RESIDUAL NOISE AFTER STRIPPING ON FADING MULTIPATH CHANNELS

R. G. Gallager, M.I.T. 5/9/94

1) INTRODUCTION: We are interested in transmitting pseudo-noise (CDMA) signals over a multipath channel, measuring the channel from the output, and using that measurement to subtract the effect of the transmitted signal from the received waveform. The reason for wanting to do this is to take advantage of stripping for multiaccess communication, but we can study the channel measurement and the noise introduced by stripping without reference to the multiple sources. Thus here we look at only a single source.

LOW PASS EQUIVALENTS: Consider M-ary signalling with the M signals $u_1(t),...,u_M(t)$. Let T be the intersymbol duration, so that each T seconds, one of the signals $\{u_m(t), 1 \leq m \leq M\}$ is transmitted. We assume that $u_m(t)$ is essentially non-zero only for $0 \leq t < T$ so that successive signals do not overlap. After passing through the multipath channel there will be some overlap which we discuss later. The signals all have bandwidth W, centered around some carrier frequency $f_0 >> W$. Letting $U_m(f) = \int u_m(t)e^{-j2\pi ft}\,dt$ be the Fourier transform of $u_m(t)$ for each m, we take the low pass equivalent waveforms $x_m(t)$ with Fourier transforms $X_m(f)$ to be defined by $X_m(f) = U_m(f+f_0)$, $f > -f_0$ and $X_m(f) = 0$ otherwise (see Fig. 1).

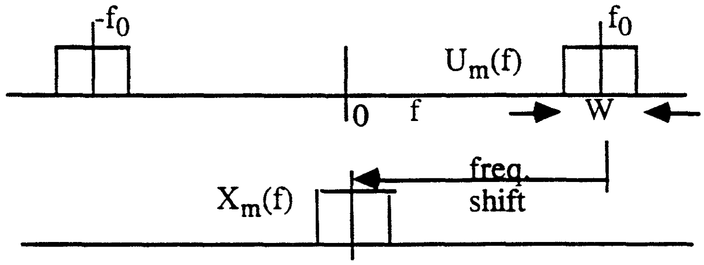

Figure 1: Low pass equivalent; $U_m(f)$ and $X_m(f)$ are actually complex.

This particular way of going from passband to baseband is not entirely conventional. It has the peculiar aspect that

$$u_m(t) = 2\mathrm{Re}[x_m(t)e^{j2\pi f_0 t}] = 2\mathrm{Re}[x_m(t)]\cos(2\pi f_0 t) - 2\mathrm{Im}[x_m(t)]\sin(2\pi f_0 t) \qquad (1)$$

An even more peculiar aspect is that if E is the energy in the signal $u_m(t)$, so that

$$\int_0^T [u_m(t)]^2\,dt = \int_{-\infty}^{\infty} [u_m(t)]^2\,dt = \int_{-\infty}^{\infty} |U_m(f)|^2\,df = E \qquad (2)$$

then the low pass waveform (since it is a frequency translate only of the positive frequency part of u) contains only half the signal energy E:

$$\int_{-\infty}^{\infty} |x_m(t)|^2\,dt = \int_{-\infty}^{\infty} |X_m(f)|^2\,df = \frac{E}{2} \qquad (3)$$

The desirable feature of this scaling is that if u(t) is passed through a linear filter of impulse response g(t) to get output v(t), and if x(t), h(t), and y(t) are the corresponding low pass equivalents of u(t), g(t), and v(t), then

$$v(t) = u(t)*g(t); \quad V(f) = G(f)V(f)\,; \quad y(t) = h(t)*x(t); \quad Y(f) = H(f)X(f) \qquad (4)$$

Since the arguments to follow depend critically on being able to view signals and filters interchangeably, we have defined low pass waveforms so that (4) is satisfied, and by necessity this forces the peculiar energy scaling in (3).

The particular signals used in multiaccess systems are usually pseudo-noise signals (often called CDMA signals). They have the property that $|U_m(f)|$ is essentially constant over the signalling bandwidth $|f-f_0| \leq W/2$. It follows that $|X_m(f)|$ is essentially constant over $|f| \leq W/2$ and 0 elsewhere. For simplicity, we henceforth assume that $|X_m(f)|$ is exactly constant over $|F| \leq W/2$ and zero elsewhere. Using (3) then,

$$|X_m(f)|^2 = \frac{E}{2W} \text{ for } |f| \leq W/2\,; \quad |X_m(f)|^2 = 0 \text{ for } |f| > W/2 \qquad (5)$$

Since $|X_m(f)|^2$ and $R_m(\tau) = \int x_m^*(t)x_m(t+\tau)\,d\tau$ are Fourier transforms, it follows from (5) that $R_m(\tau)$ has a (sin x)/x form. Thus, if we view $x_m(t)$ in terms of its samples at rate W, we have $\sum_i x^*_{m,i}\, x_{m,i+j} = (E\,W/2)\,\delta(j)$. Eq.(5) appears to be more intuitively satisfying for those who don't work with sampled data systems constantly, so we use that in what follows.

It is not possible to find waveforms x(t) that are both time limited to the signal interval T and low pass limited to the band W/2. CDMA systems, however, have a relatively large time bandwidth product, WT >>1 (which is why they are called spread spectrum systems), and for this reason, waveforms can be found that are both approximately time limited and frequency limited. Finding such waveforms with desirable cross correlation properties is a large and very well studied field, but studying this would draw us away from our main purpose. Thus in what follows, we simply assume (5) to be valid, and recognize that the approximation can be quite good for $WT >> 1$.

THE EFFECT OF MULTIPATH: Let $a_i(t)$ be the strength of the i^{th} propagation path at time t, and let $\tau_i(t)$ be the propagation delay of that path. Both a_i and τ_i change slowly with time. The impulse response of the channel, i.e., the output at time t due to an impulse τ seconds earlier is then $g(\tau,t) = \sum_i a_i(t)\, \delta(\tau-\tau_i(t))$. Thus the response to a signal u(t) is

$$v(t) = \int u(t-\tau)g(\tau,t)d\tau = \sum_i u(t-\tau_i(t))a_i(t) \qquad (6)$$

Defining $G(f,t) = \int g(\tau,t)e^{-j2\pi f\tau}\, d\tau$, we have $G(f,t) = \sum_i a_i(t)\, e^{-j2\pi f\tau_i(t)}$. Since the input to the channel is bandlimited, we are only interested in G(f,t) in the band $|f-f_0| \leq W/2$. Thus we define $H(f,t) = G(f+f_0,t)$ for $|f|\leq W/2$. Thus, for $|f| \leq W/2$,

$$H(f,t) = \sum_i a_i(t)\, e^{-j2\pi f_0\tau_i(t)}e^{-j2\pi f\tau_i(t)} = \sum_i \alpha_i(t)e^{-j2\pi f\tau_i(t)} \quad \text{where } \alpha_i(t) = a_i(t)e^{-j2\pi f_0\tau_i(t)}$$

Inverse Fourier transforming, the low pass equivalent filter, for $|f| \leq W/2$, is

$$h(\tau,t) = \sum_i \alpha_i(t) \frac{\sin[\pi W(t-\tau_i(t))]}{\pi(t-\tau_i(t))} \qquad (7)$$

Note that $h(\tau,t)$ has one filtered impulse for each path, and that the sin x over x pulse representing the filtered impulse has a peak that increases with W and a width that decreases, thus keeping unit area. The multipath structure does not change as the bandwidth of the input is changed, but the filter $h(\tau,t)$ does change, since $h(\tau,t)$ represents only the effect of the channel over the given bandwidth. This is an important point, since the effect of the channel is typically very complex, but we need measure its effect only on the signals in the given band. Since we want to measure the channel over the bandwidth W, we want to characterize it in the simplest way over that band.

Define L as the multipath spread of the channel; this is the difference in propagation delay between the longest and shortest path. Thus $h(\tau,t)$ is approximately 0 for $\tau<0$ and for $\tau>L$ (in communication, one usually adjusts the time reference at the receiver to be delayed from that at the transmitter by the shortest (or sometimes the strongest) propagation delay). For cellular mobile communication, L is typically between a few microseconds and 30 microseconds, and for the personal communication systems of the future, L is typically much smaller, on the order of 100 nsec. If L = 10 μsec, and W = 10^6 H, then $h(\tau,t)$ could be represented (through the sampling theorem) by 10 samples in τ; each sample is complex, so measuring h at a given t corresponds to measuring 20 real numbers.

Define B as the Doppler spread of the channel; this is the difference between the largest and the smallest Doppler shift. Typical values in a mobile system are around 100 H. B determines how quickly $h(\tau,t)$ can change with t. The phases in the path strengths $\alpha_i(t)$ can change significantly over the interval 1/B, so that measurements of the channel become outdated over intervals of duration 1/B. We will assume in what follows that the signalling interval T is very much smaller than 1/B, and thus we assume that $h(\tau,t)$ is constant as a function of t over a signal interval T. Thus $h(\tau,t)$ is modeled as a linear time invariant filter over individual signal intervals, allowing one to play all the games of elementary linear systems. One must recognize, of course, that $h(\tau,t)$ changes significantly over many signalling intervals, so that one cannot simply measure h once and for all.

ESTIMATING $h(\tau,t)$: First ignore noise, assume that $x_m(\tau)$ is transmitted, and consider passing the channel output, $x_m(\tau)*h(\tau,t)$ through a filter matched to x_m (i.e., a filter with impulse response $x^*_m(-\tau)$).

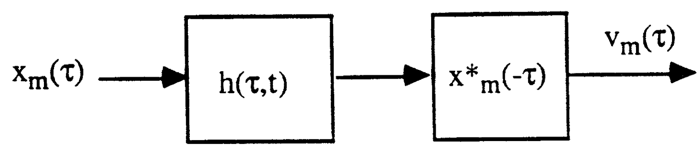

Figure 2

Taking Fourier transforms, we have

$$V_m(f) = X_m(f)\, H(f,t)\, X^*_m(f) = |X_m(f)|^2\, H(f,t) = \frac{E}{2W} H(f,t) \tag{8}$$

where we have used (5). Taking the inverse Fourier transform, $v_m(\tau) = \frac{E}{2W} h(\tau,t)$. Since we are looking at an input in the interval (0,T), and we are assuming that $h(\tau,t)$ does not change in t over intervals of duration T, the parameter t can be taken to be 0. This suggests that the output should be attenuated by $2W/E$ in order to obtain an estimate of $h(\tau,t)$.

We now put the white noise back in the picture and look at the output of the attenuated matched filter including noise (see Fig 3). Assume the noise has spectral density $N_0/2$. Filtering the noise to $|f-f_0| \leq W/2$, and defining the base band equivalent noise, as before, as the upper sideband shifted down by f_0, the baseband equivalent noise process is Gaussian and has the spectral density $N_0/2$ for $|f| \leq W/2$. It follows that the noise power of the baseband waveform is $N_0W/2$, which is half the noise power of the band pass waveform. Thus we have scaled the noise in the same way as the signal. Physically, when one demodulates a passband waveform into quadrature baseband components, one can scale those baseband waveforms arbitrarily, but the signal and noise must be scaled the same way.

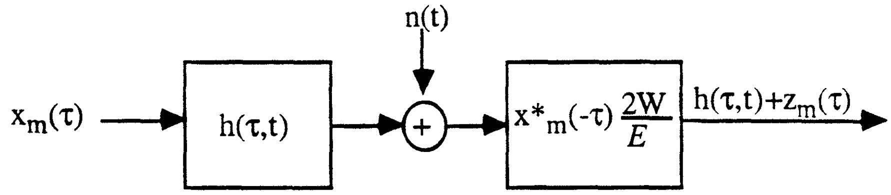

Figure 3

We have seen that the component of the output due to the signal $x_m(\tau)$ is $h(\tau,t)$. To analyze a sample $z_m(t)$ of the output noise process, note that the Fourier transform of the attenuated matched filter is $X^*_m(f)2W/E$. The spectral density of the filter, i.e., the magnitude squared of the Fourier transform, is $(E/(2W))[2W/E]^2 = 2W/E$. Since the input process has the spectral density $N_0/2$, the output process $\{Z_m(\tau)\}$ has the spectral density N_0W/E. Thus the noise power of $\{Z_m(t)\}$ is N_0W^2/E and

$$E[Z_m(t)^2] = N_0W^2/E. \tag{9}$$

Now consider a rake receiver (see Figure 4). Assume that $h(\tau,t)$ has been well estimated, and thus that the output of the filter matched to $h(\tau,t)$ allows a correct decision to be made on m. Given this decision, the output of the m^{th} filter matched to the signal yields a new estimate of h plus additive Gaussian noise. The device to estimate h then uses the decision on m to accept the input from the m^{th} filter to update the old estimate of h.

To avoid worrying about the optimal estimate of h, we can get a good approximate answer by simply assuming that the estimate of h is simply the linear average of the previous n measurements (where each measurement comes from appropriate m). Since the filter

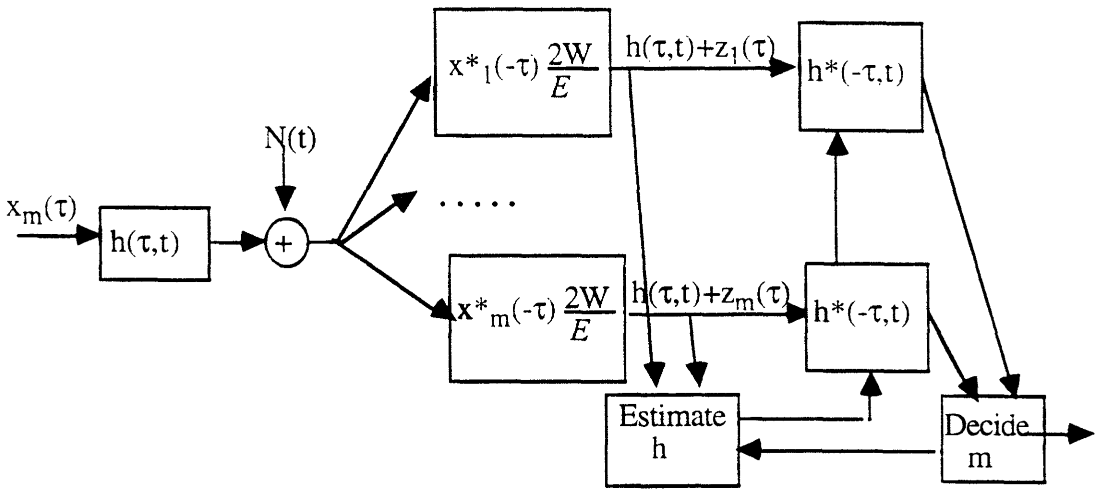

Figure 4: Rake Receiver

remains almost constant for a time on the order of 1/B, we take n = 1/(BT). Since taking an average over n measurements reduces the noise variance by a factor of n, we have

$$\hat{h}(\tau,t) = h(\tau,t) + z'(\tau) \;;\quad E[(Z'(\tau))^2] = \frac{N_0W^2}{nE} = \frac{BTN_0W^2}{E} \qquad 0\le\tau\le L \qquad (10)$$

where $z'(\tau)$ is a sample function of the Gaussian random process $\{Z'(\tau)\}$. We assume that the multipath spread L is known, and thus that $\hat{h}(\tau,t)$ is taken to be 0 for $\tau<0$ and $\tau>L$. Since all of the noise processes being averaged are white over $|f| \le W/2$, $\{Z'(\tau)\}$ is also white over $|f|\le W/2$.

RESIDUAL NOISE: Finally we have the problem of determining the residual noise if the effect of the detected signal is subtracted from the received waveform (again assuming the correct signal was detected). The effect of the signal $x_m(\tau)$ on the received signal is $x_m(\tau)*h(\tau,t)$. The quantity subtracted from the received signal in stripping is $x_m(\tau)*\hat{h}(\tau,t)$, and thus the residual noise after stripping is $\phi(\tau) = x_m(\tau)*z'(\tau)$. Taking Fourier transforms,

$$\Phi(f) = X_m(f)Z'(f) \;;\qquad |\Phi(f)|^2 = |X_m(f)|^2|Z'(f)|^2 = \frac{E}{2W}|Z'(f)|^2 \qquad (11)$$

Thus,

$$\int |\Phi(f)|^2 df = \int \frac{E}{2W}|Z'(f)|^2 df = \int |\phi(\tau)|^2 d\tau = \int \frac{E}{2W}|z'(\tau)|^2 d\tau \quad (12)$$

Taking the expected value of both sides of (12),

$$\int E[|\phi(\tau)|^2]d\tau = \int \frac{E}{2W} E[|Z'(\tau)|^2]d\tau = \frac{BTN_0WL}{2} \quad (13)$$

This is the baseband expected energy of the residual noise in the band $|f|\leq W$ and over the interval $0\leq t\leq T$. Since $Z'(\tau)$ is white over the band $|f|\leq W/2$, $\phi(\tau)$ is also white over $|f|\leq W$. Thus the spectral density of this noise power (averaged over the time interval (0,T)) is $BLN_0/2$. Since the spreading product BL is small for most wireless situations, this indicates that the residual noise is small relative to the ordinary additive noise of spectral density $N_0/2$.

When multiaccess communication is taken into account, the noise that effects the filter measurement becomes not only the white noise but also the other users signals, which have been passed through their own multipath filters before contributing to the measurement of the filter in question.

The argument here does not necessarily mean that stripping will work. It has assumed that decisions can be made correctly, and has not adjusted for the effect of incorrect decisions. The argument has also assumed that correct decisions can be made without delay (which is not necessarily correct in coded systems). Finally, perfect timing recovery has been assumed. What we have shown, however, is that there is no inherent reason why stripping can not work on multipath channels.