An Elegant Lambert Algorithm for Multiple Revolution Orbits

by

Laura A. Loechler

S.B. Massachusetts Institute of Technology (1984)

Submitted to the Department of
Aeronautics and Astronautics in
Partial Fulfillment of the
Requirements for the Degree of

Master of Science
in Aeronautics and Astronautics
at the
Massachusetts Institute of Technology

May 1988

Signature of Author ______________________________________
Department of Aeronautics and Astronautics
May 18, 1988

Certified by ______________________________________
Dr. Richard H. Battin, Adjunct Professor
Thesis Supervisor

Accepted by ______________________________________
Professor Harold Y. Wachman
Chairman, Departmental Graduate Committee

An Elegant Lambert Algorithm for Multiple Revolution Orbits

by

Laura A. Loechler

Submitted to the Department of Aeronautics and Astronautics on May 18, 1988 in partial fulfillment of the requirements for the Degree of Master of Science in Aeronautics and Astronautics.

Abstract

A simple efficient algorithm to solve the non-unique multiple revolution time-constrained two-body two-point orbital boundary value Lambert's problem is of great interest for spacecraft interception and rendezvous and interplanetary transfer. The multiple revolution transfer orbits permit greater navigational flexibility and result in lower energy orbits than the unique direct transfer orbit. Two algorithms, based on the elegant algorithm for direct orbital transfer developed by R. H. Battin from Gauss' orbit determination method, definitively separate the low and high energy solutions for each complete revolution within the transfer. Both retain the excellent convergence characteristics of the direct transfer method, requiring only one additional iteration to converge to an additional four significant figures and having nearly uniform convergence over a wide range of problems.

First the derivation of the direct transfer algorithm is summarized. A simple approximation of the maximum number of complete revolutions that can be made within the given transfer time is suggested. Both the high and low energy algorithms are derived along the lines of the direct transfer algorithm. Tables display the number of iterations required for convergence for a large number of transfer problems, demonstrating the capability of these algorithms.

Thesis Supervisor: Dr. Richard H. Battin
Title: Adjunct Professor of Aeronautics and Astronautics

Acknowledgements

Thanks to my advisor, Prof. Battin, for his wonderful improvements on Gauss' method which made this thesis possible and his help and suggestions in pursuing various, fruitful and fruitless, lines of thought.

Special thanks to Brent Lander for his help with my Macintosh and his LaserWriter.

The concluding year of my Master's studies has been sponsored by my employer, the Naval Research Laboratory of Washington, D.C., through its Select Graduate Study Program. Thanks to my supervisor Tom Lawton and coworker Jim Young for their support and efforts in helping me obtain funding and time for my education.

Table of Contents

List of Figures

List of Tables

Chapter 1 Introduction

The time-constrained two-body two-point orbital boundary value problem, known as Lambert's problem, has historically been of interest in determining the elements of the orbits of planets and comets. Johann Heinrich Lambert (1728-1779) discovered that the time to transverse an elliptic arc, actually any conic, depends on only the semimajor axis and the distances from the initial and final points to the center of force and to each other. Leonhard Euler (1707-1783) developed the theorem for the parabolic case and Joseph-Louis Lagrange (1736-1813) proved the theorem for elliptic orbits. It is now of practical interest in spacecraft interception and rendezvous and interplanetary transfer.

Carl Friedrich Gauss (1777-1855) was interested in determination of the orbit of the asteroid Ceres from observations taken over a few days. He developed an elegant computational method for orbits with a small transfer angle between terminal points which is singular for a 180° transfer angle.

The limitations of Gauss' method make it impractical for a wide range of modern navigation problems. Richard H. Battin made a geometric transformation of the orbit that removed the singularity for the 180° transfers and with Robin M. Vaughan developed an algorithm similar to Gauss' that has good convergence for 0° to 360° transfers. Like Gauss, Battin makes no assumptions regarding the type of orbit: method works elegantly for hyperbolic, parabolic, and elliptic orbits as constrained by geometry and time-of-flight.

We are interested in building on this algorithm, hereafter called the Direct Transfer Algorithm, to exploit problems in which the time-of-flight is long enough that there exist elliptic orbits for which the time-of-flight is greater than one period. Then the orbit determination problem is not unique. In general each complete revolution adds two solutions. Utilizing these multiple revolution transfer orbits provides some navigational flexibility and results in lower energy transfer orbits.

The two solutions for each possible complete revolution can easily be determined from two algorithms which are simple extensions of the direct transfer algorithm, using the same geometric transformation and a successive substitution iterative method, and which have the same or better convergence than the direct transfer algorithm.

Chapter 2 Direct Transfer Algorithm

2.1 Lambert's Problem Definitions

Because the multiple revolution algorithms are extensions of the direct transfer algorithm, consider a brief derivation of the direct transfer algorithm.

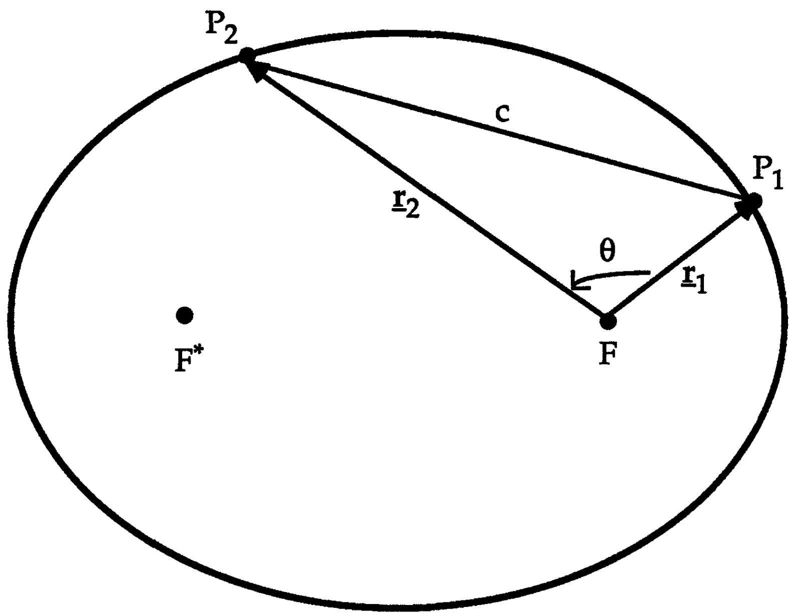


Figure 1 Geometry of Elliptic Lambert Problem

For simplicity consider only elliptic orbits as shown if Figure 1, although the direct transfer algorithm is valid for all orbits. Define:

Δt	time-of-flight
$\left.\begin{matrix}\underline{r}_1 \\ \underline{r}_2\end{matrix}\right\}$	position vectors of the initial point P_1 and the terminal point P_2 from the center of force F
$\left.\begin{matrix}r_1 = \lvert\underline{r}_1\rvert \\ r_2 = \lvert\underline{r}_2\rvert\end{matrix}\right\}$	distance from F to P_1 and P_2 respectively
$c = \lvert\underline{r}_2 - \underline{r}_1\rvert$	distance from P_1 to P_2
a	semimajor axis of orbit

a_m — smallest possible semimajor axis of any orbit connecting P_1 and P_2

$s = \left\{ \begin{array}{c} \dfrac{(r_1 + r_2 + c)}{2} \\ 2a_m \end{array} \right\}$ — semiperimeter of the triangle FP_1P_2

θ — angle from $\underline{r}_1$ to $\underline{r}_2$

For convenience define the non-dimensional parameter λ $(-1 < \lambda < 1)$ to describe the Lambert's problem geometry,

$$\lambda \equiv \frac{\sqrt{r_1 r_2}}{s} \cos\frac{\theta}{2} = \pm\sqrt{\frac{s-c}{s}}$$

where λ is positive for $0 < \theta < \pi$ and negative for $\pi < \theta < 2\pi$. Also define a time-of-flight T normalized to the minimum period,

$$T \equiv \sqrt{\frac{\mu}{a_m^3}}\Delta t = \sqrt{\frac{8\mu}{s^3}}\Delta t$$

where $\mu \equiv G(m_1 + m_2)$. λ and T are the algorithm inputs.

2.2 Geometrical Transformation of Orbit

Lambert's Theorem states that the time-of-flight is a function solely of the semimajor axis, the sum of the distances of the initial and final points from the center of force, and the length of the chord joining these points.[1]

$$\sqrt{\mu}\,\Delta t = F(a, r_1 + r_2, c)$$

Therefore any transformation that changes the shape of the orbit but fixes a, $r_1 + r_2$, and c is valid.

The transformation that is used moves the focus so that the mean point, the point at which the orbital tangent is parallel to the chord, coincides with pericenter or apocenter as shown in Figure 2.

[1]Battin, *Introduction to the Mathematics and Methods of Astrodynamics*, p. 276.

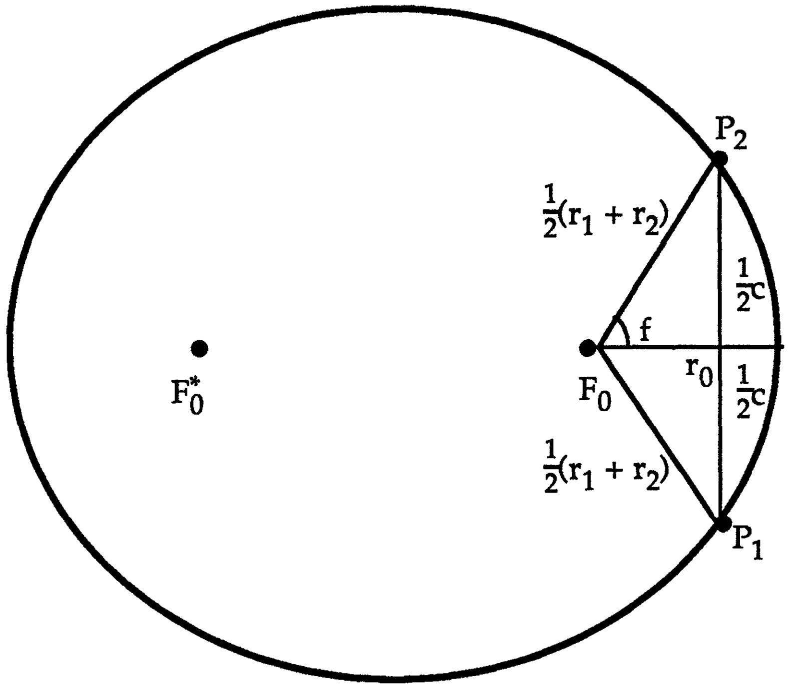


Figure 2 Transformed Elliptical Orbit

The equation for the pericenter radius must now equal that of the mean point r_0,[1]

$$r_0 = a\,(1 - e_0) = r_{0p} \sec^2 \frac{1}{4}(E_2 - E_1) \tag{2.1}$$

where e_0 is the eccentricity of the transformed orbit and r_{0p} is the mean point of the parabolic orbit connecting P_1 and P_2,[2]

$$r_{0p} = \frac{1}{4}(r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\frac{\theta}{2}) = \frac{1}{4} s\,(1 + \lambda)^2 \tag{2.2}$$

The true anomaly of the transformed orbit is a function of constants of the transfer,[3]

$$\sin f = \frac{\frac{1}{2}c}{\frac{1}{2}(r_1 + r_2)} = \frac{(1 - \lambda^2)}{(1 + \lambda^2)} \qquad \cos f = \frac{\sqrt{r_1 r_2}\cos\frac{\theta}{2}}{\frac{1}{2}(r_1 + r_2)} = \frac{2\lambda}{(1 + \lambda^2)} \tag{2.3}$$

---

[1] *ibid.*, eqn. 6.79
[2] *ibid.*, eqn. 6.77.
[3] *ibid.*, eqn. 7.79.

Since the eccentric anomaly of the mean point is the arithmetic mean of E_1 and E_2, the eccentric anomaly E corresponding to f must clearly be half the difference of E_2 and E_1[1]

$$E = \frac{1}{2}(E_2 - E_1)$$

and must be related to f by the identity

$$\tan\frac{f}{2} = \sqrt{\frac{1+e_0}{1-e_0}}\tan\frac{E}{2} \tag{2.4}$$

The time-of-flight can be written as Kepler's equation[2]

$$\frac{1}{2}\sqrt{\frac{\mu}{a^3}}\Delta t = E - e_0 \sin E \tag{2.5}$$

2.3 Transformation of Time-of-Flight to Cubic Equation

The next objective is to rewrite the time-of-flight equation 2.5 in terms of E and transfer constants. Replace the semimajor axis using the mean point expression, equation 2.1,

$$\frac{r_{0p}}{a} = (1-e_0)\cos^2\frac{E}{2}$$

Then the time-of-flight is

$$\frac{1}{2}\sqrt{\frac{\mu}{8r_{0p}^3}}\Delta t\left[2(1-e_0)\cos^2\frac{E}{2}\right]^{\frac{3}{2}} = E - \sin E + (1-e_0)\sin E$$

Replace $(1-e_0)$ using the identity 2.4 relating f to E,

$$(1-e_0) = \frac{2\tan^2\frac{E}{2}}{\tan^2\frac{f}{2} + \tan^2\frac{E}{2}}$$

---

[1] *ibid.*, eqn. 7.81.
[2] *ibid.*, eqn. 7.80.

$$\sqrt{\frac{\mu}{8r_{0p}^3}}\,\Delta t\,\frac{4\tan^3\frac{E}{2}}{\left[\left(\tan^2\frac{f}{2}+\tan^2\frac{E}{2}\right)\left(1+\tan^2\frac{E}{2}\right)\right]^{\frac{3}{2}}} \tag{2.6}$$

$$= E - \frac{2\tan\frac{E}{2}}{1+\tan^2\frac{E}{2}} + \frac{4\tan^3\frac{E}{2}}{\left(\tan^2\frac{f}{2}+\tan^2\frac{E}{2}\right)\left(1+\tan^2\frac{E}{2}\right)}$$

Define the dependent variable for this method as

$$x \equiv \tan^2\frac{E}{2}$$

This is convenient because $\frac{E}{2}$ ranges from 0 to $\frac{\pi}{2}$ exclusive so there are no sign ambiguities. Also define the geometric parameter

$$\ell \equiv \tan^2\frac{f}{2} = \frac{1-\cos f}{1+\cos f} = \left(\frac{1-\lambda}{1+\lambda}\right)^2$$

The normalized time-of-flight equation is then

$$T = \frac{(1+\lambda)^3}{2x}\sqrt{\frac{(\ell+x)(1+x)}{x}}\left[(\ell+x)(1+x)\tan^{-1}\sqrt{x} - \sqrt{x}\,(\ell-x)\right] \tag{2.7}$$

Figure 3 contains a graph of T versus x for various λ. Note the change of curvature in the $\lambda = -0.99$ curve. The simple Newton's method will not work for λ approaching -1, or transfer angles near 360°.

Define the time-of-flight parameter,

$$m \equiv \frac{\mu\Delta t^2}{8r_{0p}^3} = \frac{8\mu\Delta t^2}{s^3(1+\lambda)^6} = \frac{T^2}{(1+\lambda)^6}$$

substituting for r_{0p} from equation 2.2, so that the time-of-flight equation 2.6 can be written

$$\left[\frac{m}{(\ell+x)(1+x)}\right]^{\frac{3}{2}} = \frac{m}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) + \left[\frac{m}{(\ell+x)(1+x)}\right] \tag{2.8}$$

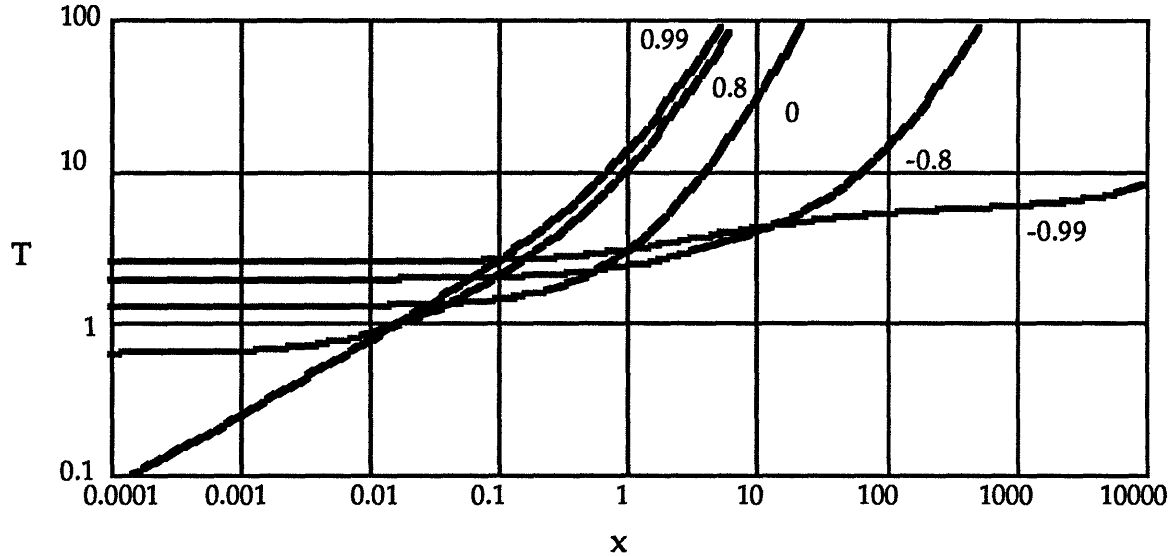


Figure 3 Time-of-Flight as a Function of x for Direct Transfer

The equation can be written as a cubic by defining

$$y^2 \equiv \frac{m}{(\ell + x)(1 + x)} \tag{2.9}$$

All quantities, ℓ, m, and x, are always positive so there are no problems in taking the square root. Then the time-of-flight equation 2.8 is

$$y^3 - y^2 - \frac{m}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1 + x}\right) = 0 \tag{2.10}$$

The equation can be solved using a successive substitution algorithm:

(1) Make an initial estimate of x.

(2) Calculate $\frac{m}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1 + x}\right)$.

(3) Solve cubic equation 2.10 for y.

(4) Determine a new value for x by inverting the definition of y, using the positive sign for the radical since x must be positive.

$$x = \sqrt{\left(\frac{1-\lambda}{2}\right)^2 + \left(\frac{m}{y^2}\right)} - \left(\frac{1+\lambda}{2}\right) \qquad (2.11)$$

(5) Go back to step (2) and repeat until x no longer changes within a specified tolerance.

A graphic interpretation is shown is Figure 4. An initial value of x is chosen and used to solve the cubic equation, represented by the vertical line from the x axis to the thick solid line which is the locus of the positive real solution to the cubic equation. That value of y is used to solve for a new value of x, represented by the horizontal line from the locus of the cubic solution to the thick dotted line which is the locus of the definition of y. The process is repeated until the solution, the intersection of the solution to the cubic equation and the definition of y, is reached. Depending on the shape of the curves, the iteration process may converge quickly or may require many steps.

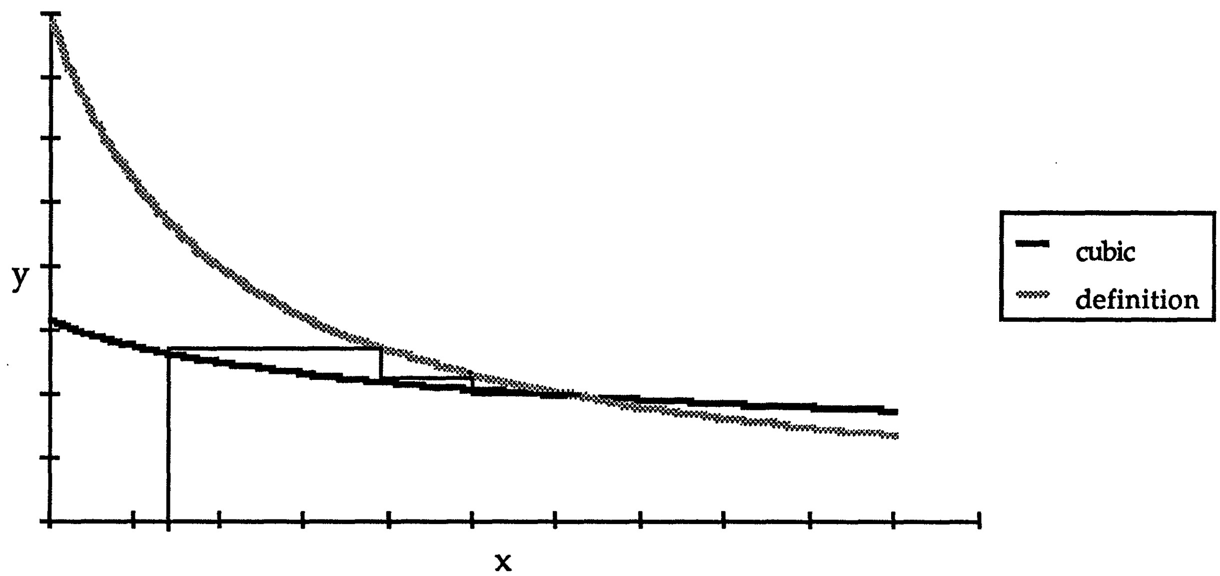


Figure 4 Solution Path for Direct Transfer *

---

* The examples in this thesis were generated using $\lambda = 0$ and $T = 16$.

2.4 Use of Free Parameter to Flatten Cubic Equation

To improve the convergence of the successive substitution algorithm introduce a free parameter, h_1, that can be chosen to flatten the root of the cubic equation at the solution point. Using the definition of y, equation 2.9, rewrite the cubic equation 2.10,

$$y^3 - (1 + h_1)\, y^2 - m\left[\frac{1}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) + \frac{h_1}{(\ell + x)(1+x)}\right] = 0 \qquad (2.12)$$

Differentiate

$$[3y^2 - 2(1 + h_1)\, y]\,\frac{dy}{dx} - \left[y^2 - \frac{m}{(\ell + x)(1+x)}\right]\frac{dh_1}{dx}$$

$$- m\left\{\frac{d}{dx}\left[\frac{1}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right)\right] - h_1\,\frac{d}{dx}\frac{1}{(\ell + x)(1+x)}\right\} = 0$$

At the solution point the second term is zero. The free parameter is chosen so that $\frac{dy}{dx}$ is zero. Therefore h_1 is given by

$$\frac{d}{dx}\left[\frac{1}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right)\right] - h_1\,\frac{d}{dx}\frac{1}{(\ell + x)(1+x)} = 0$$

$$h_1 = \frac{(\ell + x)^2}{4x^2(1 + 2x + \ell)}\left[3\,(1 + x)^2\,\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - (3 + 5x)\right] \qquad (2.13)$$

Let

$$h_2 \equiv m\left[\frac{1}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) + \frac{h_1}{(\ell + x)(1+x)}\right]$$

$$h_2 = \frac{m}{4x^2(1 + 2x + \ell)}\left\{[x^2 - (1 + \ell)x - 3\ell]\,\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} + (3\ell + x)\right\} \qquad (2.14)$$

The cubic equation 2.12, which is now only "correct" at the solution point, is of the form

$$y^3 - (1 + h_1)\, y^2 - h_2 = 0 \qquad (2.15)$$

The successive substitution algorithm is the essentially the same:

(1) Make an initial estimate of x.

(2) Calculate h_1 and h_2 from equations 2.13 and 2.14.

(3) Solve cubic equation 2.15 for y.

(4) Determine a new value for x from equation 2.11.

(5) Go back to step (2) and repeat until x no longer changes within a specified tolerance.

A graphic interpretation of a successive substitution algorithm is shown in Figure 5. In the vicinity of the solution the locus of the cubic solution is flat, making fewer iterations necessary.

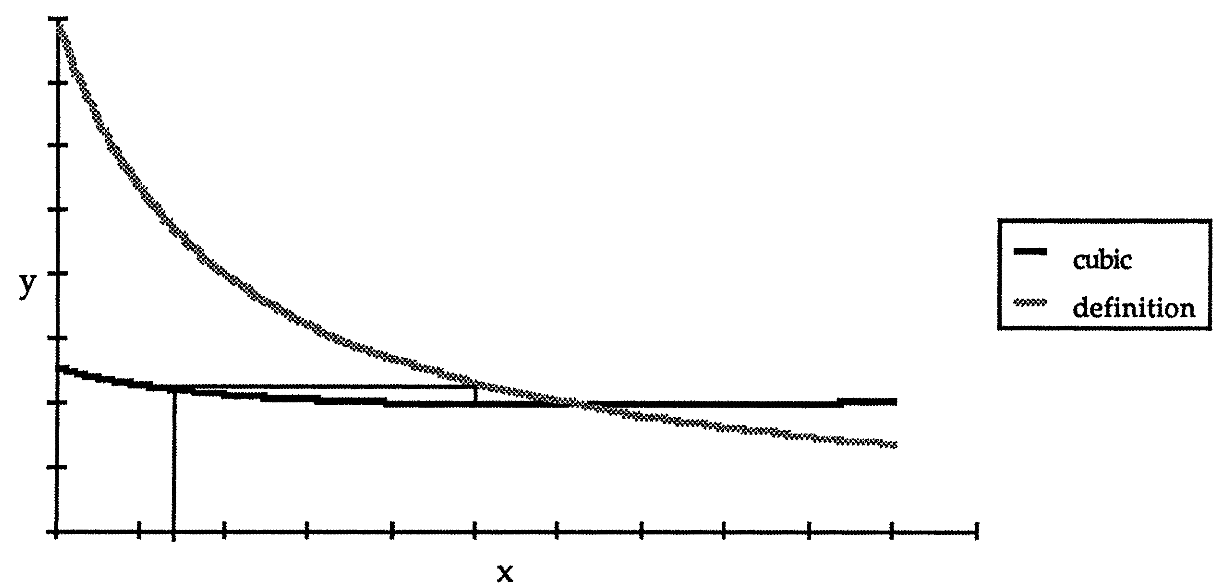


Figure 5 Solution Path for Direct Transfer Using Free Parameter

Battin uses a continued fraction expansion of $\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}$ to remove the singularity at x = 0, which is the solution for a parabolic orbit.[1]

2.5 Solution of Cubic Equation

A convenient transformation to use in solving the cubic equation 2.15

$$y^3 - (1 + h_1)\, y^2 - h_2 = 0$$

is

$$y = \frac{2}{3}(1 + h_1)\left(\frac{b}{z} + 1\right) \qquad (2.16)$$

The cubic is converted to the canonical form

$$z^3 - 3z = 2b$$

if

$$b = \sqrt{\frac{27h_2}{4(1 + h_1)^3} + 1}$$

Let

$$B \equiv \frac{27h_2}{4(1 + h_1)^3} \qquad (2.17)$$

The solution to the canonical form can be calculated using the identities

$$4\cos^3\frac{2}{3}\alpha - 3\cos\frac{2}{3}\alpha = \cos 2\alpha$$
$$4\cosh^3\frac{2}{3}\alpha - 3\cosh\frac{2}{3}\alpha = \cosh 2\alpha$$

by

---

[1] *ibid.*, pp. 327-328, 334-337.

$$z = \begin{Bmatrix} 2\cos\frac{2}{3}\alpha \\ 2\cosh\frac{2}{3}\alpha \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} \cos 2\alpha = b \\ \cosh 2\alpha = b \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} b \le 1 \\ b \ge 1 \end{Bmatrix}$$

or

$$z = \begin{Bmatrix} 2\cos\left[\frac{1}{3}\cos^{-1}\sqrt{B+1}\right] \\ 2\cosh\left[\frac{1}{3}\cosh^{-1}\sqrt{B+1}\right] \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} -1 \le B \le 0 \\ B \ge 0 \end{Bmatrix} \tag{2.18}$$

Battin further manipulates the above expressions and finds a unified continued fraction solution that is appropriate for small positive and negative B.[1]

2.6 Calculation of Orbital Parameters

Once x has been found, the semimajor axis, orbital parameter, and velocity at P_1 can be determined. From the definition of x

$$\tan^2\frac{E}{2} = x$$

$$\cos^2\frac{E}{2} = \frac{1}{1+\tan^2\frac{E}{2}} = \frac{1}{1+x}$$

$$\cos E = 2\cos^2\frac{E}{2} - 1 = \frac{1-x}{1+x}$$

From the mean point expression, equation 2.1, the semimajor axis is

$$a = \frac{r_{0p}}{(1-e_0)}\sec^2\frac{E}{2} = \frac{s(1+\lambda)^2(1+x)(\ell+x)}{8x} \tag{2.19}$$

In terms of the eccentric anomaly difference, the orbital parameter is[2]

---

[1]*ibid.*, pp. 338-339.
[2]*ibid.*, eqn. 6.107.

$$p = \frac{2r_1r_2\sin^2\frac{\theta}{2}}{r_1 + r_2 - 2\sqrt{r_1r_2}\cos\frac{\theta}{2}\cos E} = \frac{2r_1r_2\sin^2\frac{\theta}{2}}{s(1+\lambda)^2}\frac{(1+x)}{(\ell + x)} \qquad (2.20)$$

The velocity at P_1 required to make the transfer, $\underline{v}_1$, can be written as[1]

$$\underline{v}_1 = \frac{\mu}{r_1}\left(\sigma_1 \hat{\underline{i}}_{r_1} + \sqrt{p}\,\hat{\underline{i}}_h \times \hat{\underline{i}}_{r_1}\right)$$

where $\hat{\underline{i}}_{r_1}$ is the unit vector in the direction of $\underline{r}_1$, $\hat{\underline{i}}_h$ is the unit vector normal to the orbital plane, and the quantity σ_1 is defined as $\frac{\underline{r}_1 \bullet \underline{v}_1}{\mu}$. In terms of p and the eccentric anomaly difference[2]

$$\sigma_1 = \frac{\sqrt{p}}{\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - \sqrt{\frac{r_1}{r_2}}\cos E\right) = \frac{\sqrt{p}}{\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - \sqrt{\frac{r_1}{r_2}\frac{1-x}{1+x}}\right)$$

Then

$$\underline{v}_1 = \frac{\sqrt{\mu p}}{r_1 \sin\frac{\theta}{2}}\left[\left(\cos\frac{\theta}{2} - \sqrt{\frac{r_1}{r_2}\frac{1-x}{1+x}}\right)\hat{\underline{i}}_{r_1} + \sin\frac{\theta}{2}\,\hat{\underline{i}}_h \times \hat{\underline{i}}_{r_1}\right]$$

Substituting for p and $\cos\frac{\theta}{2}$

$$\underline{v}_1 = \frac{1}{(1+\lambda)}\sqrt{\frac{2\mu(1+x)}{s(\ell + x)}}\left[\frac{\lambda s(1+x) - r_1(1-x)}{r_1(1+x)}\hat{\underline{i}}_{r_1} + \sqrt{\frac{r_2}{r_1}}\sin\frac{\theta}{2}\,\hat{\underline{i}}_h \times \hat{\underline{i}}_{r_1}\right]$$

The velocity $\underline{v}_1$ can be written in terms of $\underline{r}_1$ and $\underline{r}_2$ by using $\underline{r}_1$ and $\underline{r}_2$ to calculate $\hat{\underline{i}}_h$. However, this creates a singularity if θ is 180° at which point $\underline{r}_1$ and $\underline{r}_2$ are colinear and cannot define a plane.

$$\hat{\underline{i}}_h = \frac{\underline{r}_1 \times \underline{r}_2}{r_1 r_2 \sin\theta} = \frac{1}{\sin\theta}\left(\hat{\underline{i}}_{r_1} \times \hat{\underline{i}}_{r_2}\right) \qquad \hat{\underline{i}}_h \times \hat{\underline{i}}_{r_1} = \frac{1}{\sin\theta}\left(\hat{\underline{i}}_{r_2} - \cos\theta\,\hat{\underline{i}}_{r_1}\right)$$

---

[1] *ibid.*, eqn. 7.33.
[2] *ibid.*, p. 307.

$$\underline{v}_1 = \frac{1}{\lambda(1+\lambda)}\sqrt{\frac{\mu(1+x)}{2s^3(\ell+x)}}\left[(\underline{r}_2 - \underline{r}_1) + \frac{s(1+\lambda)^2(\ell+x)}{r_1(1+x)}\underline{r}_1\right]$$

2.7 Convergence Behavior

	T									
λ	8	10	12	14	16	18	20	22	24	26
-0.999	7/8	5/6	5/5	4/5	4/4	3/4	4/4	4/5	4/5	4/5
-0.997	7/7	5/6	5/5	4/5	4/4	3/4	4/4	4/5	4/5	4/5
-0.995	7/7	6/6	5/5	4/5	4/4	3/4	4/4	4/5	4/5	4/5
-0.993	7/7	5/6	5/5	4/5	4/4	3/4	4/4	4/4	4/5	4/5
-0.991	7/7	5/6	5/5	4/5	4/4	3/4	4/4	4/5	4/5	4/5
-0.99	7/7	5/6	5/5	4/5	4/4	3/4	4/4	4/5	4/5	4/5
-0.97	6/7	5/6	5/5	4/5	4/4	4/4	4/4	4/5	4/5	4/4
-0.95	6/6	5/6	5/5	4/5	3/4	4/4	4/4	4/4	4/4	4/4
-0.93	5/6	5/5	4/5	4/4	3/4	4/4	4/4	4/4	4/4	4/5
-0.91	5/6	5/5	4/5	4/4	4/4	4/4	4/4	4/4	4/5	4/5
-0.9	5/6	5/5	4/5	4/4	4/4	4/4	4/4	4/4	4/5	4/5
-0.8	4/5	4/4	3/4	4/4	3/4	4/4	4/4	4/4	4/4	4/5
-0.7	4/4	3/3	3/4	3/4	4/4	4/4	4/5	4/5	4/5	4/5
-0.6	3/4	3/4	4/4	4/5	4/5	4/5	4/5	4/5	5/5	5/5
-0.5	3/4	4/4	4/5	4/5	4/5	5/5	5/5	5/5	5/5	5/5
-0.4	4/5	4/5	4/5	4/5	5/5	5/5	5/5	5/5	5/5	5/6
-0.3	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6
-0.2	4/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6
-0.1	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.1	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.2	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.3	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.4	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.5	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.6	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.7	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6	5/6
0.8	5/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6	5/6
0.9	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6
0.91	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6	5/6
0.93	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6
0.95	4/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6
0.97	4/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6	5/6
0.99	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6
0.991	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6
0.993	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6
0.995	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6
0.997	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6
0.999	4/5	5/5	5/5	5/5	5/5	5/5	5/5	5/6	5/6	5/6

Table 1 Number of Iterations to Compute Direct Transfer Solution to Eight/Twelve Significant Figures

Table 1 contains the number of iterations required for the direct transfer algorithm to converge to at least eight and twelve significant figures for various T and λ. Note that only one additional iteration is required to solve for the additional four significant figures. The initial estimate of x used was ℓ as suggested by Battin and Vaughan.[1] The only difficult problem seems to be for transfers near 360°, $\lambda = -1$, for smaller T that approach the minimum energy period of 2π.[2]

[1]Vaughan, Robin M. *An Improvement of Gauss' Method for Solving Lambert's Problem,* p. 47.

[2]*ibid.*, Table 8.

Chapter 3 Time-of-Flight Equation with Multiple Revolution Orbits

3.1 Addition of Multiple Periods

Because the orbital period is just a function of the semimajor axis, $P = 2\pi\sqrt{a^3/\mu}$, Lambert's theorem can be extended to include multiple revolutions by the inclusion of the number, N, of complete revolutions in the orbital transfer,

$$\sqrt{\mu}\,\Delta t \;=\; F\,(N, a, r_1 + r_2, c)$$

and the geometrical orbit transformation for the direct transfer case remains valid.

Kepler's equation for the multiple revolution orbit must be

$$\frac{1}{2}\sqrt{\frac{\mu}{a^3}}\,\Delta t \;=\; (N\pi + E) - e_0 \sin E$$

Writing a and e_0 in terms of r_{0p}, E, and f as in section 2.3 yields

$$\sqrt{\frac{\mu}{8r_{0p}^3}}\,\Delta t \frac{4\tan^3\frac{E}{2}}{\left[\left(\tan^2\frac{f}{2} + \tan^2\frac{E}{2}\right)\left(1 + \tan^2\frac{E}{2}\right)\right]^{\frac{3}{2}}} \tag{3.1}$$

$$= (N\pi + E) - \frac{2\tan\frac{E}{2}}{1 + \tan^2\frac{E}{2}} + \frac{4\tan^3\frac{E}{2}}{\left(\tan^2\frac{f}{2} + \tan^2\frac{E}{2}\right)\left(1 + \tan^2\frac{E}{2}\right)}$$

Using the same definitions as for the direct transfer

$$T \;=\; \frac{(1+\lambda)^3}{2x}\sqrt{\frac{(\ell + x)(1 + x)}{x}}\;\left[(\ell + x)(1 + x)\left(N\frac{\pi}{2} + \tan^{-1}\sqrt{x}\right) - \sqrt{x}\,(\ell - x)\right] \tag{3.2}$$

which, of course, reduces to the direct transfer case, equation 2.7, if $N = 0$.

From Figure 6 of T versus x for $N = 0$ and $N = 1$ note that there are two solutions given T, λ, and N greater than 0 for $T > T_m$ and one solution for

$T = T_m$, where T_m is the minimum of the T versus x curve. It would appear that solutions for T near T_m will be difficult to obtain for λ near ±1 because the locus of T is flat for a wide range of x.

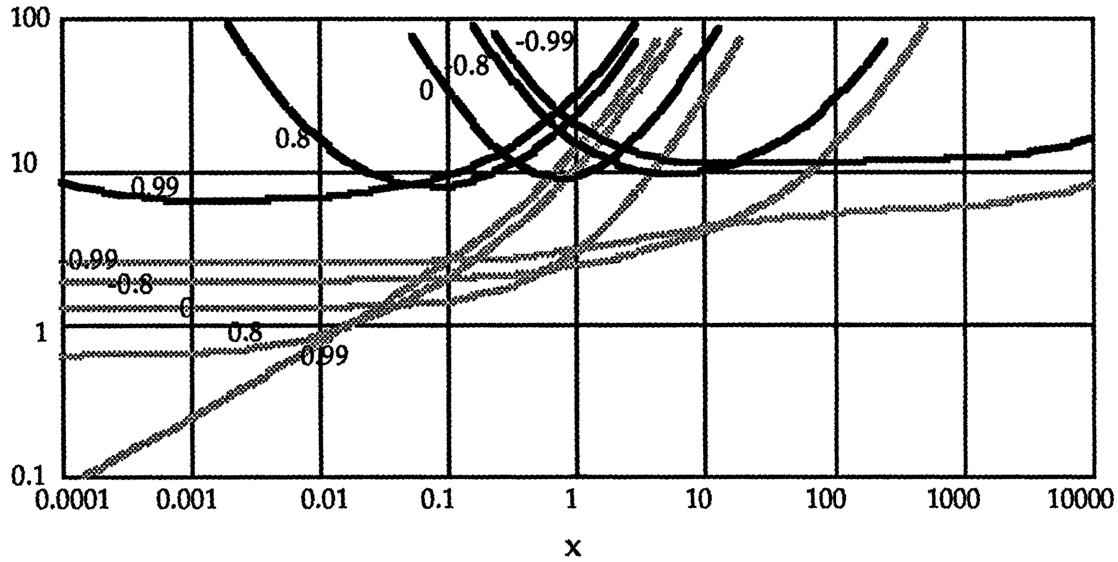


Figure 6 Time-of-Flight as a Function of x for Multiple Revolution Orbits

3.2 Estimation of Maximum Number of Revolutions

To know if a solution will be possible within the time constraint,it is necessary to be able to determine T_m given λ and N. Differentiate the time-of-flight equation 3.2,

$$\frac{dT}{dx} = \frac{(1+\lambda)^3}{4x^3}\sqrt{\frac{x}{(\lambda + x)(1+x)}}$$

$$\times \left\{ \sqrt{x}\,[3x^3 + (2+3\lambda)x^2 + \lambda(3+2\lambda)x + 3\lambda^2] - 3(\lambda - x^2)(\lambda + x)(1+x)\left(N\frac{\pi}{2} + \tan^{-1}\sqrt{x}\right) \right\}$$

and set equal to zero to solve for x_m where $T(x_m) = T_m$,

$$\sqrt{x_m}\,[3x_m^3 + (2+3\lambda)x_m^2 + \lambda(3+2\lambda)x_m + 3\lambda^2] - 3(\lambda - x_m^2)(\lambda + x_m)(1+x_m)\left(N\frac{\pi}{2} + \tan^{-1}\sqrt{x_m}\right) = 0$$

Examining the signs of the terms of the equation, there can be a solution only if

$$x_m \leq \sqrt{\ell}$$

Also consider that if $\lambda = 1$ ($\theta = 0°$), T_m must be N multiples of the period of the minimum energy orbit, and if $\lambda = -1$ ($\theta = 360°$), T_m must be (N + 1) multiples of the minimum period (2π in normalized time units). Therefore the range of T_m is

$$2N\pi < T_m (N,\lambda) < 2(N+1)\pi$$

It is necessary to use some iterative method, such as inverse linear interpolation, to get x_m. Newton's method will not work for λ near -1.

Making a visual approximation from x_m versus λ curves

$$x_m \approx \frac{2N+1}{2N+2}\sqrt{\ell} \tag{3.3}$$

The actual and approximate x_m versus λ are shown in Figure 7. The approximation is good except near the singular points $\lambda = \pm 1$ but still requires calculation of T.

A simpler visual approximation is

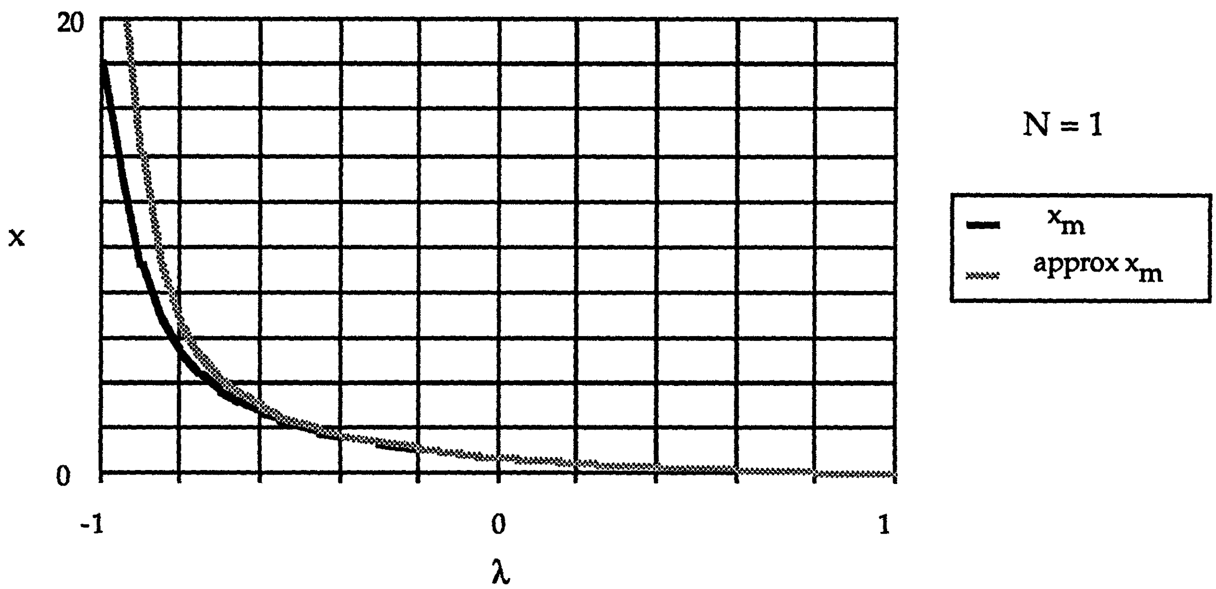
$$T_m \approx \pi\,[(2N+1) - \lambda^5] \tag{3.4}$$

Figure 8 shows T_m, T of approximate x_m, and approximate T_m versus λ. Both approximations become better as N increases. For small N the simple approximation of T_m is larger than the actual T_m for most λ, so a solution might actually be possible when the approximation indicates that it is not.

Using the simple formula for the approximation of T_m, solve for the maximum possible N

$$T \geq T_m (N_{max}, \lambda) \approx \pi\,[(2N_{max}+1) - \lambda^5]$$

$$N_{max} \lesssim \frac{T - \pi\,(1-\lambda^5)}{2\pi} \tag{3.5}$$



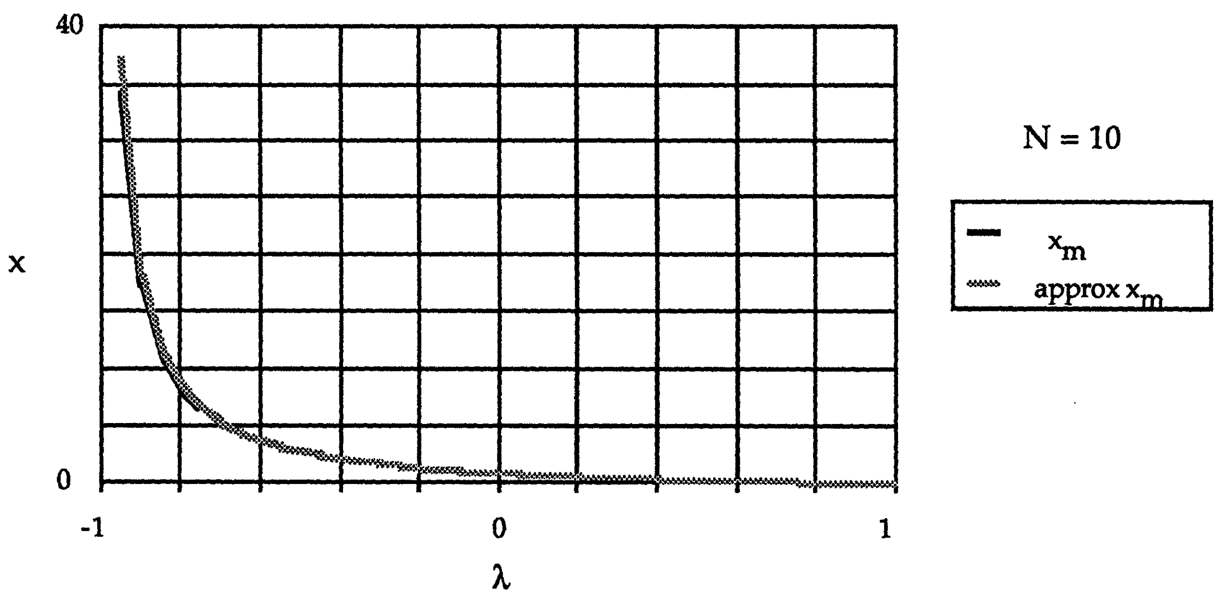


Figure 7 x_m and Approximation as a Function of λ

This certainly may be inaccurate if the right-hand expression is close to an integer, but it gives an approximation of N_{max}.

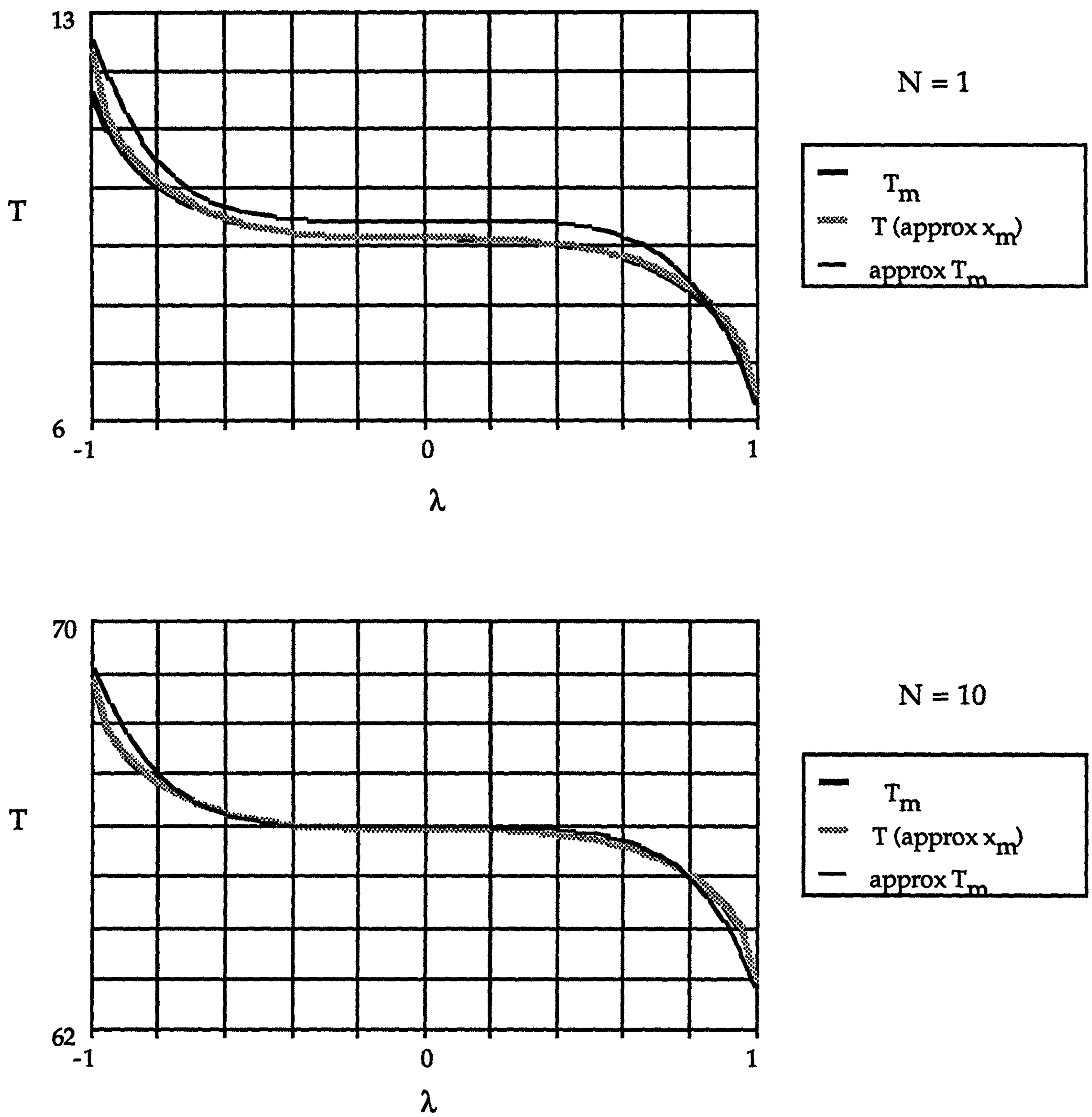


Figure 8 T_m and Approximations as a Function of λ

In sum there are $2N_{max}$ or $(2N_{max}+1)$ possible solutions to a given problem. For every N from 1 to $(N_{max}-1)$ there are two solutions. There is one solution for the direct transfer, N = 0, case, and there may be one or two solutions for $N = N_{max}$, depending on whether T is greater than or equal to T_m. Obviously the transfer orbit with the largest semimajor axis (and energy) results from a direct transfer and the orbit with the smallest semimajor axis (and energy) is achieved by making as many complete revolutions as possible within the transfer.

3.3 Relative Energy of the Two Solutions per Revolution

Classification of the two solutions by their relative energy is convenient since the total energy is only a function of the semimajor axis, $-\mu/2a$. In terms of x,

$$\frac{a}{a_m} = (1+\lambda)^2 \frac{(\swarrow + x)(1+x)}{4x}$$

A graph of a/a_m versus x is shown in Figure 9.

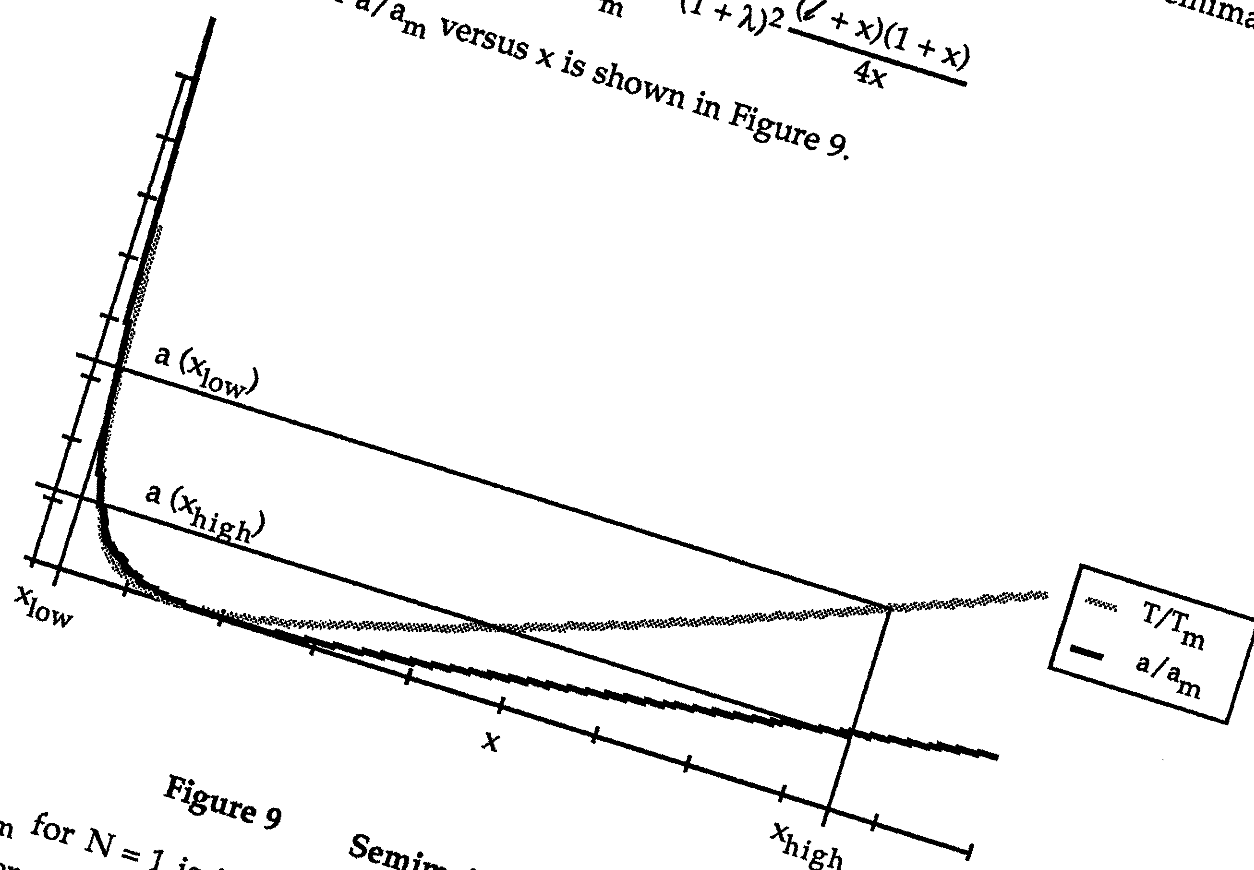


Figure 9 Semimajor Axis as a Function of x*

T/T_m for $N = 1$ is included so one can see that the higher x solution has a smaller semimajor axis and is therefore the lower energy orbit: the horizontal line intercepts of the T/T_m curve give the two solution points; vertical lines dropped to the a/a_m curve gives the semimajor axis for each solution point.

* Generated using $\lambda = 0$, $N = 1$.

Chapter 4 Algorithm to Determine Low Energy Solution

4.1 Revised Cubic Equation

The time-of-flight equation 3.1 is

$$\left[\frac{m}{(\ell + x)(1 + x)}\right]^{\frac{3}{2}} = \frac{m}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1 + x}\right) + \left[\frac{m}{(\ell + x)(1 + x)}\right]$$

As before let

$$y^2 \equiv \frac{m}{(\ell + x)(1 + x)}$$

Then the cubic equation is

$$y^3 - y^2 - \frac{m}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1 + x}\right) = 0 \qquad (4.1)$$

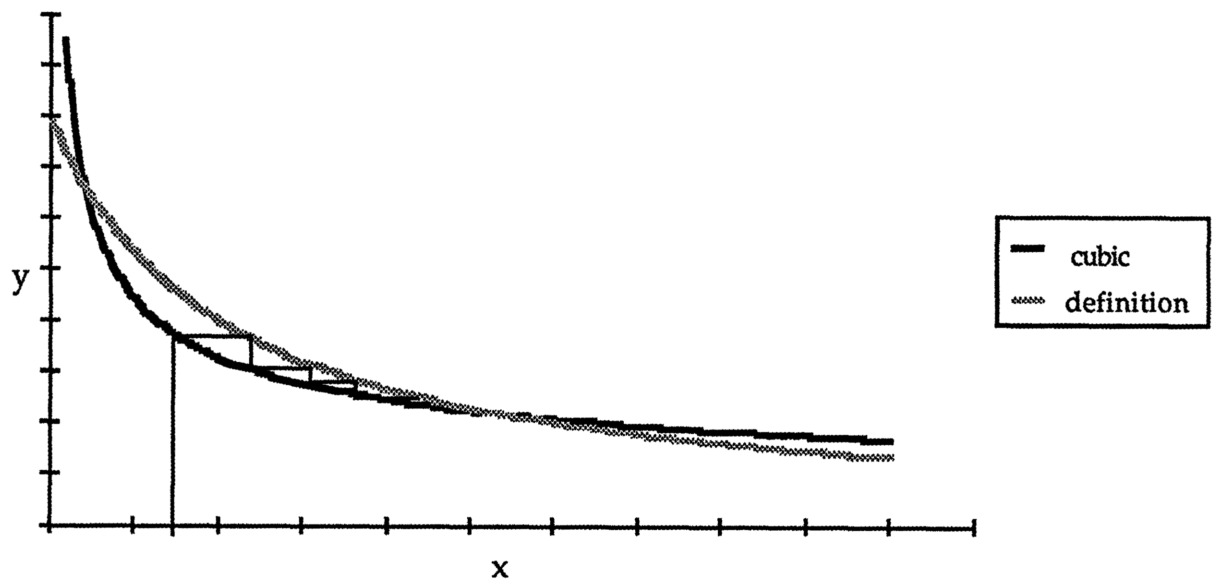


Figure 10 Solution Path for Multiple Revolutions*

* The multiple revolution examples were generated using $\lambda = 0$, $T = 16$, and $N = 1$.

which reduces to the direct transfer equation 2.10 when N = 0.

Figure 10 shows the solution of the cubic equation 4.1 and the definition of y, equation 2.9, versus x. Note that the solution of the cubic goes to infinity as x approaches zero. Also the inversion of the definition of y yields only one positive x. This makes it impossible to use successive substitution to get the lower x solution. This algorithm can only yield the higher x solution. The shape of the curves may make the successive substitution solution more difficult to obtain than for the direct transfer.

4.2 Choice of Free Parameter to Flatten Cubic Equation

To flatten the cubic solution in the vicinity of a root introduce the free parameter h_1 as in section 2.4,

$$y^3 - (1 + h_1)\, y^2 - m\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) + \frac{h_1}{(\ell + x)(1 + x)}\right] = 0 \qquad (4.2)$$

Differentiate

$$[3y^2 - 2(1 + h_1)\, y]\,\frac{dy}{dx} - \left[y^2 - \frac{m}{(\ell + x)(1 + x)}\right]\frac{dh_1}{dx}$$
$$- m\left\{\frac{d}{dx}\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right)\right] - h_1\frac{d}{dx}\frac{1}{(\ell + x)(1 + x)}\right\} = 0$$

At the solution point the second term is zero. Choose h_1 so that $\frac{dy}{dx}$ is zero.

$$\frac{d}{dx}\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right)\right] - h_1\frac{d}{dx}\frac{1}{(\ell + x)(1 + x)} = 0$$

$$h_1 = \frac{(\ell + x)^2}{4x^2(1 + 2x + \ell)}\left[3\,(1 + x)^2\,\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - (3 + 5x)\right] \qquad (4.3)$$

Let

$$h_2 \equiv m\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) + \frac{h_1}{(\ell + x)(1+x)}\right]$$

$$h_2 = \frac{m}{4x^2(1+2x+\ell)}\left\{[x^2 - (1+\ell)x - 3\ell]\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} + (3\ell + x)\right\} \quad (4.4)$$

As expected, the equations reduce to those of the direct transfer case, equations 2.13 and 2.14.

The cubic equation 4.2 is of the form

$$y^3 - (1 + h_1)\, y^2 - h_2 = 0 \quad (4.5)$$

A graphic interpretation of a successive substitution algorithm using this equation is shown in Figure 11. The solution of the cubic equation 4.5 is shown as zero if the cubic does not have a real positive solution.

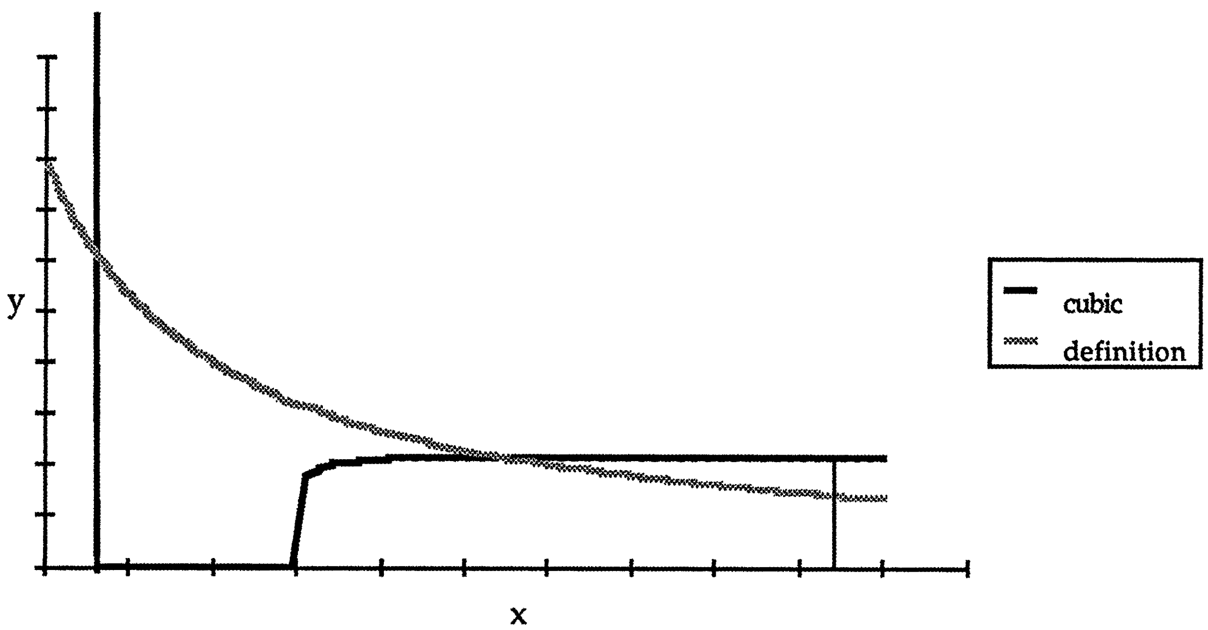


Figure 11 Solution Path for Multiple Revolutions Using Free Parameter

Note the "dead" area where there is no positive real solution to the cubic equation. Near $x = 0$ the root of the cubic is very large, eliminating the low x solution completely from the algorithm. An initial estimate of x that is large enough to avoid this "dead" area is needed. The locus of the solution of the cubic is very flat, similar to the direct transfer, requiring fewer iterations of the successive substitution algorithm to reach the solution.

Chapter 5 Algorithm to Determine High Energy Solution

5.1 Redefinition of y and Revised Cubic Equation

In order to find the higher energy, smaller x solution, the root of the cubic equation must be flatter for small x. To that end try defining

$$y^2 \equiv \frac{mx}{(\ell + x)(1 + x)} \tag{5.1}$$

Then rewrite the time-of-flight equation 3.1

$$\left[\frac{mx}{(\ell + x)(1 + x)}\right]^{\frac{3}{2}} = \frac{m\sqrt{x}}{2}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1 + x}\right) + \sqrt{x}\left[\frac{mx}{(\ell + x)(1 + x)}\right]$$

$$y^3 - \sqrt{x}\,y^2 - \frac{m\sqrt{x}}{2}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1 + x}\right) = 0 \tag{5.2}$$

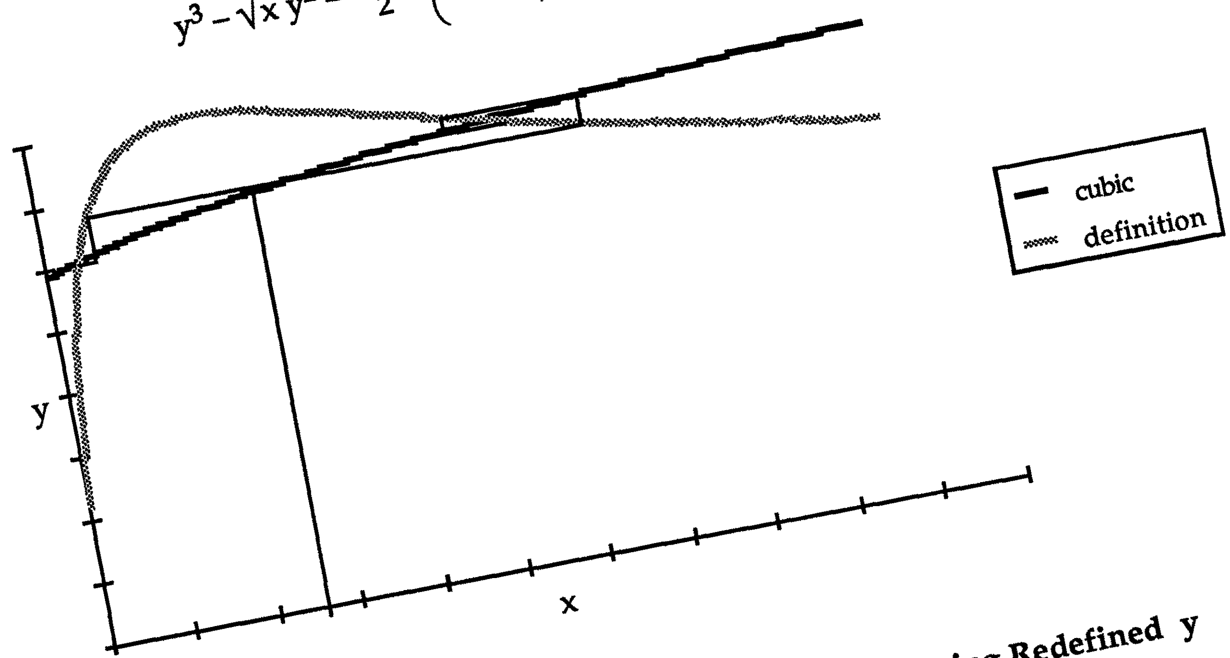


Figure 12 Solution Path for Multiple Revolutions Using Redefined y

Using the new definition of y, equation 5.1, x can be calculated as

$$x = \frac{1}{2}\left\{\left[\frac{m}{y^2} - (1+\ell)\right] \pm \sqrt{\left[\frac{m}{y^2} - (1+\ell)\right]^2 - 4\ell}\right\} \tag{5.3}$$

Figure 12 shows new y versus x curves. Both solutions are possible using successive substitution by using the "+" to get the higher x estimate and "–" to get the lower x estimate. Of greatest importance, however, is that the cubic solution curve appears to be "flattenable" in the vicinity of the lower x solution.

5.2 Choice of Free Parameter to Flatten Cubic Equation

Following same procedure as before, introduce a free parameter h_1,

$$y^3 - \sqrt{x}\,(1+h_1)\,y^2$$

$$- mx\sqrt{x}\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) + \frac{h_1}{(\ell+x)(1+x)}\right] = 0 \tag{5.4}$$

Differentiate

$$[3y^2 - 2\sqrt{x}\,(1+h_1)\,y]\frac{dy}{dx} - \sqrt{x}\left[y^2 - \frac{m}{(\ell+x)(1+x)}\right]\frac{dh_1}{dx} - (1+h_1)\,y^2\,\frac{d\sqrt{x}}{dx}$$

$$- mx\sqrt{x}\left\{\frac{d}{dx}\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right)\right] - h_1\frac{d}{dx}\frac{1}{(\ell+x)(1+x)}\right\}$$

$$- m\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) - \frac{h_1}{(\ell+x)(1+x)}\right]\frac{dx\sqrt{x}}{dx} = 0$$

Use the definition of y, equation 5.1, to eliminate terms and set $\frac{dy}{dx}$ to zero

$$\frac{m\sqrt{x}}{2}\left[\frac{3}{2x}\left(\frac{N\frac{\pi}{2}+\tan^{-1}\sqrt{x}}{\sqrt{x}}-\frac{1}{1+x}\right)-\frac{(2h_1-1)}{(\ell+x)(1+x)}\right]$$
$$+\,mx\sqrt{x}\left\{\frac{d}{dx}\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2}+\tan^{-1}\sqrt{x}}{\sqrt{x}}-\frac{1}{1+x}\right)\right]-h_1\frac{d}{dx}\frac{1}{(\ell+x)(1+x)}\right\}=0$$

Carrying through the algebra, the result is a very simple neat formula for h_1,

$$h_1 = \frac{(\ell+x)(1+2x+\ell)}{2(\ell-x^2)} \qquad (5.5)$$

Obviously there is a singularity at $\sqrt{\ell}$. However, since the smaller x solution must be less than $\sqrt{\ell}$, the singularity is outside the range of interest of x.

Solve for h_2

$$h_2 \equiv mx\sqrt{x}\left[\frac{1}{2x}\left(\frac{N\frac{\pi}{2}+\tan^{-1}\sqrt{x}}{\sqrt{x}}-\frac{1}{1+x}\right)-\frac{h_1}{(\ell+x)(1+x)}\right]$$

$$h_2 = \frac{m\sqrt{x}}{2(\ell-x^2)}\left[(\ell-x^2)\frac{N\frac{\pi}{2}+\tan^{-1}\sqrt{x}}{\sqrt{x}}-(\ell+x)\right] \qquad (5.6)$$

As desired, the cubic equation has a finite solution at zero.

Figure 13 displays the y versus x curves using the free parameter to flatten the cubic. Again for ranges of x for which there is no positive real solution of the cubic, the cubic solution is shown as zero. The above definition of h_1 eliminates the possibility of using this definition of y to find both solutions. In the vicinity of the lower x solution, the locus of the cubic solution is very flat, requiring very few iterations for solution if a small enough initial estimate of x is used. Near $\sqrt{\ell}$ there are positive real roots to the cubic equation, but they are very large and irrelevant.

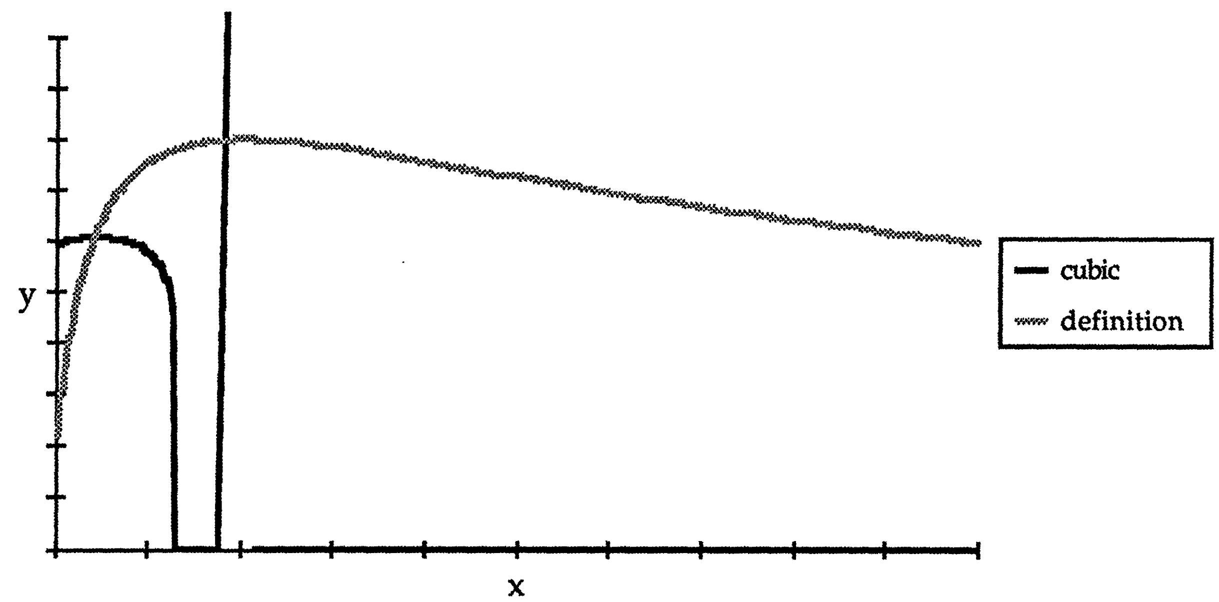


Figure 13 Solution Path for Multiple Revolutions Using Redefined y and Free Parameter

5.3 Solution of Cubic Equation

The cubic equation is now of the form

$$y^3 - \sqrt{x}\,(1 + h_1)\,y^2 - h_2 = 0 \tag{5.7}$$

Use the transformation

$$y = \frac{2}{3}\sqrt{x}\,(1 + h_1)\left(\frac{b}{z} + 1\right) \tag{5.8}$$

but note that it introduces a singularity at zero. The cubic is converted to the canonical form

$$z^3 - 3z = 2b$$

if

$$b = \sqrt{\frac{27h_2}{4[\sqrt{x}\,(1 + h_1)]^3} + 1}$$

Define

$$B \equiv \frac{27h_2}{4[\sqrt{x}\,(1 + h_1)]^3} \tag{5.9}$$

From section 2.5,

$$z = \begin{Bmatrix} 2\cos\left[\frac{1}{3}\cos^{-1}\sqrt{B+1}\right] \\ 2\cosh\left[\frac{1}{3}\cosh^{-1}\sqrt{B+1}\right] \end{Bmatrix} \text{ for } \begin{Bmatrix} -1 \le B \le 0 \\ B \ge 0 \end{Bmatrix}$$

For the high energy solution B is almost always positive and is often quite large, except for T near T_m. For $B \ge 0$ consider

$$z = 2\cosh\frac{2}{3}\alpha \text{ where } \cosh 2\alpha = b$$

Using identities for hyperbolic functions write

$$2\alpha = \cosh^{-1} b = \ln(b + \sqrt{b^2 - 1})$$

$$z = 2\cosh[\tfrac{1}{3}\ln(b + \sqrt{b^2 - 1})] = e^{\ln(b + \sqrt{b^2-1})^{\frac{1}{3}}} + e^{-\ln(b + \sqrt{b^2-1})^{\frac{1}{3}}}$$

$$z = (\sqrt{B} + \sqrt{B+1})^{\frac{1}{3}} + (\sqrt{B} + \sqrt{B+1})^{-\frac{1}{3}}$$

The root of the high-energy cubic is

$$y = \frac{2}{3}\sqrt{x}\,(1 + h_1)\left[\frac{\sqrt{B+1}}{A + 1/A} + 1\right] = \frac{2}{3}\sqrt{x}\,(1 + h_1)\left[\frac{A^2 + \sqrt{B+1}\,A + 1}{A^2 + 1}\right] \tag{5.10}$$

where

$$A = (\sqrt{B} + \sqrt{B+1})^{\frac{1}{3}}$$

Chapter 6 Behavior of Multiple Revolution Algorithms

6.1 Determination of Low and High Initial Estimates

A benefit of using these two algorithms is that the two solutions for each N are completely separated. Both algorithms have a "dead" zone in the middle x range in which the coefficients of the cubic equation change sign so that there is no single real positive root for y. A composite of the y versus x curves for both solutions is shown in Figure 14.

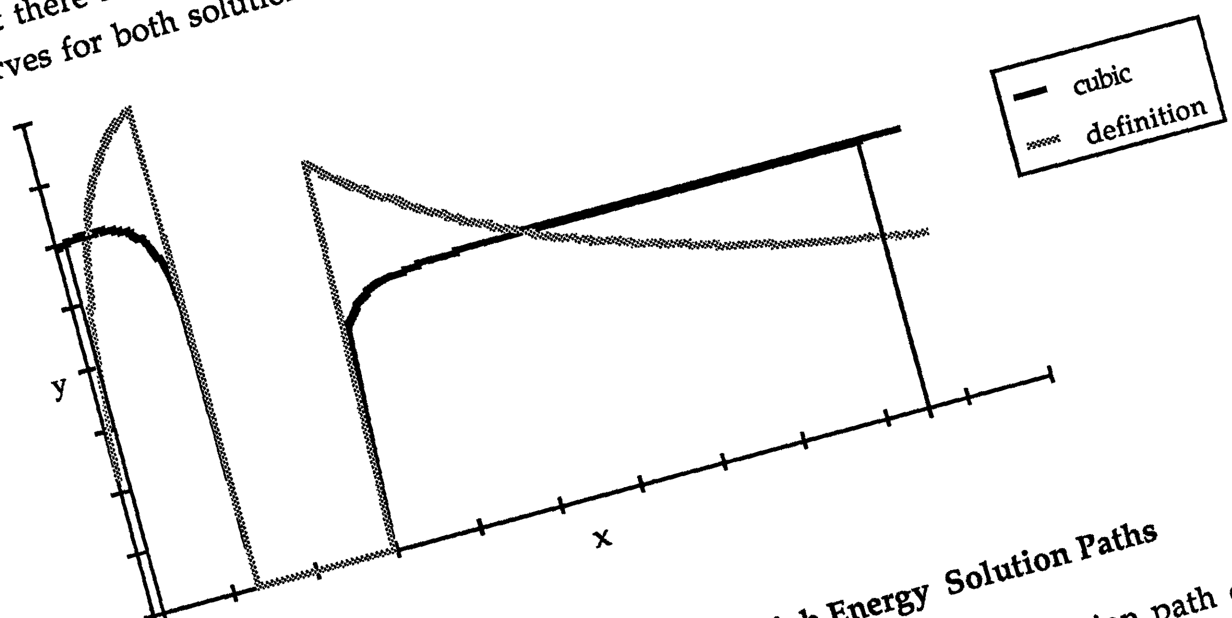


Figure 14 Composite Low and High Energy Solution Paths

Therefore the initial estimates must be chosen so that the solution path does not enter the "dead" zone. Fortunately this just requires a "small enough" estimate for the high energy, low x algorithm and a "large enough" estimate for the low energy, high x algorithm. From transformation of cubic to canonical form, $B \geq -1$ is needed for a solution.

For the low energy, high x algorithm

$$B = \frac{27h_2}{4(1 + h_1)^3}$$

$$h_1 = \frac{(\ell + x)^2}{4x^2(1 + 2x + \ell)}\left[3\,(1 + x)^2 \frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} - (3 + 5x)\right]$$

$$h_2 = \frac{m}{4x^2(1 + 2x + \ell)}\left\{[x^2 - (1 + \ell)x - 3\ell]\frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} + (3\ell + x)\right\}$$

so

$(1 + h_1) =$

$$\frac{1}{4x^2(1 + 2x + \ell)}\left\{3(\ell + x)^2(1 + x)^2 \frac{N\frac{\pi}{2} + \tan^{-1}\sqrt{x}}{\sqrt{x}} + [3x^3 + (1-6\ell)x^2 - \ell\,(6+5\ell)x - 3\ell^2]\right\}$$

The easiest way to guarantee that $B \geq -1$ is to require that $(1 + h_1)$ and h_2 are both positive. h_2 will be definitely positive if

$$x^2 - (1 + \ell)x - 3\ell \geq 0$$

$$x \geq \left(\frac{1 + \ell}{2}\right)\left\{1 + \sqrt{1 + \frac{12\ell}{(1 + \ell)^2}}\right\}$$

Use the binomial series expansion of the square root

$$x \geq \left(\frac{1 + \ell}{2}\right)\left\{1 + 1 + \frac{1}{2}\left[\frac{12\ell}{(1 + \ell)^2}\right] - \frac{1}{8}\left[\frac{12\ell}{(1 + \ell)^2}\right]^2 + \ldots\right\} \geq 1 + \ell + \frac{3\ell}{1 + \ell} \geq 1 + 4\ell$$

$$x_0 = 1 + 4\ell \tag{6.1}$$

Now check that this value will yield a positive $(1 + h_1)$ by substituting into the only term that could be negative

$$3(1 + 4\ell)^3 - (1 - 6\ell)(1 + 4\ell)^2 - \ell\,(6 + 5\ell)(1 + 4\ell) - 3\ell^2 \overset{?}{\geq} 0$$

$$4 + 32\ell + 80\ell^2 + 76\ell^3 \geq 0$$

Therefore $x_0 = 1 + 4\ell$ is an acceptable initial estimate for the low energy algorithm.

For the high energy algorithm h_1 will be positive for all x less than $\sqrt{\ell}$, but determining at what x h_2 would be positive is difficult. Instead consider that at x = 0, which is definitely smaller than the solution, y^3 equals h_2.

$$y\,(x{=}0) \;=\; \left(mN\frac{\pi}{4}\right)^{\frac{1}{3}}$$

From equation 5.3 find the next value of x,

$$x_0 = \left[\left(\frac{2m}{N^2\pi^2}\right)^{\frac{1}{3}} - \left(\frac{1+\ell}{2}\right)\right] - \sqrt{\left[\left(\frac{2m}{N^2\pi^2}\right)^{\frac{1}{3}} - \left(\frac{1+\ell}{2}\right)\right]^2 - 4\ell} \qquad (6.2)$$

6.2 Summary of Algorithms

The multiple revolution algorithms can be summarized, with the appropriate equation numbers, as follows:

	Low Energy	High Energy
(1) Make an initial estimate of x.	6.1	6.2
(2) Calculate h_1 and h_2.	4.3 and 4.4	5.5 and 5.6
(3) Calculate B.	2.17	5.9
(4) Calculate z.	2.18	2.18
(5) Calculate y.	2.16	2.16 or 5.10
(6) Determine a new value for x.	2.11	5.3
(7) Go back to step (2) and repeat until x no longer changes within a specified tolerance.		

6.3 Convergence Characteristics

Tables 2 and 3 display the number of iterations required to solve for the low energy solution to at least eight and twelve significant figures for N = 1 and N = 2 respectively. As with the direct transfer algorithm, only one additional iteration is required to find the additional four significant figures. In general, solution is more difficult near T_m, but especially so if λ is near ±1.

λ	T: 8	10	12	14	16	18	20	22	24	26
-0.999			34/35	9/10	7/7	6/6	5/6	5/5	4/5	5/5
-0.997			28/28	9/9	7/7	6/6	5/6	5/5	4/5	5/5
-0.995			24/25	8/9	7/7	6/6	5/6	5/5	4/5	5/5
-0.993			21/22	8/9	7/7	6/6	5/6	5/5	4/5	5/5
-0.991			19/20	8/9	7/7	6/6	5/6	5/5	4/5	5/5
-0.99			18/19	8/9	7/7	6/6	5/6	5/5	4/5	5/5
-0.97			11/12	7/8	6/7	5/6	5/5	4/4	5/5	5/5
-0.95			9/10	7/7	6/6	5/6	5/5	4/5	5/5	5/5
-0.93			8/9	6/7	6/6	5/6	4/4	5/5	5/5	5/5
-0.91			8/8	6/7	5/6	5/5	4/5	5/5	5/5	5/5
-0.9			7/8	6/7	5/6	4/5	5/5	5/5	5/5	5/5
-0.8			6/6	5/5	5/5	5/5	5/5	5/5	5/5	5/5
-0.7		7/8	5/5	5/5	5/5	5/5	5/5	5/5	5/5	5/5
-0.6		6/7	5/5	5/5	5/5	5/5	5/5	4/5	4/5	4/5
-0.5		5/6	5/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4
-0.4		5/6	5/5	5/5	5/5	5/5	4/5	4/5	4/4	4/5
-0.3		5/6	5/5	5/5	5/5	4/5	4/5	4/4	4/5	4/5
-0.2		5/6	5/5	5/5	4/5	4/5	4/4	4/4	4/5	4/5
-0.1		5/6	4/5	4/5	4/5	4/5	4/4	4/4	4/4	4/4
0		5/6	4/5	4/5	4/5	4/4	4/4	4/4	4/4	4/4
0.1		5/6	4/5	4/4	4/4	4/4	3/4	3/4	3/4	3/4
0.2		6/6	4/5	4/5	4/4	3/4	3/4	3/4	3/4	3/4
0.3		6/6	5/5	4/5	4/4	3/4	3/3	3/3	3/4	3/4
0.4		6/6	5/5	4/5	4/4	3/4	3/4	3/3	3/3	3/4
0.5		6/6	5/5	4/5	4/4	4/4	3/4	3/4	3/4	3/4
0.6		6/6	5/5	4/5	4/5	4/4	3/4	3/4	3/4	3/4
0.7		6/6	5/6	5/5	4/5	4/4	4/4	3/4	3/4	3/4
0.8		6/6	5/6	5/5	4/5	4/5	4/4	4/4	3/4	3/4
0.9	8/9	6/7	5/6	5/5	5/5	4/5	4/5	4/4	4/4	3/4
0.91	8/9	6/7	5/6	5/6	5/5	4/5	4/5	4/5	4/4	4/4
0.93	8/9	6/7	5/6	5/6	5/5	4/5	4/5	4/5	4/4	4/4
0.95	8/9	6/7	5/6	5/6	5/5	4/5	4/5	4/5	4/4	4/4
0.97	8/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/4	4/4
0.99	8/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4
0.991	8/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4
0.993	8/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4
0.995	8/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4
0.997	9/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4
0.999	9/9	6/7	6/6	5/6	5/5	5/5	4/5	4/5	4/5	4/4

Table 2 Number of Iterations to Compute Low Energy Solution for N = 1 to Eight/Twelve Significant Figures

					T					
λ	8	10	12	14	16	18	20	22	24	26
-0.999							11/12	8/9	7/7	6/7
-0.997							11/11	8/8	7/7	6/7
-0.995							10/11	8/8	7/7	6/7
-0.993							10/11	8/8	7/7	6/7
-0.991							10/10	8/8	7/7	6/7
-0.99							10/10	8/8	7/7	6/7
-0.97						14/15	8/9	7/8	6/7	6/6
-0.95						11/11	8/8	7/7	6/7	6/6
-0.93						9/10	7/8	6/7	6/6	5/6
-0.91						9/9	7/8	6/7	5/6	5/6
-0.9						8/9	7/7	6/7	5/6	5/6
-0.8						7/7	6/6	5/6	4/5	5/5
-0.7						6/7	5/6	4/5	5/5	5/5
-0.6					8/9	5/6	4/5	5/5	5/5	5/5
-0.5					7/8	5/6	4/5	5/5	5/5	5/5
-0.4					7/7	5/6	4/5	5/5	5/5	5/5
-0.3					6/7	5/6	4/5	4/5	5/5	4/5
-0.2					6/7	5/6	4/5	4/5	4/5	4/5
-0.1					6/7	5/6	4/5	4/5	4/5	4/5
0					6/7	5/6	5/5	4/5	4/4	3/4
0.1					7/7	5/6	5/5	4/5	4/5	4/4
0.2					7/7	5/6	5/5	4/5	4/5	4/4
0.3					7/7	5/6	5/5	4/5	4/5	4/4
0.4					7/7	5/6	5/5	4/5	4/5	4/4
0.5					7/7	5/6	5/5	5/5	4/5	4/5
0.6					7/7	5/6	5/6	5/5	4/5	4/5
0.7					7/7	6/6	5/6	5/5	5/5	4/5
0.8					7/7	6/6	5/6	5/6	5/5	5/5
0.9					7/8	6/7	6/6	5/6	5/6	5/5
0.91					7/8	6/7	6/6	5/6	5/6	5/5
0.93				11/12	7/8	6/7	6/6	5/6	5/6	5/5
0.95				10/11	7/8	6/7	6/6	5/6	5/6	5/6
0.97				10/11	8/8	6/7	6/7	6/6	5/6	5/6
0.99				11/11	8/8	7/7	6/7	6/6	5/6	5/6
0.991				11/11	8/8	7/7	6/7	6/6	5/6	5/6
0.993				11/11	8/8	7/7	6/7	6/6	5/6	5/6
0.995				11/12	8/9	7/7	6/7	6/6	5/6	5/6
0.997				11/12	8/9	7/7	6/7	6/6	5/6	5/6
0.999				11/12	8/9	7/7	6/7	6/6	5/6	5/6

Table 3 Number of Iterations to Compute Low Energy Solution for N = 2 to Eight/Twelve Significant Figures

Tables 4 and 5 display the number of iterations required to solve for the high energy solution to at least eight and twelve significant figures for N = 1 and N = 2 respectively. As with the direct transfer algorithm and the low energy algorithm, only one additional iteration is required to find the additional four significant figures. Unlike the low energy algorithm, solution is not more difficult for T near T_m or for λ near ± 1.

					T					
λ	8	10	12	14	16	18	20	22	24	26
-0.999			6/6	4/5	4/4	4/5	4/4	4/4	4/4	4/4
-0.997			5/6	4/*	4/5	4/4	4/5	4/4	4/4	4/4
-0.995			5/6	4/5	4/5	4/4	4/4	4/4	4/4	4/4
-0.993			5/6	4/5	4/5	4/4	4/4	4/4	4/4	4/4
-0.991			5/6	4/5	4/5	4/4	4/4	4/4	4/4	4/4
-0.99			5/6	4/5	4/5	4/4	4/4	4/*	4/4	4/4
-0.97			5/6	4/5	4/4	4/4	4/4	4/4	4/4	3/4
-0.95			5/5	4/5	4/4	4/4	4/4	4/4	4/4	3/4
-0.93			5/5	4/5	4/4	4/4	4/4	4/4	4/4	3/4
-0.91			5/5	4/5	4/4	4/4	4/4	4/4	3/4	3/4
-0.9			4/5	4/5	4/4	4/4	4/4	4/4	3/4	3/4
-0.8			4/5	4/4	4/4	4/4	4/4	3/4	3/4	3/4
-0.7		5/6	4/5	4/4	4/4	4/4	3/4	3/4	3/4	3/4
-0.6		5/5	4/4	4/4	4/4	3/4	3/4	3/4	3/4	3/4
-0.5		5/5	4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4
-0.4		4/5	4/4	4/4	4/4	3/4	3/4	3/4	3/4	3/4
-0.3		5/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
-0.2		5/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
-0.1		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0.1		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0.2		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0.3		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0.4		4/5	4/4	4/4	4/4	3/4	3/4	3/4	3/4	3/4
0.5		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0.6		4/5	4/4	4/4	3/4	3/4	3/4	3/4	3/4	3/4
0.7		4/5	4/4	3/4	3/4	3/4	3/4	3/4	3/4	3/4
0.8		4/4	3/4	3/4	3/4	3/4	3/4	3/4	3/4	3/3
0.9	5/5	4/4	3/4	3/4	3/4	3/3	3/3	3/3	3/3	3/3
0.91	4/5	4/4	3/4	3/4	3/4	3/3	3/3	3/3	3/3	3/3
0.93	4/5	3/4	3/4	3/4	3/3	3/3	3/3	3/3	3/3	3/3
0.95	4/5	3/4	3/4	3/4	3/3	3/3	3/3	3/3	3/3	3/3
0.97	4/4	3/4	3/3	3/3	3/3	3/3	3/3	3/3	3/3	3/3
0.99	3/4	3/3	3/3	3/3	3/3	2/3	2/3	2/3	2/3	2/4
0.991	3/4	3/3	3/3	3/3	2/3	2/3	2/3	2/4	2/3	2/3
0.993	3/4	3/3	3/3	2/3	2/3	2/3	2/3	2/3	2/3	2/3
0.995	3/3	3/3	2/3	3/3	2/3	2/3	2/3	2/3	2/4	2/3
0.997	3/3	3/4	2/3	2/3	2/3	2/3	2/3	2/3	2/3	2/3
0.999	3/3	2/3	2/3	2/4	2/2	2/2	2/2	2/2	2/2	2/3

Table 4 Number of Iterations to Compute High Energy Solution for N = 1 to Eight/Twelve Significant Figures*

* Does not converge to twelve significant figures with current implementation.

λ	T 8	10	12	14	16	18	20	22	24	26
-0.999							5/5	4/6	4/5	4/4
-0.997							5/5	4/5	4/4	4/4
-0.995							5/5	4/5	4/4	4/4
-0.993							5/5	4/5	4/4	4/4
-0.991							4/5	4/5	4/4	4/4
-0.99							4/5	4/5	4/4	4/4
-0.97						5/6	4/5	4/5	4/4	4/4
-0.95						5/6	4/5	4/4	4/4	4/4
-0.93						5/5	4/5	4/4	4/4	4/4
-0.91						5/5	4/5	4/4	4/4	4/4
-0.9						5/5	4/5	4/4	4/4	4/4
-0.8						4/5	4/4	4/4	4/4	3/4
-0.7						4/5	4/4	4/4	3/4	4/4
-0.6					5/6	4/5	4/4	3/4	4/4	3/4
-0.5					5/6	4/4	4/4	4/4	3/4	3/4
-0.4					5/5	4/5	4/4	4/4	3/4	3/4
-0.3					5/5	4/5	4/4	4/4	3/4	3/4
-0.2					5/5	4/5	4/4	4/4	3/4	3/4
-0.1					5/5	4/4	4/4	4/4	3/4	3/4
0					5/5	4/4	4/4	3/4	3/4	3/4
0.1					5/5	4/4	4/4	3/4	3/4	3/4
0.2					5/5	4/4	4/4	3/4	3/4	3/4
0.3					5/5	4/4	4/4	3/4	3/4	3/4
0.4					5/5	4/4	4/4	3/4	3/4	3/4
0.5					4/5	4/4	4/4	4/4	3/4	3/4
0.6					4/5	4/4	4/4	3/4	3/4	3/4
0.7					4/5	4/4	4/4	3/4	3/4	3/4
0.8					4/5	4/4	3/4	3/4	3/4	3/4
0.9					4/4	3/4	3/4	3/4	3/4	3/3
0.91					4/4	3/4	3/4	3/4	3/4	3/3
0.93				5/6	4/4	3/4	3/4	3/4	3/3	3/3
0.95				4/5	3/4	3/4	3/4	3/3	3/3	3/3
0.97				4/4	3/4	3/4	3/3	3/3	3/3	3/3
0.99				3/4	3/3	3/3	3/3	3/3	3/3	3/3
0.991				3/4	3/3	3/3	3/3	3/3	3/3	2/3
0.993				3/4	3/3	3/3	3/3	3/3	2/3	2/3
0.995				3/4	3/3	3/3	2/3	2/3	2/3	2/3
0.997				3/3	3/3	3/3	2/3	2/3	2/4	2/3
0.999				3/3	2/4	2/3	2/3	2/3	2/3	2/2

Table 5 Number of Iterations to Compute High Energy Solution for $N = 2$ to Eight/Twelve Significant Figures

Table 6 displays the number of iterations required to compute both solutions to at least eight and twelve significant figures for $T = 1.01T_m$. The property that only one additional iteration is needed to compute the additional four significant figures is maintained. There are no difficulties in finding the high energy solution, but many iterations are required to find the low energy solution near the singular points at $\lambda = \pm 1$.

λ	T_m	Low Energy	High Energy
-0.999	11.63781258943	33/34	6/7
-0.997	11.60361802781	28/29	6/7
-0.995	11.57018940617	26/27	6/7
-0.993	11.53751862029	24/25	6/7
-0.991	11.50559482845	23/24	6/7
-0.99	11.48990898153	23/23	6/7
-0.97	11.21121489822	16/17	6/7
-0.95	10.98572795637	14/14	6/7
-0.93	10.79726396256	12/13	6/7
-0.91	10.63549866068	11/12	6/7
-0.9	10.56251463024	11/12	6/7
-0.8	10.02008404139	9/10	6/7
-0.7	9.68146547180	8/9	6/6
-0.6	9.45927663312	8/8	6/6
-0.5	9.31413909263	7/8	6/6
-0.4	9.22304335083	7/8	6/6
-0.3	9.17032549577	7/8	6/6
-0.2	9.14412122311	7/8	6/6
-0.1	9.13466385734	7/8	6/6
0	9.13332658859	7/8	6/6
0.1	9.13198931985	7/8	6/6
0.2	9.12253195403	7/8	6/6
0.3	9.09632767791	8/8	6/6
0.4	9.04360975307	8/8	6/6
0.5	8.95251322580	8/8	6/6
0.6	8.80736926187	8/9	6/6
0.7	8.58513508118	8/9	6/6
0.8	8.24619104536	9/10	6/6
0.9	7.70058452852	10/11	5/6
0.91	7.62652569540	10/11	5/6
0.93	7.46118463150	11/11	5/6
0.95	7.26508215591	11/12	5/6
0.97	7.02000399780	13/13	5/6
0.99	6.66866780554	16/17	5/5
0.991	6.64486144792	16/17	5/5
0.993	6.59356093535	17/18	5/5
0.995	6.53561938625	18/19	4/5
0.997	6.46700406156	21/21	4/5
0.999	6.37505540838	25/26	4/4

Table 6 **Number of Iterations to Compute Solutions for N = 1 at 101% of T_m to Eight/Twelve Significant Figures**

Chapter 7 Conclusion

Multiple revolution transfer orbits are considered as solutions to Lambert's problem by adding the number of complete revolutions to Kepler's equation. This creates a double-valued time-of-flight with a clearly defined minimum which permits the solutions to be distinguished by their relative energy. Therefore two solutions exist for each complete revolution if the given transfer time is greater than the minimum time-of-flight; one solution exists if the given transfer time is equal to the minimum time-of-flight; and no solutions exist if the given transfer time is less than the minimum time-of-flight. In general there are $(2N_{max}+1)$ possible solutions to a given problem with the direct transfer orbit having the most possible energy or largest semimajor axis and the lower energy, smaller semimajor axis, solution for the N_{max} revolution orbit having the least possible energy.

The multiple revolution algorithms are based on Battin's direct transfer algorithm which transforms the mean point, the point at which the orbital tangent is parallel to the chord connecting the two terminal points, to a pericenter or apocenter and which uses as a dependent variable $x \equiv \frac{1}{4}(E_2 - E_1)$. A variable y is defined so that Kepler's equation can be formulated as a cubic equation in y. A successive substitution method is used by estimating x, solving the cubic equation for y, and calculating the next estimate of x from the definition of y. The method is quick and efficient because of the addition of a free parameter to the cubic equation which is chosen to make the slope zero at the solution point. This guides the estimation of x to the solution point, requiring few iterations to converge.

The low energy algorithm is derived using the same definitions as for the direct transfer solution and, in fact, reduces to the direct transfer algorithm if zero is used as the number of complete revolutions. The low energy algorithm can only converge to the low energy solution. The high energy algorithm is derived using a definition of y that makes the cubic solution finite for small x. The choice of the free parameter makes the high energy algorithm converge to only the high energy solution.

The convergence of both the low and high energy algorithms compares favorably with that of the direct transfer. The characteristic of improving four significant figures with one iteration once the solution point has been reached is retained. Less than ten iterations, and often only five, are required for almost all problems. Convergence is nearly completely uniform for the higher energy algorithm while the lower energy algorithm requires a large number of iterations only if the given transfer time is slightly greater than the minimum time-of-flight and the transfer angle is either slightly larger than 0° or slightly smaller than 360°.

Computational issues must still be considered, but the multiple revolution algorithms' convergence behaviors indicate great capability for handling a wide variety of navigational problems.

References

BATTIN, RICHARD H. *An Introduction to the Mathematics and Methods of Astrodynamics,* American Institute of Aeronautics and Astronautics, Inc., 1987.

BATTIN, RICHARD H. and VAUGHAN, ROBIN M. "An Elegant Lambert Algorithm," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 6, Nov.-Dec. 1984, pp. 662-670.

SUN, FANG-TOH, VINH, NGUYEN X., and CHERN, TZE-JANG. "Analytic Study of the Solution Families of the Extended Godal's Time Equation for Lambert's Problem," *The Journal of the Astronautical Sciences,* Vol. 35, No. 2, April-June 1987, pp. 213-234.

SUN, FANG-TOH, and VINH, NGUYEN X. "Lambertian Invariance and Application to the Problem of Optimal Fixed-Time Impulsive Orbital Transfer," *Acta Astronautica,* Vol. 10, No. 5-6, 1983, pp. 319-350.

VAUGHAN, ROBIN M. *An Improvement of Gauss' Method for Solving Lambert's Problem,* M.I.T. Master's Thesis, 1983.