exponent of (-1) gives for (A3.9)

$$\sqrt{(2J_0+1)(2J_0'+1)}\sum_{m_c}(-1)^{j_c+j_b+J_0+J_0'}\begin{pmatrix} J_0 & j_c & J \\ M-m_c & m_c & -M \end{pmatrix}$$

$$\times \sqrt{(2J+1)(2J'+1)} \sum_{m_b} (-1)^{j_a+j_b+J_0'+m_c+m_b}$$

$$\times \begin{pmatrix} j_a & j_b & J_0 \\ M-m_c-m_b & m_b & -M+m_c \end{pmatrix} \begin{pmatrix} J' & j_b & J_0' \\ M' & -m_b & -M+m_b \end{pmatrix}$$

$$\times \begin{pmatrix} j_a & j_c & J_0' \\ -M+m_c+m_b & -m_c & M-m_b \end{pmatrix} \delta_{j_b j_c'} \delta_{j_c j_b'} \delta_{j_a j_a'} \qquad \text{(A3.12)}$$

This is simplified by the relationship from Reference 29)

$$\sum_{M_1 M_2 M_3} (-1)^{l_1+l_2+l_3+M_1+M_2+M_3} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & M_2 & -M_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -M_1 & m_2 & M_3 \end{pmatrix}$$

$$\times \begin{pmatrix} l_1 & l_2 & j_3 \\ M_1 & -M_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$

(A3.13)

where $\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$ is a 6-j symbol. Using (A3.13), (A3.12) then becomes

$$\sqrt{(2J+1)(2J'+1)(2J_0+1)(2J_0'+1)}\,(-1)^{j_b+j_c+J_0+J_0'} \sum_{m_c} \begin{pmatrix} J_0 & j_c & J \\ M-m_c & m_c & -M \end{pmatrix}$$

$$\times \begin{pmatrix} J' & j_c & J_0 \\ M' & -m_c & -M+m_c \end{pmatrix} \begin{Bmatrix} J & j_c & J_0 \\ j_a & j_b & J_0' \end{Bmatrix} \delta_{j_b j_c'} \delta_{j_c j_b'} \delta_{j_a j_a'} \qquad \text{(A3.14)}$$

The orthonormality condition for 3-j symbols is[29]

$$\sum_{m_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & -m_3-m_1 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & -m_3'-m_1 & m_3' \end{pmatrix} = (2j_3+1)^{-1} \delta_{j_3 j_3'} \delta_{m_3 m}$$

(A3.15)

Finally, (A3.11) and (A3.15) simplify (A3.14) to give the contribution due to term (2)

$$- \sqrt{(2J_0+1)(2J_0'+1)}\,(-1)^{j_b+j_c+J_0+J_0'} \begin{Bmatrix} J & j_c & J_0 \\ j_a & j_b & J_0' \end{Bmatrix}$$

$$\times\, \delta_{j_b j_c'} \delta_{j_c j_b'} \delta_{j_a j_a'} \delta_{JJ'} \delta_{MM'} \qquad \text{(A3.16)}$$

The contribution of terms (6) and (4) may be lumped together as

$$\sqrt{(2J_0'+1)(2J_0+1)} \begin{Bmatrix} J & j_c & J_0 \\ j_b & j_a & J_0' \end{Bmatrix} \delta_{j_a j_c'} \Big(\delta_{j_b j_b'} \delta_{j_c j_a'}$$

$$- (-1)^{j_a'+j_b'+J_0'} \delta_{j_c j_b'} \delta_{j_b j_a'} \Big) \delta_{JJ'} \delta_{MM'} \qquad \text{(A3.17)}$$

and that due to (2) and (5) as

$$-(-1)^{j_a+j_b+J_0} \delta_{JJ'} \delta_{MM'} \sqrt{(2J_0'+1)(2J_0+1)} \begin{Bmatrix} J & j_c & J_0 \\ j_a & j_b & J_0' \end{Bmatrix} \delta_{j_b j_c'}$$

$$\times \Big(\delta_{j_a j_b'} \delta_{j_c j_a'} - (-1)^{j_a'+j_b'+J_0'} \delta_{j_c j_b'} \delta_{j_a j_a'} \Big) \qquad \text{(A3.18)}$$

Equation (A3.17) is due to the coupling of j_b with the odd particle's angular momentum, j_c , while (A3.18) represents the coupling of j_a with j_c. As a check one sees that (A3.18) is just $-(-1)^{j_a+j_b+J_0}$ times (A3.17) provided j_a and j_b are interchanged. The minus sign comes from the anti-commutation rule (I20') for operators, and $(-1)^{j_a+j_b+J_0}$ from the interchange of a and b in the Clebsch-Gordon coefficient, $\langle j_a m_a j_b m_b | J_0 M_0 \rangle$ (A1.8) Hence, combining (A3.7), (A3.17), and (A3.18), one gets for (A3.2)

$$\langle \hat{0} | \left([\beta^{+}{}_{j_a'} \beta^{+}{}_{j_b'}]^{J_0'} \beta^{+}{}_{j_c'} \right)^{+J'}_{M'} \left([\beta^{+}{}_{j_a} \beta^{+}{}_{j_b}]^{J_0} \beta^{+}{}_{j_c} \right)^{J}_{M} | \hat{0} \rangle$$

$$= \delta_{J_0 J_0'} \delta_{j_c j_c'} \left(\delta_{j_a j_a'} \delta_{j_b j_b'} - (-1)^{j_a+j_b+J_0} \delta_{j_a j_b'} \delta_{j_b j_a'} \right)$$

$$\times \delta_{JJ'} \delta_{MM'}$$

$$+ \delta_{JJ'} \delta_{MM'} \sqrt{(2J_0'+1)(2J_0+1)} \left[\begin{Bmatrix} J & j_c & J_0 \\ j_b & j_a & J_0' \end{Bmatrix} \delta_{j_a j_c'} \right.$$

$$\times \left(\delta_{j_b j_b'} \delta_{j_c j_a'} - (-1)^{j_a'+j_b'+J_0'} \delta_{j_c j_b'} \delta_{j_b j_a'} \right)$$

$$- (-1)^{j_a+j_b+J_0} \begin{Bmatrix} J & j_c & J_0 \\ j_a & j_b & J_0' \end{Bmatrix} \delta_{j_b j_c'} \left(\delta_{j_a j_b'} \delta_{j_c j_a'} \right.$$

$$\left. \left. - (-1)^{j_a'+j_b'+J_0'} \delta_{j_c j_b'} \delta_{j_a j_a'} \right) \right] \qquad \text{(A3.18)}$$

This result says that the set of states (A3.1) is not orthogonal. This is due to the intermediate angular momentum, J_0 , and can easily be seen, e.g., by letting $j_a = j_a' = j_b = j_b' = j_c = j_c'$, J = J', and M = M'. Both J_0' and J_0 must be even and (A3.18) becomes

$$2\,\delta_{J_0 J_0'} + 4\sqrt{(2J_0'+1)(2J_0+1)}\begin{Bmatrix} J & j_a & J_0 \\ j_a & j_a & J_0' \end{Bmatrix} \tag{A3.19}$$

which does not necessarily vanish for $J_0 \neq J_0'$.

The normalization factor for (A3.1) is readily obtained by letting $j_a = j_a'$, $j_b = j_b'$, $j_c = j_c'$, $J_0 = J_0'$, $J = J'$, and M = M' in (A3.18) as

$$\Big[(1+\delta_{j_a j_b}) + (2J_0+1)\Big(\begin{Bmatrix} J & j_c & J_0 \\ j_a & j_c & J_0 \end{Bmatrix}\delta_{j_b j_c} + \begin{Bmatrix} J & j_c & J_0 \\ j_b & j_c & J_0 \end{Bmatrix}\delta_{j_a j_c} + 2\begin{Bmatrix} J & j_a & J_0 \\ j_a & j_a & J_0 \end{Bmatrix}\delta_{j_a j_b}\,\delta_{j_b j_c}\,\delta_{j_c j_a}\Big]^{-1/2} \tag{A3.20}$$

4. Quasi-Boson:

From equation (I47a), the creation operator for a quasi-Boson of angular momentum, k , and z component, m , is $\Gamma^k_m(n)$ where

$$\Gamma^k_m(n) = \sum_{j_{a_n} \geq j_{b_n}} \left\{ r(j_{a_n} j_{b_n})\, B^{+k}_m(j_{a_n} j_{b_n}) + s(j_{a_n} j_{b_n})\, C^k_m(j_{a_n} j_{b_n}) \right. \tag{I47a}$$

and the annihilation operator, $\Gamma^{+k}_m(n)$, is given by

$$\Gamma^{+k}_m(n) = (-1)^{k+m} \sum_{j_{a_n} \geq j_{b_n}} \left[r(j_{a_n} j_{b_n})\, C^k_{-m}(j_{a_n} j_{b_n}) + s(j_{a_n} j_{b_n})\, B^{+k}_{-m}(j_{a_n} j_{b_n}) \right. \tag{I47b}$$

The operators, $B^{+k}_{m}(j_{a_3} j_{b_3})$ and $C^{k}_{m}(j_{a_3} j_{b_3})$, are defined in (I38), and the sets, $r(j_{a_3} j_{b_3})$ and $s(j_{a_3} j_{b_3})$, are given by (I49).

To check for orthonormality of the set (I47a), look at

$$\langle \tilde{\hat{0}}_3 | \Gamma^{+k'}_{m'}(3) \; \Gamma^{k}_{m}(3) | \tilde{\hat{0}}_3 \rangle \qquad \text{(A4.1)}$$

where $|\tilde{\hat{0}}_3\rangle$ is the quasi-Boson vacuum, i.e.,

$$\Gamma^{+k'}_{m'}(3) \, | \tilde{\hat{0}}_3 \rangle = 0 \qquad \text{(A4.2)}$$

Equation (A4.1) may be written, then, as

$$\langle \tilde{\hat{0}}_3 | \Gamma^{+k'}_{m'}(3) \; \Gamma^{k}_{m}(3) | \tilde{\hat{0}}_3 \rangle = \langle \tilde{\hat{0}}_3 | [\Gamma^{+k'}_{m'}(3), \Gamma^{k}_{m}(3)] | \tilde{\hat{0}}_3 \rangle \qquad \text{(A4.3)}$$

Inserting (I47a) and (I47b) into (A4.3) gives

$$(-1)^{k'+m'} \sum_{j_{a_3} \geq j_{b_3}} \; \sum_{j'_{a_3} \geq j'_{b_3}} \Big(r(j'_{a_3} j'_{b_3}) \, r(j_{a_3} j_{b_3})$$
$$\times \left[B^{+k'}_{-m'}(j'_{a_3} j'_{b_3}), B^{+k}_{m}(j_{a_3} j_{b_3}) \right] + s(j'_{a_3} j'_{b_3}) \, s(j_{a_3} j_{b_3})$$
$$\times \left[B^{+k'}_{-m'}(j'_{a_3} j'_{b_3}), C^{k}_{m}(j_{a_3} j_{b_3}) \right] + r(j'_{a_3} j'_{b_3}) \, r(j_{a_3} j_{b_3})$$
$$\times \left[C^{k'}_{-m'}(j'_{a_3} j'_{b_3}), B^{+k}_{m}(j_{a_3} j_{b_3}) \right] + r(j'_{a_3} j'_{b_3}) \, s(j_{a_3} j_{b_3})$$
$$\times \left[C^{k'}_{-m'}(j'_{a_3} j'_{b_3}), C^{k}_{m}(j_{a_3} j_{b_3}) \right] \Big) \qquad \text{(A4.4)}$$

Using the commutation rules (I40) the first and fourth terms of (A4.4) vanish. Noting that

$$\sum_{j_{a_3} \geq j_{b_3}} = \sum_{j_{a_3} j_{b_3}} \frac{1+\delta_{j_{a_3} j_{b_3}}}{2} \qquad \text{(A4.5)}$$

and using the Boson commutation part of (I41), (A4.4) becomes

$$\frac{1}{4} \delta_{mm'} \delta_{kk'} \sum_{j_{a_3} j_{b_3}} \sum_{j_{a_3}' j_{b_3}'} (1+\delta_{j_{a_3} j_{b_3}})(1+\delta_{j_{a_3}' j_{b_3}'})$$

$$\times \Big[-\frac{1}{1+\delta_{j_{a_3} j_{b_3}'}} s(j_{a_3}' j_{b_3}') s(j_{a_3} j_{b_3}) \big(\delta_{j_{a_3} j_{a_3}'} \delta_{j_{b_3} j_{b_3}'}$$

$$- (-1)^{j_{a_3}+j_{b_3}+k} \delta_{j_{a_3} j_{b_3}'} \delta_{j_{b_3} j_{a_3}'} \big)$$

$$+ \frac{1}{1+\delta_{j_{a_3}' j_{b_3}}} r(j_{a_3}' j_{b_3}') r(j_{a_3} j_{b_3}) \big(\delta_{j_{a_3} j_{a_3}'} \delta_{j_{b_3} j_{b_3}'}$$

$$- (-1)^{j_{a_3}+j_{b_3}+k} \delta_{j_{a_3} j_{b_3}'} \delta_{j_{b_3} j_{a_3}'} \big) \Big] =$$

$$\frac{1}{4} \delta_{mm'} \delta_{kk'} \sum_{j_{a_3} j_{b_3}} (1+\delta_{j_{a_3} j_{b_3}}) \big(\delta_{j_{a_3} j_{a_3}'} \delta_{j_{b_3} j_{b_3}'} - (-1)^{j_{a_3}+j_{b_3}+k}$$

$$\times \delta_{j_{a_3} j_{b_3}'} \delta_{j_{b_3} j_{a_3}'} \big) \big[r(j_{a_3}' j_{b_3}') r(j_{a_3} j_{b_3}) - s(j_{a_3}' j_{b_3}') s(j_{a_3} j_{b_3}) \big] \qquad \text{(A4.6)}$$

This shows that $\Gamma_m^k(\eta)$ is an orthogonal set. The form of (A4.6) may be simplified, since $\Gamma_m^k(\eta)$ (I47a) may be rewritten using (A4.5) as

$$\Gamma_m^k(\eta) = \frac{1}{2} \sum_{j_{a_3} j_{b_3}} (1+\delta_{j_{a_3} j_{b_3}}) \big[r(j_{a_3} j_{b_3}) B_m^{+k}(j_{a_3} j_{b_3}) \qquad \text{(A4.7)}$$

$$+ s(j_{a_3} j_{b_3}) C_m^k(j_{a_3} j_{b_3}) \big]$$

Interchanging the dummy indices, j_{a_η}, j_{b_η}, gives

$$\Gamma^{k}_{m}(\eta)=\frac{1}{2}\sum_{j_{a_\eta} j_{b_\eta}}(1+\delta_{j_{a_\eta} j_{b_\eta}})\Big[r(j_{b_\eta} j_{a_\eta})B^{+k}_{m}(j_{b_\eta} j_{a_\eta})+s(j_{b_\eta} j_{a_\eta}) \times C^{k}_{m}(j_{b_\eta} j_{a_\eta})\Big] \quad \text{(A4.8)}$$

Equation (I39) involving $B^{+k}_{m}(j_{a_\eta} j_{b_\eta})$ and $C^{k}_{m}(j_{a_\eta} j_{b_\eta})$ enables (A4.7) to be rewritten as

$$\Gamma^{k}_{m}(\eta)=\frac{1}{2}\sum_{j_{a_\eta} j_{b_\eta}}(1+\delta_{j_{a_\eta} j_{b_\eta}})\Big[-(-1)^{j_{a_\eta}+j_{b_\eta}+k}\, r(j_{a_\eta} j_{b_\eta})B^{+k}_{m}(j_{a_\eta} j_{b_\eta}) -(-1)^{j_{a_\eta}+j_{b_\eta}+k} s(j_{a_\eta} j_{b_\eta})\, C^{k}_{m}(j_{a_\eta} j_{b_\eta})\Big] \quad \text{(A4.9)}$$

so that comparing (A4.9) and (A4.7) yields for any k

$$r(j_{b_\eta} j_{a_\eta})=-(-1)^{j_{a_\eta}+j_{b_\eta}+k}\, r(j_{a_\eta} j_{b_\eta}) \quad \text{(A4.10a)}$$

$$s(j_{b_\eta} j_{a_\eta})=-(-1)^{j_{a_\eta}+j_{b_\eta}+k}\, s(j_{a_\eta} j_{b_\eta}) \quad \text{(A4.10b)}$$

Inserting (A4.10) into (A4.6) gives

$$\frac{1}{2}\delta_{mm'}\delta_{kk'}\sum_{j_{a_\eta} j_{b_\eta}}(1+\delta_{j_{a_\eta} j_{b_\eta}})\Big[r^2(j_{a_\eta} j_{b_\eta})-s^2(j_{a_\eta} j_{b_\eta})\Big] \quad \text{(A4.11)}$$

Using (A4.5), (A4.11), and (A4.3) one gets finally for (A4.1)

$$\langle \tilde{0}_\eta | \Gamma^{+k'}_{m'}(\eta)\,\Gamma^{k}_{m}(\eta) | \tilde{0}_\eta \rangle = \delta_{mm'}\delta_{kk'} \times \sum_{j_{a_\eta} \geq j_{b_\eta}}\Big[r^2(j_{a_\eta} j_{b_\eta})-s^2(j_{a_\eta} j_{b_\eta})\Big] \quad \text{(A4.12)}$$

so that for $k=k'$ and $m=m'$, the normalization condition (A4.12) becomes

$$\langle \tilde{0}_\eta | \Gamma^{+k}_{m}(\eta)\,\Gamma^{k}_{m}(\eta) | \tilde{0}_\eta \rangle = \sum_{j_{a_\eta} \geq j_{b_\eta}}\Big[r^2(j_{a_\eta} j_{b_\eta})-s^2(j_{a_\eta} j_{b_\eta})\Big]$$

5. Quasi-Boson (made up from quasi-neutrons) Plus one Quasi-Proton:

The set of states of total angular momentum, J, and z component, M, obtained by coupling an odd quasi-proton to the quasi-Boson (made up from quasi-neutrons) is

$$\left[\Gamma^{k}(N)\, \beta^{+j_c}(P) \right]^{J}_{M} \,|\, \tilde{\hat{0}}_N ; \hat{0}_P \rangle$$

(I53)

The quasi-Boson vacuum, $|\tilde{\hat{0}}_N\rangle$, and quasi-proton vacuum, $|\hat{0}_P\rangle$, are separable, and j_{cp} is the angular momentum of the quasi-proton. Equation (I53) may be rewritten as

$$\sum_{m\, m_{cp}} \langle k\, m\, j_{cp}\, m_{cp} | J M \rangle\, \Gamma^{k}_{m}(N) |\tilde{\hat{0}}_N\rangle \times \beta^{+j_c}_{m_c}(P) |\hat{0}_P\rangle$$

(A5.1)

Since

$$\left[\Gamma^{k}_{m}(N) ,\ \beta^{+j_c}_{m_c}(P) \right] = 0$$

(A5.2)

the orthogonality condition for (I53) becomes

$$\langle \tilde{\hat{0}}_N ; \hat{0}_P | \left[\Gamma^{k'}(N)\ \beta^{+j_c'}(P) \right]^{J' +}_{M'} \left[\Gamma^{k}(N) \beta^{+j_c}(P) \right]^{J}_{M}$$

$$\times\, |\tilde{\hat{0}}_N ; \hat{0}_P\rangle = \sum_{\substack{m\, m_{cp} \\ m'\, m_{cp}'}} \langle k\, m\, j_{cp}\, m_{cp} | J M \rangle \langle k' m' j_{cp}' m_{cp}' | J' M' \rangle$$

$$\times \langle \tilde{\hat{0}}_N | \Gamma^{k'}_{m'}(N)\, \Gamma^{k}_{m}(N) |\tilde{\hat{0}}_N\rangle \langle \hat{0}_P | \beta^{j_c'}_{m_c'}(P)\, \beta^{+j_c}_{m_c}(P) |\hat{0}_P\rangle$$

(A5.3)

From (A4.12), (A2.3), and the orthonormality of Clebsch-Gordon coefficients (A5.3) becomes

$$\langle \tilde{\hat{O}}_N ; \tilde{\hat{O}}_P | \left[\Gamma^{k'}(N)\, \beta^{+ j_c}(P) \right]^{J'\,+}_{M'} \left[\Gamma^{k}(N)\, \beta^{+ j_c}(P) \right]^{J}_{M} | \tilde{\hat{O}}_N ; \tilde{\hat{O}}_P \rangle$$

$$= \delta_{JJ'}\, \delta_{MM'}\, \delta_{kk'}\, \delta_{j_{cP} j_{cP}'}$$

$$\times \sum_{j_{a_N} \geq j_{b_N}} \left[\lambda^2(j_{a_N} j_{b_N}) - \Delta^2(j_{a_N} j_{b_N}) \right]$$

(A5.4)

which verifies the orthogonality of the set (I53). The expansion coefficient sets, $\lambda(j_{a_N} j_{b_N})$ and $\Delta(j_{a_N} j_{b_N})$, are given by (I49).

To obtain the normalization condition, let $J = J'$, $M = M'$, $k = k'$, and $j_{cP} = j_{cP}'$ in (A5.4) giving

$$\langle \tilde{\hat{O}}_N ; \tilde{\hat{O}}_P | \left[\Gamma^{k}(N)\, \beta^{+ j_c}(P) \right]^{J\,+}_{M} \left[\Gamma^{k}(N)\, \beta^{+ j_c}(P) \right]^{J}_{M} | \tilde{\hat{O}}_N ; \tilde{\hat{O}}_P \rangle$$

$$= \sum_{j_{a_N} \geq j_{b_N}} \left[\lambda^2(j_{a_N} j_{b_N}) - \Delta^2(j_{a_N} j_{b_N}) \right] = 1$$

(A5.5)

This is the same normalization as for a single quasi-Boson (A4.13).

APPENDIX B

The Arbitrariness in the Intermediate Coupling for Two Quasi-Neutrons and an Odd Quasi-Proton

Consider two quasi-neutrons, one of angular momentum, j_{a_N}, and the other of angular momentum, j_{b_N}. Let them couple with each other and with an odd quasi-proton of angular momentum, j_{c_P}. If there are many possible states, j_{a_N}, j_{b_N}, and j_{c_P}, then, the state of total angular momentum, J, and z component, M, may be written as

$$\sum_{(j_{a_N}, j_{b_N}) J_0 j_{c_P}} a^{J}_{(j_{a_N} j_{b_N}) J_0 j_{c_P}} \left(\left[\beta^{+j_a}_{(N)} \beta^{+j_b}_{(N)} \right]^{J_0}_{M_0} \beta^{+j_c}_{(P)} \right)^{J}_{M} \left| \tilde{0}_N ; \tilde{0}_P \right\rangle \tag{B1}$$

The normalization is contained in the coefficients, $a^{J}_{(j_{a_N} j_{b_N}) J_0 j_{c_P}}$.

An alternate representation of the state of total angular momentum, J, and z component, M, is

$$\sum_{\substack{(j_{a_N} j_{c_p}) J_0' j_{b_N} \\ \text{with } j_{a_N} \geq j_{b_N}}} c^{J}_{(j_{a_N} j_{c_p}) J_0' j_{b_N}} \left(\left[\beta^{+j_a}(N) \beta^{+j_c}(P) \right]^{J_0'}_{M_0'} \beta^{+j_b}_{(N)} \right)^{J}_{M} \left| \tilde{0}_N ; \tilde{0}_P \right>$$

(B2)

where $c^{J}_{(j_{a_N} j_{c_p}) J_0' j_{b_N}}$ is another set of expansion coefficients including the normalization. From (I21') the quasi-proton and quasi-neutron operators commute, and from (I20'b) the quasi-neutron creation operators anti-commute. Equation (B2) is then

$$-\sum_{(j_{a_N} j_{c_p}) J_0' j_{b_N}} c^{J}_{(j_{a_N} j_{c_p}) J_0' j_{b_N}} \left(\beta^{+j_b}_{(N)} \left[\beta^{+j_a}(N) \beta^{+j_c}(P) \right]^{J_0'}_{M_0'} \right)^{J}_{M} \left| \tilde{0}_N ; \tilde{0}_P \right>$$

(B3)

with $j_{a_N} \geq j_{b_N}$

From Edmonds[29)] (B3) may be rewritten as

$$-\sum_{\substack{(j_{a_N} j_{c_p}) J_0' j_{b_N} \\ \text{with } j_{a_N} \geq j_{b_N}}} c^{J}_{(j_{a_N} j_{c_p}) J_0' j_{b_N}} \sum_{J_0''} (-)^{j_{b_N}+j_{a_N}+j_{c_p}+J} \sqrt{(2J_0''+1)(2J_0'+1)} \begin{Bmatrix} j_{b_N} & j_{a_N} & J_0'' \\ j_{c_p} & J & J_0' \end{Bmatrix}$$
$$\times \left(\left[\beta^{+j_b}(N) \beta^{+j_a}(N) \right]^{J_0''}_{M_0''} \beta^{+j_c}_{(P)} \right)^{J}_{M} \left| \tilde{0}_N ; \tilde{0}_P \right>$$

(B4)

Using the anti-commutation of the quasi-neutron creation operators (I20'b), and rewriting the Clebsch-Gordon coefficients coupling j_{a_N} and j_{b_N} (A1.8), equation (B4) becomes

$$\sum_{(j_{a_N} j_{c_p}) J_0' j_{b_N}} c^{J}_{(j_{a_N} j_{c_p}) J_0' j_{b_N}} \sum_{J_0''} (-1)^{j_{c_p}+J+J_0''} \sqrt{(2J_0''+1)(2J_0'+1)} \begin{Bmatrix} j_{b_N} & j_{a_N} & J_0'' \\ j_{c_p} & J & J_0' \end{Bmatrix}$$

$$\times \left(\left[\beta^{+j_{a(N)}} \beta^{+j_{b(N)}} \right]^{J_0''}_{M_0''} \beta^{+j_{c(P)}} \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_P \rangle \qquad \text{(B5)}$$

with $j_{a_N} \geq j_{b_N}$

Performing the sum over J_0' first, (B5) may be written as

$$\sum_{(j_{a_N} \geq j_{b_N}) J_0'' j_{c_p}} a''^{J}_{(j_{a_N} j_{b_N}) J_0'' j_{c_p}} \left(\left[\beta^{+j_{a(N)}} \beta^{+j_{b(N)}} \right]^{J_0''}_{M_0''} \beta^{+j_{c(P)}} \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_P \rangle \qquad \text{(B6)}$$

where

$$a''^{JM}_{(j_{a_N} j_{b_N}) J_0'' j_{c_p}} = (-1)^{j_{c_p}+J+J_0''} \sqrt{2J_0''+1} \sum_{J_0'} \sqrt{2J_0'+1}\, c^{J}_{(j_{a_N} j_{c_p}) J_0' j_{b_N}}$$

$$\times \begin{Bmatrix} j_{b_N} & j_{a_N} & J_0'' \\ j_{c_p} & J & J_0' \end{Bmatrix} \qquad \text{(B7)}$$

Since J_0'' is a dummy index, (B6) may be written as

$$\sum_{(j_{a_N} \geq j_{b_N}) J_0 j_{c_p}} a^{J}_{(j_{a_N} j_{b_N}) J_0 j_{c_p}} \left(\left[\beta^{+j_{a(N)}} \beta^{+j_{b(N)}} \right]^{J_0}_{M_0} \beta^{+j_{c(P)}} \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_P \rangle \qquad \text{(B8)}$$

which is the same as (B1).

APPENDIX C

Calculation of

The Matrix Elements of the Long-Range Interaction

Introduction:

From equation (II1) and the fact that only one quasi-proton is ever present, one may write

$$H^{k}_{N,P} = \frac{1}{2}\left[\left(\mathcal{H}^{k}_{\underset{(22)}{NP}} + \mathcal{H}^{k}_{\underset{(31)}{NP}}\right) + \left(\mathcal{H}^{k}_{\underset{(22)}{PN}} + \mathcal{H}^{k}_{\underset{(31)}{PN}}\right)\right] \tag{C1}$$

The only terms from (I34) that will contribute to (C1) are given by (I34i,j,k,l) for $\mathcal{H}^{k}_{\underset{(31)}{NP}}$ and $\mathcal{H}^{k}_{\underset{(31)}{PN}}$, and by (I34e,f,g,h) for $\mathcal{H}^{k}_{\underset{(22)}{NP}}$ and $\mathcal{H}^{k}_{\underset{(22)}{PN}}$.

By appropriate interchange of the dummy indices 1 and 2 for 3 and 4 respectively, it is easy to show that

$$\mathcal{H}^{k}_{\underset{(22)}{NP}} = \mathcal{H}^{k}_{\underset{(22)}{PN}} \tag{C2a}$$

and

$$\mathcal{H}^{k}_{\underset{(31)}{NP}} = \mathcal{H}^{k}_{\underset{(31)}{PN}} \tag{C2b}$$

so that

$$H^{k}_{\underset{(31)}{N,P}} \equiv \frac{1}{2}\left[\mathcal{H}^{k}_{\underset{(31)}{NP}} + \mathcal{H}^{k}_{\underset{(31)}{PN}}\right] = \mathcal{H}^{k}_{\underset{(31)}{NP}} \tag{C3a}$$

and

$$H^{k}_{N,P\,(22)} \equiv \frac{1}{2}\left[\mathcal{H}^{k}_{NP\,(22)} + \mathcal{H}^{k}_{PN\,(22)}\right] = \mathcal{H}^{k}_{NP\,(22)}$$

(C3b)

The actual expressions for $H^{k}_{NP\,(31)}$ and $H^{k}_{N,P\,(22)}$ are then simply obtained from (I34) by just calculating $\mathcal{H}^{k}_{NP\,(31)}$ and $\mathcal{H}^{k}_{NP\,(22)}$ respectively.

The Hermiticity of $H^{k}_{N,P\,(22)}$ (II5) will be shown in Appendices (C1b) and (C2b), while that of $H^{k}_{N,P\,(31)}$ (II4 or II15) will now be verified.

From Chapter II, one has

$$H^{k}_{N,P\,(31)} = (-1)^{k}\frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} q^{k}(j_{1N} j_{2N})\, q^{k}(j_{3P} j_{4P}) (-1)^{l_{1N}} v_{1N}\, u_{2N}$$

$$\times \left\{\left(-\left[\gamma^{j_1}(N)\gamma^{j_2}(N)\right] + (-1)^{k}\left[\beta^{+j_1}(N)\beta^{+j_2}(N)\right]\right)^{k}\left[\beta^{+j_3}(P)\gamma^{j_4}(P)\right]^{k}\right\}^{0}_{0}$$

(II15a)

The contribution of the $-\left[\gamma^{j_1}(N)\gamma^{j_2}(N)\right]^{k}$ part of (II15a) may be written in expanded form using (C1a.3) as

$$-(-1)^{k}\frac{4\pi\, F^{k}}{\sqrt{2k+1}} \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} q^{k}(j_{1N} j_{2N})\, q^{k}(j_{3P} j_{4P}) (-1)^{l_{1N}} v_{1N}\, u_{2N} \sum_{\substack{m_{1N} m_{2N} \\ m_{3P} m_{4P}}} \langle j_{1N} m_{1N} j_{2N} m_{2N} | k q \rangle$$

$$\times \langle j_{3P} m_{3P} j_{4P} m_{4P} | k -q \rangle \frac{(-1)^{k-q}}{\sqrt{2k+1}} \left(\gamma^{j_1}_{m_1}(N)\gamma^{j_2}_{m_2}(N)\, \beta^{+j_3}_{m_3}(P)\, \gamma^{j_4}_{m_4}(P)\right) \quad \text{(C4)}$$

Using (I4) gives for the operators of (C4)

$$(-1)^{j_{1N}-m_{1N}+j_{2N}-m_{2N}+j_{4P}-m_{4P}} \left(\beta^{j_1}_{-m_1}(N)\, \beta^{j_2}_{-m_2}(N)\, \beta^{+j_3}_{m_3}(P)\, \beta^{j_4}_{-m_4}(P) \right) \qquad (C5)$$

The Hermitian conjugate of (C4) using (C5), (I20'b), (A1.8), the appropriate interchange of dummy indices, and the expression governing reversal of signs in a Clebsch-Gordon coefficient[29] becomes

$$(-1)^k \frac{4\pi}{\sqrt{2k+1}} F^k \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} g^k(j_{1N}\, j_{2N})\, g^k(j_{3P}\, j_{4P})\, (-1)^{l_{1N}} v_{1N}\, u_{2N}$$

$$\times (-1)^k \left\{ \left[\beta^{+j_1}(N)\, \beta^{+j_2}(N) \right]^k \left[\beta^{+j_3}(P)\, \gamma^{j_4}(P) \right]^k \right\}^0_0 \qquad (C6)$$

Now (C6) is just the contribution of the $(-1)^k\left[\beta^{+j_1}(N)\beta^{+j_2}(N)\right]^k$ term of (II5a) so that Hermiticity is verified.

1. Quasi-Boson (of quasi-neutron origin) Plus Quasi-Proton

a) $\underset{(31)}{H^k_{N,P}}$

The long-range Hamiltonian term, $\underset{(31)}{H^k_{N,P}}$, (c.f. Introduction to Appendix C) is

$$\underset{(31)}{H^k_{N,P}} = (-1)^k \frac{4\pi}{\sqrt{2k+1}} F^k \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} \sqrt{1+\delta_{j_{1N} j_{2N}}}\, g^k(j_{1N}\, j_{2N})\, g^k(j_{3P}\, j_{4P})\, (-1)^{l_{1N}}$$

$$\times v_{1N}\, u_{2N} \left\{ \left[C(j_{1N}\, j_{2N}) + (-1)^k B^+(j_{1N}\, j_{2N}) \right]^k \left[\beta^{+j_3}_{(P)}\, \gamma^{j_4}_{(P)} \right]^k \right\}^0 \qquad (II4)$$

The set of states of total angular momentum, J , and z component, M , is given by

$$|JM\rangle_{0,k} = a_{0J}^{J}\, \beta^{+J}_{M}(p)|\hat{\tilde{0}}_N ; \tilde{0}_p\rangle + \sum_{j_{cp}} a^{J}_{k j_{cp}} \times \left[\Gamma^{k}(N)\, \beta^{+j_c}(p)\right]^{J}_{M} |\hat{\tilde{0}}_N ; \hat{0}_p\rangle \qquad \text{(I57)}$$

One wishes to calculate

$$\langle (\hat{\tilde{0}}_N ; \tilde{j}_{cp})\, j_{cp} m_{cp} |\, H^{k}_{N,p\,(31)} \,| (\hat{\tilde{k}}_N ; \tilde{j}'_{cp})\, JM\rangle \qquad \text{(C1a.1)}$$

where the notation follows (I53).

In II4)

$$\left\{ \left[c(j_{1N} j_{2N}) + (-1)^k B^+(j_{1N} j_{2N})\right]^k \left[\beta^{+j_3}(p)\, \gamma^{j_4}(p)\right]^k \right\}^{c}_{0}$$

$$= \sum_q \frac{(-1)^{k-q}}{\sqrt{2k+1}} \left[c(j_{1N} j_{2N}) + (-1)^k B^+(j_{1N} j_{2N})\right]^k_q \sum_{m_{3p} m_{4p}} \langle j_{3p} m_{3p} j_{4p} m_{4p} | k\, -q\rangle \times \beta^{+j_3}_{m_3}(p)\, \gamma^{j_4}_{m_4}(p) \qquad \text{(C1a.2)}$$

where use has been made of

$$\langle k q\, k\, -q | 00\rangle = \frac{(-1)^{k-q}}{\sqrt{2k+1}} \qquad \text{(C1a.3)}$$

Equation (C1a.1) becomes (using (I4)) just R x S with R defined by

$$R \equiv (-1)^k \frac{4\pi}{\sqrt{2k+1}} F^k \sum_{j_{3p} j_{4p}} g^k(j_{3p} j_{4p}) \sum_{m_{3p} m_{4p}} \langle j_{3p} m_{3p} j_{4p} m_{4p} | k\, -q\rangle \times \sum_{m\, m'_{cp}} \langle k\, m\, j'_{cp}\, m'_{cp} | JM\rangle \sum_q \frac{(-1)^{k-q}}{\sqrt{2k+1}} \langle \hat{0}_p | (-1)^{j_{cp}-m_{cp}} \times \gamma^{j_c}_{-m_c}(p)\, \beta^{+j_3}_{m_3}(p)\, \gamma^{j_4}_{m_4}(p)\, \beta^{+j'_c}_{m_c}(p) | \hat{0}_p\rangle \qquad \text{(C1a.4)}$$

and S defined by

$$S \equiv \sum_{j_{1N} j_{2N}} \sqrt{1+\delta_{j_{1N} j_{2N}}}\, q^{k}(j_{1N} j_{2N})\, (-1)^{l_{1N}}\, \nu_{1N}\, u_{2N} < \tilde{\hat{0}}_N \,|\, [\, C(j_{1N} j_{2N}) + (-1)^{k} B^{+}(j_{1N} j_{2N})]^{k}_{q}\, \Gamma^{k}_{m}(N)\, |\, \tilde{\hat{0}}_N > \qquad \text{(C1a.5)}$$

Using the Fermion commutation rule (I20'c), equation (C1a.4) becomes

$$R = \frac{4\pi}{2k+1} F^{k} \sum_{j_{3p} j_{4p}} q^{k}(j_{3p} j_{4p}) \sum_{\substack{m_{3p} m_{4p} \\ m\ m'_{cp} \\ q}} (-1)^{-j_{cp}-m_{cp}-q}$$

$$\times < j_{3p}\, m_{3p}\, j_{4p}\, m_{4p} \,|\, k\, -q > < k\, m\, j'_{cp}\, m'_{cp} \,|\, JM >$$

$$\times\ \delta_{j_{3p} j_{cp}}\, \delta_{m_{3p} m_{cp}}\, (-1)^{j_{3p}+m_{3p}}\, \delta_{j'_{cp} j_{4p}}\, \delta_{m'_{cp}, -m_{4p}}\, (-1)^{j'_{cp}+m'_{cp}}$$

$$= \frac{4\pi}{2k+1} F^{k}\, q^{k}(j_{cp}\, j'_{cp}) \sum_{m\, m'_{cp}\, q} < j_{cp}\, m_{cp}\, j'_{cp}\, -m'_{cp} \,|\, k\, -q >$$

$$\times < k\, m\, j'_{cp}\, m'_{cp} \,|\, JM > (-1)^{-q+j'_{cp}+m'_{cp}}$$

(C1a.6)

Equations (II10) and (II11) allow one to write (C1a.5) as

$$S = \sum_{j_{1N} j_{2N}} \sqrt{1+\delta_{j_{1N} j_{2N}}}\, q^{k}(j_{1N}\, j_{2N}) (-1)^{l_{1N}}\, \nu_{1N}\, u_{2N} (-1)^{k+m}\, \delta_{m,-q}$$

$$\times \left[r(j_{1N}\ j_{2N}) - (-1)^{k}\, s(j_{1N} j_{2N}) \right]$$

(C1a.7)

Now multiplying R (C1a.6) by S (C1a.7) enables one to use

$$\sum_{m\, m'_{cp}\, q} (-1)^{-q+j'_{cp}+m'_{cp}} \langle j_{cp}\, m_{cp}\, j'_{cp}\, -m'_{cp} | k\, -q \rangle \langle k\, m\, j'_{cp}\, m'_{cp} | J M \rangle$$

$$\times (-1)^{k+m} \delta_{m,-q} = \sum_{m'_{cp}, m} (-1)^{j'_{cp}+m'_{cp}+k} \langle j_{cp}\, m_{cp}\, j'_{cp}\, -m'_{cp} | k\, m \rangle$$

$$\times \langle k\, m\, j'_{cp}\, m'_{cp} | J M \rangle \qquad \text{(Cla.8)}$$

The Clebsch-Gordon coefficients imply that

$$m_{cp} = M \quad \text{and} \quad m = M - m'_{cp}$$

(Cla.9)

(Cla.8) then becomes

$$\delta_{m_{jp}, M} \sum_{m'_{cp}} (-1)^{j'_{cp}+m'_{cp}+k} \langle j_{cp}\, M\, j'_{cp}\, -m'_{cp} | k\, M-m'_{cp} \rangle$$

$$\times \langle k\, M-m'_{cp}\, j'_{cp}\, m'_{cp} | J M \rangle \qquad \text{(Cla.10)}$$

From Edmonds[29] one may write

$$\langle j_{cp}\, M\, j'_{cp}\, -m'_{cp} | k\, M-m'_{cp} \rangle = (-1)^{k+j_{cp}-m'_{cp}} \left(\frac{2k+1}{2j_{cp}+1}\right)^{1/2}$$

$$\times \langle k\, M-m'_{cp}\, j'_{cp}\, m'_{cp} | j_{cp}\, M \rangle \qquad \text{(Cla.11)}$$

The orthonormality of Clebsch-Gordon coefficients then gives for equation (Cla.10)

$$\delta_{m_{cp} M}\, \delta_{j_{cp} J}\, (-1)^{j_{cp}+j'_{cp}} \left(\frac{2k+1}{2J+1}\right)^{1/2}$$

(Cla.12)

Utilizing (I30) to permute the indices of $q^k(j_{cp} j'_{cp})$, one gets for (Cla.1)

$$\frac{-4\pi F^k q^k(j'_{cp}\, J)}{\sqrt{(2k+1)(2J+1)}}\, \delta_{j_{cp} J}\, \delta_{m_{cp} M} \left(\sum_{j_{1\nu} j_{2\nu}} \sqrt{1+\delta_{j_{1\nu} j_{2\nu}}}\, q^k(j_{1\nu}\, j_{2\nu}) \right.$$

$$\times (-1)^{l_{1\nu}} \, n_{j_{1\nu}} \, w_{j_{2\nu}} \left[r(j'_{1\nu} \, j_{2\nu}) - (-1)^{k} \right.$$

$$\left. \left. \times \, s(j_{1\nu} \, j_{2\nu}) \right] \right)$$

(Cla.13)

The quantity in parentheses may be written in terms of $\sum_{j_{1\nu} \geq j_{2\nu}}$ to give finally for (Cla.1)

$$\langle (\tilde{\hat{o}}_{\nu} ; \hat{j}_{cp}) j_{cp} m_{cp} | H^{k}_{\nu,p\,(31)} | (\tilde{\hat{k}}_{\nu} ; \hat{j}'_{cp}) JM \rangle$$

$$= -4\pi F^{k} q^{k}(j'_{cp} J) \delta_{j_{cp}J} \delta_{m_{cp}M} \sum_{j'_{1\nu} \geq j'_{2\nu}} \frac{1}{\sqrt{1+\delta_{j'_{1\nu} j'_{2\nu}}}}$$

$$\times q^{k}(j'_{1\nu} j'_{2\nu}) (-1)^{l_{1\nu}} \sin(\tau_{1\nu} + \tau_{2\nu})$$

$$\times \left[r(j'_{1\nu} j'_{2\nu}) - (-1)^{k} s(j'_{1\nu} j'_{2\nu}) \right] \quad \text{(Cla.14)}$$

where (II3b) has been used.

Equation (Cla.14) is readily checked if one calculates

$$\langle (\tilde{\hat{k}}_{\nu} ; \hat{j}'_{cp}) JM | H^{k}_{\nu,p\,(31)} | (\tilde{\hat{o}}_{\nu} ; \hat{j}_{cp}) j_{cp} m_{cp} \rangle$$

(Cla.15)

The procedure is so similar to that just carried out that the details will not be shown. Equations (Cla.14) and (Cla.15) combine to give (II12).

b) $H^{k}_{\nu,p\,(22)}$

The long-range Hamiltonian term, $H^{k}_{\nu,p\,(22)}$ is given (c.f. Introduction to Appendix C) by

$$H^{k}_{\substack{N,P\\(22)}} = -(-1)^{k}\frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1N} j_{2N}\\ j_{3P} j_{4P}}} g^{k}(j_{1N} j_{2N})\, q^{k}(j_{3P} j_{4P})$$

$$\cos(\tau_{1N}+\tau_{2N}) \left\{ \left[\beta^{+j_1(N)}\gamma^{j_2(N)}\right]^{k} \left[\beta^{+j_3(P)}\gamma^{j_4(P)}\right]^{k} \right\}^{0}_{0}$$

(II5)

According to the reasoning of Chapter II, Section A. (just prior to (II13)), the quasi-Boson vacuum, $|\tilde{\tilde{0}}_N\rangle$, is approximated by the quasi-neutron vacuum, $|\tilde{0}_N\rangle$. In essence this is due to the presence of single quasi-neutron operators.

The set of states used in calculating matrix elements of $H^{k}_{\substack{N,P\\(22)}}$ in the approximation mentioned above is

$$a^{J}_{0J}\, \beta^{+J}_{M}(P)\, |\tilde{\tilde{0}}_N ; \tilde{0}_P\rangle + \sum_{j_{cP}'} a^{J}_{k\, j_{cP}'} \Big[\sum_{j_{5N}' \geq j_{6N}'} \lambda(j_{5N}' j_{6N}')$$
$$\times B^{+k}(j_{5N}' j_{6N}')\beta^{+j_c'}_{(P)}\Big]^{J} |\tilde{0}_N ; \tilde{0}_P\rangle \qquad \text{(C1b.1)}$$

The use of $|\tilde{\tilde{0}}_N\rangle \simeq |\tilde{0}_N\rangle$ has caused the $C^{k}(j_{5N}' j_{6N}')$ components of $\Gamma^{k}_{(N)}$ (I47a) to give zero contribution. Non-vanishing matrix elements only result from the one quasi-Boson plus quasi-proton components. Hence, the matrix element to be evaluated is

$$\langle (\tilde{\tilde{k}}_N ; \tilde{j}_{cP}) JM |\, H^{k}_{\substack{N,P\\(22)}} \,| (\tilde{\tilde{k}}_N ; \tilde{j}_{cP}') J'M' \rangle$$

(C1b.2)

where

$$|(\tilde{\tilde{k}}_N ; \tilde{j}_{cP}') JM\rangle \equiv \left[\Gamma^{k}(N)\,\beta^{+j_c'}(P)\right]^{J}_{M} |\tilde{\tilde{0}}_N ; \tilde{0}_P\rangle$$

$$\simeq \Big[\sum_{j_{5N}' \geq j_{6N}'} \lambda(j_{5N}' j_{6N}')\, B^{+k}(j_{5N}' j_{6N}')\, \beta^{+j_c'}(P)\Big]^{J}_{M} |\tilde{0}_N ; \tilde{0}_P\rangle$$

The calculation is made easier if one replaces $\sum_{j'_{5_N} \geq j'_{6_N}} \lambda(j'_{5_N} j'_{6_N})$ by $\frac{1}{2} \sum_{j'_{5_N} j'_{6_N}} (1+\delta_{j'_{5_N} j'_{6_N}}) \lambda(j'_{5_N} j'_{6_N})$. Using the definition of B^{+k} (I38a), the state (Clb.3) becomes

$$| (\tilde{\tilde{k}}_N ; \tilde{j}_{cp}') J'M' \rangle \cong \frac{1}{2} \sum_{j'_{5_N} j'_{6_N}} \sqrt{1+\delta_{j'_{5_N} j'_{6_N}}} \, \lambda(j'_{5_N} j'_{6_N})$$

$$\times \sum_{\substack{m'_{5_N} m'_{6_N} \\ m' m_{j_{cp}}}} \langle j'_{5_N} m'_{5_N} j'_{6_N} m'_{6_N} | k m' \rangle \langle k m' j'_{cp} m'_{cp} | J'M' \rangle \beta^{+j'_5}_{m'_5}(N)$$

$$\times \beta^{+j'_6}_{m'_6}(N) \, \beta^{+j'}_{m'_j}(P) | \tilde{0}_N ; \tilde{0}_P \rangle \qquad \text{(Clb.4)}$$

Similarly

$$\langle (\tilde{\tilde{k}}_N ; j_{cp}) JM | \cong \frac{1}{2} \sum_{j_{5_N} j_{6_N}} \sqrt{1+\delta_{j_{5_N} j_{6_N}}} \, \lambda(j_{5_N} j_{6_N})$$

$$\times \sum_{\substack{m_{5_N} m_{6_N} \\ m \, m_{cp}}} \langle j_{5_N} m_{5_N} j_{6_N} m_{6_N} | k m \rangle \langle k m j_{cp} m_{cp} | JM \rangle (-1)^{-j_{cp}-m_{cp}}$$

$$\times (-1)^{-j_{6_N}-m_{6_N}-j_{5_N}-m_{5_N}} \langle \tilde{0}_N ; \tilde{0}_P | \gamma^{j}_{-m_j}(P) \, \gamma^{j_6}_{-m_6}(N) \, \gamma^{j_5}_{-m_5}(N) \qquad \text{(Clb.5)}$$

where (I4) is used.

Combining (II5), (Clb.4), (Clb.5), and (Cla.3) gives for (Clb.2)

$$-\frac{\pi}{2k+1} F^k \sum_{\substack{j_{1_N} j_{2_N} \\ j_{3_P} j_{4_P} \\ j_{5_N} j_{6_N} \\ j'_{5_N} j'_{6_N}}} q^k(j_{1_N} j_{2_N}) q^k(j_{3_N} j_{4_N}) \cos(\varphi_{j_{1_N}} + \varphi_{j_{2_N}}) \sqrt{1+\delta_{j'_{5_N} j'_{6_N}}}$$

$$\times \sqrt{1+\delta_{j_{5_N} j_{6_N}}} \; \lambda(j'_{5_N} j'_{6_N}) \lambda(j_{5_N} j_{6_N}) \sum_{\substack{q \, m_{1_N} m_{2_N} \\ m_{3_N} m_{4_N} m_{j_P}}} \; \sum_{\substack{m' m'_{5_N} m'_{6_N} \\ m_{j_P} m \, m_{5_N} m_{6_N}}} (-1)^{-q}$$

$$\times \langle j_{1_N} m_{1_N} j_{2_N} m_{2_N} | k q \rangle \langle j_{3_P} m_{3_P} j_{4_P} m_{4_P} | k -q \rangle \langle j'_{5_N} m'_{5_N} j'_{6_N} m'_{6_N} | k m' \rangle$$

$$\times \langle k\, m'\, j_{cp}'\, m_{cp}' \mid J' M' \rangle \langle j_{5_N}\, m_{5_N}\, j_{6_N}\, m_{6_N} \mid k\, m \rangle$$

$$\times \langle k\, m\, j_{cp}\, m_{cp} \mid J M \rangle (-1)^{-j_{cp} - m_{cp} - j_{6_N} - m_{6_N} - j_{5_N} - m_{5_N}}$$

$$\times \langle \tilde{0}_p \mid \gamma^{j_c}_{-m_c}(p)\, \beta^{+j_3}_{m_3}(p)\, \gamma^{j_4}_{m_4}(p)\, \beta^{+j_c'}_{m_c'}(p) \mid \tilde{0}_p \rangle$$

$$\times \langle \tilde{0}_N \mid \gamma^{j_6}_{-m_6}(N)\, \gamma^{j_5}_{-m_5}(N)\, \beta^{+j_1}_{m_1}(N)\, \gamma^{j_2}_{m_2}(N)\, \beta^{+j_5'}_{m_5'}(N)\, \beta^{+j_6'}_{m_6'}(N) \mid \tilde{0}_N \rangle$$

(Clb.6)

One notes the separability into a product of a term consisting of quasi-proton operators and a term made up of quasi-neutron operators. This results from the commutability of quasi-protons and quasi-neutrons (I21'). The commutation rules for $\beta^{+}(j)$ and $\gamma(j)$ (I20'c) are now used for the terms of (Clb.6) involving these operators. After maneuvering all annihilation operators to the right, one gets in the notation of (Al.4)

$$\langle \tilde{0}_p \mid \gamma^{j_c}_{-m_c}(p)\, \beta^{+j_3}_{m_3}(p)\, \gamma^{j_4}_{m_4}(p)\, \beta^{+j_c'}_{m_c'}(p) \mid \tilde{0}_p \rangle$$

$$\times \langle \tilde{0}_N \mid \gamma^{j_6}_{-m_6}(N)\, \gamma^{j_5}_{-m_5}(N)\, \beta^{+j_1}_{m_1}(N)\, \gamma^{j_2}_{m_2}(N)\, \beta^{+j_5'}_{m_5'}(N)\, \beta^{+j_6'}_{m_6'}(N) \mid \tilde{0}_N \rangle =$$

$$(-1)^{j_{3p} + m_{3p} + j_{cp}' + j_{1_N} + m_{1_N} + j_{5_N}' + m_{5_N}' + j_{6_N}' + m_{6_N}' + m_{cp}'}\, \delta_{cp', -4p}\, \delta_{3p, cp}$$

$$\times \left[\left(\delta_{6_N' 6_N} \delta_{1_N 5_N} - \delta_{1_N 6_N} \delta_{6_N' 5_N} \right) \delta_{5_N', -2_N} - \left(\delta_{5_N' 6_N} \delta_{1_N 5_N} - \delta_{1_N 6_N} \delta_{5_N' 5_N} \right) \right]$$

(Clb.7)

Inserting (Clb.7) into (Clb.6) yields after performing some summations

$$-\frac{\pi}{2k+1} F^k q^k(j_{cp} j_{cp}') \sum_{\substack{j_{1_N} j_{2_N} j_{5_N} j_{6_N} \\ j_{5_N}' j_{6_N}'}} \sum q^k(j_{1_N} j_{2_N}) \cos(\chi_{1_N} + \chi_{2_N})$$

$$\times \sqrt{1+\delta_{j_{5_N}' j_{6_N}'}}\, \sqrt{1+\delta_{j_{5_N} j_{6_N}}}\; r(j_{5_N}' j_{6_N}')\, r(j_{5_N} j_{6_N}) \sum_{\substack{m' m \\ m_{1_N} m_{2_N}}} \sum_{\substack{m_{5_N}' m_{6_N}' \\ m_{5_N} m_{6_N}}}$$

$$\times \langle j_{1_N}\, m_{1_N}\, j_{2_N}\, m_{2_N} | k\; m-m' \rangle \langle j'_{5_N}\, m'_{5_N}\, j'_{6_N}\, m'_{6_N} | k\; m' \rangle$$

$$\times \langle j_{5_N}\, m_{5_N}\, j_{6_N}\, m_{6_N} | k\; m \rangle (-1)^{-j_{6_N}-m_{6_N}-j_{5_N}-m_{5_N}+j_{1_N}+m_{1_N}+j'_{5_N}+m'_{5_N}}$$

$$\times (-1)^{j'_{6_N}+m'_{6_N}-m+m'} \langle j_{cp}\, M-m\; j'_{cp}\, -M+m' | k\; -m+m' \rangle$$

$$\times \langle k\; m'\; j'_{cp}\; M'-m' | J'M' \rangle \langle k\; m\; j_{cp}\; M-m | JM \rangle (-1)^{j'_{cp}+M'-m'}$$

$$\times \Big[\left(\delta_{6'_N 6_N}\delta_{1_N 5_N} - \delta_{1_N 6_N}\delta_{6'_N 5_N}\right)\delta_{5'_N, -2_N} - \left(\delta_{5'_N 6_N}\delta_{1_N 5_N}\right.$$

$$\left. - \delta_{1_N 6_N}\delta_{5'_N 5_N}\right)\delta_{6'_N, -2_N}\Big] \equiv (a) + (b) + (c) + (d)$$

(Clb.8)

(a) is the term due to $\delta_{6'_N 6_N}\delta_{1_N 5_N}\delta_{5'_N, -2_N}$

(b) is the term due to $-\delta_{1_N 6_N}\delta_{6'_N 5_N}\delta_{5'_N, -2_N}$

(c) is the term due to $-\delta_{5'_N 6_N}\delta_{1_N 5_N}\delta_{6'_N, -2_N}$ and

(d) is the term due to $\delta_{1_N 6_N}\delta_{5'_N 5_N}\delta_{6'_N, -2_N}$

Since $j'_{5_N}, j'_{6_N}, j_{5_N}, j_{6_N}, m'_{5_N}, m'_{6_N}, m_{5_N}, m'_{5_N}$ are dummy indices, an enormous simplification of (Clb.8) is possible. By interchanging j'_{5_N}, m'_{5_N} with j'_{6_N}, m'_{6_N} in (c) and then utilizing (A4.10a) and (A1.8) one sees that

$$(c) = (a)$$

(Clb.9)

A similar procedure involving (d) shows that

$$(d) = (b)$$

(Clb.10)

If now j_{5_N}, m_{5_N} and j_{6_N}, m_{6_N} are interchanged in (b) and again use is made of (A4.12) and (A1.8) the result is

$$(b) = (a)$$

(Clb.11)

Hence

$$(a) = (b) = (c) = (d) \tag{C1b.12}$$

which may be readily verified by direct calculation of (C1b.8).

Only one arbitrary term in (C1b.8), e.g., (a) need be evaluated. In (a) the number of summation indices may be reduced since not all these indices are independent. If, in addition, the Clebsch-Gordon coefficients are transformed to 3-j symbols (A3.10) one has

$$\begin{aligned}(a) = &-\frac{\pi}{2k+1} F^k q^k(j_{cp} j'_{cp}) \sum_{j_{1\nu} j_{2\nu} j_{b\nu}} q^k(j_{1\nu} j_{2\nu}) \cos(\tau_{1\nu}+\tau_{2\nu}) \sqrt{1+\delta_{j_{2\nu} j_{b\nu}}} \\ &\sqrt{1+\delta_{j_{1\nu} j_{b\nu}}}\; r(j_{2\nu} j_{b\nu})\, r(j_{1\nu} j_{b\nu}) \Big[-(2k+1)^2 \sqrt{(2J+1)(2J'+1)} \\ &\times \sum_{m'm} (-1)^{j'_{cp}-M} \begin{pmatrix} j_{cp} & j'_{cp} & k \\ M-m & +m'-M & m-m' \end{pmatrix} \begin{pmatrix} k & j'_{cp} & J' \\ m' & M-m' & -M \end{pmatrix} \\ &\times \begin{pmatrix} k & j_{cp} & J \\ m & M-m & -M \end{pmatrix}\Big] \Big[\sum_{m_{b\nu}} (-1)^{j_{2\nu}-m_{b\nu}} \begin{pmatrix} j_{1\nu} & j_{2\nu} & k \\ m-m_{b\nu} & m_{b\nu}-m' & -m+m' \end{pmatrix} \\ &\times \begin{pmatrix} k & j_{2\nu} & j_{b\nu} \\ -m' & -m_{b\nu}+m' & m_{b\nu} \end{pmatrix} \begin{pmatrix} j_{1\nu} & k & j_{b\nu} \\ -m+m_{b\nu} & m & -m_{b\nu} \end{pmatrix}\Big] \delta_{MM'}\end{aligned} \tag{C1b.13}$$

Application of (A3.13) to the sum over the last three 3-j symbols in (C1b.13) produces

$$\begin{aligned}&\sum_{m_{b\nu}} (-1)^{j_{2\nu}-m_{b\nu}} \begin{pmatrix} j_{1\nu} & j_{2\nu} & k \\ m-m_{b\nu} & m_{b\nu}-m' & -m+m' \end{pmatrix} \begin{pmatrix} k & j_{2\nu} & j_{b\nu} \\ -m' & -m_{b\nu}+m' & m_{b\nu} \end{pmatrix} \\ &\times \begin{pmatrix} j_{1\nu} & k & j_{b\nu} \\ -m+m_{b\nu} & m & -m_{b\nu} \end{pmatrix} = -(-1)^{-j_{1\nu}-j_{b\nu}-m-m'} \begin{pmatrix} k & k & k \\ -m' & m & -m+m' \end{pmatrix} \\ &\times \begin{Bmatrix} k & k & k \\ j_{1\nu} & j_{2\nu} & j_{b\nu} \end{Bmatrix}\end{aligned} \tag{C1b.14}$$

Inserting (Clb.14) into (Clb.13), and applying the orthogonality of 3-j symbols (A3.15) gives the expression for (a). Multiplying this expression by 4, one has (Clb.8), and hence (Clb.2). The result is

$$-(2k+1)4\pi F^k q^k(j_{cp}\, j_{cp}')(-1)^{j_{cp}+J}\begin{Bmatrix} k & k & k \\ j_{cp} & j_{cp}' & J\end{Bmatrix}\sum_{j_{1N} j_{2N} j_{bN}}$$

$$q^k(j_{1N}\, j_{2N})\cos(\alpha_{1N}+\alpha_{2N})\sqrt{1+\delta_{j_{2N} j_{bN}}}\ \sqrt{1+\delta_{j_{1N} j_{bN}}}\ r(j_{2N}\, j_{bN})$$

$$r(j_{1N}\, j_{bN})(-1)^{j_{1N}+j_{bN}+k}\begin{Bmatrix} k & k & k \\ j_{1N} & j_{2N} & j_{bN}\end{Bmatrix}\delta_{JJ'}\,\delta_{MM'}$$

(Clb.15)

Rewriting $r(j_{1N} j_{bN})$ via (A4.10a) and letting the dummy indices $1_N, 2_N, b_N$ be replaced by a_N, b_N, c_N gives finally

$$\langle(\tilde{\hat{k}}_N ; \tilde{j}_{cp})JM|\, H^k_{N,p\,(22)}\, |(\tilde{\hat{k}}_N ; \tilde{j}_{cp}')J'M'\rangle \cong (2k+1)4\pi F^k$$

$$\times\, q^k(j_{cp}\, j_{cp}')(-1)^{j_{cp}+J}\begin{Bmatrix} k & k & k \\ j_{cp} & j_{cp}' & J\end{Bmatrix}\delta_{JJ'}\,\delta_{MM'}$$

$$\times \sum_{j_{a_N} j_{b_N} j_{c_N}} q^k(j_{a_N}\, j_{b_N})\cos(\alpha_{a_N}+\alpha_{b_N})\sqrt{(1+\delta_{j_{b_N} j_{c_N}})(1+\delta_{j_{a_N} j_{c_N}})}\ r(j_{b_N}\, j_{c_N})$$

$$\times\, r(j_{c_N}\, j_{a_N})\begin{Bmatrix} k & k & k \\ j_{a_N} & j_{b_N} & j_{c_N}\end{Bmatrix}$$

(Clb.16)

Since $q^k(j_{cp}'\, j_{cp})(-1)^{j_{cp}'} = (-1)^{j_{cp}-j_{cp}'} q^k(j_{cp}\, j_{cp}')(-1)^{j_{cp}'} = q^k(j_{cp}\, j_{cp}')(-1)^{j_{cp}}$

one easily verifies that

$$\langle(\tilde{\hat{k}}_N ; \tilde{j}_{cp}')J'M'|\, H^k_{N,p\,(22)}\, |(\tilde{\hat{k}}_N ; \tilde{j}_{cp})JM\rangle$$

$$= \langle(\tilde{\hat{k}}_N ; \tilde{j}_{cp})JM|\, H^k_{N,p\,(22)}\, |(\tilde{\hat{k}}_N ; \tilde{j}_{cp}')J'M'\rangle$$

(Clb.17)

Equations (Clb.16) and (Clb.17) thus verify (II13).

2. Two Quasi-Neutrons Plus Quasi-Proton:

a) $H^{k}_{N,P\,(31)}$

The best expression to use for the long-range Hamiltonian term, $H^{k}_{N,P\,(31)}$ (c.f. Introduction to Appencix C) is

$$H^{k}_{N,P\,(31)} = (-1)^{k} \frac{4\pi}{\sqrt{2k+1}} F^{k} \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} q^{k}(j_{1N}\, j_{2N})\, q^{k}(j_{3P}\, j_{4P}) (-1)^{l_{1N}} N_{j_{1N}} N_{j_{2N}}$$

$$\times \left\{ \left(-\left[\gamma^{j_1}_{(N)} \gamma^{j_2}_{(N)}\right] + (-1)^{k} \left[\beta^{+j_1}{}_{(N)}\, \beta^{+j_2}{}_{(N)}\right] \right)^{k} \left[\beta^{+j_3}_{(P)}\, \gamma^{j_4}_{(P)}\right]^{k} \right\}^{0}_{0}$$

(II15a)

with

$$\left(-\left[\gamma^{j_1}_{(N)}\ \gamma^{j_2}_{(N)}\right] + (-1)^{k} \left[\beta^{+j_1}{}_{(N)}\ \beta^{+j_2}{}_{(N)}\right] \right)^{k}$$

$$= -\left[\gamma_{(N)}{}^{j_1} \gamma_{(N)}{}^{j_2}\right]^{k} + (-1)^{k} \left[\beta^{+j_1}{}_{(N)}\ \beta^{+j_2}{}_{(N)}\right]^{k}$$

(II15b)

The set of states with respect to which the diagonalization is to take place is

$$|JM\rangle = \Big[a^{J}_{0J}\, \beta^{+J}_{M}(P) + \sum_{\substack{j_{aN} \geq j_{bN} \\ J_0, j_{cP}}} c^{J}_{(j_{aN} j_{bN}) J_0 j_{cP}} \frac{1}{\sqrt{1+\delta_{j_{aN} j_{bN}}}}$$

$$\times \left(\left[\beta^{+j_a}{}_{(N)}\, \beta^{+j_b}{}_{(N)} \right]^{J_0} \beta^{+j_c}{}_{(P)} \right)^{J}_{M} \Big] |\hat{0}_N ; \hat{0}_P\rangle$$

(I67)

One wants to calculate

$$\langle \tilde{0}_N ; \tilde{j}_{cp} \tilde{m}_{cp} | H^{k}_{N,P\,(31)} | [(\tilde{j}_{a_N}' \tilde{j}_{b_N}') J_0' \tilde{j}_{cp}'] JM \rangle$$

(C2a.1)

with

$$| [(\tilde{j}_{a_N}' \tilde{j}_{b_N}') J_0' j_{cp}'] JM \rangle \equiv \frac{1}{\sqrt{1+\delta_{j_{a_N}' j_{b_N}'}}} \times \left(\left[\beta^{+j_a'}_{(N)} \beta^{+j_b'}_{(N)} \right]^{J_0'} \beta^{+j_c'}_{(P)} \right)^{J}_{M} | \tilde{0}_N ; \tilde{0}_P \rangle$$

(C2a.2)

Only the $\left[\gamma^{j_1}_{(N)} \gamma^{j_2}_{(N)} \right]$ term of (II15) will contribute to (C2a.1). Inserting (II15), (C2a.2), (I4), and (C1a.3) into (C2a.1) gives

$$- \frac{1}{\sqrt{1+\delta_{j_{a_N}' j_{b_N}'}}} \frac{4\pi F^k}{2k+1} \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} q^k(j_{1N} j_{2N}) q^k(j_{3P} j_{4P}) (-1)^{l_{1N}} N_{1N} u_{2N} \sum_{\substack{q, m_{1N}, m_{2N} \\ m_{3P}, m_{4P} \\ m_{cP}, m' \\ m_{a_N}', m_{b_N}'}}$$

$$\times (-1)^{-q - j_{cp} - m_{cp}} \langle j_{1N} m_{1N} j_{2N} m_{2N} | k q \rangle$$

$$\times \langle j_{3P} m_{3P} j_{4P} m_{4P} | k -q \rangle \langle j_{a_N}' m_{a_N}' j_{b_N}' m_{b_N}' | J_0' m' \rangle$$

$$\times \langle J_0' m' j_{cp}' m_{cp}' | JM \rangle \langle \tilde{0}_P | \gamma^{j_c}_{-m_c}(P) \beta^{+j_3}_{m_3}(P) \gamma^{j_4}_{m_4}(P) \beta^{+j_c'}_{m_c'}(P) | \tilde{0}_P \rangle$$

$$\times \langle \tilde{0}_N | \gamma^{j_1}_{m_1}(N) \gamma^{j_2}_{m_2}(N) \beta^{+j_a'}_{m_a'}(N) \beta^{+j_b'}_{m_b'}(N) | \tilde{0}_N \rangle$$

(C2a.3)

By using the anti-commutation rule (I20'c), applying the resultant Kronecker δ's involving quasi-protons, and using 3-j notation (A3.10), one may write (C2a.3) as

$$\frac{1}{\sqrt{1+\delta_{j'_{a_N} j'_{b_N}}}} 4\pi \sqrt{2J'_0+1}\; F^k q^k(j_{cp} j'_{cp})(-1)^{j'_{cp}-j_{cp}} \sum_{j_{1N} j_{2N}} q^k(j_{1N} j_{2N})$$

$$\times (-1)^{l_{1N}} n_{1N} w_{2N} \sqrt{2J+1} \sum_{m'} \begin{pmatrix} j_{cp} & j'_{cp} & k \\ m_{cp} & -M+m' & -m_{cp}+M-m' \end{pmatrix} \begin{pmatrix} J'_0 & j'_{cp} & J \\ m' & M-m' & -M \end{pmatrix}$$

$$\times (-1)^{J'_0} \sum_{\substack{m_{1N} m_{2N} \\ m'_{a_N} m'_{b_N}}} (-1)^{j_{1N}+j_{2N}+m_{cp}-M} \begin{pmatrix} j_{1N} & j_{2N} & k \\ m_{1N} & m_{2N} & m_{cp}-M+m' \end{pmatrix}$$

$$\times \begin{pmatrix} j'_{a_N} & j'_{b_N} & J'_0 \\ m'_{a_N} & m'_{b_N} & -m' \end{pmatrix} \left(\delta_{b'_N,-1_N}\, \delta_{a'_N,-2_N} - \delta_{a'_N,-1_N}\, \delta_{b'_N,-2_N}\right)$$

(C2a.4)

where the notation of (A1.4) is used. The orthogonality of 3-j symbols (A3.15) applied to the sums over m_{1N}, m_{2N}, m'_{a_N}, m'_{b_N} requires $\delta_{k J'_0}$. By again applying (A3.15), the parity rule (II2), and the rule for index permutation in $q^k(j'_{a_N} j'_{b_N})$ (I30), one gets for the $\delta_{b'_N,-1_N}\delta_{a'_N,-2_N}$ terms of (C2a.4)

$$-\frac{1}{\sqrt{1+\delta_{j'_{a_N} j'_{b_N}}}} \frac{4\pi}{\sqrt{2k+1}} F^k q^k(j'_{cp} J)\, q^k(j'_{a_N} j'_{b_N}) (-1)^{l'_{a_N}} n'_{b_N} w'_{a_N} \frac{1}{\sqrt{2J+1}}$$

$$\times\, \delta_{j_{cp} J}\, \delta_{m_{cp} M}\, \delta_{J'_0 k} \qquad \text{(C2a.5)}$$

One may easily verify that the $-\delta_{a'_N,-1_N}\delta_{b'_N,-2_N}$ term of (C2a.4) should be just $-(-1)^{j'_{a_N}+j'_{b_N}+k}$ times (C2a.5) provided one interchanges j'_{a_N} and j'_{b_N} in (C2a.5). The $(-1)^{j'_{a_N}+j'_{b_N}+k}$ comes from permutation in a 3-j symbol. Thus this term is

$$-\frac{\delta_{j_{cp} J}\, \delta_{m_{cp} M}\, \delta_{J'_0 k}}{\sqrt{1+\delta_{j'_{a_N} j'_{b_N}}}} \frac{4\pi}{\sqrt{2k+1}} F^k q^k(j'_{cp} J)\, q^k(j'_{b_N} j'_{a_N}) \frac{(-1)^{l'_{b_N}} n'_{a_N} w'_{b_N}}{\sqrt{2J+1}} \times(-)(-1)^{j'_{a_N}+j'_{b_N}+k} \qquad \text{(C2a.6)}$$

Invoking the parity rule (II2), and the permutation rule (I30) in (C2a.6) yields

$$-\frac{1}{\sqrt{1+\delta_{j'_{a_N} j'_{b_N}}}} \frac{4\pi}{\sqrt{2k+1}} F^k q^k(j'_{c_p} J) q^k(j'_{a_N} j'_{b_N})$$

$$\times (-1)^{\ell'_{a_N}} \nu_{j'_{b_N}} \mu_{j'_{a_N}} \frac{1}{\sqrt{2J+1}} \delta_{j'_{c_p} J} \delta_{m_{c_p} M} \delta_{J'_0 k}$$

(C2a.7)

Combining (C2a.5) and (C2a.7) one has for (C2a.1)

$$\langle \tilde{0}_N ; \tilde{j}_{c_p} \tilde{m}_{c_p} | \underset{(31)}{H^k_{N,P}} | [(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J'_0 \tilde{j}'_{c_p}] JM \rangle$$

$$= -\frac{1}{\sqrt{1+\delta_{j'_{a_N} j'_{b_N}}}} \frac{4\pi F^k}{\sqrt{(2k+1)(2J+1)}} q^k(j'_{c_p} J) q^k(j'_{a_N} j'_{b_N})$$

$$\times (-1)^{\ell'_{a_N}} \sin(\chi'_{a_N} + \chi'_{b_N}) \delta_{J j'_{c_p}} \delta_{M m_{c_p}} \delta_{k J'_0}$$

(C2a.8)

b) $\underset{(22)}{H^k_{N,P}}$

This long-range interaction term is given by

$$\underset{(22)}{H^k_{N,P}} = -(-1)^k \frac{4\pi}{\sqrt{2k+1}} F^k \sum_{\substack{j_{1N} j_{2N} \\ j_{3P} j_{4P}}} q^k(j_{1N} j_{2N}) q^k(j_{3P} j_{4P})$$

$$\times \cos(\chi_{1N} + \chi_{2N}) \left\{ [\beta^{+ j_1}_{(N)} \gamma^{j_2 (N)}]^k \, [\beta^{+ j_3 (P)} \gamma^{j_4 (P)}]^k \right\}^0_0$$

(II5)

The set of states (I67) used in Appendix (C2a) is also used here. In particular one calculates

$$\langle [(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{c_p}] JM | \underset{(22)}{H^k_{N,P}} | [(\tilde{j}'_{a_N} \tilde{j}'_{b_N}) J'_0 \tilde{j}'_{c_p}] J'M' \rangle$$

(C2b.1)

The bra and ket notation of (C2b.1) are defined via (C2a.2) and its adjoint. Inserting (II5) into (C2b.1) involves the calculation of

$$\langle \tilde{0}_\rho | \gamma^{j_c}_{-m_c}(\rho) \beta^{+j_3}_{m_3}(\rho) \gamma^{j_4}_{m_4}(\rho) \beta^{+j_c'}_{m_c'}(\rho) | \tilde{0}_\rho \rangle \times$$

$$\times \langle \tilde{0}_\nu | \gamma^{j_b}_{(\nu)-m_b} \gamma^{j_a}_{(\nu)-m_a} \beta^{+j_1}_{(\nu)m_1} \gamma^{j_2}_{(\nu)m_2} \beta^{+j_a'}_{(\nu)m_a'} \beta^{+j_b'}_{(\nu)m_b'} | \tilde{0}_\nu \rangle \quad \text{(C2b.2)}$$

where use has been made of equation (I4). Now (C2b.2) has been evaluated previously in Appendix (C1b) is from (C1b.7)

$$(-1)^{j_{3\rho}+m_{3\rho}+j_{c\rho}'+m_{c\rho}'+j_{1\nu}+m_{1\nu}+j_{a\nu}'+m_{a\nu}'+j_{b\nu}'+m_{b\nu}'}$$

$$\times \delta_{c\rho',-4\rho}\,\delta_{c\rho,3\rho}\Big[\big(\delta_{b_\nu' b_\nu}\delta_{1_\nu a_\nu} - \delta_{1_\nu b_\nu}\delta_{b_\nu' a_\nu}\big)\delta_{a_\nu',-2_\nu}$$

$$-\big(\delta_{a_\nu' b_\nu}\delta_{1_\nu a_\nu} - \delta_{a_\nu b_\nu}\delta_{a_\nu' a_\nu}\big)\Big]$$

(C2b.3)

where the notation of (A1.4) is used. Applying $\delta_{c\rho',-4\rho}\delta_{c\rho,3\rho}$ of (C2b.3) and somewhat simplifying the Clebsch-Gordon coefficients enable one to write (C2b.1) as

$$-\frac{1}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j_{a_\nu}' j_{b_\nu}'})}}\;\frac{4\pi}{2k+1}F^k g^k(j_{c\rho}j_{c\rho}')\sum_{j_{1\nu}j_{2\nu}} g^k(j_{1\nu}j_{2\nu})$$

$$\times \cos(\varphi_{1\nu}+\varphi_{2\nu}) \sum_{m'm}\;\sum_{m_{1\nu}m_{2\nu}}\;\sum_{\substack{m_{a_\nu}'m_{b_\nu}'\\ m_{a_\nu}m_{b_\nu}}} \langle j_{1\nu}m_{1\nu}j_{2\nu}m_{2\nu}|k\; m-m'\rangle$$

$$\times \langle j_{a_\nu}' m_{a_\nu}' j_{b_\nu}' m_{b_\nu}' | J_0' m'\rangle \langle j_{a_\nu} m_{a_\nu} j_{b_\nu} m_{b_\nu} | J_0 m\rangle$$

$$\times (-1)^{-j_{b_\nu}-m_{b_\nu}-j_{a_\nu}-m_{a_\nu}+j_{1\nu}+m_{1\nu}+j_{a_\nu}'+m_{a_\nu}'+j_{b_\nu}'+m_{b_\nu}'-m+m'}$$

$$\times \langle j_{c\rho}\; M-m\; j_{c\rho}'\; -M+m' \,|\, k\; -m+m'\rangle$$

$$\times \langle J_0' m' \, j_{cp}' \, M-m' | J' M \rangle \langle J_0 \, m \, j_{cp} \, M-m | J M \rangle$$

$$\times (-1)^{j_{cp}' + M - m'} \delta_{MM'} \Big[\left(\delta_{b_\nu' b_\nu} \delta_{1_\nu a_\nu} - \delta_{1_\nu b_\nu} \delta_{b_\nu' a_\nu} \right) \delta_{a_\nu', -2_\nu}$$

$$- \left(\delta_{a_\nu' b_\nu} \delta_{1_\nu a_\nu} - \delta_{1_\nu b_\nu} \delta_{a_\nu' a_\nu} \right) \delta_{b_\nu', -2_\nu} \Big] \qquad \text{(C2b.4)}$$

One need only evaluate the $\delta_{b_\nu' b_\nu} \delta_{1_\nu a_\nu} \delta_{a_\nu', -2_\nu}$ term since the other groups of Kronecker deltas are merely permutations of this term. This $\delta_{b_\nu' b_\nu} \delta_{1_\nu a_\nu} \delta_{a_\nu', -2_\nu}$ term is evaluated in a completely analogous way to term (a) of (C1b.8) with two exceptions. First, in the present case there is a sum over only two dummy angular momenta j_{1_ν} and j_{2_ν}, rather than over j_{1_ν}, j_{2_ν}, j_{a_ν}, j_{b_ν}, $j_{a_\nu'}$, and $j_{b_\nu'}$. The other exception is the absence of the $\mathcal{N}(j_{a_\nu} j_{b_\nu})$ and $\mathcal{N}(j_{a_\nu'} j_{b_\nu'})$ coefficients, which are only germaine to the quasi-Boson calculation. The $\delta_{b_\nu' b_\nu} \delta_{1_\nu a_\nu} \delta_{a_\nu', -2_\nu}$ term becomes

$$-4\pi (-1)^k F^k \sqrt{(2J_0+1)(2J_0'+1)} \, q^k(j_{cp} j_{cp}') (-1)^{j_{cp}+J}$$

$$\times \begin{Bmatrix} J_0' & J_0 & k \\ j_{cp} & j_{cp}' & J \end{Bmatrix} q^k(j_{a_\nu} j_{a_\nu}') \cos(\tau_{a_\nu} + \tau_{a_\nu'}) (-1)^{j_{a_\nu} + j_{b_\nu}}$$

$$\times \begin{Bmatrix} J_0' & J_0 & k \\ j_{a_\nu} & j_{a_\nu}' & j_{b_\nu} \end{Bmatrix} \delta_{j_{b_\nu} j_{b_\nu}'} \frac{\delta_{JJ'} \delta_{MM'}}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j_{a_\nu}' j_{b_\nu}'})}} \qquad \text{(C2b.5)}$$

The second group of Kronecker deltas, $-\delta_{1_\nu b_\nu}\delta_{b_\nu' a_\nu}$ $\times\delta_{a_\nu',-2_\nu}$, in (C2b.4) leads to simply $-(-1)^{j_{a_\nu}+j_{b_\nu}}$ $\times(-1)^{J_0}$ times (C2b.5) provided one interchanges j_{a_ν} and j_{b_ν} in (C2b.5). The result is

$$4\,\eta\,(-1)^k F^k \sqrt{(2J_0+1)(2J_0'+1)}\; q^k(j_{cp}\,j_{cp}')(-1)^{j_{cp}+J}\begin{Bmatrix} J_0' & J_0 & k \\ j_{cp} & j_{cp}' & J\end{Bmatrix}$$

$$\times\, q^k(j_{b_\nu}\,j_{a_\nu}')\cos(\tau_{b_\nu}+\tau_{a_\nu'})\begin{Bmatrix} J_0' & J_0 & k \\ j_{b_\nu} & j_{a_\nu}' & j_{a_\nu}\end{Bmatrix}(-1)^{J_0}\,\delta_{j_{a_\nu} j_{b_\nu}'}$$

$$\times\,\frac{\delta_{JJ'}\,\delta_{MM'}}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j_{a_\nu}' j_{b_\nu}'})}} \qquad \text{(C2b.6)}$$

The third group of Kronecker deltas, $-\delta_{a_\nu' b_\nu}\delta_{1_\nu a_\nu}$ $\times\delta_{b_\nu',2_\nu}$ in (C2b.4) implies just $-(-1)^{j_{a_\nu}'+j_{b_\nu}'+J_0'}$ times (C2b.5) provided j_{a_ν}' and j_{b_ν}' are interchanged in (C2b.5). The result is

$$-\,4\,\eta\,(-1)^k F^k \sqrt{(2J_0+1)(2J_0'+1)}\; q^k(j_{cp}\,j_{cp}')(-1)^{j_{cp}+J}$$

$$\times\begin{Bmatrix} J_0' & J_0 & k \\ j_{cp} & j_{cp}' & J\end{Bmatrix} q^k(j_{a_\nu}\,j_{b_\nu}')\cos(\tau_{j_{a_\nu}}+\tau_{j_{b_\nu}'})(-1)^{j_{a_\nu}+j_{b_\nu}'}$$

$$\times\begin{Bmatrix} J_0' & J_0 & k \\ j_{a_\nu} & j_{b_\nu}' & j_{b_\nu}\end{Bmatrix}(-1)^{J_0'}\,\delta_{j_{b_\nu} j_{a_\nu}'}\;\frac{\delta_{JJ'}\,\delta_{MM'}}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j_{a_\nu}' j_{b_\nu}'})}} \qquad \text{(C2b.7)}$$

The last group of Kronecker deltas, $\delta_{1_\nu b_\nu}\delta_{a_\nu' a_\nu}\delta_{b_\nu',-2_\nu}$, in (C2b.4) gives $-(-1)^{j_{a_\nu}'+j_{b_\nu}'+J_0'}$ times (C2b.6) with j_{a_ν}' and j_{b_ν}' interchanged in (C2b.6). Also this last group of deltas yields $-(-1)^{j_{a_\nu}+j_{b_\nu}+J_0}$ times

(C2b.7) with $j_{a\nu}$ and $j_{b\nu}$ interchanged in (C2b.7). Either way the result is

$$-4\pi(-1)^k F^k \sqrt{(2J_0+1)(2J_0'+1)}\; g^k(j_{cp}\, j_{cp}')$$
$$\times (-1)^{j_{cp}+J}\begin{Bmatrix} J_0' & J_0 & k \\ j_{cp} & j_{cp}' & J\end{Bmatrix} g^k(j_{b\nu}\, j_{b\nu}')$$
$$\times \cos(\tau_{b\nu}+\tau_{b\nu}')\,(-1)^{j_{a\nu}+j_{b\nu}'}\begin{Bmatrix} J_0' & J_0 & k \\ j_{b\nu} & j_{b\nu}' & j_{a\nu}\end{Bmatrix}$$
$$\times (-1)^{J_0+J_0'}\;\frac{\delta_{j_{a\nu}j_{a\nu}'}\,\delta_{JJ'}\,\delta_{MM'}}{\sqrt{(1+\delta_{j_{a\nu}j_{b\nu}})(1+\delta_{j_{a\nu}'j_{b\nu}'})}} \qquad \text{(C2b.8)}$$

To check for Hermiticity one interchanges $j_a \leftrightarrow j_a'$, $j_b \leftrightarrow j_b'$, $J_0 \leftrightarrow J_0'$, and finally $j_{cp} \leftrightarrow j_{cp}'$, in equations (C2b.5), (C2b.6), (C2b.7), and (C2b.8). By invoking the rule for permutation of the indices of g^k (İ30), one easily gets that:

(C2b.5) remains (C2b.5)

(C2b.6) becomes (C2b.7)

(C2b.7) becomes (C2b.6)

and

(C2b.8) remains (C2b.8)

This means that combining (C2b.5), (C2b.6), (C2b.7), and (C2b.8) produces

$$\langle [(\tilde{j}_{a_N}\tilde{j}_{b_N})J_0\tilde{j}_{c_P}]JM|H^{k}_{N,P\,(22)}|[(\tilde{j}_{a_N}'\tilde{j}_{b_N}')J_0'\tilde{j}_{c_P}']J'M'\rangle$$

$$=\langle [(\tilde{j}_{a_N}'\tilde{j}_{b_N}')J_0'\tilde{j}_{c_P}']JM|H^{k}_{N,P\,(22)}|[(\tilde{j}_{a_N}\tilde{j}_{b_N})J_0\tilde{j}_{c_P}]JM\rangle$$

$$=(-1)^{k}(\delta_{JJ'}\delta_{MM'})\,4\pi F^{k}\,q^{k}(j_{c_P}j_{c_P}')(-1)^{j_{c_P}+J}\begin{Bmatrix} J_0' & J_0 & k\\ j_{c_P} & j_{c_P}' & J\end{Bmatrix}$$

$$\times\frac{\sqrt{(2J_0+1)(2J_0'+1)}}{\sqrt{(1+\delta_{j_{a_N}j_{b_N}})(1+\delta_{j_{a_N}'j_{b_N}'})}}\Bigg[-q^{k}(j_{a_N}j_{a_N}')\cos(\tau_{a_N}+\tau_{a_N}')$$

$$\times(-1)^{j_{a_N}+j_{b_N}}\begin{Bmatrix} J_0' & J_0 & k\\ j_{a_N} & j_{a_N}' & j_{b_N}\end{Bmatrix}\delta_{j_{b_N}j_{b_N}'}+q^{k}(j_{b_N}j_{a_N}')\cos(\tau_{b_N}+\tau_{a_N}')$$

$$\times(-1)^{J_0}\begin{Bmatrix} J_0' & J_0 & k\\ j_{b_N} & j_{a_N}' & j_{a_N}\end{Bmatrix}\delta_{j_{a_N}j_{b_N}'}-q^{k}(j_{a_N}j_{b_N}')\cos(\tau_{a_N}+\tau_{b_N}')$$

$$\times(-1)^{j_{a_N}+j_{b_N}'+J_0'}\begin{Bmatrix} J_0' & J_0 & k\\ j_{a_N} & j_{b_N}' & j_{b_N}\end{Bmatrix}\delta_{j_{b_N}j_{a_N}'}-q^{k}(j_{b_N}j_{b_N}')$$

$$\times\cos(\tau_{b_N}+\tau_{b_N}')(-1)^{j_{a_N}+j_{b_N}'+J_0+J_0'}\begin{Bmatrix} J_0' & J_0 & k\\ j_{b_N} & j_{b_N}' & j_{a_N}\end{Bmatrix}$$

$$\times\,\delta_{j_{a_N}j_{a_N}'}\Bigg]$$

(II21)

c) $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 1)

This long-range Hamiltonian term is

$$\mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class 1}) = \frac{4\pi F^k}{\sqrt{2k+1}} \sum_{\substack{j_{1N} j_{2N}\\ j_{3N} j_{4N}}} q^k(j_{1N} j_{2N}) q^k(j_{3N} j_{4N})$$

$$\times\, u_{1N} v_{2N} u_{3N} v_{4N} (-1)^{l_{1N}+l_{3N}} \Big\{ \big[\beta^{+j_1}_{(N)} \beta^{+j_2}_{(N)}\big]^k$$

$$\times \big[\gamma(N)^{j_3} \gamma(N)^{j_4}\big]^k + \big[\gamma^{j_1}(N) \gamma^{j_2}(N)\big]^k \big[\beta^{+j_3}_{(N)} \beta^{+j_4}_{(N)}\big]^k \Big\}^0_0$$

(II18)

The matrix elements to be considered are with respect to the set (I67) and are typified by

$$\big\langle \big[(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0\, \tilde{j}_{c_p}\big] JM \big| \mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class 1}) \big| \big[(\tilde{j}_{a_N}{}' \tilde{j}_{b_N}{}') J_0{}'$$

$$\times\, \tilde{j}_{c_p}{}'\big] J'M' \big\rangle \qquad \text{(C2c.1)}$$

with $[(\tilde{j}_{a_N}{}' \tilde{j}_{b_N}{}') J_0{}' \tilde{j}_{c_p}{}'] J'M'$ defined in (C2a.2) and with $\langle [(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{c_p}] JM$ the adjoint of (C2a.2).

Equation (II18) may conveniently be broken up into two parts (A) and (B) where

$$(A) \equiv \frac{4\pi F^k}{\sqrt{2k+1}} \sum_{\substack{j_{1N} j_{2N}\\ j_{3N} j_{4N}}} q^k(j_{1N} j_{2N}) q^k(j_{3N} j_{4N}) u_{1N} v_{2N} u_{3N} v_{4N} (-1)^{l_{1N}+l_{3N}}$$

$$\times \Big\{ \big[\beta^{+j_1}_{(N)} \beta^{+j_2}_{(N)}\big]^k \big[\gamma^{j_3}(N) \gamma^{j_4}_{(N)}\big]^k \Big\}^0_0$$

(C2c.2)

and

$$(B) = \frac{4\pi F^k}{\sqrt{2k+1}} \sum_{\substack{j_{1N} j_{2N} \\ j_{3N} j_{4N}}} q^k(j_{1N}\, j_{2N})\, q^k(j_{3N}\, j_{4N})\, w_{1N}\, v_{2N}\, w_{3N}\, v_{4N}$$

$$\times (-1)^{l_{1N}+l_{3N}} \left\{ \left[\gamma^{j_1}{(N)}\, \gamma^{j_2}(N) \right]^k \left[\beta^{+j_3}(N)\, \beta^{+j_4}(N) \right]^k \right\}^0_0$$

(C2c.3)

Inserting (A) into (C2c.1), and using (C2a.2), (I4), and (C1a.3) gives

$$\frac{4\pi F^k}{2k+1} (-1)^k \sum_{\substack{j_{1N} j_{2N} \\ j_{3N} j_{4N}}} q^k(j_{1N}\, j_{2N})\, q^k(j_{3N}\, j_{4N}) (-1)^{l_{1N}+l_{3N}}$$

$$\times w_{1N}\, v_{2N}\, w_{3N}\, v_{4N} \sum_{\substack{m_{1N} m_{2N} \\ q\, m_{a_N}' m_{b_N}'}} \sum_{\substack{m_{3N} m_{4N} \\ m' m_{cp}'}} \sum_{\substack{m_b\, m\, m_{cp} \\ m_a}} \langle j_{1N}\, m_{1N}\, j_{2N}\, m_{2N} | k\, q \rangle$$

$$\times \langle j_{3N}\, m_{3N}\, j_{4N}\, m_{4N} | k\, -q \rangle \langle j_{a_N}'\, m_{a_N}'\, j_{b_N}'\, m_{b_N}' | J_0'\, m' \rangle$$

$$\times \langle J_0'\, m'\, j_{cp}'\, m_{cp}' | J' M' \rangle \langle j_a\, m_a\, j_b\, m_b | J_0\, m \rangle$$

$$\times \langle J_0\, m\, j_{cp}\, m_{cp} | J M \rangle (-1)^{-j_a - m_a - j_b - m_b - j_{cp} - m_{cp} - q}$$

$$\times \langle \hat{0}_P | \gamma^{j_c}_{-m_c}(P)\, \beta^{+j_c'}_{m_c'}(P) | \hat{0}_P \rangle \frac{1}{\sqrt{(1+\delta_{j_{a_N} j_{b_N}})(1+\delta_{j_{a_N}' j_{b_N}'})}}$$

$$\times \langle \hat{0}_N | \gamma^{j_b}_{-m_b}(N)\, \gamma^{j_a}_{-m_a}(N)\, \beta^{+j_1}_{m_1}(N)\, \beta^{+j_2}_{m_2}(N)\, \gamma^{j_3}_{m_3}(N)\, \gamma^{j_4}_{m_4}(N)$$

$$\times \beta^{+j_a'}_{m_a'}(N)\, \beta^{+j_b'}_{m_b'}(N) | \hat{0}_N \rangle$$

(C2c.4)

By repeated use of the commutation rule (I20'c) all annihilation operators may be brought to the right. This leads to

$$- (-1)^k \frac{4\pi}{2k+1} F^k \sum_{\substack{j_{1\nu} j_{2\nu} \\ j_{3\nu} j_{4\nu}}} q^k(j_{1\nu} j_{2\nu})\, q^k(j_{3\nu} j_{4\nu})\, (-1)^{l_{1\nu}+l_{3\nu}}$$

$$\times u_{1\nu}\, v_{2\nu}\, u_{3\nu}\, v_{4\nu} \frac{1}{\sqrt{(1+\delta_{j_{a\nu} j_{b\nu}})(1+\delta_{j'_{a\nu} j'_{b\nu}})}} \sum_{\substack{m_{1\nu} m_{2\nu} \\ m_{3\nu} m_{4\nu}}} \sum_{\substack{m'_{a\nu} m'_{b\nu} \\ m_{a\nu} m_{b\nu} m \\ q}}$$

$$\times \langle j_{a\nu} m_{a\nu} j_{b\nu} m_{b\nu} | J_0 m \rangle (-1)^{-m_{1\nu} - m_{2\nu}}$$

$$\times \langle J_0' \; m-M+M' \; j_{cp} \; M-m | J'M' \rangle \langle J_0 m \; j_{cp} \; M-m | JM \rangle$$

$$\times \delta_{j'_{cp} j_{cp}} \left(\delta_{2_\nu a_\nu} \delta_{1_\nu b_\nu} - \delta_{1_\nu a_\nu} \delta_{2_\nu b_\nu} \right) \left(\delta_{a'_\nu, -4_\nu} \delta_{b'_\nu, -3_\nu} \right.$$

$$\left. - \delta_{b'_\nu, -4_\nu} \delta_{a'_\nu, -3_\nu} \right) (-1)^{j'_{a\nu} + m'_{a\nu} + j'_{b\nu} + m'_{b\nu}}$$

$$\times \langle j_{1\nu} m_{1\nu} j_{2\nu} m_{2\nu} | k\, q \rangle \langle j_{3\nu} m_{3\nu} j_{4\nu} m_{4\nu} | k \, -q \rangle$$

$$\times \langle j'_{a\nu} m'_{a\nu} j'_{b\nu} m'_{b\nu} | J_0' \; m-M+M' \rangle$$

(C2c.5)

in the notation of (A1.4).

Only one term of (C2c.5) involving Kronecker deltas need be evaluated. If, for example, one calculates the term involving $\delta_{2_\nu a_\nu} \delta_{1_\nu b_\nu} \delta_{a'_\nu, -4_\nu} \delta_{b'_\nu, -3_\nu}$ then one notes that the other terms differ only in permutations of indices.

Calling the $\delta_{2_\nu a_\nu}\delta_{1_\nu b_\nu}\delta_{a'_\nu,-4_\nu}\delta_{b'_\nu,-3_\nu}$ term, λ, there follows:

The $-\delta_{1_\nu a_\nu}\delta_{2_\nu b_\nu}\delta_{a'_\nu,-4_\nu}\delta_{b'_\nu,-3_\nu}$ term $\longrightarrow -(-1)^{j_{a_\nu}+j_{b_\nu}+J_0}\lambda$, provided j_{a_ν} and j_{b_ν} are interchanged in λ.

The $-\delta_{2_\nu a_\nu}\delta_{1_\nu b_\nu}\delta_{b'_\nu,-4_\nu}\delta_{a'_\nu,-3_\nu}$ term $\longrightarrow -(-1)^{j'_{a_\nu}+j'_{b_\nu}+J'_0}\lambda$, provided j'_{a_ν} and j'_{b_ν} are interchanged in λ.

The $\delta_{1_\nu a_\nu}\delta_{2_\nu b_\nu}\delta_{b'_\nu,-4_\nu}\delta_{a'_\nu,-3_\nu}$ term $\longrightarrow (-1)^{j_{a_\nu}+j_{b_\nu}+j'_{a_\nu}+j'_{b_\nu}+J_0+J'_0}\lambda$, provided j_a and j_b in addition to j'_{a_ν} and j'_{b_ν} are interchanged in λ. These conditions are just due to the Clebsch-Gordon coefficients.

The determination of term λ is quite easy in that these coefficients may be grouped in pairs. Thus λ becomes

$$\frac{-4\pi}{2k+1}F^k g^k(j_{a_\nu}j_{b_\nu})\,g^k(j'_{a_\nu}j'_{b_\nu})(-1)^{l_{b_\nu}+l'_{b_\nu}}w_{b_\nu}N_{a_\nu}\,w'_{b_\nu}N'_{a_\nu}\,\delta_{J_0J'_0}\delta_{J'_0k}\delta_{j_{cp}j'_{cp}}\delta_{MM'}\delta_{JJ'}\frac{1}{\sqrt{(1+\delta_{j_{a_\nu}j_{b_\nu}})(1+\delta_{j'_{a_\nu}j'_{b_\nu}})}} \qquad \text{(C2c.6)}$$

Combination of λ as given by (C2c.6) and the other three terms mentioned in the above paragraph gives for the matrix element of (A)

$$\langle[(\tilde{j}_{a_\nu}\hat{j}_{b_\nu})J_0\hat{j}_{cp}]JM|(A)|[(\tilde{j}'_{a_\nu}\tilde{j}'_{b_\nu})J'_0\hat{j}'_{cp}]J'M'\rangle$$

$$= \frac{-4\pi}{2k+1}F^k g^k(j_{a_\nu}j_{b_\nu})\,g^k(j'_{a_\nu}j'_{b_\nu})(-1)^{l_{b_\nu}+l'_{b_\nu}}$$

$$\times \sin(x_{a_\nu}+x_{b_\nu})\sin(x'_{a_\nu}+x'_{b_\nu})\,\delta_{J_0J'_0}\delta_{J'_0k}\delta_{j_{cp}j'_{cp}}$$

$$\times \frac{1}{\sqrt{(1+\delta_{j_{a_\nu}j_{b_\nu}})(1+\delta_{j'_{a_\nu}j'_{b_\nu}})}}\,\delta_{MM'}\delta_{JJ'} \qquad \text{(C2c.7)}$$

By interchanging j_{a_N} & j_{a_N}', j_{b_N} & j_{b_N}', l_{b_N} & l_{b_N}', J_0 & J_0', and finally j_{cp} and j_{cp}', Hermiticity is verified, i.e.,

$$\langle [(\tilde{j}_{a_N}' \tilde{j}_{b_N}')J_0' \tilde{j}_{cp}'] J'M' | (A) | [(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{cp}] JM \rangle$$

$$= \langle [(\tilde{j}_{a_N} \tilde{j}_{b_N}) J_0 \tilde{j}_{cp}] JM | (A) | [(\tilde{j}_{a_N}' \tilde{j}_{b_N}') J_0' \tilde{j}_{cp}'] J'M' \rangle$$

(C2c.8)

Inserting (B) (C2c.3) into (C2c.1), using (C2a.2) and its adjoint, (I4), and (C1a.3), and finally summing over quasi-proton indices gives

$$-(-1)^k \frac{4\pi}{2k+1} F^k \frac{1}{\sqrt{(1+\delta_{j_{a_N} j_{b_N}})(1+\delta_{j_{a_N}' j_{b_N}'})}} \sum_{\substack{j_{1N} j_{2N} \\ j_{3N} j_{4N}}} q^k(j_{1N} j_{2N})$$

$$\times q^k(j_{3N} j_{4N}) (-1)^{l_{1N}+l_{3N}} u_{1N} v_{2N} u_{3N} v_{4N} \sum_{\substack{m_{1N} m_{2N} \\ m_{3N} m_{4N} \\ m_{a'N} m_{b'N} \\ m_{a_N} m_{b_N} m\, q}}$$

$$\times \langle j_{1N} m_{1N} j_{2N} m_{2N} | k\, q \rangle \langle j_{3N} m_{3N} j_{4N} m_{4N} | k\, -q \rangle$$

$$\times \langle j_{a_N}' m_{a_N}' j_{b_N}' m_{b_N}' | J_0' \; m-M+M' \rangle \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | J_0 m \rangle$$

$$\times (-1)^{-j_{a_N} - m_{a_N} - j_{b_N} - m_{b_N} - m_{1N} - m_{2N}} \langle J_0' \; m-M+M' \; j_{cp} \; M-m | J'M' \rangle$$

$$\times \langle J_0 \, m \, j_{cp} \, M-m | J M \rangle \delta_{j_{cp} j_{cp}'} \langle \tilde{0}_N | \gamma^{j_a}_{-m_a}(N) \gamma^{j_b}_{-m_b}(N)$$

$$\times \gamma^{j_1}_{m_1}(N) \gamma^{j_2}_{m_2}(N) \beta^{+j_3}_{m_3}(N) \beta^{+j_4}_{m_4}(N) \beta^{+j_a'}_{m_a'}(N) \beta^{+j_b'}_{m_b'}(N) | \tilde{0}_N \rangle \qquad \text{(C2c.9)}$$

The evaluation of the matrix element involving operators in (C2c.9) leads to (in the notation of (A1.4))

$$\langle \tilde{O}_N | \gamma^{j_a}_{-m_a}(N)\, \gamma^{j_b}_{-m_b}(N)\, \gamma^{j_1}_{m_1}(N)\, \gamma^{j_2}_{m_2}(N)\, \beta^{+j_3}_{m_3}(N)\, \beta^{+j_4}_{m_4}(N)\, \beta^{+j_a'}_{m_a'}(N)$$
$$\times\, \beta^{+j_b'}_{m_b'}(N) | \tilde{O}_N \rangle = \langle \tilde{O}_N | \gamma^{j_a}_{-m_a}(N)\, \gamma^{j_b}_{-m_b}(N)\, \beta^{+j_1}_{m_1}(N)$$
$$\times\, \beta^{+j_2}_{m_2}(N)\, \gamma^{j_3}_{m_3}(N)\, \gamma^{j_4}_{m_4}(N)\, \beta^{+j_5'}_{m_5'}(N)\, \beta^{+j_6'}_{m_6'}(N) | \tilde{O}_N \rangle$$
$$+ \Big(\langle \tilde{O}_N | \gamma^{j_a}_{-m_a}\, \gamma^{j_b}_{-m_b}(N)\, \gamma^{j_3}_{m_3}(N)\, \beta^{+j_2}_{m_2}(N)\, \beta^{+j_a'}_{m_a'}(N)\, \beta^{+j_b'}_{m_b'}(N) | \tilde{O}_N \rangle$$
$$\times\, \delta_{1_N,-4_N} (-1)^{j_{1_N}+m_{1_N}} - \langle \tilde{O}_N | \gamma^{j_a}_{-m_a}(N)\, \gamma^{j_b}_{-m_b}(N)\, \gamma^{j_3}_{m_3}(N)$$
$$\times\, \beta^{+j_1}_{m_1}(N)\, \beta^{+j_a'}_{m_a'}(N)\, \beta^{+j_b'}_{m_b'}(N) | \tilde{O}_N \rangle \delta_{2_N,-4_N} (-1)^{j_{2_N}+m_{2_N}} \Big)$$
$$+ \Big(\langle \tilde{O}_N | \gamma^{j_a}_{-m_a}(N)\, \gamma^{j_b}_{-m_b}(N)\, \beta^{+j_2}_{m_2}(N)\, \gamma^{j_4}_{m_4}(N)\, \beta^{+j_a'}_{m_a'}(N)\, \beta^{+j_b'}_{m_b'}(N)$$
$$\times\, | \tilde{O}_N \rangle \delta_{1_N,-3_N} (-1)^{j_{1_N}+m_{1_N}} - \langle \tilde{O}_N | \gamma^{j_a}_{-m_a}(N)\, \gamma^{j_b}_{-m_b}(N)$$
$$\times\, \beta^{+j_1}_{m_1}(N)\, \gamma^{j_4}_{m_4}(N)\, \beta^{+j_a'}_{m_a'}(N)\, \beta^{+j_b'}_{m_b'}(N) | \tilde{O}_N \rangle \delta_{2_N,-3_N} (-1)^{j_{2_N}+m_{3_N}} \Big)$$
$$\equiv (r) + (s) + (t)$$

(C2c.10)

where (r) is the first term of (C2c.10) and contains eight operators. The symbol, (s), stands for the second term and consists of two matrix elements each of which has six operators. The two elements differ only in the permutation of $j_{1\nu}, m_{1\nu}$ and $j_{2\nu}, m_{2\nu}$. The symbol (t) stands for the third and last term of (C2c.10). It also consists of two matrix elements of six operators each, with the two elements differing in the permutation of $j_{1\nu}, m_{1\nu}$ and $j_{2\nu}, m_{2\nu}$.

The symbol, (s), is just the quasi-neutron operator matrix element appearing in the matrix element of (A) (C2c.4), and leads to the same result as (C2c.7). One sees that this is the direct result of placing $[\gamma_{(\nu)}^{j_1}\gamma_{(\nu)}^{j_2}]^k_q$ to the right of $[\beta^{+j_3}_{(\nu)}\beta^{+j_4}{}_{(\nu)}]^k_{-q}$ in (II18). Or in the language of $B^{+k}_{-q}(j_{3\nu}j_{4\nu})$ and $C^k_q(j_{1\nu}j_{2\nu})$ (I38), one puts the C^k_q to the right of B^{+k}_q .

The term (s) (C2c.10) may be converted to the form of (t)(C2c.10) by a single use of the commutation rule (I20'c). If this rule is invoked several more times the contribution of part of (s) and all of (t) to the evaluation of (C2c.9) becomes

$$- (-1)^k \frac{4\pi}{2k+1} F^k \sum_{\substack{j_{1\nu} j_{2\nu} \\ j_{3\nu} j_{4\nu}}} g^k(j_{1\nu} j_{2\nu}) g^k(j_{3\nu} j_{4\nu}) (-1)^{l_{1\nu}+l_{3\nu}}$$

$$\times \sin(\nu_{1\nu} + \nu_{2\nu}) \sum_{\substack{m_{1\nu} m_{2\nu} \\ m_{3\nu} m_{4\nu}}} \sum_{\substack{m_{a\nu}' m_{b\nu}' \\ m_{a\nu} m_{b\nu} m}} \langle j_{1\nu} m_{1\nu} j_{2\nu} m_{2\nu} | k q \rangle$$

$$\times \langle j_{3\nu} m_{3\nu} j_{4\nu} m_{4\nu} | k -q \rangle \frac{1}{\sqrt{(1+\delta_{j_{a\nu} j_{b\nu}})(1+\delta_{j_{a\nu}' j_{b\nu}'})}}$$

$$\times \langle j'_{a_\nu} m'_{a_\nu} j'_{b_\nu} m'_{b_\nu} | J'_0 m-M+m' \rangle \langle j_{a_\nu} m_{a_\nu} j_{b_\nu} m_{b_\nu} | J_0 m \rangle$$

$$\times \langle J'_0 m-M+M' j_{cp} M-m | J'M' \rangle \langle J_0 m j_{cp} M-m | JM \rangle$$

$$\times (-1)^{-j_{a_\nu} - m_{a_\nu} - j_{b_\nu} - m_{b_\nu} + j_{1_\nu} + j_{2_\nu} + j'_{a_\nu} + m'_{a_\nu} + j'_{b_\nu} + m'_{b_\nu}}$$

$$\times \delta_{j_{cp} j'_{cp}} \sin(\gamma_{3_\nu} + \gamma_{4_\nu}) \Big[(\delta_{b'_\nu a_\nu} \delta_{2_\nu b_\nu} - \delta_{2_\nu a_\nu} \delta_{b'_\nu b_\nu})$$

$$\times \delta_{a'_\nu, -4_\nu} - (\delta_{a'_\nu a_\nu} \delta_{2_\nu b_\nu} - \delta_{2_\nu a_\nu} \delta_{a'_\nu b_\nu})$$

$$\times \delta_{b'_\nu, -4_\nu} \Big] \delta_{1_\nu, -3_\nu}$$

(C2c.11)

The notation of (A1.4) is again invoked.

As has occurred many times before, one need only evaluate one set of Kronecker deltas in (C2c.11). Doing this and combining all of the groups of Kronecker deltas in (C2c.11) leads to

$$4\pi F^k \delta_{JJ'} \delta_{MM'} \delta_{J_0 J'_0} \delta_{j_{cp} j'_{cp}} (1 + \delta_{j_{a_\nu} j_{b_\nu}}) \delta_{j_{a_\nu} j'_{a_\nu}}$$

$$\times \delta_{j_{b_\nu} j'_{b_\nu}} \sum_{j_{1_\nu}} \left(\frac{[q^k(j_{1_\nu} j_{b_\nu})]^2}{2j_{b_\nu}+1} \sin^2(\gamma_{1_\nu} + \gamma_{b_\nu}) \right.$$

$$\left. + \frac{[q^k(j_{1_\nu} j_{a_\nu})]^2}{2j_{a_\nu}+1} \sin^2(\gamma_{1_\nu} + \gamma_{a_\nu}) \right) \frac{1}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j'_{a_\nu} j'_{b_\nu}})}}$$

(C2c.12)

The part from (Δ) that has not yet been used is

$$\langle \tilde{0}_N | \gamma^{j_a}_{-m_a}(N)\, \gamma^{j_b}_{-m_b}(N)\, \beta^{+j_a'}_{m_a'}(N)\, \beta^{+j_b'}_{m_b'}(N) | \tilde{0}_N \rangle$$

$$\times\, \delta_{1_N,-4_N} (-1)^{j_{1_N}+m_{1_N}}\, \delta_{2_N,-3_N} (-1)^{j_{2_N}+m_{2_N}} \qquad \text{(C2c.13)}$$

The contribution of (C2c.13) to (C2c.9) is similar to (C2c.11). The important difference is that

$$u_{3_N} v_{4_N} (\delta_{6_N' 5_N} \delta_{5_N' 6_N} - \delta_{5_N' 5_N} \delta_{6_N' 6_N})\, \delta_{1_N,-4_N}\, \delta_{2_N,-3_N}$$

replaces

$$\Delta (v_{3_N}+v_{4_N}) \big[(\delta_{b_N' a_N} \delta_{2_N b_N} - \delta_{2_N a_N} \delta_{b_N' b_N})\, \delta_{a_N',-4_N}$$

$$- (\delta_{a_N' a_N} \delta_{2_N b_N} - \delta_{2_N a_N} \delta_{a_N' b_N})\, \delta_{b_N',-4_N} \big]\, \delta_{1_N,-3_N}$$ in (C2c.11)

The result is

$$-4\pi F^k \delta_{JJ'} \delta_{MM'} \frac{(1+\delta_{j_{a_N} j_{b_N}})}{\sqrt{(1+\delta_{j_{a_N} j_{b_N}})(1+\delta_{j_{a_N}' j_{b_N}'})}}$$

$$\times\, \delta_{J_0 J_0'}\, \delta_{j_{cp} j_{cp}'}\, \delta_{j_{a_N} j_{a_N}'}\, \delta_{j_{b_N} j_{b_N}'} \sum_{j_{1_N} \geq j_{2_N}} [q^k(j_{1_N} j_{2_N})]^2$$

$$\times\, \sin^2(v_{1_N}+v_{2_N}) \frac{1}{1+\delta_{j_{1_N} j_{2_N}}}$$

(C2c.14)

The Hermiticity of (C2c.12) and (C2c.14) is easily seen under interchange of $j_{a_N} \leftrightarrow j_{a_N}'$, $j_{b_N} \leftrightarrow j_{b_N}'$, $J_0 \leftrightarrow J_0'$, and finally $j_{cp} \leftrightarrow j_{cp}'$.

There is an important distinction between (C2c.14) and (C2c.12). The former results from (C2c.13) which is simply a set of Kronecker deltas sandwiched between two different two quasi-particle states. The latter resulted from single quasi-neutron operators being between the two states. This means that $\left[\gamma^{j_1}(N)\gamma^{j_2}(N)\right]^k_q$ and $\left[\beta^{+j_3}(N)\beta^{+j_4}(N)\right]^k_{-q}$ (or $C^k_q(j_{1N} j_{2N})$ and $B^{+k}_{-q}(j_{3N} j_{4N})$ (I38)) upon commutation with each other give rise to a set of Kronecker deltas and also single particle operator terms. The dropping of the single particle scattering terms, and hence the neglecting of (C2c.12) is what has been called the Boson Approximation (c.f. (I41)).

In summary, doubling (C2c.7), and combining with (C2c.14) and (C2c.12) gives

$$\langle[(\tilde{j}_{a_N}\tilde{j}_{b_N})J_0\tilde{j}_{cp}]JM|\mathcal{H}^k_{NN(22)}(\text{class }1)|[(\tilde{j}_{a_N}'\tilde{j}_{b_N}')J_0'\tilde{j}_{cp}']J'M'\rangle$$

$$= \langle[(\tilde{j}_{a_N}'\tilde{j}_{b_N}')J_0'\tilde{j}_{cp}']J'M'|\mathcal{H}^k_{NN(22)}(\text{class }1)|$$

$$\times[(\tilde{j}_{a_N}\tilde{j}_{b_N})J_0\tilde{j}_{cp}]JM\rangle = \delta_{JJ'}\delta_{MM'}\Big\{\frac{-8\pi}{2k+1}F^k q^k(j_{a_N}j_{b_N})$$

$$\times q^k(j_{a_N}'j_{b_N}')(-1)^{l_{b_N}+l_{b_N}'}\sin(\gamma_{a_N}+\gamma_{b_N})\sin(\gamma_{a_N}'+\gamma_{b_N}')$$

$$\delta_{J_0J_0'}\delta_{J_0k}\delta_{j_{cp}j_{cp}'} - 4\pi F^k(1+\delta_{j_{a_N}j_{b_N}})\delta_{J_0J_0'}\delta_{j_{cp}j_{cp}'}$$

$$\delta_{j_{a_N}j_{a_N}'}\delta_{j_{b_N}j_{b_N}'}\sum_{j_{1N}\geq j_{2N}}[q^k(j_{1N}j_{2N})]^2\frac{\sin^2(\gamma_{1N}+\gamma_{2N})}{1+\delta_{j_{1N}j_{2N}}}$$

$$+4\pi F^k\delta_{J_0J_0'}\delta_{j_{cp}j_{cp}'}(1+\delta_{j_{a_N}j_{b_N}})\delta_{j_{a_N}j_{a_N}'}\delta_{j_{b_N}j_{b_N}'}\sum_{j_{1N}}\Big([q^k(j_{1N}j_{b_N})]^2$$

$$\times\frac{\sin^2(\gamma_{1N}+\gamma_{b_N})}{2j_{b_N}+1}+[q^k(j_{1N}j_{a_N})]^2\times\frac{\sin^2(\gamma_{1N}+\gamma_{a_N})}{2j_{a_N}+1}\Big)\Big\}\frac{1}{\sqrt{(1+\delta_{j_{a_N}j_{b_N}})(1+\delta_{j_{a_N}'j_{b_N}'})}}$$

(II 23)

d) $\mathcal{H}^{k}_{\substack{NN\\(22)}}$ (Class 2)

This long-range interaction term is

$$\mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class } 2) = -(-1)^{k}\frac{4\pi F^{k}}{\sqrt{2k+1}} \sum_{\substack{j_{1N} j_{2N}\\ j_{3N} j_{4N}}} q^{k}(j_{1N} j_{2N})\, q^{k}(j_{3N} j_{4N})$$

$$\times \cos(\tau_{1N}+\tau_{2N})\cos(\tau_{3N}+\tau_{4N}) \left\{ \left[\beta^{+j_1(N)}\gamma^{j_2(N)}\right]^{k} \times \left[\beta^{+j_3(N)}\gamma^{j_4(N)}\right]^{k} \right\}^{0}_{0}$$

(II19)

One wants to calculate

$$\langle \left[(\tilde{j}_{a_N}\ \tilde{j}_{b_N})J_0\, \tilde{j}_{cp}\right] JM \,|\, \mathcal{H}^{k}_{\substack{NN\\(22)}}(\text{class } 2)\,| \left[(\tilde{j}_{a_N}'\ \tilde{j}_{b_N}')J_0'\, \tilde{j}_{cp}'\right] J'M' \rangle \qquad \text{(C2d.1)}$$

The set of states involved in the diagonalization is given by (I67). In (C2d.1) the notation $\left[(\tilde{j}_{a_N}'\ \tilde{j}_{b_N}')J_0'\,\tilde{j}_{cp}'\right]J'M'\rangle$ is defined in (C2a.2) and $\langle\left[(\tilde{j}_{a_N}\ \tilde{j}_{b_N})J_0\, j_{cp}\right]JM$ is the adjoint of (C2a.2).

Using (I4) and (C1a.3), the insertion of (II19) into (C2d.1) produces

$$-\frac{4\pi}{2k+1} F^{k} \sum_{\substack{j_{1N} j_{2N}\\ j_{3N} j_{4N}}} q^{k}(j_{1N} j_{2N})\, q^{k}(j_{3N} j_{4N}) \cos(\tau_{1N}+\tau_{2N})$$

$$\times \cos(\tau_{3N}+\tau_{4N}) \frac{1}{\sqrt{(1+\delta_{j_{a_N} j_{b_N}})(1+\delta_{j_{a_N}' j_{b_N}'})}}$$

$$\times \sum_{\substack{m_{1_N} m_{2_N} \\ m_{3_N} m_{4_N}}} \sum_{\substack{q\, m_{a_N}' m_{b_N}' \\ m' m_{cp}'}} \sum_{\substack{m_a m_b \\ m\, m_{cp}}} \langle j_{1_N} m_{1_N} j_{2_N} m_{2_N} | k q \rangle \langle j_{3_N} m_{3_N} j_{4_N} m_{4_N} | k -q \rangle$$

$$\times \langle j_{a_N}' m_{a_N}' j_{b_N}' m_{b_N}' | J_0' m' \rangle \langle J_0' m' j_{cp}' m_{cp}' | J' M' \rangle$$

$$\times \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | J_0 m \rangle \langle J_0 m j_{cp} m_{cp} | J M \rangle (-1)^{-q - j_{a_N} - m_{a_N}}$$

$$\times (-1)^{j_{b_N} - m_{b_N} - j_{cp} - m_{cp}} \langle \tilde{0}_p | \gamma^{j_c}_{-m_c}(p) \beta^{+j_c'}_{m_c'}(p) | \tilde{0}_p \rangle$$

$$\times \langle \tilde{0}_N | \gamma^{j_b}_{-m_b}(N) \gamma^{j_a}_{-m_a}(N) \beta^{+j_1}_{m_1}(N) \gamma^{j_2}_{m_2}(N) \beta^{+j_3}_{m_3}(N) \gamma^{j_4}_{m_4}(N) \beta^{+j_a'}_{m_a'}(N) \beta^{+j_b'}_{m_b'}(N) | \tilde{0}_N \rangle$$

(C2d.2)

Now the use of the commutation rule (I 20c) shows that

$$\langle \tilde{0}_p | \gamma^{j_c}_{-m_c}(p) \beta^{+j_c'}_{m_c'}(p) | \tilde{0}_p \rangle = \delta_{j_{cp}' j_{cp}} \delta_{m_{cp} m_{cp}'} (-1)^{j_{cp}' + m_{cp}'}$$

(C2d.3)

and

$$\langle \tilde{0}_N | \gamma^{j_b}_{-m_b}(N) \gamma^{j_a}_{-m_a}(N) \beta^{+j_1}_{m_1}(N) \gamma^{j_2}_{m_2}(N) \beta^{+j_3}_{m_3}(N) \gamma^{j_4}_{m_4}(N)$$

$$\times \beta^{+j_a'}_{m_a'}(N) \beta^{+j_b'}_{m_b'}(N) | \tilde{0}_N \rangle = (-1)^{j_{1_N} + m_{1_N} + j_{3_N} + m_{3_N} + j_{a_N}'}$$

$$\times (-1)^{m_{a_N}' + j_{b_N}' + m_{b_N}'} \Big((R) + (S) \Big)$$

(C2d.4)

where, in the notation of (A1.4)

$$(R) \equiv \Big[\left(\delta_{1_N a_N} \delta_{b_N b_N'} - \delta_{1_N b_N} \delta_{a_N b_N'} \right) \delta_{a_N', -4_N}$$

$$- \left(\delta_{1_N a_N} \delta_{b_N a_N'} - \delta_{1_N b_N} \delta_{a_N a_N'} \right) \delta_{b_N', -4_N} \Big] \delta_{3_N, -2_N} \qquad \text{(C2d.5)}$$

and

$$(S) \equiv \left(\delta_{1_\nu a_\nu} \delta_{3_\nu b_\nu} - \delta_{1_\nu b_\nu} \delta_{3_\nu a_\nu} \right) \left(\delta_{a_\nu', -2_\nu} \delta_{b_\nu', -4_\nu} - \delta_{b_\nu', -2_\nu} \delta_{a_\nu', -4_\nu} \right)$$

(C2d.6)

Since any one group of Kronecker deltas in (C2d.5) is obtained by permutation of the appropriate indices in another group, only one group need be explicitly calculated. As shown in preceding sections, this is due to the fact that a pair of permuted indices appear together in a Clebsch-Gordon coefficient. The same statements hold for the groups of Kronecker deltas in (C2d.6).

For example, one may put (C2d.4) and the group of Kronecker deltas, $\left(\delta_{1_\nu a_\nu} \delta_{b_\nu b_\nu'} - \delta_{1_\nu b_\nu} \delta_{a_\nu b_\nu'} \right) \delta_{a_\nu', -4_\nu} \delta_{3_\nu, -2_\nu}$, from (R)(C2d.5) into (C2d.2). After reducing the number of z component summation indices this gives

$$\frac{1}{\sqrt{(1+\delta_{j a_\nu j b_\nu})(1+\delta_{j a_\nu' j b_\nu'})}} \frac{4\pi}{2k+1} F^k \sum_{j_{2_\nu}} g^k(j_{a_\nu} j_{2_\nu}) g^k(j_{2_\nu} j_{a_\nu'})$$

$$\times \cos(\tau_{a_\nu} + \tau_{2_\nu}) \cos(\tau_{2_\nu} + \tau_{a_\nu'}) \delta_{MM'} \sum_{m_{cp}} \langle J_0' \; M - m_{cp} \; j_{cp} m_{cp} | J' M \rangle$$

$$\times \langle J_0 \; M - m_{cp} \; j_{cp} \; m_{cp} | J M \rangle \sum_{m_{b_\nu}} \langle j_{a_\nu'} \; M - m_{cp} - m_{b_\nu} \; j_{b_\nu} m_{b_\nu} |$$

$$J_0' \; M - m_{cp} \rangle \langle j_{a_\nu} \; M - m_{cp} - m_{b_\nu} \; j_{b_\nu} \; m_{b_\nu} | J_0 \; M - m_{cp} \rangle$$

$$\times \sum_{m_{2_\nu}} \langle j_{a_\nu} \; M - m_{cp} - m_{b_\nu} \; j_{2_\nu} m_{2_\nu} | k \; m_{2_\nu} + M - m_{cp} - m_{b_\nu} \rangle$$

$$\times \langle j_{2_\nu} - m_{2_\nu} \; j_{a_\nu'} - M + m_{cp} + m_{b_\nu} | k - m_{2_\nu} - M + m_{cp} + m_{b_\nu} \rangle (-1)^{j_{2_\nu} + j_{a_\nu'}}$$

(C2d.7)

For simplicity, one should use 3-j symbols (A3.10). The summation over $m_{2\nu}$ is proportional to $\delta_{j_{a\nu} j_{a\nu}'}$. The application of $\delta_{j_{a\nu} j_{a\nu}'}$ to the summation over $m_{b\nu}$ yields $\delta_{J_0 J_0'}$, and finally the use of $\delta_{J_0 J_0'}$ in the sum over m_{cp} gives $\delta_{JJ'}$. In this way (C2d.7) becomes

$$-\frac{4\pi F^k}{2j_{a\nu}+1}\;\frac{\delta_{JJ'}\delta_{MM'}}{\sqrt{(1+\delta_{j_{a\nu} j_{b\nu}})(1+\delta_{j_{a\nu}' j_{b\nu}'})}}\sum_{j_{2\nu}}\left[q^k(j_{2\nu} j_{a\nu})\right]^2\cos^2(\tau_{2\nu}+\tau_{a\nu})$$

$$\times\,\delta_{j_{cp} j_{cp}'}\,\delta_{j_{b\nu} j_{b\nu}'}\,\delta_{j_{a\nu} j_{a\nu}'}\,\delta_{J_0 J_0'}$$

(C2d.8)

One then combines (C2d.8) with the results from the other groups of Kronecker deltas of (C2d.5). This gives for the total contribution of (R) (C2d.5) to the matrix element (C2d.2)

$$\frac{-4\pi F^k\delta_{JJ'}\delta_{MM'}}{\sqrt{(1+\delta_{j_{a\nu} j_{b\nu}})(1+\delta_{j_{a\nu}' j_{b\nu}'})}}(1+\delta_{j_{a\nu} j_{b\nu}})\,\delta_{j_{a\nu} j_{a\nu}'}\,\delta_{j_{b\nu} j_{b\nu}'}$$

$$\times\,\delta_{j_{cp} j_{cp}'}\,\delta_{J_0 J_0'}\sum_{j_{1\nu}}\left(\frac{\left[q^k(j_{1\nu} j_{b\nu})\right]^2}{2j_{b\nu}+1}\cos^2(\tau_{1\nu}+\tau_{b\nu})\right.$$

$$\left.+\,\frac{\left[q^k(j_{1\nu} j_{a\nu})\right]^2}{2j_{a\nu}+1}\cos^2(\tau_{2\nu}+\tau_{a\nu})\right)$$

(C2d.9)

Under interchange of j_{a_ν} and j_{a_ν}', j_{b_ν} and j_{b_ν}', J_0 and J_0', and finally j_{cp} and j_{cp}', expression (C2d.9) remains the same. This confirms Hermiticity.

To find the contribution of (5)(C2d.6) to (C2d.2) one may, e.g., just use the group of Kronecker deltas $\delta_{1_\nu a_\nu}\delta_{3_\nu b_\nu}\delta_{a_\nu',-2_\nu}\delta_{b_\nu',-4_\nu}$. Inserting this group and (C2d.3) into (C2d.2) gives upon reducing the number of z component summation indices and using $3-j$ symbols (A3.10)

$$\frac{4\pi}{2k+1} F^k q^k(j_{a_\nu} j_{a_\nu}')\, q^k(j_{b_\nu} j_{b_\nu}') \cos(\gamma_{a_\nu}+\gamma_{a_\nu}')$$

$$\times \cos(\gamma_{b_\nu}+\gamma_{b_\nu}')\,\delta_{MM'}\frac{1}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j_{a_\nu}' j_{b_\nu}'})}} \times (2k+1)$$

$$\times \sqrt{(2J+1)(2J'+1)(2J_0+1)(2J_0'+1)}\sum_{m_{cp}}\begin{pmatrix} J_0' & j_{cp} & J \\ M-m_{cp} & m_{cp} & -M \end{pmatrix}$$

$$\times \begin{pmatrix} J_0 & j_p & J \\ M-m_{cp} & m_{cp} & -M \end{pmatrix}\sum_{m_{a_\nu}}\begin{pmatrix} j_{a_\nu} & j_{b_\nu} & J_0 \\ m_{a_\nu} & M-m_{cp}-m_{a_\nu} & -M+m_{cp} \end{pmatrix}$$

$$\sum_{m_{a_\nu}'}\begin{pmatrix} j_{a_\nu}' & j_{b_\nu}' & J_0' \\ m_{a_\nu}' & M-m_{cp}-m_{a_\nu}' & -M+m_{cp} \end{pmatrix}\begin{pmatrix} j_{a_\nu} & j_{a_\nu}' & k \\ m_{a_\nu} & -m_{a_\nu}' & -m_{a_\nu}+m_{a_\nu}' \end{pmatrix}$$

$$\times \begin{pmatrix} j_{b_\nu} & j_{b}' & k \\ M-m_{cp}-m_{a_\nu} & -M+m_{cp}+m_{a_\nu}' & m_{a_\nu}-m_{a_\nu}' \end{pmatrix}(-1)^{j_{a_\nu}'+j_{b_\nu}'+M+m_{cp}-m_{a_\nu}+m_{a_\nu}}$$

(C2d.10)

First, the sum over m'_{a_ν} via (A3.13) is performed, and then one does the sums over m_{a_ν} and $m_{c\rho}$ in that order. The orthogonality of 3-j symbols thus produces for (C2d.10)

$$-\delta_{JJ'}\delta_{MM'}\frac{1}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j'_{a_\nu} j'_{b_\nu}})}}$$
$$\times 4\pi F^k q^k(j_{a_\nu} j'_{a_\nu}) q^k(j_{b_\nu} j'_{b_\nu}) \cos(\gamma_{a_\nu}+\gamma'_{a_\nu})$$
$$\times \cos(\gamma_{b_\nu}+\gamma'_{b_\nu}) \delta_{J_0 J_0'} \begin{Bmatrix} j_{b_\nu} & j_{a_\nu} & J_0 \\ j'_{a_\nu} & j'_{b_\nu} & k \end{Bmatrix}$$
$$\times (-1)^{j'_{a_\nu}+j_{b_\nu}+J_0}\,\delta_{j_{c\rho} j'_{c\rho}} \tag{C2d.11}$$

Now using (C2d.11) and the results of the other groups of Kronecker deltas (C2d.6), one gets the total contribution of (5) (C2d.6) to (C2d.2) to be

$$-\delta_{JJ'}\delta_{MM'}\frac{1}{\sqrt{(1+\delta_{j_{a_\nu} j_{b_\nu}})(1+\delta_{j'_{a_\nu} j'_{b_\nu}})}} \times 8\pi F^k \delta_{j_{c\rho} j'_{c\rho}}$$
$$\times \delta_{J_0 J_0'}(-1)^{j_{b_\nu}+j'_{a_\nu}}\Big[(-1)^{J_0} q^k(j_{a_\nu} j'_{a_\nu}) q^k(j_{b_\nu} j'_{b_\nu}) \cos(\gamma_{a_\nu}+\gamma'_{a_\nu})$$
$$\times \cos(\gamma_{b_\nu}+\gamma'_{b_\nu}) \begin{Bmatrix} j_{b_\nu} & j_{a_\nu} & J_0 \\ j'_{a_\nu} & j'_{b_\nu} & k \end{Bmatrix} + q^k(j_{b_\nu} j'_{a_\nu}) q^k(j_{a_\nu} j'_{b_\nu}) \cos(\gamma_{b_\nu}+\gamma'_{a_\nu})$$
$$\times \cos(\gamma_{a_\nu}+\gamma'_{b_\nu}) \begin{Bmatrix} j_{a_\nu} & j_{b_\nu} & J_0 \\ j'_{a_\nu} & j'_{b_\nu} & k \end{Bmatrix}\Big] \tag{C2d.12}$$

Interchange of $j_{a_\nu} \leftrightarrow j'_{a_\nu}$, $j_{b_\nu} \leftrightarrow j'_{b_\nu}$, $J_0 \leftrightarrow J_0'$, and lastly $j_{c\rho} \leftrightarrow j'_{c\rho}$ show that (C2d.12) is Hermitian.

Combining (C2d.9) and (C2d.12) gives the final result

$$\langle [(\tilde{j}_{a\nu}\ \tilde{j}_{b\nu})J_0\ \tilde{j}_{cp}]JM \,|\, \mathcal{W}^{k}_{\substack{\nu\nu\\(22)}}(\text{class }2) \,|\, [(\tilde{j}'_{a\nu}\ \tilde{j}'_{b\nu})J_0'\ \tilde{j}'_{cp}]J' M'\rangle = \langle [(\tilde{j}'_{a\nu}\ \tilde{j}'_{b\nu})J_0'\ \tilde{j}'_{cp}]J'M' \,|\, \mathcal{W}^{k}_{\substack{\nu\nu\\(22)}}(\text{class }2) \,|\, [(\tilde{j}_{a\nu}\ \tilde{j}_{b\nu})J_0\ \tilde{j}_{cp}]JM\rangle = \delta_{JJ'}\delta_{MM'}\Big\{-4\pi F^{k}$$

$$\times (1+\delta_{j_{a\nu}j_{b\nu}})\,\delta_{j_{a\nu}j'_{a\nu}}\,\delta_{j_{b\nu}j'_{b\nu}}\,\delta_{j_{cp}j'_{cp}}\,\delta_{J_0J_0'}\sum_{j_{1\nu}}\Big(\frac{[q^{k}(j_{1\nu}j_{b\nu})]^2}{2j_{b\nu}+1}$$

$$\times\cos^2(\chi_{1\nu}+\chi_{b\nu}) + \frac{[q^{k}(j_{1\nu}j_{a\nu})]^2}{2j_{a\nu}+1}\cos^2(\chi_{1\nu}+\chi_{a\nu})\Big)$$

$$-8\pi F^{k}\,\delta_{j_{cp}j'_{cp}}\,\delta_{J_0J_0'}\,(-1)^{j_{b\nu}+j'_{a\nu}}\Big[(-1)^{J_0}q^{k}(j_{a\nu}j'_{a\nu})$$

$$\times q^{k}(j_{b\nu}j'_{b\nu})\cos(\chi_{a\nu}+\chi'_{a\nu})\cos(\chi_{b\nu}+\chi'_{b\nu})\begin{Bmatrix} j_{b\nu} & j_{a\nu} & J_0\\ j'_{a\nu} & j'_{b\nu} & k\end{Bmatrix}$$

$$+q^{k}(j_{b\nu}j'_{a\nu})\,q^{k}(j_{a\nu}j'_{b\nu})\cos(\chi_{b\nu}+\chi'_{a\nu})\cos(\chi_{a\nu}+\chi'_{b\nu})$$

$$\times\begin{Bmatrix} j_{a\nu} & j_{b\nu} & J_0\\ j'_{a\nu} & j'_{b\nu} & k\end{Bmatrix}\Big]\Big\}\frac{1}{\sqrt{(1+\delta_{j_{a\nu}j_{b\nu}})(1+\delta_{j'_{a\nu}j'_{b\nu}})}}$$

APPENDIX D

B(Ek), The Reduced Electric Transition Probability

Introduction

The reduced electric transition probability of multipole order k is given in Chapter III by

$$\underset{J_i \to J_f}{B(Ek)} = \frac{1}{2J_i+1} \left| \langle J_f \| \frac{1}{e} \sum_{m,\eta} e_\eta r^k_{m_\eta} Y_k(\theta_{m_\eta} \phi_{m_\eta}) \| J_i \rangle \right|^2$$

(III 1)

The ground and excited state angular momenta are respectively J_f and J_i, and the transition indicated in (III 1) is from J_i to J_f. The reduced matrix element in (III 1) is given by

$$\langle J_f \| \frac{1}{e} \sum_{m,\eta} e_\eta r^k_{m_\eta} Y_k(\theta_{m_\eta} \phi_{m_\eta}) \| J_i \rangle = \sqrt{2J_f+1}\, (-1)^{k+J_i-J_f}$$

$$\times \left(\frac{1}{\langle k\, q\, J_i M_i | J_f\, q+M_i \rangle} \right) \langle J_f\, M_i+q | \frac{1}{e} \sum_{m,\eta} e_\eta r^k_{m_\eta} Y_k(\theta_{m_\eta} \phi_{m_\eta}) | J_i M_i \rangle$$

(III3)

and the procedure will be to calculate the non-reduced matrix element in (III3).

The sum over η is broken up into separate sums over protons and neutrons, i.e.,

$$\frac{1}{e}\sum_{n,\tau} e_\tau r_{n_\tau}^{k} Y_k(\theta_{n_\tau}\phi_{n_\tau}) = \sum_{n} r_{n_p}^{k} Y_{kq}(\theta_{n_p}\phi_{n_p}) + \frac{e_N}{e}\sum_{n} r_{n_N}^{k} Y_{kq}(\theta_{n_N}\phi_{n_N})$$

(III5)

For the case of one proton outside of a major closed shell

$$\sum_{n} r_{n_p}^{k} Y_{kq}(\theta_{n_p}\phi_{n_p}) = \sum_{j_{a_p} j_{b_p}} q^k(j_{a_p} j_{b_p}) \sum_{m_{a_p} m_{b_p}}$$

$$\times \langle j_{a_p} m_{a_p} j_{b_p} m_{b_p} | k q \rangle \beta_{m_a}^{+j_a}(p)\, \gamma_{m_b}^{j_b}(p)$$

$$= \sum_{j_{a_p} j_{b_p}} q^k(j_{a_p} j_{b_p}) \left[\beta^{+j_a}_{(p)}\, \gamma^{j_b}_{(p)}\right]^k_q \qquad \text{(III7)}$$

where (III7) is written in the quasi-proton occupation number representation.

1. Quasi-Boson Plus Quasi-Proton:

The neutron term of (III5) is best written as

$$\frac{e_N}{e}\sum_{n} r_{n_N}^{k} Y_{kq}(\theta_{n_N}\phi_{n_N}) = \frac{e_N}{e}\sum_{\substack{j_{a_N}\\ j_{b_N}}} q^k(j_{a_N} j_{b_N})$$

$$\times \Big\{ -(-1)^{\ell_{b_N}} u_{a_N} v_{b_N} \left[(-1)^k C_q^k(j_{a_N} j_{b_N}) + B_q^{+k}(j_{a_N} j_{b_N})\right]$$

$$\times \sqrt{1+\delta_{j_{a_N} j_{b_N}}} + \cos(\gamma_{a_N} + \gamma_{b_N})\, \beta_{m_a}^{+j_a}(N)\, \gamma_{m_b}^{j_b}(N) \Big\}$$

(III 10)

where the quasi-neutron occupation number representation is used and $C_q^k(j_{a_N} j_{b_N})$ and $B_q^{+k}(j_{a_N} j_{b_N})$ are defined in (I38) in terms of the quasi-neutron annihilation and creation operators respectively.

The state, $|J_i M_i\rangle$,is written as $|J_i M_i\rangle_{0,k}$ and is given by

$$|J_i M_i\rangle_{0,k} = a^{J_i}_{0 J_i} \beta^{+J_i}_{M_i}(p) |\tilde{\tilde{0}}_N ; \tilde{0}_p\rangle + \sum_{j_{cp}} a^{J_i}_{k j_{cp}} \times \left[\Gamma^k(N) \beta^{+j_c}(p) \right]^{J_i}_{M_i} |\tilde{\tilde{0}}_N ; \tilde{0}_p\rangle \tag{I57}$$

The expansion coefficients, $a^{J_i}_{0 J_i}$ and $a^{J_i}_{k j_{cp}}$, are determined by the appropriate diagonalizations of Chapter II. From Chapter I, $\Gamma^k_m(N)$ is the quasi-Boson creation operator given by

$$\Gamma^k_m(N) = \sum_{j_{1N} \geq j_{2N}} \left[r(j_{1N} j_{2N}) B^{+k}_m(j_{1N} j_{2N}) + s(j_{1N} j_{2N}) C^k_m(j_{1N} j_{2N}) \right] \tag{I47a}$$

where $r(j_{a_N} j_{b_N})$ and $s(j_{a_N} j_{b_N})$ are expansion coefficients (I49).

The state, $\langle J_f M_i + q|$, is written as ${}_{0,k}\langle J_f M_i + q|$ and is

$$ {}_{0,k}\langle J_f M_i + q| = (-1)^{-J_f - M_i - q} a^{J_f}_{0 J_f} \langle \tilde{\tilde{0}}_N ; \tilde{0}_p | \gamma^{J_f}_{-M_i - q}(p) + \sum_{j'_{cp}} a^{J_f}_{k j'_{cp}} \langle \tilde{\tilde{0}}_N ; \tilde{0}_p | \left[\Gamma^k(N) \beta^{+j'_c}(p) \right]^{+J_f}_{M_i+q} \tag{III11}$$

with $\Gamma^{+k}_{m'}(N)$ being the quasi-Boson annihilation operator given by

$$\Gamma^{+k}_{m'}(N) = (-1)^{k+m'} \sum_{j'_{1N} \geq j'_{2N}} \left[r(j'_{1N} j'_{2N}) C^k_{-m'}(j'_{1N} j'_{2N}) + s(j'_{1N} j'_{2N}) B^{+k}_{-m'}(j'_{1N} j'_{2N}) \right] \tag{A4.2}$$

The matrix element to be evaluated is then from (III7) and (III 10)

$$
{}_{o,k}\langle J_f \; M_i+q | \frac{1}{\ell} \sum_{m,\eta} \ell_\eta r_{m_\eta}^2 Y_k(\theta_{m_\eta} \phi_{m_\eta}) | J_i M_i \rangle
$$

$$
= {}_{o,k}\langle J_f \; M_i+q | \sum_{j_{a_p} j_{b_p}} q^k(j_{a_p} j_{b_p}) \left[\beta^{+j_a}(p) \gamma^{j_b}(p) \right]^k_q
$$

$$
+ \frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N}) \Big\{ -(-1)^{\ell_{b_N}} \mu_{a_N} \nu_{b_N} \Big[(-1)^k C^k_q (j_{a_N} j_{b_N})
$$

$$
+ B^{+k}_q (j_{a_N} j_{b_N}) \Big] \sqrt{1+\delta_{j_{a_N} j_{b_N}}} + \cos(\gamma_{a_N} + \gamma_{b_N}) \beta^{+j_a}_{m_a}(N)
$$

$$
\times \gamma^{j_b}_{m_b}(N) \Big\} | J_i M_i \rangle_{o,k} \qquad \text{(III 12)}
$$

with ${}_{o,k}\langle J_f \; M_i+q|$ and $|J_i \; M_i\rangle_{o,k}$ given by (III 11) and (I57) respectively.

The matrix element (III 12) may be written as the sum of four terms

$$
\langle k q J_i \; M_i | J_f \; q+M_i \rangle \left[A_P + B_P + C_N + D_N \right] \qquad \text{(D1.1)}
$$

with

$$
\langle k q J_i M_i | J_f \; q+M_i \rangle A_P \equiv (-1)^{-J_f - M_i - q} a_{o J_f}^{J_f}
$$

$$
\times \langle \tilde{\tilde{o}}_N ; \tilde{o}_P | \gamma^{J_f}_{-M_i-q}(P) \sum_{j_{a_P} j_{b_P}} q^k(j_{a_P} j_{b_P}) \left[\beta^{+j_a}_{(P)} \gamma^{j_b}(P) \right]^k_q a_{o J_i}^{J_i} \beta^{+J_i}_{M_i}(P) | \tilde{\tilde{o}}_N ; \tilde{o}_P \rangle
$$

$$
\text{(D1.2)}
$$

$$
\langle k q J_i M_i | J_f \; q+M_i \rangle B_P \equiv \sum_{j_{c_P}'} a^{J_f}_{k j_{c_P}'} \langle \tilde{\tilde{o}}_N ; \tilde{o}_P | \left[\Gamma^k_{(N)} \beta^{+j_c'}_{(P)} \right]^{J_f}_{M_i+q}
$$

$$
\times \sum_{j_{a_P} j_{b_P}} q^k(j_{a_P} j_{b_P}) \left[\beta^{+j_a}_{(P)} \gamma^{j_b}_{(P)} \right]^k_q \sum_{j_{c_P}} a^{J_i}_{k j_{c_P}} \left[\Gamma^k_{(N)} \beta^{+j_c}_{(P)} \right]^{J_i}_{M_i} | \tilde{\tilde{o}}_N ; \tilde{o}_P \rangle
$$

$$
\text{(D1.3)}
$$

$$\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle C_N \equiv \langle k\, q\, J_i\, M_i | J_f\; q+M_i \rangle \times \left(C_N^{(1)} + C_N^{(2)} \right) \tag{D1.4}$$

such that

$$\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle C_N^{(1)} \equiv \sum_{j_{cp}'} a^{J_f}_{k\, j_{cp}'} \langle \tilde{o}_N ; \tilde{o}_p | \left[\Gamma^{k}(N)\, \beta^{+ j_{c}'}(p) \right]^{J_f}_{M_i+q}$$
$$\times \frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^{k}(j_{a_N} j_{b_N}) \Big\{ -(-1)^{\ell_{b_N}} u_{a_N} v_{b_N} \Big[(-1)^{k}\, c^{k}_{q}(j_{a_N} j_{b_N})$$
$$+ \beta^{+k}_{q}(j_{a_N} j_{b_N}) \Big] \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \Big\}\, a^{J_i}_{0 J_i}\, \beta^{+ J_i}_{M_i}(p) | \tilde{o}_N ; \tilde{o}_p \rangle \tag{D1.5}$$

and

$$\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle C_N^{(2)} \equiv (-1)^{-J_f - M_i - q}\, a^{J_f}_{0 J_f}$$
$$\times \langle \tilde{o}_N ; \tilde{o}_p | \gamma^{J_f}_{-M_i-q}(p)\, \frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^{k}(j_{a_N} j_{b_N}) \Big\{ -(-1)^{\ell_{b_N}} u_{a_N} v_{b_N}$$
$$\times \Big[(-1)^{k}\, c^{k}_{q}(j_{a_N} j_{b_N}) + \beta^{+k}_{q}(j_{a_N} j_{b_N}) \Big] \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \Big\}$$
$$\times \sum_{j_{cp}} a^{J_i}_{k\, j_{cp}} \left[\Gamma^{k}(N)\, \beta^{+ j_c}(p) \right]^{J_i}_{M_i} | \tilde{o}_N ; \tilde{o}_p \rangle \tag{D1.6}$$

and finally

$$\langle k\, q\, J_i\, M_i | J_f\; q+M_i \rangle D_N \equiv \sum_{j_{cp}'} a^{J_f}_{k\, j_{cp}'} \langle \tilde{o} ; \tilde{o}_p |$$
$$\times \left[\Gamma^{k}(N)\, \beta^{+ j_c'}(p) \right]^{+ J_f}_{M_i+q}\, \frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^{k}(j_{a_N} j_{b_N}) \cos(\gamma_{a_N} + \gamma_{b_N})$$
$$\times \beta^{+ j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) \sum_{j_{cp}} a^{J_i}_{k\, j_{cp}}$$

$$\times \left[\Gamma^{k}(n)\ \beta^{+\,j_c}(p) \right]^{J_i}_{M_i} \left| \tilde{0}_n^2 ; \tilde{0}_p \right\rangle$$

(D1.7)

a) Evaluation of A_p From Equation (D1.2) -

From (D1.2) and the commutation rule for $\beta^{+}(\eta)$ and $\gamma(\eta)$ (I20'c)

$$\langle k q\ J_i M_i | J_f\ q+M_i \rangle A_p =$$

$$a_0{}^{J_i}_{J_i}\ a_{0J_f}^{J_f} (-1)^{J_f - M_i - q} \sum_{j_{a_p} j_{b_p}} q^{k}(j_{a_p} j_{b_p})$$

$$\times \sum_{m_{a_p} m_{b_p}} \langle j_{a_p} m_{a_p}\ j_{b_p} m_{b_p} | k q \rangle\ \delta_{a_p, J_f}\ \delta_{b_p, -J_i} (-1)^{J_i + M_i + J_f + M_i + q}$$

(D1a.1)

where the notation for the Kronecker deltas follows (A1.4). Equation (D1a.1) is simplified by permuting the indices in the Clebsch-Gordon coefficient via (A1.8) to give

$$\langle k q\ J_i M_i | J_f\ q+M_i \rangle A_p = (-1)^{k} a_{0J_i}^{J_i}\ a_{0J_f}^{J_f}\ q^{k}(J_f J_i)(-1)^{J_i + J_f}$$

$$\times \sqrt{\frac{2k+1}{2J_f+1}}\ \langle k q\ J_i M_i | J_f\ M_i + q \rangle$$

(D1a.2)

so that

$$A_p = (-1)^{k} a_{0J_i}^{J_i}\ a_{0J_f}^{J_f}\ q^{k}(J_f J_i)(-1)^{J_i + J_f} \sqrt{\frac{2k+1}{2J_f+1}}$$

(III 13a)

b) Evaluation of B_p From Equation (D1.3) -

Writing (D1.3) in terms of Clebsch-Gordon coefficients and using (I4) yields

$$\sum_{j_{cp}'} a^{J_f}_{k j_{cp}'} \sum_{j_{cp}} a^{J_i}_{k j_{cp}} \sum_{j_{ap} j_{bp}} q^{k}(j_{ap} j_{bp}) \sum_{\substack{m' m_{cp}' \\ m\, m_{cp} \\ m_{ap} m_{bp}}} \langle k m' j_{cp}' m_{cp}' | J_f M_i + q \rangle$$

$$\times \langle k m\, j_p m_{j_p} | J_i M_i \rangle \langle j_{ap} m_{ap}\, j_{bp} m_{bp} | k q \rangle (-1)^{j_{cp}' - m_{cp}'}$$

$$\times \langle \tilde{0_p} | \gamma^{j_c'}_{-m_c'}(p)\, \beta^{+ j_a}_{m_a}(p)\, \gamma^{j_b}_{m_b}(p)\, \beta^{+ j_c}_{m_c}(p) | \tilde{0_p} \rangle \delta_{m m'}$$

(D1b.1)

with $\delta_{m m'}$ resulting from (A4.12) and (A4.13) via

$$\langle \tilde{0}_N | \Gamma^{+k}_{m'}(N)\, \Gamma^{k}_{m}(N) | \tilde{0}_N \rangle = \delta_{m m'}$$

(D1b.1')

Now using the commutation rule (I20'c) in (D1b.1) gives in the notation of (A1.4)

$$\langle \tilde{0_p} | \gamma^{j_c'}_{-m_c'}(p)\, \beta^{+ j_a}_{m_a}(p)\, \gamma^{j_b}_{m_b}(p)\, \beta^{+ j_c}_{m_c}(p) | \tilde{0_p} \rangle$$

$$= \delta_{j_{cp}, -b_p}\, \delta_{j_{cp}', a_p} (-1)^{j_{cp} + m_{cp} + j_{cp}' + m_{cp}'}$$

(D1b.2)

One may now write the coupling coefficients in (D1b.1) as 3-j symbols (A3.10). Invoking (D1b.2) and introducing the 6-j symbol via (A3.13) may be easily shown to give

$$\langle k q\ J_i M_i | J_f\ q+M_i \rangle B_\rho = (-1)^k \sqrt{(2J_i+1)(2k+1)}$$
$$\times \sum_{j'_{c\rho} j_{c\rho}} a^{J_f}_{k j'_{c\rho}} a^{J_i}_{k j_{c\rho}} q^k (j'_{c\rho}\ j_{c\rho}) (-1)^{j_{c\rho}-J_i} \begin{Bmatrix} k & J_i & J_f \\ k & j'_{c\rho} & j_{c\rho} \end{Bmatrix}$$
$$\times \langle k q\ J_i M_i | J_f\ q+M_i \rangle \qquad \text{(D1b.3)}$$

so that

$$B_\rho = (-1)^k \sqrt{(2J_i+1)(2k+1)} \sum_{j_{c\rho} j'_{c\rho}} a^{J_f}_{k j'_{c\rho}} a^{J_i}_{k j_{c\rho}} q^k (j'_{c\rho}\ j_{c\rho})$$
$$\times (-1)^{j_{c\rho}-J_i} \begin{Bmatrix} k & J_i & J_f \\ k & j'_{c\rho} & j_{c\rho} \end{Bmatrix}$$

(III 13b)

c) B(Ek) of the Adjacent Even-Even Nucleus -

This calculation is presented to show that C_N (D1.4, D1.5, and D1.6) contains the very matrix element that gives this B(Ek). Also this result will make possible the comparison of B(Ek)'s of even-odd and adjacent even-even nuclei.

The expression for B(Ek) is obtained from (III 1) by letting $J_i = 0$ and $J_f = k$ as

$$B_{0\to k}(Ek) = \left| \langle \tilde{\tilde{k}}_N \| \frac{1}{e} \sum_{m,\eta} e_\eta r_{m_\eta}^k Y_k(\theta_{m_\eta} \phi_{m_\eta}) \| \tilde{\tilde{0}}_N \rangle \right|^2 \qquad \text{(D1c.1)}$$

where $\langle \tilde{\tilde{k}} \|$ is the one quasi-Boson state. The matrix in (D1c.1) is obtained by

$$\langle \tilde{\tilde{k}}_N \| \frac{1}{e} \sum_{m,\eta} e_\eta r_{m_\eta}^k Y_k(\theta_{m_\eta} \phi_{m_\eta}) \| \tilde{\tilde{0}}_N \rangle$$
$$= \sqrt{2k+1}\ \langle \tilde{\tilde{k}} q | \frac{1}{e} \sum_{m,\eta} e_\eta r_{m_\eta}^k Y_{kq}(\theta_{m_\eta} \phi_{m_\eta}) | \tilde{\tilde{0}}_N \rangle \qquad \text{(D1c.2)}$$

which is the analog of (III3) for the case of no odd proton. The state, $\langle k\tilde{\tilde{q}}|$, is that of one quasi-Boson and is given by

$$\langle k\tilde{\tilde{q}}| = \langle \tilde{\tilde{o}}_N| \Gamma^{+k}_{q}(N) \tag{Dlc.3}$$

The only nucleons present are neutrons and hence from (III 10)

$$\langle k_N \tilde{\tilde{q}}_N | \frac{e_N}{e} \sum_{n} r^k_{n_N} Y_{kq}(\theta_{n_N} \phi_{n_N}) | \tilde{\tilde{o}}_N \rangle = \langle \tilde{\tilde{o}}_N | \Gamma^{+k}_{q}(N)$$

$$\times \; -\frac{e_N}{e} \sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N}) (-1)^{l_{b_N}} u_{a_N} v_{b_N} \left[(-1)^k C^k_q(j_{a_N} j_{b_N}) \right.$$

$$\left. + B^{+k}_q(j_{a_N} j_{b_N}) \right] \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \, | \tilde{\tilde{o}}_N \rangle = -\frac{e_N}{e} \sum_{\substack{j_{a_N} \\ j_{b_N}}} q^k(j_{a_N} j_{b_N})$$

$$\times (-1)^{l_{b_N}} u_{a_N} v_{b_N} \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \; \langle \tilde{\tilde{o}}_N | \Gamma^{+k}_{q}(N) \left[(-1)^k C^k_q(j_{a_N} j_{b_N}) \right.$$

$$\left. + B^{+k}_q(j_{a_N} j_{b_N}) \right] | \tilde{\tilde{o}}_N \rangle \tag{Dlc.4}$$

Interchanging the dummy indices, j_{a_N} and j_{b_N} , in (Dlc.4) gives for (Dlc.4)

$$-\frac{e_N}{e} \sum_{\substack{j_{a_N} \\ j_{b_N}}} q^k(j_{a_N} j_{b_N}) (-1)^{l_{a_N}} v_{a_N} u_{b_N} \sqrt{1+\delta_{j_{a_N} j_{b_N}}}$$

$$\times \langle \tilde{\tilde{o}}_N | \Gamma^{+k}_{q}(N) \left[C^k_q(j_{a_N} j_{b_N}) + (-1)^k B^{+k}_q(j_{a_N} j_{b_N}) \right] | \tilde{\tilde{o}}_N \rangle \tag{Dlc.5}$$

Minus $\frac{\ell}{\ell_N}$ times this very expression comes about in the calculation of the matrix elements, $\langle (\tilde{\tilde{k}}_N ; \tilde{j}'_{cp}) JM | H^{k}_{N,p\,(31)} | \times (\tilde{\tilde{0}}_N ; \tilde{j}_{cp}) j_{cp} m_{cp} \rangle$ (Cla.15). The reason for the minus sign is that the long-range force includes a minus sign to indicate the attractive nature of the long-range interaction. Because of (A 4.2), equation (Dlc.5) may be rewritten in terms of $\left[\Gamma^{+k}_{q}(N) , \left(C^{k}_{q}(j_{a_N} j_{b_N}) + (-1)^{k} B^{+k}_{q}(j_{a_N} j_{b_N}) \right) \right]$ so that using (I47b) and the Boson part of (I4la) then gives

$$\langle \tilde{\tilde{k}}_N q_N | \frac{\ell_N}{\ell} \sum_{m} r^{k}_{m_N} Y_{kq}(\theta_{m_N} \phi_{m_N}) | \tilde{\tilde{0}}_N \rangle$$

$$= -(-1)^{k} \sum_{j_{a_N} \geq j_{b_N}} q^{k}(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}} \sin(\tau_{a_N} + \tau_{b_N})$$

$$\times \sqrt{1 + \delta_{j_{a_N} j_{b_N}}} \left[r(j_{a_N} j_{b_N}) - (-1)^{k} s(j_{a_N} j_{b_N}) \right] \tag{Dlc.6}$$

Hence, from (Dlc.1), (Dlc.2), and (Dlc.6)

$$B(Ek)_{0 \to k} = (2k+1) \left(\frac{\ell_N}{\ell} \right)^2 \left(\sum_{j_{a_N} \geq j_{b_N}} q^{k}(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}} \sin(\tau_{a_N} + \tau_{b_N}) \right.$$

$$\left. \times \sqrt{1 + \delta_{j_{a_N} j_{b_N}}} \left[r(j_{a_N} j_{b_N}) - (-1)^{k} s(j_{a_N} j_{b_N}) \right] \right)^2 \tag{Dlc.7}$$

One may also calculate from (III 1)

$$B(Ek)_{k \to 0} = \frac{1}{2k+1} \left| \langle \tilde{\tilde{0}}_N \| \frac{1}{\ell} \sum_{m,3} \ell_3 r^{k}_{m_3} Y_{k}(\theta_{m_3} \phi_{m_3}) \| \tilde{\tilde{k}}_N \rangle \right|^2 \tag{Dlc.8}$$

This will serve to verify equation (III2). Again the only nucleons present are neutrons (III 10)

From (III3) and (Cla.3) there follows

$$\langle \tilde{\tilde{0}}_N \| \frac{\ell_N}{\ell} \sum_m r^k_{m_N} Y_k(\theta_{m_N} \phi_{m_N}) \| \tilde{\tilde{k}}_N \rangle \frac{(-1)^{k-q}}{\sqrt{2k+1}}$$

$$= \langle \tilde{\tilde{0}}_N | \frac{\ell_N}{\ell} \sum_m r^k_{m_N} Y_{kq}(\theta_{m_N} \phi_{m_N}) | k_N \tilde{\tilde{-}} q_N \rangle \quad \text{(Dlc.9)}$$

since

$$| k_N \tilde{\tilde{-}} q_N \rangle = \Gamma^k_{-q}(N) | \tilde{\tilde{0}}_N \rangle \quad \text{(Dlc.10)}$$

one has, using (Dlc.10) and (III 10) with the indices, j_{a_N} and j_{b_N}, interchanged

$$\langle \tilde{\tilde{0}}_N | \frac{\ell_N}{\ell} \sum_m r^k_{m_N} Y_{kq}(\theta_{m_N} \phi_{m_N}) | k_N \tilde{\tilde{-}} q_N \rangle$$

$$= -\frac{\ell_N}{\ell} \sum_{j_{a_N} \geq j_{b_N}} q^k(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}} \sin(\gamma_{a_N} + \gamma_{b_N}) \sqrt{1+\delta_{j_{a_N} j_{b_N}}}$$

$$\times \langle \tilde{\tilde{0}}_N | \left[C^k_q(j_{a_N} j_{b_N}) + (-1)^k B^{+k}_q(j_{a_N} j_{b_N}) \right] \Gamma^k_{-q}(N) | \tilde{\tilde{0}}_N \rangle$$

(Dlc.11)

This is simply $\frac{-\ell_N}{\ell}$ times S, where S is defined by (Cla.5) in the evaluation of the matrix elements of $H^k_{N,P\,(31)}$ in Appendix Cla. Hence

$$\langle \tilde{\tilde{0}}_N | \frac{\ell_N}{\ell} \sum_m r^k_{m_N} Y_{kq}(\theta_{m_N} \phi_{m_N}) | k_N \tilde{\tilde{-}} q_N \rangle$$

$$= -(-1)^{k+q} \frac{\ell_N}{\ell} \sum_{j_{a_N} \geq j_{b_N}} \sqrt{1+\delta_{j_{a_N} j_{b_N}}}\, q^k(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}}$$

$$\times \sin(\alpha_{a_N}+\alpha_{b_N}) \left[r(j_{a_N} j_{b_N}) - (-1)^k s(j_{a_N} j_{b_N})\right]$$

(Dlc.12)

so that from (Dlc.9)

$$\langle \tilde{\tilde{a}}_N \| \frac{e_N}{e} \sum_n e_N r^k_{n_N} Y_k(\theta_{n_N} \phi_{n_N}) \| \tilde{\tilde{b}}_N \rangle$$

$$= -\sqrt{2k+1} \left(\frac{e_N}{e}\right) \sum_{j_{a_N} j_{b_N}} \sqrt{1+\delta_{j_{a_N} j_{b_N}}}\, q^k(j_{a_N} j_{b_N}) (-1)^{l_{a_N}}$$

$$\times v_{j_{a_N}} u_{j_{b_N}} \left[r(j_{a_N} j_{b_N}) - (-1)^k s(j_{a_N} j_{b_N})\right]$$

(Dlc.13)

Then using (Dlc.8)

$$\underset{k\to 0}{B(Ek)} = \left(\frac{e_N}{e}\right)^2 \left(\sum_{j_{a_N} j_{b_N}} \sqrt{1+\delta_{j_{a_N} j_{b_N}}}\, q^k(j_{a_N} j_{b_N}) (-1)^{l_{a_N}}\right.$$

$$\left.\times v_{j_{a_N}} u_{j_{b_N}} \left[r(j_{a_N} j_{b_N}) - (-1)^k s(j_{a_N} j_{b_N})\right]\right)^2$$

(Dlc.14)

Comparing (Dlc.14) and (Dlc.7)

$$\underset{0\to k}{B(Ek)} = (2k+1) \underset{k\to 0}{B(Ek)}$$

which corresponds to (III2).

d) Evaluation of C_N From Equations (D1.4),(D1.5), and (D1.6) -

Equation (D1.5) is just

$$\sum_{j_{cp}'} a^{J_f}_{k j_{cp}'} a^{J_i}_{0 J_i} \sum_{m' m_{cp}'} \langle k m' j_{cp}' m_{cp}' | J_f M_i + q \rangle (-1)^{-j_{cp}' - m_{cp}'}$$

$$\times \langle \tilde{0}_p | \gamma^{j'}_{-m_j'}(p) \beta^{+J_i}_{M_i}(p) | \hat{0}_p \rangle \times \frac{-\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N})$$

$$\times (-1)^{\ell_{a_N}} v_{a_N} u_{b_N} \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \langle \tilde{\tilde{0}}_N | \Gamma^k_{m'}(N) [C^k_q(j_{a_N} j_{b_N})$$

$$+ (-1)^k B^{+k}_q (j_{a_N} j_{b_N})] | \hat{\tilde{0}}_N \rangle \tag{D1d.1}$$

where j_{a_N} and j_{b_N}, the dummy indices, have been interchanged. The quasi-Boson portion of this matrix element has already been calculated (D1c.6). Using the commutation rule (I20'c) then gives

$$C_N^{(1)} = (-1)^k a^{J_f}_{k J_i} a^{J_i}_{0 J_i} \Big\{ -\frac{\ell_N}{\ell} \sum_{j_{a_N} \geq j_{b_N}} q^k(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}} \sin(\tau_{a_N} + \tau_{b_N})$$

$$\times \sqrt{1+\delta_{j_{a_N} j_{b_N}}} [r(j_{a_N} j_{b_N}) - (-1)^k s(j_{a_N} j_{b_N})] \Big\} \tag{D1d.2}$$

Equation (D1.6) may be written as

$$(-1)^{-J_f - M_i - q} a_0{}^{J_f}_{J_f} \sum_{j_{cp}} a^{J_i}_{k j_{cp}} \sum_{m m_{cp}} \langle k m j_{cp} m_{cp} | J_i M_i \rangle$$

$$\times \langle \tilde{0}_p | \gamma^{J_f}_{-M_i - q}(p) \beta^{+j_c}_{m_c}(p) | \hat{0}_p \rangle \times -\frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}}$$

$$\times v_{a_N} u_{b_N} \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \langle \tilde{\tilde{0}}_N | [C^k_q(j_{a_N} j_{b_N}) + (-1)^k$$

$$\times B^{+k}_q (j_{a_N} j_{b_N})] \Gamma^k_m(N) | \tilde{\tilde{0}}_N \rangle \tag{D1d.3}$$

where the indices, j_{a_ν} and j_{b_ν}, have been interchanged. The quasi-Boson part of this has been evaluated (Dlc.12). Upon using the commutation rule (I20'c), equation (Dld.3) becomes

$$a_{0J_f}^{J_f} \; a_{kJ_f}^{J_i} \langle k\,-q\; J_f\, M_i+q \mid J_i\, M_i\rangle$$
$$\times -(-1)^{k+q} \frac{l_\nu}{l} \sum_{j_{a_\nu} \geq j_{b_\nu}} \sqrt{1+\delta_{j_{a_\nu} j_{b_\nu}}}\; q^k(j_{a_\nu} j_{b_\nu})$$
$$\times (-1)^{l_{a_\nu}} \sin(\tau_{a_\nu}+\tau_{b_\nu}) \qquad \text{(Dld.4)}$$
$$\times \left[r(j_{a_\nu} j_{b_\nu}) - (-1)^k s(j_{a_\nu} j_{b_\nu}) \right]$$

Rewriting the Clebsch-Gordon coefficient in (Dld.4) produces

$$C_\nu^{(2)} = a_{0J_f}^{J_f} \; a_{kJ_f}^{J_i} \sqrt{\frac{2J_i+1}{2J_f+1}} \; (-1)^{J_i+J_f+k}$$
$$\times \frac{l_\nu}{l} \sum_{j_{a_\nu} \geq j_{b_\nu}} \sqrt{1+\delta_{j_{a_\nu} j_{b_\nu}}}\; q^k(j_{a_\nu} j_{b_\nu}) (-1)^{l_{a_\nu}} \sin(\tau_{a_\nu}+\tau_{b_\nu})$$
$$\times \left[r(j_{a_\nu} j_{b_\nu}) - (-1)^k s(j_{a_\nu} j_{b_\nu}) \right]$$

(Dld.5)

so that from (Dl.4), (Dld.2), and (Dld.5)

$$C_\nu = (-1)^k \frac{l_\nu}{l} \left[-a_{kJ_i}^{J_f} \, a_{0J_i}^{J_i} + (-1)^{J_i+J_f} \left(\frac{2J_i+1}{2J_f+1} \right)^{1/2} a_{0J_f}^{J_f} a_{kJ_f}^{J_i} \right]$$

$$\times \sum_{j_{a_N} \geq j_{b_N}} \sqrt{1+\delta_{j_{a_N} j_{b_N}}} \; q^k(j_{a_N} j_{b_N}) (-1)^{\ell_{a_N}} \sin(\tau_{a_N} + \tau_{b_N})$$

$$\times \left[s(j_{a_N} j_{b_N}) - (-1)^k s(j_{a_N} j_{b_N}) \right] \qquad \text{(III 13c)}$$

e) Evaluation of D_N From Equation (D1.7)

This follows very closely the work of Appendix C1b with the approximation

$$\left[\Gamma^k_{(N)} \beta^{+j_c}{}_{(P)} \right]^{J_i}_{M_i} |\tilde{0}_N ; \tilde{0}_P\rangle \simeq \left[\sum_{j_{1_N} \geq j_{2_N}} s(j_{1_N} j_{2_N}) B^{+k}(j_{1_N} j_{2_N}) \beta^{+j_c}{}_{(P)} \right]^{J_i}_{M_i} |\tilde{0}_N ; \hat{0}_P\rangle \qquad \text{(D1e.1)}$$

Now expand (D1.7) in terms of the individual quasi-neutron operators via (I38) and invoke (I4). The calculation is simplified if for the time being $\sum_{j_{1_N} \geq j_{2_N}} s(j_{1_N} j_{2_N})$ and $\sum_{j'_{1_N} j'_{2_N}} s(j'_{1_N} j'_{2_N})$ are replaced by $\frac{1}{2} \sum_{j_{1_N} j_{2_N}} (1+\delta_{j_{1_N} j_{2_N}}) s(j_{1_N} j_{2_N})$ and $\frac{1}{2} \sum_{j'_{1_N} j'_{2_N}} (1+\delta_{j'_{1_N} j'_{2_N}}) s(j'_{1_N} j'_{2_N})$ respectively. Equation (D1.7) is then

$$\frac{\ell_N}{\ell} \sum_{j'_{cP}} a^{J_f}_{k j'_{cP}} \sum_{j'_{1_N} j'_{2_N}} \frac{\sqrt{1+\delta_{j'_{1_N} j'_{2_N}}}}{2} s(j'_{1_N} j'_{2_N}) \times \sum_{\substack{m'_{1_N} m'_{2_N} \\ m' m'_{cP}}}$$

$$\times \langle j'_{1_N} m'_{1_N} j'_{2_N} m'_{2_N} | k m' \rangle \langle k m' j'_{cP} m'_{cP} | J_f M_i + q \rangle$$

$$\times (-1)^{-j'_{cP} - m'_{cP} - j'_{2_N} - m'_{2_N} - j'_{1_N} - m'_{1_N}} \sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N})$$

$$\times \sum_{\substack{m_{a_N} \\ m_{b_N}}} \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | k q \rangle \cos(\tau_{a_N} + \tau_{b_N})$$

$$\times \sum_{j_{cP}} a^{J_i}_{k j_{cP}} \sum_{\substack{j_{1_N} \\ j_{2_N}}} \frac{\sqrt{1+\delta_{j_{1_N} j_{2_N}}}}{2} s(j_{1_N} j_{2_N})$$

$$\times \sum_{\substack{m_{1N} m_{2N} \\ m\, m_{cp}}} \langle j_{1N} m_{1N}\, j_{2N} m_{2N} | k\, m \rangle \langle k\, m\, j_{cp} m_{cp} | J_i M_i \rangle$$

$$\times \langle \tilde{0}_N | \gamma^{j_2'}_{-m_2'}(N)\, \gamma^{j_1'}_{-m_1'}(N)\, \beta^{+j_a}_{m_a}(N)\, \gamma^{j_f}_{m_f}(N)\, \beta^{+j_1}_{m_1}(N)\, \beta^{+j_2}_{m_2}(N) | \tilde{0}_N \rangle$$

$$\times \langle \tilde{0}_p | \gamma^{j_c'}_{-m_c'}(p)\, \beta^{+j_c}_{m_c}(p) | \tilde{0}_p \rangle$$

(Dle.2)

The procedure is so similar to Appendix Clb that it will not be presented here. In essence, after commutating the operators via (I20'c), use is made of 3-j symbols (A3.10) and the sum rule (A3.13). The result is

$$D_N = \frac{\ell_N}{\ell} (-1)^k (2k+1)^{3/2} \sqrt{2J_i+1} \sum_{j_{cp}} a^{J_f}_{k j_{cp}}\, a^{J_i}_{k j_{cp}} \begin{Bmatrix} k & k & k \\ J_i & J_f & j_{cp} \end{Bmatrix}$$

$$\times (-1)^{j_{cp} - J_f} \sum_{j_{a_N} j_{b_N} j_{c_N}} q^k(j_{a_N}, j_{b_N}) \sqrt{1+\delta_{j_{a_N} j_{c_N}}}\, \sqrt{1+\delta_{j_{b_N} j_{c_N}}}\, r(j_{a_N} j_{c_N})$$

$$\times r(j_{b_N} j_{c_N}) \cos(\theta_{a_N} + \theta_{b_N}) \begin{Bmatrix} k & k & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix} (-1)^{j_{a_N} + j_{c_N}}$$

(III 13d)

f) Summary -

From equation (III 3), (D1.1), and (III 12) there follows

$$\langle J_f \| \frac{1}{e} \sum_{n,\eta} e_\eta\, r_{n_\eta}^{k} Y_k(\theta_{n_\eta}\phi_{n_\eta}) \| J_i \rangle$$

$$= \sqrt{2J_f+1}\,(-1)^{k+J_i-J_f}\left[A_p+B_p+C_N+D_N\right] \quad \text{(D1f.1)}$$

and finally from (III 1)

$$B(Ek) = \frac{2J_f+1}{2J_i+1}\left|A_p+B_p+C_N+D_N\right|^2 \quad \text{(III 13)}$$

where A_p, B_p, C_N and D_N are given by (III 13a), (III 13b), (III 13c), and (III 13d) respectively.

If J_f and J_i are interchanged in (III3), and use is made of the permutation rule for $q^k(J_f J_i)$ (I30), then it is not difficult to show that

$$\underset{J_i\to J_f}{B(Ek)}\,(2J_i+1) = \underset{J_f\to J_i}{B(Ek)}\,(2J_f+1) \quad \text{(III2)}$$

2. Two Quasi-Neutrons Plus Quasi-Proton

The neutron term of (III5) is best written as

$$\frac{e_N}{e}\sum_{j_{a_N} j_{b_N}} q^k(j_{a_N} j_{b_N}) \sum_{m_{a_N} m_{b_N}} \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | k\, q\rangle \Big[(-1)^{l_{b_N}} u_{a_N} v_{b_N}$$

$$\times\left((-1)^k \gamma_{m_a}^{j_a}(N)\,\gamma_{m_b}^{j_b}(N) - \beta_{m_a(N)}^{+j_a}\,\beta_{m_b(N)}^{+j_b}\right) + \cos(\chi_{a_N}+\chi_{b_N})\,\beta_{m_a}^{+j_a}(N)\,\gamma_{m_b(N)}^{j_b}\Big] \quad \text{(III9)}$$

The state, $|J_i M_i\rangle$,is given by

$$|J_i M_i\rangle = \sum_{(j_{1N} \geq j_{2N}) J_0 j_{cp}} c^{J_i}_{(j_{1N} j_{2N}) J_0 j_{cp}} \frac{1}{\sqrt{1+\delta_{j_{1N} j_{2N}}}} \left(\left[\beta^{+j_1}{}_{(N)} \beta^{+j_2}_{(N)} \right]^{J_0} \right.$$

$$\left. \times \beta^{+j_c}_{(P)} \right)^{J_i}_{M_i} |\tilde{0}_N ; \tilde{0}_P\rangle + a^{J_i}_{0 J_i} \beta^{+J_i}_{M_i}(P) |\tilde{0}_N ; \tilde{0}_P\rangle \qquad (I67)$$

while $\langle J_f \; M_i + q|$ is written as

$$\langle J_f \; M_i + q| = \sum_{j_{1N}' j_{2N}' (J_0') j_{cp}'} c^{J_f}_{(j_{1N}' j_{2N}') J_0' j_{cp}'} \frac{1}{\sqrt{1+\delta_{j_{1N}' j_{2N}'}}}$$

$$\times \langle \tilde{0}_N ; \tilde{0}_P| \left(\left[\beta^{+j_1'}_{(N)} \beta^{+j_2'}_{(N)} \right]^{J_0'} \beta^{+j_c'}_{(P)} \right)^{+J_f}_{M_i+q} + a^{J_f}_{0 J_f} (-1)^{-J_f - M_i - q}$$

$$\times \langle \tilde{0}_N ; \tilde{0}_P| \gamma^{J_f}_{-M_i-q}(P) \qquad (III19)$$

A useful relationship involving the coefficients of the two quasi-neutrons plus quasi-proton coefficients is

$$c^{J}_{(j_{bN} j_{aN}) J_0 j_{cp}} = (-1)^{j_{bN} - j_{aN} + J_0} c^{J}_{(j_{aN} j_{bN}) J_0 j_{cp}} \qquad (D2.1)$$

This may be seen as follows. First rewrite the summation notation as

$$\sum_{(j_{aN} \geq j_{bN}) J_0 j_{cp}} c^{J}_{(j_{aN} j_{bN}) J_0 j_{cp}} \left(\left[\beta^{+j_a}{}_{(N)} \beta^{+j_b}{}_{(N)} \right]^{J_0} \beta^{+j_c}{}_{(P)} \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_P\rangle$$

$$= \frac{1}{2} \sum_{(j_{aN} j_{bN}) J_0 j_{cp}} (1+\delta_{j_{aN} j_{bN}}) c^{J}_{(j_{aN} j_{bN}) J_0 j_{cp}} \left(\left[\beta^{+j_a}_{(N)} \beta^{+j_b}_{(N)} \right]^{J_0} \beta^{+j_c}_{(P)} \right)^{J}_{M} |\tilde{0}_N ; \tilde{0}_P\rangle \qquad (D2.2)$$

By permuting the individual quasi-neutron creation operators (I20'b) and interchanging the indices in the Clebsch-Gordon coefficient (A1.8), equation (D2.2) becomes

$$\frac{1}{2} \sum_{(j_{a_N} j_{b_N}) J_0 j_{c_P}} (1+\delta_{j_{a_N} j_{b_N}}) (-1)^{j_{b_N} - j_{a_N} + J_0} C^{J}_{(j_{a_N} j_{b_N}) J_0 j_{c_P}}$$

$$\times \left(\left[\beta^{+j_b}_{(N)} \beta^{+j_a}_{(N)} \right]^{J_0} \beta^{+j_c}_{(P)} \right)^{J}_{M} | \tilde{0}_N ; \tilde{0}_P \rangle \qquad \text{(D2.3)}$$

If the dummy indices, j_{a_N} and j_{b_N}, are interchanged in (D2.2) then one gets

$$\frac{1}{2} \sum_{(j_{a_N} j_{b_N}) J_0 j_{c_P}} (1+\delta_{j_{a_N} j_{b_N}}) C^{J}_{(j_{b_N} j_{a_N}) J_0 j_{c_P}} \left(\left[\beta^{+j_b}_{(N)} \beta^{+j_a}_{(N)} \right]^{J_0} \beta^{+j_c}_{(P)} \right)^{J}_{M}$$

$$\times | \tilde{0}_N ; \tilde{0}_P \rangle \qquad \text{(D2.4)}$$

Now comparing (D2.4) and (D2.3) gives (D2.1)

The matrix element to be evaluated is then from (III7) and (III9)

$$\langle J_f \; M_i+q | \frac{1}{\ell} \sum_{m,\eta} \ell_\eta \, r^k_{m_\eta} Y_k(\theta_{m_\eta} \phi_{m_\eta}) | J_i M_i \rangle$$

$$= \langle J_f \; M_i + q | \sum_{j_{a_P} j_{b_P}} q^k (j_{a_P} j_{b_P}) \left[\beta^{+j_a(P)} \gamma^{j_b(P)} \right]^k_q$$

$$+ \frac{\ell_N}{\ell} \sum_{j_{a_N} j_{b_N}} q^k (j_{a_N} j_{b_N}) \sum_{m_{a_N} m_{b_N}} \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | k q \rangle$$

$$\times \Big[(-1)^{\ell_{b_N}} u_{a_N} v_{b_N} \Big((-1)^k \gamma^{j_a}_{m_a}(N) \gamma^{j_b}_{m_b}(N) \qquad \text{(III20)}$$

$$- \beta^{+j_a}_{m_a}{}^{(N)} \beta^{+j_b}_{m_b}{}^{(N)} \Big) + \cos(\tau_{a_N} + \tau_{b_N}) \beta^{+j_a}_{m_a}{}^{(N)} \gamma^{j_b}_{m_b}(N) \Big] | J_i M_i \rangle$$

with $\langle J_f M_i + q |$ and $| J_i M_i \rangle$ given by (III 19) and (I67) respectively.

The above matrix (III20) may then be written as the sum of four terms

$$\langle k q \; J_i M_i | J_f \; q + M_i \rangle \left[A_p + B_p' + C_n' + D_n' \right] \tag{D2.5}$$

The first term, $\langle k q J_i M_i | J_f \; q + M_i \rangle A_p$, has already been mentioned (D1.2), and in Appendix D1a one finds

$$A_p = (-1)^k \, a^{J_i}_{0 J_i} \, a^{J_f}_{0 J_f} \, q^k (J_f J_i) (-1)^{J_i + J_f} \sqrt{\frac{2k+1}{2J_f+1}} \tag{III 13a}$$

The second term, $\langle k q J_i M_i | J_f \; q + M_i \rangle$ B_p', of (D2.5) is given by

$$\langle k q \; J_i M_i | J_f \; q + M_i \rangle B_p' =$$

$$\sum_{(j_{1n}' \geq j_{2n}') J_0' j_{cp}'} c^{J_f}_{(j_{1n}' j_{2n}') J_0' j_{cp}'} \frac{1}{\sqrt{1+\delta_{j_{1n}' j_{2n}'}}} \langle \tilde{0}_n ; \tilde{0}_p |$$

$$\times \left(\left[\beta^{+ j_1'}_{(n)} \beta^{+ j_2'}_{(n)} \right]^{J_0'} \beta^{+ j_c'}_{(p)} \right)^{J_f +}_{M_i + q}$$

$$\times \sum_{j_{ap} j_{bp}} q^k (j_{ap} j_{bp}) \left[\beta^{+ j_a}_{(p)} \gamma^{j_b}_{(p)} \right]^k_q$$

$$\times \sum_{(j_{1n} \geq j_{2n}) J_0 j_{cp}} c^{J_i}_{(j_{1n} j_{2n}) J_0 j_{cp}} \frac{1}{\sqrt{1+\delta_{j_{1n} j_{2n}}}} \left(\left[\beta^{+ j_1}_{(n)} \beta^{+ j_2}_{(n)} \right]^{J_0} \beta^{+ j_c}_{(p)} \right)^{J_i}_{M_i} | \tilde{0}_n ; \tilde{0}_p \rangle \tag{D2.6}$$

The third term, $\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle\; C_N'$, of (D2.5) may be broken up into two parts such that

$$C_N' \equiv C_N'^{(1)} + C_N'^{(2)} \tag{D2.7}$$

with $C_N'^{(1)}$ defined by

$$\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle\, C_N'^{(1)} \equiv$$

$$\sum_{(j_{1\nu}' \geq j_{2\nu}') J_0' j_{c\rho}'} c^{J_f}_{(j_{1\nu}' j_{2\nu}') J_0' j_{c\rho}'} \frac{1}{\sqrt{1+\delta_{j_{1\nu}' j_{2\nu}'}}} \langle \tilde{0}_\nu ; \tilde{0}_\rho | \left(\left[\beta^{+j_1'}_{(\nu)} \beta^{+j_2'}_{(\nu)} \right]^{J_0'}\right.$$

$$\times \left. \beta^{+j_c'}(\rho) \right)^{+J_f}_{M_i+q} \frac{l_\nu}{l} \sum_{j_{a\nu} j_{b\nu}} q^k (j_{a\nu} j_{b\nu}) \sum_{m_{a\nu} m_{b\nu}} \langle j_{a\nu} m_{a\nu}\, j_{b\nu} m_{b\nu} | k\, q \rangle$$

$$\times \left[(-1)^{l_{b\nu}} u_{a\nu} v_{b\nu} \beta^{+j_a}_{m_a}(\nu) \beta^{+j_b}_{m_b}(\nu) \right] a_{0\,J_i}^{J_i}\, \beta^{+J_i}_{M_i}(\rho) | \tilde{0}_\nu ; \tilde{0}_\rho \rangle \tag{D2.8}$$

and $C_N'^{(2)}$ defined by

$$\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle\, C_N'^{(2)} \equiv (-1)^{-J_f - M_i - q}\, a_{0\,J_f}^{J_f} \langle \tilde{0}_\nu ; \tilde{0}_\rho |$$

$$\times \gamma^{J_f}_{-M_i-q}(\rho) \frac{l_\nu}{l} \sum_{j_{a\nu} j_{b\nu}} q^k (j_{a\nu} j_{b\nu}) \sum_{m_{a\nu} m_{b\nu}} \langle j_{a\nu} m_{a\nu}\, j_{b\nu} m_{b\nu} | k\, q \rangle$$

$$\times \left[(-1)^{l_{b\nu}} u_{a\nu} v_{b\nu} (-1)^k \gamma^{j_a}_{m_a}(\nu) \gamma^{j_b}_{m_b}(\nu) \right] \sum_{(j_{1\nu} \geq j_{2\nu}) J_0 j_{c\rho}} c^{J_i}_{(j_{1\nu} j_{2\nu}) J_0 j_{c\rho}} \frac{1}{\sqrt{1+\delta_{j_{1\nu} j_{2\nu}}}}$$

$$\times \left(\left[\beta^{+j_1}_{(\nu)} \beta^{+j_2}_{(\nu)} \right]^{J_0} \beta^{+j}_{(\rho)} \right)^{J_i}_{M_i} | \tilde{0}_\nu ; \tilde{0}_\rho \rangle \tag{D2.9}$$

The last term, $\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle\; D_N'$, from (D2.5) is defined by

$$\langle k\, q\, J_i M_i | J_f\; q+M_i \rangle D_N' \equiv \sum_{j_{1\nu}' j_{2\nu}' (J_0') j_{c\rho}'} c^{J_f}_{(j_{1\nu}' j_{2\nu}') J_0' j_{c\rho}'} \frac{1}{\sqrt{1+\delta_{j_{1\nu}' j_{2\nu}'}}} \langle \tilde{0}_\nu ; \tilde{0}_\rho |$$

$$\times \left(\left[\beta^{+j_1'}_{(\nu)} \beta^{+j_2'}_{(\nu)} \right]^{J_0'} \beta^{+j_c'}_{(\rho)} \right)^{+J_f}_{M_i+q} \frac{l_\nu}{l} \sum_{j_{a\nu} j_{b\nu}} q^k (j_{a\nu} j_{b\nu}) \sum_{m_{a\nu} m_{b\nu}} \langle j_{a\nu} m_{a\nu}\, j_{b\nu} m_{b\nu} | k\, q \rangle$$

$$\times \cos(\tau_{a_N} + \tau_{b_N}) \beta^{+j_a}_{m_a}(N)\, \gamma^{j_b}_{m_b}(N) \sum_{(j_{1N} \geq j_{2N}) J_0 j_{cp}} c^{J_i}_{(j_{1N} j_{2N}) J_0 j_{cp}} \frac{1}{\sqrt{1+\delta_{j_{1N} j_{2N}}}}$$

$$\times \left(\left[\beta^{+j_1}_{(N)} \beta^{+j_2}_{(N)} \right]^{J_0} \beta^{+j_c}_{(P)} \right)^{J_i}_{M_i} |\tilde{0}_N ; \tilde{0}_P \rangle$$

(D2.10)

a) Evaluation of B_p' from equation (D2.6)

Writing (D2.6) out in detail, and using $\sum_{j_{1N} j_{2N}} \frac{1}{2}(1+\delta_{j_{1N} j_{2N}})$ and $\sum_{j_{1N}' j_{2N}'} \frac{1}{2}(1+\delta_{j_{1N}' j_{2N}'})$ in place $\sum_{j_{1N} \geq j_{2N}}$ and $\sum_{j_{1N}' \geq j_{2N}'}$ respectively, gives

$$\frac{1}{4} \sum_{j_{1N} j_{2N} (J_0) j_{cp}} \sum_{j_{1N}' j_{2N}' (J_0') j_{cp}'} c^{J_i}_{(j_{1N} j_{2N}) J_0 j_{cp}} c^{J_f}_{(j_{1N}' j_{2N}') J_0' j_{cp}'}$$

$$\times \sqrt{1+\delta_{j_{1N}' j_{2N}'}} \sqrt{1+\delta_{j_{1N} j_{2N}}} \sum_{\substack{m_{1N}' m_{2N}' \\ m_{1N} m_{2N}}} \langle j_{1N}' m_{1N}' j_{2N}' m_{2N}' | J_0' m' \rangle$$

$$\times \langle j_{1N} m_{1N} j_{2N} m_{2N} | J_0 m \rangle (-1)^{-j_{2N}' - m_{2N}' - j_{1N}' - m_{1N}'}$$

$$\times \langle \tilde{0}_N | \gamma^{j_2'}_{-m_2'}(N) \gamma^{j_1'}_{-m_1'} \beta^{+j_1}_{m_1}(N) \beta^{+j_2}_{m_2}(N) | \tilde{0}_N \rangle \sum_{j_{ap} j_{bp}} q^{k}(j_{ap} j_{bp})$$

$$\times \sum_{m' m_{cp}'} \sum_{m\, m_{cp}} \sum_{m_{ap} m_{bp}} \langle J_0' m' j_{cp}' m_{cp}' | J_f M_i + q \rangle \langle J_0 m j_{cp} m_{cp} | J_i M_i \rangle$$

$$\times \langle j_{ap} m_{ap} j_{bp} m_{bp} | k q \rangle (-1)^{-j_{cp}' - m_{cp}'} \langle \tilde{0}_P | \gamma^{j_c'}_{-m_c'}(P)$$

$$\times \beta^{+j_a}_{m_a}(P) \gamma^{j_b}_{m_b}(P) \beta^{+j_c}_{m_c}(P) | \tilde{0}_P \rangle$$

(D2a.1)

The quasi-neutron operators upon application of the commutation rule (I20'c) give

$$\langle \tilde{0}_N | \gamma^{j_2'}_{-m_2'}(N)\, \gamma^{j_1'}_{-m_1'}(N)\, \beta^{+j_1}_{m_1}(N)\, \beta^{+j_2}_{m_2}(N) | \tilde{0}_N \rangle$$

$$= \left(\delta_{2_N 2_N'} \delta_{1_N 1_N'} - \delta_{1_N 2_N'} \delta_{2_N 1_N'} \right) (-1)^{j_{1N}+m_{1N}+j_{2N}+m_{2N}}$$

(D2a.2)

The Kronecker deltas are in the notation of (A1.4). In a similar way

$$\langle \tilde{0}_P | \gamma^{j_c'}_{-m_c'}(P)\, \beta^{+j_a}_{m_a}(P)\, \gamma^{j_b}_{m_b}(P)\, \beta^{+j_c}_{m_c}(P) | \tilde{0}_P \rangle = \delta_{j_{cP}, -b_P}\, \delta_{j_{cP}' a_P}$$
$$\times (-1)^{j_{cP}+m_{cP}+j_{cP}'+m_{cP}'}$$ (D2a.3)

Inserting (D2a.2) and (D2a.3) into (D2a.1), one may utilize the separability of the quasi-neutron and quasi-proton parts of (D2a.1). The former involves just the orthonormality of Clebsch-Gordon coefficients. The latter is a little more complicated, but can be readily obtained by using 3-j symbols (A3.10) and introducing the 6-j symbol (A3.13). The result is

$$B_P' = \sqrt{(2J_i+1)(2k+1)} \sum_{\substack{j_{1N} \geq j_{2N} \\ J_0\, j_{cP}\, j_{cP}'}} c^{J_i}_{(j_{1N} j_{2N}) J_0 j_{cP}}\; c^{J_f}_{(j_{1N} j_{2N}) J_0 j_{cP}'}$$

$$\times\, q^k(j_{cP}'\ j_{cP})\, (-1)^{j_{cP}-J_i+J_0} \begin{Bmatrix} k & J_i & J_f \\ J_0 & j_{cP}' & j_{cP} \end{Bmatrix}$$

(III22a)

b) Evaluation of C_N' from equations (D2.7), (D2.8), and (D2.9) -

Writing $C_N'^{(1)}$ (D2.8) out in explicit detail, and temporarily replacing $\sum_{j_{1N}' \geq j_{2N}'}$ by $\sum_{j_{1N} j_{2N}} \frac{1}{2}(1+\delta_{j_{1N} j_{2N}})$ gives

$$\frac{1}{2} \sum_{(j_{1N}' j_{2N}') J_0' j_{cP}'} c^{J_f}_{(j_{1N}' j_{2N}') J_0' j_{cP}'} \sqrt{1+\delta_{j_{1N}' j_{2N}'}} \sum_{\substack{m_{1N}' m_{2N}' \\ m' m_{cP}'}} \langle j_{1N}' m_{1N}' j_{2N}' m_{2N}' | J_0' m' \rangle$$

$$\times \langle J_0' \, m' \, j_{cP}' \, m_{cP}' | J_f \, M_i + q \rangle (-1)^{-j_{cP}' - m_{cP}' - j_{2N}' - m_{2N}' - j_{1N}' - m_{1N}'}$$

$$\times -\frac{\ell_N}{\ell} \sum_{j_{aN} j_{bN}} q^k (j_{aN} j_{bN}) \sum_{\substack{m_{aN} \\ m_{bN}}} \langle j_{aN} m_{aN} j_{bN} m_{bN} | k q \rangle (-1)^{\ell_{bN}} u_{j_{aN}} v_{j_{bN}}$$

$$\times a_{0 J_i}^{J_i} \langle \tilde{0}_N | \gamma^{j_2'}_{-m_2'}(N) \gamma^{j_1'}_{-m_1'}(N) \beta^{+j_a}_{m_a}(N) \beta^{+j_b}_{m_b}(N) | \hat{0}_N \rangle$$

$$\times \langle \tilde{0}_P | \gamma^{j_c'}_{m_c'}(P) \beta^{+J_i}_{M_i}(P) | \hat{0}_P \rangle \qquad \text{(D2b.1)}$$

The commutation rule (I20'c) yields in the notation of (A1.4)

$$\langle \tilde{0}_N | \gamma^{j_2'}_{-m_2'}(N) \gamma^{j_1'}_{-m_1'}(N) \beta^{+j_a}_{m_a}(N) \beta^{+j_b}_{m_b}(N) | \hat{0}_N \rangle$$

$$= (\delta_{b_N, 2_N'} \delta_{a_N, 1_N'} - \delta_{a_N, 2_N'} \delta_{b_N, 1_N'}) (-1)^{j_{aN} + m_{aN} + j_{bN} + m_{bN}}$$

$$\text{(D2b.2)}$$

and

$$\langle \tilde{0}_P | \gamma^{j'}_{-m_j'}(P) \beta^{+J_i}_{M_i}(P) | \hat{0}_P \rangle = \delta_{J_i j_{cP}'} \delta_{M_i m_{cP}'} (-1)^{J_i + M_i} \qquad \text{(D2b.3)}$$

Inserting (D2b.2) and (D2b.3) into (D2b.1) and invoking orthonormality of Clebsch-Gordon coefficients gives

$$c_N'^{(1)} = -\frac{\ell_N}{\ell} a_0{}^{J_i}_{J_i} \sum_{j_{aN} j_{bN}} c^{J_f}_{(j_{aN} j_{bN}) k J_i} \sqrt{1+\delta_{j_{aN} j_{bN}}} \, q^k (j_{aN} j_{bN}) (-1)^{\ell_{bN}} u_{a_N} v_{b_N}$$

$$\text{(D2b.4)}$$

which may be written as

$$C_N^{\prime(1)} = -(-1)^{k} \frac{\ell_N}{\ell} a_{0 J_i}^{J_i} \sum_{j_{a_N} \geq j_{b_N}} C^{J_f}_{(j_{a_N} j_{b_N}) k J_i} \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}} q^{k}(j_{a_N} j_{b_N})$$

$$\times (-1)^{\ell_{a_N}} \sin(\tau_{a_N} + \tau_{b_N}) \qquad \text{(D2b.5)}$$

Now also writing $C_N^{\prime(2)}$ (D2.9) in detail, and temporarily replacing $\sum_{j_{1_N} \geq j_{2_N}}$ by $\sum_{j_{1_N} j_{2_N}} \frac{1}{2}(1+\delta_{j_{1_N} j_{2_N}})$ gives

$$\frac{1}{2} a_{0 J_f}^{J_f} (-1)^{-J_f - M_i - q} \frac{\ell_N}{\ell} \sum_{\substack{j_{a_N} \\ j_{b_N}}} q^{k}(j_{a_N} j_{b_N}) \sum_{\substack{m_{a_N} \\ m_{b_N}}} \langle j_{a_N} m_{a_N} j_{b_N} m_{b_N} | k q \rangle$$

$$\times (-1)^{\ell_{b_N}} u_{a_N} v_{b_N} \sum_{(j_{1_N} j_{2_N}) J_0 j_{c_P}} C^{J_i}_{(j_{1_N} j_{2_N}) J_0 j_{c_P}} \sqrt{1+\delta_{j_{1_N} j_{2_N}}}$$

$$\times \sum_{\substack{m_{1_N} m_{2_N} \\ m\, m_{c_P}}} \langle j_{1_N} m_{1_N} j_{2_N} m_{2_N} | J_0 m \rangle \langle J_0 m\, j_{c_P} m_{c_P} | J_i M_i \rangle$$

$$\times (-1)^{k} \langle \tilde{0}_N | \gamma^{j_a}_{m_a}(N) \gamma^{j_b}_{m_b}(N) \beta^{+ j_1}_{m_1}(N) \beta^{+ j_2}_{m_2}(N) | \tilde{0}_N \rangle$$

$$\times \langle \tilde{0}_P | \gamma^{J_f}_{M_i - q}(P) \beta^{+ j_c}_{m_c}(P) | \tilde{0}_P \rangle$$

(D2b.6)

Using the commutation rule (I20'c) in the notation of (A1.4) shows that

$$\langle \tilde{0}_N | \gamma^{j_a}_{m_a}(N) \gamma^{j_b}_{m_b}(N) \beta^{+ j_1}_{m_1}(N) \beta^{+ j_2}_{m_2}(N) | \tilde{0}_N \rangle$$

$$= \left(\delta_{2_N, -a_N} \delta_{1_N, -b_N} - \delta_{1_N, -a_N} \delta_{2_N, -b_N} \right) (-1)^{j_{1_N} + m_{1_N} + j_{2_N} + m_{2_N}}$$

(D2b.7)

and

$$\langle \tilde{0}_p | \gamma^{J_f}_{-M_i, -q}(p)\, \beta^{+\,j_c}_{m_c}(p) | \tilde{0}_p \rangle = \delta_{j_{cp} J_f}\, \delta_{m_{cp}, M_i+q} (-1)^{j_{cp}+m_{cp}}$$

(D2b.8)

Now placing (D2b.8) and (D2b.7) into (D2b.6) and using the orthonormality of the Clebsch-Gordon coefficients leads to

$$C_N^{\prime(2)} = (-1)^k \frac{l_N}{l} (-1)^{J_i+J_f} \left(\frac{2J_i+1}{2J_f+1}\right)^{1/2} a_{0\,J_f}^{J_f}$$

$$\times \sum_{j_{a_N} \geq j_{b_N}} c^{J_i}_{j_{a_N} j_{b_N}(k) J_f} \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}}\, q^k(j_{a_N} j_{b_N}) (-1)^{l_{a_N}} \sin(\tau_{a_N}+\tau_{b_N})$$

(D2b.9)

Finally, combining (D2b.5) and (D2b.9) produces from (D2.7)

$$C_N' = \frac{l_N}{l}(-1)^k \sum_{j_{a_N} \geq j_{b_N}} \Big[-a_{0\,J_i}^{J_i}\, c^{J_f}_{(j_{a_N} j_{b_N}) k J_i} + (-1)^{J_i+J_f}$$

$$\times \left(\frac{2J_i+1}{2J_f+1}\right)^{1/2} a_{0\,J_f}^{J_f}\, c^{J_i}_{(j_{a_N} j_{b_N}) k J_f} \Big] \frac{1}{\sqrt{1+\delta_{j_{a_N} j_{b_N}}}}\, q^k(j_{a_N} j_{b_N})$$

$$\times (-1)^{l_{a_N}} \sin(\tau_{a_N}+\tau_{b_N}) \qquad \text{(III22b)}$$

The quantity not in parentheses is just what one would have in the case of the adjacent even-even nucleus.

c) Evaluation of D_N' From Equation (D2.10) -

This calculation is quite similar to that of the matrix elements of $H^{k}_{N,P\,(22)}$ in Appendix C2b. The details will not be presented here, but one may infer the answer by comparison to Appendix D1e. In particular in (D1e.2) replace

$$\sum_{j_{1_N} j_{2_N}} \sum_{j_{c_P}} a^{J_i}_{k\, j_{c_P}} r(j_{1_N} j_{2_N}) \longrightarrow \sum_{j_{1_N} j_{2_N} (J_0) j_{c_P}} c^{J_i}_{(j_{1_N} j_{2_N}) J_0 j_{c_P}} \qquad \text{(D2c.1)}$$

and

$$\sum_{j'_{1_N} j'_{2_N}} \sum_{j'_{c_P}} a^{J_f}_{k\, j'_{c_P}} r(j'_{1_N} j'_{2_N}) \longrightarrow \sum_{j'_{1_N} j'_{2_N} (J_0') j'_{c_P}} c^{J_f}_{(j'_{1_N} j'_{2_N}) J_0' j'_{c_P}} \qquad \text{(D2c.2)}$$

One must also in (D1ℓ.2) replace the intermediate angular momentum, k, of two quasi-neutron states by J_0 or J_0' whichever is appropriate. The result is

$$D_N' = \frac{\ell_N}{\ell} (-1)^{k} \sqrt{(2k+1)(2J_0+1)(2J_0'+1)(2J_i+1)}$$

$$\times \sum_{\substack{j_{a_N} j_{b_N} j_{c_N} \\ J_0 J_0' j_{c_P}}} c^{J_f}_{(j_{a_N} j_{b_N}) J_0' j_{c_P}}\, c^{J_i}_{(j_{b_N} j_{c_N}) J_0 j_{c_P}} (-1)^{j_{c_P} - J_f} \begin{Bmatrix} k & J_0' & J_0 \\ j_{c_P} & J_i & J_f \end{Bmatrix}$$

$$\times q^{k}(j_{a_N} j_{b_N}) \sqrt{(1+\delta_{j_{a_N} j_{c_N}})(1+\delta_{j_{b_N} j_{c_N}})}\, \cos(\tau_{a_N} + \tau_{b_N}) \begin{Bmatrix} J_0 & J_0' & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix}$$

$$\times (-1)^{j_{c_N} + j_{a_N} + J_0 + J_0'} \qquad \text{(III22c)}$$

d) Summary -

From equations (III 3), (D2.5), and (III20) there follows

$$\langle J_f \| \frac{1}{e} \sum_{n_3} e_3 \, r_{n_3}^{k} \, Y_k (\theta_{n_3} \phi_{n_3}) \| J_i \rangle = \sqrt{2J_f+1}\,(-1)^{k+J_i-J_f}\left[A_p + B_p' + C_N' + D_N'\right] \quad \text{(D2d.1)}$$

and hence from (III 1)

$$B(Ek) = \frac{2J_f+1}{2J_i+1} \left| A_p + B_p' + C_N' + D_N' \right|^2 \quad \text{(III21)}$$

with A_p, B_p', C_N', and D_N' given by (III 13a), (III22a), (III22b), and (III22c) respectively.

By invoking the permutation rule for $q^k(J_f J_i)$ (I30) there follows after a little work

$$B(Ek)_{J_i \to J_f} (2J_i+1) = B(Ek)_{J_f \to J_i} (2J_f+1) \quad \text{(III2)}$$

APPENDIX E

Sum Rules

1. Orthonormality Relations:

For a particular total angular momentum, J_i , one may write the orthonormality relation

$$\sum_{\lambda_{J_i}} a_{k\, j_{cp}}^{J_i}(\lambda_{J_i})\, a_{k'\, j'_{cp}}^{J_i}(\lambda_{J_i}) = \delta_{kk'}\,\delta_{j_{cp} j'_{cp}} \tag{E1}$$

where $a_{k\, j'_{cp}}^{J_i}(\lambda_{J_i})$ is an expansion coefficient in the eigenfunction of state, (J_i, M_i) , with eigenvalue, $E_{\lambda_{J_i}}$, satisfying

$$\sum_{k' j'_{cp}} \left| a_{k'\, j'_{cp}}^{J_i}(\lambda_{J_i}) \right|^2 = 1 \tag{E2}$$

The index, λ_{J_i} , (usually implicit) indicates the λ_{J_i}'th state of total angular momentum, J_i .

In the quasi-Boson (of angular momentum, k) plus quasi-proton scheme, k' may equal either k or 0 , and the eigenfunction for state, J_i , M_i , λ_{J_i} , is

$$|JM\,\lambda_{J_i}\rangle_{0,k} = a_{0\,J_i}^{J_i}(\lambda_{J_i})\,\beta^{+\,J_i}_{M_i}(p)\,|\tilde{0}_N,\hat{0}_p\rangle$$

$$+\sum_{j_{cp}} a_{k\,j_{cp}}^{J_i}(\lambda_{J_i})\left[\Gamma^{k}(N)\,\beta^{+j_c}(p)\right]^{J_i}_{M_i}|\tilde{0}_N;\hat{0}_p\rangle$$

(E3)

so that equation (E1) implies

$$\sum_{\lambda_{J_i}} a_{k\,j_{cp}}^{J_i}(\lambda_{J_i})\,a_{k\,j'_{cp}}^{J_i}(\lambda_{J_i})=\delta_{j_{cp}j'_{cp}}$$

(E4a)

$$\sum_{\lambda_{J_i}} a_{0\,j_{cp}}^{J_i=j_{cp}}(\lambda_{J_i})\,a_{k\,j'_{cp}}^{J_i}(\lambda_{J_i}) = 0$$

(E4b)

and

$$\sum_{\lambda_{J_i}}\left|a_{0\,J_i}^{J_i}(\lambda_{J_i})\right|^2 = 1$$

(E4c)

2. Results:

The expression (in the quasi-Boson plus quasi-proton scheme) for $B(Ek)_{J_i,\lambda_{J_i}\to J_f}$ (III 13) where J_i and J_f are respectively the angular momenta of the initial and final states may be expanded to give

$$B(Ek)_{J_i\lambda_{J_i}\to J_f} = \frac{2J_f+1}{2J_i+1}\Big[|A_p|^2+|B_p|^2+|C_N|^2+|D_N|^2$$

(E5)

$$+(A_p\,B_p)+(A_p\,C_N)+(A_p\,D_N)+(B_p\,C_N)+(B_p\,D_N)+(C_N\,D_N)\Big]$$

In (E5), A_p, B_p, C_N, and D_N are given by equations (III 13a,b,c,d) respectively and all are functions of α_{J_i}. Summing over α_{J_i}, and applying the orthnormality relations (E4a,b,c) gives for the individual terms on the right hand side of (E5)

$$\frac{2J_f+1}{2J_i+1}\sum_{\alpha_{J_i}}|A_p|^2 = |a_{0J_f}^{J_f}|^2\left[q^k(J_f J_i)\right]^2\left(\frac{2k+1}{2J_i+1}\right)$$

$$= |a_{0J_f}^{J_f}|^2\, B(Ek)_{S.P.\; J_i\to J_f} \qquad \text{(E6a)}$$

where (III 14) has been used

$$\frac{2J_f+1}{2J_i+1}\sum_{\alpha_{J_i}}|B_p|^2 = (2J_f+1)(2k+1)\left[\sum_{j'_{cp},j_{cp}} a_{k\,j'_{cp}}^{J_f}\, q^k(j'_{cp}\, j_{cp})(-1)^{j_{cp}-J_i}\begin{Bmatrix} k & J_i & J_f\\ k & j'_{cp} & j_{cp}\end{Bmatrix}\right]$$

$$\times\left[\sum_{j'_{dp}j_{dp}} a_{k\,j'_{dp}}^{J_f}\, q^k(j'_{dp}\, j_{dp})(-1)^{j_{dp}-J_i}\begin{Bmatrix} k & J_i & J_f\\ k & j'_{dp} & j_{dp}\end{Bmatrix}\right]$$

$$\times\sum_{\alpha_{J_i}} a_{k\,j_{cp}}^{J_i}(\alpha_{J_i})\, a_{k\,j_{dp}}^{J_i}(\alpha_{J_i}) = (2J_f+1)(2k+1)\sum_{j'_{cp}j_{cp}j'_{dp}} a_{k\,j'_{cp}}^{J_f}\, a_{k\,j'_{dp}}^{J_f}$$

$$\times q^k(j'_{cp}\, j_{cp})\, q^k(j'_{dp}\, j_{cp})\begin{Bmatrix} k & J_i & J_f\\ k & j'_{cp} & j_{cp}\end{Bmatrix} \qquad \text{(E6b)}$$

$$\times\begin{Bmatrix} k & J_i & J_f\\ k & j'_{dp} & j_{cp}\end{Bmatrix}$$

$$\frac{2J_f+1}{2J_i+1}\sum_{\alpha_{J_i}}|C_N|^2 = \left[|a_{k\,J_i}^{J_f}|^2\left(\frac{2J_f+1}{2J_i+1}\right) + |a_{0\,J_f}^{J_f}|^2\right] B(Ek)_{k\to 0^+} \qquad \text{(E6c)}$$

where use has been made of (III 15)

$$\frac{2J_f+1}{2J_i+1}\sum_{aJ_i}|D_N|^2 = \left(\frac{\ell_N}{\ell}\right)^2(2k+1)^3(2J_f+1)$$
$$\times\left[\sum_{j_{cp}}\left|a^{J_f}_{k\,j_{cp}}\right|^2\begin{Bmatrix}k & k & k\\ J_i & J_f & j_{cp}\end{Bmatrix}^2\right]\left[\sum_{j_{aN}j_{bN}j_{cN}} q^k(j_{aN}\,j_{bN})\right.$$
$$\times\sqrt{1+\delta_{j_{aN}j_{bN}}}\sqrt{1+\delta_{j_{bN}j_{cN}}}\;r(j_{aN}\,j_{cN})\,r(j_{bN}\,j_{cN})$$
$$\left.\times\cos(\tau_{aN}+\tau_{bN})\begin{Bmatrix}k & k & k\\ j_{aN} & j_{bN} & j_{cN}\end{Bmatrix}(-1)^{j_{aN}+j_{cN}}\right]^2$$

(E6d)

$$\frac{2J_f+1}{2J_i+1}\sum_{aJ_i}(A_P B_P) = 0$$

(E6e)

$$\frac{2J_f+1}{2J_i+1}\sum_{aJ_i}(A_P C_N) = -(-1)^k\, g^k(J_f\,J_i)$$
$$\times(-1)^{J_i+J_f}\frac{\sqrt{(2k+1)(2J_f+1)}}{2J_i+1}\sqrt{\underset{k\to 0^+}{B}(Ek)}$$
$$\times\, a^{J_f}_{0\,J_f}\, a^{J_f}_{k\,J_i}$$

(E6f)

$$\frac{2J_f+1}{2J_i+1}\sum_{aJ_i}(A_P D_N) = 0$$

(E6g)

$$\frac{2J_f+1}{2J_i+1}\sum_{\partial_{J_i}}(B_p C_N) = -(-1)^k\sqrt{(2k+1)(2J_f+1)}\sum_{j'_{cp}} a^{J_f}_{k\,j'_{cp}}$$

$$\times\, q^k(j'_{cp}\,J_f)\begin{Bmatrix} k & J_i & J_f \\ k & j'_{cp} & J_f \end{Bmatrix} a^{J_f}_{0\,J_f}\sqrt{B(Ek)_{k\to 0^+}}$$

(E6h)

$$\frac{2J_f+1}{2J_i+1}\sum_{\partial_{J_i}}(B_p D_N) = (2k+1)^2(2J_f+1)\frac{l_N}{l}$$

$$\times\sum_{j'_{cp}\,j_{cp}} a^{J_f}_{k\,j'_{cp}}\, q^k(j'_{cp}\,j_{cp})(-1)^{J_f-J_i}\begin{Bmatrix} k & J_i & J_f \\ k & j'_{cp} & j_{cp} \end{Bmatrix}$$

$$\times\, a^{J_f}_{k\,j_{cp}}\begin{Bmatrix} k & k & k \\ J_i & J_f & j_{cp} \end{Bmatrix}\sum_{j_{a_N}j_{b_N}j_{c_N}} q^k(j_{a_N}\,j_{b_N})\sqrt{1+\delta_{j_{a_N}j_{c_N}}}$$

$$\times\sqrt{1+\delta_{j_{b_N}j_{c_N}}}\; r(j_{a_N}\,j_{c_N})\, r(j_{b_N}\,j_{c_N})\cos(\tau_{a_N}+\tau_{b_N})$$

$$\times\begin{Bmatrix} k & k & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix}(-1)^{j_{a_N}+j_{c_N}}$$

(E6i)

and finally

$$\frac{2J_f+1}{2J_i+1}\sum_{\partial_{J_i}}(C_N D_N) = \frac{l_N}{l}(-1)^k(2k+1)^{3/2}\sqrt{2J_f+1}\; a^{J_f}_{0\,J_f}\sqrt{B(Ek)_{k\to 0^+}}$$

$$\times\, a^{J_f}_{k\,J_f}\begin{Bmatrix} k & k & k \\ J_i & J_f & J_f \end{Bmatrix}\sum_{j_{a_N}j_{b_N}j_{c_N}} q^k(j_{a_N}\,j_{b_N})\sqrt{1+\delta_{j_{a_N}j_{b_N}}}\sqrt{1+\delta_{j_{b_N}j_{c_N}}}$$

$$r(j_{a_N}\,j_{c_N})\, r(j_{b_N}\,j_{c_N})\cos(\tau_{a_N}+\tau_{b_N})\begin{Bmatrix} k & k & k \\ j_{a_N} & j_{b_N} & j_{c_N} \end{Bmatrix}(-1)^{j_{a_N}+j_{c_N}}$$

(E6j)

The terms (E6b,d,i, and part of c) are proportional to sums over the products of two coefficients that indicate quasi-Boson plus quasi-proton contributions to the ground state, while (E6f,h,j) relate to one such coefficient and $\sqrt{B(E\ell)_{\ell\to 0^+}}$. Thus, such terms can be expected to contribute little to a sum over λ_{J_i} on the right hand side of (E5). If each term in the equations (E6) was multiplied by $E_{\lambda_{J_i}}$ prior to summing over λ_{J_i} , the orthonormality rules (E4a,b,c) could not be used. However, the resultant terms corresponding to (E6b,d,i, part of c,f,h,j) would again be almost negligible. Hence, in both cases the only significant contributions come from the "core" term (the $|a_{0 J_f}^{J_f}|^2$ part of E6c) and the single proton term (E6a). Finally, summing over J_i gives

$$\sum_{J_i} \sum_{\lambda_{J_i}} B(E\ell)_{J_i,\lambda_{J_i}\to J_f} \simeq |a_{0 J_f}^{J_f}|^2 \sum_{J_i} \left[B(E\ell)_{J_i\to J_f\,s.p.} + B(E\ell)_{\ell\to 0^+} \right] \tag{V1}$$

and (after first multiplying by $2J_i+1$)

$$\sum_{J_i,\lambda_{J_i}} (2J_i+1)\, E_{\lambda_{J_i}}\, B(E\ell)_{J_i,\lambda_{J_i}\to J_f} \simeq |a_{0 J_f}^{J_f}|^2$$

$$\times \sum_{J_i\lambda_{J_i}} (2J_i+1) E_{\lambda_{J_i}} \Big[|a_{0 J_i}^{J_i}(\lambda_{J_i})|^2\, B(E\ell)_{s.p.\; J_i\to J_f}$$

$$+ |a_{\ell J_f}^{J_i}(\lambda_{J_i})|^2\, B(E\ell)_{\ell\to 0^+} \Big] \tag{V2}$$

T A B L E I

QUASI-NEUTRON PARAMETERS "A"

Single Neutron State	Relative Energy mev	Quasi-neutron Transformation Coefficients μ_N	ν_N	Quasi-Neutron Energy E_N mev	μ_N^2	ν_N^2
$2p_{3/2}^-$	0	0.450	0.892	1.43	0.203	0.797
$1f_{5/2}^-$	0.78	0.716	0.698	1.15	0.513	0.487
$2p_{1/2}^-$	1.56	0.901	0.434	1.46	0.812	0.188
$1g_{9/2}^+$	4.52	0.990	0.141	4.09	0.980	0.020

TWO QUASI-NEUTRON PARAMETERS "B"

j_{a_N}	j_{b_N}	$E_{a_N} + E_{b_N}$ mev	$\sin(\gamma_{a_N} + \gamma_{b_N})$	$\cot(\gamma_{a_n} + \gamma_{b_n})$	$q^k(j_{a_N} j_{b_N})$ units of $(4\pi)^{-1/2}\left(\frac{\hbar}{m\omega_{H-O}}\right)^{k/2}$
			(k = 2)		
$3/2^-$	$1/2^-$	2.89	0.999	0.018	-4.02
$3/2^-$	$3/2^-$	2.86	0.802	-0.596	4.02
$5/2^-$	$1/2^-$	2.61	0.939	0.342	4.10
$5/2^-$	$3/2^-$	2.58	0.953	-0.301	2.19
$5/2^-$	$5/2^-$	2.30	0.999	0.026	5.26
$9/2^+$	$9/2^+$	8.18	0.279	0.960	8.55
			(k = 3)		
$9/2^+$	$3/2^-$	5.52	0.948	0.319	-16.4
$9/2^+$	$5/2^-$	5.24	0.792	0.612	-5.96

T A B L E II

QUASI-BOSON EXPANSION COEFFICIENTS

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

Ni^{62}

j_{1_N}	j_{2_N}	$\lambda(j_{1_N} j_{2_N})$	$\Delta(j_{1_N} j_{2_N})$	$\lambda^2(j_{1_N} j_{2_N})$	$\Delta^2(j_{1_N} j_{2_N})$	$(\lambda/\Delta)^2(j_{1_N} j_{2_N})$	$\lambda(j_{1_N} j_{2_N})-\Delta(j_{1_N} j_{2_N})$
$3/2^-$	$1/2^-$	$\pm$ 0.480	$\mp$ 0.204	0.230	0.0416	5.58	$\pm$ 0.684
$3/2^-$	$3/2^-$	$\mp$ 0.278	$\pm$ 0.117	0.0773	0.0138	5.68	$\mp$ 0.395
$5/2^-$	$1/2^-$	$\mp$ 0.545	$\pm$ 0.208	0.297	0.0434	6.89	$\mp$ 0.753
$5/2^-$	$3/2^-$	$\mp$ 0.304	$\pm$ 0.114	0.0924	0.0130	7.01	$\mp$ 0.418
$5/2^-$	$5/2^-$	$\mp$ 0.675	$\pm$ 0.220	0.455	0.0485	9.42	$\mp$ 0.895
$9/2^+$	$9/2^+$	$\pm$ 0.0496	$\mp$ 0.0372	0.00246	0.00138	1.77	$\pm$ 0.0868

TABLE III

MATRIX ELEMENTS IN THE QUASI-BOSON PLUS QUASI-PROTON SCHEME

(c.f. equations (II 12) and (II 13))

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

Cu^{63} $\quad k_N = 2^+$

"A"

j_p'	J	$\langle ((\tilde{\tilde{0}}_N ; \tilde{j}_p = J) JM \mid H^{(2)}_{N,P\,(31)} \mid (\tilde{\tilde{k}}_N ; \tilde{j}_p') JM \rangle$
$1/2^-$	$3/2^-$	$\mp$ 0.448 mev
	$5/2^-$	$\mp$ 0.372
$3/2^-$	$1/2^-$	$\pm$ 0.634
	$3/2^-$	$\mp$ 0.448
	$5/2^-$	$\pm$ 0.198
$5/2^-$	$1/2^-$	$\mp$ 0.646
	$3/2^-$	$\mp$ 0.246
	$5/2^-$	$\mp$ 0.477

"B"

j_p	j_p'	J	$\langle ((\tilde{\tilde{k}}_N ; \tilde{j}_p) JM \mid H^{(2)}_{N,P\,(22)} \mid (\tilde{\tilde{k}}_N ; \tilde{j}_p') JM \rangle$
$3/2^-$	$1/2^-$	$3/2^-$	-0.0302 mev
		$5/2^-$	0.0161
$3/2^-$	$3/2^-$	$1/2^-$	-0.0302
		$3/2^-$	0
		$5/2^-$	0.0215
		$7/2^-$	-0.00865
$5/2^-$	$1/2^-$	$3/2^-$	-0.0165
		$5/2^-$	-0.0270
$5/2^-$	$3/2^-$	$1/2^-$	0.00880
		$3/2^-$	0.0118
		$5/2^-$	0.00515
		$7/2^-$	-0.00925

(cont.)

TABLE III (cont.)

MATRIX ELEMENTS IN THE QUASI-BOSON PLUS QUASI-PROTON SCHEME

(c.f. equations (II 12) and (II 13))

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

"B"

Cu^{63} $k_N = 2^+$

j_p	j_p'	J	$\langle (\tilde{\tilde{k}}_N ; \tilde{j}_p) JM \mid H^{(2)}_{N,p\,(22)} \mid (\tilde{\tilde{k}}_N ; \tilde{j}_p') JM \rangle$
$5/2^-$	$5/2^-$	$1/2^-$	-0.0344
		$3/2^-$	-0.0123
		$5/2^-$	0.0123
		$7/2^-$	0.0209
		$9/2^-$	-0.00919

TABLE IV

QUASI-BOSON PLUS QUASI-PROTON SCHEME

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

Cu^{63}

Total Angular Momentum J	Energy in mev	QUASI-BOSON PLUS QUASI-PROTON EIGENFUNCTION			
		$a^{J}_{0^+,J}$	$a^{J}_{2^+,\frac{1}{2}^-}$	$a^{J}_{2^+,3/2^-}$	$a^{J}_{2^+,5/2^-}$
$1/2^-$	0.797	-0.642	0	-0.702	0.309
	1.78	-0.396	0	0.648	0.651
	2.77	0.657	0	-0.296	0.694
$3/2^-$	0	-0.937	0.145	0.300	0.105
	1.54	-0.279	0.0689	-0.949	0.127
	2.21	0.131	-0.0527	0.0895	0.986
	3.04	0.164	0.986	0.0271	0.0284
$5/2^-$	0.740	-0.903	0.156	-0.265	0.299
	1.48	-0.226	0.0486	0.962	0.144
	2.36	-0.305	0.142	0.0622	-0.940
	3.04	0.200	0.976	-0.0146	0.0818
$7/2^-$	1.40	0	0	1.000	0.0114
	2.21	0	0	0.0114	-1.000
$9/2^-$	1.94	0	0	0	1

TABLE V

QUASI-BOSON PLUS QUASI-PROTON SCHEME

Cu^{63} B(E2)

Total Angular Momentum J	Energy in mev	Terms contributing to B(E2)↓: Single Proton	One quasi-Boson exc. st. to one quasi-Boson gd. st. (due to Proton)	One quasi-Boson exc. st. to one quasi-Boson gd. st. (due to neutrons)	Zero quasi-Boson exc. st. to one quasi-Boson gd. st. (due to neutrons)	One quasi-Boson exc. st. to zero quasi-Boson gd. st. (due to neutrons)	B(E2)↓ units of 10^{-52} sq. cm.	Ratio of B(E2)↓ to pure single Proton
$1/2^-$	0.797	-2.70	-0.450	-0.119	-1.01	-5.07	226	8.67
	1.78	-1.67	-0.567	0.0822	-0.625	4.68	9.35	0.359
	2.77	2.77	-0.782	-0.0665	1.04	-2.14	1.74	0.0666
$3/2^-$	1.54	-1.17	-0.578	-0.0003	-0.912	9.70	63.8	2.45
	2.21	0.552	-0.639	0.060	0.429	-0.914	0.338	0.0130
	3.04	0.692	-0.896	-0.104	0.537	-0.276	0.00282	0.0001
$5/2^-$	0.740	-2.07	-0.286	0.0315	-1.04	-3.31	38.3	1.47
	1.48	-0.519	0.243	-0.194	-0.259	12.0	110	4.22
	2.36	-0.699	-0.0328	0.00585	-0.349	0.778	0.0758	0.0029
	3.04	0.459	-1.07	-0.0679	0.230	-0.182	0.338	0.0130
$7/2^-$	1.40	0	-1.54	-0.0900	0	-14.4	167	6.40
	2.21	0	0.124	0.0605	0	-0.165	0.0002	0.0000
$9/2^-$	1.16	0	0	0	0	0	0	0

TABLE VI

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

BASES FOR Cu^{63} STATES OF SPINS $1/2^-$, $3/2^-$, $5/2^-$, $7/2^-$

	$J = 1/2^-$				$J = 3/2^-$				$J = 5/2^-$				$J = 7/2^-$			
NO.	j_{a_N}	j_{b_N}	J_0	j_{c_p}	j_{a_N}	j_{b_N}	J_0	j_{c_p}	j_{a_N}	j_{b_N}	J_0	j_{c_p}	j_{a_N}	j_{b_N}	J_0	j_{c_p}
1	----	----	0^+	$1/2^-$	----	----	0^+	$3/2^-$	----	----	0^+	$5/2^-$	$3/5^-$	$1/2^-$	1^+	$5/2^-$
2	$3/2^-$	$1/2^-$	1^+	$1/2^-$	$3/2^-$	$1/2^-$	1^+	$1/2^-$	$3/2^-$	$1/2^-$	1^+	$3/2^-$	$5/2^-$	$3/2^-$	1^+	$5/2^-$
3	$3/2^-$	$1/2^-$	1^+	$3/2^-$	$3/2^-$	$1/2^-$	1^+	$3/2^-$	$3/2^-$	$1/2^-$	1^+	$5/2^-$	$3/2^-$	$1/2^-$	2^+	$3/2^-$
4	$5/2^-$	$3/2^-$	1^+	$1/2^-$	$3/2^-$	$1/2^-$	1^+	$5/2^-$	$5/2^-$	$3/2^-$	1^+	$3/2^-$	$3/2^-$	$1/2^-$	2^+	$5/2^-$
5	$5/2^-$	$3/2^-$	1^+	$3/2^-$	$5/2^-$	$3/2^-$	1^+	$1/2^-$	$5/2^-$	$3/2^-$	1^+	$5/2^-$	$3/2^-$	$3/2^-$	2^+	$3/2^-$
6	$3/2^-$	$1/2^-$	2^+	$3/2^-$	$5/2^-$	$3/2^-$	1^+	$3/2^-$	$3/2^-$	$1/2^-$	2^+	$1/2^-$	$3/2^-$	$3/2^-$	2^+	$5/2^-$
7	$3/2^-$	$1/2^-$	2^+	$5/2^-$	$5/2^-$	$3/2^-$	1^+	$5/2^-$	$3/2^-$	$1/2^-$	2^+	$3/2^-$	$5/2^-$	$1/2^-$	2^+	$3/2^-$
8	$3/2^-$	$3/2^-$	2^+	$3/2^-$	$3/2^-$	$1/2^-$	2^+	$1/2^-$	$3/2^-$	$1/2^-$	2^+	$5/2^-$	$5/2^-$	$1/2^-$	2^+	$5/2^-$
9	$3/2^-$	$3/2^-$	2^+	$5/2^-$	$3/2^-$	$1/2^-$	2^+	$3/2^-$	$3/2^-$	$3/2^-$	2^+	$1/2^-$	$5/2^-$	$3/2^-$	2^+	$3/2^-$
10	$5/2^-$	$1/2^-$	2^+	$3/2^-$	$3/2^-$	$1/2^-$	2^+	$5/2^-$	$3/2^-$	$3/2^-$	2^+	$3/2^-$	$5/2^-$	$3/2^-$	2^+	$5/2^-$
11	$5/2^-$	$1/2^-$	2^+	$5/2^-$	$3/2^-$	$3/2^-$	2^+	$1/2^-$	$3/2^-$	$3/2^-$	2^+	$5/2^-$	$5/2^-$	$5/2^-$	2^+	$3/2^-$
12	$5/2^-$	$3/2^-$	2^+	$3/2^-$	$3/2^-$	$3/2^-$	2^+	$3/2^-$	$5/2^-$	$1/2^-$	2^+	$1/2^-$	$5/2^-$	$5/2^-$	2^+	$5/2^-$
13	$5/2^-$	$3/2^-$	2^+	$5/2^-$	$3/2^-$	$3/2^-$	2^+	$5/2^-$	$5/2^-$	$1/2^-$	2^+	$3/2^-$	$9/2^+$	$9/2^+$	2^+	$3/2^-$
14	$5/2^-$	$5/2^-$	2^+	$3/2^-$	$5/2^-$	$1/2^-$	2^+	$1/2^-$	$5/2^-$	$1/2^-$	2^+	$5/2^-$	$9/2^+$	$9/2^+$	2^+	$5/2^-$
15	$5/2^-$	$5/2^-$	2^+	$5/2^-$	$5/2^-$	$1/2^-$	2^+	$3/2^-$	$5/2^-$	$3/2^-$	2^+	$1/2^-$	$5/2^-$	$1/2^-$	3^+	$1/2^-$
16	$9/2^+$	$9/2^+$	2^+	$3/2^-$	$5/2^-$	$1/2^-$	2^+	$5/2^-$	$5/2^-$	$3/2^-$	2^+	$3/2^-$	$5/2^-$	$1/2^-$	3^+	$3/2^-$
17	$9/2^+$	$9/2^+$	2^+	$5/2^-$	$5/2^-$	$3/2^-$	2^+	$1/2^-$	$5/2^-$	$3/2^-$	2^+	$5/2^-$	$5/2^-$	$1/2^-$	3^+	$5/2^-$
18	$5/2^-$	$1/2^-$	3^+	$5/2^-$	$5/2^-$	$3/2^-$	2^+	$3/2^-$	$5/2^-$	$5/2^-$	2^+	$1/2^-$	$5/2^-$	$3/2^-$	3^+	$1/2^-$
19	$5/2^-$	$3/2^-$	3^+	$5/2^-$	$5/2^-$	$3/2^-$	2^+	$5/2^-$	$5/2^-$	$5/2^-$	2^+	$3/2^-$	$5/2^-$	$3/2^-$	3^+	$3/2^-$
20	$9/2^+$	$1/2^-$	4^-	$9/2^+$	$5/2^-$	$5/2^-$	2^+	$1/2^-$	$5/2^-$	$5/2^-$	2^+	$5/2^-$	$5/2^-$	$3/2^-$	3^+	$5/2^-$
21	$9/2^+$	$3/2^-$	4^-	$9/2^+$	$5/2^-$	$5/2^-$	2^+	$3/2^-$	$9/2^+$	$9/2^+$	2^+	$1/2^-$	$5/2^-$	$3/2^-$	4^+	$1/2^-$
22	$9/2^+$	$5/2^-$	4^-	$9/2^+$	$5/2^-$	$5/2^-$	2^+	$5/2^-$	$9/2^+$	$9/2^+$	2^+	$3/2^-$	$5/2^-$	$3/2^-$	4^+	$3/2^-$

(cont.)

T A B L E VI (cont.)

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

BASES FOR Cu^{63} STATES OF SPINS $1/2^-$, $3/2^-$, $5/2^-$, $7/2^-$

	$J = 1/2^-$				$J = 3/2^-$				$J = 5/2^-$				$J = 7/2^-$			
NO.	j_{a_N}	j_{b_N}	J_0	j_{c_p}	j_{a_N}	j_{b_N}	J_0	j_{c_p}	j_{a_N}	j_{b_N}	J_0	j_{c_p}	j_{a_n}	j_{b_N}	J_0	j_{c_p}
23	$9/2^+$	$1/2^-$	5^-	$9/2^+$	$9/2^+$	$9/2^+$	2^+	$1/2^-$	$9/2^+$	$9/2^+$	2^+	$5/2^-$	$5/2^-$	$3/2^-$	4^+	$5/2^-$
24	$9/2^+$	$3/2^-$	5^-	$9/2^+$	$9/2^+$	$9/2^+$	2^+	$3/2^-$	$5/2^-$	$1/2^-$	3^+	$1/2^-$	$5/2^-$	$5/2^-$	4^+	$1/2^-$
25	$9/2^+$	$5/2^-$	5^-	$9/2^+$	$9/2^+$	$9/2^+$	2^+	$5/2^-$	$5/2^-$	$1/2^-$	3^+	$3/2^-$	$5/2^-$	$5/2^-$	4^+	$3/2^-$
26					$5/2^-$	$1/2^-$	3^+	$3/2^-$	$5/2^-$	$1/2^-$	3^+	$5/2^-$	$5/2^-$	$5/2^-$	4^+	$5/2^-$
27					$5/2^-$	$1/2^-$	3^+	$5/2^-$	$5/2^-$	$3/2^-$	3^+	$1/2^-$	$9/2^+$	$9/2^+$	4^+	$1/2^-$
28					$5/2^-$	$3/2^-$	3^+	$3/2^-$	$5/2^-$	$3/2^-$	3^+	$3/2^-$	$9/2^+$	$9/2^-$	4^+	$3/2^-$
29					$5/2^-$	$3/2^-$	3^+	$5/2^-$	$5/2^-$	$3/2^-$	3^+	$5/2^-$	$9/2^+$	$9/2^+$	4^+	$5/2^-$
30					$5/2^-$	$3/2^-$	4^+	$5/2^-$	$5/2^-$	$3/2^-$	4^+	$3/2^-$	$9/2^+$	$9/2^+$	6^+	$5/2^-$
31					$5/2^-$	$5/2^-$	4^+	$5/2^-$	$5/2^-$	$3/2^-$	4^+	$5/2^-$	$9/2^+$	$5/2^-$	2^-	$9/2^+$
32					$9/2^+$	$9/2^+$	4^+	$5/2^-$	$5/2^-$	$5/2^-$	4^+	$3/2^-$	$9/2^+$	$3/2^-$	3^-	$9/2^+$
33					$9/2^+$	$3/2^-$	3^-	$9/2^+$	$5/2^-$	$5/2^-$	4^+	$5/2^-$	$9/2^+$	$5/2^-$	3^-	$9/2^+$
34					$9/2^+$	$5/2^-$	3^-	$9/2^+$	$9/2^+$	$9/2^+$	4^+	$3/2^-$	$9/2^+$	$1/2^-$	4^-	$9/2^+$
35					$9/2^+$	$1/2^-$	4^-	$9/2^+$	$9/2^+$	$9/2^+$	4^+	$5/2^-$	$9/2^+$	$3/2^-$	4^-	$9/2^+$
36					$9/2^+$	$3/2^-$	4^-	$9/2^+$	$9/2^+$	$5/2^-$	2^-	$9/2^+$	$9/2^+$	$5/2^-$	4^-	$9/2^+$
37					$9/2^+$	$5/2^-$	4^-	$9/2^+$	$9/2^+$	$3/2^-$	3^-	$9/2^+$	$9/2^+$	$1/2^-$	5^-	$9/2^+$
38					$9/2^+$	$1/2^-$	5^-	$9/2^+$	$9/2^+$	$5/2^-$	3^-	$9/2^+$	$9/2^+$	$3/2^-$	5^-	$9/2^+$
39					$9/2^+$	$3/2^-$	5^-	$9/2^+$	$9/2^+$	$1/2^-$	4^-	$9/2^+$	$9/2^+$	$5/2^-$	5^-	$9/2^+$
40					$9/2^+$	$5/2^-$	5^-	$9/2^+$	$9/2^+$	$3/2^-$	4^-	$9/2^+$	$9/2^+$	$3/2^-$	6^-	$9/2^+$
41					$9/2^+$	$3/2^-$	6^-	$9/2^+$	$9/2^+$	$5/2^-$	4^-	$9/2^+$	$9/2^+$	$5/2^-$	6^-	$9/2^+$
42					$9/2^+$	$5/2^-$	6^-	$9/2^+$	$9/2^+$	$1/2^-$	5^-	$9/2^+$	$9/2^+$	$5/2^-$	7^-	$9/2^+$
43									$9/2^+$	$3/2^-$	5^-	$9/2^+$				
44									$9/2^+$	$5/2^-$	5^-	$9/2^+$				
45									$9/2^+$	$3/2^-$	6^-	$9/2^+$				
46									$9/2^+$	$5/2^-$	6^-	$9/2^+$				
47									$9/2^+$	$5/2^-$	7^-	$9/2^+$				

TABLE VII

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

MATRIX ELEMENTS OF $H^{k=2}_{n,p\,(31)}$ (c.f. equation (II 22)

LONG RANGE QUADRUPOLE-QUADRUPOLE INTERACTION

Cu^{63}

j_{a_N}	j_{b_N}	J_o	j_{c_p}	$\langle[(\tilde{j}_{a_N}\tilde{j}_{b_N})J_o\,\tilde{j}_{c_p}]JM\,\vert H^{(2)}_{N,p\,(31)}\vert\,\tilde{0}_M;\tilde{j}_{c_p}=J, m_{c_p}=M\rangle$ J = 1/2⁻	J = 3/2⁻	J = 5/2⁻
3/2⁻	1/2⁻	2⁺	1/2⁻	0 mev	0.165 mev	0.137 mev
3/2⁻	3/2⁻	↓	↓	↓	-0.132	-0.110
5/2⁻	1/2⁻				-0.158	-0.131
5/2⁻	3/2⁻				-0.0857	-0.0715
5/2⁻	5/2⁻				-0.216	-0.180
9/2⁺	9/2⁺				0.0979	0.0814
3/2⁻	1/2⁻		3/2⁻	-0.233	0.165	-0.0733
3/2⁻	3/2⁻		↓	0.187	-0.132	0.0589
5/2⁻	1/2⁻			0.223	-0.158	0.0702
5/2⁻	3/2			0.121	-0.0857	0.0384
5/2⁻	5/2⁻			0.306	-0.216	0.0960
9/2⁺	9/2⁺			-0.138	0.0979	-0.0435
3/2⁻	1/2⁻		5/2⁻	0.238	0.0898	0.176
3/2⁻	3/2⁻		↓	-0.191	-0.0721	-0.141
5/2⁻	1/2⁻			-0.228	-0.0860	-0.168
5/2⁻	3/2⁻			-0.124	-0.0465	-0.0915
5/2⁻	5/2⁻			-0.311	-0.118	-0.230
9/2⁺	9/2⁺			0.141	0.0533	0.104

T A B L E VIII

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

MATRIX ELEMENTS OF $H^{(2)}_{NN\,(22)}$ (Class 1) (c.f. equations (II23) and (II34a))

Cu^{63} LONG RANGE QUADRUPOLE-QUADRUPOLE INTERACTION

$J = 1/2^-,\ 3/2^-,\ 5/2^-,\ \text{or}\ 7/2^-$

$J_0 = J_0' = 2 \qquad j_{cp} = j'_{cp} = 1/2^-,\ 3/2^-,\ \text{or}\ 5/2^-$

j_{a_N}	j_{b_N}	j'_{a_N}	j'_{b_N}	$\langle [(j_{a_N} j_{b_N})J_0 j_{c_p}]JM \mid H^{(2)}_{NN\,(22)}\ \text{(class 1)} \mid [(j'_{a_N} j'_{b_N})J'_0 j'_{c_p}]JM \rangle$ (Part (a))
3/2⁻	1/2⁻	3/2⁻	1/2⁻	0.296 mev
↓	↓	3/2⁻	3/2⁻	-0.237
		5/2⁻	1/2⁻	-0.282
		5/2⁻	3/2⁻	-0.153
		5/2⁻	5/2⁻	-0.386
		9/2⁺	9/2⁺	0.175
3/2⁻	3/2⁻	3/2⁻	3/2⁻	0.190
↓	↓	5/2⁻	1/2⁻	0.227
		5/2⁻	3/2⁻	0.123
		5/2⁻	5/2⁻	0.310
		9/2⁺	9/2⁺	-0.141
5/2⁻	1/2⁻	5/2⁻	1/2⁻	0.270
↓	↓	5/2⁻	3/2⁻	0.147
		5/2⁻	5/2⁻	0.370
		9/2⁺	9/2⁺	-0.168
5/2⁻	3/2⁻	5/2⁻	3/2⁻	0.0796
↓	↓	5/2⁻	5/2⁻	0.201
		9/2⁺	9/2⁺	-0.0910
5/2⁻	5/2⁻	5/2⁻	5/2⁻	0.506
↓	↓	9/2⁺	9/2⁺	-0.230
9/2⁺	9/2⁺	9/2⁺	9/2⁺	0.104

Part (a) indicates that only the term due to $\sum_{j_{1_N} j_{2_N}} 2\left[B^{+(2)}(j_{1_N} j_{2_N}) C^{(2)}(j_{3_N} j_{4_N})\right]^0_0$ is considered.

T A B L E IX

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

MATRIX ELEMENTS OF $H^{(2)}_{NN(22)}$ (Class 1)(c.f. equations (II 23),(II34b), & (II 34c))

LONG RANGE QUADRUPOLE-QUADRUPOLE INTERACTION

Cu^{63} $J = 1/2^-, 3/2^-, 5/2^-$ or $7/2^-$

$J_0 = J_0'$ $j_{c_p} = j_{c_p}' = 1/2^-, 3/2^-$, or $5/2^-$

j_{a_N}	j_{b_N}	j'_{a_N}	j'_{b_N}	J_0	$\langle [(j_{a_N} j_{b_N}) J_0 j_{c_p}] JM \mid \mathcal{H}^{(2)}_{NN(22)}(\text{class 1}) \mid [(j'_{a_N} j'_{b_N}) J'_0 j'_{c_p}] JM \rangle$ Part (b)	Part (c)
$3/2^-$	$1/2^-$	$3/2^-$	$1/2^-$	1^+	-2.62 mev	1.06 mev
$5/2^-$	$3/2^-$	$5/2^-$	$3/2^-$	1^+	-2.62	0.713
$3/2^-$	$1/2^-$	$3/2^-$	$1/2^-$	2^+	-2.62	1.06
$3/2^-$	$3/2^-$	$3/2^-$	$3/2^-$	2^+	-5.24	1.42
$5/2^-$	$1/2^-$	$5/2^-$	$1/2^-$	2^+	-2.62	1.06
$5/2^-$	$3/2^-$	$5/2^-$	$3/2^-$	2^+	-2.62	0.714
$5/2^-$	$5/2^-$	$5/2^-$	$5/2^-$	2^+	-5.24	1.43
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	2^+	-5.24	0.0521
$5/2^-$	$1/2^-$	$5/2^-$	$1/2^-$	3^+	-2.62	1.06
$5/2^-$	$3/2^-$	$5/2^-$	$3/2^-$	3^+	-2.62	0.714
$5/2^-$	$3/2^-$	$5/2^-$	$3/2^-$	4^+	-2.62	0.714
$5/2^-$	$5/2^-$	$5/2^-$	$5/2^-$	4^+	-5.24	1.43
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	4^+	-5.24	0.0521
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	6^+	-5.24	0.0521
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	8^+	-5.24	0.0521

Part (b) indicates that only the term due to $\sum_{j_{1N} j_{2N}} \frac{1}{1+\delta_{j_{1N} j_{2N}}} \left[\delta_{j_{2N} j_{4N}} \delta_{j_{1N} j_{3N}} - (-1)^{j_{1N}+j_{2N}} \delta_{j_{1N} j_{4N}} \delta_{j_{2N} j_{3N}}\right]$ is considered.
while **Part (c)** considers only the contribution due to single quasi-neutron scattering terms (neglected in the Boson-approximation).

T A B L E X

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

MATRIX ELEMENTS OF $\mathcal{H}^{(2)}_{NN\,(22)}$ (class 2) (c.f. equation (II 24))

Cu^{63}

LONG RANGE QUADRUPOLE-QUADRUPOLE INTERACTION

$J_o = J_o'$ $\quad j_{c_p} = j_{c_p}' = 1/2^-,\ 3/2^-,\ \text{or}\ 5/2^-$

$J = 1/2^-,\ 3/2^-,\ 5/2^-\ \text{or}\ 7/2^-$

j_{a_N}	j_{b_N}	j_{a_N}'	j_{b_N}'	J_0	$\langle [(j_{a_N} j_{b_N})J_o j_{c_p}]JM \vert \mathcal{H}^{(2)}_{NN\,(22)}(\text{class 2}) \vert [(j_{a_N}' j_{b_N}')J_o' j_{c_p}']JM \rangle$
3/2⁻	1/2⁻	3/2⁻	1/2⁻	1⁺	-0.114 mev
3/2⁻	1/2⁻	5/2⁻	3/2⁻	1⁺	0.0273
5/2⁻	3/2⁻	5/2⁻	3/2⁻	1⁺	-0.084
3/2⁻	1/2⁻	3/2⁻	1/2⁻	2⁺	-0.114
3/2⁻	1/2⁻	3/2⁻	3/2⁻	2⁺	-0.00318
3/2⁻	1/2⁻	5/2⁻	1/2⁻	2⁺	0.00186
3/2⁻	1/2⁻	5/2⁻	3/2⁻	2⁺	-0.0568
3/2⁻	1/2⁻	5/2⁻	5/2⁻	2⁺	-0.0276
3/2⁻	3/2⁻	3/2⁻	3/2⁻	2⁺	-0.132
3/2⁻	3/2⁻	5/2⁻	3/2⁻	2⁺	-0.0288
3/2⁻	3/2⁻	5/2⁻	5/2⁻	2⁺	0.00174
5/2⁻	1/2⁻	3/2⁻	3/2⁻	2⁺	0.00163
5/2⁻	1/2⁻	5/2⁻	1/2⁻	2⁺	-0.066
5/2⁻	1/2⁻	5/2⁻	3/2⁻	2⁺	-0.00579
5/2⁻	1/2⁻	5/2⁻	5/2⁻	2⁺	-0.00350
5/2⁻	3/2⁻	5/2⁻	3/2⁻	2⁺	-0.0971
5/2⁻	3/2⁻	5/2⁻	5/2⁻	2⁺	-0.00188
5/2⁻	5/2⁻	5/2⁻	5/2⁻	2⁺	-0.080
9/2⁺	9/2⁺	9/2⁺	9/2⁺	2⁺	-1.97
5/2⁻	1/2⁻	5/2⁻	1/2⁻	3⁺	-0.030
5/2⁻	1/2⁻	5/2⁻	3/2⁻	3⁺	0.0135
5/2⁻	3/2⁻	5/2⁻	3/2⁻	3⁺	-0.0928

(cont.)

T A B L E X

(cont.)

j_{a_N}	j_{b_N}	j'_{a_N}	j'_{b_N}	J_0	$\left\langle \left[(j_{a_n} j_{b_n}) J_0 j_{c_p} \right] JM \middle\vert \mathcal{H}^{(2)}_{NN(22)} (\text{class } 2) \middle\vert \left[(j'_{a_n} j'_{b_n}) J'_0 j'_{c_p} \right] JM \right\rangle$
$5/2^-$	$3/2^-$	$5/2^-$	$3/2^-$	4^+	-0.0875
$5/2^-$	$3/2^-$	$5/2^-$	$5/2^-$	4^+	0.00166
$5/2^-$	$5/2^-$	$5/2^-$	$5/2^-$	4^+	-0.080
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	4^+	-1.19
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	6^+	-0.574
$9/2^+$	$9/2^+$	$9/2^+$	$9/2^+$	8^+	-0.946

T A B L E XI

TWO QUASI-NEUTRONS (COUPLED TO ARBITRARY ANGULAR MOMENTUM) PLUS QUASI-PROTON

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

$J = 5/2^-$ Cu^{63}

No.	Energy in mev	TERMS CONTRIBUTING TO $B(E2)\downarrow$ Single Proton	Two quasi-neutrons to two quasi-neutrons (due to proton)	two quasi-neutrons to zero quasi-neutrons (due to neutrons)	B(E2) units of 10^{-52} sq. cm.	ratio of B(E2) to pure single proton
1	0.440	0.0265	-0.00144	-0.0402	0.000196	0.000008
2	0.526	-0.234	-0.175	1.72	1.48	0.0568
3	0.601	-0.0590	-0.0590	0.351	0.0467	0.00179
4	0.667	0.256	0.0579	-0.770	0.179	0.00686
5	0.686	0.0797	0.0284	-0.287	0.0276	0.00106
6	0.770	-1.59	-0.0652	1.03	0.334	0.0128
7	0.784	1.65	0.0226	-0.469	1.25	0.0478
8	0.944	0.00593	0.00164	0.0617	0.00413	0.000158
9	1.11	-0.0228	-0.0205	0.0116	0.000868	0.000033
10	1.18	0.0751	-0.0249	0.654	0.426	0.0164
11	1.23	-0.0302	0.0312	-0.122	0.0126	0.000483
12	1.31	-0.0405	0.0118	-0.253	0.0681	0.00261
13	1.32	-0.240	-0.0286	0.221	0.00196	0.000075

(cont.)

TABLE XI (cont.)

NO.	Energy in mev	Single Proton	Two quasi-neutrons to two quasi-neutrons (due to proton)	two quasi-neutrons to zero quasi-neutrons (due to neutrons)	B(E2) units of 10^{-52} sq. cm.	ratio of B(E2) to pure single proton
14	1.38	-0.00711	0.0268	-0.0561	0.00114	0.000044
15	1.44	0.0943	-0.00646	0.150	0.0487	0.00187
16	1.49	0.0724	0.000420	0.0787	0.0197	0.000758
17	1.55	0.0696	0.00244	-0.0875	0.000207	0.000008
18	1.72	-0.000917	-0.0306	0.00446	0.000631	0.000024
19	1.89	-0.0192	0.0527	-0.0694	0.00111	0.000042
20	1.97	0.0916	-0.0885	0.304	0.0811	0.00311
21	2.04	0.0814	-0.0753	1.86	2.99	0.115
22	2.05	0.219	-0.121	7.98	56.2	2.16
23	2.12	0.0683	-0.356	-0.258	0.256	0.00983
24	2.24	0.0210	-0.0183	0.413	0.149	0.00572
25	2.27	0.0742	-0.229	-0.816	0.811	0.0311
26	2.50	0.0114	-0.0115	-0.0365	0.00116	0.000044
27	2.67	-0.00584	0.00273	-0.0224	0.000560	0.000022
28	2.76	0.149	-0.127	-0.205	0.0289	0.00111
29	2.89	0.310	0.00952	0.202	0.235	0.00900
30	3.64	0.215	-0.479	0.0492	0.0397	0.00152
31	4.59	0.0233	-0.00581	2.77	6.70	0.257
32	5.12	-0.00614	0.0221	0.580	0.305	0.0117
33	5.43	-0.0477	0.00707	-0.0912	0.0149	0.000573
34	5.88	-0.00714	0.0242	-0.240	0.0429	0.00164
35	6.46	0.0400	-0.0684	0.128	0.00859	0.000330

TABLE XII a

TWO QUASI-NEUTRONS (COUPLED TO 2^+ ANGULAR MOMENTUM ONLY) PLUS QUASI-PROTON

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

$J = 1/2^-$ Cu^{63}

NO.	Energy in mev	TERMS CONTRIBUTING TO B(E2)			B(E2) units of 10^{-52} cm^4	ratio of B(E2) to pure single proton
		Single Proton	two quasi-neutrons to two quasi-neutrons (due to proton)	two quasi-neutrons to zero quasi-neutrons (due to neutrons)		
1	0.523	-0.305	0.0838	0.896	1.17	0.0451
2	0.677	0.550	-0.0941	-1.36	2.10	0.0805
3	0.941	-0.0670	0.00449	0.135	0.0135	0.000518
4	1.18	1.99	-0.184	-1.83	0.00221	0.000085
5	1.28	-1.48	0.234	0.522	1.35	0.0520
6	1.35	2.13	-0.0748	-1.29	1.52	0.0583
7	1.52	-1.72	-0.179	1.33	0.820	0.0315
8	1.72	0.0875	0.0225	-0.0889	0.00115	0.000044
9	2.01	-0.235	-0.119	0.597	0.152	0.00584
10	2.25	-1.76	0.0584	-3.87	80.2	3.08
11	3.00	1.42	0.493	0.467	14.6	0.561
12	4.94	0.167	0.00299	1.55	7.65	0.294
13	5.75	-0.155	-0.0856	-0.101	0.302	0.0116

TABLE XII b

TWO QUASI-NEUTRONS (COUPLED TO 2^+ ANGULAR MOMENTUM ONLY) PLUS QUASI-PROTON

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION $J = 3/2^-$ Cu^{63}

NO.	Energy in mev	Terms contributing to B(E2): Single Proton	Terms contributing to B(E2): two quasi-neutrons to two quasi-neutrons (due to Proton)	Terms contributing to B(E2): two quasi-neutrons to zero quasi-neutrons (due to Neutrons)	B(E2) Units of 10^{-52} sq. cm.	Ratio of B(E2) to Pure Single Proton
1	0	-------	-------	-------	-------	-------
2	0.544	-0.410	0.198	-0.976	1.82	0.0699
3	0.692	0.301	-0.134	0.909	1.50	0.0574
4	0.936	0.00851	-0.00669	0.0501	0.00348	0.000134
5	1.21	-0.113	0.0567	-0.810	1.04	0.0398
6	1.32	0.0895	0.154	-0.0560	0.0453	0.00174
7	1.47	-0.0582	-0.159	0.311	0.0114	0.000436
8	1.72	0.00193	0.00257	0.0500	0.00426	0.000163
9	1.99	-0.0746	0.149	-1.48	2.55	0.0979
10	2.08	-0.609	0.00556	-5.74	51.9	1.99
11	2.11	0.113	-0.259	2.23	5.60	0.215
12	2.26	-0.0327	0.230	-1.59	2.52	0.0965
13	2.50	0.00624	0.0145	-0.0429	0.000631	0.000024
14	2.77	-0.0966	-0.198	0.425	0.0219	0.000842
15	2.86	-0.215	-0.302	-0.139	0.556	0.0214
16	3.64	-0.362	-0.432	-0.0308	0.877	0.0337
17	4.73	-0.0803	0.0220	-2.21	6.63	0.254
18	5.60	-0.0238	-0.0546	0.227	0.0284	0.00109
19	6.35	-0.0759	-0.0701	-0.205	0.159	0.00610

TABLE XII c

TWO QUASI-NEUTRONS (COUPLED TO 2^{+} ANGULAR MOMENTUM ONLY) PLUS QUASI-PROTON

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

$J = 5/2^{-}$ Cu^{63}

NO.	Energy in Mev	Terms contributing to B(E2): Single Proton	Terms contributing to B(E2): two-quasi-neutrons to two-quasi-neutrons (due to proton)	Terms contributing to B(E2): two quasi-neutrons to two quasi-neutrons (due to neutrons)	B(E2) Units of 10^{-52} sq cm.	Ratio of B(E2) to Pure Single Proton
1	0.530	0.236	-0.181	-1.74	2.44	0.0936
2	0.692	0.415	-0.0796	-1.04	0.424	0.0163
3	0.779	-2.27	0.0557	0.980	1.32	0.0505
4	0.942	-0.00306	0.00175	-0.0539	0.00262	0.000101
5	1.19	0.0912	0.0342	0.709	0.598	0.0230
6	1.32	0.241	-0.0268	-0.173	0.00145	0.000056
7	1.48	0.135	-0.00690	0.000027	0.0140	0.000538
8	1.72	0.00267	-0.0277	0.0271	0.000004	0.000000
9	1.97	-0.0932	-0.0854	-0.305	0.201	0.00772
10	2.05	-0.235	-0.145	-8.20	63.3	2.43
11	2.11	-0.0682	-0.356	0.390	0.000990	0.000038
12	2.25	0.0749	0.238	-0.866	0.263	0.0101
13	2.50	0.0109	0.0112	-0.0365	0.000179	0.000007
14	2.76	0.143	0.124	-0.206	0.00322	0.000124
15	2.89	-0.313	0.00651	-0.199	0.220	0.00843
16	3.64	0.216	0.480	0.0524	0.481	0.0185
17	4.63	0.0222	0.00194	2.84	7.06	0.271
18	5.44	0.0454	0.00211	0.0408	0.00670	0.000257
19	6.36	-0.0431	-0.0748	-0.145	0.0593	0.00228

T A B L E XII d

TWO QUASI-NEUTRONS (COUPLED TO 2^+ ANGULAR MOMENTUM ONLY) PLUS QUASI-PROTON

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION

$J = 7/2^-$ Cu^{63}

NO.	Energy in mev	Terms contributing to B(E2): Single Proton	Terms contributing to B(E2): two quasi-neutrons to two quasi-neutrons (due to Proton)	Terms contributing to B(E2): two quasi-neutrons to zero quasi-neutrons (due to Neutrons)	B(E2) Units of 10^{-52} sq. cm.	Ratio of B(E2) to Pure Single Proton
1	0.542	0	0.371	1.74	2.86	0.110
2	0.692	0	0.318	1.71	2.65	0.102
3	0.938	0	-0.0134	-0.124	0.0122	0.000467
4	1.21	0	+0.183	-1.43	1.68	0.0645
5	1.31	0	+0.0556	0.00755	0.00149	0.000057
6	1.48	0	0.0132	0.264	0.0495	0.00190
7	1.72	0	+0.0106	-0.100	0.00796	0.000305
8	1.97	0	-0.133	-1.55	1.83	0.0703
9	2.06	0	-0.718	-9.42	66.7	2.56
10	2.83	0	0.116	0.0323	0.0143	0.000547
11	4.80	0	0.120	3.14	6.84	0.265
12	5.41	0	-0.0404	-0.273	0.0635	0.00244

T A B L E XIII a

TWO QUASI-NEUTRONS (COUPLED TO 3^- ANGULAR MOMENTUM ONLY) PLUS QUASI-PROTON

k = 3 (OCTUPOLE INTERACTION) Cu^{63}

j_{a_N}	j_{b_N}	j'_{a_N}	j'_{b_N}	(A) (a)	(A) (b)	(A) (c)	(B)	DIAGONAL TERMS $E_{j_{a_N}} + E_{j_{b_N}}$	DIAGONAL TERMS $E_{j_{a_N}} + E_{j_{b_N}}$ + (A) + (B)
$9/2^+$	$3/2^-$	$9/2^+$	$3/2^-$	-1.38 mev	-5.28 mev	-1.74 mev	-0.221 mev	5.52 mev	0.38 mev
$9/2^+$	$5/2^-$	$9/2^+$	$5/2^-$	-0.128	-5.28	0.602	-0.135	5.24	0.30
$9/2^+$	$3/2^-$	$9/2^+$	$5/2^-$	-0.420	0	0	-0.00364	------	------

T A B L E XIII b

k = 2 (QUADRUPOLE INTERACTION)

j_{a_N}	j_{b_N}	j'_{a_N}	j'_{b_N}	(A) (a)	(A) (b)	(A) (c)	(B)	DIAGONAL TERMS $E_{j_{a_N}} + E_{j_{b_N}}$	DIAGONAL TERMS $E_{j_{a_N}} + E_{j_{b_N}}$ + (A) + (B)
$9/2^+$	$3/2^-$	$9/2^+$	$3/2^-$	0	-2.62	0.366	-0.192	5.52	3.08
$9/2^+$	$5/2^-$	$9/2^+$	$5/2^-$	0	-2.62	0.369	-0.316	5.24	2.67
$9/2^+$	$3/2^-$	$9/2^+$	$5/2^-$	0	0	0	-0.0459	----	----

(A) $\equiv$ (a) + (b) + (c) $\equiv \langle [(j_{a_N} j_{b_N})3^- \, j_{c_p}] JM | \mathcal{H}^{(2)}_{NN\,(22)}$ (class 1) $| [(j'_{a_N} j'_{b_N})3^- \, j_{c_p}] JM \rangle$ (c.f. equation (II 23))

(a) = term due to $\sum_{j_{1_N} j_{2_N}} 2 [B^+(3) (j_{1_N} j_{2_N}) B^{(3)} (j_{3_N} j_{4_N})]^0_0$ (c.f. equation (III 34a))

(b) = term due to $\sum_{j_{1_N} j_{2_N}} \frac{1}{1+\delta_{j_{1N}\, j_{2N}}} [\delta_{j_{2_N} j_{4_N}} \delta_{j_{1_N} j_{3_N}} + \delta_{j_{1_N} j_{4_N}} \delta_{j_{2_N} j_{3_N}} (-1)^{j_{1_N}+j_{2_N}}]$ (c.f. equation (III 34b))

(c) = terms dropped in the Boson approximation (c.f. equation (III 34c))

(B) $\equiv \langle [(j_{a_N} j_{b_N})3^- \, j_{c_p}] JM | \mathcal{H}^{(2)}_{NN\,(22)}$ (class 2) $| [(j'_{a_N} j'_{b_N})3^- \, j_{c_p}] JM \rangle$ (c.f. equation (II 24))

TABLE XIV

TWO QUASI-NEUTRONS (COUPLED TO ANGULAR MOMENTUM 3^- ONLY) plus quasi-proton

Cu^{63}

QUADRUPOLE-QUADRUPOLE INTERACTION ONLY

J	Energy in mev	a^J_{oJ}	$\kappa^J (9/2^+, 3/2^-)3^- j_{c_p}$: $j_{c_p} = 1/2^-$	$j_{c_p} = 3/2^-$	$j_{c_p} = 5/2^-$	$\kappa^J (9/2, 5/2^-)3^- j_{c_p}$: $j_{c_p} = 1/2^-$	$j_{c_p} = 3/2^-$	$j_{c_p} = 5/2^-$	B(E3) units of 10^{-78} cm^6	Ratio of B(E3) to pure single proton
1/2	3.43	0 →	0	0	0.0864	0	0	0.996	----	---
	3.78		0	0	0.996	0	0	-0.0864	----	----
3/2	2.67		0	-0.150	0.0109	0	-0.989	0.00571	242	0.247
	3.05		0	-0.986	0.0662	0	0.150	0.0182	1055	1.08
	3.47		0	-0.0280	-0.138	0	-0.00301	-0.990	0.996	0.00102
	3.86		0	0.0638	0.988	0	0.000460	-0.140	4.88	0.00498
5/2	2.77		-0.0142	-0.107	0.00816	-0.0548	-0.992	0.0377	199	0.203
	3.19		-0.0834	-0.989	0.0542	-0.00487	0.109	0.00696	1089	1.11
	3.53		0.0201	0.0144	0.109	0.0795	0.0323	0.990	0.699	0.000714
	3.95		0.125	0.0422	0.985	0.0202	-0.00363	-0.113	2.01	0.00205
	4.32		-0.0928	0.0217	0.0399	-0.990	0.0569	0.0748	1.81	0.00185
	4.76		0.984	-0.0890	-0.119	-0.0988	0.0000369	0.00225	9.43	0.00962
7/2	2.81		0.00909	-0.0832	0.00315	0.0383	-0.996	0.0128	178	0.182

(cont.)

TABLE XIV (cont.)

J	Energy in mev	a^J_{0J}	$c^J_{(1/2^+, 3/2^-)3^-\, j_{c_p}}$: $j_{c_p} = 1/2^-$	$j_{c_p} = 3/2^-$	$j_{c_p} = 5/2^-$	$c^J_{(1/2^+, 5/2^-)3^-\, j_{c_p}}$: $j_{c_p} = 1/2^-$	$j_{c_p} = 3/2^-$	$j_{c_p} = 5/2^-$	B(E3) units of 10^{-78} cm^6	Ratio of B(E3) to pure Single Proton
7/2	3.27	0	0.0618	-0.994	0.0239	0.00645	0.0841	0.00771	1118	1.14
	3.58	↓	-0.0228	-0.0133	-0.0903	-0.109	-0.0163	-0.989	0.400	0.000408
	4.02	↓	-0.172	-0.0339	-0.979	-0.0356	-0.00194	0.0977	1.42	0.00145
	4.33	↓	-0.0866	-0.0125	0.0622	-0.988	-0.0362	0.106	0.662	0.000676
	4.76	↓	0.979	0.0562	-0.170	-0.0970	0.0000071	0.00289	3.76	0.00384
9/2	2.72	↓	0	0.127	0.0112	0	0.991	0.0415	220	0.224
	3.12	↓	0	-0.989	-0.0658	0	0.128	-0.0104	1077	1.10
	3.48	↓	0	0.0247	-0.137	0	0.0398	-0.989	1.62	0.00165
	3.87	↓	0	0.0639	-0.988	0	-0.00285	0.139	4.74	0.00483
	4.61	1	0	0	0	0	0	0	934	0.953
11/2	3.48	0	0	0	0.131	0	0	0.991	----	----
	3.89	0	0	0	0.991	0	0	-0.131	----	----

T A B L E XV a

INELASTIC ALPHA SCATTERING CROSS SECTIONS (AT A FIXED ANGLE) FOR NEGATIVE PARITY LEVELS

Spin	Energy in mev: experiment (ref. 10)	Energy in mev: quasi-boson plus quasi-proton scheme	Energy in mev: two quasi-neutrons (coupled to 2+ only) plus quasi-proton scheme	dσ/dΩ at diffraction max. via experiment (ref. 10), mb/ster: 26°	36°	46°	Born Approx. dσ/dΩ in mb/ster: quasi-boson plus quasi-proton scheme (max. at ~15°)	Born Approx.: two quasi-neutrons (coupled to 2+ only) plus quasi-proton scheme	DWBA dσ/dΩ in mb/ster via two quasi-neutrons plus quasi-proton scheme (diffraction maxima are predicted at about 27.5°, 36.5°, and 46°, c.f. Fig. 11): 26°	36°	46°
$1/2^-$	0.668	0.797	2.25	1.6	0.55	0.27	1.6	0.57	1.64	0.852	0.268
$3/2^-$	1.547	1.54	2.08	0.25	0.080	0.045	0.94	0.74	2.00	1.04	0.328
$5/2^-$	0.961	1.48	2.05	3.8	1.7	0.8	2.4	1.3	3.35	1.75	0.558
$7/2^-$	1.327	1.40	2.06	4.0	2.0	0.9	4.9	1.9	4.24	2.20	0.704

TABLE XV b

INELASTIC ALPHA SCATTERING CROSS SECTIONS (fixed angle) FOR POSITIVE PARITY LEVELS

Energy via Experiment Reference 10 mev.	$d\sigma/d\Omega$ at diffraction max. via experiment (ref. 10) mb./ster.				Two quasi-neutrons (coupled to 3⁻ only) plus quasi-proton scheme					
					Spin	Energy mev	DWBA $d\sigma/d\Omega$ in mb./ster (diffraction max. are predicted at about 22°, 32°, 41°, and 50° c.f. Fig. 12)			
	22°	32°	40°	50°			22°	32°	40°	50°
2.50	4.2	1.4	0.65	0.35	3/2	2.67	0.466	0.172	0.0531	0.0196
3.32	2.7	0.9	0.4	0.2	9/2	2.72	1.04	0.388	0.119	0.0441
3.51	2.5	0.9	0.4	0.2	5/2	2.77	0.551	0.204	0.0624	0.0230
3.74	1.7	~0.8	~0.35	0.2	7/2	2.81	0.653	0.241	0.0735	0.0272
3.83 (may be doublet)	1.9	0.9	0.3	?	3/2	3.05	2.84	1.07	0.342	0.122
					9/2	3.12	7.25	2.73	0.869	0.310
					5/2	3.19	4.35	1.64	0.520	0.187
					7/2	3.27	5.95	2.26	0.710	0.254

TABLE XVI

Cu^{63} COMPARISON OF THE TWO QUASI-NEUTRONS (COUPLED TO 2^+ ONLY) PLUS QUASI-PROTON (REFERRED TO AS 2) AND QUASI-BOSON PLUS QUASI-PROTON (REFERRED TO AS 1) SCHEMES WITH EACH OTHER AND WITH EXPERIMENT

Properties of Negative Parity "Collective" States	Scheme 1.		Scheme 2.		Experiment (ref. 8)(fig. 1b)		Experiment (ref. 10)	
Number of Such States	4 (figs. 1a,b)		4 (fig. 1c)		3		3	
Spins and Energies	1/2⁻	0.797 mev (figs. 1a,b)	5/2⁻	2.05 (Tables XII a,b,c,d)	1/2⁻	0.668	1/2⁻	0.668 (Table keV)
	7/2⁻	1.40	7/2⁻	2.06	5/2⁻	0.961	5/2⁻	0.961
	5/2⁻	1.48	3/2⁻	2.08	7/2⁻	1.33	7/2⁻	1.327
	3/2⁻	1.54	1/2⁻	2.25			3/2⁻	1.547 not "collective"
Maximum Energy Minus Minimum Energy	0.74 mev		0.20 mev		0.66 mev		0.66 mev	
B(E2)'s in Single Proton Units	1/2⁻	8.67 (Figs. 1a, b)	5/2⁻	2.43 (Tables XII a,b,c,d)	1/2⁻	8.46	----	
	7/2⁻	6.34	7/2⁻	2.56	5/2⁻	8.46		
	5/2⁻	4.22	3/2⁻	1.99	7/2⁻	9.62		
	3/2⁻	2.45	1/2⁻	3.08				

(cont.)

TABLE XVI (cont.)

Properties of Negative Parity "Collective" States	Scheme 1.		Scheme 2.		Experiment		
Number of Levels Below the Lowest "Collective" Level	1 ($5/2^-$ level at 0.740 mev) (Figs. 1a,b)		34 (Fig. 1c)		0	0	
Inelastic Alpha Scattering Differential Cross Sections Evaluated at the Angle of Largest Maximum	Born Approx. (Figs. 6a,b, c,d and Table XV)		Born Approx. (Figs. 7a,b,c,d and Table XV)		--------	(26°) (Table XV)	
	$1/2^-$	1.6 mb./ster.	$5/2^-$	1.3 mb./ster.		$1/2^-$	1.6 mb./ster
	$7/2^-$	4.9	$7/2^-$	1.9		$5/2^-$	3.8
	$5/2^-$	2.4	$3/2^-$	0.74		$7/2^-$	4.0
	$3/2^-$	0.94	$1/2^-$	0.57		$3/2^-$	0.25 non "collective"
PHASE RULES 1. out of phase with the elastic 2. out of phase with the positive parity levels of ref. 10	Both phase rules obeyed (Inferred from the similarity of the $5/2^-$ scheme 1. and $5/2^-$ scheme 2. form factors) (Fig. 14)		Both phase rules obeyed. (Figs. 9, 10, 11)		------	Both phase rules obeyed (Figs. 9, 10 and Table XV)	

QUASI - BOSON PLUS QUASI - PROTON SCHEME QUADRUPOLE - QUADRUPOLE INTERACTION

spin	Energy	B(E2) units of 10^{-52} sq.cm.	Ratio of B(E2)↓ to pure single proton
5/2⁻	3.04	0.338	0.0130
3/2⁻		0.00282	0.0001
1/2⁻	2.77	1.74	0.0666
5/2⁻	2.36	0.0758	0.0029
3/2⁻	2.21	0.338	0.0130
7/2⁻		0.0002	0.0000
1/2⁻	1.78	9.35	0.359
3/2⁻	1.54	63.8	2.45
5/2⁻	1.48	110	4.22
7/2⁻	1.40	167	6.40
9/2⁻	1.16	0	0
1/2⁻	0.797	226	8.67
5/2⁻	0.740	38.3	1.47
3/2⁻	0		

Cu^{63}

Figure 1a

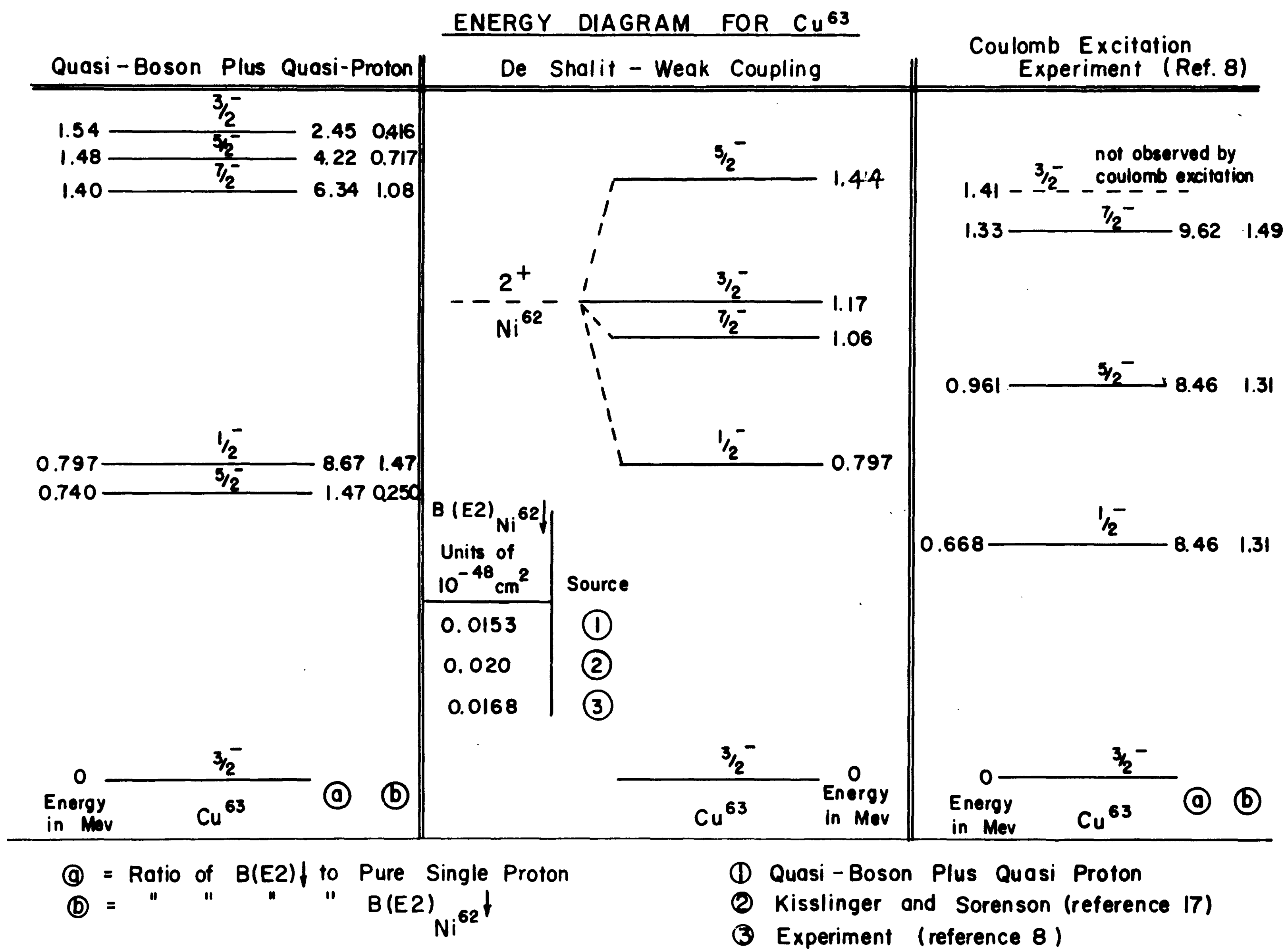
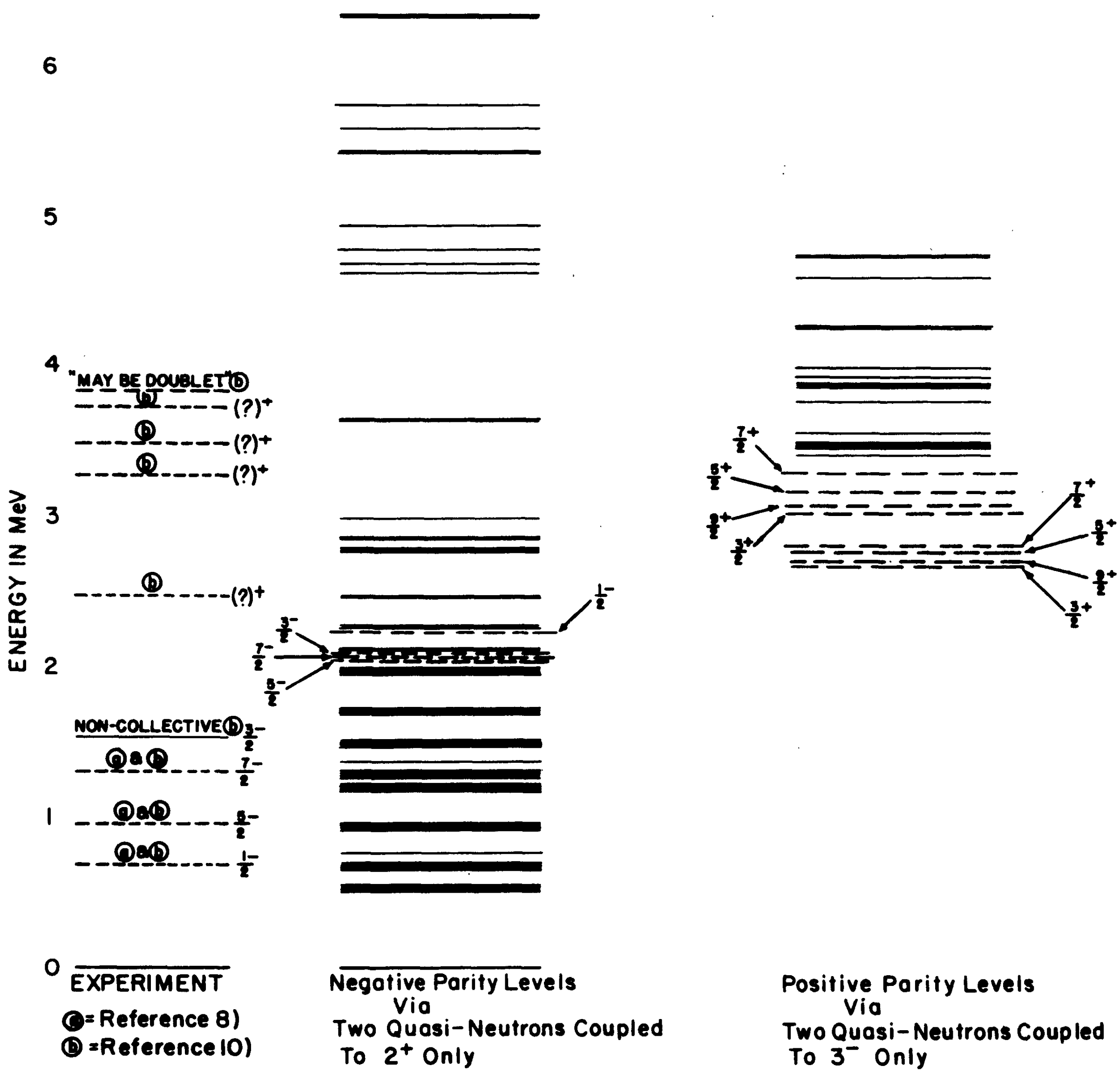

ENERGY DIAGRAM FOR Cu63
Quasi-Boson Plus Quasi-Proton
De Shalit - Weak Coupling
Coulomb Excitation Experiment (Ref. 8)
1.54
1.48
1.40
3/2-
5/2-
7/2-
2.45 0.416
4.22 0.717
6.34 1.08
0.797
0.740
1/2-
5/2-
8.67 1.47
1.47 0.250
0
Energy in Mev
3/2-
Cu63
ⓐ ⓑ
2+
Ni62
5/2-
3/2-
7/2-
1/2-
1.44
1.17
1.06
0.797
B(E2) Ni62↓
Units of 10-48 cm2
Source
0.0153
0.020
0.0168
①
②
③
Cu63
Energy in Mev
not observed by coulomb excitation
1.41
1.33
0.961
0.668
9.62 1.49
8.46 1.31
8.46 1.31
ⓐ = Ratio of B(E2)↓ to Pure Single Proton
ⓑ = " " " " B(E2) Ni62↓
① Quasi-Boson Plus Quasi Proton
② Kisslinger and Sorenson (reference 17)
③ Experiment (reference 8)

Fig 1b

CU^{63}

TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME

QUADRUPOLE-QUADRUPOLE LONG RANGE INTERACTION



DASHED LINES INDICATE "COLLECTIVE LEVELS"

Fig. 1c

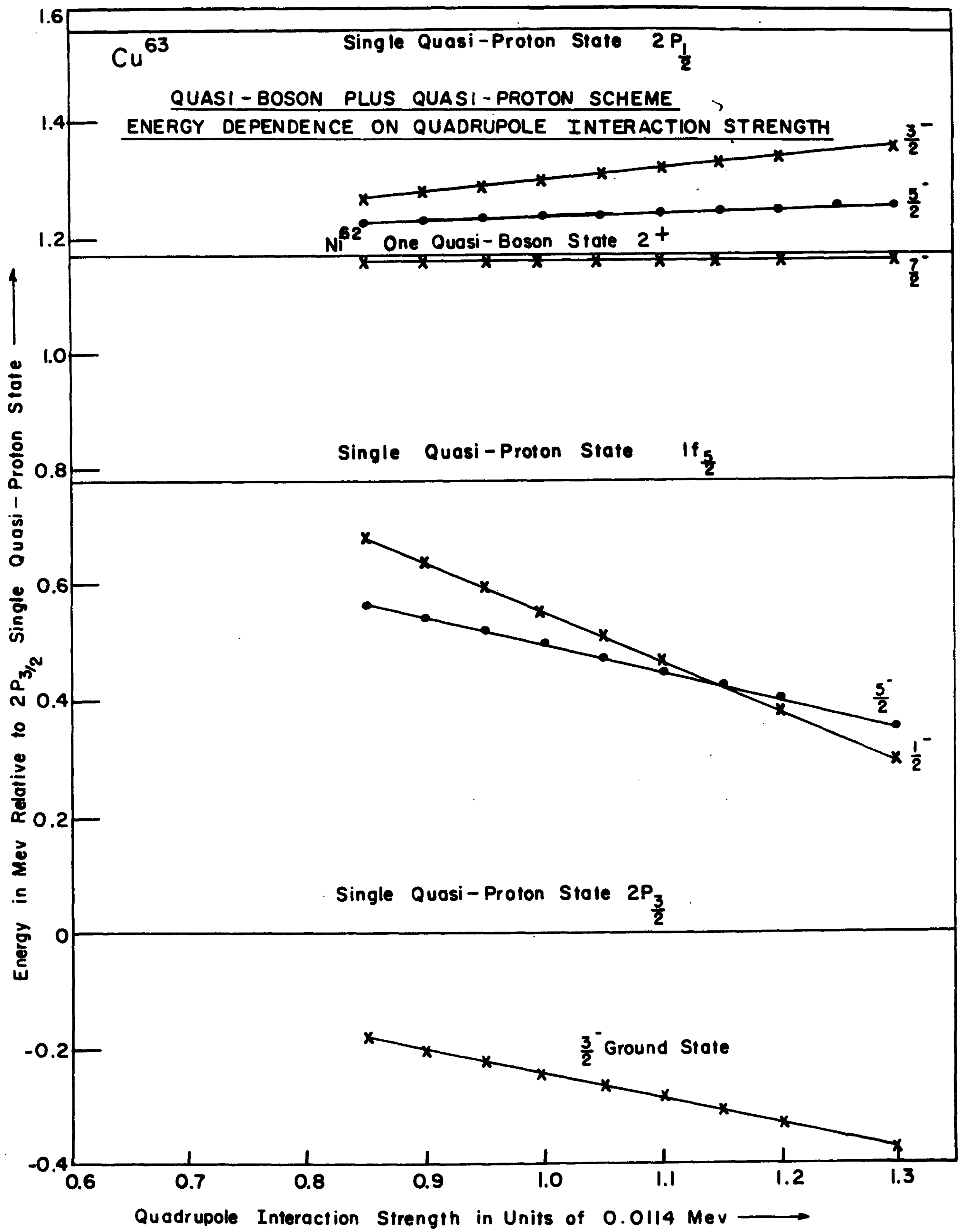
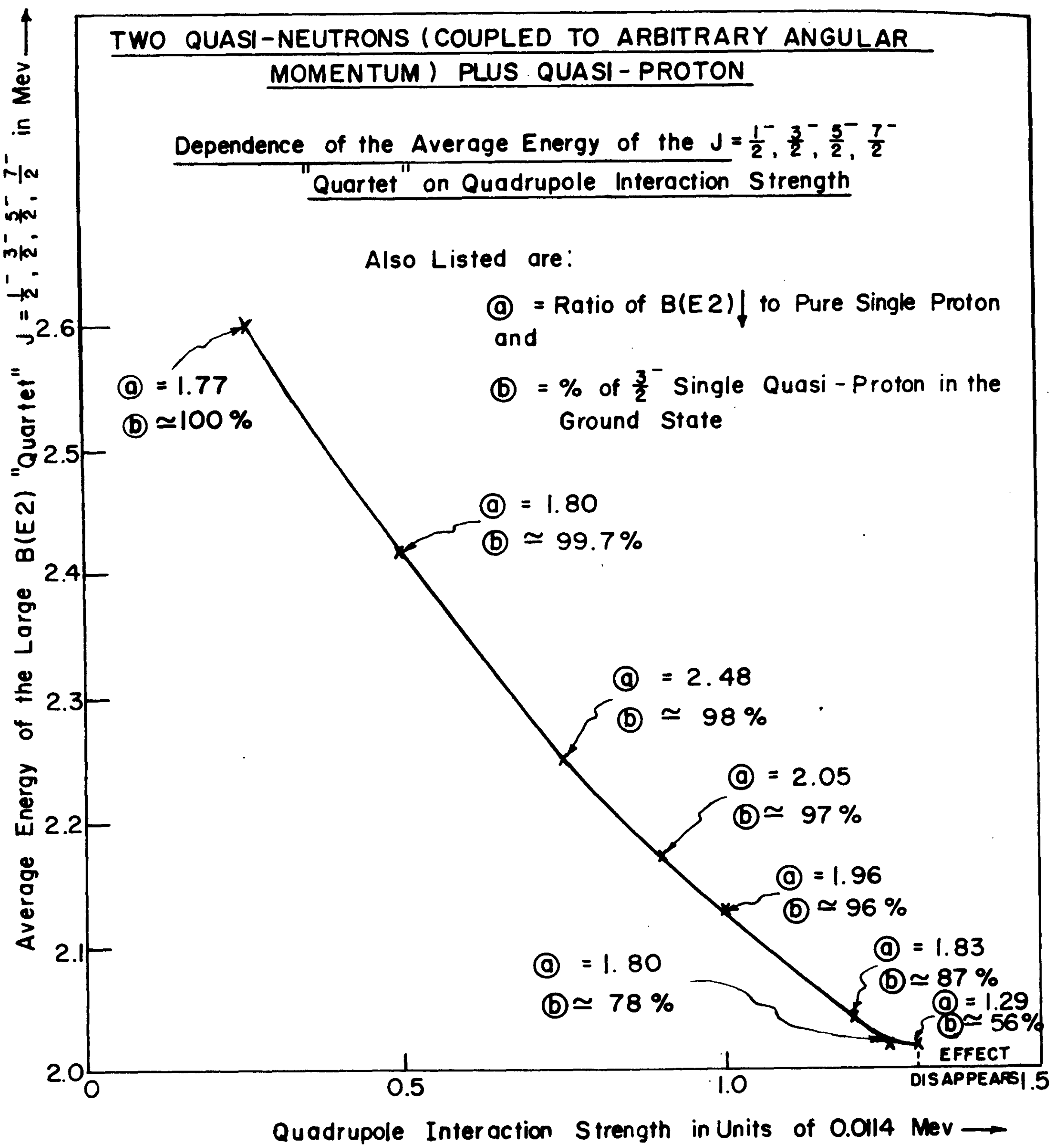
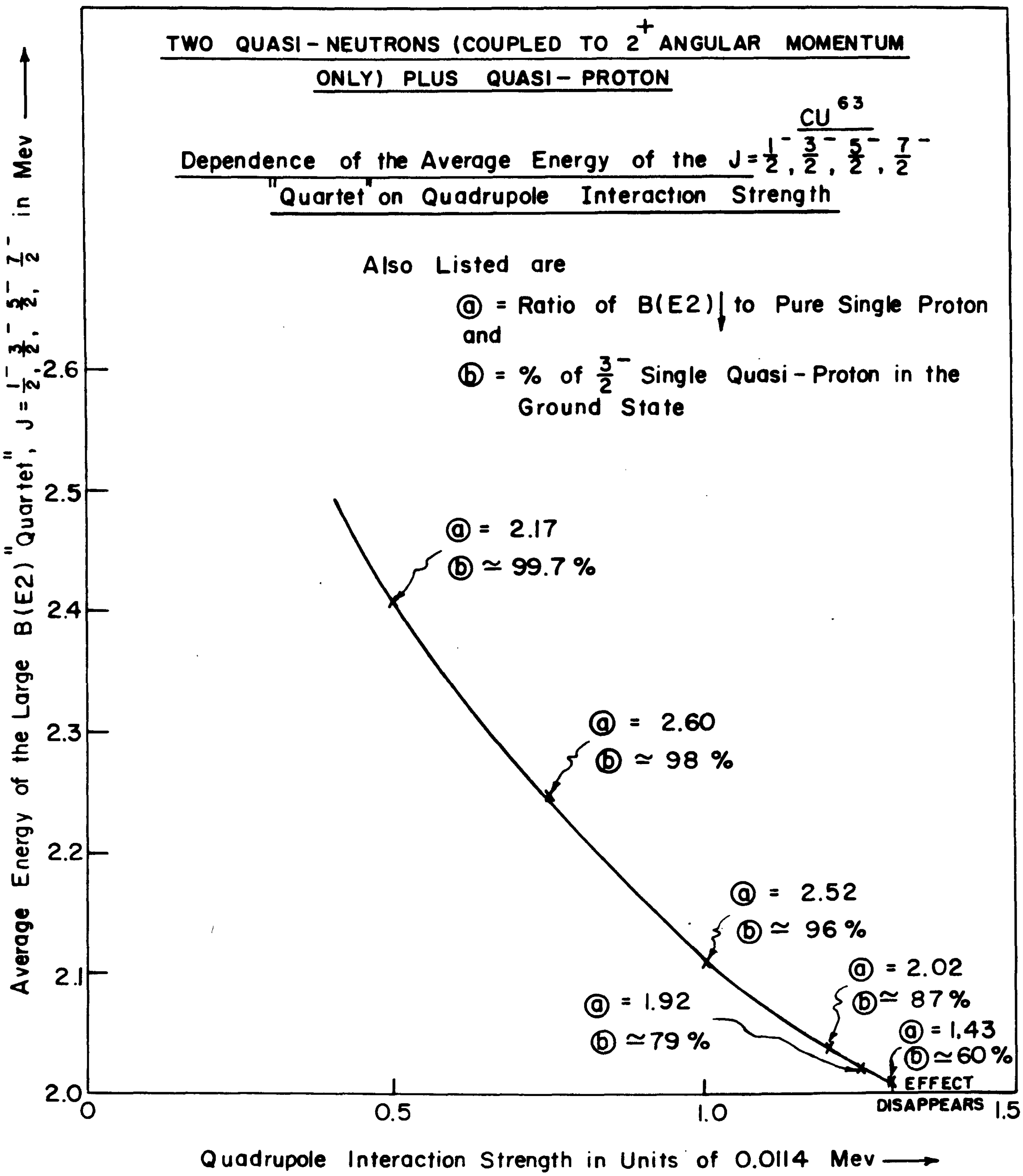

Cu^{63}
Single Quasi-Proton State $2P_{\frac{1}{2}}$
QUASI-BOSON PLUS QUASI-PROTON SCHEME
ENERGY DEPENDENCE ON QUADRUPOLE INTERACTION STRENGTH
$\frac{3}{2}^-$
$\frac{5}{2}^-$
Ni^{62} One Quasi-Boson State 2+
$\frac{7}{2}^-$
Single Quasi-Proton State $1f_{\frac{5}{2}}$
$\frac{5}{2}^-$
$\frac{1}{2}^-$
Single Quasi-Proton State $2P_{\frac{3}{2}}$
$\frac{3}{2}^-$ Ground State
Energy in Mev Relative to $2P_{3/2}$ Single Quasi-Proton State
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
Quadrupole Interaction Strength in Units of 0.0114 Mev

FIG. 2



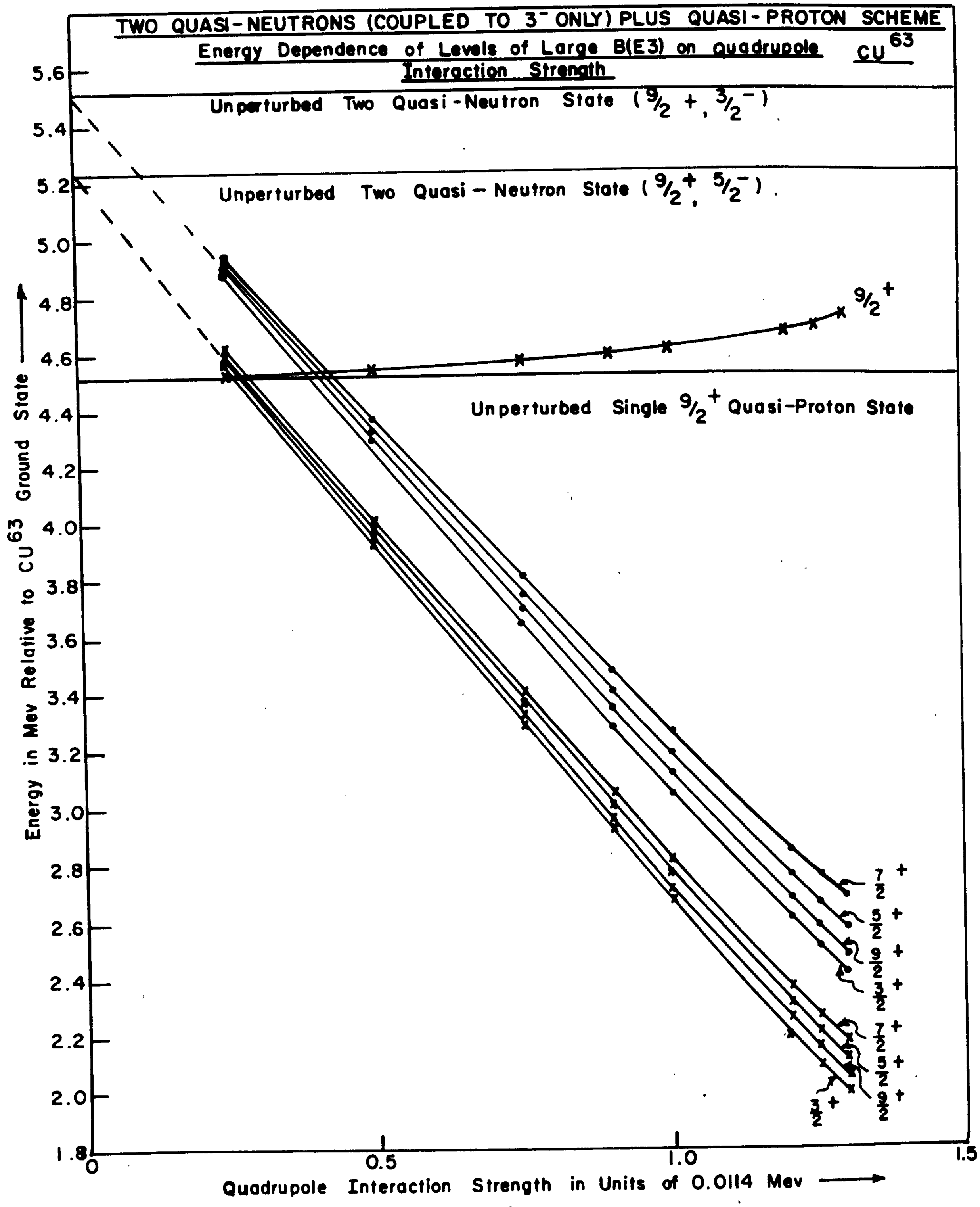
TWO QUASI-NEUTRONS (COUPLED TO ARBITRARY ANGULAR MOMENTUM) PLUS QUASI-PROTON
Dependence of the Average Energy of the $J = \frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-$ "Quartet" on Quadrupole Interaction Strength
Also Listed are:
ⓐ = Ratio of B(E2)↓ to Pure Single Proton and
ⓑ = % of $\frac{3}{2}^-$ Single Quasi-Proton in the Ground State
ⓐ = 1.77
ⓑ ≃ 100 %
ⓐ = 1.80
ⓑ ≃ 99.7 %
ⓐ = 2.48
ⓑ ≃ 98 %
ⓐ = 2.05
ⓑ ≃ 97 %
ⓐ = 1.96
ⓑ ≃ 96 %
ⓐ = 1.83
ⓑ ≃ 87 %
ⓐ = 1.80
ⓑ ≃ 78 %
ⓐ = 1.29
ⓑ ≃ 56 %
EFFECT DISAPPEARS
Average Energy of the Large B(E2) "Quartet" $J = \frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-$ in Mev
2.0
2.1
2.2
2.3
2.4
2.5
2.6
0
0.5
1.0
1.5
Quadrupole Interaction Strength in Units of 0.0114 Mev

Fig. 3



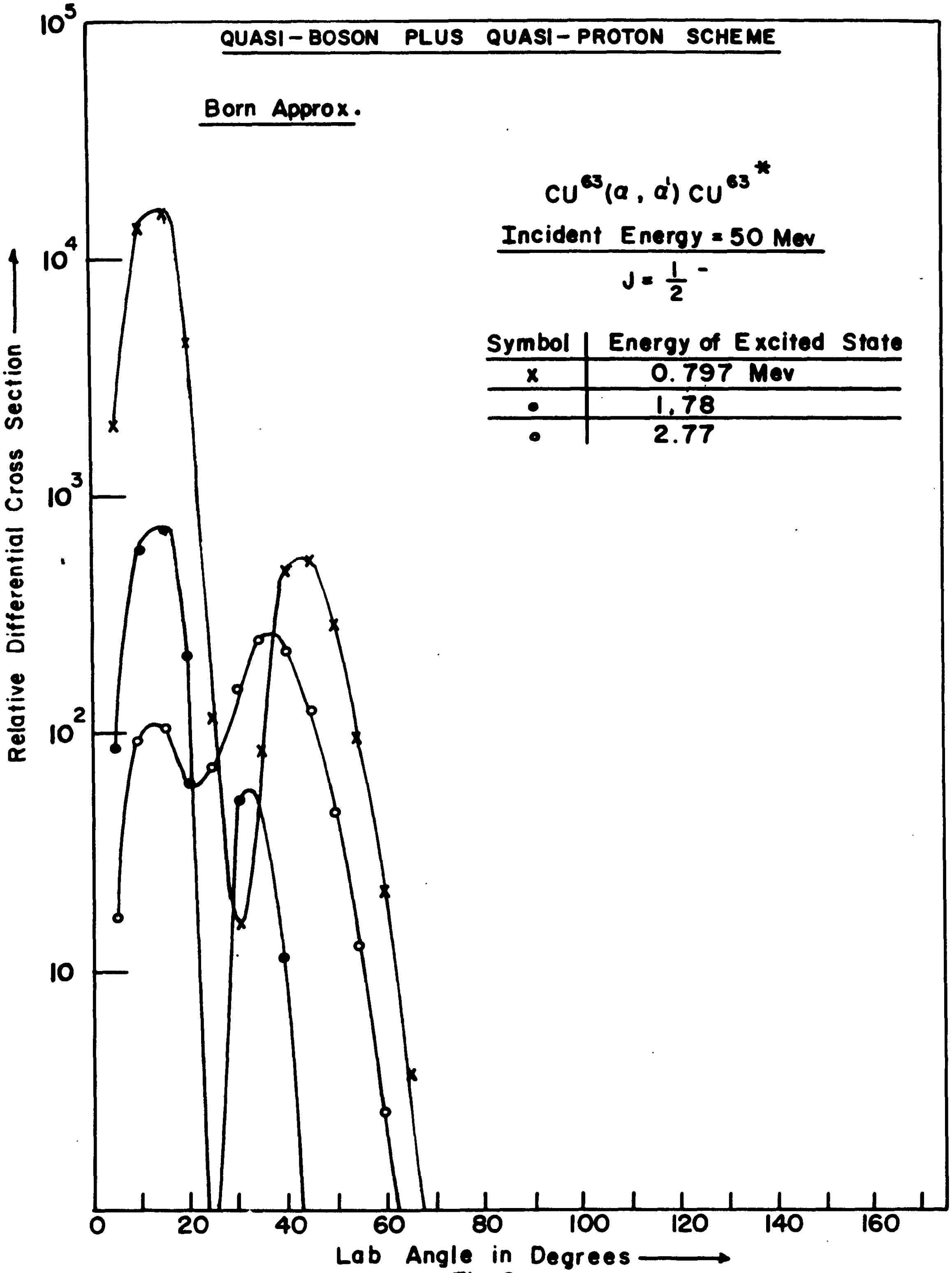
TWO QUASI-NEUTRONS (COUPLED TO 2+ ANGULAR MOMENTUM ONLY) PLUS QUASI-PROTON
CU 63
Dependence of the Average Energy of the J = 1/2⁻, 3/2⁻, 5/2⁻, 7/2⁻ "Quartet" on Quadrupole Interaction Strength
Also Listed are
ⓐ = Ratio of B(E2)↓ to Pure Single Proton
and
ⓑ = % of 3/2⁻ Single Quasi-Proton in the Ground State
ⓐ = 2.17
ⓑ ≃ 99.7 %
ⓐ = 2.60
ⓑ ≃ 98 %
ⓐ = 2.52
ⓑ ≃ 96 %
ⓐ = 2.02
ⓑ ≃ 87 %
ⓐ = 1.92
ⓑ ≃ 79 %
ⓐ = 1.43
ⓑ ≃ 60 %
EFFECT DISAPPEARS
2.0
2.1
2.2
2.3
2.4
2.5
2.6
0
0.5
1.0
1.5
Average Energy of the Large B(E2) "Quartet", J = 1/2⁻, 3/2⁻, 5/2⁻, 7/2⁻ in Mev
Quadrupole Interaction Strength in Units of 0.0114 Mev

Fig. 4



TWO QUASI-NEUTRONS (COUPLED TO 3⁻ ONLY) PLUS QUASI-PROTON SCHEME
Energy Dependence of Levels of Large B(E3) on Quadrupole Interaction Strength
CU 63
Unperturbed Two Quasi-Neutron State (9/2 +, 3/2 −)
Unperturbed Two Quasi-Neutron State (9/2 +, 5/2 −)
9/2 +
Unperturbed Single 9/2 + Quasi-Proton State
7/2 +
5/2 +
9/2 +
3/2 +
7/2 +
5/2 +
9/2 +
3/2 +
Energy in Mev Relative to CU 63 Ground State
Quadrupole Interaction Strength in Units of 0.0114 Mev
5.6
5.4
5.2
5.0
4.8
4.6
4.4
4.2
4.0
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
0
0.5
1.0
1.5

Fig 5



QUASI-BOSON PLUS QUASI-PROTON SCHEME
Born Approx.
CU^{63}(α, α') CU^{63*}
Incident Energy = 50 Mev
J = $\frac{1}{2}^-$
Symbol | Energy of Excited State
x | 0.797 Mev
• | 1.78
∘ | 2.77
Relative Differential Cross Section
10^5
10^4
10^3
10^2
10
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

Fig 6a

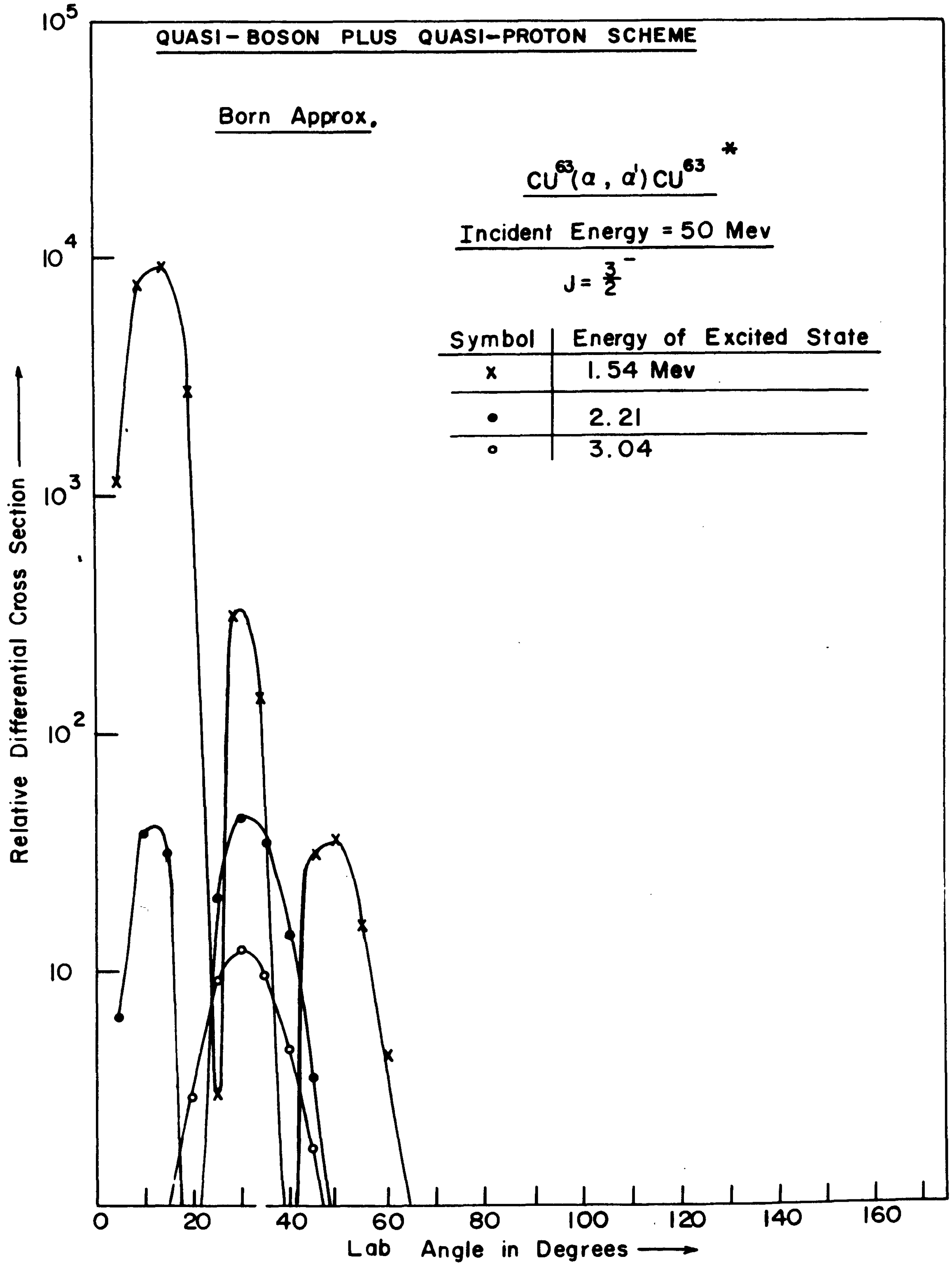
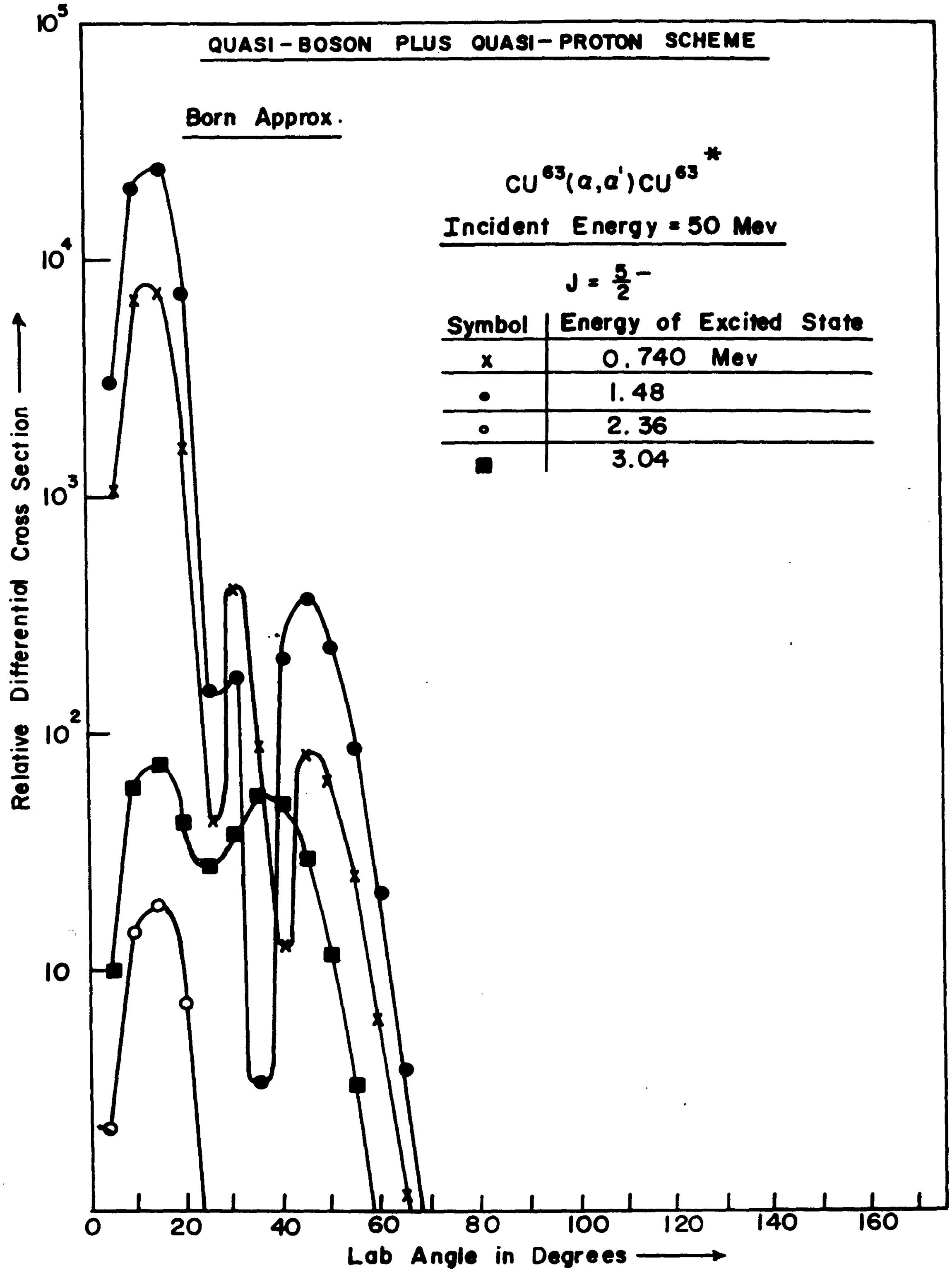
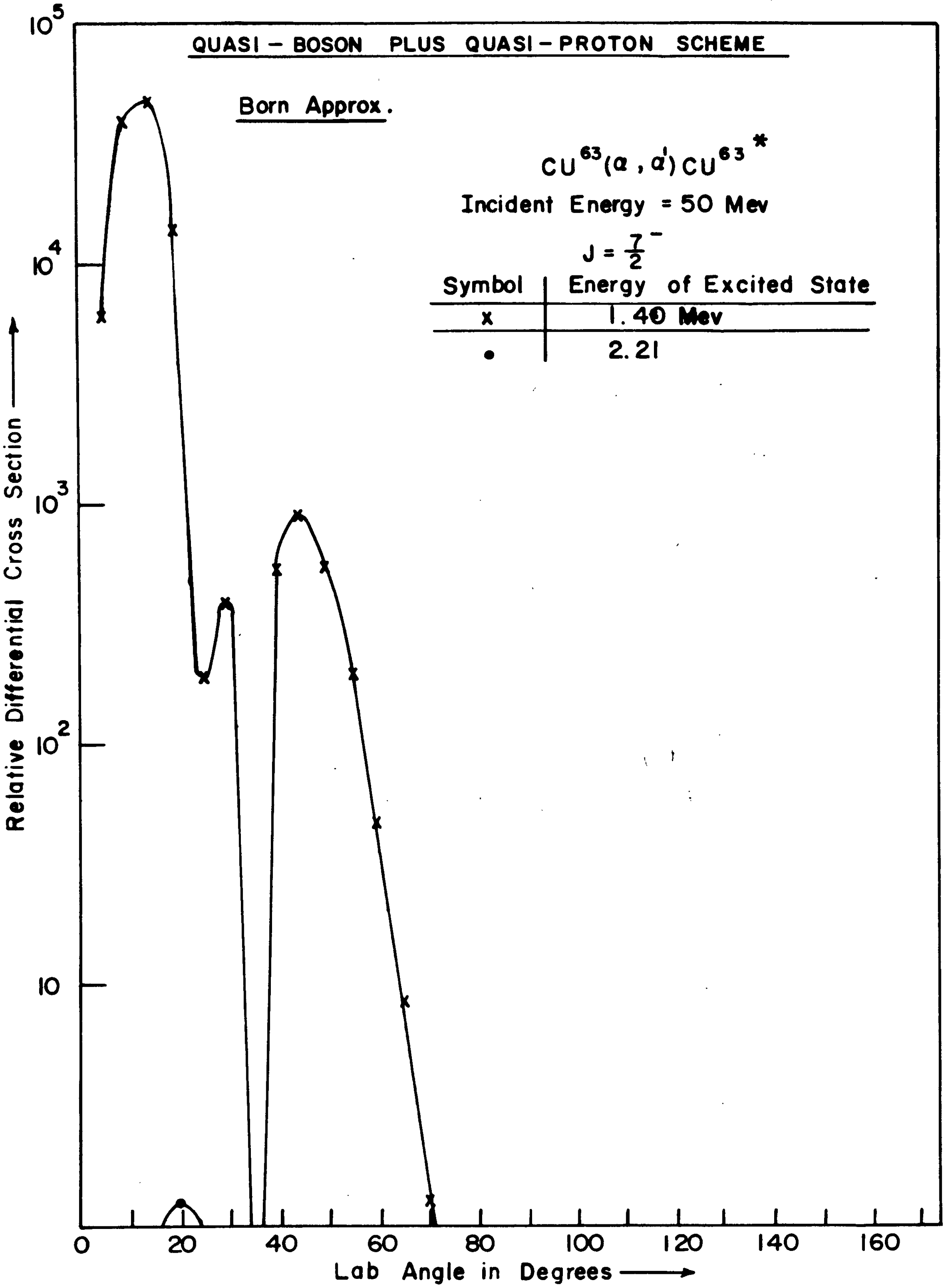

QUASI-BOSON PLUS QUASI-PROTON SCHEME
Born Approx.
CU63(α, α')CU63 *
Incident Energy = 50 Mev
J = 3/2 −
Symbol | Energy of Excited State
x | 1.54 Mev
• | 2.21
∘ | 3.04
10^5
10^4
10^3
10^2
10
Relative Differential Cross Section
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

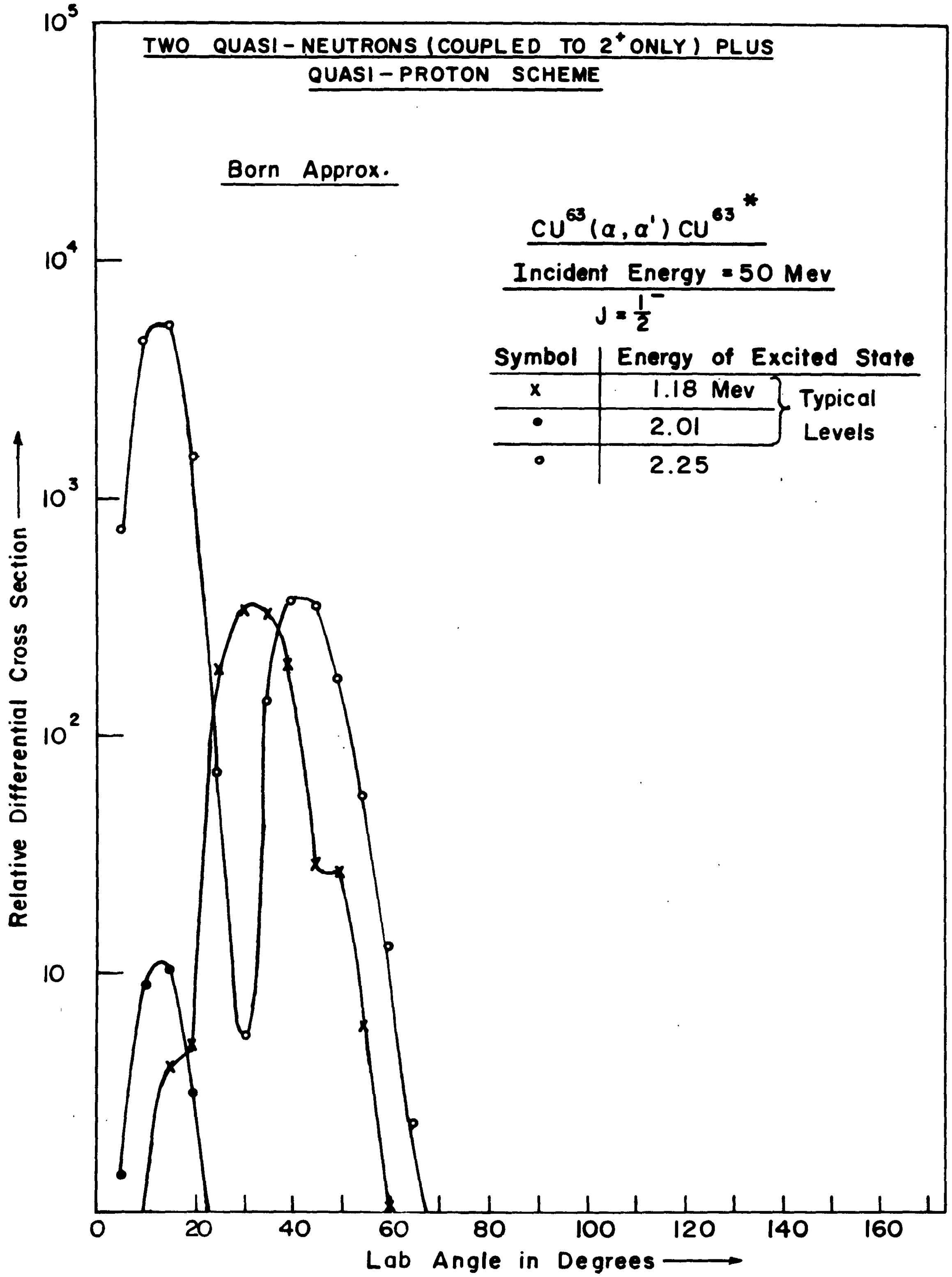
Fig. 6b



QUASI-BOSON PLUS QUASI-PROTON SCHEME
Born Approx.
CU63(a,a')CU63*
Incident Energy = 50 Mev
J = 5/2 −
Symbol | Energy of Excited State
x | 0.740 Mev
• | 1.48
∘ | 2.36
■ | 3.04
Relative Differential Cross Section
10^5
10^4
10^3
10^2
10
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

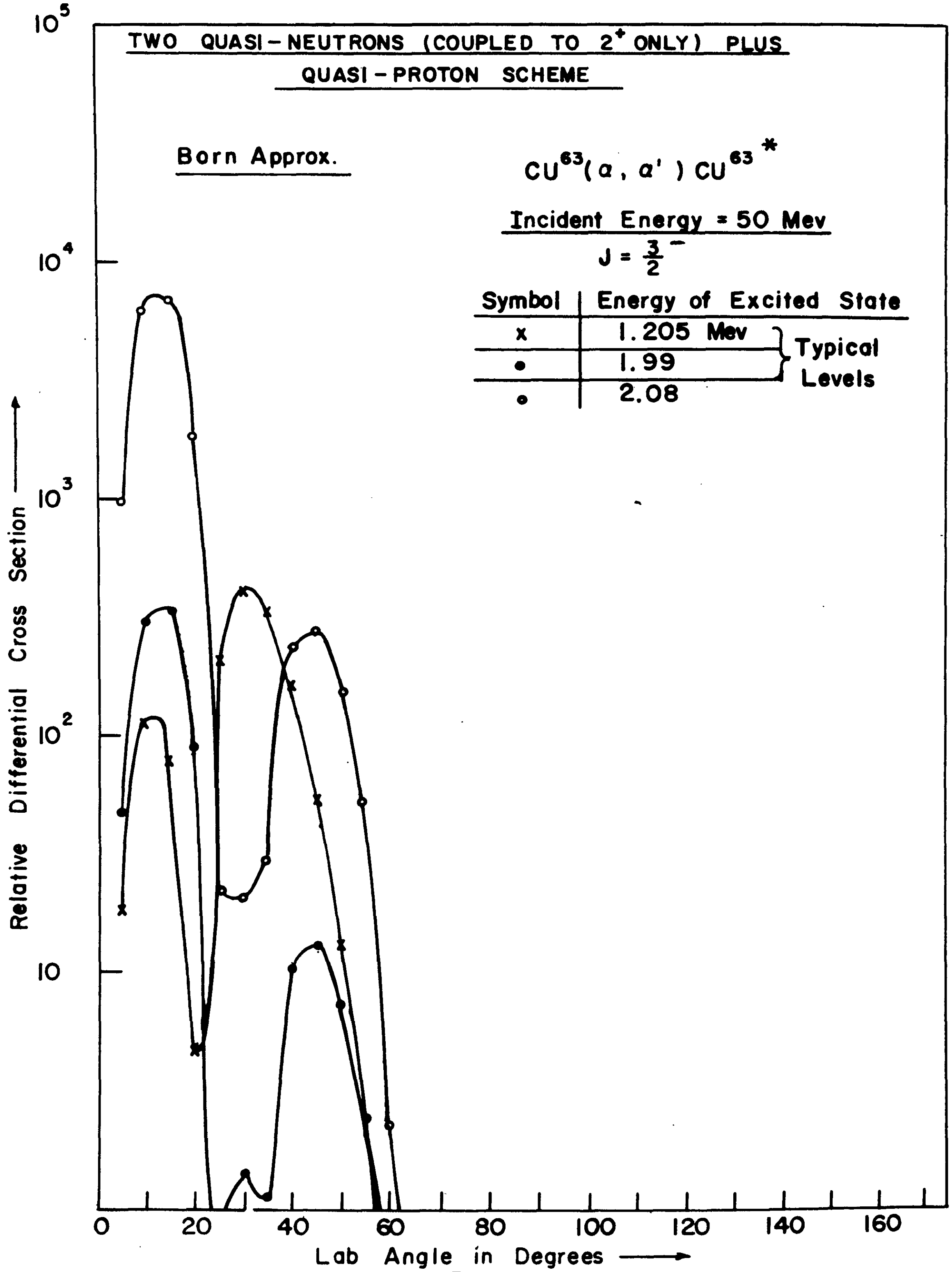
Fig 6c


QUASI - BOSON PLUS QUASI - PROTON SCHEME
Born Approx.
CU^63 (α, α') CU^63 *
Incident Energy = 50 Mev
J = 7/2 ^-
Symbol | Energy of Excited State
x | 1.40 Mev
• | 2.21
10^5
10^4
10^3
10^2
10
Relative Differential Cross Section
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees
Fig 6d



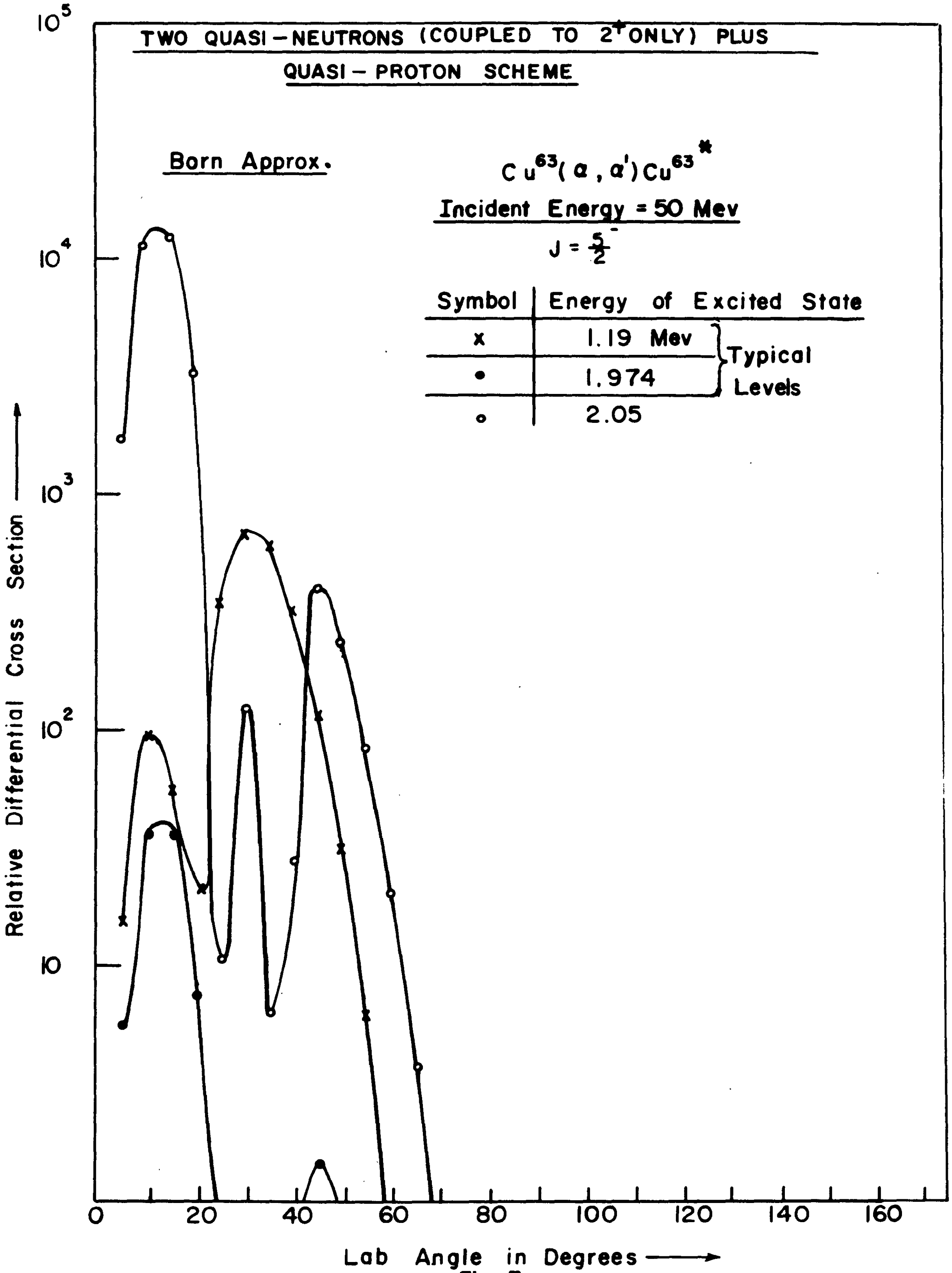
TWO QUASI-NEUTRONS (COUPLED TO 2+ ONLY) PLUS
QUASI-PROTON SCHEME
Born Approx.
CU63 (α, α') CU63*
Incident Energy = 50 Mev
J = 1/2−
Symbol | Energy of Excited State
x | 1.18 Mev
• | 2.01
o | 2.25
Typical Levels
Relative Differential Cross Section
10
10^2
10^3
10^4
10^5
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

Fig 7a



TWO QUASI-NEUTRONS (COUPLED TO 2+ ONLY) PLUS
QUASI-PROTON SCHEME
Born Approx.
CU63(α, α') CU63*
Incident Energy = 50 Mev
J = 3/2−
Symbol | Energy of Excited State
x | 1.205 Mev
• | 1.99
∘ | 2.08
Typical Levels
Relative Differential Cross Section
10^5
10^4
10^3
10^2
10
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

Fig. 7b


TWO QUASI-NEUTRONS (COUPLED TO 2+ ONLY) PLUS
QUASI-PROTON SCHEME
Born Approx.
Cu63(α, α')Cu63*
Incident Energy = 50 Mev
J = 5/2-
Symbol
Energy of Excited State
x
1.19 Mev
•
1.974
o
2.05
Typical Levels
Relative Differential Cross Section
10^5
10^4
10^3
10^2
10
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees
Fig. 7c

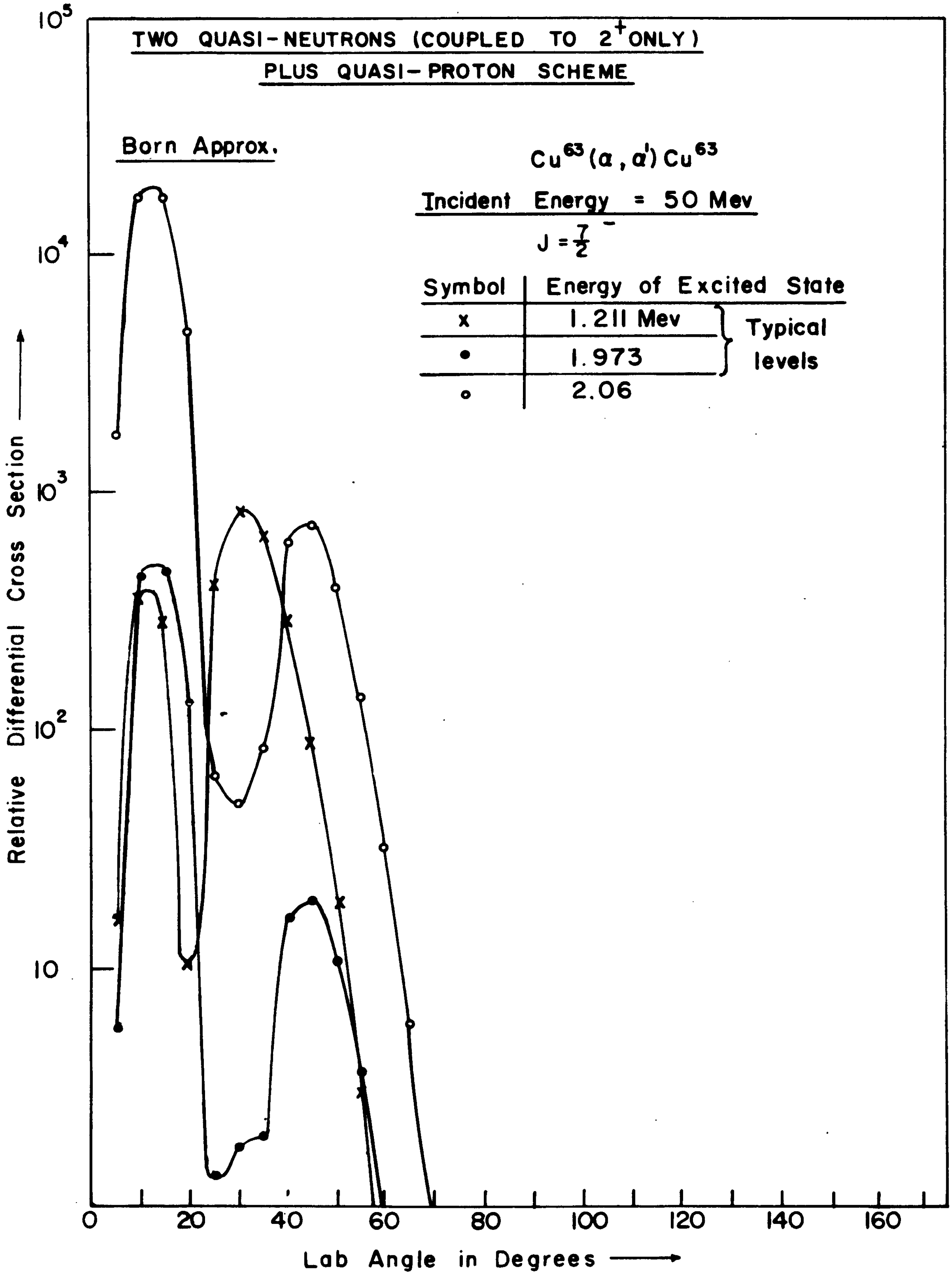

TWO QUASI-NEUTRONS (COUPLED TO 2^+ ONLY)
PLUS QUASI-PROTON SCHEME
Born Approx.
$Cu^{63}(\alpha, \alpha') Cu^{63}$
Incident Energy = 50 Mev
$J = \frac{7}{2}^-$
Symbol | Energy of Excited State
x | 1.211 Mev
• | 1.973
o | 2.06
Typical levels
Relative Differential Cross Section
Lab Angle in Degrees
10^5
10^4
10^3
10^2
10
0
20
40
60
80
100
120
140
160

Fig 7d

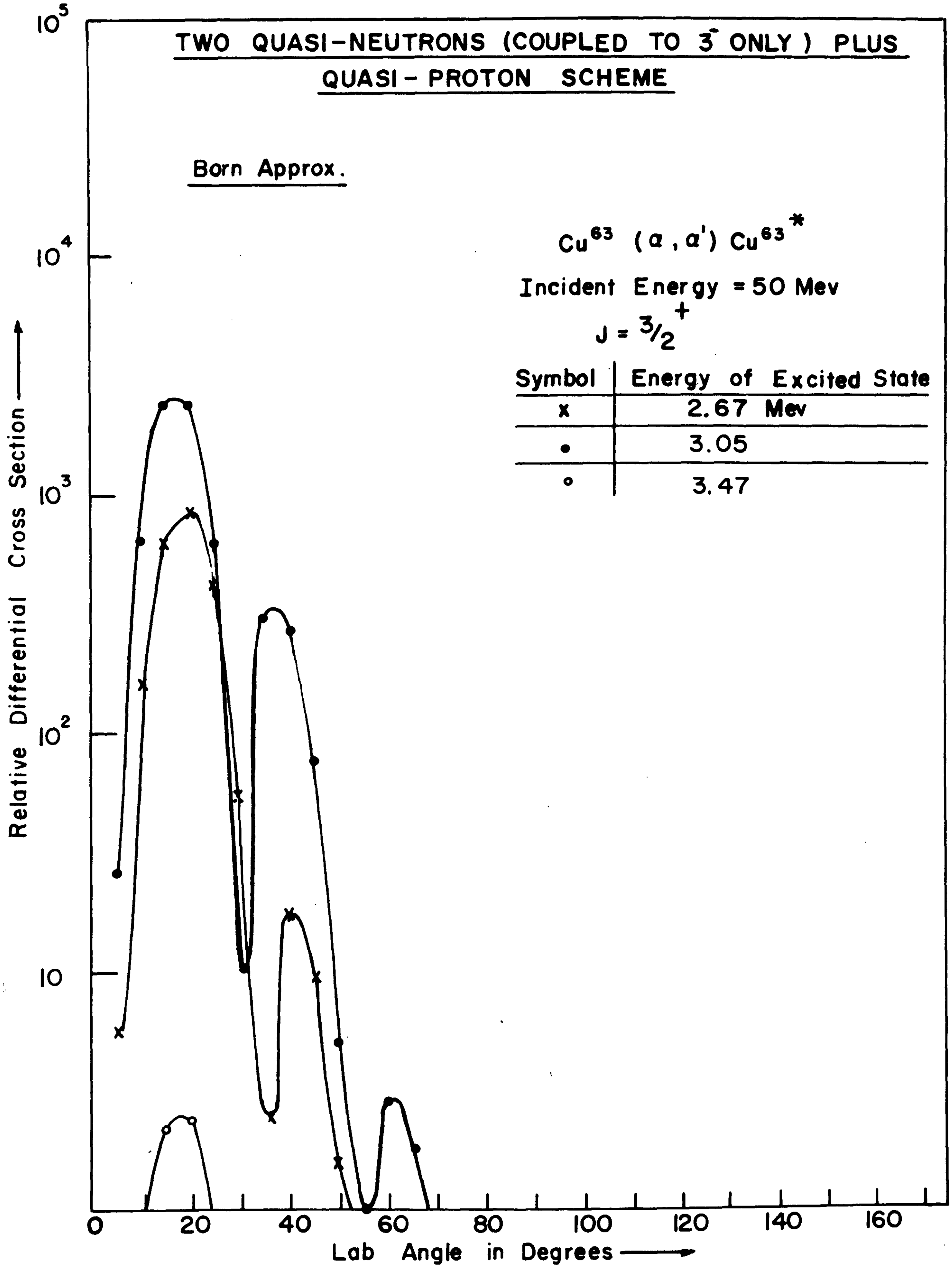

TWO QUASI-NEUTRONS (COUPLED TO 3⁻ ONLY) PLUS
QUASI-PROTON SCHEME
Born Approx.
Cu^{63} (α, α') Cu^{63*}
Incident Energy = 50 Mev
$J = 3/2^{+}$
Symbol | Energy of Excited State
x | 2.67 Mev
• | 3.05
∘ | 3.47
10^5
10^4
10^3
10^2
10
Relative Differential Cross Section
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

Fig 8a

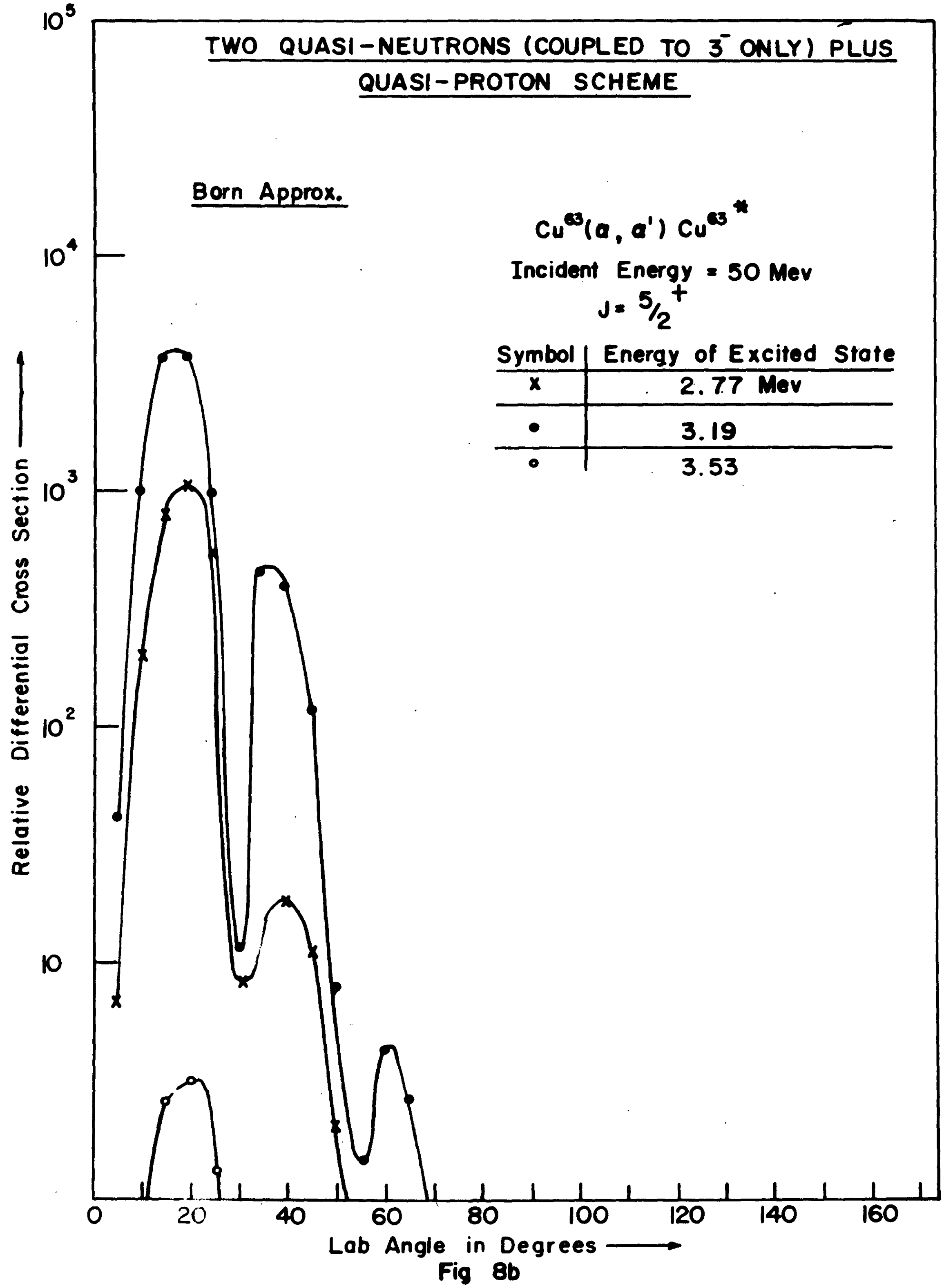
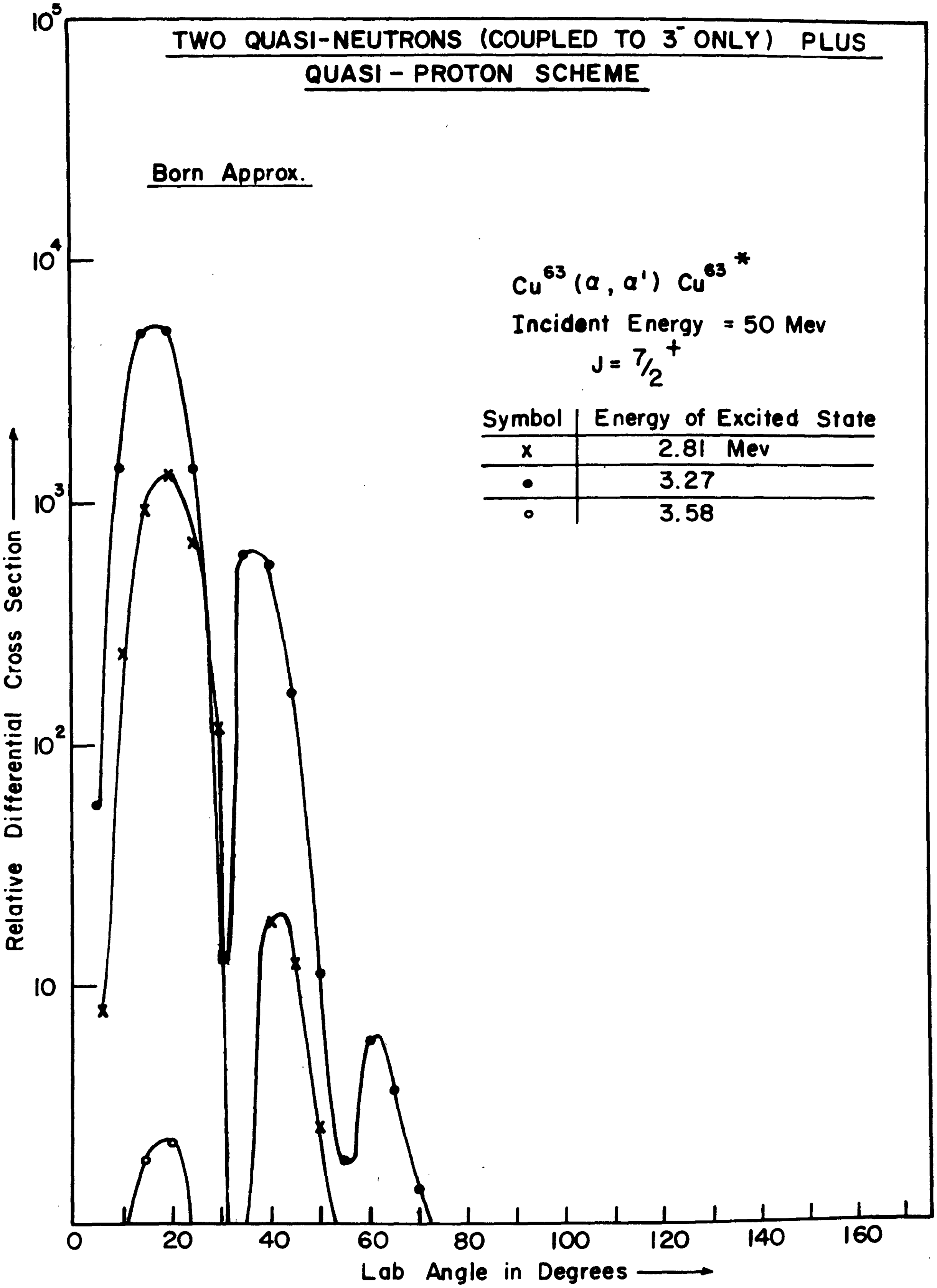

TWO QUASI-NEUTRONS (COUPLED TO 3⁻ ONLY) PLUS QUASI-PROTON SCHEME
Born Approx.
Cu63(α, α') Cu63*
Incident Energy = 50 Mev
J = 5/2+
Symbol | Energy of Excited State
x 2.77 Mev
• 3.19
o 3.53
Relative Differential Cross Section
10⁵
10⁴
10³
10²
10
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

Fig 8b



TWO QUASI-NEUTRONS (COUPLED TO 3⁻ ONLY) PLUS QUASI-PROTON SCHEME
Born Approx.
$Cu^{63}(\alpha, \alpha') Cu^{63*}$
Incident Energy = 50 Mev
$J = 7/2^+$
Symbol | Energy of Excited State
x | 2.81 Mev
• | 3.27
o | 3.58
10^5
10^4
10^3
10^2
10
Relative Differential Cross Section
0
20
40
60
80
100
120
140
160
Lab Angle in Degrees

Fig. 8c

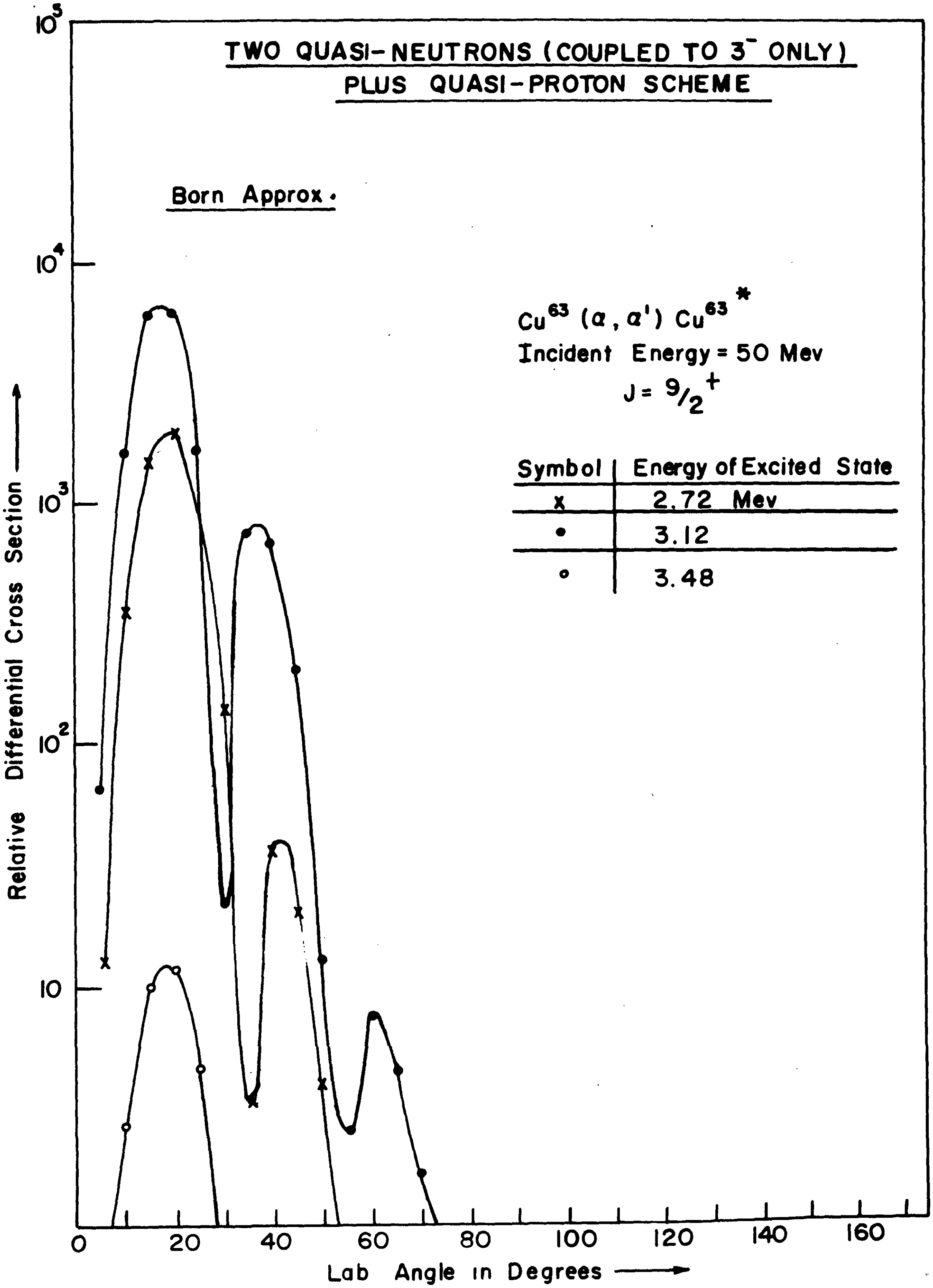

TWO QUASI-NEUTRONS (COUPLED TO 3⁻ ONLY)
PLUS QUASI-PROTON SCHEME
Born Approx.
Cu63 (α, α') Cu63*
Incident Energy = 50 Mev
J = 9/2+
Symbol
Energy of Excited State
x
2.72 Mev
•
3.12
∘
3.48
Relative Differential Cross Section
Lab Angle in Degrees
10^5
10^4
10^3
10^2
10
0
20
40
60
80
100
120
140
160

Fig. 8d

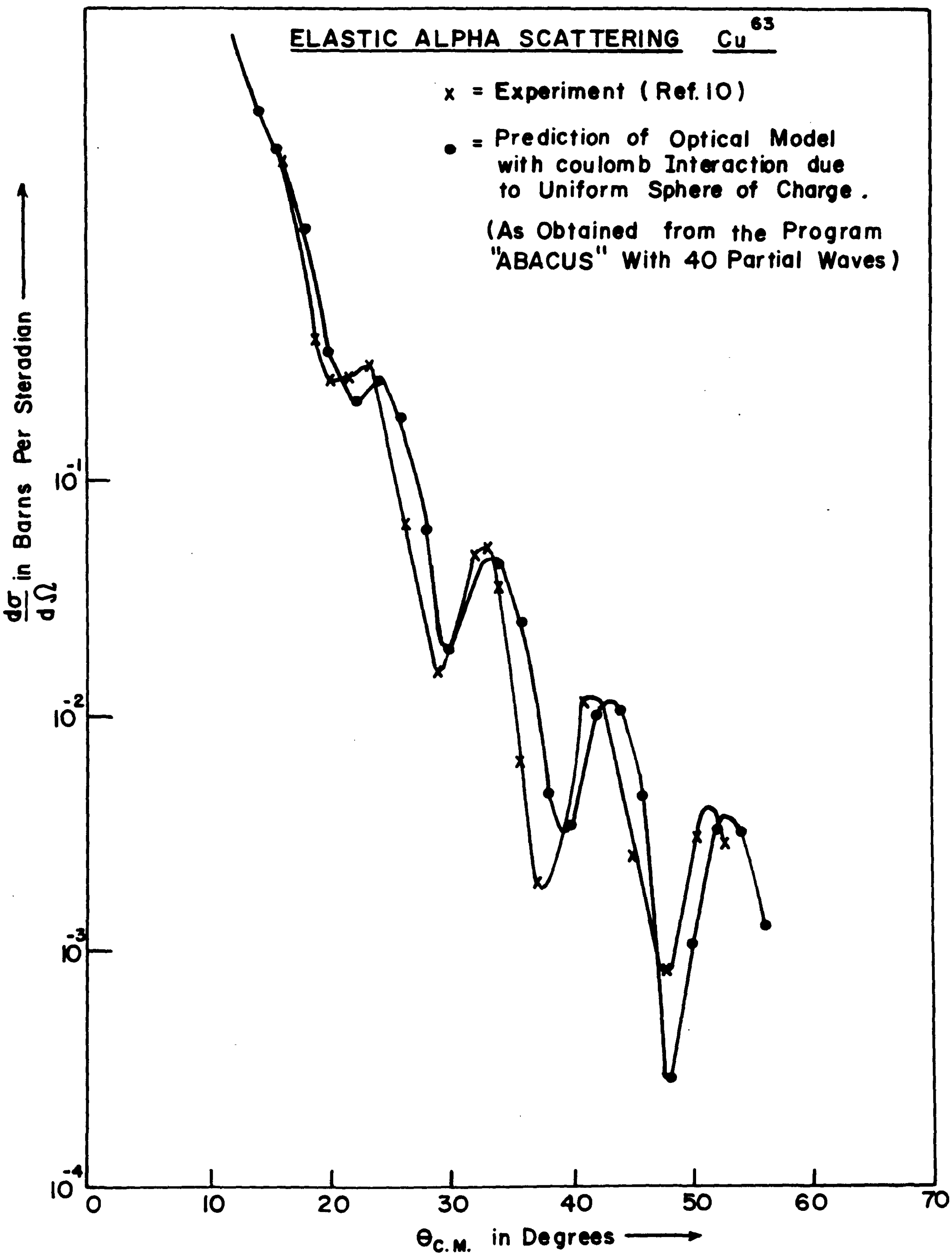

ELASTIC ALPHA SCATTERING Cu^{63}
x = Experiment (Ref. 10)
● = Prediction of Optical Model with coulomb Interaction due to Uniform Sphere of Charge.
(As Obtained from the Program "ABACUS" With 40 Partial Waves)
$\frac{d\sigma}{d\Omega}$ in Barns Per Steradian
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
10
20
30
40
50
60
70
$\Theta_{C.M.}$ in Degrees

Fig 9

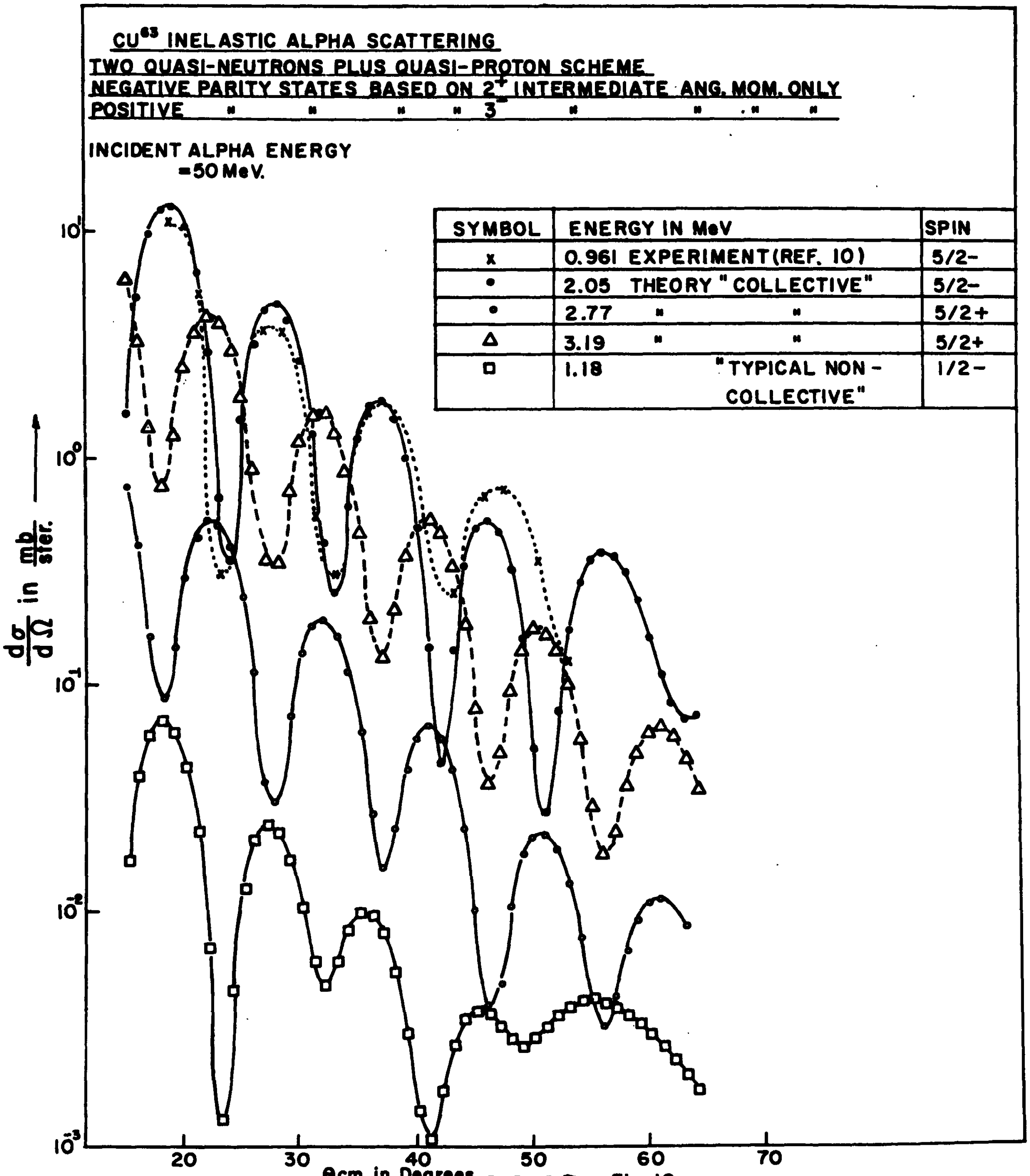
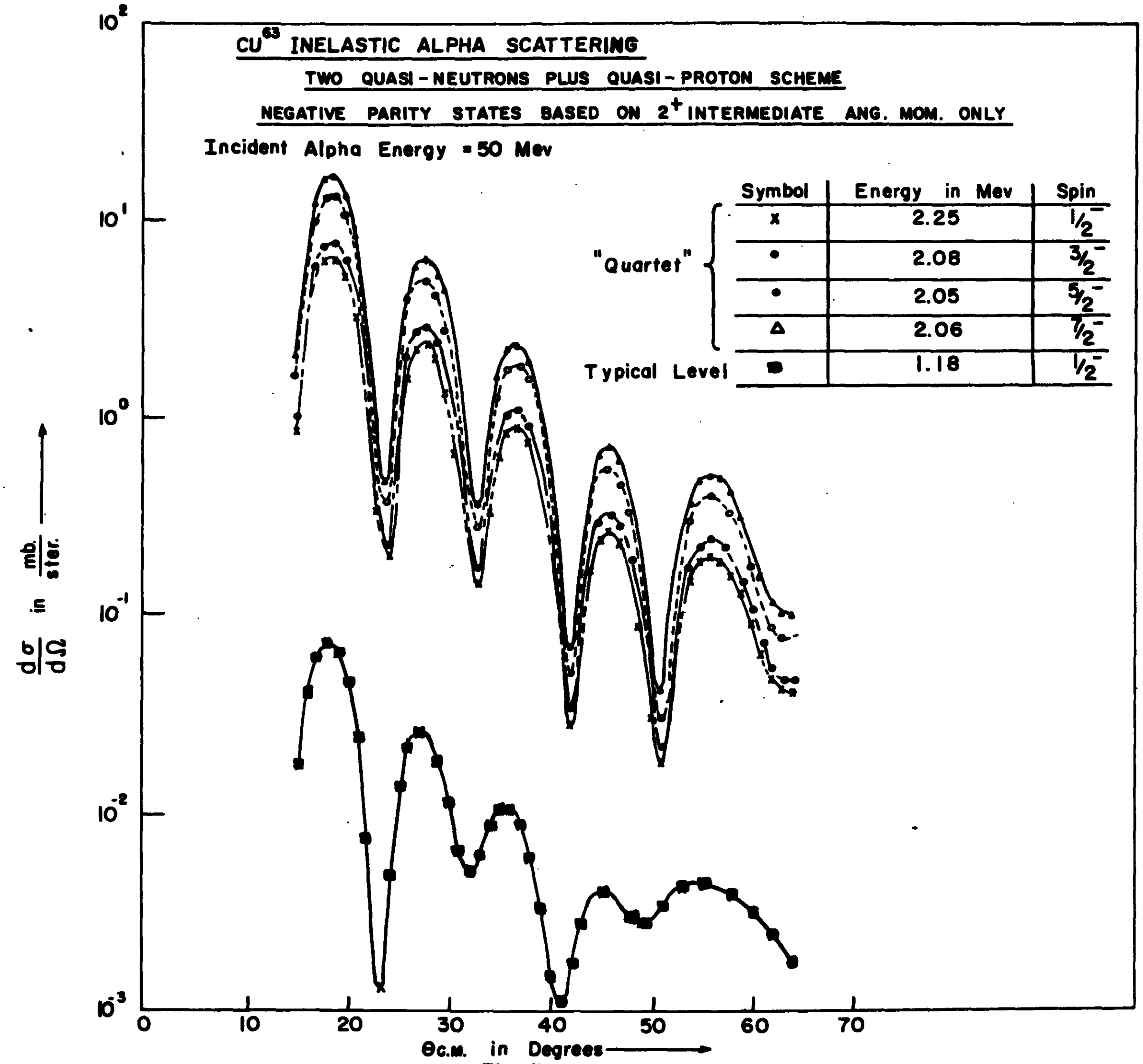

CU63 INELASTIC ALPHA SCATTERING
TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME
NEGATIVE PARITY STATES BASED ON 2+ INTERMEDIATE ANG. MOM. ONLY
POSITIVE " " " " 3- " " " "
INCIDENT ALPHA ENERGY =50 MeV.
SYMBOL | ENERGY IN MeV | SPIN
x | 0.961 EXPERIMENT (REF. 10) | 5/2-
• | 2.05 THEORY "COLLECTIVE" | 5/2-
• | 2.77 " " | 5/2+
Δ | 3.19 " " | 5/2+
□ | 1.18 "TYPICAL NON-COLLECTIVE" | 1/2-
dσ/dΩ in mb/ster.
θcm in Degrees
20
30
40
50
60
70

Fig. 10



CU63 INELASTIC ALPHA SCATTERING
TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME
NEGATIVE PARITY STATES BASED ON 2+ INTERMEDIATE ANG. MOM. ONLY
Incident Alpha Energy = 50 Mev
Symbol | Energy in Mev | Spin
"Quartet"
x 2.25 1/2-
• 2.08 3/2-
• 2.05 5/2-
Δ 2.06 7/2-
Typical Level 1.18 1/2-
dσ/dΩ in mb./ster.
θc.m. in Degrees
10^2
10^1
10^0
10^-1
10^-2
10^-3
0
10
20
30
40
50
60
70

Fig 11

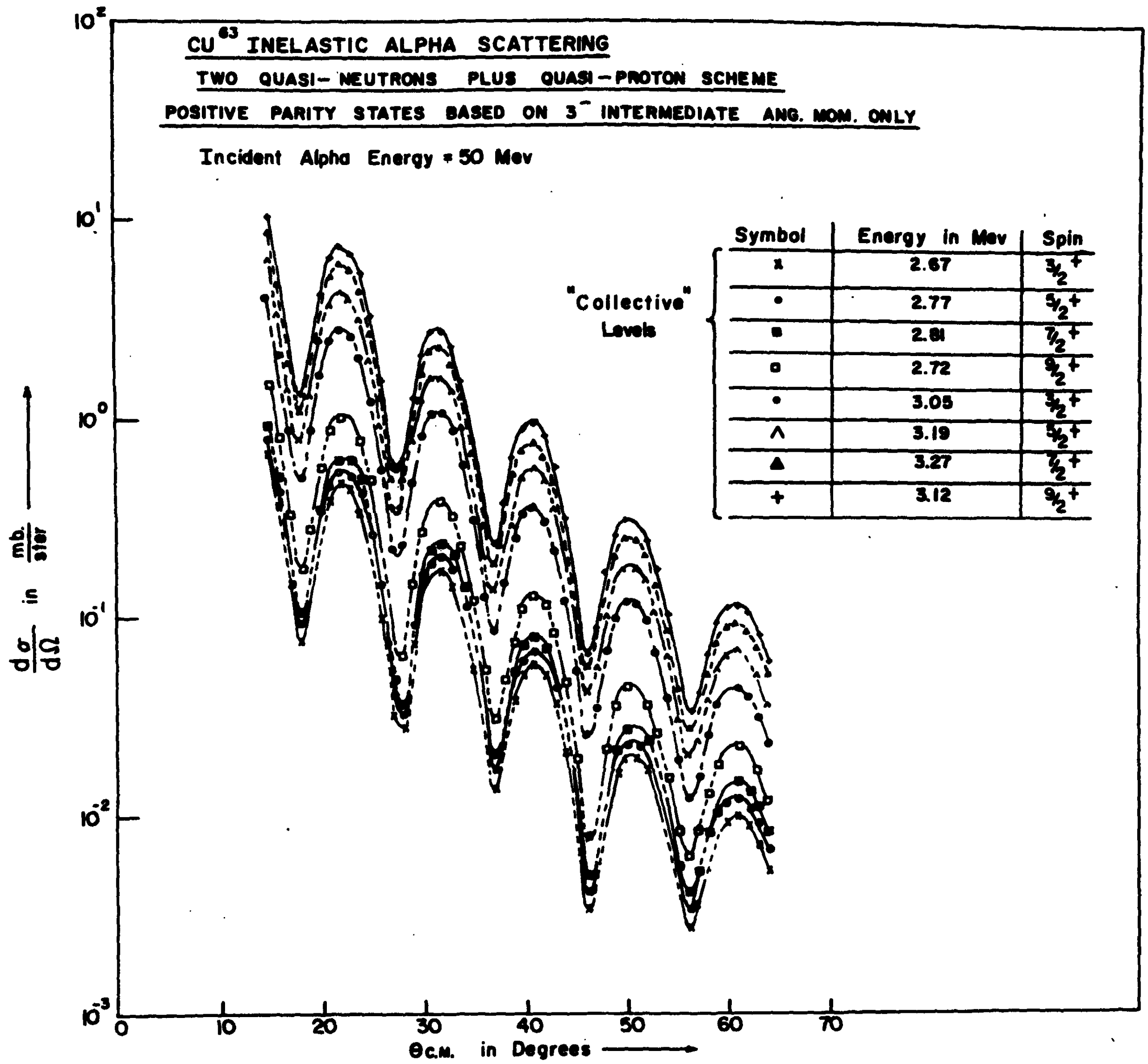

CU 63 INELASTIC ALPHA SCATTERING
TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME
POSITIVE PARITY STATES BASED ON 3⁻ INTERMEDIATE ANG. MOM. ONLY
Incident Alpha Energy = 50 Mev
"Collective" Levels
Symbol | Energy in Mev | Spin
x 2.67 3/2+
• 2.77 5/2+
■ 2.81 7/2+
□ 2.72 9/2+
• 3.05 3/2+
∧ 3.19 5/2+
▲ 3.27 7/2+
+ 3.12 9/2+
dσ/dΩ in mb./ster
θc.m. in Degrees
10², 10¹, 10⁰, 10⁻¹, 10⁻², 10⁻³
0 10 20 30 40 50 60 70

Fig 12

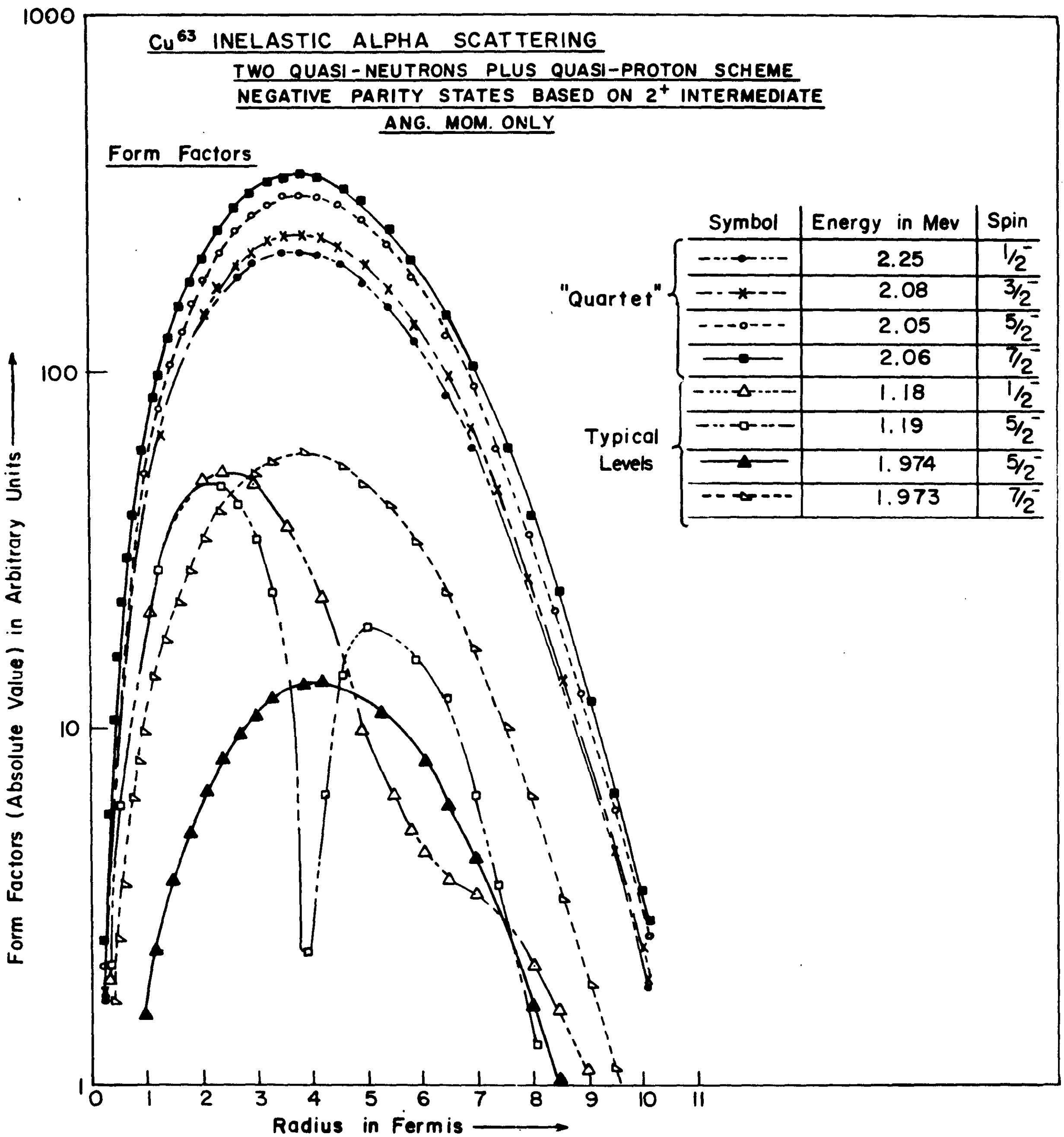

Cu63 INELASTIC ALPHA SCATTERING
TWO QUASI-NEUTRONS PLUS QUASI-PROTON SCHEME
NEGATIVE PARITY STATES BASED ON 2+ INTERMEDIATE
ANG. MOM. ONLY
Form Factors
Symbol
Energy in Mev
Spin
"Quartet"
2.25
1/2-
2.08
3/2-
2.05
5/2-
2.06
7/2-
Typical Levels
1.18
1/2-
1.19
5/2-
1.974
5/2-
1.973
7/2-
Form Factors (Absolute Value) in Arbitrary Units
1000
100
10
1
0
1
2
3
4
5
6
7
8
9
10
11
Radius in Fermis

Fig 13

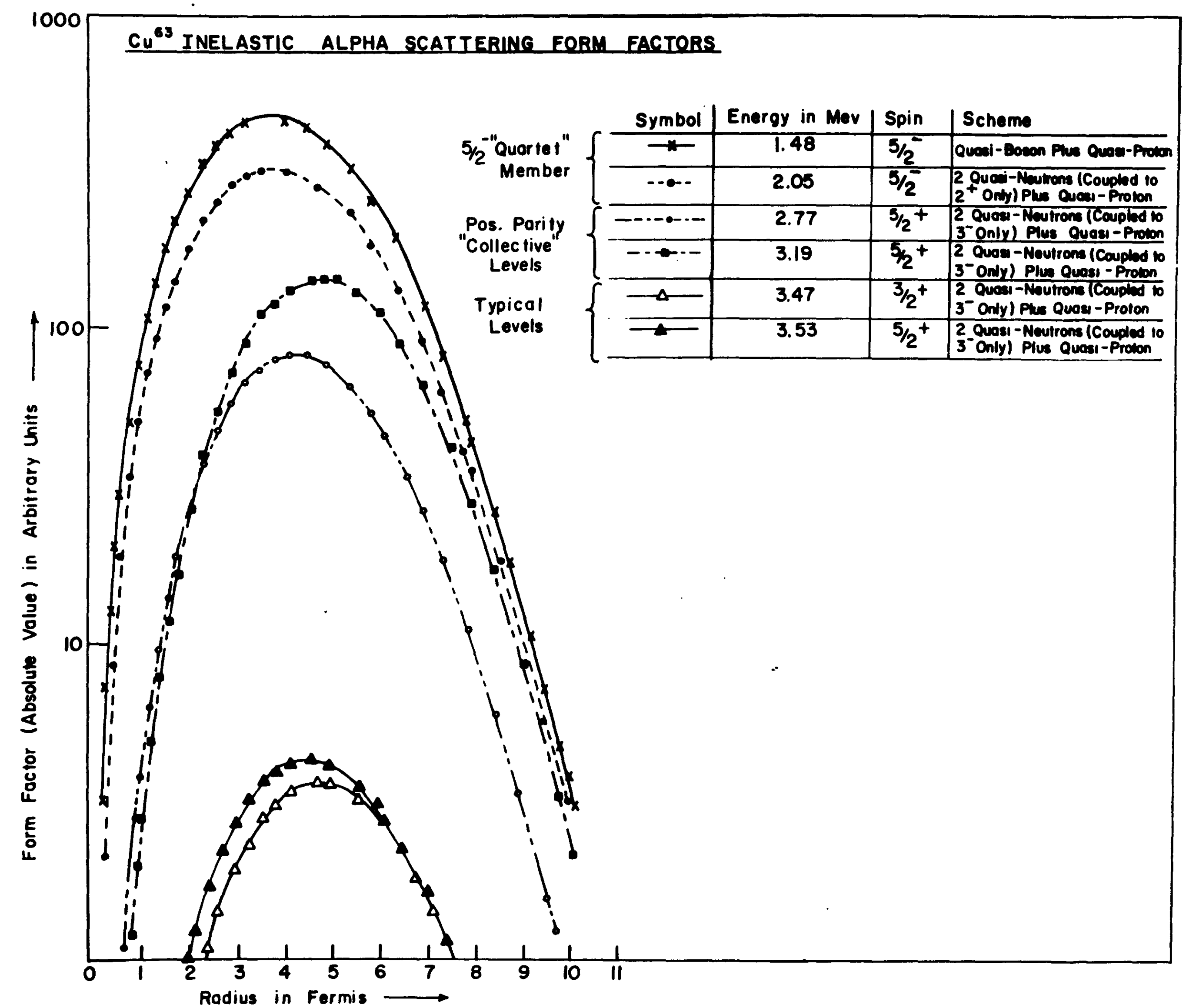

Cu63 INELASTIC ALPHA SCATTERING FORM FACTORS
Symbol
Energy in Mev
Spin
Scheme
5/2- "Quartet" Member
1.48
5/2-
Quasi-Boson Plus Quasi-Proton
2.05
5/2-
2 Quasi-Neutrons (Coupled to 2+ Only) Plus Quasi-Proton
Pos. Parity "Collective" Levels
2.77
5/2+
2 Quasi-Neutrons (Coupled to 3- Only) Plus Quasi-Proton
3.19
5/2+
2 Quasi-Neutrons (Coupled to 3- Only) Plus Quasi-Proton
Typical Levels
3.47
3/2+
2 Quasi-Neutrons (Coupled to 3- Only) Plus Quasi-Proton
3.53
5/2+
2 Quasi-Neutrons (Coupled to 3- Only) Plus Quasi-Proton
1000
100
10
Form Factor (Absolute Value) in Arbitrary Units
0 1 2 3 4 5 6 7 8 9 10 11
Radius in Fermis

Fig 14

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BIOGRAPHICAL NOTE

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