LOCALIZATION IN DISORDERED PERIODIC STRUCTURES

by

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ABSTRACT

Disorder in periodic structures is known to cause spatial localization of normal modes and attenuation of waves in all frequency bands. This thesis uses a traveling wave perspective to investigate these effects on one-dimensional periodic structures of interest to the engineer. Relevant work in the fields of solid state physics, mathematics and engineering is reviewed. A transfer matrix formalism including wave transfer matrices is used to model disordered periodic structures. A limit theorem of Furstenberg for products of random matrices is exploited to calculate localization effects as a function of frequency. The approach presented is applicable to virtually any disordered periodic system carrying a single pair of waves. Localization is studied on three disordered periodic systems using both theoretical calculations and Monte Carlo simulations. Localization is found to be quite pronounced at frequencies near the stopbands of the perfectly periodic counterparts. The problem of localization in one-dimensional systems carrying a multiplicity of wave types is examined using the theorem of Oseledets on products of random matrices. A new result is presented - the multiwave localization factor as a function of the transmission properties of the system.

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Acknowledgement

I have had a lot of fun studying the localization phenomenon during the past two years. I have been fortunate to share that fun with my thesis supervisor, Professor Andy von Flotow. He has been kind enough to listen to me talk about my evolving understanding of localization week after week, and I have seen his creativity and energy at work as he helped me with this fascinating subject.

Dr. Richard Battin is the professor at MIT I have known the longest. From him I took some very enjoyable classes, and from him I first learned about symplectic matrices which have played such an important part in this thesis. I wish him all the best in his semi-retirement.

I learned a lot of what I know about stochastic processes from Professor Wallace Vander Velde. It is appropriate that he should be on my committee because of our common interests in flexible space structures and probability theory.

Professor George Verghese has had an interest in random matrices for quite a while, and it was through him that I learned of one of my most important references. I am grateful for his many useful suggestions on a draft of this thesis, and I am glad that I could share my insights with him.

I have received assistance in my work from other quarters as well. Professor Dick Dudley of the Mathematics Department helped me in my interpretation of some of the esoteric aspects of Furstenberg's theorem. Professor Patrick Lee of the Physics Department gave me a suggestion that led to the simplification of the derivation of the multiwave localization factor in Chapter 5.

During my sojourn in Cambridge there are many people that have helped me in many ways. One of those people is Eugene Gath. Eugene not only led me through the intricacies of Irish politics, but also guided me through the mysteries of Green's functions, p-forms and much more that I encountered in this research.

This research was supported in part by the Air Force Office of Scientific Research with Dr. Tony Amos as contract monitor.

Having been localized in Cambridge, it is time to propagate, but before I go, I must dedicate this thesis to Mom, Dad and Deb.

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Nomenclature

a	element of Cayley matrix
A	cross-sectional area of rod
$\overleftarrow{A}$	amplitude of left traveling wave
b	element of Cayley matrix
$\overrightarrow{B}$	amplitude of right traveling wave
$\mathbf{C}$	Cayley matrix
C^{2d}	complex Euclidean vector space of dimension $2d$
$diag\{*\}$	diagonal matrix
E	Young's modulus
H	(superscript) hermitian transpose
$\bar{H}$	nondimensional transfer function
i	$i^2 = -1$
I	area moment of inertia
k	wave number
k_s	spring constant
$\bar{k}_s, \tilde{k}_s$	nondimensional spring constant
l	length of a bay
$\bar{l}, \tilde{l}$	nondimensional length of a bay
m_j	random mass of jth bay

m	average mass, mass of perfectly periodic structure
n	number of bays
$\bar{N}_j$	nondimensional internal force, jth point
$o(*)$	terms of order greater than the argument
$p(*)$	probability density function of the indicated argument
r_j	reflection coefficient of jth bay
$\mathbf{r}$	reflection matrix of a bay
$\hat{\mathbf{r}}$	reflection matrix of a bay
R^{2d}	real Euclidean vector space
sup	supremum
t_j	transmission coefficient of jth bay
$\mathbf{t}$	transmission matrix of a bay
$\hat{\mathbf{t}}$	transmission matrix of a bay
tr	trace of a matrix
T	(superscript) matrix transpose
$\mathbf{T}$	transfer matrix
$\mathbf{T}_j$	random transfer matrix, jth bay
$\mathbf{T}(\alpha)$	transfer matrix, function of random variable α
u_j	displacement of jth mass
$\bar{U}_j$	nondimensional longitudinal displacement, jth point

$w.p.\ 1$	with probability one
$\mathbf{W}_j$	wave transfer matrix, jth bay
$\mathbf{x}$	a real state vector
$\bar{\mathbf{x}}$	normalized real state vector or direction of state vector
$\mathbf{X}$	eigenvector matrix
$\mathbf{z}$	a complex state vector
$\bar{\mathbf{z}}$	normalized complex state vector
α	random variable or vector
γ	localization factor
γ_j	jth Lyapunov exponent
λ	eigenvalue
Λ^p	p-form operator
$\mu_j, \bar{\mu}_j$	nondimensional jth mass
ρ	reflection matrix of n bays or mass density per unit volume
ρ_n	reflection coefficient of n bays
σ	singular value of a matrix
σ_α^2	variance of random variable or vector α
τ	transmission matrix of n bays
τ_n	transmission coefficient of n bays
ω	radian frequency

$\bar{\omega}$ nondimensional radian frequency

$*$ (superscript) complex conjugate

$< \alpha >$ average of a random variable or vector α

$1_A(*)$ indicator function, its value is 1 when the argument lies on A and 0 otherwise

$|_{<\alpha>}$ evaluate at $< \alpha >$.

$|_{<\alpha_j>}$ evaluated at $< \alpha_j >$

$\subset$ subset of

$\in$ an element of

Chapter 1

Introduction

1.1 Introduction to Localization

This thesis describes some of the dynamic consequences of disorder in what are normally spatially periodic structures. Periodic structures are frequently encountered in many fields of engineering and physics. Periodic electromagnetic waveguides, crystalline structures and periodic truss structures are some examples that come to mind.

The periodic structures examined here are systems having repetitive bays along one linear dimension. Those of interest to the structural dynamicist include beams on evenly spaced supports, the skin-stringer panels found in airplane fuselages and truss beams that will form the support structure of the space station. See Figure 1.1.

The dynamics of perfectly periodic systems have special characteristics. Most notably they are characterized by frequency bands that alternately pass and stop traveling waves (assuming no damping) with the natural frequencies of the structure lying within the passbands. See Figure 1.2 In addition, the normal mode shapes of periodic structures are themselves periodic. See Figure 1.3.

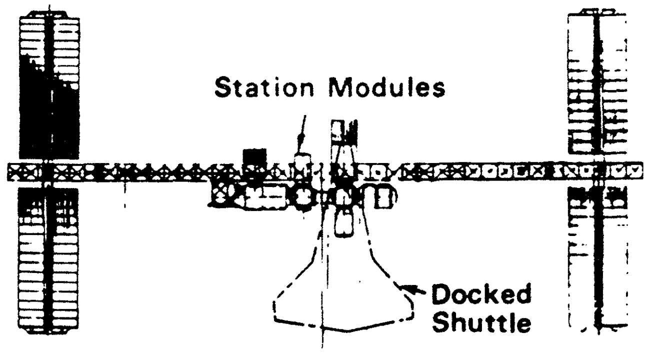

Figure 1.1: Periodic truss stucture along the length of the space station from [Covault 86]

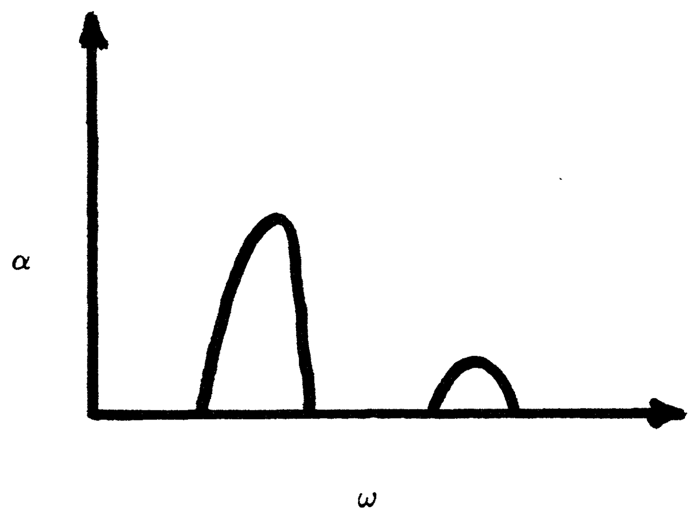

Figure 1.2: Alternating pass and stopbands of a perfectly periodic structure. The attenuation coefficient, α, represents the decay per bay

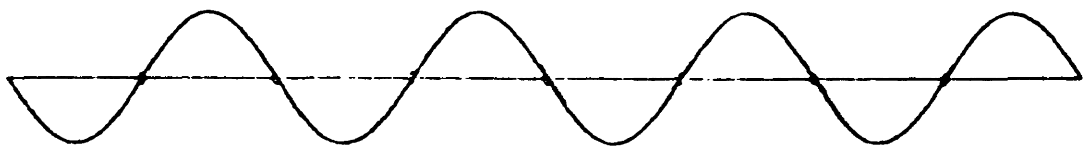

Figure 1.3: Mode of a perfectly periodic structure from [Hodges and Woodhouse 83]

Because of manufacturing or assembly defects, no structure will be perfectly periodic. Disorder can occur in the length of bays and in the material and mass properties of the structure. The disorder is assumed to be distributed among all the bays and not confined to just a few. Recently, [Hodges 82,Hodges and Woodhouse 83] demonstrated with simple examples that this disorder in periodicity can have some amazing consequences. Disruption in the periodicity will lead to attenuation of waves in all frequency bands independent of any dissipation in the system! See Figure 1.4 This is a result of the multiple scattering effects from the randomized bays. Equivalently, each normal mode, whose amplitude is periodic along the length of a perfectly periodic structure, will have its amplitude spatially localized in the disordered counterpart. See Figure 1.5.

This localized behavior of the mode shapes, or equivalently the attenuation of all the traveling waves, means that energy injected into one end of a disordered structure

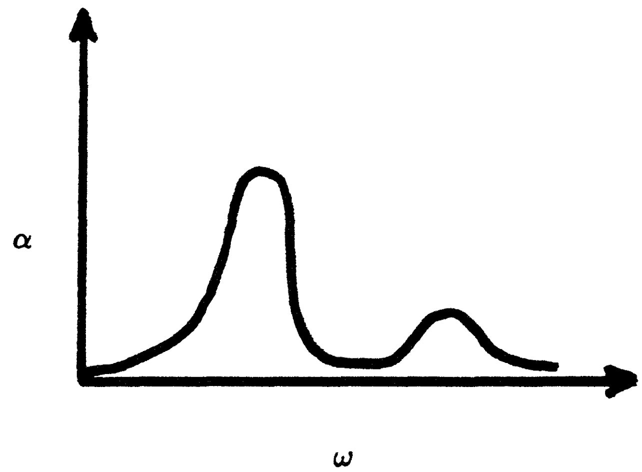

Figure 1.4: Attenuation in all frequency bands of a disordered periodic structure

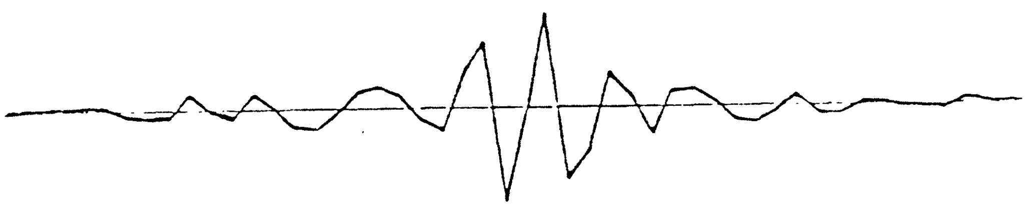

Figure 1.5: Mode of a disordered periodic structure from [Dean and Bacon 63]

will not be able to propagate arbitrarily far, but will be confined to the region near the input. Because such behavior can impact disturbance propagation in and control of structures, as well as complicate schemes to identify the dynamic characteristics of a system, engineers should be aware of the localization phenomenon.

A simple example from [Hodges 82] will intuitively illustrate the localization phenomenon. Consider an infinite chain of equivalent pendula with nearest neighbors connected by identical springs. This is an example of a perfectly periodic structure and so its mode shapes will be periodic. Now consider disordering this system by replacing each pendulum by one with a random length. In this way the natural frequency of each pendulum is random, and so we no longer have a perfectly periodic system. First assume that the spring constant between each pendulum is zero, so that each pendulum vibrates independently. This is a trivial example of mode localization. Now consider adding a tiny amount of the same spring constant between each pendulum. In this case each pendulum will vibrate at a frequency different from its neighbor, and, with the spring stiffness being so small, its amplitude will not couple significantly with its neighbor. Indeed, if there is only a small probability of encountering within a short distance a pendulum with the same natural frequency as the one under consideration, we can understand how the vibrational amplitude of this pendulum will be localized.

Though the appellation "localization" comes from the fact that normal modes are spatially localized, we will be studying the phenomenon from a traveling wave perspective. Very few analytical results are available dealing directly with normal modes in disordered systems. Our approach is consistent with that in the field of solid state physics, where the phenomenon was originally discovered.

The localization phenomenon makes for a particularly attractive field of study. From the perspective of structural dynamics this is true because it seems to manifest itself as a damping mechanism even though vanishingly small damping may be present. The study of localization is frequently referred to as one of great mathematical richness and subtlety and this has made for a challenging course of research, especially as we have made use of the mathematics for products of random matrices. The fact that the localization phenomenon has been studied for many years by solid state physicists allows us to borrow the insights and avoid the mistakes from their analogous work. In addition, any new results generated by this research have immediate applicability to virtually any disordered periodic system, even outside the structural dynamics field. Finally, connections between localization theory and the rapidly developing fields of fractals [Rubin 84], chaos [Ikeda and Matsumoto 86] and superconductivity [Lee and Ramakrishnan 85] have been noted.

1.2 History of Localization Studies

The study of the localization phenomenon has a colorful history spanning three decades, with major contributions from researchers in the United States, United Kingdom, Japan, France and the Soviet Union in the fields of solid state physics, mathematics and only lately in engineering. In this section we review some of that history in order that we can place the contribution of this thesis in proper context.

1.2.1 Solid State Physics, Mathematics and the Localization Phenomenon

Two notable papers in the 1950's [Dyson 53,Schmidt 57] explored the effects of disorder on the eigenvalues of an infinite mass-spring chain in one linear dimension. Though they did not examine the effects of disorder on the eigenvectors or on wave propagation, some of their results help explain the mathematics of wave transmission in such randomized systems.

The scientist to first describe eigenstate localization was solid state physicist Philip W. Anderson, whose 1958 paper [Anderson 58] showed that an electron in a three-dimensional disordered lattice of infinite extent had a finite probability of not being transported from its original site as time tended to infinity. In honor of his original contribution, the phenomenon is sometimes called Anderson localization. Localization was at first not well understood or even believed by many people. But through the efforts of researchers like [Mott and Twose 61] it gained acceptance in the solid state physics community. Meanwhile [Borland 63] examined the one-dimensional localization problem from a nonrigorous probabilistic perspective and [Dean and Bacon 63] did numerical simulations of disordered mass-spring chains of finite length showing that eigenmode localization was much more pronounced at high frequency than at low frequency. The solid state physics literature on localization has become quite extensive over the years and much of it is not relevant to this thesis. The reader is referred to [Ziman 79,Erdős and Herndon 82,Lee and Ramakrishnan 85] for extensive bibliographies relevant to that field. The remainder of this review will encompass those physics, mathematics and engineering papers that have had some impact on the thesis.

The pioneering work of [Furstenberg 63] on products of random matrices has provided rigorous results that have immediate applicability to the one-dimensional localization problem. This is so because each bay of a disordered periodic structure can be modeled with a random transfer matrix, and, as a result, the entire structure can be modeled with a product of random matrices. The researchers [McCoy and Wu 68] were apparently the first to recognize the importance of Furstenberg's therorem to disordered physical systems when they studied random Ising models of ferromagnetic systems. However, [Matsuda and Ishii 70] and [Ishii 73] were the first to bring Furstenberg's work to bear on the localization problem. They carefully related Furstenberg's results to eigenmode localization and wave propagation in disordered chains and some simple quantum mechanical models.

In [Oseledets 68] a Russian mathematician proved a multiplicative ergodic theorem that has enhanced our understanding of the asymptotic behavior of products of random matrices. This theorem has important applications to the study of the localization phenonmenon in systems carrying a multiplicity of wave types at a given frequency. Lately, [Pichard and Sarma 81-1], [Pichard and Sarma 81-2] and [Pichard and André 86] have examined localization in solid state multiwave systems. In analyzing these systems they have exploited the work of Oseledets on products of random matrices. Mathematicians have taken renewed interest in the theory of products of random matrices as indicated by two recent publications, [Bougerol and Lacroix 85,AMS 86].

The work of [Herbert and Jones 71,Thouless 72] provides another perspective as far as the calculation of localization effects are concerned. They derived a formula for the localization factor (defined below) which is a function of the spectrum of the disordered system. Their approach is nearly as rigorous as that using products of random matrices.

In 1977 when Anderson and Mott (and Van Vleck) were awarded the Nobel Prize in physics, they were cited in part for their work on localization. In his speech in Stockholm, Anderson [Anderson 78] made the following comment:

> Localization ...has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

While it is still true that we must use numerical simulations to confirm our analytical insights about localization, we will argue in this thesis that mathematical tools are available which allow us to answer some very important questions about localization in one-dimensional systems.

Despite a Nobel Prize, [Czycholl and Kramer 79] raised serious questions with their numerical work about even the existence of localization in one-dimensional systems. This prompted [Anderson et al 80] to do some fundamental work on the localization problem in one dimension. They derived what they called a scaling variable for one-dimensional disordered systems carrying a single pair of waves. This variable, involving $\ln|t|$[1] where t is the transmission coefficient for a bay, was argued to be the statistically meaningful quantity to average when examining one-dimensional random systems. They also argued that the variable satisfied a central limit theorem. Earlier [O'Connor 75] had made an important contribution toward establishing a central limit theorem for disordered periodic systems. Subsequently [Abrahams and Stephen 80], [Andereck and Abrahams 80] and [Stone 83] provided numerical evidence to support the central limit theorem ideas of [Anderson et al 80]. Apparently [Le Page 82] has provided the definitive mathematical work supporting a central limit theorem contention.

1.2.2 Structural Dynamic and Acoustical Applications of Localization Theory

Solid state physicist C. H. Hodges [Hodges 82] was the first to recognize the relevance of localization theory to disordered periodic systems of interest to the structural dynamicist. He used wave arguments to calculate localization effects at high frequency for a beam on randomly spaced supports. His work raised the possibility that disorder could have a dramatic impact on the dynamics of what are normally spatially periodic structures. Unfortunately, his analysis provided little indication of how localization effects varied with frequency, and his techniques were not applicable to a broad range of periodic structures. Both the insights and shortcomings of his work motivated research leading to this thesis.

[1] The precise scaling variable they used was $\ln\frac{1}{|t|^2}$, which is simply $-2\ln|t|$. They use the term scaling variable in the sense that the mean value of the variable for two bays is the sum of the mean values of the variable for each bay individually. Also the variance of this variable scales at least according to a weak law of large numbers.

In a later paper [Hodges and Woodhouse 83] attempted to apply the work of [Herbert and Jones 71,Thouless 72] to estimate localization effects in two passbands for a taut wire with unevenly spaced masses. They also conducted an experiment on the wire-mass system which qualitatively confirmed the localization effects.

More recently [Bendiksen 86,Bendiksen and Valero 87,Cornwell and Bendiksen 87] have examined mode localization in closed disordered periodic structures, like compressor rotors and dish antennas. These closed systems are not mathematically equivalent to the linear one-dimensional structures under consideration here. [Pierre et al 86] and [Pierre 87,Pierre and Dowell 87] have also examined localization, but only with the aid of deterministically disordered systems with as few as three bays. None of the engineering papers so far provided analytical calculations for localization effects over any significant frequency range. This thesis and [Kissel 87] are the first publications to calculate frequency dependent localization factors for disordered periodic systems of interest to the structural dynamicist.

The most rigorous examination of localization in an acoustical setting has been that by [Baluni and Willemsen 85]. They effectively used Furstenberg's work to calculate frequency dependent localization effects; however, their application was for layers of sandstone and shale with random thicknesses. The paper [Sheng et al 86] also examined localization with geophysical applications in mind.

Recently, more solid state physicists [Anderson 85,Flesia et al 87] have recognized that localization manifests itself in acoustical and optical systems. They append the term "classical localization" to the phenomenon when it occurs outside the context of quantum mechanics.

1.3 Goals, Approach and Contribution of Thesis

The ultimate goal of this research is to provide the analyst and experimentalist with the tools to decide (given some engineering judgement of the disorder) how significant the dynamic effects of disorder will be on a periodic structure as a function of frequency and the properties of the structure. This thesis is a major step toward the goal of providing tools to rigorously examine the localization phenomenon in one-dimensional disordered periodic structures. We present the tools for mono-coupled disordered periodic structures (structures in which one bay is connected to its neighboring bays through one coupling coordinate) to calculate, analytically and numerically, localization effects over a wide frequency range at moderate levels of disorder. In addition, an important new tool is presented here to guide localization work on multiwave systems.

The approach of the thesis is probabilistic, as opposed to the deterministic analysis of [Bansal 80,Pierre and Dowell 87]. The methods of probability theory allow us to model our uncertainty in a way that yields meaningful answers. This is particularly true when we make use of theory on products of random matrices, which puts us on a firm mathematical footing.

What had the most profound impact on the direction of the research was the observation of confusion about localization in the late 1970's in one-dimensional disordered systems. In this instance the confusion could have been avoided had more researchers availed themselves of the appropriate mathematical tools. The very important observation about the $\ln|t|$ being the key statistical variable in the study of localization can be easily deduced in a few algebraic steps by making use of wave transfer matrices and Furstenberg's theorem. This is explained in Chapter 3.

It is the philosophy of this thesis that the transfer matrix formalism accompanied by the appropriate theories on products of random matrices can lead to a better understanding of the localization phenomenon. This philosophy has been needlessly neglected in most of the theoretical localization literature to date. The reason that theorems on products of random matrices have generally received scant attention from physicists working on localization is that they have relied on their own heuristic techniques, and that they have been more interested in two- and three-dimensional systems, which cannot be as easily handled with transfer matrices.

The first principal contribution of the thesis is the explanation of how random transfer matrix techniques can be used to model disordered systems, deduce transmission properties and calculate localization effects. This includes a discussion of the important transformation to wave transfer matrix form and the relevance of the theorems of Furstenberg and Oseledets to the one-dimensional localization problem.

The second principal contribution is the calculation of localization effects as a function of frequency for three disordered periodic models of interest to the structural dynamicist. In most instances the localization effects are found to be strongest at frequencies near the stopbands of the normally perfectly periodic structures. Localization effects are also pronounced when the length of a bay is disordered.

The third principal contribution is the derivation of the localization factor for multiwave one-dimensional systems as a function of the transmission matrix. This at last allows a rigorous treatment of localization in multiwave systems. Because transfer matrix methods can be used to model almost any disordered periodic system in one dimension, including systems of interest to the solid state physicist, the results here will be of interest outside the engineering field as well.

In addition to these principal contributions, we will note in the body of the thesis instances where previously published results are extended and where mistaken approaches and conclusions exist in the literature.

1.4 Preview of Thesis

Before studying the effects of disorder on periodic structures, Chapter 2 presents a brief discussion of the modeling and dynamics of perfectly periodic structures. Here the transfer matrix formalism is introduced and the important passband and stopband property is discussed. The modeling of disordered periodic structures is next presented and the very important wave transfer form of the transfer matrix is introduced.

This serves as a prelude to Chapter 3 in which we discuss Furstenberg's theorem on products of random matrices. This is the tool used to study localization for mono-coupled periodic structures. With Furstenberg's theorem in hand, we are able to deduce the asymptotic behavior of the transmission coefficient, τ_n, of the n bay disordered periodic structure. We will show that the wave intensity, $|\tau_n|$, decays as $e^{-\gamma n}$, where γ is the localization factor. We are also able, using the same theorem, to estimate the localization factor as a function of the level of disorder, frequency and physical properties of the system.

This theory is demonstrated on three examples in Chapter 4. The first and simplest example is a linear chain of springs and masses. Initially only the masses are disordered and then only the springs, followed by masses and springs disordered simultaneously. All calculations are confirmed by Monte Carlo simulations. Similarly, a rod with attached resonators is studied. First the masses, springs and lengths are disordered individually, after which all three variables are disordered. The last mono-coupled example is a Bernoulli-Euler beam on simple supports with random lengths between the supports.

Most real structures carry more than a single pair of wave types at a given frequency, so localization in these multiwave systems should be investigated. Unfortunately, Furstenberg's theorem will be of little use for investigating localization effects in multiwave structures; however, the theorem of Oseledets is precisely suited to multiwave analysis. In Chapter 5, after discussing Oseledets' theorem, we present a new result – the localization factor for multiwave systems in terms of the transmission matrix, τ. The significance of the result is discussed, and an analytical technique for calculating the localization factor for multiwave systems is suggested.

Concluding remarks and suggestions for future research are made in Chapter 6.

Several appendices are included and are referred to frequently in the body of the thesis. Appendix A discusses some definitions and properties from matrix theory and group theory used in the thesis. The derivation of the wave transfer matrix for mono-coupled systems is discussed in Appendix B. In Appendix C the modeling of a mass-spring chain, a rod with attached resonators and a beam on simple supports is discussed, both when they are periodic and when they are disordered. A simple method to calculate localization factors, not depending on theories for products of random matrices, is discussed in Appendix D. In the final appendix, Appendix E, we examine some properties of scattering and wave transfer matrices that will be useful in Chapter 5. The reader should at least scan these appendices before proceeding with the rest of the thesis.

Chapter 2

Transfer Matrix Models of Periodic and Disordered Periodic Structures

2.1 Introduction

In this chapter we will describe the nature of periodic structures of interest in the thesis and show how transfer matrices are used to model these structures. Some of the properties of periodic structures are mentioned, including the important passband and stopband characteristic. The modeling of disordered periodic structures via a product of random transfer matrices is then discussed, along with the very important transformation of these matrices to wave transfer form.

2.2 Perfectly Periodic Structures

2.2.1 One-Dimensional Periodic Structures

In Chapter 1 we described the kinds of periodic structures of interest in the thesis as those with repetitive bays in one linear dimension. These identical bays are connected in identical ways to form what is intended to be a perfectly periodic structure. Because we are looking at structures in a *linear* dimension, our discussion excludes closed periodic structures like a compressor rotor or a dish antenna which can be modeled as one-dimensional periodic structures [Bendiksen 86]. We will not be examining periodic structures in two or three dimensions as they are much more difficult to model with transfer matrices, and, in addition, the localization effects are understood to be much less pronounced in these higher dimensions than in the one-dimensional case.

2.2.2 Modeling of Perfectly Periodic Structures

The key modeling tool used throughout the thesis is the transfer matrix. Each bay of the periodic structure is modeled with a linear transformation, $\mathbf{T}$, which relates a state vector of one cross-section to the state vector of the succeeding cross-section, namely:

$$\mathbf{x}_j = \mathbf{T}\mathbf{x}_{j-1}$$

This is a difference equation, where the matrix $\mathbf{T}$ can be thought of as a spatial state transition matrix evaluated between the points j and $j-1$. One transfer matrix is associated with each bay in the structure. The state vector may consist of generalized displacements and forces, for example, or it might consist of the generalized displacements of neighboring bays. The transfer matrix can be found by manipulating the dynamic equations of motion of a bay, possibly derived with the finite element method. The derivation of transfer matrices is discussed at length in [Pestel and Leckie 63]. The formulation of the transfer matrix assumes a sinusoidal time dependence ($e^{i\omega t}$) in the equations of motion. No damping[1] is included in the models so that the effects of disorder can later be highlighted.

The transfer matrix will always be of even dimension, as will the state vector. For most of the thesis we will confine our discussion to bays modeled with 2×2 transfer matrices, which in turn means each state vector is 2×1. These structures are called mono-coupled periodic structures because each bay is connected to its neighboring bays through one coupling coordinate. Mono-coupled periodic structures carry only a single pair of waves.

Because each bay is identical, the state vector after n bays is simply related [Faulkner and Hong 85] to the state vector at the beginning by

$$\mathbf{x}_n = \mathbf{T}^n \mathbf{x}_0$$

Because we are raising a transfer matrix to the nth power, we need only examine the transfer matrix $\mathbf{T}$ to understand the dynamic properties of the periodic structure.

The three transfer matrices describing the three example periodic structures in this thesis can be found in Appendix C. These structures comprise a chain of springs and masses, a rod in longitudinal compression with attached resonators and a Bernoulli-Euler beam on simple supports.

2.2.3 Properties of Perfectly Periodic Structures

To appreciate the consequences of disorder in periodic structures we must first examine the modal and wave properties of periodic structures without disorder. There is extensive literature on perfectly periodic systems and the reader is referred to

[1]The analogous assumption in the solid state localization problem is to neglect inelastic scattering mechanisms.

[Brillouin 46,Miles 56,Mead 70,Cremer et al 73,Mead 75-1,Elachi 76,Engels 80] and [Faulkner and Hong 85,Mead 86]. The literature specifically examining periodic systems carrying a multiplicity of wave types is much less abundant, [Mead 73], [Mead 75-2], [Signorelli 87], [Bernelli et al 87]. The properties we are examining below are for structures with transfer matrices of dimension 2×2.

In a periodic structure the vibrational mode shapes are themselves periodic, i.e., have amplitude equally strong along any section of the structure. As we will see shortly, the natural frequencies at which these modes vibrate tend to occur in clumps along the frequency axis.

Dual to the modal properties of the structure are the wave properties. Two types of waves, traveling waves and attenuating waves, occur in alternating frequency bands known as passbands and stopbands, respectively. In the passbands waves travel according to $e^{\pm ik}$, where k is the real wave number and the positive sign indicates negative-going waves and the negative sign positive-going waves. The wave number $k = \frac{2\pi}{\lambda}$ is a spatial frequency which refers to the phase difference of motions in adjacent bays. Here λ is the wavelength and k varies in magnitude from 0 to π or some multiple thereof. In the stopbands, waves propagate according to $e^{\pm\alpha}$ or $e^{\pm(\alpha+i\pi)}$. The real exponent α implies nontraveling or attenuating waves. The $\alpha + i\pi$ exponent implies adjacent bays vibrating out of phase with each other, in addition to wave attenuation. Both k and α are functions of frequency. Only in the passbands of the perfectly periodic mono-coupled structure can energy be transmitted along the structure [Mead 75-1]. Another type of wave - a complex traveling wave - can occur, but only for systems modeled by transfer matrices of dimension 4×4 or greater [Mead 75-2,Signorelli 87,Bernelli et al 87].

The frequency ranges of passbands for mono-coupled periodic structures can be found by determining those frequencies at which the eigenvalues of its transfer matrix are complex, $e^{\pm ik}$. By examining the characteristic equation of the 2×2 transfer matrix $\mathbf{T}$, where $\det(\mathbf{T}) = 1$ because we have assumed no damping, we readily deduce that passbands occur at frequencies where $|tr(\mathbf{T})| < 2$. Otherwise, the eigenvalues are real, $e^{\pm\alpha}$ or $e^{\pm\alpha+i\pi}$, and we are in a stopband.

This passband and stopband property is characteristic of any periodic system, whether it be an electrical network, a periodic truss structure, a layered acoustic medium or a periodic potential along which electrons might propagate. It is important to remember that in the frequency ranges of the passbands of the perfectly periodic system there is perfect transmission of waves and energy.

The connection between the wave description and modal description for a finite structure is formally made with the phase closure principle [Cremer et al 73,Mead 75-1] and [Signorelli 87]. This principle says that at a natural frequency, the total phase change of a wave as it travels backwards and forwards once through the entire structure, including the phase changes at the boundaries, is an integral multiple of 2π. The connection between this wave description of a periodic system and a modal description becomes more apparent by noting that the natural frequencies of the periodic structure lie within the passbands. For a periodic structure of infinite extent an infinite number of natural frequencies lie densely in each passband. For an n bay periodic structure, n natural frequencies lie within each passband. (This result is strictly true only when each bay can be modeled as having symmetry of mass and stiffness about its midpoint. If the bay is unsymmetric, one frequency will occur in the stopband [Mead 75-1]).

Another property of mono-coupled periodic systems to note is the order in which the passbands and stopbands occur. For periodic systems connected to the ground, a stopband will occur first as a function of frequency followed by a passband after which the pattern is repeated. This makes sense because clearly the low frequency motion is constrained by the connection to the ground. For periodic systems not connected to the ground, this pattern is reversed, with a passband occurring first followed by a stopband and so on.

No real structure will have an infinite number of bays, but frequently a structure with a finite number of bays can mimic quite well the properties of an infinite structure, especially if it is long. But surprisingly, [Roy and Plunkett 86] note good agreement between passband/stopband properties of a theoretically infinite dissipationless beam with attached cantilevers and their experimental results for such a system with only 15 cantilevers.

2.3 Disordered Periodic Structures

Now that we have described the kinds of periodic structures of interest, and some properties they possess, we turn our attention to disordered periodic structures.

2.3.1 Nature of the Disorder

The term disorder refers to each bay of the structure having one or more of its properties departing in a random fashion from the average. We assume here that the disorder is distributed equally among all the bays and not scattered in a few. (In some literature [Toda 66] the term localization refers to the effect of disordering two well separated bays out of an otherwise perfectly periodic system. We are taking a more general definition of localization which encompasses a finite to an infinite number of disordered bays without any intervening perfectly periodic section of bays.) With this kind of disorder, the properties of the bay being disordered, whether masses, springs or lengths, can be modeled as independent identically distributed random variables. Note here that we do not model continuously disordered systems like a turbulent atmosphere [Wenzel 83] or a beam with mass that is a random function of length [Howe 72]. Rather, our disorder is discrete in that it occurs from bay to bay.

When several variables of a bay are disordered we assume that the random variables are mutually independent. Because the randomness for any variable will not be considered too large, we will make use of "narrow" uniform probability density functions from which to draw the random variables. This is also in conformity with the practice in the solid state localization literature.

2.3.2 Modeling of Disordered Periodic Structures

For the disordered periodic structure we will continue to use the transfer matrix formalism established in Section 2.2.2. For each bay now the transfer matrix, $\mathbf{T}_j$, is simply a function of one or more random variables, $\mathbf{T}_j(\alpha_1, \ldots, \alpha_q)$. See Appendix C for the random transfer matrices of our three periodic structures. Because the random variables are independent and identically distributed, so also are the random transfer matrices.

The disordered periodic structure with n bays cannot be modeled as $\mathbf{T}^n$, but is modeled as a product of random transfer matrices:

$$\prod_{j=1}^{n} \mathbf{T}_j = \mathbf{T}_n \cdots \mathbf{T}_1$$

This is the key modeling assumption of the entire thesis. We will examine one important asymptotic property of products of random matrices and deduce from that the nature of the localization phenomenon.

2.3.2.1 Wave Transfer Matrix

Because strong wave attenuation already occurs in the stopbands, our focus is on the effects of disorder in the passbands of the normally perfectly periodic structure. Unlike the case for the perfectly periodic structure, we cannot simultaneously diagonalize each $\mathbf{T}_j$ with the same eigenvector similarity transformation. However, we can transform each random transfer matrix, $\mathbf{T}_j$, forming the product into a *wave transfer matrix*, $\mathbf{W}_j$ seen in the following equation (see Appendix B):

$$\begin{bmatrix} \overleftarrow{A}_j \\ \overrightarrow{B}_j \end{bmatrix} = \begin{bmatrix} \frac{1}{t_j} & -\frac{r_j}{t_j} \\ -\frac{r_j^*}{t_j^*} & \frac{1}{t_j^*} \end{bmatrix} \begin{bmatrix} \overleftarrow{A}_{j-1} \\ \overrightarrow{B}_{j-1} \end{bmatrix} \tag{2.1}$$

where $\overleftarrow{A}$ is the amplitude of the left traveling wave and $\overrightarrow{B}$ is the amplitude of the right traveling wave.

This is a wave transfer matrix for one random bay inserted in the middle of an otherwise perfectly periodic structure carrying a pair of traveling waves. The wave amplitudes in Equation 2.1 are those supported by the periodic system surrounding the disordered bay. The transmission coefficient, t_j, is the complex amplitude of a wave emerging from the right of this random bay when a wave of amplitude 1 is incident at the left. The reflection coefficient, r_j, is the complex amplitude of the reflected wave when a wave of amplitude 1 is incident from the left. Physically, $|t_j|^2$ represents the ratio of transmitted energy to incident energy, and $|r_j|^2$ the ratio of reflected energy to incident energy. Energy conservation implies that $|t_j|^2 + |r_j|^2 = 1$.

Some readers may be more familiar with r_j and t_j appearing in a *scattering matrix.* The scattering matrix corresponding to Equation 2.1 appears in the following equation:

$$\begin{bmatrix} \overleftarrow{A}_{j-1} \\ \overrightarrow{B}_j \end{bmatrix} = \begin{bmatrix} r_j & t_j \\ t_j & r_j \end{bmatrix} \begin{bmatrix} \overrightarrow{B}_{j-1} \\ \overleftarrow{A}_j \end{bmatrix}$$

The scattering matrix relates wave amplitudes leaving a bay (which are on the left of the equation) to those entering the bay (which premultiply the scattering matrix). The disadvantage in using the scattering matrix to analyze a disordered periodic system is that it is not a transfer matrix. This means that the scattering matrix for two or more bays cannot be realized by simple multiplication of the respective scattering matrices. The scattering matrix for two or more bays is realized through a complicated "star product" described in [Redheffer 61].

We will use the wave transfer matrix precisely because it is a transfer matrix and because it allows us to model disordered periodic structures by pure matrix multiplication. So the wave transfer matrix for the n disordered bays is:

$$\prod_{j=1}^{n} \mathbf{W}_j = \begin{bmatrix} \frac{1}{\tau_n} & -\frac{\rho_n}{\tau_n} \\ -\frac{\rho_n^*}{\tau_n^*} & \frac{1}{\tau_n^*} \end{bmatrix}$$

where τ_n is the transmission coefficient of the n bay disordered system, and ρ_n is the reflection coefficient of the n bay disordered system. Here $|\tau_n|^2$ is the ratio of transmitted energy to incident energy for the disordered structure.

2.3.2.2 Properties of the Wave Transfer Matrix

The wave transfer matrix has some special properties that will be exploited to simplify our analysis of the localization phenomenon. First, because we will always use transfer matrices of determinant one to model our disordered bays (this is true because no dissipation is included in the models), the corresponding wave transfer matrix will have unit determinant. Thus the wave transfer matrix is an element of (see Appendix A) $SU(1,1)$ and $Sp(1,C)$.

Recall that the original transfer matrix, $\mathbf{T}$, was real and of unit determinant, and so was an element of the group $SL(2,R)$. What has happened in going from $\mathbf{T}$ to $\mathbf{W}$ is that we have taken advantage of an isomorphism between $SL(2,R)$ and $SU(1,1)$.

Chapter 3

Furstenberg's Theorem and Calculation of Localization Factors for Mono-Coupled Disordered Periodic Structures

3.1 Introduction

As has been discussed earlier, disordered periodic structures can be modeled via a product of random transfer matrices. In this section we will exploit the mathematical theory of products of random matrices to reveal an important transmission property of disordered periodic systems. It is precisely this transmission property that we associate with the localization phenomenon. In the chapter we will formally state Furstenberg's theorem, then restate it in more familiar terms. We then relate the localization factor to the transmission coefficient of the long disordered system, after which we will find an approximate analytical expression to calculate the localization factor.

3.2 Furstenberg's Theorem

A rigorous statement about the properties of a product of a finite number of random matrices is difficult to make; however, we can come to some rigorous conclusions on properties when the number of matrices in the product becomes very large. We will focus on one property that was originally proved in [Furstenberg 63] and which we specialize to 2×2 matrices. One formal statement of this limiting behavior of products of random matrices is as follows:

Theorem 1 (Furstenberg's Theorem, original form) *Let $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ be independent identically distributed 2×2 random matrices with distribution μ. Let G be the smallest closed subgroup of $SL(2, R)$ containing the support of μ. If G is a noncompact subgroup of $SL(2, R)$ such that no subgroup of G of finite index is irreducible and if*

$$E[max(\ln \|\mathbf{T}_j\|, 0)] < +\infty$$

then there exists $\gamma > 0$ such that for each $\mathbf{x}_0 \neq \mathbf{0}$

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\mathbf{T}_n \cdots \mathbf{T}_1 \mathbf{x}_0\| = \gamma \qquad w.p.\ 1$$

and

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\mathbf{T}_n \cdots \mathbf{T}_1\| = \gamma \qquad w.p.\ 1.$$

and if ν is a μ-invariant distribution on $P(R^2)$($P(R^2)$ is the projective space of R^2, namely half of the unit circle), then

$$\gamma = \iint \ln \|\mathbf{T}\bar{\mathbf{x}}\| d\mu(\mathbf{T}) d\nu(\bar{\mathbf{x}}) \tag{3.1}$$

where $\bar{\mathbf{x}}$ is in $P(R^2)$.

The condition of invariance for ν is stated mathematically in many ways including:

$$\int 1_A(\bar{\mathbf{x}}) d\nu(\bar{\mathbf{x}}) = \iint 1_A\left(\frac{\mathbf{T}\bar{\mathbf{x}}}{\|\mathbf{T}\bar{\mathbf{x}}\|}\right) d\mu(\mathbf{T}) d\nu(\bar{\mathbf{x}})$$

where $1_A(*)$ is the indicator function; its value is one when the argument lies on A and 0 otherwise.

There is one special direction for the initial vector $\mathbf{x}_0$ for which the Furstenberg result will not hold for a given realization of $\mathbf{T}_n \cdots \mathbf{T}_1$. Namely if $\hat{\mathbf{x}}_0$ is along this special direction, then

$$\lim_{n\to\infty} \frac{1}{n} \ln \|\mathbf{T}_n \cdots \mathbf{T}_1 \hat{\mathbf{x}}_0\| = -\gamma$$

This is a consequence of the theorem of Oseledets which will be discussed in Chapter 5.

The above theorem can be modified and restated in more familiar terms with just a few assumptions and some explanation.

As stated in Chapter 2 we are considering our random matrices to be functions of one or more random variables, where the random variables are drawn from some probability density function (Dirac delta functions are permissible in our definition of probability density functions, so probability mass functions are possible in the above). We exclude Bernoulli random variables (random variables having probability density functions with mass at only two points) in our transfer matrices because they can result in the distribution ν having neither mass nor density. The distribution for ν would be a so-called continuous singular probability measure. So now probability measures μ and ν become $p(\alpha)$ and $p(\bar{\mathbf{x}})$, respectively.

The subgroup G can now be interpreted as the set of all matrices generated by the probability density functions of the random variables plus the inverses of those matrices, plus the identity matrix, plus any products of the above matrices. The conditions concerning noncompactness and irreducibility of G have been shown by [Matsuda and Ishii 70] to be equivalent to requiring that G contain two elements in $SL(2, R)$ with no common eigenvectors. In addition, [Goda 82] has shown that the Furstenberg result will hold for matrices in $GL(2, R)$ as long as

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\prod_{j=1}^{n} |\det \mathbf{T}_j|\right) = 0 \qquad (3.2)$$

Finally, the theorem will also hold for matrices with complex entries [Bougerol and Lacroix 85].

Now that we have clarified some of the conditions under which Furstenberg's theorem holds, let us examine why the Furstenberg result, Equation 3.1, is reasonable. As the deterministic vector $\mathbf{x_0}$ is propagated by the random matrices, its direction, $\bar{\mathbf{x}}$, begins to take on a probability density of its own. In fact as $n \to \infty$, the probability density of this direction becomes invariant with respect to the probability density for the random transfer matrices. Specifically, the invariance condition means if

$$\mathbf{x}_n = \mathbf{T}_n \mathbf{x}_{n-1}$$

then as $n \to \infty$

$$p(\bar{\mathbf{x}}_n) = p(\bar{\mathbf{x}}_{n-1})$$

This condition of invariance is frequently called the Dyson-Schmidt self-consistency condition in the solid state physics literature [Ziman 79]. This condition of invariance does not hold for systems in two or three dimensions or for closed periodic structures in one dimension [Ziman 79, page 309]. Therefore, as $n \to \infty$ the two relevant probability distributions are those for $\mathbf{T}$ and $\bar{\mathbf{x}}$, and the double integral of $\ln \|\mathbf{T}\bar{\mathbf{x}}\|$ over these two distributions seems reasonable.

With these points in mind, we can restate Furstenberg's theorem as follows:

Theorem 2 (Furstenberg's Theorem, modified form) *Let* $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n$ *be complex valued, invertible, independent identically distributed* 2×2 *matrices where* $\mathbf{W}_j = \mathbf{W}_j(\alpha)$ *is a function of the random vector* α *with probability density* $p(\alpha)$*. If at least two of the random transfer matrices do not have common eigenvectors, and if*

$$\lim_{n\to\infty} \frac{1}{n} \ln(\prod_{j=1}^{n} |\det \mathbf{W}_j|) = 0$$

and if

$$E[\max(\ln \|\mathbf{W}_j\|, 0)] < +\infty$$

then there exists $\gamma > 0$ *such that for each* $\mathbf{z}_0 \neq 0$

$$\lim_{n\to\infty} \frac{1}{n} \ln \|\mathbf{W}_n \cdots \mathbf{W}_1 \mathbf{z}_0\| = \gamma \qquad w.p.\ 1$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln \|\mathbf{W}_n \cdots \mathbf{W}_1\| = \gamma \qquad w.p.\ 1$$

where

$$\gamma = \iint \ln \|\mathbf{W}(\alpha)\bar{\mathbf{z}}\| p(\alpha) d\alpha p(\bar{\mathbf{z}}) d\bar{\mathbf{z}} \tag{3.3}$$

where $p(\bar{\mathbf{z}})$ *is invariant with respect to the probability density function* $p(\alpha)$ *for the random transfer matrices, i.e.*

$$\int 1_A(\bar{\mathbf{z}}) p(\bar{\mathbf{z}}) d\bar{\mathbf{z}} = \iint 1_A\left(\frac{\mathbf{W}(\alpha)\bar{\mathbf{z}}}{\|\mathbf{W}(\alpha)\bar{\mathbf{z}}\|}\right) p(\alpha) d\alpha p(\bar{\mathbf{z}}) d\bar{\mathbf{z}}$$

where A *is any arc along the half unit circle.*

A number of other properties for products of random matrices can be shown [Bougerol and Lacroix 85]; however, Furstenberg's theorem gives the one property which, as we will see, is relevant to localization in a disordered periodic system.

Furstenberg's theorem is a law of large of numbers for products of random matrices. More recently a central limit theorem [Le Page 82,Bougerol and Lacroix 85] has been proved for products of random matrices. The central limit theorem tells us that

$$\frac{1}{\sqrt{n}}(\ln \|\mathbf{W_n} \cdots \mathbf{W_1}\| - n\gamma) \stackrel{distribution}{\longrightarrow} N(0, \sigma^2)$$

The conditions on the random matrices are a little more restrictive than the ones for Furstenberg's theorem, but determining whether they apply to the transfer matrices considered here is left for future research.

3.3 Localization Factor as a Function of the Transmission Coefficient

Now we will relate the Furstenberg limit theorem for products of random matrices to a transmission property of disordered periodic systems. From Furstenberg's theorem we know

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \ln \|\mathbf{W}_n \cdots \mathbf{W}_1\| \quad w.p.\ 1 \tag{3.4}$$

Recall that a product of n wave transfer matrices is of the form:

$$\prod_{j=1}^{n} \mathbf{W}_j = \begin{bmatrix} \frac{1}{\tau_n} & -\frac{\rho_n}{\tau_n} \\ -\frac{\rho_n^*}{\tau_n^*} & \frac{1}{\tau_n^*} \end{bmatrix} \tag{3.5}$$

To apply Equation 3.4 we first take a matrix norm of Equation 3.5. Here we choose the maximum singular value (see Appendix A); so a little algebra gives:

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \ln(\frac{1+|\rho_n|}{|\tau_n|})$$

or

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \ln(1+|\rho_n|) - \frac{1}{n} \ln|\tau_n|$$

Knowing that $0 \leq |\rho_n| < 1$, the first term vanishes, and we are left with

$$\gamma = -\lim_{n\to\infty} \frac{1}{n} \ln|\tau_n| \tag{3.6}$$

Now we can understand the relevance of γ to the dynamic properties of a disordered periodic structure. Asymptotically, Equation 3.6 says that the absolute value of the transmission coefficient decays exponentially with n, the number of bays. The rate of decay per bay is governed by γ which will be called the localization factor. Thus traveling waves will no longer be propagated perfectly, but will tend to be confined near their point of origin according to the localization factor γ. This result says that $|\tau_n|^2 \sim e^{-2\gamma n}$, the transmitted energy decays exponentially with n. It has been argued

[Matsuda and Ishii 70,Pastawski et al 85] that the now spatially localized modes are governed by an exponential envelope of the form $e^{-\gamma n}$.

We observe that $\ln|\tau_n|$ is a statistically well behaved variable, namely we have derived an asymptotic relation for it based on a law of large numbers for products of random matrices. Notice also that we are not taking an expectation of $\ln|\tau_n|$ to find γ; the result holds as $n \to \infty$. Random variables with this property are called self-averaging [Pastur 80,van Hemmen 82].

The notion that the $\ln|\tau_n|$ is statistically well-behaved is further strengthened if one applies these same manipulations to the central limit theorem for products of random matrices. Thus using the available mathematical tools, we confirm in just a few steps the conjecture about the statistical behavior of $\ln|\tau_n|$ by [Anderson et al 80] and [Stone et al 81].

3.4 Calculation of Localization Factors via an Approximation to Furstenberg's Theorem

In this section we will simplify Equation 3.3 of Furstenberg's Theorem; this will lead to an approximation for the localization factor, γ. First recall Equation 3.3

$$\gamma = \int\int \ln \|\mathbf{W}(\alpha)\bar{\mathbf{z}}\| p(\alpha) d\alpha p(\bar{\mathbf{z}}) d\bar{\mathbf{z}}$$

then without loss of generality we have:

$$\bar{\mathbf{z}} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\theta} \\ e^{-i\theta} \end{bmatrix}$$

$$\mathbf{W}(\alpha) = \begin{bmatrix} \frac{1}{t} & -\frac{r}{t} \\ -\frac{r^*}{t^*} & \frac{1}{t^*} \end{bmatrix}$$

where for the moment we suppress the dependence of t and r on α. So,

$$\mathbf{W}(\alpha)\bar{\mathbf{z}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{e^{i\theta}}{t} - \frac{re^{-i\theta}}{t} \\ -\frac{r^* e^{i\theta}}{t^*} + \frac{e^{-i\theta}}{t^*} \end{bmatrix}$$

After some complex algebra we find:

$$\|\mathbf{W}(\alpha)\bar{\mathbf{z}}\| = |\frac{1}{t} - \frac{r}{t}e^{-i2\theta}|$$

Now the equation for γ is

$$\gamma = \iint \ln |\frac{1}{t} - \frac{r}{t}e^{-i2\theta}| p(\alpha) d\alpha p(\theta) d\theta \tag{3.7}$$

where $p(\theta)d\theta$ must satisfy the invariance condition:

$$\int 1_A(\theta)p(\theta)d\theta = \iint 1_A(\frac{\mathbf{W}(\alpha)\bar{\mathbf{z}}(\theta)}{\|\mathbf{W}(\alpha)\bar{\mathbf{z}}(\theta)\|})p(\alpha)d\alpha p(\theta)d\theta \tag{3.8}$$

Because $p(\theta)$ can only be found in rare instances [Pincus 80], we will find an approximation to γ by taking a Taylor series expansion about $< \alpha >$, recalling that α is a vector, of the terms in Equations 3.7 and 3.8 and retaining terms to first order in σ_α^2. This approach has been discussed in [Baluni and Willemsen 85]. Let us first recall the form of the Taylor series expansion for a multivariable function. The first three terms are:

$$\begin{aligned} f(\alpha) &= f(\alpha)|_{<\alpha>} + \sum_{l=1}^{q}(\alpha_l - <\alpha_l>)\frac{\partial f(\alpha)}{\partial \alpha_l}|_{<\alpha>} + \\ &\quad \frac{1}{2}\sum_{l=1}^{q}\sum_{l=1}^{q}(\alpha_l - <\alpha_l>)(\alpha_i - <\alpha_i>)\frac{\partial^2 f(\alpha)}{\partial \alpha_l \partial \alpha_i}|_{<\alpha>} + \cdots \end{aligned}$$

We now examine the expansion of $\ln |\frac{1}{t(\alpha)} - \frac{r(\alpha)}{t(\alpha)}e^{-i2\theta}|$. The first term in the expansion is simply that for the undisordered or perfectly periodic system. Recall that

$$\begin{bmatrix} \frac{1}{t} & -\frac{r}{t} \\ -\frac{r^*}{t^*} & \frac{1}{t^*} \end{bmatrix} = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix}$$

for the perfectly periodic system in the passband. Therefore the first term is:

$$\ln |e^{ik}| = 0$$

The second term in the expansion will not be needed because the terms $(\alpha_l - <\alpha_l>)$ vanish when integrated over $p(\alpha)$.

Finally, the third term is examined. Because the terms in the random vector are mutually independent, we know that $(\alpha_l - <\alpha_l>)(\alpha_i - <\alpha_i>)$ $i \neq l$ will vanish after integrating over $p(\alpha)$. We are left with

$$\frac{1}{2}\sum_{l=1}^{q}(\alpha_l - <\alpha_l>)^2 \frac{\partial^2 \ln|\frac{1}{t(\alpha)} - \frac{r(\alpha)}{t(\alpha)}e^{-i2\theta}|}{\partial\alpha_l^2}|_{<\alpha>}$$

So γ to first order in the variance of the $\alpha_l s$ is:

$$\gamma = \frac{1}{2}\sum_{l=1}^{q}\sigma_{\alpha_l}^2 \int \frac{\partial^2 \ln|\frac{1}{t(\alpha)} - \frac{r(\alpha)}{t(\alpha)}e^{-i2\theta}|}{\partial\alpha_l^2}|_{<\alpha>} p^0(\theta)d\theta \tag{3.9}$$

where we now must find $p^0(\theta)$ which is $p(\theta)$ to the zeroth order in the variance of α.

To find $p^0(\theta)$ we examine Equation 3.8 where we only look at terms to zeroth order in σ_α^2, namely:

$$\int 1_A(\theta)p^0(\theta)d\theta = \int\int 1_A(\frac{\mathbf{W}(\alpha)\bar{\mathbf{z}}(\theta)}{\|\mathbf{W}(\alpha)\bar{\mathbf{z}}(\theta)\|}|_{<\alpha>})p(\alpha)d\alpha p^0(\theta)d\theta$$

but

$$\mathbf{W}(\alpha)\bar{\mathbf{z}}|_{<\alpha>} = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\theta} \\ e^{-i\theta} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i(k+\theta)} \\ e^{-i(k+\theta)} \end{bmatrix}$$

We therefore require $p^0(\theta)$ to satisfy

$$\int 1_A(\theta)p^0(\theta)d\theta = \int 1_A(k+\theta)p^0(\theta)d\theta$$

Because k can take on any value between 0 and π, or some multiple thereof, we must have that $p^0(\theta) = \frac{1}{\pi}$, which is a uniform probability density function.

To further simplify Equation 3.9 we note that

$$\ln|\frac{1}{t(\alpha)} - \frac{r(\alpha)}{t(\alpha)}e^{-i2\theta}| = \ln|\frac{1}{t(\alpha)}| + \ln|1 - r(\alpha)e^{-i2\theta}|$$

The term $\ln|1 - r(\alpha)e^{-i2\theta}|$ can be expanded in a series, and recalling $e^{-i2\theta} = \cos 2\theta - i\sin 2\theta$, the term vanishes after integrating.

Therefore we are left with

$$\gamma = \frac{1}{2}\sum_{l=1}^{q}\sigma_{\alpha_l}^2 \frac{\partial^2 \ln\left|\frac{1}{t(\alpha)}\right|}{\partial\alpha_l^2}\Big|_{<\alpha>} + o(\sigma_\alpha^2)$$

or

$$\gamma = -\frac{1}{2}\sum_{l=1}^{q}\sigma_{\alpha_l}^2 \frac{\partial^2 \ln|t(\alpha)|}{\partial\alpha_l^2}\Big|_{<\alpha>} + o(\sigma_\alpha^2)$$

This is also equivalent to

$$\gamma = \frac{1}{2}\sum_{l=1}^{q}\sigma_{\alpha_l}^2 \frac{\partial^2 \ln\left|\frac{1}{t(<\alpha\acute{>})}\right|}{\partial\alpha_l^2}\Big|_{<\alpha_l>} + o(\sigma_\alpha^2) \tag{3.10}$$

where

$$<\alpha\acute{>} = \begin{bmatrix} <\alpha_1> \\ \vdots \\ <\alpha_{l-1}> \\ \alpha_l \\ <\alpha_{l+1}> \\ \vdots \\ <\alpha_q> \end{bmatrix}$$

The prime indicates that all but the lth term is evaluated at the mean value. This latter result says we can calculate localization effects by disordering one variable at a time in a transfer matrix.

Notice that the localization factor to first order in the variance is simply a sum of the localization factors for each variable randomized individually. We suspect that as the variance of the disordered variables increases the estimate of γ will be poorer because we have retained terms only to first order in the variances.

We also note that Furstenberg's theorem has been shown to be robust to uncertainty in the probability law of the random transfer matrices The paper [Slud 86] shows that if the postulated probability measure for the transfer matrices is "close" to the actual one then the asymptotic behaviors will be arbitrarily close.

A technique to approximately calculate localization factors without resorting to theories on products of random matrices is presented in Appendix D.

Chapter 4

Calculation of Localization Factors for Three Mono-Coupled Disordered Periodic Structures

4.1 Introduction

This chapter will illustrate localization calculations for three periodic structures that can be modeled with 2×2 transfer matrices. The results will show dramatically how localization effects can vary with frequency. The analytical results are compared to Monte Carlo simulations of these systems. We provide, where possible, a physical explanation for the observed localization effects.

The first system examined is a chain of spring and masses. This simplest possible system provides a convenient vehicle to illustrate the calculation of localization effects. Indeed, this thesis provides the first comprehensive examination of the localization effects of a disordered mass-spring chain.

The second example is a rod in longitudinal compression with attached resonators which mimics some of the important dynamic behavior of a real truss structure. Unlike the mass-spring system which has only a single passband, the rod with resonators has an infinite number of passbands. We examine localization effects over several of these passbands.

The final example is a Bernoulli-Euler beam on simple supports. When we disorder the distances between the supports we will see a very pronounced effect near the stopbands of the underlying periodic structure.

In our analysis we will consider the random variables, α_l disordered $\pm p\%$ from the average value $< \alpha_l >$. A disorder of $\pm p\%$ from the average value $< \alpha_l >$ translates into a uniform probability density function with width of $\frac{2p<\alpha_l>}{100}$ and height of $\frac{100}{2p<\alpha_l>}$. The uniform probability density function will be centered around $< \alpha_l >$. Note that the variance of any random variable with a uniform probability density function is always $\frac{width^2}{12}$.

4.2 Localization in a Mass-Spring Chain

We will examine at length the localization effects in a chain of springs and masses. The mass-spring chain is an excellent example to begin our discussion of localization, not only because of its simplicity, but also because this system and its analogs have been studied over the years, giving us the opportunity to directly compare our results with those already published. Even though the mass-spring chain and its equivalents have received a lot of attention in the literature, amazingly it has not received exhaustive treatment. For example, in the literature the chain is examined with only the mass disordered and the localization factor calculated is generally valid over only the first half of the passband. In this thesis we will study localization in this chain where masses and springs are disordered and the equation we present for the localization factor will be valid over virtually the entire passband.

In the next sections we will first examine the chain with only masses disordered and present our analytical approximation of the localization factor based on Equation 3.10. These results will be confirmed by a Monte Carlo simulation. We will also compare this analytical result with the one usually found in the literature. The localization factor will be studied for three levels of disorder: masses with a .1%,1% and 10% variation above or below the nominal value. We will see that the localization factor depends only on frequency and the level of disorder. Throughout the thesis any such dependencies are suppressed when writing the localization factor, γ.

Next we examine the chain with only springs disordered and show that this disorder is dual to the mass disorder. Finally both masses and springs are disordered and we again confirm the analytical results with a Monte Carlo simulation.

4.2.1 Only Masses Disordered

We first examine a chain with disordered masses, which in the physics literature is identified as isotopic disorder, referring to atomic systems with various isotopes. This chain with masses disordered has been examined in [Matsuda and Ishii 70,Rubin 84]; its electrical circuit analog was studied in [Akkermans and Maynard 84], and the solid state analog in [Stone et al 81] and elsewhere.

The mass-spring model and its transfer matrix are presented in Appendix C.1. We make use of a nondimensional frequency, $\bar{\omega}$, in the transfer matrix and our analysis :

$$\bar{\omega} = \frac{\omega}{2\sqrt{\frac{k_t}{m}}}$$

The condition for the existence of a passband (see Chapter 2) tells us that only one passband exists and occurs for

$$0 < \bar{\omega} < 1$$

In the passband waves and energy travel with perfect transmission. However when the system is disordered, the transmission is disrupted; and the resulting disruption in the passband is what we are examining.

In our analysis we will first consider masses disordered from their average value on the chain. We make use of nondimensional quantities wherever possible. Here the nondimensional mass is μ_j, where

$$\mu_j = \frac{m_j}{m} \quad and \quad m =< m_j >$$

and m is the mass for the perfectly periodic system. The transfer matrix for one bay of this chain with a disordered mass is shown in Appendix C.1 and the $(1,1)$ term of that transfer matrix is $\frac{1}{t(\mu_j)}$, Equation C.1, which can be used in Equation 3.10 to calculate the localization factor, γ_μ. For this mass-spring system with masses disordered we will go through the calculation of the localization factor; this will serve as an example of the steps necessary to do the calculation for any disordered system. Equation 3.10 now becomes

$$\gamma_\mu \doteq \frac{1}{2}\sigma_\mu^2 \frac{\partial^2 (ln|\frac{1}{t(\mu_j)}|)}{\partial \mu_j^2}|_{<\mu_j>}$$

where $\doteq$ indicates we are neglecting terms of order greater than the variance. From Appendix C.1.1 we have

$$\frac{1}{t(\mu_j)} = e^{ik}(1 - i\delta_j)$$

where

$$\delta_j = \frac{2\bar{\omega}^2(1-\mu_j)}{\sin k}$$

and where

$$\cos k = 1 - 2\bar{\omega}^2$$

Suppressing the subscript j, we now have

$$\gamma_\mu \doteq \frac{1}{4}\sigma_\mu^2 \frac{\partial^2 \ln(1+\delta^2)}{\partial \mu^2}|_{<\mu>}$$

Letting

$$\delta' = \frac{\partial \delta}{\partial \mu} = \frac{-2\bar{\omega}^2}{\sin k}$$

we have the first partial derivative

$$\frac{\partial \ln(1+\delta^2)}{\partial \mu} = \frac{2\delta\delta'}{1+\delta^2}$$

Taking a partial derivative again we have

$$\frac{\partial^2 \ln(1+\delta^2)}{\partial \mu^2} = -\frac{2\delta\delta'}{(1+\delta^2)^2} + \frac{2\delta'}{1+\delta^2} + \frac{2\delta\delta''}{1+\delta^2}$$

We have to evaluate the above terms at $< \mu >$. Note that δ evaluated at $< \mu >$ is zero, so now we have

$$\gamma_\mu \doteq \frac{1}{4}\sigma_\mu^2 2\delta'^2|_{<\mu>}$$

or

$$\gamma_\mu \doteq \frac{2\bar{\omega}^4\sigma_\mu^2}{\sin^2 k}$$

Knowing $\cos k$ from above, we can calculate $\sin^2 k$ and so finally for the mass-spring system in the passband, $0 < \bar{\omega} < 1$:

$$\gamma_\mu \doteq \frac{\bar{\omega}^2\sigma_\mu^2}{2[1-\bar{\omega}^2]} \tag{4.1}$$

We observe that the localization factor is a function of the nondimensional frequency $\bar{\omega}$ and the variance of the nondimensional mass. Clearly the localization effects increase with frequency and also with the amount of disorder. At low frequency

$$\gamma_\mu \doteq \frac{\bar{\omega}^2\sigma_\mu^2}{2} \quad (\bar{\omega} \to 0) \tag{4.2}$$

indicating that γ_μ is proportional to $\bar{\omega}^2$ at low frequency. The low frequency estimate of the localization factor for a chain with disordered masses is the one usually seen in the literature [Matsuda and Ishii 70], but in the following dimensional form(and derived through much more torturous methods than are used here):

$$\gamma_m \doteq \frac{\omega^2\sigma_m^2}{8k_s m} \quad (\omega \to 0)$$

The nondimensional analytical results of Equation 4.1 and Equation 4.2 for masses disordered $\pm.1\%$ from the average value are plotted in Figure 4.1 with the nondimensional frequency as the abscissa and $\log_{10}(\gamma_\mu)$ as the ordinate.

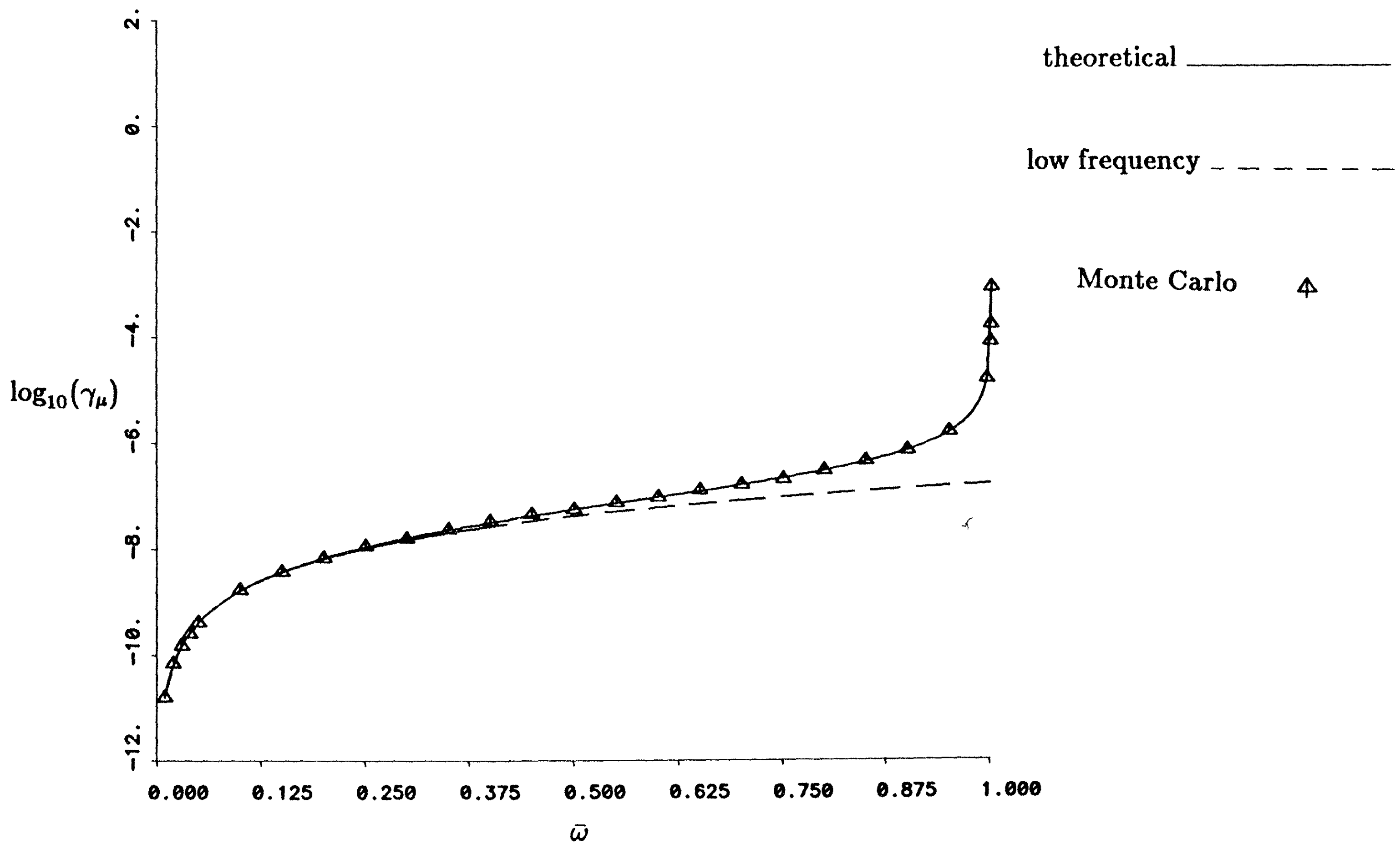

Figure 4.1: Localization factor for mass-spring chain with masses disordered $\pm.1\%$ from their average value.

The frequency dependence of γ can now be seen explicitly. Note the rising value of γ as it approaches the stopband near $\bar{\omega} = 1$. This makes physical sense, because the masses are vibrating at higher frequencies as we approach the end of the passband, so we expect that disorder will have a greater impact than it would for masses vibrating at lower frequencies. The dashed line represents the nondimensional low frequency estimate of γ_μ. Clearly it provides an adequate estimate of γ_μ for about half the passband, while it grossly underestimates the localization effects at the highest frequencies of the passband.

As an example, let us consider the effects of localization at $\bar{\omega} = .9995$, where $\gamma_\mu = .1665 \times 10^{-3}$. This result tells us that on average the transmitted energy, $|\tau_n|^2$, after 1000 bays will be $e^{-2.1665\times10^{-3}1000} = .72$ of the incident energy even though no damping is present. A modal amplitude for a normal mode at that frequency would be confined to an exponential envelope governed by $e^{-\gamma n}$ with $\gamma = .1665 \times 10^{-3}$. The localization will be less pronounced at lower frequencies, but is nonetheless present. The attenuation caused by the disorder is unlike that of dissipation. Here localization prevents the wave from traveling along the structure, unlike the case for a perfectly periodic system, where the wave would travel without attenuation. Localization tends to confine the wave near its point of origin, where it is eventually dissipated by the damping that inevitably exists in all real structures.

Our localization result in Equation 4.2 is one-half the result[1] presented by [Chow and Keller 72]. In their work they calculated the effects of randomness on the coherent portion of waves traveling through a random chain. We can reproduce their results with the aid of Appendix D. If in Appendix D we proceed to find the mean value of τ_n and then take the natural log, instead of averaging $\ln|\tau_n|$ directly, we will get twice the result of Equation 4.2. Clearly, they are averaging the wrong variable, τ_n. In addition to making the statistical arguments about averaging of proper variables, we can

[1]Note that the relevant result in [Chow and Keller 72] has a typographical error on the bottom of page 1412. It should read $Imk(\omega, \epsilon^2) = \frac{\epsilon^2\omega^2<\mu^2>}{d\omega_0^2}$.

make the following physical argument to explain their results. By examining only the coherent or mean wave, as was also done by [Eatwell 83], they really neglect the incoherent portion of the wave which can also carry energy. When we average over $\ln|\tau_n|$ we are taking into account all the energy transmitted, because by definition $|\tau_n|$ is the ratio of transmitted energy to incident energy. Other authors have pointed out the invalidity of averaging other quanitities like $|\tau_n|$ [Hodges 82], ρ_n [Baluni and Willemsen 85] and $\frac{|\rho_n|^2}{|\tau_n|^2}$ [Stone et al 81].

The validity of the analytical result should be verified by some numerical simulation. Specifically, we want to see whether the analytical result is valid for the entire frequency range of interest and for increasing levels of disorder. The obvious simulation is to multiply a huge number of random transfer matrices at a given frequency to see if indeed

$$\gamma = -\lim_{n\to\infty} \frac{1}{n} \ln|\tau_n|$$

Because we cannot really take an infinite number of products, we must resort to averaging $\ln|\tau_n|$ over an ensemble of realizations of the chain. The question arises whether to use a large number of matrices per ensemble or a large number of ensembles and a few matrices. Upon examining this issue numerically, we found that we did not even have to take a product of random matrices to get Monte Carlo results that matched our analytical results. Rather, averaging $\ln|t_j|$ from an individual matrix over a sufficient number of realizations (in our case 1001) gave excellent agreement with the analytical results over large frequency ranges. A similar observation was made by [Pastawski et al 85]. The agreement was good in the sense that the mean value of the Monte Carlo simulation tracked our analytical results well (as can be seen in the numerous figures), but also in the sense that the standard error was consistently one to two orders of magnitude smaller than the mean value.

Recall that in a Monte Carlo simulation [Hammersley and Handscomb 64] the unbiased estimator of the mean value is

$$< \ln |t_j| >= \frac{1}{r}\sum_{j=1}^{r} \ln |t_j|$$

where r is the number of realizations in the simulation. We estimate the standard error as:

$$standard\ \ error = \frac{s}{\sqrt{r}}$$

where s^2 is the unbiased estimator of the variance:

$$s^2 = \frac{1}{r-1}\sum_{j=1}^{r}(\ln |t_j| - < \ln |t_j| >)^2$$

Again the standard error was one to two orders of magnitude smaller than the mean value for the simulations of all three structures. The results of the Monte Carlo simulation are indicated with the small triangles in Figure 4.1. They confirm the validity of the analytical result over the entire passband.

Now we choose to increase the disorder in the masses so that they vary ±1% from some nominal value, which means that the uniform probability density function has width of .02. Examining Equation 4.1, we would simply expect our localization factor to be scaled by the new σ_μ^2 compared to the previous result. Indeed this is what we confirm with our simulation illustrated in Figure 4.2.

Finally we examine the chain with a 10% variation in the masses. Such a highly disordered state would probably not occur through unintentional assembly or manufacturing error, but rather we look at this highly disordered situation to see if the theory accurately predicts the localization effect. Because of the increased disorder we will clearly have greater localization, as is pictured in Figure 4.3.

Notice, however, at high frequency that our theoretical result overpredicts the localization factor. For example, at $\bar{\omega} = .9999$, at the very edge of the passband, $\gamma_{MonteCarlo} = .1178$ and $\gamma_{theoretical} = 8.332$. This discrepancy can probably be attributed to the neglecting of higher order terms in our Taylor series expansion performed in

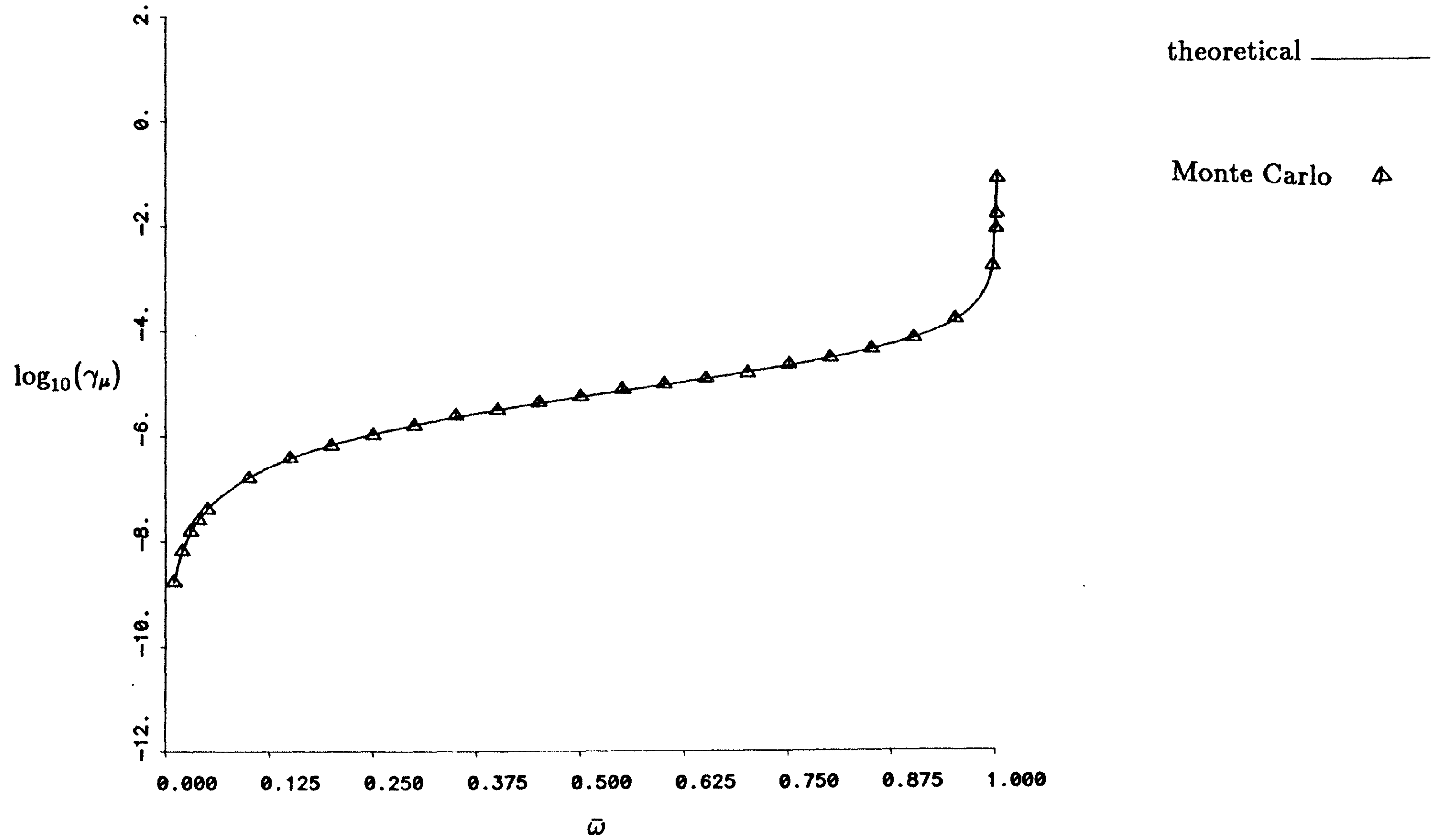

Figure 4.2: Localization factor for mass-spring chain with masses disordered ±1% from their average value.

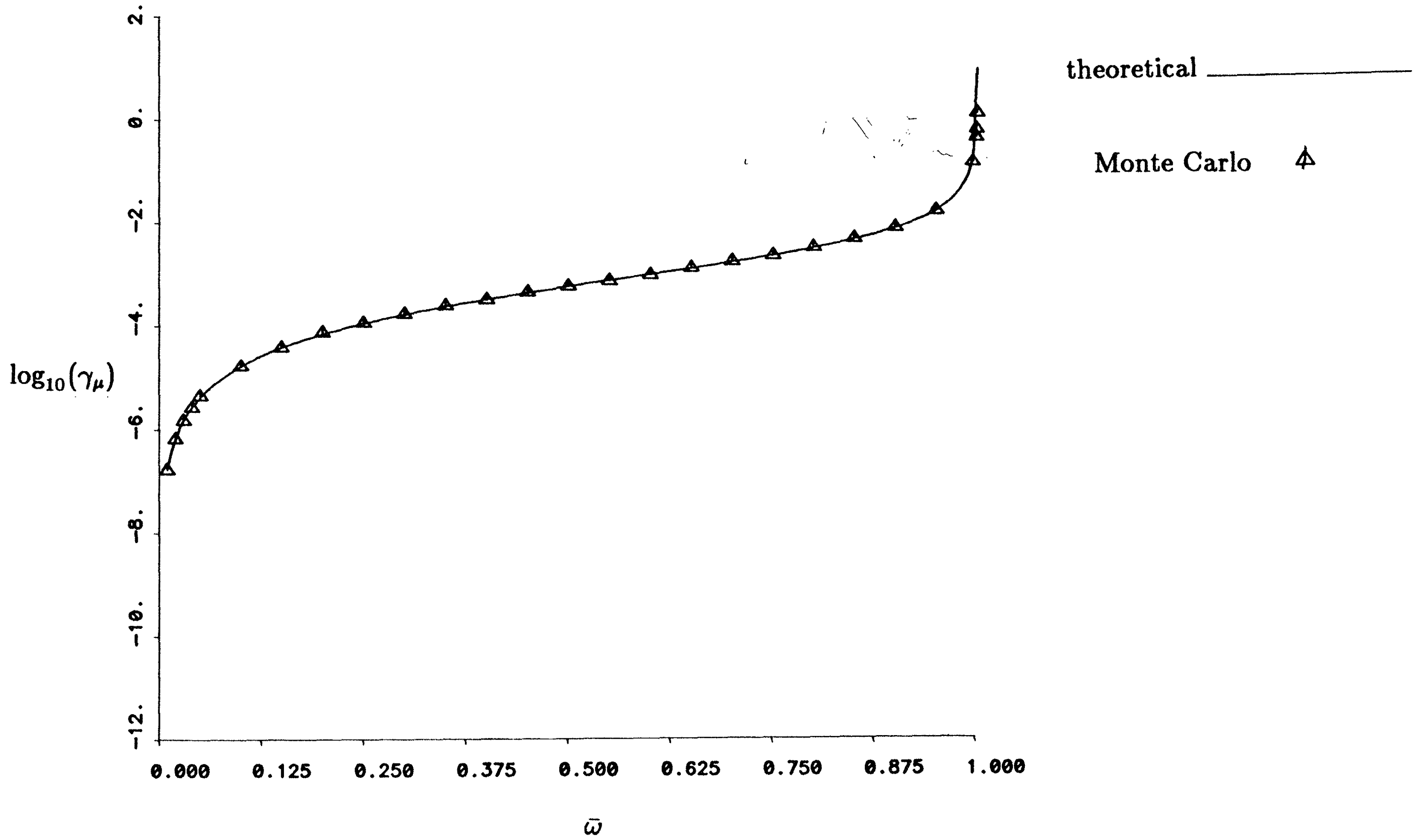

Figure 4.3: Localization factor for mass-spring chain with masses disordered ±10% from their average value.

Chapter 3. These higher order terms could become significant as we approach the edge of the passband ($\bar{\omega} = 1$), just as the term to first order in σ_μ^2 in Equation 4.1 becomes significant as $\bar{\omega} \to 1$.

4.2.2 Only Springs Disordered

Consider a chain in which every mass is exactly the same, but each spring, k_s, varies in a random fashion from some average value. This localization problem for the mass-spring system has been rarely discussed in the literature. One researcher, [Goda 82] (citing [Toda 66]) argues that the localization problem with only springs disordered is exactly dual to the localization problem with only masses disordered. Our calculations support this contention. Duality [Toda 66] here means that each mass of a mass-spring system can be replaced by a certain spring and each spring can be replaced by a certain mass such that the new system behaves in the same way as the old and in particular has the same natural frequency.

To examine the problem, we begin with the transfer matrix for the chain with only springs disordered, which is in Appendix C.1. Here $\tilde{k}_{sj}$ is the nondimensional spring constant. Identifying $\frac{1}{t(\tilde{k}_{sj})}$ in Appendix C.1, we again use Equation 3.10 to calculate $\gamma_{\tilde{k}_s}$ and find:

$$\gamma_{\tilde{k}_s} \doteq \frac{\bar{\omega}^2 \sigma_{\tilde{k}_s}^2}{2[1 - \bar{\omega}^2]}$$

So indeed this is the same as Equation 4.1 with σ_μ^2 replaced by $\sigma_{\tilde{k}_s}^2$, and confirms Goda's contention that the localization problem with masses disordered is dual to that with springs disordered. This means that all the localization results displayed in Figures 4.1-4.3 will apply to the problem of springs disordered by simply replacing the word mass by the word spring.

4.2.3 Masses and Springs Disordered

Finally we consider the situation where both masses and springs are disordered. As was stated in Chapter 3 the localization factor for this situation is simply the sum of the individual localization factors when a single variable is randomized. So

$$\gamma_{\mu\tilde{k}_s} = \gamma_\mu + \gamma_{\tilde{k}_s} \doteq \frac{\bar{\omega}^2(\sigma_\mu^2 + \sigma_{\tilde{k}_s}^2)}{2[1-\bar{\omega}^2]}$$

We check this result with a Monte Carlo simulation in which both masses and springs are randomly varied form their average values by $\pm 1\%$. The Monte Carlo results again track the analytical results. See Figure 4.4.

Before closing this section, one final note is in order. When [Goda 82] originally considered the localization problem of masses and springs disordered (without solving for $\gamma_{\mu\tilde{k}_s}$), his transfer matrix did not have unit determinant, so he knew he could not use Furstenberg's original theorem which requires unit determinant for the random matrix. As a consequence he spent most of the paper proving that the Furstenberg result will hold even if the determinant is not unity, so long as Equation 3.2 is satisfied. Apparently Goda was not aware that the transfer matrix could be reformulated so that even when both masses and springs were disordered the transfer matrix would still have unit determinant. The transfer matrix Goda used had the state vector containing two adjacent generalized displacements:

$$\begin{bmatrix} d_n \\ d_{n-1} \end{bmatrix}$$

while the state vector we use contains a generalized displacement and a generalized nondimensional force at the same point.

$$\begin{bmatrix} d_{n-1} \\ \bar{f}_{n-1} \end{bmatrix}$$

resulting in a unit determinant transfer matrix.

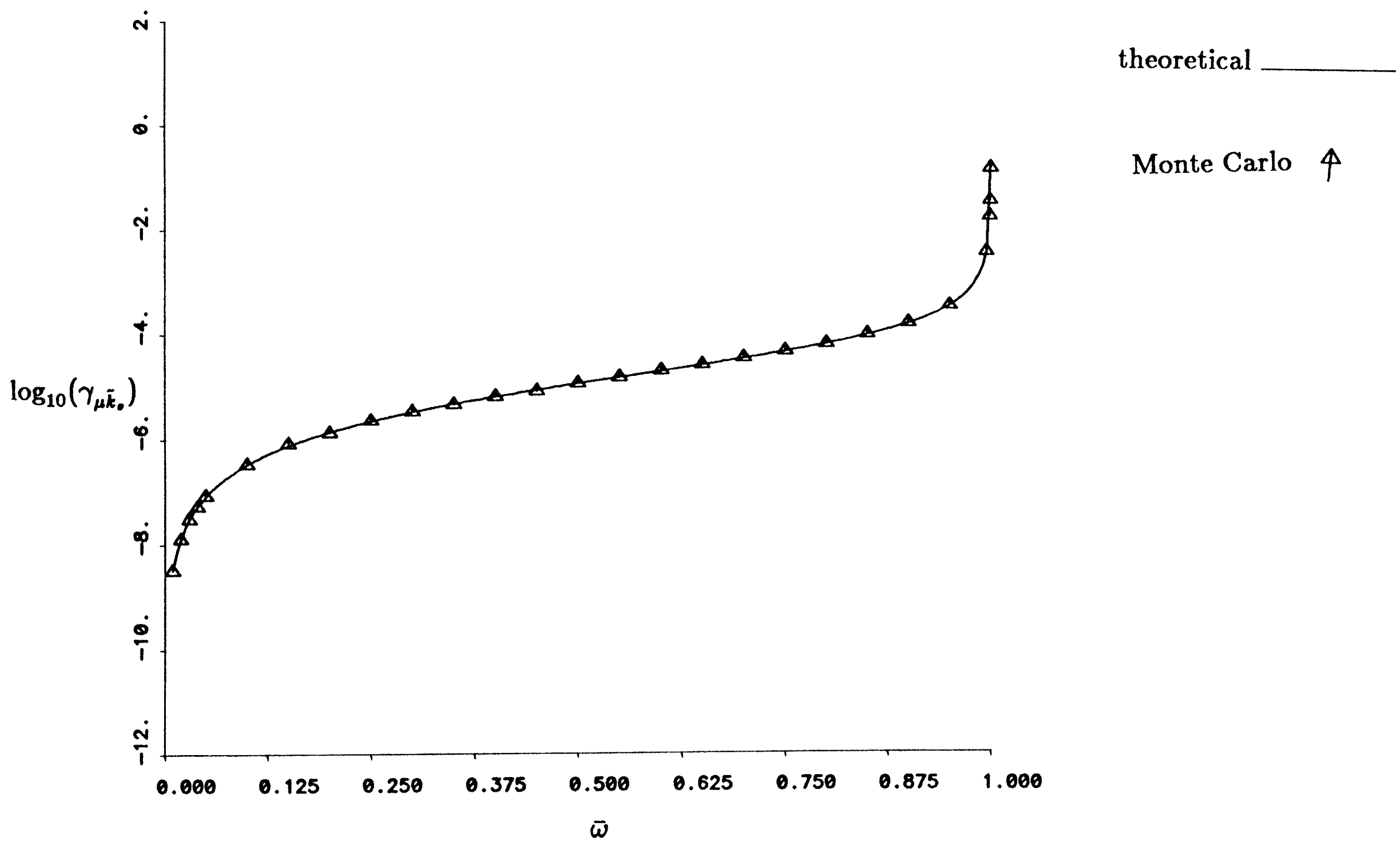

Figure 4.4: Localization factor for mass-spring chain with masses and springs disordered $\pm 1\%$ from their average values.

4.3 Localization in a Rod with Attached Resonators

In this section we investigate localization factors for a model proposed by [von Flotow 82] which mimics some of the important behavior of a periodic truss structure. The model is a longitudinal wave carrying rod with attached resonators, where the attached resonators represent the vibrating cross-members present in a real truss structure and the continuous rod models compression, bending, shear or any continuous deformation of the truss member. This simple model allows us to gain some insight into the dynamic behavior of truss structures without having to deal with models of real truss members involving transfer matrices of dimensions possibly 12×12 or greater. The model and relevant properties are discussed in Appendix C.2.

We will explore the localization phenomenon when the attached masses, the attached springs and the distances between the attached resonators are individually disordered. Finally we examine the system when all three variables are disordered. Our results will indicate that the most pronounced localization effects will occur at frequencies near the stopbands.

4.3.1 Only Masses Disordered

We first consider disordering only the masses on the attached resonators and evaluate the effects on the transmission properties of the system. The transfer matrix and wave transfer matrix when the attached masses are disordered are presented in Appendix C.2. Note our use of the nondimensional mass, $\bar{\mu}_j$, where $< \bar{\mu}_j >= \bar{\mu}$. In all of our examples for the rod with attached resonators $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$. These values allow for the ease of presentation of results and are consistent with [von Flotow 82].

Now we use the equation for $\frac{1}{t(\bar{\mu}_j)}$, appearing in Appendix C.2, in Equation 3.10 and find that

$$\gamma_{\bar{\mu}} = \frac{(\sin^2 \pi\bar{\omega})\sigma_{\bar{\mu}}^2}{8(\sin^2 k)(\pi\bar{\omega})^6\bar{\mu}^4((1/\bar{k}_s) - (1/\bar{\omega}^2\pi^2\bar{\mu}))^4} \tag{4.3}$$

Clearly the dependence of $\gamma_{\bar{\mu}}$ on frequency is much more complicated than we found for the simple mass-spring system. An analysis of the localization factor shows that it is proportional to $\bar{\omega}^2$ at very low frequency, as was the case for the mass-spring chain.

We now examine the localization factor over the frequencies of the passbands of the periodic system. Our first analytical and numerical results are for the masses disordered $\pm.1\%$ from the average value of $\bar{\mu} = .2$. As can be seen in Figure 4.5 we have excellent agreement between the analytical and Monte Carlo results even when the localization factor varies by seven orders of magnitude over one passband.

Some distinguishing features are noticeable for this type of disorder. First, the localization factor is largest in the vicinity of the first stopband. This first stopband occurs around $\bar{\omega} = (\frac{\bar{k}_s}{\bar{\mu}})^{\frac{1}{2}}/\pi = .5033$, the natural frequency at which the average attached resonator vibrates. Adding even more resonators would compound this effect. Second, we notice that the localization effects generally decrease with increasing frequency. This result seems reasonable because we suspect that at higher frequency, the attached mass vibrates less and less because of its inertia.

Notice that near the second and higher stopbands (each of which begins at integer values of $\bar{\omega}$) the localization factor decreases with frequency approaching the beginning of the stopband, while on the other side of the passband the localization factor is clearly amplified near the stopband. One explanation for this behavior is that the frequency at the beginning of the second and higher stopbands ($\bar{\omega} = 1, 2, \cdots$) coincides with the frequencies in the perfectly periodic system at which the rod of length l between the resonators vibrates as if it had fixed-fixed boundary conditions [Mead 75-1]. Some calculations confirm this effect. Therefore, at these integral frequencies, the rod does not vibrate at the points of attachment of the resonator, thus the fact that the mass on the resonator is disordered would have little impact on the dynamic behavior. On

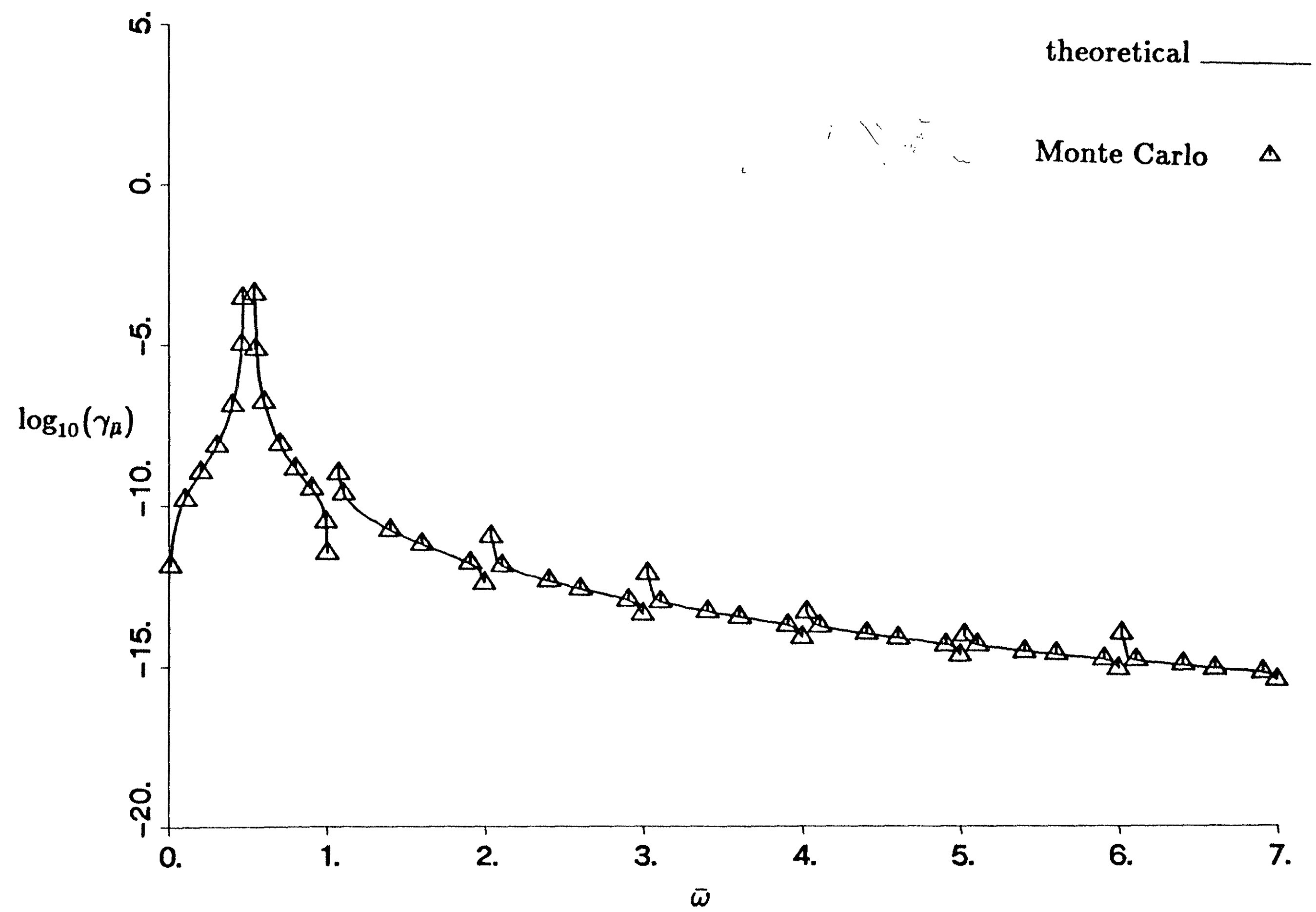

Figure 4.5: Localization factor for rod and attached resonators with masses disordered $\pm.1\%$ from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

the upper end of each stopband each segment of rod no longer vibrates in a fixed-fixed condition and so the disordered mass can now influence the transmission properties.

To give us some idea what the nondimensional frequencies might correspond to in reality, we have substituted some values for the physical parameters. From Appendix C.2 we have

$$\bar{\omega} = \frac{\omega l (\frac{\rho}{E})^{\frac{1}{2}}}{\pi}$$

We choose a length l of 9 feet (2.74 m), E of 45×10^6 lb/in^2 (3.103×10^8 kN/m^2) and ρ of .063 lb/in^3 (1.7×10^3 kg/m^3). This corresponds to a graphite epoxy rod, and the bay length was suggested at one time for the space station. With these values we find that $\bar{\omega} = 1$, the beginning of the second stopband, corresponds to $\omega = 15,491$ rad/s or a frequency of 2465.5 Hz.

We next consider the attached masses disordered with a 1% variation from the average value. In this case the localization effects are increased proportionately through $\sigma_{\bar{\mu}}^2$ in Equation 4.3. We show the localization factor as a function of frequency in four passbands for this level of disorder in Figure 4.6. We essentially see the same pattern we saw for the lower level of disorder.

Finally we increase the disorder of the mass to ±10% of the average value of the nondimensional mass. The results are presented in Figure 4.7. Again we see the familiar behavior of the localization factor as a function of frequency. As we did in the previous section on the mass-spring chain, we notice that the theoretical prediction diverges from our Monte Carlo simulation when the localization factor has a value of .1 or greater. Again this must be a result of only calculating $\gamma_{\bar{\mu}}$ to first order in $\sigma_{\bar{\mu}}^2$.

In summary, we conclude that the localization effects are strongest in the vicinity of the stopband associated with the natural frequency of the average attached resonator, while the effects become less and less significant at higher frequencies.

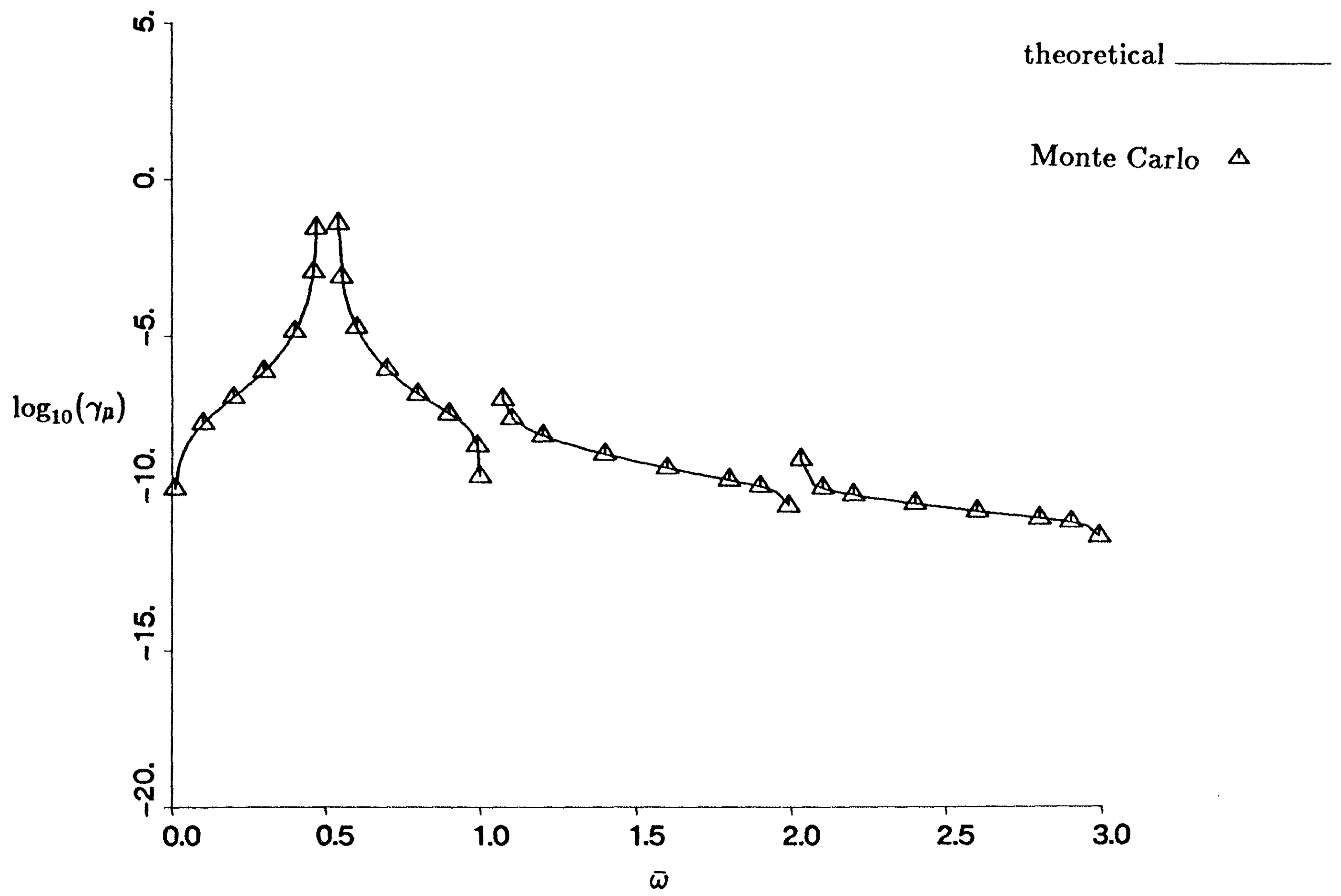

Figure 4.6: Localization factor for rod and attached resonators with masses disordered $\pm 1\%$ from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

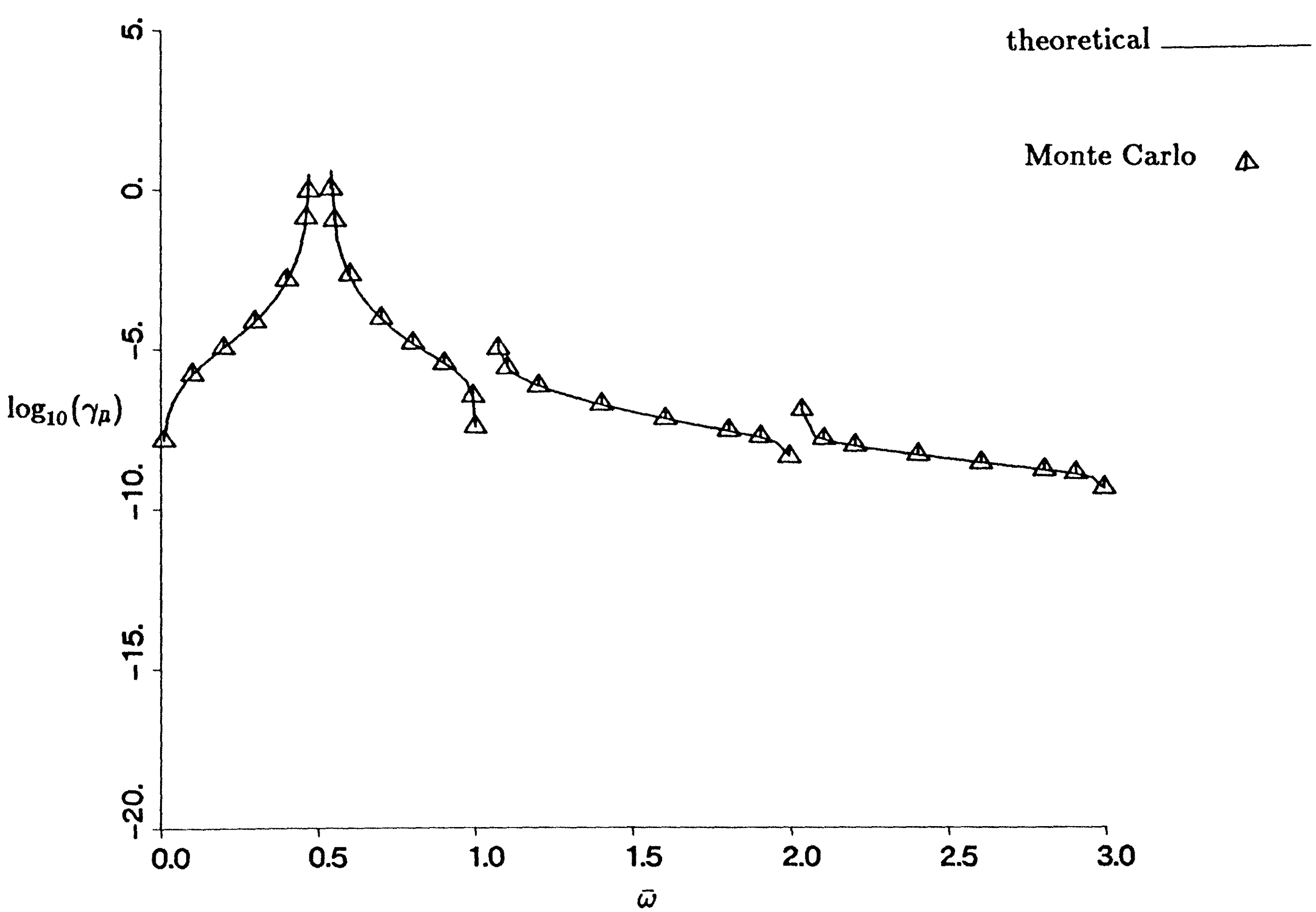

Figure 4.7: Localization factor for rod and attached resonators with masses disordered $\pm 10\%$ from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

4.3.2 Only Springs Disordered

We next disorder only the springs of the attached resonators, where the average nondimensional spring constant, $< \bar{k}_{sj} >= \bar{k}_s = .5$. We will examine springs disordered $\pm.1\%, \pm 1\%$ and $\pm 10\%$ from the average value. In all instances the localization effects follow, as a function of frequency, a pattern very similar to that seen for only masses disordered. One difference we will note though is that at the same levels of disorder, the localization effects that are due to the mass disorder are greater than those due to the spring disorder in the first passband. In the second and higher passbands the trend is reversed and we find that disorder in the springs has greater localizing effects than does the comparable disorder in the masses.

The transfer matrix and wave transfer matrix with the springs disordered is discussed in Appendix C.2. By using Equation 3.10 we find the localization factor for only springs disordered:

$$\gamma_{\bar{k}_s} = \frac{(\sin^2 \pi\bar{\omega})\sigma^2_{\bar{k}_s}}{8(\sin^2 k)(\pi\bar{\omega})^2\bar{k}_s^4((1/\bar{k}_s^4) - (1/\bar{\omega}^2\pi^2\bar{\mu}))^4} \tag{4.4}$$

Note that $\gamma_{\bar{k}_s}$ is very similar to $\gamma_{\bar{\mu}}$, though they are not dual to each other.

This localization factor is plotted in Figure 4.8 for $\pm.1\%$ variation in the springs. The results of the Monte Carlo simulation are also plotted at several frequencies and follow very closely the analytical results. One discrepancy between analytical and Monte Carlo results occurs at the lowest frequency shown. This is a consequence of working with numbers that are too low even for double precision simulations. Note the frequency dependent pattern is very similar to that for the case when the masses were disordered. Again we see the most pronounced localization effects occurring around the first stopband. In addition the localization effects become less pronounced with increasing frequency. We again see that on the immediate left hand side of the second and higher passbands the localization factor is diminished while on the immediate right hand sides it is amplified.

Comparing this localization factor with the one for masses only disordered we find that $\gamma_{\bar{\mu}}$ is consistently larger than $\gamma_{\bar{k}_s}$ in the first passband. This difference can be as much as one or two orders of magnitude at the very lowest frequencies plotted. The lower the frequency the greater the difference. On the other hand, for the second and higher passbands the localization effects due to spring disorder are consistently greater than those due to mass disorder. These differences can be as great as four orders of magnitude at the highest frequencies seen in Figure 4.8. The effect is more pronounced with increasing frequency. Similar effects are noted for the higher levels of disorder. These effects seem reasonable if one considers the effect of wiggling the end of a spring with a mass on the other end of it (this is essentially what the rod is doing to the attached resonator). At low frequency, most of the motion is associated with the movement of the mass, while the spring stretches and compresses very little. Therefore we expect that disorder in the mass will have a greater impact at low frequency than will disorder in the spring. This is indeed what we observe. At higher frequencies, as we move past resonance, $\bar{\omega} = .5033$, the inertia of the mass will cause it to move little while the spring will see a lot of motion. So disorder in the springs should give a much larger contribution to localization effects at high frequency than should disorder in the masses. This too was observed.

Finally, the localization factor is plotted for variations of $\pm 1\%$ and $\pm 10\%$ in the nondimensional spring constant in Figures 4.9 and 4.10, respectively. With increasing disorder the localization effects are amplified, and we again see that our theoretical results mispredict the localization factor near the first stopband for the highest level of disorder.

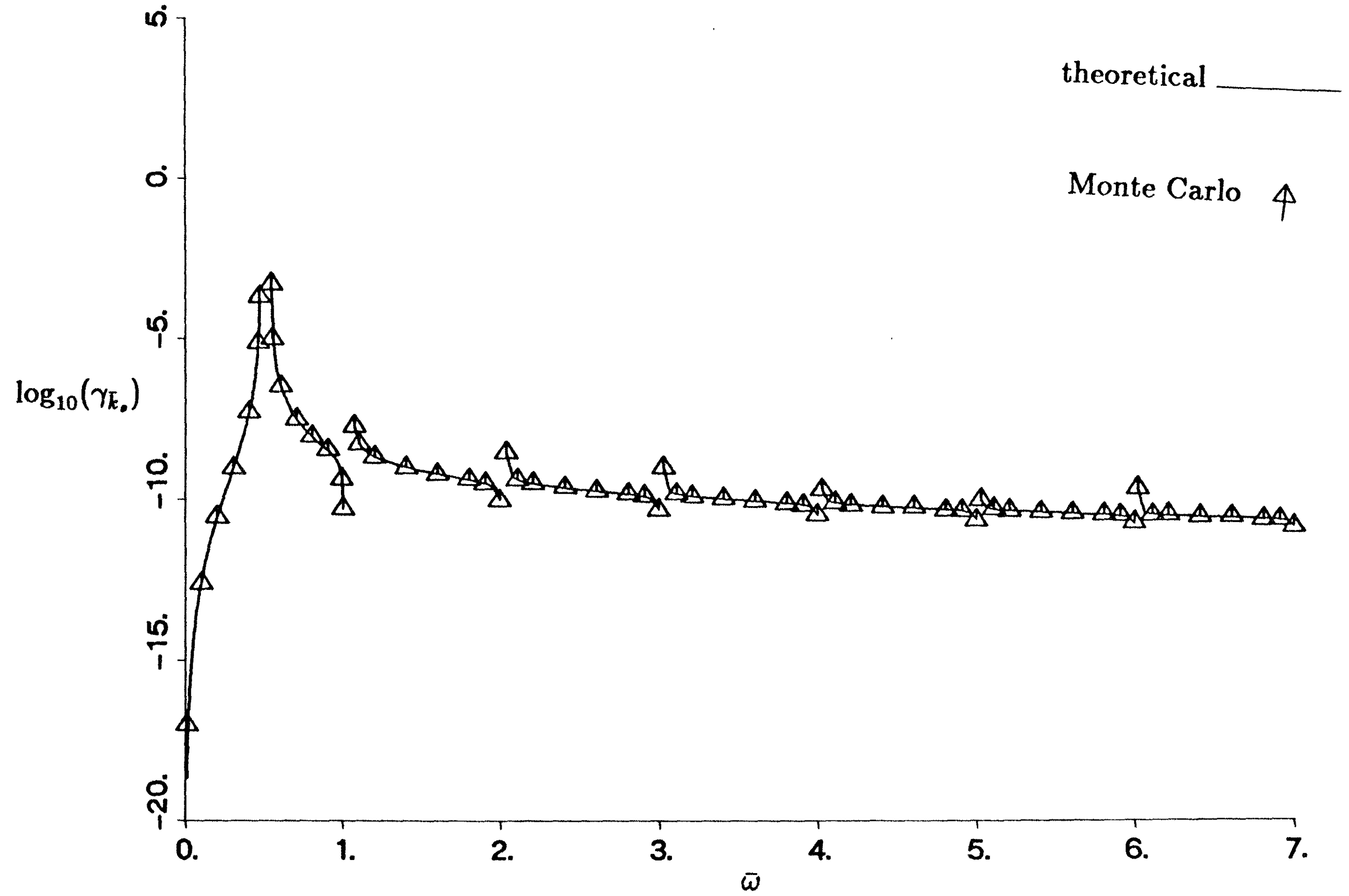

Figure 4.8: Localization factor for rod and attached resonators with springs disordered $\pm.1\%$ from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

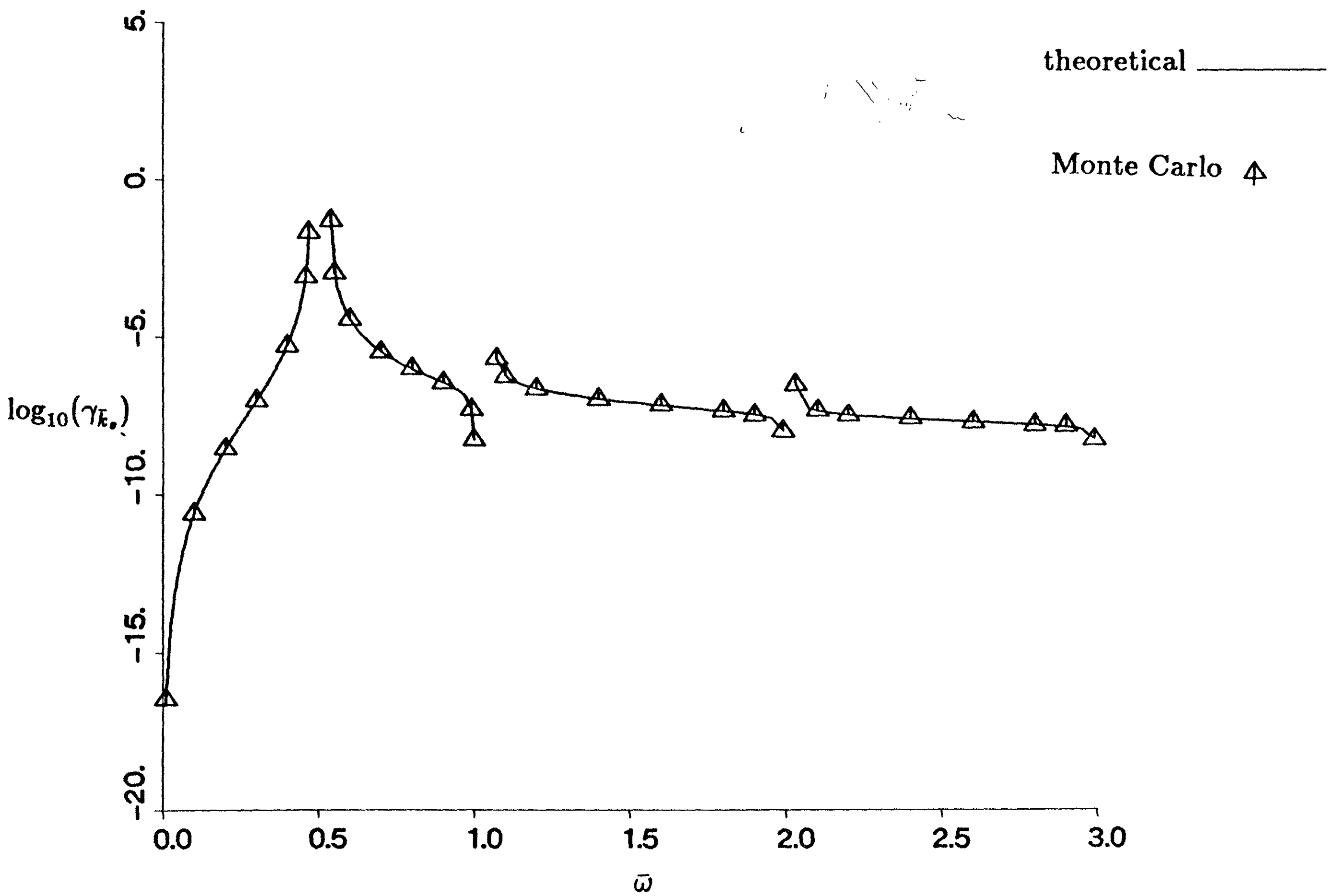

Figure 4.9: Localization factor for rod and attached resonators with springs disordered ±1% from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

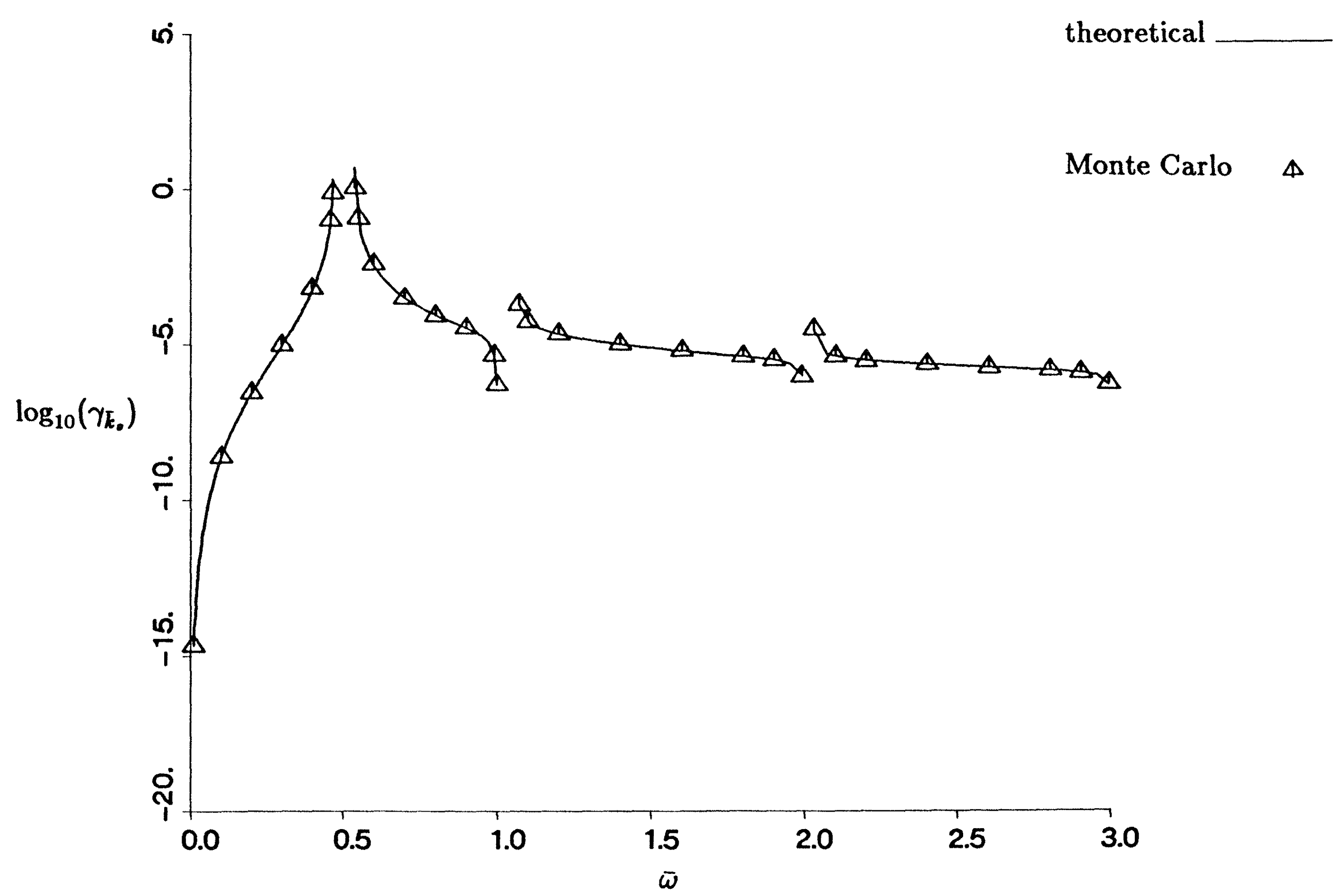

Figure 4.10: Localization factor for rod and attached resonators with springs disordered $\pm 10\%$ from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

4.3.3 Only Lengths Between Resonators Disordered

Now we allow all the resonators to be the same with $\bar{\mu} = .2$ and $\bar{k}_s = .5$, while we will disorder the lengths between the attached resonators by $\pm .1\%$, $\pm 1\%$ and $\pm 10\%$ from the nominal value. The transfer matrix and wave transfer matrix for lengths disordered are discussed in Appendix C.2. Note that we use the nondimensional variable $\bar{l}_j = \frac{l_j}{<l_j>}$, so that $<\bar{l}_j>= 1$. With the lengths only disordered we will find a startling change in behavior of the localization factor as a function of frequency compared to the cases where only the masses or springs were disordered.

The calculation of the localization factor for lengths only disordered, $\gamma_{\bar{l}}$, is much more complicated than that for the previous two cases. Applying Equation 3.10 we find:

$$\begin{aligned}\gamma_{\bar{l}} \doteq & \frac{\sigma_{\bar{l}}^2}{4}[-\pi\bar{\omega}\sin(\pi\bar{\omega})\bar{H}\cos(\pi\bar{\omega}) \\ & -(\pi\bar{\omega})^2\cos^2(\pi\bar{\omega}) - \frac{\bar{H}^2\sin^2(\pi\bar{\omega})}{4} \\ & + \frac{\cos^2 k}{2\sin^2 k}\{2(\pi\bar{\omega})^2\sin^2(\pi\bar{\omega}) - 2(\pi\bar{\omega})\bar{H}\cos(\pi\bar{\omega}) \\ & + \frac{\bar{H}^2}{2} + \frac{\bar{H}^2\cos^2(\pi\bar{\omega})}{2}\}] \end{aligned} \tag{4.5}$$

At low frequency $\gamma_{\bar{l}}$ behaves as $\bar{\omega}^2$, as was the case for $\gamma_{\bar{\mu}}$ and $\gamma_{\bar{k}_s}$.

This localization factor is plotted in the first eight passbands of the underlying perfectly periodic system in Figure 4.11. Here we immediately notice some striking differences from our previous localization plots for the rod with attached resonators. We notice that the localization effects are amplified on either side of all stopbands. We also notice that for a narrow band of frequencies in each passband, the localization factor is greatly diminished. Note that it was difficult for the Monte Carlo simulation to reproduce the extremely small localization factors seen in the plot in the middle of the passbands. We do not believe that this is a result of numerical problems at these low values, but rather is a result of our neglecting higher order terms in our Taylor series expansion that apparently make a significant enough contribution at those frequencies. At higher disorders the effect is even more pronounced. The fact that these discrepancies do not show up in the fifth and eighth passbands is because we have not found that frequency where the localization factor takes it smallest values in those passbands.

These effects seem reasonable because the wavelength of the traveling wave at the end and beginning of each stopband is some multiple of the length between the resonators. Thus we would expect that disorder in the length between resonators would have its greatest effect at those frequencies as opposed to other frequencies where the wavelengths are not so correlated with the bay length. Why $\gamma_{\bar{l}}$ becomes so extremely small in the middle of the passbands is not clear.

Similar effects are noted when the disorder in length is increased to $\pm 1\%$ and $\pm 10\%$ from the nominal value. The corresponding localization factors as a function of frequency are shown in Figures 4.12 and 4.13.

4.3.4 All Three Parameters Disordered

Finally we examine what might be the most realistic situation in which the masses, springs and lengths between the resonators are disordered. The transfer matrix as a function of $\bar{\mu}_j$, $\bar{k}_{sj}$ and $\bar{l}_j$ is presented in Appendix C.2. Again we assume that $< \bar{\mu}_j >= .2$, $< \bar{k}_{sj} >= .5$ and $< \bar{l}_j >= 1$. Here we will disorder both the masses and springs $\pm 1\%$ from their average values, while we will only disorder the lengths by $\pm .1\%$ from its average value. As was explained in Chapter 3, the localization factor with several variables disordered is simply the sum of the localization factors when each

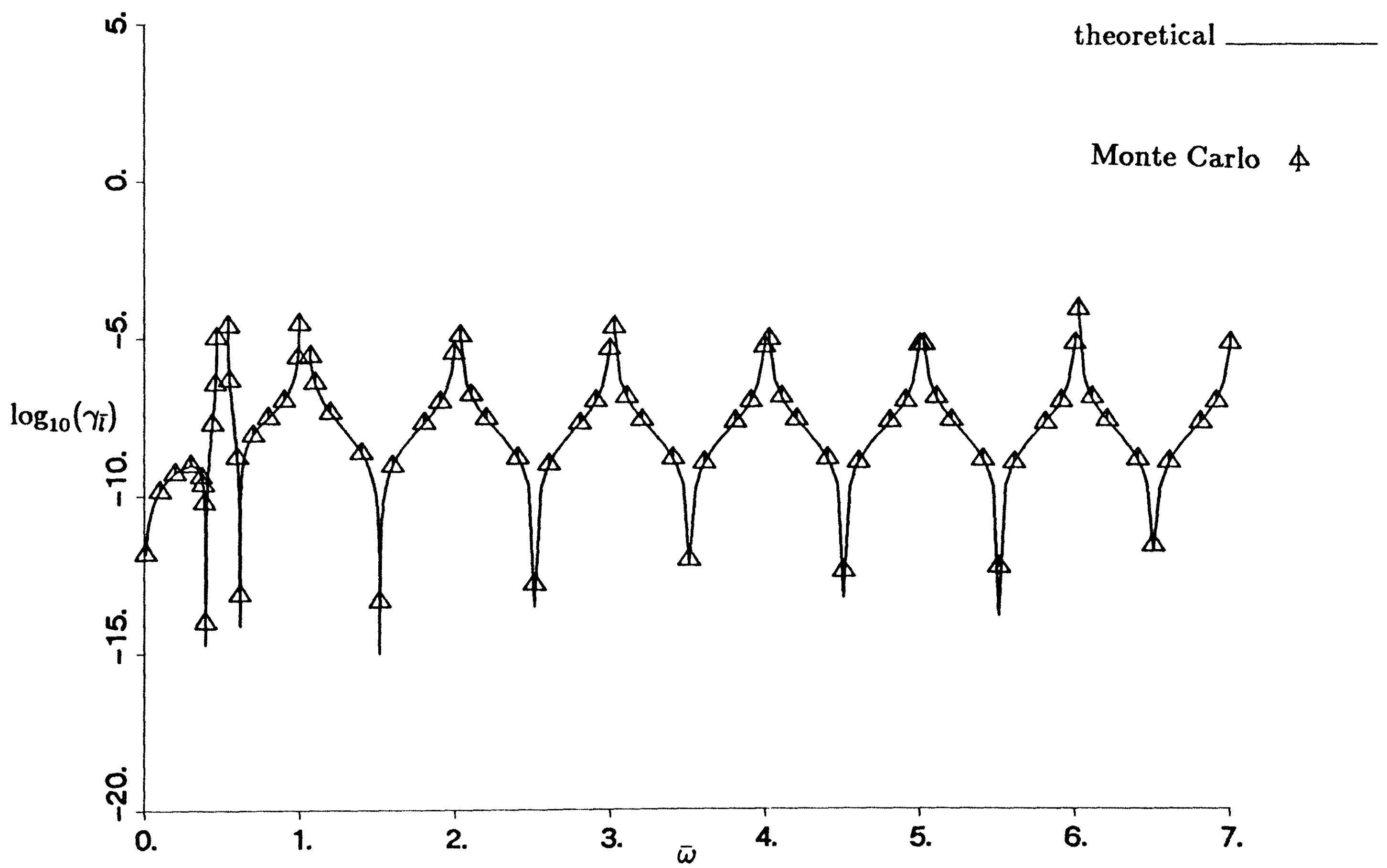

Figure 4.11: Localization factor for rod and attached resonators with lengths between resonators disordered ±.1% from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

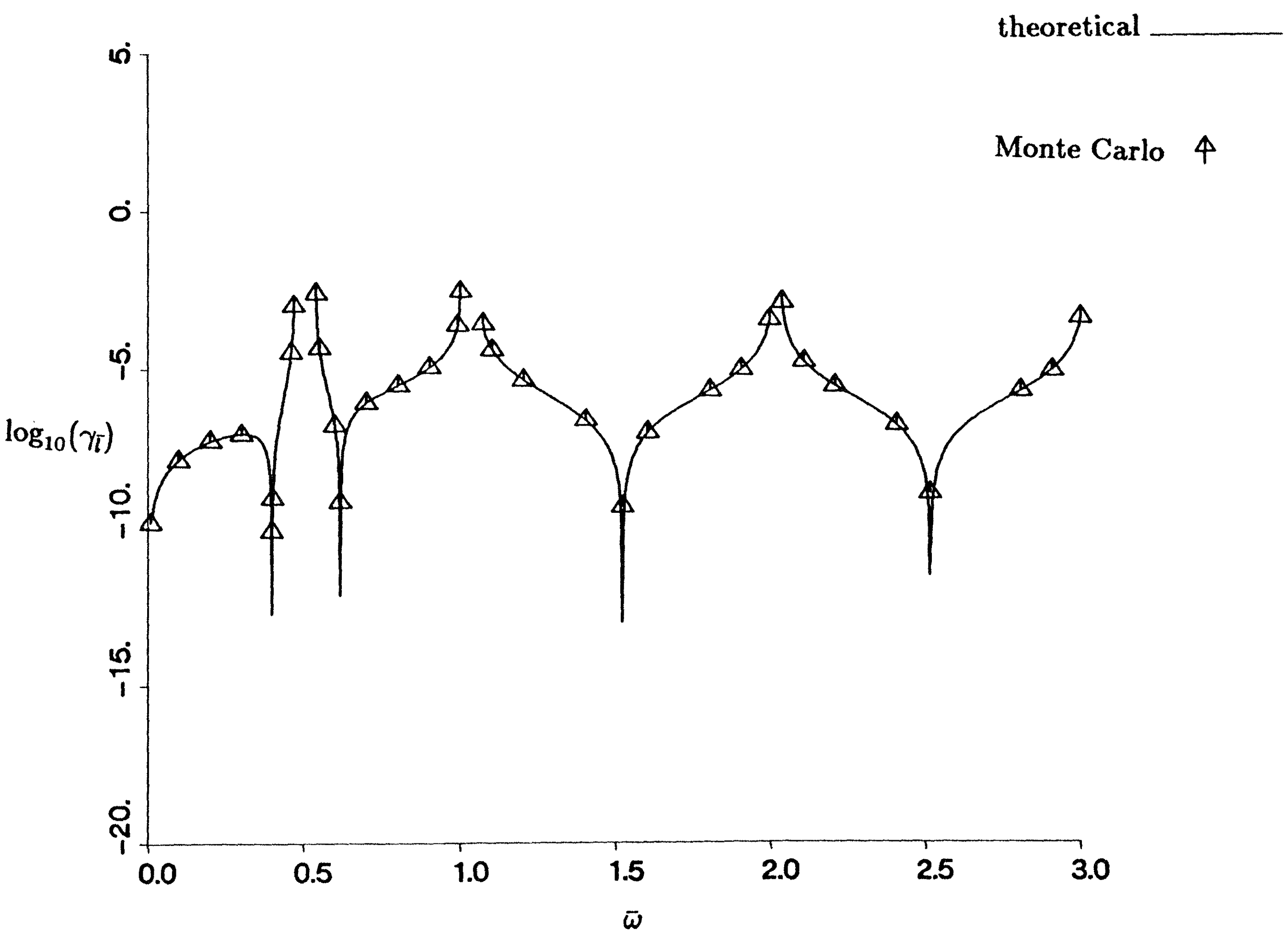

Figure 4.12: Localization factor for rod and attached resonators with lengths between resonators disordered ±1% from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

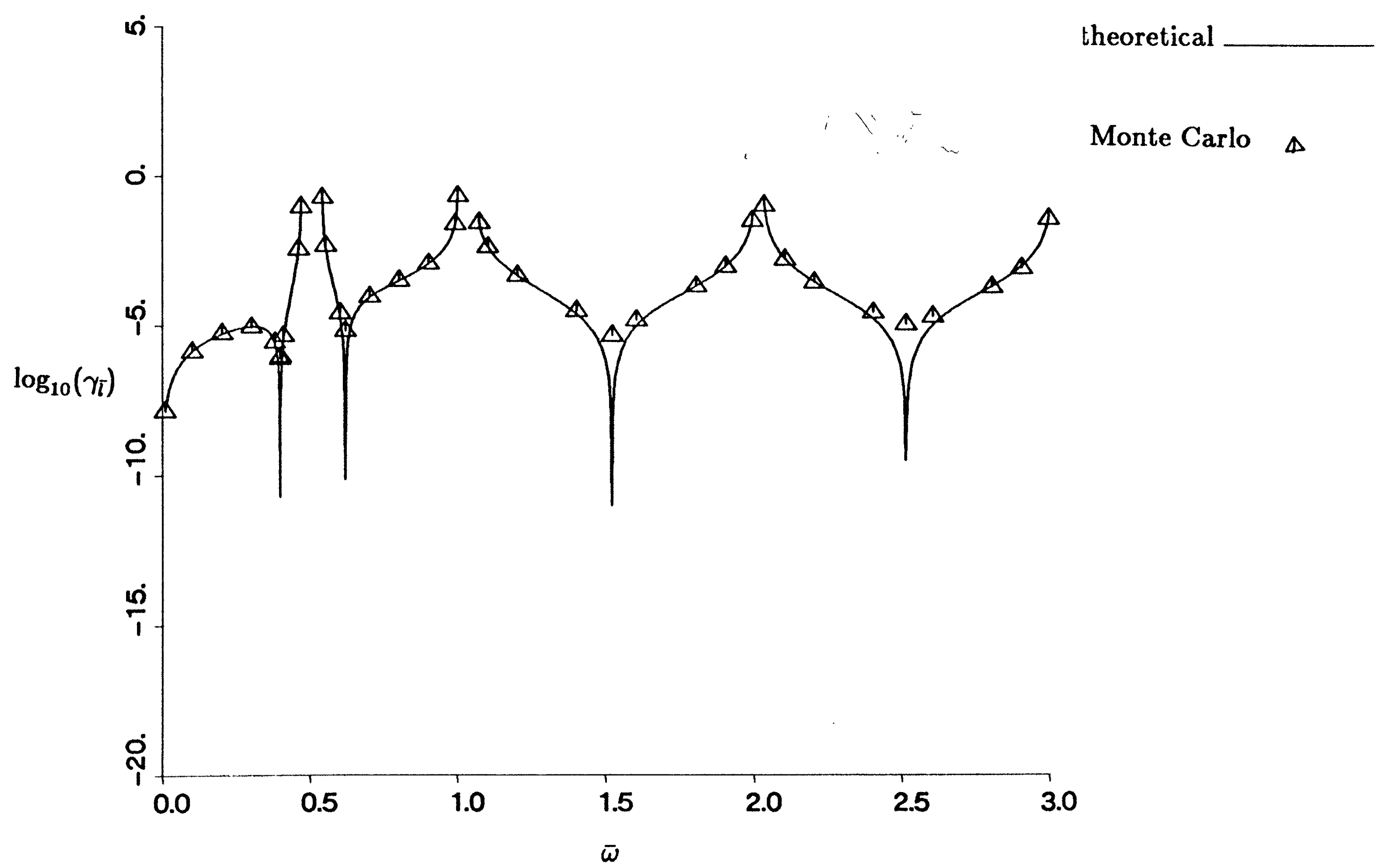

Figure 4.13: Localization factor for rod and attached resonators with lengths between resonators disordered ±10% from their average value with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

variable is disordered individually,

$$\gamma_{\tilde{\mu}\tilde{k}_s\tilde{l}} = \gamma_{\tilde{\mu}} + \gamma_{\tilde{k}_s} + \gamma_{\tilde{l}}$$

In Figure 4.14 we see that our analytical results and Monte Carlo simulation agree very well over eight passbands for these levels of disorder. In this case the localization factor is greatly amplified around all the stopbands, particularly the first one which is associated with the natural frequency of the average attached resonator. The localization effects tend to diminish with increasing frequency. For these levels of disorder, we find that the localization effects are predominately caused by the mass disorder in most of the first passband, while in most of the second passband and in the middle of the subsequent passbands the disorder in the springs has the greatest contribution to $\gamma_{\tilde{\mu}\tilde{k}_s\tilde{l}}$. Only near the second and subsequent stopbands does the disorder in the length predominate in the localization factor. The physical reasoning given earlier when each parameter was disordered individually helps to explain these effects.

4.4 Localization in a Beam on Simple Supports

The final example concerns a Bernoulli-Euler beam on evenly spaced simple supports in the perfectly periodic case, and on randomly spaced simple supports in the disordered case. The perfectly periodic system is presented in Appendix C.3 and its dynamics have been discussed extensively by [Miles 56,Mead 70].

The beam on randomly spaced supports has been discussed in [Yang and Lin 75] and [Lin 76]. There they considered a beam on up to six supports and numerically averaged frequency response functions when the beam was under point loading or convected loading. Their results were consistent with what one would expect from localized dynamics. Unfortunately this approach gives very little insight into the underlying mechanisms associated with disruption of periodicity. Our approach is analytically rig-

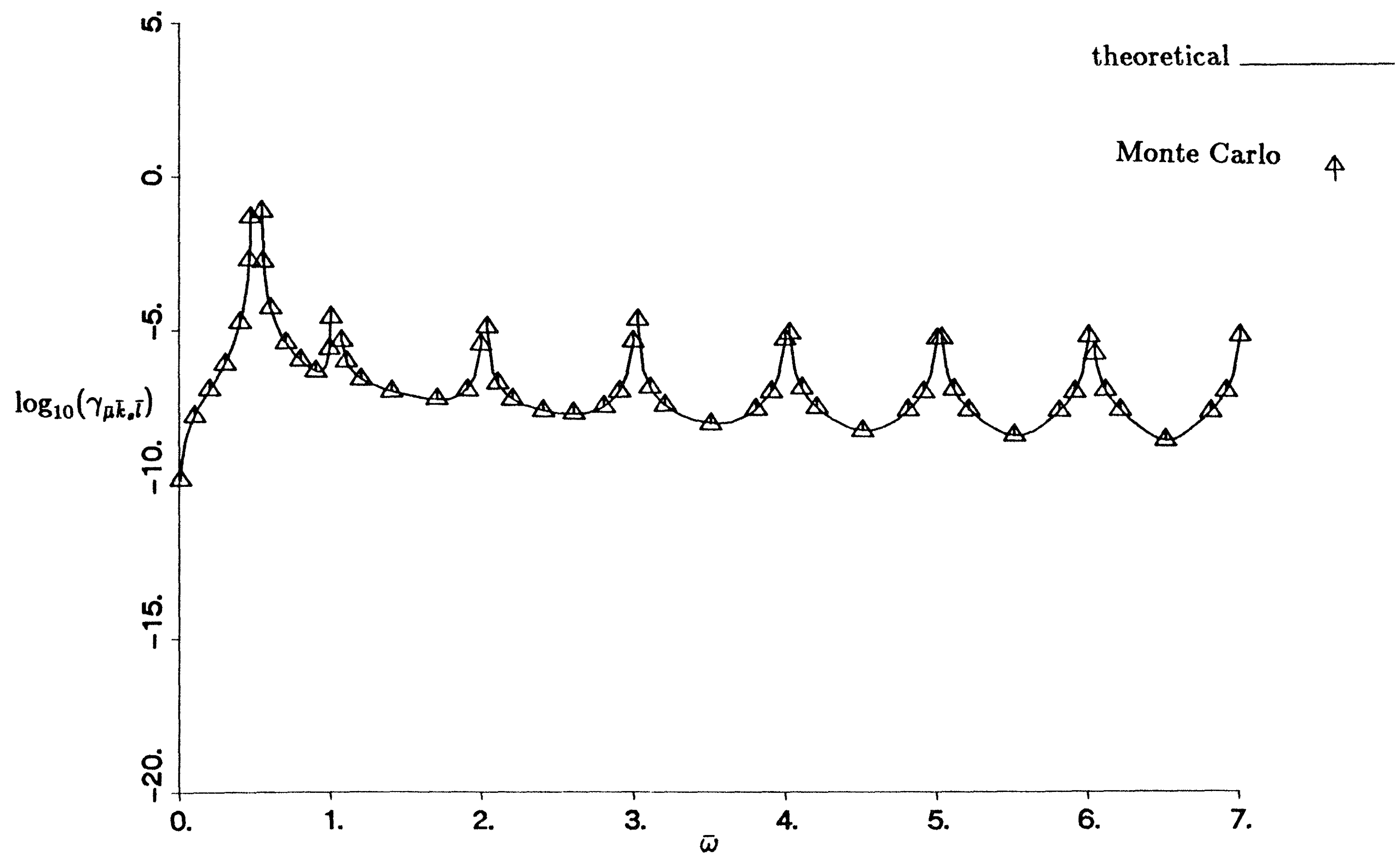

Figure 4.14: Localization factor for rod and attached resonators with masses, springs and lengths between resonators disordered ±1%, ±1% and ±.1%, respectively, from their average values with $\bar{\mu} = 0.2$ and $\bar{k}_s = 0.5$.

orous and centers on a variable which is known to be statistically well-behaved and has physical meaning. In [Bansal 80] a situation similar to ours was considered in which a disordered segment of beam was inserted between perfectly periodic beams on supports. However, the analysis was for deterministically disordered segments.

The transfer matrices for the perfectly periodic system, as well as for the disordered system are presented in Appendix C.3. The random length is nondimensionalized so that $\hat{l}_j = \frac{l_j}{<l_j>}$ and $< \hat{l}_j >= 1$. From the equation for $\frac{1}{t_{\hat{l}_j}}$ and Equation 3.10 we can calculate the localization factor. The calculation is quite involved and many of the terms needed in the calculation are presented in [Yang and Lin 75,Lin 76][2] After extensive calculation we find

$$\begin{aligned} \gamma_{\hat{l}} &\doteq \frac{1}{4}\sigma_{\hat{l}}^2[2g_r'^2 + 2g_r g_r'' + 2g_i'^2 \\ &\quad + 2g_i g_i'' - (2g_r g_r' + 2g_i g_i')^2] \end{aligned}$$

where

$$g_r = \cos k$$

$$g_r' = -\sqrt{\bar{\omega}}[s_4 + \cos k c_2]/s_3$$

$$g_r'' = -\bar{\omega}\cos k[2c_4 + s_1 s_3]/s_3^2$$

$$g_i = \sin k$$

$$\begin{aligned} g_i' &= \frac{\sin k}{2\alpha}\sqrt{\bar{\omega}}[\cos k - \frac{c_4 c_2}{s_3^2}] \\ &\quad + \frac{\alpha\sqrt{\bar{\omega}}}{4\sin k}[\sinh^2\sqrt{\bar{\omega}}\cos\sqrt{\bar{\omega}} - \cosh\sqrt{\bar{\omega}}\sin^2\sqrt{\bar{\omega}}]/s_3^2 \end{aligned}$$

[2] We believe one term in Appendix A of [Lin 76] and Appendix I of [Yang and Lin 75] should read

$$\begin{aligned} b_{12}'' &= -(\xi/EI)\{[s_4(l) + 2\cos\theta c_2(l)]/s_3(l) \\ &\quad - c_4(l)[2c_2^2(l) - s_1(l)s_3(l)]/s_3^3\} \end{aligned}$$

$$
\begin{aligned}
g_i'' &= -\frac{\bar{\omega}\sin k}{2\alpha}[(s_4 + 2c_2\cos k)/s_3 \\
&\quad - c_4(2c_2^2 - s_1 s_3)/s_3^3] \\
&\quad + \frac{\bar{\omega}\alpha}{2\sin k}[2s_3\cos^2 k - s_1 s_4]/s_3^2
\end{aligned}
$$

Clearly, we will have to look at a plot of γ_f in the passbands to make some sense of the above equation. This has been done in Figure 4.15 where we have disordered the nondimensional length by $\pm.1\%$ from the average value.

In the eight passbands we clearly see that the maximum localization effects occur in the immediate vicinity of the stopbands, while in the middle of the nominal passbands the localization factor is greatly diminished. These results seem reasonable because in the perfectly periodic system at the beginning of the stopbands it is well known [Mead 70] that each span of the beam vibrates as if it were clamped on both ends, while at the end of each stopband it vibrates as if it were pinned on both ends. Indeed, the traveling waves become standing waves at the edges of the stopbands. Thus, the dynamics of the system are very sensitive to the distances between supports at frequencies near the beginning and ends of the stopbands. This explains the large localization factors at those frequencies. At all other frequencies the wave motion is not so physically correlated with the span lengths.

To give some meaning to our nondimensional frequency, we choose some properties for our physical parameters corresponding to those given in [Yang and Lin 75]. From Appendix C.3 we have

$$\bar{\omega} = \omega\sqrt{\frac{\mu l^4}{EI}}$$

If we let the thickness of the beam be .05 inches and the width be 1 inch, E be 10.5×10^6 lb/in^2 and μ be 2.616×10^{-4} lbs^2/in^2, we find that for $\bar{\omega} = 100$, we have $\omega = 1530.5$ rad/s or 243 Hz.

Finally, we examine the case of extreme disorder where the distances between the

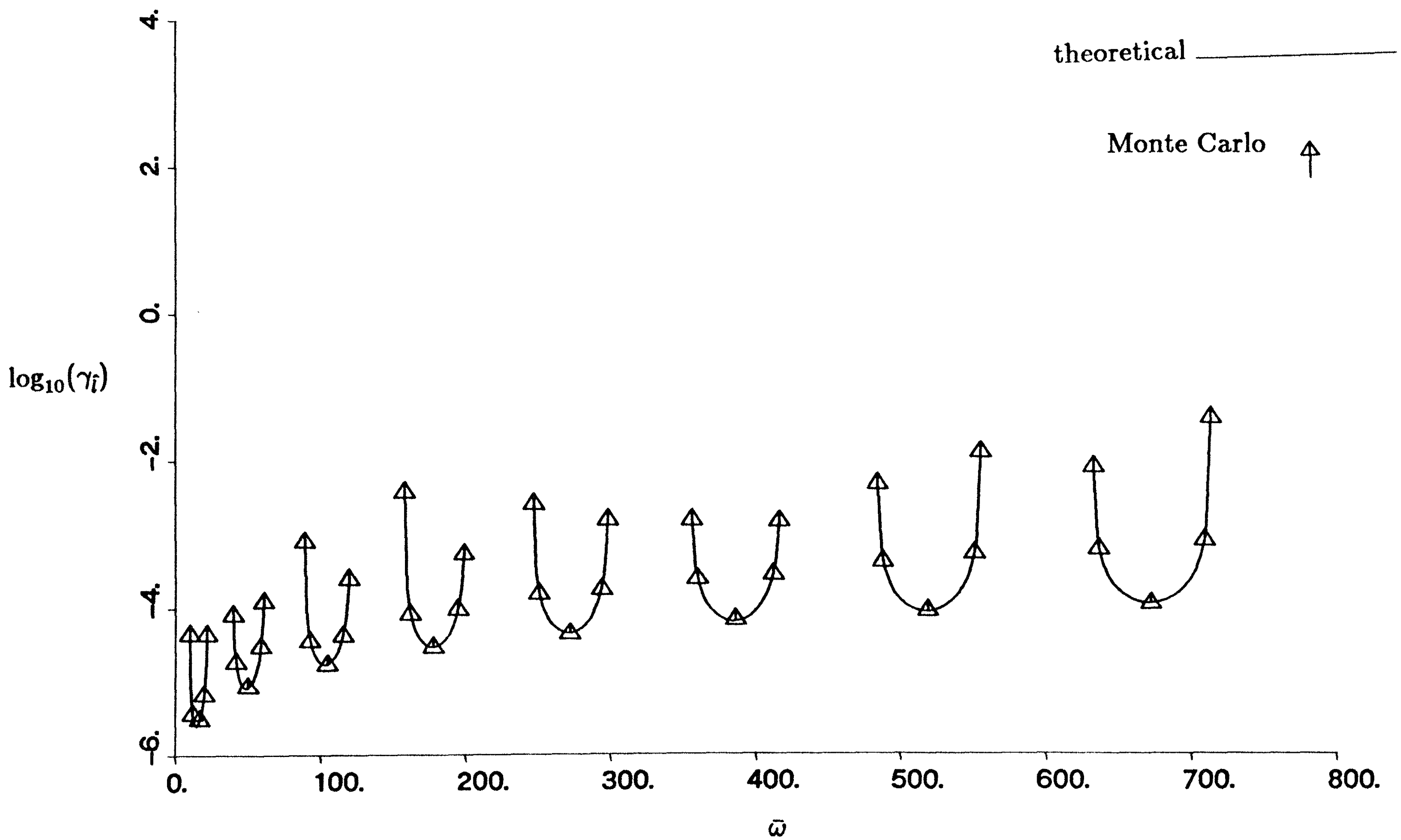

Figure 4.15: **Localization factor for beam on simple supports with lengths between supports disordered ±.1% from their average value.**

supports are randomized by ±10% from their average value. The localization results are plotted in Figure 4.16. Our localization factors take on very high values and we see that the theoretical result overpredicts the Monte Carlo simulation. Yet, the simulation clearly shows the same pattern observed at the lower disorder. The localization effects are most pronounced near the stopbands.

We also notice that the localization effects seem to become stationary with increasing frequency in that the pattern of the localization factor as a function of frequency does not change substantially. This may be a function of the phase randomness ideas discussed by [Hodges 82,Lambert and Thorpe 82,Lambert and Thorpe 83] and [Baluni and Willemsen 85]. The argument here is that at high enough frequency complete phase uncertainty in the wave sets in leading to a particularly simple calculation of the localization factor. The calculation leads to the conclusion that the localization factor will be a constant as a function of frequency. In [Hodges 82] it is found that

$$\gamma_{\hat{l}} = \ln |t_{support}|$$

where $t_{support}$ is the transmission coefficient for one support on an infinitely long beam. From [Cremer et al 73, page 321] we find that $|t_{support}|^2 = .5$ This gives a value of the localization factor that is .347. Clearly, though, we do not observe the localization factor becoming a constant as a function of frequency. Instead it is noticeably amplified in the vicinity of the stopbands. Therefore the notion that the localization factor becomes a constant with frequency must be considered misleading for this kind of system. However, the fact that the localization factor behaves in the same manner from passband to passband at high frequency could be a consequence of these phase randomness ideas.

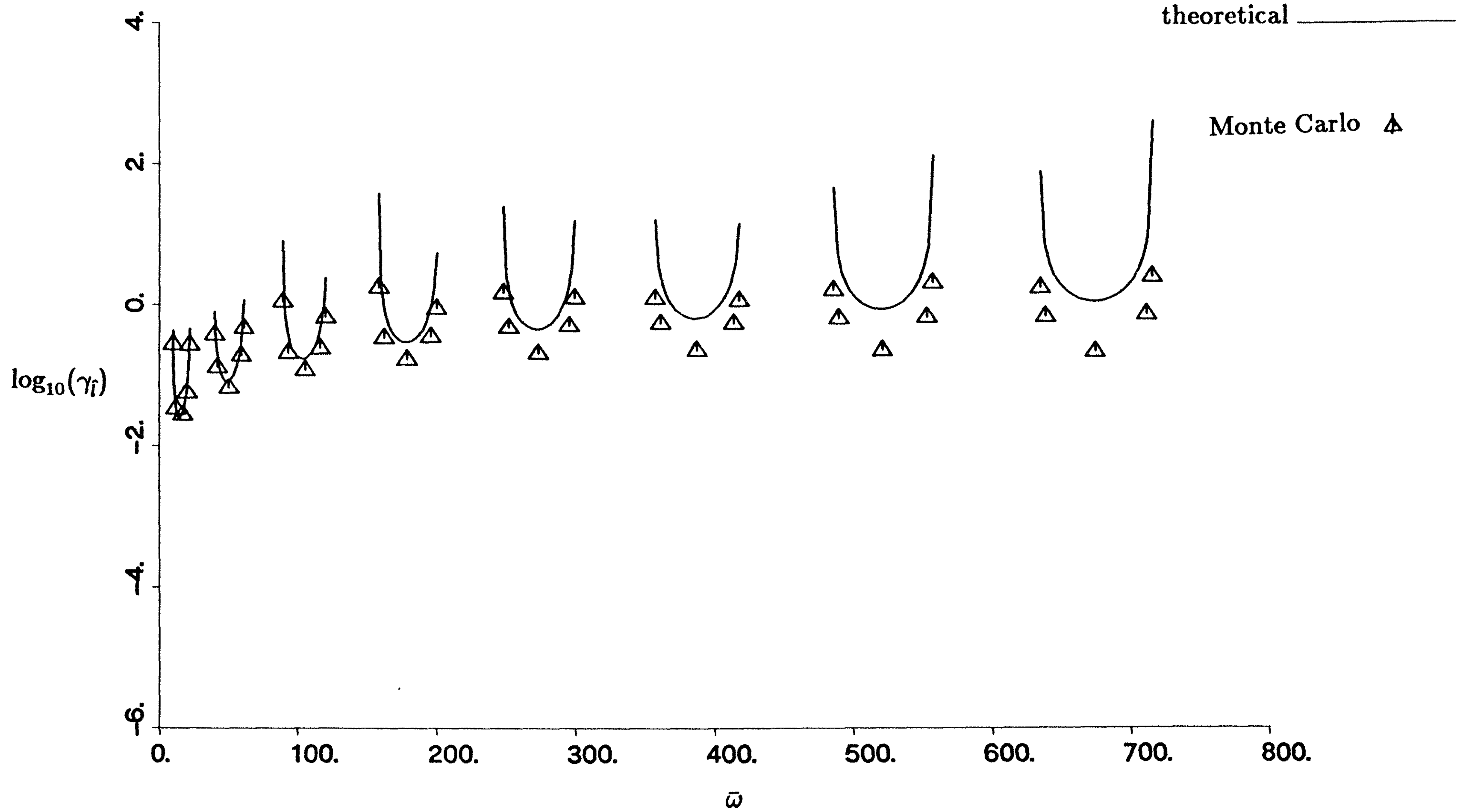

Figure 4.16: Localization factor for beam on simple supports with lengths between supports disordered ±10% from their average value.

4.5 Observations

We have collected a lot of results in this chapter on localization effects in some useful structural dynamic examples, so we need to reflect on some of the insights we have gained.

Clearly the localization effects increase with greater amounts of disorder, though our theoretical results have difficulty tracking the localization factor when it becomes greater than $\gamma = .1$. More importantly, localization effects are strongly varying functions of frequency. Whenever the first frequency band is a passband, we notice that the localization factor is proportional to frequency squared. The most dramatic frequency effect we see is that the localization effects can be quite pronounced around the stopbands. The localization factor was particularly high in the vicinity of the stopband associated with the natural frequency of the attached resonator on the rod. This result indicates that localization effects could be quite important on periodic truss structures which have a number of cross-members. Real periodic truss structures are really multiwave systems which will be investigated in Chapter 5; however, we suspect that the insights we have generated with the mono-coupled systems should generalize to the multiwave systems. We also notice that disorder in the lengths of bays result in quite pronounced localization effects in the vicinity of stopbands as well. Specifically, we see that the localization factor when lengths are disordered consistently take on high values at the edges of the passbands, while they are consistently small in the middles of the passbands. This is in contrast to disorder in masses and springs where the localization factor does not vary so dramatically over any but the first passband. Because localization can become quite pronounced in the vicinity of stopbands, experimental measurements on real periodic structures in those frequency regimes could be susceptible to the effects of disorder.

In addition, our analytical and numerical work has clarified some of the few, yet misleading, results that have appeared in the literature. Most published results up to this point have simply indicated that the localization effects increase with frequency and take on constant values at high frequency. Clearly these results are mistaken. Our work indicates that the importance of localization effects can vary greatly over even a single passband and generally become quite pronounced near the stopbands.

Chapter 5

Localization in Multiwave Systems

For the bulk of this thesis we have considered the localization phenomenon in mono-coupled disordered periodic structures, i.e., systems modeled with 2×2 random transfer matrices. However, most real structures are better modeled with transfer matrices that are of dimension 4×4 or greater. This implies the structures can carry a multiplicity of wave types at a single frequency as opposed to the one wave type in the mono-coupled case. Periodic structures of this kind are called multiwave or multichannel systems. Frequently in the solid state physics literature the term "wire" is used to describe these systems in contrast to the term "chain" used to describe mono-coupled systems. Just as there are many complications in going from single-input single-output to multiple-input multiple-output control system design and analysis, there are analogous complications in going from disordered one-dimensional systems carrying a single pair of waves to disordered one-dimensional systems carrying a multiplicity of waves.

Before embarking on our analysis of multiwave systems, let us review the territory we have covered for mono-coupled disordered systems. After briefly summarizing some relevant properties of periodic systems, we demonstrated that disordered periodic structures can be modeled via a product of random transfer matrices. That product of random matrices was then transformed to a wave transfer matrix involving transmission and reflection coefficients. By employing Furstenberg's theorem on products of random matrices, we were able to show that the transmission coefficient, τ_n, is well-behaved in the sense that

$$\gamma = -\lim_{n\to\infty} \frac{1}{n} \ln |\tau_n|, \quad \gamma > 0$$

Furstenberg's theorem also provides us with a closed-form solution for γ involving a double integral over two probability density functions. Because one of the probability density functions is virtually impossible to find, we were forced to approximate the double integral to first order in the variances of the disordered variables.

We then examined the localization factor for three one-dimensional disordered mono-coupled periodic structures. For reasonable levels of disorder our analytical solution to γ provided a good approximation to the Monte Carlo calculations of the localization factor. We noticed that the localization factor was a strongly varying function of frequency taking on its greatest values at frequencies near the stopbands of the underlying perfectly periodic system.

We believe the approach followed in the study of mono-coupled disordered periodic systems should be followed in the study of multiwave systems to yield the best results. Indeed, as we will see below, this approach has already been successful in giving us the multiwave localization factor as a function of the transmission matrix.

Perfectly periodic multiwave structures have been examined by [Mead 73,Mead 75-1] and [Roy and Plunkett 86,Signorelli 87,Signorelli and von Flotow 87,Bernelli et al 87]. Just as mono-coupled periodic structures have passbands and stopbands, so do multiwave periodic systems. However, in the passbands of multiwave systems, both traveling and attenuating waves, frequently called evanescent waves, can exist simultaneously. Indeed, even complex waves, those which propagate according to $e^{ik+\alpha}$, are known to exist, yet these act as if they were evanescent waves. Because evanescent waves are already strongly localized, our focus in this chapter will be on the effects of disorder on the traveling waves [Büttiker et al 85].

The localization phenomenon in multiwave systems has received much less attention than its single wave counterpart. Anderson [Anderson 81] derived a scaling variable for multiwave systems from an analysis of the scattering matrix. Several researchers [Pichard and Sarma 81-1,Pichard and Sarma 81-2,Pichard 86,Pichard and André 86] and [Imry 86] have used the transfer matrix formalism and theory on products of random matrices to study multiwave systems, though mainly with the intention of extending the results to two- and three-dimensional systems. In [Johnston and Kunz 83-1] and [Johnston and Kunz 83-2] the localization problem of multiwave systems is examined in its own right.

In our analysis of the problem, we state our assumptions about the wave transfer matrix, which follows from certain properties of the scattering matrix. As we shall see, Furstenberg's theorem will not be of use in analyzing multiwave localization. As in [Pichard and Sarma 81-1], we will use the theroem of Oseledets to guide our work. Two subsections are devoted to discussing this important theorem. Our goal is to find the multiwave analog to our mono-coupled result:

$$\gamma = -\lim_{n\to\infty} \frac{1}{n} \ln |\tau_n|$$

We will indeed derive a multiwave analog to this and compare our result with three others that have appeared in the literature. Physically, our goal is to find that wave in the multiplicity of attenuated waves, which is attenuated the least by the disorder. This least attenuated wave carries energy the farthest and so is the one of interest when thinking about localization in multiwave systems.

5.1 Wave Transfer Matrix Assumptions

Our wave transfer matrix assumptions will follow from two properties of the scattering matrix usually found in the solid state literature [Anderson 81] and [Johnston and Kunz 83-1,Büttiker et al 85]. We assume the scattering matrix of one disordered bay sitting in an otherwise perfectly periodic system is both symmetric and unitary.[1] The symmetry of the scattering matrix follows from the symmetry of the impedance (or admittance matrix) describing the bay [Carlin and Giordano 64] and unitarity follows from assuming no dissipation and excluding any evanescent waves [Büttiker et al 85]. See Appendix E.

Our two assumptions about the scattering matrix, $\mathbf{S}$, translate into two properties of the wave transfer matrix, $\mathbf{W}$. First

$$\mathbf{S} \ \textit{symmetric} \iff \mathbf{W} \ \textit{symplectic}$$

and second

$$\mathbf{S} \ \textit{unitary} \iff \mathbf{W} \in SU(d,d)$$

These properties are discussed in Appendix E. Both properties will be important in the derivation of the multiwave localization factor in what follows. The wave transfer matrix $\mathbf{W}$ can be derived from the corresponding real transfer matrix, $\mathbf{T}$, by premultiplying $\mathbf{T}$ by the transposes of the left eigenvectors and postmultiplying by the right eigenvectors corresponding to the traveling waves.

5.2 Theorem of Oseledets

As we did for mono-coupled systems, we will in the case of multiwave systems rely on a theory for products of random matrices to guide our work. We use the theorem

[1] This corresponds to the physical assumptions of time reversal symmetry and current conservation in the solid state localization problem.

of Oseledets [Oseledets 68] specialized to symplectic matrices; however, the reader is referred to [Bougerol and Lacroix 85] and [AMS 86] to better understand its relevance to the problem at hand. We divide the relevant portions of Oseledets' theorem into several parts. First we will state a result concerning the eigenvalues of an asymptotic matrix product, then we will discuss a vector propagation interpretation of the same theorem. In the final section we will see how the Lyapunov exponents (defined below) might be calculated analytically.

5.2.1 Eigenvalues of Limiting Matrix

Let $\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_n$ form a sequence of independent identically distributed random symplectic matrices of size $2d \times 2d$. Suppose also that

$$E(\sup\{\ln \sigma_{max}(\mathbf{W}_1), 0\}) < +\infty$$

If we set $\mathbf{V}_n = \mathbf{W}_n \cdots \mathbf{W}_1$ then the sequence of matrices $(\mathbf{V}_n^H \mathbf{V}_n)^{\frac{1}{2n}}$ converges *w.p.*1 as $n \to \infty$ to a random matrix $\mathbf{B}$ with $2d$ nonrandom eigenvalues $e^{\gamma_1}, \cdots, e^{\gamma_d}, e^{-\gamma_d}, \cdots, e^{-\gamma_1}$ where $\gamma_1 \geq \cdots \geq \gamma_d > 0$ [Johnston and Kunz 83-1]. These $\gamma_i s$ are the Lyapunov exponents of the random matrix product $\mathbf{W}_n \cdots \mathbf{W}_1$. In random dynamical systems, Lyapunov exponents are considered a measure of stochasticity [Benettin and Galgani 79].

The eigenvalues physically represent d pairs of waves traveling in both directions. The theorem of Furstenberg applied to $2d \times 2d$ matrices allows us to calculate γ_1, which is the uppermost Lyapunov exponent. However, in this multiwave case with $\gamma_d \leq \gamma_1$, γ_d represents the wave with potentially the least amount of decay, and so it carries energy along the structure farther than the wave represented by γ_1. As a result, γ_d is the quantity of interest when calculating multiwave localization effects.

Note that we can also express the Lyapunov exponents of this random symplectic matrix product in terms of its singular values (see Appendix A), $\sigma_j = \sigma_j(\mathbf{V}_n)$. If we recall that the singular values of a symplectic matrix occur in reciprocal pairs: $\sigma_1, \cdots, \sigma_d, \sigma_d^{-1}, \cdots, \sigma_1^{-1}$ where $\sigma_1 \geq \cdots \geq \sigma_d \geq 1$. Then *w.p.* 1

$$\gamma_j = \lim_{n \to \infty} \frac{1}{n} \ln \sigma_j(\mathbf{V}_j) \quad 1 \leq j \leq d$$

This result [Bougerol and Lacroix 85] will be very useful in the section in which we derive γ_d as a function of the transmission properties of the system.

5.2.2 Vector Propagation Interpretation of Oseledets' Theorem

Another aspect of Oseledets' theorem involves the limiting behavior of a random matrix product premultiplied by a nonrandom vector. This aspect will help explain one of the properties mentioned in connection with Furstenberg's theorem in Chapter 3.

Given the assumptions and results of the previous section, let $\psi_1 > \psi_2 > \cdots > \psi_r$ (with $r \leq 2d$) be the strictly decreasing sequence of distinct elements of $(\gamma_1, \cdots, \gamma_d, -\gamma_d, \cdots, -\gamma_1)$. Then there exists a strictly increasing sequence of subspaces

$$\{0\} = S_{r+1} \subset S_r \subset \cdots \subset S_1 = C^{2d}$$

(known as a filtration of C^{2d}) such that if

$$\mathbf{z}_0 \in S_j \setminus S_{j+1}$$

then

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\mathbf{W}_n \cdots \mathbf{W}_1 \mathbf{z}_0\| = \psi_j \quad j \leq r$$

Here $\mathbf{z}_0 \in S_j \setminus S_{j+1}$ says that $\mathbf{z}_0$ is an element of the subspace S_j but not an element of S_{j+1}. Also we have

$$\dim S_{j+1} - \dim S_j =$$

number of elements of the sequence $(\gamma_1, \cdots, \gamma_d, -\gamma_d, \cdots, -\gamma_1)$
which are equal to ψ_j

This vector propagation property is best understood by examining the example of 2×2 real transfer matrices. In this case our sequence of Lyapunov exponents is $(\gamma_1, -\gamma_1)$ where $\gamma_1 > 0$, so $r = 2$ and $\psi_1 = \gamma_1$ and $\psi_2 = -\gamma_1$. We have the sequence of subspaces

$$\{0\} = S_3 \subset S_2 \subset S_1 = R^2$$

If

$$\acute{\mathbf{x}}_0 \in S_2 \setminus S_3$$

i.e.,

$$\acute{\mathbf{x}}_0 \in S_2 \setminus \{0\}$$

where S_2 is a particular line in R^2, then

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\mathbf{T}_n \cdots \mathbf{T}_1 \acute{\mathbf{x}}_0\| = -\gamma_1$$

What direction $\acute{\mathbf{x}}_0$ takes in R^2 will depend on the particular realization of the infinite matrix product. Likewise if $\mathbf{x}_0 \in S_1 \setminus S_2$, i.e., $\mathbf{x}_0 \in R^2 \setminus S_2$ then

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\mathbf{T}_n \cdots \mathbf{T}_1 \mathbf{x}_0\| = \gamma_1$$

These vector propagation ideas are the basis for numerical methods to calculate Lyapunov exponents of various dynamical systems [Benettin and Galgani 79] and [Pichard and Sarma 81-2,Ikeda and Matsumoto 86].

This propagation behavior is very analogous to what happens when a vector is propagated by a product of deterministic matrices, $\mathbf{T}$, whose eigenvalues are λ and $\frac{1}{\lambda}$ with $\lambda > 1$. If we choose any vector $\mathbf{v}$, so long as it has some piece along the eigenvector associated with λ, then as n becomes large the direction of $\mathbf{T}^n\mathbf{v}$ will become aligned with the eigenvector associated with the λ. If, on the other hand, the vector $\mathbf{v}$ is aligned with the eigenvector associated with $\frac{1}{\lambda}$ then $\mathbf{T}^n\mathbf{v}$ will always be aligned with that eigenvector no matter how large n is.

5.3 Localization Factor for Multiwave Systems as a Function of the Transmission Matrix

In the previous sections we have identified the dth Lyapunov exponent, γ_d, of the matrix product $\mathbf{W}_n \cdots \mathbf{W}_1$ as the localization factor for a multiwave disordered periodic system. Much as we did for mono-coupled sytems, in which we showed

$$\gamma = -\lim_{n\to\infty} \frac{1}{n} \ln |\tau_n|$$

we want to find γ_d as a function of the transmission properties of the system. Work relevant to this issue has been done by [Anderson 81], [Johnston and Kunz 83-1] and [Imry 86].

Here we assume the $2d \times 2d$ wave transfer matrix is symplectic and is an element of $SU(d,d)$, so

$$\mathbf{V}_n = \prod_{j=1}^{n} \mathbf{W}_j = \begin{bmatrix} \tau_n^{-1} & -\tau_n^{-1}\rho_n \\ -\tau_n^{-1*}\rho_n^* & \tau_n^{-1*} \end{bmatrix} \tag{5.1}$$

The form and properties of the wave transfer matrix were established in Appendix E. The two assumptions about the wave transfer matrix are those made by [Anderson 81], [Johnston and Kunz 83-1], [Imry 86], though [Anderson 81] adds more restrictive assumptions. For the rest of the discussion we will suppress the subscript n on the transmission and reflection matrices, τ and ρ, respectively.

We will show that the localization factor (or the dth Lyapunov exponent of $\mathbf{V}_n$) is

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln \sigma_{\max}(\tau)$$

or

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln[tr(\tau\tau^H)]^{\frac{1}{2}}$$

or

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln |\tau_{ij}|_{\max}$$

where τ is $d \times d$ and τ_{ij} is the ijth element of τ and all the results hold $w.p.$ 1.

The derivation of these results begins by recalling

$$\gamma_d = \lim_{n \to \infty} \frac{1}{n} \ln \sigma_d(\mathbf{V}_n)$$

Recalling that the dth singular value of $\mathbf{V}_n$ is the positive square root of the dth eigenvalue of $\mathbf{V}_n^H \mathbf{V}_n$ we have

$$\gamma_d = \lim_{n \to \infty} \frac{1}{2n} \ln \lambda_d(\mathbf{V}_n^H \mathbf{V}_n)$$

Consider the matrix

$$\mathbf{V}_n^H \mathbf{V}_n = \begin{bmatrix} 2(\tau\tau^H)^{-1} - \mathbf{I} & -\rho^T(\tau^*\tau^T)^{-1} - (\tau\tau^H)^{-1}\rho \\ -\rho^H(\tau\tau^H)^{-1} - (\tau^*\tau^T)^{-1}\rho^* & 2(\tau^*\tau^T)^{-1} - \mathbf{I} \end{bmatrix} \tag{5.2}$$

Here $\mathbf{V}_n^H \mathbf{V}_n$ is symplectic, so its eigenvalues will occur in reciprocal pairs $\lambda_1, \cdots, \lambda_d$, $\frac{1}{\lambda_d}, \cdots, \frac{1}{\lambda_1}$ where $\lambda_1 \geq \cdots \geq \lambda_d \geq 1$.

Our analysis will be simplified by recognizing the following [2]:

$$(\mathbf{V}_n^H \mathbf{V}_n) + (\mathbf{V}_n^H \mathbf{V}_n)^{-1} = \begin{bmatrix} 4(\tau\tau^H)^{-1} - 2\mathbf{I} & 0 \\ 0 & 4(\tau^*\tau^T)^{-1} - 2\mathbf{I} \end{bmatrix} \tag{5.3}$$

where each block in the matrix is $d \times d$. The matrix has repeated eigenvalues $\lambda_1 + \frac{1}{\lambda_1}, \cdots, \lambda_d + \frac{1}{\lambda_d}$ for a total of $2d$ eigenvalues. However, we notice that these eigenvalues are the eigenvalues of the two diagonal blocks of this block diagonal matrix. The eigenvalues of each block are clearly real because both blocks are Hermitian. In addition, each block is the complex conjugate of each other, and real eigenvalues being invariant with respect to complex conjugation, both blocks must have the same eigenvalues.

So the eigenvalues, μ_j, of $4(\tau\tau^H)^{-1} - 2\mathbf{I}$ are

$$\mu_1 = \lambda_1 + \frac{1}{\lambda_1}, \cdots, \mu_d = \lambda_d + \frac{1}{\lambda_d}$$

[2] [Engels 80,Pichard and André 86] recognized a similar result, though [Engels 80], working in a different context, never realized he was dealing with symplectic matrices.

where

$$\mu_1 \geq \cdots \geq \mu_d$$

Now let $\mu_j[*]$ be the jth eigenvalue of the indicated argument. So

$$\begin{aligned}
\lambda_d + \frac{1}{\lambda_d} &= \mu_d[4(\tau\tau^H)^{-1} - 2\mathbf{I}] \\
&= \mu_{\min}[4(\tau\tau^H)^{-1} - 2\mathbf{I}] \\
&= 4\mu_{\min}[(\tau\tau^H)^{-1}] - 2
\end{aligned}$$

where we have used a couple of determinant identities in the last equation. Now taking the same limit on both sides:

$$\lim_{n\to\infty} \frac{1}{2n} \ln(\lambda_d + \frac{1}{\lambda_d}) = \lim_{n\to\infty} \frac{1}{2n} \ln\{4\mu_{\min}[(\tau\tau^H)^{-1}] - 2\}$$

We notice that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{2n} \ln(\lambda_d + \frac{1}{\lambda_d}) &= \lim_{n\to\infty} \frac{1}{2n} \ln(\lambda_d)(1 + \frac{1}{\lambda_d^2}) \\
&= \lim_{n\to\infty} \frac{1}{2n} \ln(\lambda_d) + \lim_{n\to\infty} \frac{1}{2n} \ln(1 + \frac{1}{\lambda_d^2})
\end{aligned}$$

Recalling that $\lambda_d \geq 1$, the second term above must vanish in the limit. So we are left with (recalling the definition of γ_d)

$$\begin{aligned}
\gamma_d &= \lim_{n\to\infty} \frac{1}{2n} \ln(\lambda_d) \\
&= \lim_{n\to\infty} \frac{1}{2n} \ln\{4\mu_{\min}[(\tau\tau^H)^{-1}] - 2\}
\end{aligned}$$

Note that

$$\mu_{\min}[(\tau\tau^H)^{-1}] = \frac{1}{\mu_{\max}[\tau\tau^H]}$$

So we can write

$$\gamma_d = \lim_{n\to\infty} \frac{1}{2n} \ln(\frac{4}{\mu_{\max}[\tau\tau^H]} - 2)$$

or

$$\gamma_d = \lim_{n\to\infty} \frac{1}{2n} \ln(\frac{1}{\mu_{\max}[\tau\tau^H]})(4 - 2\mu_{\max}[\tau\tau^H])$$

or

$$
\begin{aligned}
\gamma_d &= \lim_{n\to\infty} \frac{1}{2n} \ln(\frac{1}{\mu_{\max}[\tau\tau^H]}) \\
&\quad + \lim_{n\to\infty} \frac{1}{2n} \ln(4 - 2\mu_{\max}[\tau\tau^H])
\end{aligned}
$$

In Appendix E we show that $0 < \mu_{\max}[\tau\tau^H] \leq 1$, so that the second term above must vanish in the limit.

We are left with:

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{2n} \ln \mu_{\max}[\tau\tau^H]$$

or recalling the definition of singular values

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln \sigma_{\max}(\tau) \tag{5.4}$$

As a byproduct of this analysis we can find all d of the Lyapunov exponents of $\mathbf{V}_n$ in terms of the transmission matrix τ. First recall from Section 5.2.1

$$\gamma_j = \lim_{n\to\infty} \frac{1}{2n} \ln(\lambda_j) \quad 1 \leq j \leq d$$

and from earlier in this section

$$
\begin{aligned}
\lambda_j + \frac{1}{\lambda_j} &= \mu_j[4(\tau\tau^H)^{-1} - 2\mathbf{I}] \\
&= 4\mu_j[(\tau\tau^H)^{-1}] - 2
\end{aligned}
$$

Note here that

$$\mu_j[(\tau\tau^H)^{-1}] = \frac{1}{\mu_{d-j+1}[\tau\tau^H]} \quad 1 \leq j \leq d$$

So taking limits on both sides and discarding vanishing terms we find:

$$\gamma_j = -\lim_{n\to\infty} \frac{1}{n} \ln \sigma_{d-j+1}(\tau)$$

This reproduces our result for γ_d, and also tells us that

$$
\begin{aligned}
\gamma_1 &= -\lim_{n\to\infty} \frac{1}{n} \ln \sigma_d(\tau) \\
&= -\lim_{n\to\infty} \frac{1}{n} \ln \sigma_{\min}(\tau)
\end{aligned}
$$

Now we return to examining γ_d and proceed to show that in addition to Equation 5.4

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln[tr(\tau\tau^H)]^{\frac{1}{2}}$$

First examine

$$-\lim_{n\to\infty} \frac{1}{2n} \ln tr(\tau\tau^H)$$

Take an eigenvector decomposition of the Hermitian matrix $\tau\tau^H$, and rewrite this as:

$$-\lim_{n\to\infty} \frac{1}{2n} \ln tr(\mathbf{U} diag\{\mu_i\} \mathbf{U}^H)$$

where $\mathbf{U}$ is a unitary matrix. Recalling that $tr(\mathbf{ABC}) = tr(\mathbf{BCA})$ for compatible matrices we see that the above limit equals

$$-\lim_{n\to\infty} \frac{1}{2n} \ln tr(diag\{\mu_i\})$$

or

$$-\lim_{n\to\infty} \frac{1}{2n} \ln(\mu_1 + \cdots + \mu_d)$$

or

$$-\lim_{n\to\infty} \frac{1}{2n} \ln(\mu_1[1 + \frac{\mu_2}{\mu_1} + \cdots + \frac{\mu_d}{\mu_1}])$$

Recalling that $\mu_1 \geq \cdots \geq \mu_d > 0$, we have that the term in brackets is finite and bounded below by 1 and above by d, so when taking the limit, we are left with

$$-\lim_{n\to\infty} \frac{1}{2n} \ln \mu_1[\tau\tau^H]$$

which is precisely equal to

$$-\lim_{n\to\infty} \frac{1}{n} \ln \sigma_{\max}(\tau) = \gamma_d$$

Thus we have indeed shown that

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln[tr(\tau\tau^H)]^{\frac{1}{2}} \tag{5.5}$$

One final simplification in our result is now possible. Starting with

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{2n} \ln[tr(\tau\tau^H)]$$

let τ_{ij} be the ijth element of the matrix τ. Now (this is the square of the Frobenius norm of τ)

$$\begin{aligned} tr(\tau\tau^H) &= \sum_{i=1}^{d}\sum_{j=1}^{d}|\tau_{ij}|^2 \\ &= |\tau_{11}|^2 + |\tau_{12}|^2 + \cdots + |\tau_{dd}|^2 \end{aligned}$$

We have that for one element of τ, $|\tau_{ij}| \geq |\tau_{kl}|, k \neq i, l \neq j$, and we will denote it $|\tau_{ij}|_{max}$. So

$$tr(\tau\tau^H) = |\tau_{ij}|^2_{max}(\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{|\tau_{ij}|^2}{|\tau_{ij}|^2_{max}})$$

So

$$\gamma_d = -\lim_{n\to\infty}\frac{1}{2n}\ln\{|\tau_{ij}|^2_{max}(\sum_i\sum_j\frac{|\tau_{ij}|^2}{|\tau_{ij}|^2_{max}})\}$$

and because the term in parentheses is finite and bounded below by 1 and above by d^2, it vanishes after taking the limit, so we are left with

$$\gamma_d = -\lim_{n\to\infty}\frac{1}{2n}\ln|\tau_{ij}|^2_{max}$$

or

$$\gamma_d = -\lim_{n\to\infty}\frac{1}{n}\ln|\tau_{ij}|_{max} \tag{5.6}$$

This result tells us that the wave that propagates the farthest is governed by the transmission coefficient with the largest absolute value, which makes perfect sense. Notice that our result agrees with our localization result in the mono-coupled case where the matrix τ is a scalar.

Now we are in a position to compare our result with three others that have appeared in the literature. In [Anderson 81], a scaling variable, mentioned in Chapter 1, is derived for multiwave systems in which Anderson tried to mimic the techniques which accurately gave him the scaling variable for mono-coupled disordered periodic systems [Anderson et al 80]. In addition to assuming that the scattering matrix was symmetric and unitary, he also assumed, in order to make the problem tractable from his point of view, that certain channels in what he called a back reflection matrix were independent. In the paper he acknowledged that this latter assumption was not correct, but guessed it would have little impact on the final result. The scaling variable he arrived at was

$$\rho_s \ln(1 + \frac{1}{\rho_s tr[\tau\tau^H]})$$

with

$$\begin{aligned} \rho_s &= 2, \; \frac{1}{tr[\tau\tau^H]} \to 0 \\ \rho_s &= 1.764, \; \frac{1}{tr[\tau\tau^H]} \to +\infty \end{aligned}$$

For us $\frac{1}{tr[\tau\tau^H]} \to +\infty$ is the relevant limit. An analysis of our results indicates that we would expect the scaling variable to be

$$\ln(\frac{1}{tr[\tau\tau^H]})$$

Apparently the difference between the results is a consequence of Anderson's extra assumptions on channel independence. Note also that Anderson's result does not reduce down to the scaling variable in the mono-coupled case.

A much more direct comparison of results can be made with [Imry 86]. Imry made exactly the same assumptions about the wave transfer matrix as we have, and, through the work of Pichard, was aware of Oseledets' theorem. In his paper, Imry makes some heuristic arguments concerning $tr(\tau\tau^H)$ leading to the inverse localization length, $\frac{1}{\xi}$, (the same thing as our multiwave localization factor) being

$$\frac{1}{\xi} = -\lim_{n\to\infty} \frac{1}{n} \ln tr(\tau\tau^H)$$

The problem with this result is the missing square root over $tr(\tau\tau^H)$.

Finally we compare our result with [Johnston and Kunz 83-1] who relied rigorously on theories of products of random matrices. In their paper, Johnston and Kunz used the work of [Tutubalin 68,Virster 70], though they were aware of Pichard's work. Arguing as we have, that the smallest Lyapunov exponent of a random symplectic matrix product is the localization factor for long multiwave systems, they derived the localization factor as:

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln |\tau_{ij}| \quad for\ any\ \tau_{ij}$$

This result differs from the one presented in Equation 5.6 in that our γ_d involves only the limit of $|\tau_{ij}|_{max}$. To evaluate whether the result of [Johnston and Kunz 83-1] makes sense, we see if it gives us the correct answer for the undisordered or perfectly periodic system. For a perfectly periodic system with n bays, the transmission matrix, τ, would look like:

$$\tau = \begin{bmatrix} e^{-ik_1 n} & & \\ & \ddots & \\ & & e^{-ik_d n} \end{bmatrix}$$

with all the off-diagonal terms zero. In [Johnston and Kunz 83-1] the claim is that we can take any element of τ and get the proper localization factor. Yet if we choose any off-diagonal term we get the following absurd result:

$$\begin{aligned} \gamma_d &= -\lim_{n\to\infty} \frac{1}{n} \ln(0) \\ &= -\lim_{n\to\infty} \frac{-\infty}{n} \end{aligned}$$

This is in contrast to Equation 5.6 which takes the element of τ with the maximum absolute value, namely, $|e^{-ik_j}| = 1$, from which we find

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln(1) = 0$$

This is precisely the result for perfectly periodic systems, i.e., there is no localization.

Note that all three of our localization results, Equations 5.4, 5.5 and 5.6 only hold as $n \to \infty$. Indeed all three must give equivalent answers in the limit. However, if we were to evaluate each of the three expressions for finite n we would likely find three different answers. This is a consequence of

$$|\tau_{ij}|_{max} \leq \sigma_{max}(\tau) < \sqrt{tr(\tau\tau^H)}$$

Clearly, when we are averaging one or a finite number of bays over an ensemble, $|\tau_{ij}|_{max}$ appears to be the variable to average, otherwise we would mispredict the value for γ_d. Indeed, we conjecture that by averaging $-\ln|\tau_{ij}|_{max}$ over a large ensemble of wave transfer matrices we could compute an accurate estimate of γ_d. This observation could lead to a method which would bypass the necessity of multiplying as many as $10,000$, $50,000$ or even $60,000$ matrices together as has been done in [Pichard and Sarma 81-1,Johnston and Kunz 83-2,García et al 86].

However, before pursuing some complicated numerical analysis, we should first try to discover an analytical solution for γ_d with which to compare any numerical result. This is the subject of the next section.

5.4 Calculation of the Multiwave Localization Factor Via p-Forms

Similar to our approach in Chapter 3, we need to examine the analytical tools to actually calculate γ_d, the multiwave localization factor. For mono-coupled systems we had Equation 3.3 that gave a closed form solution for γ. We will discuss the analogous equation for γ_d in this section.

The mathematics for calculating Lyapunov exponents for products of random $2d \times 2d$ matrices becomes increasingly complex compared to the case of 2×2 matrices. In particular, we will be making use of p-forms. The recent book, [Bougerol and Lacroix 85] is an excellent reference on the mathematics necessary to handle multiwave disordered systems. For completeness the relevant theorem is as follows and is adapted from [Bougerol and Lacroix 85, page 89]

Theorem 3 (Calculation of Lyapunov Exponents) *Let* $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n$ *be in-*

dependent identically distributed $2d \times 2d$ random symplectic matrices with distribution μ and let p be an integer in $\{1, \ldots, d\}$. Suppose that W_μ, the smallest closed semigroup in $Gl(d, C)$ containing the support of μ, is p-contracting and L_p-strongly irreducible and that $E[\ln \|\mathbf{W}_1\|]$ is finite. Then the following hold

$$\gamma_p > \gamma_{p+1}$$

For any nonzero $\mathbf{z}_0$ in L_p,

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\Lambda^p \mathbf{W}_n \cdots \mathbf{W}_1 \mathbf{z}_0\| = \sum_{j=1}^{p} \gamma_j$$

and

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\Lambda^p \mathbf{W}_n \cdots \mathbf{W}_1\| = \sum_{j=1}^{p} \gamma_j$$

There exists a unique μ-invariant probability distribution ν_p on

$$P(L_p) = \{\bar{\mathbf{z}} \in P(\Lambda^p R^{2d}); \mathbf{z} \in L_p\}$$

then

$$\sum_{j=1}^{p} \gamma_j = \iint \ln \|\Lambda^p \mathbf{W} \bar{\mathbf{z}}\| d\mu(\mathbf{W}) d\nu_p(\bar{\mathbf{z}}) \tag{5.7}$$

Clearly to calculate γ_d we do it inductively. Namely, we have to calculate from Equation 5.7

$$\gamma_1 + \cdots + \gamma_d$$

then

$$\gamma_1 + \cdots + \gamma_{d-1}$$

from which we can obtain γ_d.

To illustrate the increased complexity of this multiwave localization problem we note that for a 4×4 matrix, $\mathbf{W}$, we have that $\Lambda^1 \mathbf{W}$ is just the matrix $\mathbf{W}$ while $\Lambda^2 \mathbf{W}$ is a 6×6 matrix in an appropriate basis. This also means, when $p = 2$ in the above, that $\bar{\mathbf{z}}$ will be a 6×1 vector. We should also note that the norms of these p-forms take a particularly simple form:

$$\|\Lambda^p \mathbf{W}\| = \sigma_1 \sigma_2 \cdots \sigma_p$$

where σ_i is the ith singular value of the matrix $\mathbf{W}$.

The path of the research seems clear. First the conditions of Theorem 3 need to be clarified to show that they clearly apply to transfer matrices that would occur in practice. Then an approach similar to that in Chapter 3 could be taken. Namely, we could perform a Taylor series expansion on the relevant terms of Equation 5.7 in order to get some analytical approximation for γ_d to first order in the variance of the disordered parameter. Then we would be in position to calculate localization factors numerically and have some analytical results with which to compare them.

5.5 Summary

In this chapter we have tackled the very difficult problem of localization in one-dimensional multiwave disordered periodic systems. The multiwave nature increases the complexity of analysis considerably compared to the localization problem in mono-coupled periodic structures. Our first task was to clarify the assumptions on our wave transfer matrices, after which we appealed to the theorem of Oseledets to understand the asymptotic behavior of products of random multiwave matrices. We noted that the theorem of Furstenberg was of little use here. The principal contribution of the chapter was the derivation of the multiwave localization factor (the dth Lyapunov exponent) as a function of the transmission matrix for the disordered system. This issue has been addressed, but in our view unsatisfactorily, by a number of solid state physicists. Thus our results and insights should have some impact in the solid state field where traditionally most of the localization work has been done. In addition, the recent work of [Pichard and Sarma 81-1,Pichard and Sarma 81-2] indicates that our result may have some impact in clarifying the localization mechanism in two- and three-dimensional disordered systems. Finally, we pointed out the tools that can be used to analytically calculate the localization factor.

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

In this thesis we have explored the effects disorder has on the transmission properties of normally perfectly periodic structures. Disorder is known to spatially localize the mode shapes of disordered periodic systems, so the term localization is used to describe the various dynamic manifestations of disorder. The localization phenomenon has been most extensively studied in the context of solid state physics and only recently with disordered systems of interest to the engineer in mind.

This thesis has provided the tools with which engineers can decide the importance of the dynamic effects of disorder on mono-coupled periodic structures. The first principal contribution was the elucidation of random transfer matrix techniques to model disordered systems and calculate transmission properties. This included a discussion of the important transformation to wave transfer matrix form and the relevance of the theorems of Furstenberg and Oseledets to the one-dimensional localization problem.

The second principal contribution was the calculation of localization effects as a function of frequency for three periodic models of interest to the structural dynamicist. In most instances the localization effects were found to be strongest near the stopbands of the normally perfectly periodic structures. This result indicates that care must be taken when doing experimental work at frequencies near the stopbands of what are ostensibly periodic structures. Effects of length disorder in the bays were quite pronounced, even at high frequency.

The third principal contribution was the derivation of the localization factor for multiwave one-dimensional systems as a function of the transmission matrix.

6.2 Recommendations

The localization phenomenon is a fascinating and difficult problem to tackle. This thesis has presented some very useful tools that have allowed us to make some important progress in understanding localization effects. The primary recommendation is to continue work with random transfer matrices and theories on products of random matrices to gain further insights about the phenomenon. The tools we have discussed in this thesis have immediate applicability to many other fields of engineering that involve disordered periodic systems, as well as the field of solid state physics where localization work is traditionally done.

The analytical formula for calculating the localization factor to first order in the variance could be extended to include higher order effects. This would allow us to predict analytically the transmission behavior for highly disordered systems at frequencies where the localization phenomenon is most strongly felt. Possibly some asymptotic analysis near the stopbands would be another alternative to pinning down the transmission behavior there analytically. The issue of localization in one-dimensional systems which include damping should be addressed as well as the manifestation of the phenomenon in finitely long structures with fixed boundary conditions.

The localization phenomenon in multiwave systems with the evanescent waves included should be studied more rigorously. This, however, will require a better understanding of the wave transfer matrices in these situations, for which there is a dearth of information in the literature. Indeed, we observe that there is a need for a comprehensive study of the interrelationship of admittance, impedance, real transfer, scattering and wave transfer matrices for both periodic and disordered periodic multiwave systems.

In Chapter 5 we have presented the background that could lead to an analytical formula, analogous to the single wave case, for localization effects in multiwave systems. This is a very important area of research needed to understand localization effects in multiwave systems. Also, as [Pichard and André 86] have pointed out, the one-dimensional multiwave analysis could prove to be the key to understanding the localization phenomenon in two- and three-dimensional systems. Localization of classical waves in two-dimensional systems has recently been studied by [Flesia et al 87]. Only after the analytical issues have been explored should we proceed to examine the numerical issues in multiwave one-dimensional analysis and possible extensions to higher dimensions. The results in Chapter 5 could potentially simplify the numerical computations considerably by eliminating the need to multiply huge chains of matrices.

While we think that the transfer matrix formalism is a powerful tool to study the localization phenomenon, we also feel that the Herbert-Jones-Thouless formula should be explored to see if it can be easily applied to structural dynamic systems. Some efforts in this direction have already been made by [Hodges and Woodhouse 83]. Also [Johnston and Kunz 83-2] have developed the corresponding formula for multiwave systems.

Other important issues continue to be explored in the literature. Systems with correlated disorder among the bays, as opposed to the usual case of independent identically distributed random variables, have been studied by [Johnston and Kramer 86].

The impact of system nonlinearities on localization effects has been addressed by [Doucot and Rammal 87].[1] In the nonlinear case the transfer matrix formalism will be of little use.

An intentionally disordered periodic system could be valuable for the attenuation of propagating disturbances. However, if active control is performed on the same structure, the fact that the mode shapes are spatially localized may complicate the control effectiveness of actuators that are placed at locations where mode shapes have little amplitude.

Some experimental structural dynamic/acoustical verification of localization has been reported by [Hodges and Woodhouse 83,Pierre et al 86,Dépollier et al 86] and by [Hyde and Sybert 87], where the latter work was inconclusive. Further experimental work would clarify our analytical and numerical thinking. These experiments would have to be done with care. Initially, the dynamic characteristics of the perfectly periodic system should be understood experimentally. Clearly, the effects of damping and boundary conditions need to be taken into account when comparisons are made with our analytical results. In an actual experiment on a disordered system, the measurements would have to done over many realizations in order that the results could be compared with the theoretical prediction. The experimental techniques of [Hodges and Woodhouse 83,Roy and Plunkett 86] seem particularly attractive. In these cases a disturbance was inserted into one end of the system and the effects were measured at the other end. The beam with cantilevers of [Roy and Plunkett 86] was a perfectly periodic system but could be easily randomized and would provide an excellent structure to verify multiwave localization effects.

The study of the literature has provided invaluable insights into the localization phenomenon. Future researchers should continue to avail themselves of the work done on localization in many fields in order that maximum progress can be achieved in

[1] In solid state physics this is equivalent to considering electron-electron interactions.

understanding the effects of disorder.

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Appendix A

Matrix and Group Properties

In this appendix we collect most of the matrix and group properties mentioned in the thesis. First we note that all of the matrices in the thesis will be of even dimension, $2d \times 2d$, where d ranges from 1 to some finite value. In addition, all matrices will be invertible and so they are elements of the group $GL(2d, C)$. Here the letter G stands for the word *general* which means that the matrix is invertible. The letter L stands for the word linear. The $2d$ inside the parentheses implies the matrix dimension is $2d \times 2d$, and C tells us that in general the matrix elements are complex. If we were restricting ourselves to matrices with only real entries, C would of course be replaced by R.

Frequently we will make use of matrices which have unit determinant; these matrices are elements of $SL(2d, C)$. The letter S stands for the word *special* which means that the matrix has determinant equal to one.

Some of the more familiar matrices we will use are unitary matrices, which satisfy

$$\mathbf{W}^H\mathbf{W} = \mathbf{W}\mathbf{W}^H = \mathbf{I}$$

Note that unitary matrices are elements of $SU(2d)$. Symmetric matrices satisfy

$$\mathbf{W}^T = \mathbf{W}$$

even if they have complex entries, while Hermitian matrices satisfy

$$\mathbf{W}^H = \mathbf{W}$$

The symplectic (Sp) matrix group will be frequently encountered in the thesis. Symplectic matrices are always of even dimension and their group is identified as $Sp(d, C)$. A matrix $\mathbf{W}$ is symplectic if

$$\mathbf{W}^T\mathbf{J}\mathbf{W} = \mathbf{J}$$

where

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

where $\mathbf{I}$ is $d \times d$. Note that we take a transpose even though $\mathbf{W}$ is a complex matrix. The inverse of a symplectic matrix is easy to find:

$$\mathbf{W}^{-1} = -\mathbf{J}\mathbf{W}^T\mathbf{J}$$

An important property of symplectic matrices is that their eigenvalues occur in reciprocal pairs, λ and $\frac{1}{\lambda}$ [Bougerol and Lacroix 85]. It is also not difficult to prove that any 2×2 matrix with unit determinant is automatically symplectic. This tells us that $SL(2, C) = Sp(1, C)$.

The special unitary group, $SU(d, d)$ will be met in the thesis. A matrix $\mathbf{W}$ is an element of $SU(d, d)$ if

$$\mathbf{W}^H\mathbf{\Delta}\mathbf{W} = \mathbf{\Delta}$$

where

$$\mathbf{\Delta} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}$$

where again $\mathbf{I}$ is $d \times d$. The 2×2 matrices which are elements of $SU(1, 1)$ are of the form

$$\begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}$$

This matrix is in the so-called Cayley form [Hori 68].

We will make use of matrix singular values in the thesis. Any reader not already familiar with singular values and the singular value decomposition of a matrix is encouraged to consult [Noble and Daniel 77]. The singular values, σ_i, of a complex $2d \times 2d$ invertible matrix $\mathbf{W}$ are

$$\sigma_i(\mathbf{W}) = \{\lambda_i(\mathbf{W}^H \mathbf{W})\}^{\frac{1}{2}} \quad i = 1, \cdots, 2d$$

where we assume that the σ_i are ordered such that $\sigma_i \geq \sigma_{i+1}$. Note that the singular values of a symplectic matrix will occur in reciprocal pairs σ and $\frac{1}{\sigma}$

The maximum singular value, $\sigma_{max}(\mathbf{W})$ coincides with the spectral norm of a matrix:

$$\sigma_{max}(\mathbf{W}) = \max_{\mathbf{z} \neq 0} \frac{\|\mathbf{W}\mathbf{z}\|_2}{\|\mathbf{z}\|_2} = \|\mathbf{W}\|_2$$

where $\|\mathbf{z}\|_2$ is the usual Euclidean length of the vector $\mathbf{z}$.

Another matrix norm that is useful is the Frobenius (sometimes called Euclidean) norm:

$$\begin{aligned} \|\mathbf{W}\|_F &= \{tr(\mathbf{W}^H \mathbf{W})\}^{\frac{1}{2}} \\ &= \{\sum_{i=1}^{d} \sum_{j=1}^{d} |w_{ij}|^2\}^{\frac{1}{2}} \end{aligned}$$

Appendix B

Derivation of Mono-Coupled Wave Transfer Matrices

In this appendix we show how the wave transfer matrix, $\mathbf{W}_j$ of a bay (ordered or disordered) is calculated for frequencies in the passbands of the normally periodic system. In terms of the left and right traveling wave amplitudes, $\overleftarrow{A}$ and $\overrightarrow{B}$, we have:

$$\begin{bmatrix} \overleftarrow{A}_j \\ \overrightarrow{B}_j \end{bmatrix} = \mathbf{W}_j \begin{bmatrix} \overleftarrow{A}_{j-1} \\ \overrightarrow{B}_{j-1} \end{bmatrix}$$

where

$$\mathbf{W}_j = \begin{bmatrix} \frac{1}{t_j} & -\frac{r_j}{t_j} \\ -\frac{r_j^*}{t_j^*} & \frac{1}{t_j^*} \end{bmatrix}$$

The approach is to express our traveling wave amplitudes first in terms of a state vector involving generalized displacements, then to express the wave amplitudes in terms of a state vector which includes generalized displacements and generalized forces. This latter relationship is what we desire because all of our real transfer matrices involve a state vector which includes generalized displacements and forces, namely,

$$\begin{bmatrix} u_j \\ f_j \end{bmatrix} = \mathbf{T}_j \begin{bmatrix} u_{j-1} \\ f_{j-1} \end{bmatrix} \tag{B.1}$$

where

$$\mathbf{T}_j = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

and where u_j is a generalized displacement and f_j is a generalized force. Note that u_j and f_j may be nondimensional. When u_j is nondimensional, then the wave amplitudes will be nondimensional as well. Again $\mathbf{T}_j$ is the transfer matrix for the periodic or disordered system.

The generalized displacements of the perfectly periodic system can be expressed in terms of the wave amplitudes via

$$\begin{bmatrix} u_j \\ u_{j-1} \end{bmatrix} = \mathbf{Q} \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ e^{-ik} & e^{ik} \end{bmatrix}$$

Note that k is the wave number of the perfectly periodic system.

Now from B.1 we find

$$\begin{bmatrix} u_j \\ f_j \end{bmatrix} = \mathbf{V} \begin{bmatrix} u_j \\ u_{j-1} \end{bmatrix}$$

where

$$\mathbf{V} = \begin{bmatrix} 1 & 0 \\ T_{22}T_{12}^{-1} & T_{21} - T_{22}T_{12}^{-1}T_{11} \end{bmatrix}$$

So now we find

$$\begin{bmatrix} u_j \\ f_j \end{bmatrix} = \mathbf{X} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{B.2}$$

where

$$\mathbf{X} = \mathbf{VQ}$$

Equations B.1 and B.2 imply

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \mathbf{X}^{-1}\mathbf{T}_j\mathbf{X} \begin{bmatrix} A_{j-1} \\ B_{j-1} \end{bmatrix}$$

So the wave transfer matrix for a single bay is

$$\mathbf{W}_j = \mathbf{X}^{-1}\mathbf{T}_j\mathbf{X} \tag{B.3}$$

Note that we have used the perfectly periodic wave basis to derive our wave transfer matrix, whether the real transfer matrix is random or not. The columns of the matrix $\mathbf{X}$ are the eigenvectors of the transfer matrix of the perfectly periodic system. When the transfer matrices are random, the eigenvector matrix will be that for the *average* transfer matrix. So for the perfectly periodic system in the passband the wave transfer matrix looks like

$$\mathbf{W}_j = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix}$$

while in general

$$\mathbf{W}_j = \begin{bmatrix} \frac{1}{t_j} & -\frac{r_j}{t_j} \\ -\frac{r_j^*}{t_j^*} & \frac{1}{t_j^*} \end{bmatrix}$$

Note that both matrices are elements of $SU(1,1)$.

Finally, we note that $\frac{1}{t_j}$ can be shown to be invariant with respect to the scaling of the eigenvector similarity transformation used in Equation B.3, while $\frac{r_j}{t_j}$ will be off by at most a magnitude and a phase factor. Using the eigenvector transformation $\mathbf{X}$ defined above, though, we are guaranteed to get exactly the wave transfer matrix, $\mathbf{W}_j$.

Appendix C

Models of Three Periodic and Disordered Periodic Structures

In this appendix the three periodic structures examined in the thesis are described. The first system is a chain of springs and masses. The second structure is a rod in longitudinal compression with attached resonators. The final structure is a Bernoulli-Euler beam on simple supports. For each system the transfer matrix for a typical bay of the perfectly periodic structure is presented along with the associated state vector. Also shown are the eigenvector similarity transformations which induce a wave transfer matrix. Most variables are nondimensionalized in the transfer matrix descriptions. Then a single variable is randomized and the associated transfer matrix is presented, along with the relevant terms in the wave transfer matrix. Some general properties of transfer matrices are discussed in [Rubin 64].

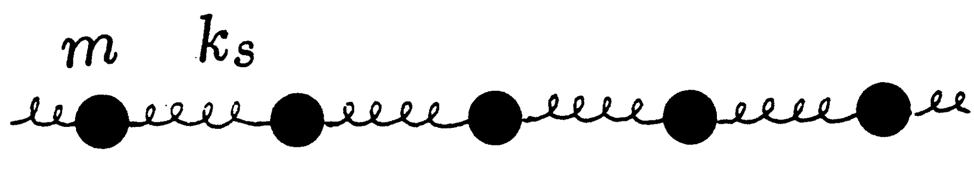

Figure C.1: Mass-spring chain.

C.1 Mass-Spring Chain

A chain of springs and masses is one of the simplest periodic structures we can examine. The system is pictured in Figure C.1 and a typical bay is shown in Figure C.2. This choice of bay (as opposed to one involving a spring and a half of two masses, for example) ensures that the det(**T**) = 1 whether m, k_s, or both are disordered. For this bay:

$$\begin{bmatrix} d_j \\ \frac{f}{k_s}_j \end{bmatrix} = \begin{bmatrix} 1-\frac{m\omega^2}{k_s} & -1 \\ \frac{m\omega^2}{k_s} & 1 \end{bmatrix} \begin{bmatrix} d_{j-1} \\ \frac{f}{k_s}_{j-1} \end{bmatrix}$$

Here d_j is the displacement of the jth mass and f_j is the force on the jth mass. Note that $\frac{f}{k_s}_j$ has units of displacement as does d_j. Let $\bar{\omega}^2 = \frac{\omega^2}{4k_s/m}$, which is the frequency at which the passband ends, then

$$\mathbf{T} = \begin{bmatrix} 1-4\bar{\omega}^2 & -1 \\ 4\bar{\omega}^2 & 1 \end{bmatrix}$$

Figure C.2: One bay of mass-spring chain used to form its transfer matrix.

From the condition that $|tr(\mathbf{T})| < 2$ in a passband (see Chapter 2), we see that a single passband exists for the perfectly periodic system at $0 < \bar{\omega} < 1$. All higher frequencies are in the stopband. The wave number (the spatial frequency of the traveling waves) for the traveling waves in the passband of the mass-spring chain is governed by

$$\cos k = 1 - 2\bar{\omega}^2$$

A more extensive discussion of the mass-spring system can be found in [Faulkner and Hong 85].

The eigenvector similarity transformation used here, which will induce a wave transfer matrix (see Appendix B), is:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ e^{-ik} - 1 & e^{ik} - 1 \end{bmatrix}$$

and its inverse is

$$\mathbf{X}^{-1} = \begin{bmatrix} (e^{ik} - 1)/(2i \sin k) & -1/(2i \sin k) \\ (1 - e^{-ik})/(2i \sin k) & 1/(2i \sin k) \end{bmatrix}$$

C.1.1 Only Masses Disordered

Now consider disordering only the masses, i.e., let the mass be a random variable and let $\mu_j = \frac{m_j}{<m_j>}$, where $< m_j >= m$ so

$$\mathbf{T}(\mu_j) = \begin{bmatrix} 1 - \mu_j 4\bar{\omega}^2 & -1 \\ \mu_j 4\bar{\omega}^2 & 1 \end{bmatrix}$$

The corresponding wave transfer matrix is, where we suppress the subscript j on the transmission and reflection coefficients.

$$\mathbf{W}(\mu_j) = \mathbf{X}^{-1}\mathbf{T}(\mu_j)\mathbf{X} = \begin{bmatrix} \frac{1}{t} & -\frac{r}{t} \\ -\frac{r^*}{t^*} & \frac{1}{t^*} \end{bmatrix}$$

where

$$\frac{1}{t} = e^{ik}(1 - i\delta_j) \tag{C.1}$$

and

$$-\frac{r}{t} = -e^{ik} i\delta_j$$

where

$$\delta_j = \frac{2\bar{\omega}^2(1-\mu_j)}{\sin k}$$

C.1.2 Only Springs Disordered

Now consider disordering only the springs, i.e., let k_s be a random variable and let $\tilde{k}_{sj} = \frac{k_{sj}}{<k_{sj}>}$, where $< k_{sj} >= k_s$. The transfer matrix is:

$$\mathbf{T}(\tilde{k}_{sj}) = \begin{bmatrix} 1 - \frac{4\bar{\omega}^2}{\tilde{k}_{sj}} & -\frac{1}{\tilde{k}_{sj}} \\ 4\bar{\omega}^2 & 1 \end{bmatrix}$$

In the corresponding wave transfer matrix

$$\frac{1}{t} = e^{ik}(1 - i\delta_j)$$

and

$$-\frac{r}{t} = i\delta_j$$

where now

$$\delta_j = \frac{2\bar{\omega}^2}{\sin k}(1 - \frac{1}{\tilde{k}_{sj}})$$

C.1.3 Masses and Springs Disordered

Finally with both the masses and springs disordered we have:

$$\mathbf{T}(\mu_j, \tilde{k}_{sj}) = \begin{bmatrix} 1 - \frac{4\bar{\omega}^2\mu_j}{\tilde{k}_{sj}} & -\frac{1}{\tilde{k}_{sj}} \\ 4\bar{\omega}^2\mu_j & 1 \end{bmatrix}$$

Note that we have no need to compute the wave transfer matrix in the calculation of the localization factor when both the masses and springs are disordered, because of the additive nature of the localization factor discussed in Chapter 3.

C.2 Rod with Attached Resonators

The second model is a longitudinal wave carrying rod with attached resonators that represent the vibrating cross-members present in a real truss structure. The model and relevant properties are shown in Figure C.3.

The transfer equation for the perfectly periodic model is:

$$\begin{bmatrix} \bar{U}_{j+1} \\ \bar{N}_{j+1} \end{bmatrix} = \begin{bmatrix} c + \frac{\bar{H}s}{2\bar{s}} & \frac{s}{\bar{s}} - \frac{\bar{H}(1-c)}{2\bar{s}^2} \\ \bar{s}s + \frac{\bar{H}(1+c)}{2} & c + \frac{\bar{H}s}{2\bar{s}} \end{bmatrix} \begin{bmatrix} \bar{U}_j \\ \bar{N}_j \end{bmatrix} \qquad \text{(C.2)}$$

where

$$\bar{U}_j = \frac{U_j}{l}$$

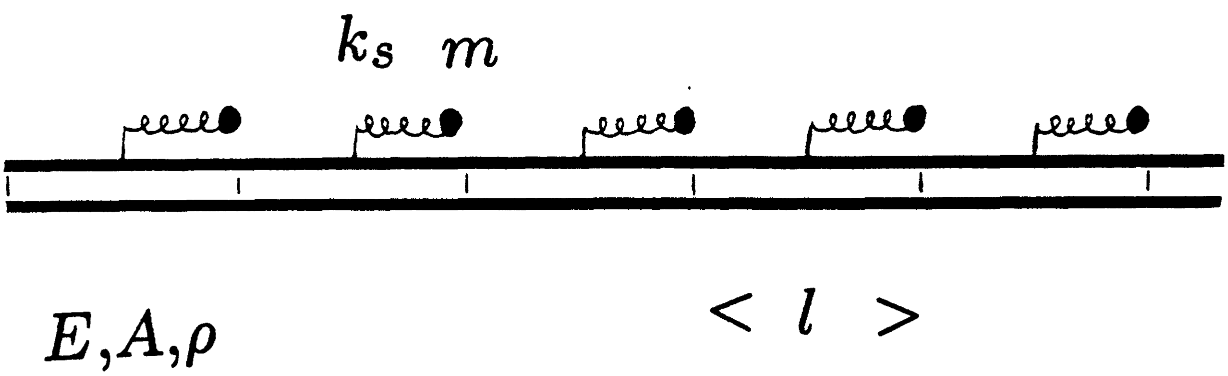

Figure C.3: Rod with attached resonators

is the nondimensional longitudinal displacement of the jth point and

$$\bar{N}_j = \frac{N_j}{EA}$$

is the nondimensional internal force at the jth point. Also

$$c = cos\pi\bar{\omega}$$

$$s = isin\pi\bar{\omega}$$

$$\bar{s} = i\pi\bar{\omega}$$

where the nondimensional transfer function of the attached resonator is:

$$\bar{H} = \left(\frac{1}{\bar{k}_s} - \frac{1}{\bar{\omega}^2\pi^2\bar{\mu}}\right)^{-1}$$

and where the nondimensional frequency, stiffness and mass are:

$$\bar{\omega} = \frac{\omega l(\frac{\rho}{E})^{\frac{1}{2}}}{\pi}$$

$$\bar{k}_s = \frac{k_s l}{EA}$$
$$\bar{\mu} = \frac{m}{(\rho A l)}$$

The transfer matrix models a bay extending across a length of rod, across a resonator, and then across another length of rod.

A discussion of the dynamic characteristics of the perfectly periodic structure can be found in [von Flotow 82]. For our work on the rod with attached resonators, we will use $\bar{\mu} = .2$ and $\bar{k}_s = .5$. These values put our first stopband around $\bar{\omega} = .5033$ which is the natural frequency of the attached resonator. This particular stopband frequency makes for ease of presentation of localization effects in the first passband. In real structures the stopband associated with the resonant frequency of a cross-member is likely to be much closer to $\bar{\omega} = 0$.

The wave number k for the passbands of the perfectly periodic structure is determined by

$$\cos k = c + \frac{\bar{H}s}{2\bar{s}}$$

The eigenvector similarity transformation that induces the wave transfer form is

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ i \sin k / (\frac{s}{\bar{s}} - \frac{\bar{H}(1-c)}{2\bar{s}^2}) & -i \sin k / (\frac{s}{\bar{s}} - \frac{\bar{H}(1-c)}{2\bar{s}^2}) \end{bmatrix}$$

and

$$\mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & -i(\frac{s}{\bar{s}} - \frac{\bar{H}(1-c)}{2\bar{s}^2})/(2 \sin k) \\ \frac{1}{2} & i(\frac{s}{\bar{s}} - \frac{\bar{H}(1-c)}{2\bar{s}^2})/(2 \sin k) \end{bmatrix}$$

C.2.1 Only Masses Disordered

Now disorder the mass, m, of the attached resonator. So let

$$\bar{\mu}_j = \frac{m_j}{\rho A l}$$

be the nondimensional random variable, $<\bar{\mu}_j>=\bar{\mu}$. Note that we do not feel compelled to divide by the average value of $\bar{\mu}$ because this variable is already nondimensional. Now the random transfer matrix $\mathbf{T}(\bar{\mu}_j)$ is found by replacing $\bar{\mu}$, which occurs in $\bar{H}$, in the nonrandom transfer matrix, $\mathbf{T}$, by $\bar{\mu}_j$. In the corresponding wave transfer matrix

$$\frac{1}{t}=e^{ik}(1-i\delta_j)$$

and

$$-\frac{r}{t}=-i\delta_j$$

where

$$\delta_j=\frac{(\sin\pi\bar{\omega})\Delta\bar{H}_j}{2(\sin k)\pi\bar{\omega}}$$

and

$$\Delta\bar{H}_j=(\frac{1}{\bar{k}_s}-\frac{1}{\bar{\omega}^2\pi^2\bar{\mu}_j})^{-1}-(\frac{1}{\bar{k}_s}-\frac{1}{\bar{\omega}^2\pi^2\bar{\mu}})^{-1}$$

C.2.2 Only Springs Disordered

Now consider disordering the springs of the attached resonators. Let

$$\bar{k}_{sj}=\frac{k_{sj}l}{EA}\ \textit{where}\ <\bar{k}_{sj}>=\bar{k}_s$$

be the nondimensional random variable. The transfer matrix $\mathbf{T}(\bar{k}_{sj})$ is the nonrandom transfer matrix with $\bar{k}_s$ replaced by $\bar{k}_{sj}$. The wave transfer matrix is the same as for the disordered masses except that

$$\Delta\bar{H}_j=(\frac{1}{\bar{k}_{sj}}-\frac{1}{\bar{\omega}^2\pi^2\bar{\mu}})^{-1}-(\frac{1}{\bar{k}_s}-\frac{1}{\bar{\omega}^2\pi^2\bar{\mu}})^{-1}$$

C.2.3 Only Lengths Disordered

Finally we examine the disordering of the bay length, i.e. the distance between the resonators. Let the nondimensional random variable be

$$\bar{l}_j=\frac{l_j}{<l_j>}$$

where $< l_j >= l$. The transfer matrix for the lengths disordered is

$$\mathbf{T}(\bar{l}_j) = \begin{bmatrix} \cos(\pi\bar{\omega}\bar{l}_j) + \frac{\bar{H}\sin(\pi\bar{\omega}\bar{l}_j)}{2\pi\bar{\omega}} & \frac{\sin(\pi\bar{\omega}\bar{l}_j)}{\pi\bar{\omega}\bar{l}_j} + \frac{\bar{H}[1-\cos(\pi\bar{\omega}\bar{l}_j)]}{2\pi^2\bar{\omega}^2\bar{l}_j} \\ -\pi\bar{\omega}\bar{l}_j\sin(\pi\bar{\omega}\bar{l}_j) + \frac{\bar{l}_j\bar{H}[1+\cos(\pi\bar{\omega}\bar{l}_j)]}{2} & \cos(\pi\bar{\omega}\bar{l}_j) + \frac{\bar{H}\sin(\pi\bar{\omega}\bar{l}_j)}{2\pi\bar{\omega}} \end{bmatrix}$$

In the corresponding wave transfer matrix

$$\frac{1}{t} = \alpha - \frac{i[\beta + \nu]}{2\sin k}$$

and

$$-\frac{r}{t} = \frac{i[-\beta + \nu]}{2\sin k}$$

where α is the (1,1) term of $\mathbf{T}(\bar{l}_j)$ and where

$$\beta = [-\pi\bar{\omega}\bar{l}_j\sin(\pi\bar{\omega}\bar{l}_j) + \frac{\bar{l}_j\bar{H}[1+\cos(\pi\bar{\omega}\bar{l}_j)]}{2}][\frac{s}{\bar{s}} - \frac{\bar{H}(1-c)}{2\bar{s}^2}]$$

and

$$\nu = [\frac{\sin(\pi\bar{\omega}\bar{l}_j)}{\pi\bar{\omega}\bar{l}_j} + \frac{\bar{H}[1-\cos(\pi\bar{\omega}\bar{l}_j)]}{2\pi^2\bar{\omega}^2\bar{l}_j}][\bar{s}s + \frac{\bar{H}(1+c)}{2}]$$

C.2.4 All Three Parameters Disordered

Finally the transfer matrix for masses, springs and lengths disordered, $\mathbf{T}(\bar{\mu}_j, \bar{k}_{sj}, \bar{l}_j)$ is simply $\mathbf{T}(\bar{l}_j)$ with $\bar{H}$ replaced by

$$\bar{H}_j = (\frac{1}{\bar{k}_{sj}} - \frac{1}{\bar{\omega}^2\pi^2\bar{\mu}_j})^{-1}$$

C.3 Bernoulli-Euler Beam on Simple Supports

The final system examined is a Bernoulli-Euler beam on simple supports shown in Figure C.4. In setting up the transfer matrix for the beam on supports we will use much of the terminology of [Yang and Lin 75,Lin 76], except we nondimensionalize

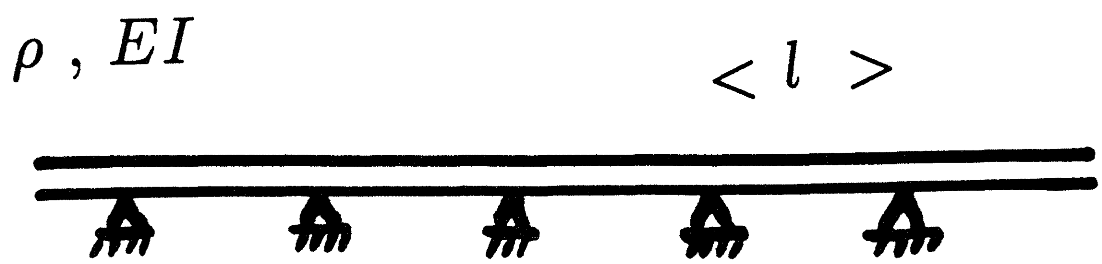

Figure C.4: Beam on simple supports.

where possible. The transfer matrix for a bay relates the slope, ϕ, and nondimensional moment at adjacent supports

$$\begin{bmatrix} \phi_j \\ \frac{lM}{EI}_j \end{bmatrix} = \begin{bmatrix} \cos k & \frac{\alpha}{\sqrt{\bar{\omega}}} \\ -\frac{\sqrt{\bar{\omega}}\sin^2 k}{\alpha} & \cos k \end{bmatrix} \begin{bmatrix} \phi_{j-1} \\ \frac{lM}{EI}_{j-1} \end{bmatrix} \tag{C.3}$$

where

$$\cos k = \frac{\sinh\sqrt{\bar{\omega}}\cos\sqrt{\bar{\omega}} - \cosh\sqrt{\bar{\omega}}\sin\sqrt{\bar{\omega}}}{\sinh\sqrt{\bar{\omega}} - \sin\sqrt{\bar{\omega}}}$$

where

$$\bar{\omega} = \omega\sqrt{\frac{\mu l^4}{EI}}$$

$$\mu = \textit{mass of beam per unit length}$$

and where (adopting the notation of [Yang and Lin 75,Lin 76])

$$\alpha = \frac{c_4}{s_3}$$

$$c_4 = (\cosh\sqrt{\bar{\omega}}\cos\sqrt{\bar{\omega}} - 1)/2$$

$$s_3 = (\sinh\sqrt{\bar\omega} - \sin\sqrt{\bar\omega})/2$$

$$s_4 = \sinh\sqrt{\bar\omega}\sin\sqrt{\bar\omega}$$

$$c_2 = (\cosh\sqrt{\bar\omega} - \cos\sqrt{\bar\omega})/2$$

The eigenvector similarity transformation which induces the wave transfer matrix is

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ i\frac{\sqrt{\bar\omega}}{\alpha}\sin k & -i\frac{\sqrt{\bar\omega}}{\alpha}\sin k \end{bmatrix}$$

and

$$\mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{i\alpha}{2\sqrt{\bar\omega}\sin k} \\ \frac{1}{2} & \frac{i\alpha}{2\sqrt{\bar\omega}\sin k} \end{bmatrix}$$

Now consider disordering the length l between each bay and let the nondimensional random length be

$$\hat{l}_j = \frac{l_j}{< l_j >}$$

where $< l_j >= l$. So the transfer matrix $\mathbf{T}(\hat{l}_j)$ can be written by simply replacing $\sqrt{\bar\omega}$ whenever it appears as an argument of sin, sinh, cos and cosh by $\sqrt{\bar\omega}\hat{l}_j$. Anywhere l appears it can be interpreted as $< l_j >$.

In the wave transfer matrix for the beam on disordered supports

$$\frac{1}{t} = \cos k(\hat{l}_j) + i[\frac{\alpha(\hat{l}_j)\sin k}{2\alpha} + \frac{\alpha\sin^2 k(\hat{l}_j)}{2\alpha(\hat{l}_j)\sin k}]$$

and

$$-\frac{r}{t} = i[\frac{\alpha\sin^2 k(\hat{l}_j)}{2\alpha(\hat{l}_j)\sin k} - \frac{\alpha(\hat{l}_j)\sin k}{2\alpha}]$$

Once again, whenever the argument $(\hat{l}_j)$ appears, it implies that the underlying circular and hyperbolic functions should have $\sqrt{\bar\omega}$ replaced by $\sqrt{\bar\omega}\hat{l}_j$.

Appendix D

A Simple Method to Calculate Localization Factors

This appendix describes the calculation of localization factors using a simple method which does not depend on theories involving products of random matrices. The method has given very good results for systems with sufficiently low randomness and over wide frequency ranges. The method is applied to systems that can be described with 2×2 transfer matrices and is a generalization of a result which appeared in [Akkermans and Maynard 84].

Briefly, the method involves taking a transfer matrix which is a function of a random variable and expanding it in terms of a Taylor series expansion about the average value of the random variable. Only the first two terms of the expansion are retained, after which they are converted to wave transfer form via the appropriate similarity transformation. Products of this low order form are taken, but only terms of order one are retained. From this low order representation of the matrix product, the transmission coefficient, τ_n, is extracted and the localization factor, γ, is calculated as

$$\gamma = -\frac{< \ln |\tau_n| >}{n}$$

First consider the bay transfer matrix which is a function of the random variable α, $\mathbf{T}(\alpha)$ or $\mathbf{T}$ for short. Now expand $\mathbf{T}$ in a Taylor series expansion about the mean value of α.

$$\mathbf{T} = \mathbf{T}|_{<\alpha>} + \delta\alpha\frac{\partial\mathbf{T}}{\partial\alpha}|_{<\alpha>} + \frac{(\delta\alpha)^2}{2!}\frac{\partial^2\mathbf{T}}{\delta\alpha^2}|_{<\alpha>} + \cdots$$

Consider retaining only the first two terms:

$$\mathbf{T} \doteq \mathbf{T}|_{<\alpha>} + \delta\alpha\frac{\partial\mathbf{T}}{\partial\alpha}|_{<\alpha>}$$

Now choose an eigenvector transformation that induces a wave transfer matrix, so

$$\mathbf{X}^{-1}\mathbf{T}\mathbf{X} \doteq \mathbf{X}^{-1}\mathbf{T}|_{<\alpha>}\mathbf{X} + \delta\alpha\mathbf{X}^{-1}\frac{\partial\mathbf{T}}{\partial\alpha}|_{<\alpha>}\mathbf{X} = \begin{bmatrix} e^{+ik} & \\ & e^{-ik} \end{bmatrix} + \delta\alpha\begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}$$

So now we have approximated the jth wave transmission matrix as

$$\mathbf{W}_j \doteq \begin{bmatrix} e^{ik} + (\delta\alpha_j)a & (\delta\alpha_j)b \\ (\delta\alpha_j)b^* & e^{ik} + (\delta\alpha_j)a^* \end{bmatrix}$$

Now let us calculate $\prod_{j=1}^{n}\mathbf{W}_j$ by retaining terms only to first order in $\delta\alpha_j$. Note for example that terms like $\delta\alpha_l\delta\alpha_j$ $l \neq j$ will vanish by mutual independence when averaging. The final result gives:

$$\prod_{j=1}^{n}\mathbf{W}_j \doteq \begin{bmatrix} e^{ink} + e^{(n-1)k}a(\sum_{j=1}^{n}\delta\alpha_j) & (2,1)^* \\ e^{-ink}[\sum_{j=1}^{n}\delta\alpha_j e^{i(2j-1)k}] & (1,1)^* \end{bmatrix}$$

The $(1,1)$ term of the above matrix product approximation is our approximation to $\frac{1}{\tau_n}$. From this, one can calculate $|\tau_n|^2$:

$$|\tau_n|^2 \doteq 1/[1 + a^*e^{ik}(\sum_{j=1}^{n}\delta\alpha_j) + ae^{-ik}(\sum_{j=1}^{n}\delta\alpha_j) + |a|^2(\sum_{j=1}^{n}\delta\alpha_j)^2]$$

Taking the natural log of $|\tau_n|^2$

$$\ln|\tau_n|^2 \doteq \ln(1) - \ln(1 + a^*e^{ik}(\sum_{j=1}^{n}\delta\alpha_j) + ae^{-ik}(\sum_{j=1}^{n}\delta\alpha_j) + |a|^2(\sum\delta\alpha_j)^2)$$

Recalling the following expansion:

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad |z| \leq 1,\ z \neq -1$$

So retaining terms to first order in z:

$$\ln|\tau_n|^2 \doteq -a^* e^{ik}(\sum_{j=1}^{n}\delta\alpha_j) - ae^{-ik}(\sum_{j=1}^{n}\delta\alpha_j) - |a|^2(\sum_{j=1}^{n}\delta\alpha_j)^2$$

Now taking the average of $\ln|\tau_n|^2$ and recalling that $<\delta\alpha_j>= 0$ and invoking independence of $\delta\alpha_j s$ we arrive at:

$$<\ln|\tau_n|^2>\doteq -|a|^2(\sum_{j=1}^{n}<(\delta\alpha_j)^2>) = -|a|^2 n\sigma_\alpha^2$$

Now

$$\gamma = -\frac{<\ln|\tau_n|>}{n}$$

or

$$\gamma = -\frac{<\ln|\tau_n|^2>}{2n} \doteq \frac{|a|^2\sigma_\alpha^2}{2}$$

which is the final result.

We find the result agrees with calculations from Equation 3.10 when $\frac{1}{t_j}$ has the forms $e^{ik}(1+i\delta_j)$ or $(1+i\delta_j)$. So this formula is valid for the mass-spring chain and the rod with disordered masses or springs on the attached resonators. The formula will not give accurate results for the rod with disordered lengths between resonators or for the beam with random lengths between supports.

Appendix E

Properties of the Scattering and Wave Transfer Matrices

In this appendix we discuss some of the properties of the scattering matrices and wave transfer matrices used in the thesis. These matrices will be used to describe the propagation of traveling waves in the passbands of periodic or disordered periodic structures. We will state the scattering and wave transfer matrices in their most general forms and then impose conditions on the scattering matrix and discuss what this implies for the wave transfer matrix. Note that we will suppress any subscripts on our transmission and reflection matrices. The scattering and wave transfer matrices are of dimension $2d \times 2d$. Scattering and wave transfer matrices are discussed in [Redheffer 61] and in [Carlin and Giordano 64,Hlawiczka 65] and for some specific disordered systems in [Osawa and Kotera 66,Omar and Schünemann 85].

The scattering matrix, $\mathbf{S}$, in its most general form is

$$\begin{bmatrix} \overleftarrow{\mathbf{A}}_{j-1} \\ \overrightarrow{\mathbf{B}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{r} & \mathbf{t} \\ \hat{\mathbf{t}} & \hat{\mathbf{r}} \end{bmatrix} \begin{bmatrix} \overrightarrow{\mathbf{B}}_{j-1} \\ \overleftarrow{\mathbf{A}}_j \end{bmatrix} \tag{E.1}$$

where $\overleftarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$ represent vectors of traveling wave amplitudes in the indicated directions. The corresponding wave transfer matrix involves a rearrangement of the state vector, so that we relate waves on the right of a bay to those on the left of a bay:

$$\begin{bmatrix} \overleftarrow{\mathbf{A}}_j \\ \overrightarrow{\mathbf{B}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{t}^{-1} & -\mathbf{t}^{-1}\mathbf{r} \\ \hat{\mathbf{r}}\mathbf{t}^{-1} & \hat{\mathbf{t}} - \hat{\mathbf{r}}\mathbf{t}^{-1}\mathbf{r} \end{bmatrix} \begin{bmatrix} \overleftarrow{\mathbf{A}}_{j-1} \\ \overrightarrow{\mathbf{B}}_{j-1} \end{bmatrix} \tag{E.2}$$

Now we require that the scattering matrix be symmetric. This means that

$$\mathbf{r} = \mathbf{r}^T$$

$$\hat{\mathbf{r}} = \hat{\mathbf{r}}^T$$

and

$$\mathbf{t} = \hat{\mathbf{t}}^T$$

These are exactly the same conditions needed for the symplecticity of the wave transfer matrix $\mathbf{W}$, namely that

$$\mathbf{W}^T\mathbf{J}\mathbf{W} = \mathbf{J}$$

be satisfied. Thus

$$\mathbf{S}\ \textit{symmetric} \Longleftrightarrow \mathbf{W}\ \textit{symplectic}$$

Now we impose the requirement that $\mathbf{S}$ be unitary, namely

$$\mathbf{S}^H\mathbf{S} = \mathbf{S}\mathbf{S}^H = \mathbf{I}$$

Now $\mathbf{S}^H\mathbf{S} = \mathbf{I}$ tells us that

$$\begin{aligned} \mathbf{r}^H\mathbf{r} + \hat{\mathbf{t}}^H\hat{\mathbf{t}} &= \mathbf{I} \\ \mathbf{t}^H\mathbf{t} + \hat{\mathbf{r}}^H\hat{\mathbf{r}} &= \mathbf{I} \\ \mathbf{r}^H\mathbf{t} + \hat{\mathbf{t}}^H\hat{\mathbf{r}} &= \mathbf{0} \end{aligned} \tag{E.3}$$

These are precisely the same conditions that must hold when $\mathbf{W}$ is an element of $SU(d,d)$ or

$$\mathbf{W}^H\mathbf{\Delta}\mathbf{W} = \mathbf{\Delta}$$

We conclude that

$$\mathbf{S}\ unitary \Longleftrightarrow \mathbf{W} \in SU(d,d)$$

Now imposing both symmetry and unitarity on the scattering matrix we have

$$\mathbf{S} = \begin{bmatrix} \mathbf{r} & \mathbf{t} \\ \mathbf{t}^T & -\mathbf{t}^{-1*}\mathbf{r}^*\mathbf{t} \end{bmatrix}$$

where $\mathbf{r} = \mathbf{r}^T$ and $-\mathbf{t}^{-1*}\mathbf{r}^*\mathbf{t} = -\mathbf{t}^T\mathbf{r}^*\mathbf{t}^{-H}$. Equivalently when the wave transfer matrix is symplectic and an element of $SU(d,d)$ we have

$$\mathbf{W} = \begin{bmatrix} \mathbf{t}^{-1} & -\mathbf{t}^{-1}\mathbf{r} \\ -\mathbf{t}^{-1*}\mathbf{r}^* & \mathbf{t}^{-1*} \end{bmatrix}$$

From the condition $\mathbf{t}^H\mathbf{t} + \hat{\mathbf{r}}^H\hat{\mathbf{r}} = \mathbf{I}$ above, we can prove that

$$0 < \mu_i[\mathbf{t}^H\mathbf{t}] \leq 1$$

where $\mu_i[*]$ is the ith eigenvalue of the indicated argument. Also note that

$$\mu_i[\mathbf{t}^H\mathbf{t}] = \mu_i[\mathbf{t}\mathbf{t}^H]$$

so that all the results stated below hold for $\mathbf{t}\mathbf{t}^H$ as well as $\mathbf{t}^H\mathbf{t}$. First we assume that $\mathbf{t}^H\mathbf{t}$ is invertible so that it is positive definite:

$$\mathbf{t}^H\mathbf{t} > 0$$

We also have that $\hat{\mathbf{r}}^H\hat{\mathbf{r}}$ is at least positive semi-definite:

$$\hat{\mathbf{r}}^H\hat{\mathbf{r}} \geq 0$$

From Equation E.3 we have

$$\mathbf{t}^H\mathbf{t} = \mathbf{I} - \hat{\mathbf{r}}^H\hat{\mathbf{r}}$$

Doing an eigenvector decomposition on the above equation we get

$$\begin{aligned} \mathbf{t}^H\mathbf{t} &= \mathbf{I} - \hat{\mathbf{r}}^H\hat{\mathbf{r}} \\ &= \mathbf{U}(\mathbf{I} - diag\{\mu_i[\hat{\mathbf{r}}^H\hat{\mathbf{r}}]\})\mathbf{U}^H \end{aligned}$$

The positive definiteness of $\mathbf{t}^H\mathbf{t}$ and the positive semi-definiteness of $\hat{\mathbf{r}}^H\hat{\mathbf{r}}$ now imply

$$0 \leq \mu_i[\hat{\mathbf{r}}^H\hat{\mathbf{r}}] < 1$$

and

$$0 < \mu_i[\mathbf{t}^H\mathbf{t}] \leq 1$$

which is the desired result.

Biography

Glen J. Kissel was born January 5, 1957, in Evansville, Indiana, where he attended grade school and three semesters at F. J. Reitz High School. He graduated from Protection High School in Protection, Kansas in 1975. His undergraduate work was done at the Oklahoma State University in Stillwater, Oklahoma, where he graduated with a B.S. in Mechanical Engineering (Aerospace Major) in 1979. There he was named the Outstanding Student in the School of Mechanical and Aerospace Engineering. He then began graduate studies at the Massachusetts Institute of Technology in Cambridge, Massachusetts, on the Donald W. Douglas Fellowship. He received his S.M. in Aeronautics and Astronautics in 1982. His research has focused on the dynamics and control of flexible space structures. Glen is a member of the AIAA, IEEE and Sigma Xi. He has published in the *Journal of Aircraft* and the *IEEE Control Systems Magazine.* He has worked summers at the Federal Aviation Administration in Washington, D.C., the John F. Kennedy Space Center in Florida and the C. S. Draper Laboratory in Cambridge, Massachusetts. Glen has published a number of articles on the U.S. space program and has appeared on television to talk about space spinoffs. He is interested in national and international politics and has debated a primary congressional candidate on the Strategic Defense Initiative. Glen's parents, Marlin and Agnes Kissel, and a sister, Debbie, reside in Pratt, Kansas. He will shortly take up employment in southern California.