A PROBABILISTIC APPROACH TO ESTIMATING DIFFERENTIAL SETTLEMENT OF FOOTINGS ON SAND

by

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ABSTRACT

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Submitted to the Department of Civil Engineering on May 14, 1971 in partial fulfillment of the requirements for the degree of Master of Science.

Two probabilistic models for estimating differential settlement of footings on sand are developed. The first model is based on the D'Appolonia method of estimating the modulus of compressibility using results of the Standard Penetration Test. The second model treats the sand beneath the footing as a layered material and the coefficient of compressibility for each layer is estimated using the Standard Penetration Test results.

Approximations of the first order probabilities associated with the total and differential settlement of the two models are found. The modulus - blowcount and coefficient of compressibility - blowcount relationships are evaluated using measured settlements. Both models are then evaluated using measured values of the mean and variance of the total and differential settlement.

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CHAPTER I

INTRODUCTION

The purpose of this study is to develop and evaluate a statistical approach to estimating total settlement and differential settlement of footings on sand. Terzaghi and Peck, Peck and Bazaraa, and Meyerhof[3] have all proposed methods of estimating the total settlement using results of the Standard Penetration Test (SPT) and various correction factors evaluated according to the ground water level and depth of the footing. However, all these methods purposely predict conservative design values of the total settlement because natural variations in the soil parameters and uncertainties associated with estimating the soil parameters using SPT have not been directly accounted for. As a result, their methods for estimating the differential settlement are usually quite conservative and do not reflect the measured variation in the SPT. For example, Terzaghi and Peck[7] write that "a study of available settlement records leads to the conclusion that the differential settlement of uniformly loaded continuous footings and of equally loaded spread footings of approximately equal size is unlikely to exceed 50% of the maximum settlement."

D'Appolonia and D'Appolonia[1],[2],[3] investigated the use of the SPT in estimating total settlement of footings on sand and proposed a new method, again using an average value of SPT, footing depth, and footing dimensions, which gives a "best estimate" of the total settlement at the location where the blowcount was measured.

While the D'Appolonia method still does not take into account the variability of the measured blowcount, it does give a "best estimate" of the total settlement at a given location.

The D'Appolonia method of estimating settlement of footings on sand can easily be modified to reflect the variation of the soil properties as indicated by the SPT results plus the uncertainty of the SPT results as an indicator of the soil properties. Such a modification involves replacing the average value of the SPT at a given location by the distribution of the values measured at that location or by the distribution of the values measured throughout the entire site if the site is thought to be nominally homogeneous.

The uncertainty indicated by the SPT can also be incorporated into the resulting predicted settlement by using the mean and the variance of the settlement. While this approach does not result in a complete description of the settlement distribution function, it does allow one to obtain relatively simple relationships between the mean, variance and the correlation structure of the measured SPT and the mean and the variance of the settlement. Once a distribution or information concerning the distribution of the settlement has been obtained, the mean of the differential settlement can also be derived. This mean value reflects the uncertainties associated with the SPT and the soil parameters it is measuring.

Resendiz and Herrera[4] investigated the probability distributions of total settlement and rotation of single footings on layered soils.

In the formulation of their solution, Resendiz and Herrera account for the random variations in the compressibility of the soil. However, when they derived the settlement and rotation distributions they not only assumed that the variation in the compressibility in the horizontal direction was random and uniformly distributed, but also that these variations were independent. That is, they assumed that the variation in the compressibility was not spatially correlated. Such an assumption may result in an unconservative estimate of the variance of the settlement. However, their approach can be modified to give a better estimate of the settlement variance.

CHAPTER II

SETTLEMENT MODELS

2.1 INTRODUCTION

Two approaches for estimating the distribution of the total settlement of footings on sand based on the Standard Penetration Test are discussed here. In the first approach, based on a settlement estimation method proposed by D'Appolonia,[(2), (3)] the soil beneath the footing is treated as if it was a homogeneous material. In this model, only a single parameter is needed to describe the compressibility of the soil. The SPT is then used to obtain a best estimate of the value of the compressibility parameter.

A second approach is to treat the soil beneath the footing as a layered material. The total settlement is then found by adding up the settlements of the individual layers. In this approach, the SPT is used to estimate the coefficient of volume change for each layer of the soil.

2.2 SINGLE PARAMETER MODEL

The D'Appolonia method of estimating the total settlement of footings on sand is based on the general displacement formulae which are derived assuming that the soil is isotropic, homogeneous and elastic. One of the simplified forms of the elastic displacement formula is given by (2.2.1).

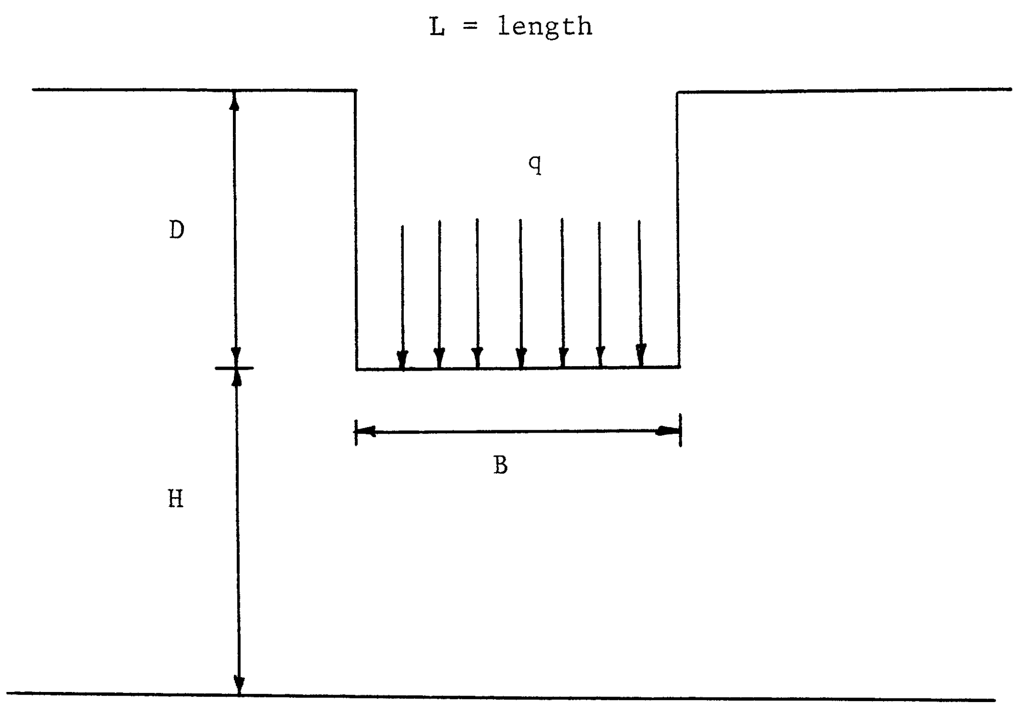


Fig. 2.2.1 Elastic Displacement Diagram

(2.2.1) $S = PBI/M = PBI/E(1-v^2)$

where

P = Load

S = Settlement

B = Typical footing dimension

I = Influence coefficient based on the footing depth, geometry, embedment, and thickness of the compressible layer

M = Modulus

E = Young's Modulus

v = Poisson's Ratio

D'Appolonia and D'Appolonia[3] point out that both the SPT and modulus are dependent upon the density of the sand and the effective overburden stress and therefore they should be related, even though the relationship would undoubtedly be approximate. To examine the relationship between the modulus and the blowcount, D'Appolonia and D'Appolonia[2] backfigured values of the modulus using measured values of the settlement. Figure (2.2.2) shows their plot of the average modulus versus the average blowcount. The blowcount used in Fig. (2.2.2) was the average value of the blowcount measured a depth B below the top of the footing.

If it is assumed that the relationship between the modulus and the blowcount is of the form

(2.2.2) $M = k_1 + k_2N + k_3U$

where

k_1, k_2, and k_3 are constants

U = independent, zero mean and unit variance variable

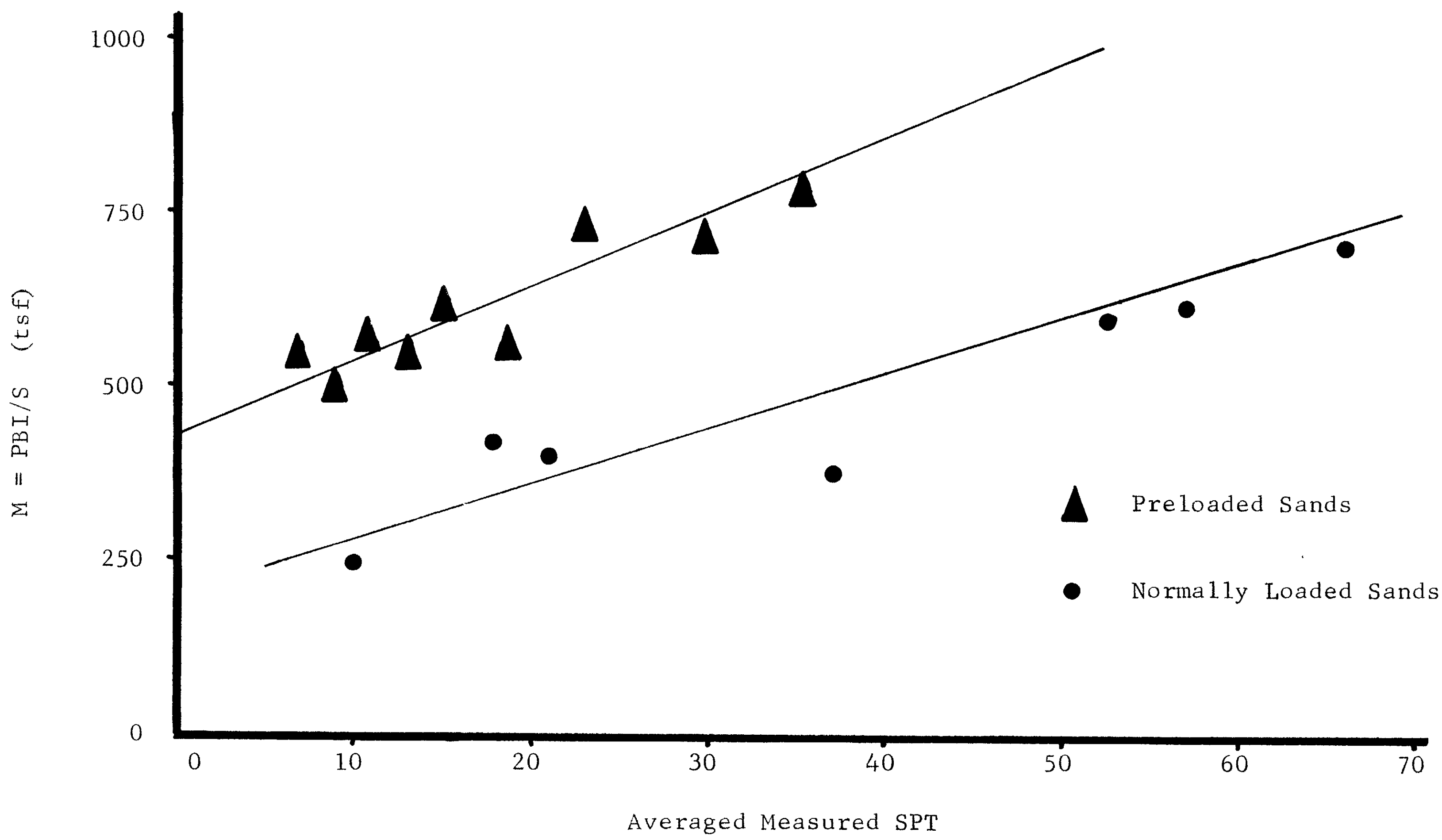


Fig. 2.2.2 Modulus - Blowcount Relationship

(from (2))

then the constants can be evaluated by obtaining the "best fit" straight line through the points. D'Appolonia and D'Appolonia [2] carried out this analysis on several sets of data to obtain values of k_1 and k_2 for preloaded and normally loaded sands.

Since the constants k_1, k_2 and k_3 can be evaluated, then eq. (2.2.2) can be substituted into the elastic displacement formula to obtain

(2.2.3) $$S = PBI/(k_1 + k_2N + k_3U)$$

where N is a random variable with mean m_N and variance s_N^2. Defining m_N as the mean value of M, it is easy to show that

(2.2.4) $$m_M = k_1 + k_2m_N$$

and its variance is

(2.2.5) $$\begin{aligned} s_M^2 &= Var(k_1 + k_2N + k_3U) \\ &= k_2^2 \; Var(N) + k_3^2 Var(U) \\ &= k_2^2 s_N^2 + k_3^2 \end{aligned}$$

The product K = PBI is treated here as a deterministic quantity. Thus, one can write

(2.2.6) $$M = K/S$$

If the probability distribution of M is known then that of S can be derived. While the mean and the variance of S cannot be found directly using "first order" probability theory, approximations of the mean and variance of M can be obtained by taking the expected value of the first few terms of a Taylor Series expansion of the function K/M about the mean of M. The Taylor Series expansion of K/M about m_M is given by eq. (2.2.7).

(2.2.7) $$S = K/M = K\left[1/m_M - (M - m_M)/m_M^2 + (M - m_M)^2/m_M^3 - \right.$$
$$\left. - (M - m_M)^3/m_M^4 + \ldots + (-1)^n(M - m_M)^n/n_M^{n+1} \right]$$

Keeping only the first three terms and taking the expected value of both sides, one obtains

(2.2.8) $$E(S) \approx K\left[E\left[1/m_M\right] - E\left[M - m_M\right]/m_M^2 + \right.$$
$$\left. + E\left[(M - m_M)^2\right] /m_M^3 \right]$$
$$\approx K\left[1/m_M + s_M^2/m_M^3 \right]$$
$$\approx K/m_M \left[1 + s_M^2/m_M^2 \right]$$

Recalling the known relationship[5]

(2.2.9) $$\mathrm{Var}\left[S\right] = E\left[S^2\right] - E\left[S\right]^2$$

The mean square of S is approximated by

(2.2.10) $$E\left[S^2\right] \approx E\left[K^2 \left[1/m_M - (M - m_M)/m_M^2 + (M - m_M)^2/m_M^3 \right]^2 \right]$$
$$= K^2/m_M^2 \left[1 - 2(M - m_M)/m_M + 3(M - m_M)^2/m_M^2 - \right.$$
$$\left. - 2(M - m_M)^3/m_M^3 + (M - m_M)^4/m_M^4 \right]$$

Substituting eq. (2.2.10) and (2.2.8) into (2.2.9) one obtains

(2.2.11) $$\mathrm{Var}\left[S\right] = K^2/m_M^2 \left[s_M^2/m_M^2 - 2U_M^{(3)}/m_M^3 + \right.$$
$$\left. + (U_M^{(4)} - s_M^4)/m_M^4 \right]$$
$$\approx \frac{K^2 V_M^2}{m_M^2}$$

where $U_M^{(n)} = E\left[(M - m_M)^n\right]$ $\quad n = 2,3$

$$V_M^2 = s_M^2/m_M^2$$

Of course, an exact solution for the moments of S can be obtained by deriving the distribution of S based on the distribution of M. This derivation is shown in Appendix A.

2.3 MULTIPARAMETER MODEL

The second method of predicting settlement is based on the concept of treating the soil beneath a foundation as if it was composed of many layers. This method was recently used by Rosendiz and Herrera.[4] The settlement of the i^{th} layer of soil, S_i, is equal to the coefficient of volume change of the i^{th} layer, V_i, times the effective vertical stress increment of the i^{th} layer, $\bar{P}_i$, times the thickness of the i^{th} layer, Z_i. The total settlement is then found by summing the settlements of the individual layers. Or,

(2.3.1) $$S_i = V_i \bar{P}_i Z_i$$

and

(2.3.2) $$S = \sum^{\text{all } i} S_i = \sum^{\text{all } i} V_i \bar{P}_i Z_i$$

where $\bar{P}_i$ can be approximated by an average value of the vertical stress which would exist assuming that the soil is elastic, homogeneous and isotropic (See Appendix D).

If it is assumed that for each layer of soil the relationship between the coefficient of volume change and the blowcount is of the form

(2.3.3) $$V_i = k_1' + k_2'/N_i + k_3'U$$

where k_1', k_2', and k_3' are constants

U is an independent, zero mean and unit variance random variable

The proposed relationship eq. (2.2.3) is further studied in Chapter III where the parameters are estimated.

The coefficient of volume change for each layer consists of a deterministic part and a random part. That is,

(2.3.4) $$V_i = V_i^o + V_i'$$

where V_i^o = expected value of the coefficient of volume change of the i^{th} layer.

V_i' = the deviation of V_i from the mean value

Using eq. (2.2.3), V_i^o and V_i' can be evaluated in terms of the blowcount. That is, (2.3.5) $$V_i^o = E\left[V_i\right] = E\left[k_1' + k_2'/N_i + k_3'U\right]$$

$$= k_1' + k_2' E\left[1/N_i\right]$$

and $$\approx k_1 + \frac{k_2'}{m_{N_i}}\left[1 + \frac{s_{N_i}^2}{m_{N_i}^2}\right]$$

$$m_{V_i} = E\left[V_i\right] = 0$$

$$s_{V_i}^2 = \mathrm{Var}\left[V_i\right] = k_2'^2 E\left[1/N^2\right] + k_2'^3 - k_2'^2 E\left[1/N_i\right]^2$$

$$\approx \frac{k_2'^2 V_{N_i}^2}{m_{N_i}^2} + k_3'^2$$

Substituting eq. (2.3.5) into eq. (2.3.1), one obtains for the settlement of the ith layer

(2.3.6) $$S_1 = (V_i^o + V_i')\bar{P}_i Z_i$$

The expected value of the settlement of the ith layer would be

(2.3.7) $$m_{S_i} = E\left[S_i\right] = V_i^o \bar{P}_i Z_i$$

and its variance is

(2.3.8) $$s_{S_i}^2 = \mathrm{Var}\left[S_i\right] = \bar{P}_i^2 Z_i^2 \mathrm{Var}\left[V_i\right]$$

Consider any linear function

$$Y = \sum_{i=1}^{n} a_i X_i$$

of n random variables X_i with means m_{x_i} and standard deviations s_{x_i}.

It is well known[5] that the mean value of Y is given by

(2.3.9) $$E\left[Y\right] = \sum_{i=1}^{n} a_i E\, X_i$$

and its variance by

(2.3.10) $$\mathrm{Var}\left[Y\right] = \sum_{i=1}^{n} a_i^2\, s_{x_i}^2 + \sum_{i=1}^{n}\sum_{j=i+1}^{n} a_i a_i \rho_{x_i x_j} s_{x_i} s_{x_j}$$

where $\rho_{x_i x_j}$ is the coefficient of correlation between x_i and x_j.

Applying the above results in eq. (2.3.2), the expected value of the total settlement becomes

(2.3.11) $$E\left[S\right] = \sum_{i=1}^{n} Z_i \bar{P}_i E\left[V_i\right] = \sum_{i=1}^{n} Z_i \bar{P}_i V_i^o$$

$$= \left(\sum_{i=1}^{n} Z_i \bar{P}_i\right) k_1' + \left(\sum_{i=1}^{n} Z_i \bar{P}_i E\left[1/N_i\right]\right) k_2'$$

And the variance of the total settlement would be

(2.3.12) $$\mathrm{Var}\ S = \sum_{i=1}^{n}\sum_{j=1}^{n} Z_i Z_j \bar{P}_i \bar{P}_j\, \rho_{V_i V_j} s_{V_i} s_{V_j}$$

In their probabilistic analysis, Resendiz and Herrera[4] assume all layer compressibilities to be statistically independent, i.e., $\rho_{V_i V_j} = 0$ for all $i \neq j$. Equation (2.3.12) then becomes

(2.3.13) $$\mathrm{Var}\left[S\right] = \sum_{i=1}^{n} Z_i^2 \bar{P}_i^2 sV_i^2$$

If the layers are completely dependent, i.e., $\rho_{V_i V_j} = 1$ for all i and j, then eq. (2.3.12) becomes

$$(2.3.14) \qquad \mathrm{Var}\Big[S\Big] = \sum_{i=1}^{n}\sum_{j=1}^{n} Z_i Z_j \bar{P}_i \bar{P}_j s_{V_i} s_{V_j}$$

In the case where there is some positive correlation the variance will lie between the two extremes corresponding to complete independence and perfect correlation.

Correlation models are needed to evaluate the variance of the settlement in the multiparameter model. While solutions have been given for the two special cases of independence and complete dependence between the layers, a more general model is needed which includes the case of partial correlation.

If one considers a correlation coefficient of the form[11]

$$(2.3.15) \qquad \rho_{X,Y} = \mathrm{Cov}(X,Y)/(s_X s_Y) = \exp\ (-d^2/a^2)$$

where d = distance between X and Y

a = constant

then it would follow that

$$\mathrm{Cov}(X,Y) = s_X s_Y \exp(-d^2/a^2)$$

Using this correlation function, the equation for the variance of S could then be written as

$$(2.3.16) \qquad \mathrm{Var}\Big[S\Big] = \sum_{i=1}^{n}\sum_{j=1}^{n} Z_i Z_j \bar{P}_i \bar{P}_j s_{m_i} s_{m_j} \exp(-d^2/a^2)$$

where d = distance between the i^{th} and the j^{th} layers.

Other correlation models are discussed in Appendix D.

2.4 DIFFERENTIAL SETTLEMENT

Once the distribution of the total settlement, or at least the first two moments of the settlement distribution have been obtained, one can proceed to make statements about the differential settlements. If the distribution of the settlements S_1 and S_2 are known and if they can be assumed to be statistically independent, then the distribution of the differential settlement $D = |S_1 - S_2|$ can be derived as shown in Appendix A. The moments of the relative settlement $\bar{D} = S_1 - S_2$ are easily obtainable.

(2.4.1) $$E\left[\bar{D}\right] = E\left[S_1 - S_2\right] = m_{S_1} - m_{S_2}$$

$$E\left[\bar{D}^2\right] = E\left[(S_1 - S_2)^2\right] = E\left[S_1^2\right] + E\left[S_2^2\right] - 2E\left[S_1 S_2\right]$$

If S_1 and S_2 are independent and identically distributed, then eq. (2.4.2) becomes

(2.4.2) $$E\left[\bar{D}\right] = 0$$

$$E\left[\bar{D}^2\right] = 2E\left[S^2\right] = 2s_S^2$$

Note that the mean square of D is equal to that of $\bar{D}$.

(2.4.3)
$$E\left[D^2\right] = E\left[|S_1 - S_2|^2\right] = E\left[|S_1 - S_2| \cdot |S_1 - S_2|\right]$$

$$= E\left[(S_1 - S_2)^2\right] = E\left[\bar{D}^2\right]$$

The standard deviation of the relative settlement can be used as an approximation for the mean of the differential settlement. In addition, the variance of the relative settlement is an upper limit to the variance of the differential settlement. Thus, approximations for the first two moments of the differential settlement would be

(2.4.4)
$$E\left[D\right] \approx (E\left[\bar{D}^2\right])^{1/2} = \sqrt{2}\, s_S$$

$$Var\left[D\right] \approx E\left[\bar{D}^2\right] = 2s_S^2$$

for $S_1 = S_2$ and S_1, S_2 independent

$$E\left[D\right] \approx (s_{S_1}^2 + s_{S_2}^2)^{1/2}$$

$$Var\left[D\right] \approx s_{S_1}^2 + s_{S_2}^2$$

for $S_1 \neq S_2$ and S_1, S_2 independent

Once an estimate of the first two moments of the differential settlement have been obtained, then Chebychev's inequality can be used to make statements about the probability of the differential settlement falling within certain bounds.

$$P\left[D \leq m_D + Cs_D \right] \leq 1 - 1/C^2 \qquad (2.4.7)$$

CHAPTER III

CASE STUDY

3.1 INTRODUCTION

In this chapter the validity and limitations of the settlement models discussed previously are evaluated by comparing values of settlements and differential settlements predicted by the models to those actually observed. The data gathered at a well documented construction site is a good source of this information. The site studied previously by D'Appolonia et al.[(1),(2)] was chosen for the case study as it has the following desirable characteristics:

A. An adequate record of the pre-construction site exploration is available so that estimates of needed soil parameters can be made.

B. A record of the post-construction foundation settlement data is available for comparison with the results predicted by the model.

C. The information presented in these records is sufficiently extensive as to provide a reasonably detailed view of the site.

D. No major spatial trends exist in the soil properties.

3.2 SITE DESCRIPTION

The construction site of interest consists of about 1300 acres of dune sand and organic marl overlying glacial till and rock. It is located near the shores of Lake Michigan in Northern Indiana and was used for the construction of a large complex of mill buildings. The entire site was precompressed under pressure of about five tons per square foot by migrating sand dunes. Figure 3.2.1 shows the soil profile of the site.

The site was prepared for construction by removing the swamp deposits between the dunes and then cutting the dunes to fill the depressions. Reference (1) gives a detailed description of the site.

The subsurface soil exploration carried out consisted of exploratory borings, borings made for the design of dewatering facilities, and 96 borings made in the vicinity of the column footings. Standard penetration tests were conducted at five foot intervals in most borings. All SPT results have been adjusted to fit the conditions that exist when the ground surface is at elevation 613 and the ground water level is at elevation 595.

The borings of interest in this case study are divided into three groups:

Control Group	Borings which are located directly under or close by a footing.
Group A	Borings in the vicinity of the west end of the plate mill.

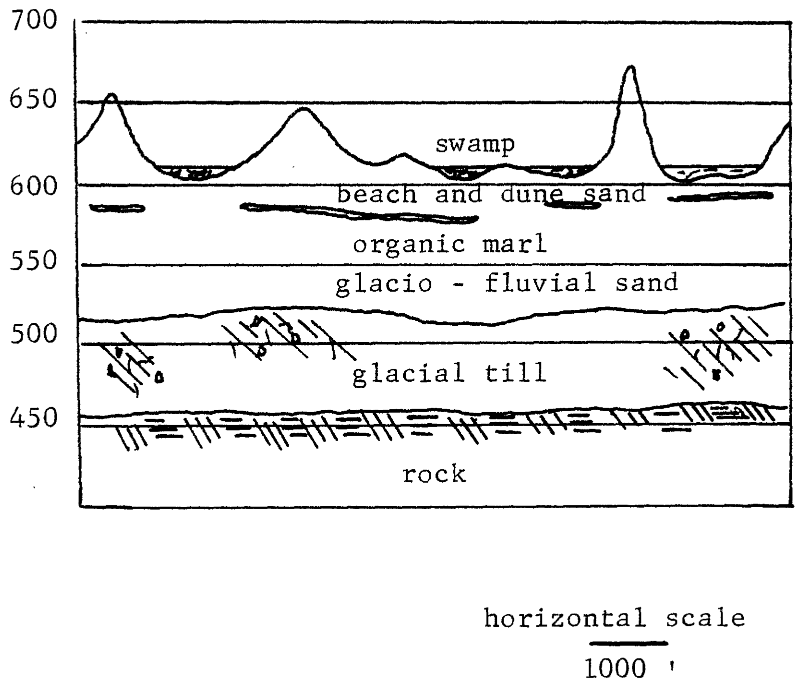


Fig. 3.2.1 Soil Profile

(from (1))

Group B Borings in the vicinity of the east end of the plate mill.

The footings considered in this case study were categorized in a similar manner. Figures 3.2.2 and 3.2.3 show the location of the borings and the footings in the vicinity of the plate mill. Tables 3.2.1 through 3.2.3 summarize the details concerning the footings, loads, and settlements for the three groups.

3.3 MODULUS - BLOWCOUNT RELATIONSHIP

The relationship between the modulus of compressibility and the blowcount[2],[3] for each data group was established by plotting computed values of the modulus of compressibility versus an average value of the blowcount for each footing in a group and then fitting a "least squares" straight line through the points. The modulus of compressibility for each footing was computed using eq. (3.3.1) which

$$S_i = B_i P_i I_i / M_i \tag{3.3.1}$$

includes the effects of the different footing geometries, applied loads and the measured settlements. The procedure used to find an average value of the blowcount is slightly different for the control group than for groups A and B. Descriptions of the methods used to determine the average blowcount can be found in sections 3.3.1 and 3.3.2.

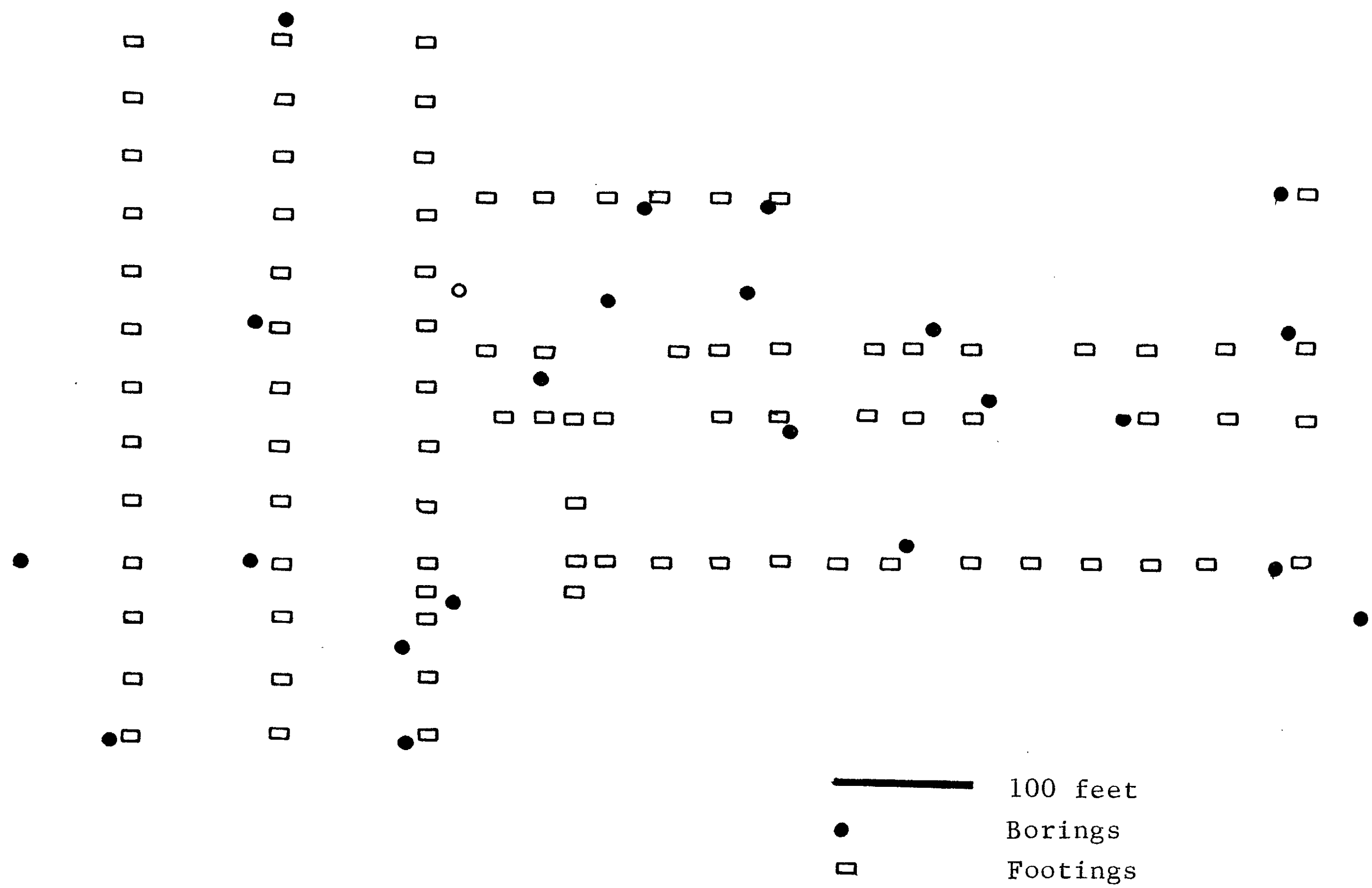


Fig. 3.2.2 Location of Borings and Footings in the Vicinity of the West End of the Plate Mill

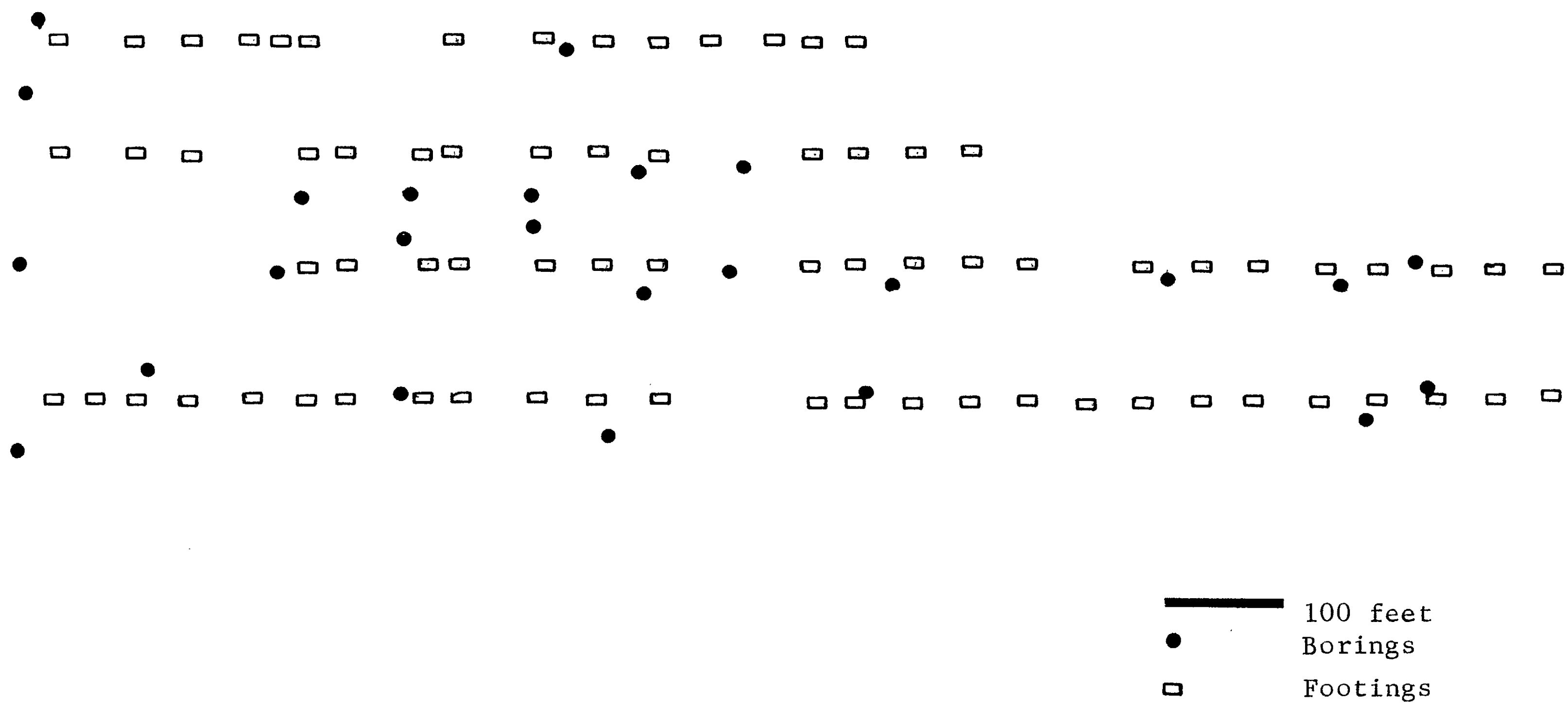


Fig. 3.2.3 Location of Borings and Footings in the Vicinity of the East End of the Plate Mill

Table 3.2.1

Control Group Footings

Footing Number	Load (kips)	Width (Feet)	Length (Feet)	Top Elevation	Settlement (inches)
MM98	760	14	18	607	.14
KK100	750	14	18	607	.25
B62	1120	18	24	602	.37
A56	900	14	24	607	.37
B50	1125	22	24	602	.16
B42	1125	22	24	602	.23
Ka44	700	14	26	597	.28
H52	780	16	26	597	.27
H70	750	16	22	607	.35
Ka56	510	13	22	607	.24
Ka58	730	13	22	607	.29
KK102	715	12	15	607	.25
LL238	750	16	26	600	.17
LL234	750	16	26	600	.17
LL133	970	16	26	594	.23
LL134	545	16	26	594	.14
KK152	660	12	16	607	.29
A1	1400	16	26	607	.33
A13	1400	16	26	607	.75
B7	980	16	26	607	.59
C1	1300	12	20	607	.44
C13	980	16	23	607	.46
C19	980	12	20	601	.34
UU3	1100	16	30	592	.29
WW3	1040	14	30	592	.40
A33	915	14	24	607	.26
A43	930	14	26	607	.28
A57	915	14	24	607	.16
B51	960	14	26	607	.32
C35	800	12	16	607	.29
C53	940	14	24	603	.25
C57	940	14	24	603	.18
H9	620	10	14	607	.44
H13	620	10	14	607	.28
E20	2110	23	40	590	.35
E24	2110	23	40	590	.30
J33	1100	12	19	607	.65
E64	770	15	25	596	.20
H63	1160	13	21	607	.36

Table 3.2.1 Continued

H82	800	11	16	607	.48
F84	1850	21	31	603	.57
E63	770	15	25	596	.32
E72	1160	15	25	595	.32
K82	820	12	17	607	.59
K91	640	10	15	607	.31
E109	640	10	15	607	.31
E101	720	10	15	607	.28
E103	640	10	15	607	.26
E84	1800	20	30	603	.40
E93	770	14	20	605	.44
K93	640	10	15	607	.34

Table 3.2.2

Area A Footings

Footing Number	Load (kips)	Width (Feet)	Length (Feet)	Top Elevation	Settlement (inches)
A33	915	14	24	607	.26
A35	915	14	24	607	.13
A37	800	14	26	607	.12
A41	930	14	26	607	.24
A43	930	14	26	607	.28
A45	930	14	26	607	.31
A47	930	14	26	607	.30
A51	915	14	24	607	.15
A53	915	14	24	607	.12
A55	915	14	24	607	.22
A57	915	14	24	607	.16
B33	960	14	26	607	.30
B35	960	14	26	607	.30
B37	800	14	26	607	.43
B39	960	14	26	607	.47
B41	960	14	26	607	.43
B43	960	14	26	607	.38
B45	960	14	26	607	.34
B47	960	14	26	607	.46
B49	960	14	26	607	.36
B51	960	14	26	607	.32
B53	960	14	26	607	.30
B55	960	14	26	607	.29
C33	800	12	16	607	.27
C35	800	12	16	607	.29
C39	980	18	22	587	.11
C47	920	18	22	603	.28
C49	910	16	24	597	.34
C51	910	16	24	597	.28
C53	940	14	24	603	.25
C54	640	10	16	603	.22
C55	940	14	24	603	.19
C57	940	14	24	603	.18
E2	740	11	16	604	.42
E4	740	12	18	600	.40
E5	760	15	22	607	.19
E20	2110	23	40	590	.35
E24	2110	23	40	590	.30
E26	1330	17	28	604	.26
E28	1580	18	31	603	.36
F8	1340	15	19	600	.48

Table 3.2.2 Continued

F13	1065	10	17	607	.43
F17	1380	16	27	607	.28
F18	1010	16	27	607	.33
F20	1450	16	27	607	.34
F24	1430	15	25	607	.38
F26	1150	15	25	607	.27
F28	1310	15	25	607	.38
F31	1180	15	25	607	.28
H7	770	14	21	588	.48
H9	620	10	14	607	.44
H11	620	10	14	607	.40
H13	620	10	14	607	.28
J20	1100	12.5	19	607	.31
J22	1100	12.5	19	607	.42
J24	1100	12.5	19	607	.48
J26	1100	12.5	19	607	.29
J28	1190	12.5	22	607	.48
J31	1190	12.5	22	607	.54
J33	1100	12.5	19	607	.65

Table 3.2.3

Area B Footings

Footing Number	Load (kips)	Width (Feet)	Length (Feet)	Top Elevation	Settlement (inches)
E63	770	15	25	596	.32
E64	770	15	25	596	.20
E66	1490	19	32	596	.32
E72	1160	15	25	595	.32
E73	915	15	25	595	.26
E76	960	20	25	584	.12
E77	1210	20	30	584	.23
H55	460	11	16	607	.14
H57	1160	13	21	607	.23
H63	1160	13	21	607	.36
H66	770	11	16.5	607	.25
H68	650	10	15	607	.25
H70	550	9	13.5	607	.19
H71	375	8	12	607	.28
H72	930	12	18	607	.22
H77	900	12.5	19	607	.38
K57	600	10	15	600	.25
K59	600	10	15	600	.25
K61	600	10	15	600	.20
K63	590	10	15	600	.13
K64	650	10	15	600	.19
K66	650	10	15	600	.56
K68	650	10	15	600	.55
K70	650	10	15	600	.30
K72	550	10	15	600	.31
K73	660	10	15	600	.55
K76	660	10	15	607	.48
K77	650	10	15	607	.65
E99	925	10	15	607	.28
E101	720	10	15	607	.28
E103	640	10	15	607	.28
E105	640	10	15	607	.24
E107	640	10	15	607	.29
E109	640	10	15	607	.31
E111	730	15	25	596	.32
K99	640	10	15	607	.24
K101	640	10	15	607	.12
K103	640	10	15	607	.10
K105	640	10	15	607	.11

Table 3.2.3 Continued

K107	640	10	15	607	.26
K109	640	10	15	607	.16
E84	1800	20	30	603	.40
E90	1580	14	20	603	.32
E91	800	14	20	605	.58
E93	770	12	18	605	.44
E95	530	12	18	605	.43
E97	925	12	18	607	.20
F84	1850	21	31	603	.57
F90	1940	21	31	603	.42
F91	810	14	20	605	.35
F93	750	14	20	605	.35
F95	765	17	24	605	.14
H80	790	11	16	607	.29
H82	800	11	16	607	.48
H84	640	10	15	607	.45
H86	640	10	15	607	.35
H88	640	10	15	607	.11
H90	640	10	15	607	.20
H91	640	10	15	607	.12
K80	780	10	16	607	.53
K82	820	12	17	607	.59
K84	1160	14	21	607	.62
K90	1160	14	21	607	.58
K91	640	10	15	607	.31
K93	640	10	15	607	.34
K95	580	10	15	607	.37
K97	640	10	15	607	.28

3.3.1 Control Group

Footings in the control group ranged in size from 10 x 14 feet to 23 x 40 feet. The length to width ratio of the footings was between 1.1 and 2.1. Design loads ranged from 1.9 kips per square foot to 5.4 kips per square foot. Figure 3.3.1 shows the correlation between the blowcount and the modulus for the control group. The blowcount used for a given footing was the average of the values of the blowcount measured between the top of the footing and the depth B in the boring directly under or close by the footing.

3.3.2 Groups A and B

The footings in groups A and B are similar to those in the control group. However, since there is not a boring corresponding to each footing in these groups a different method had to be used to calculate an average blowcount for each footing. While each group of borings was initially separated into subgroups consisting of those borings on cut areas and those on fill areas, subsequent tests indicated that there was no noticeable difference between the SPT in these areas. Therefore all the borings in a particular group were considered to be of the same population.

An average value of the blowcount versus depth for each of groups A and B was determined as follows:

A. All data in a two foot layer of soil were lumped together.

B. A mean, variance and coefficient of variation was then

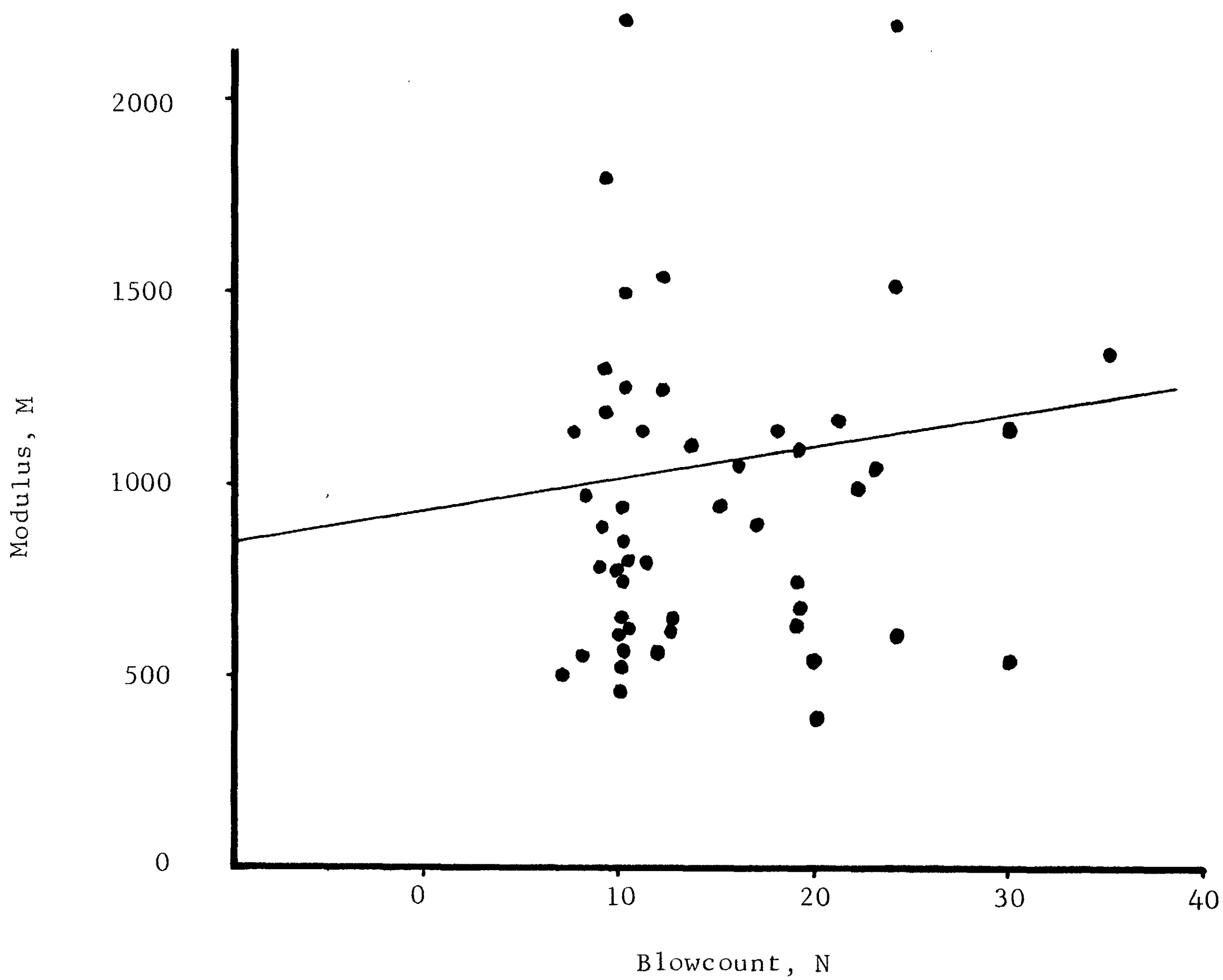


Fig. 3.3.1 Modulus - Blowcount Correlation for the Control Group

calculated for each layer of each group.

C. "Best fit" second degree polynomials were determined for the means and coefficients of variation versus depth for each of the two groups. Figures 3.3.2 through 3.3.5 show these points and the "best fit" curve through them for each of groups A and B.

The correlation between modulus and blowcount for the two data groups is shown in Figs. 3.3.6 and 3.3.7. Since there is not a boring for each footing in groups A and B, the blowcount was determined by averaging the "best fit" blowcount versus depth curve over a distance B below a footing. That is:

$$N_i = 1/B_i \int_{E_i}^{E_i + B_i} f(x)dx \qquad (3.3.1)$$

where E_i is the elevation of the i^{th} footing

$f(x)$ is the "best fit" polynomial

Since most of the footings in both groups A and B have been plotted against a small range of the blowcount, this data is not appropriate for checking the relationship between M and N. However, the variability of the modulus associated with a given value of the blowcount can be determined. Figure 3.3.8 is a histogram of the value of the modulus for a blowcount of 17 for group A. Figure 3.3.9 is a similar histogram for group B.

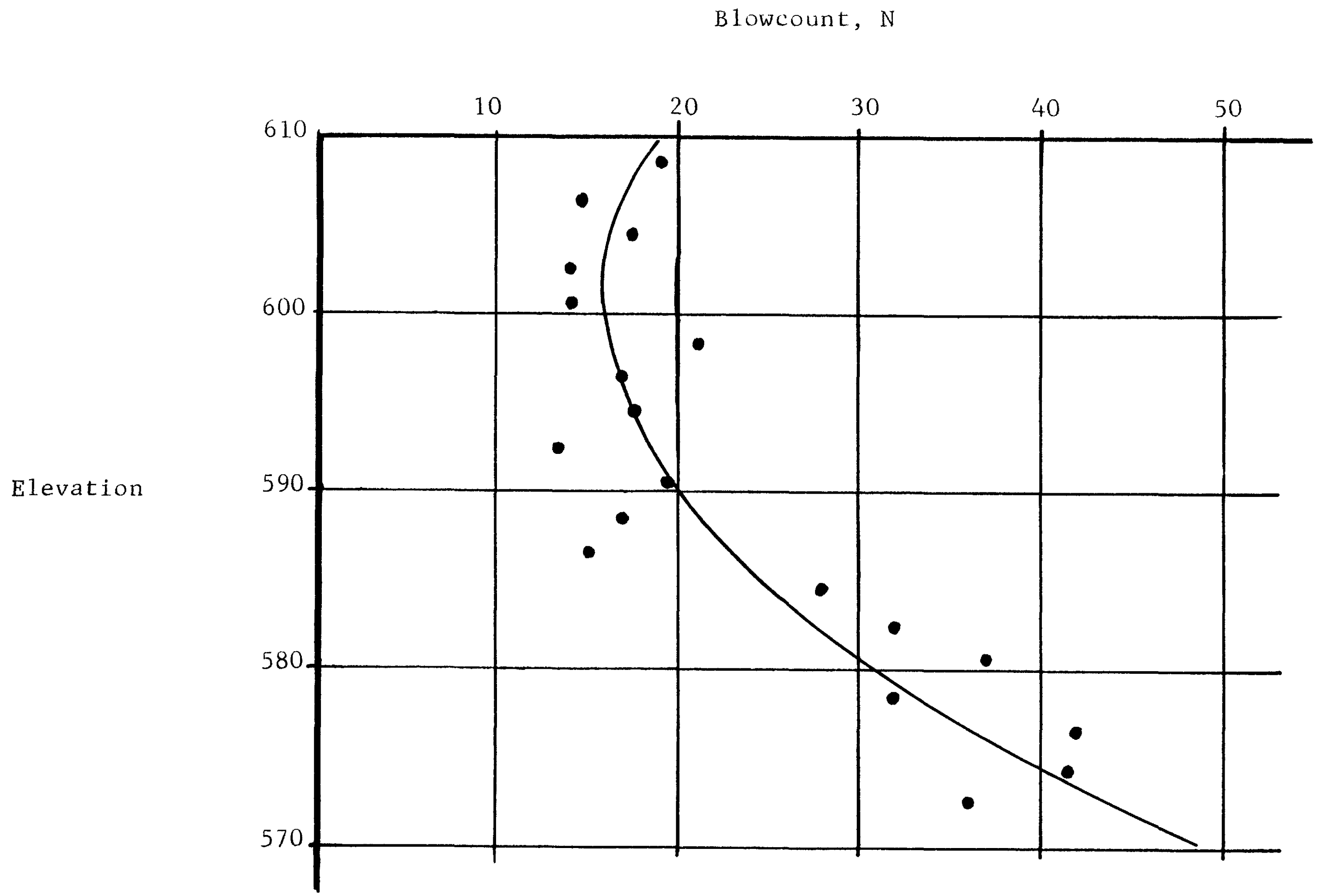


Fig. 3.3.2 Average Blowcount vs. Depth of Area A

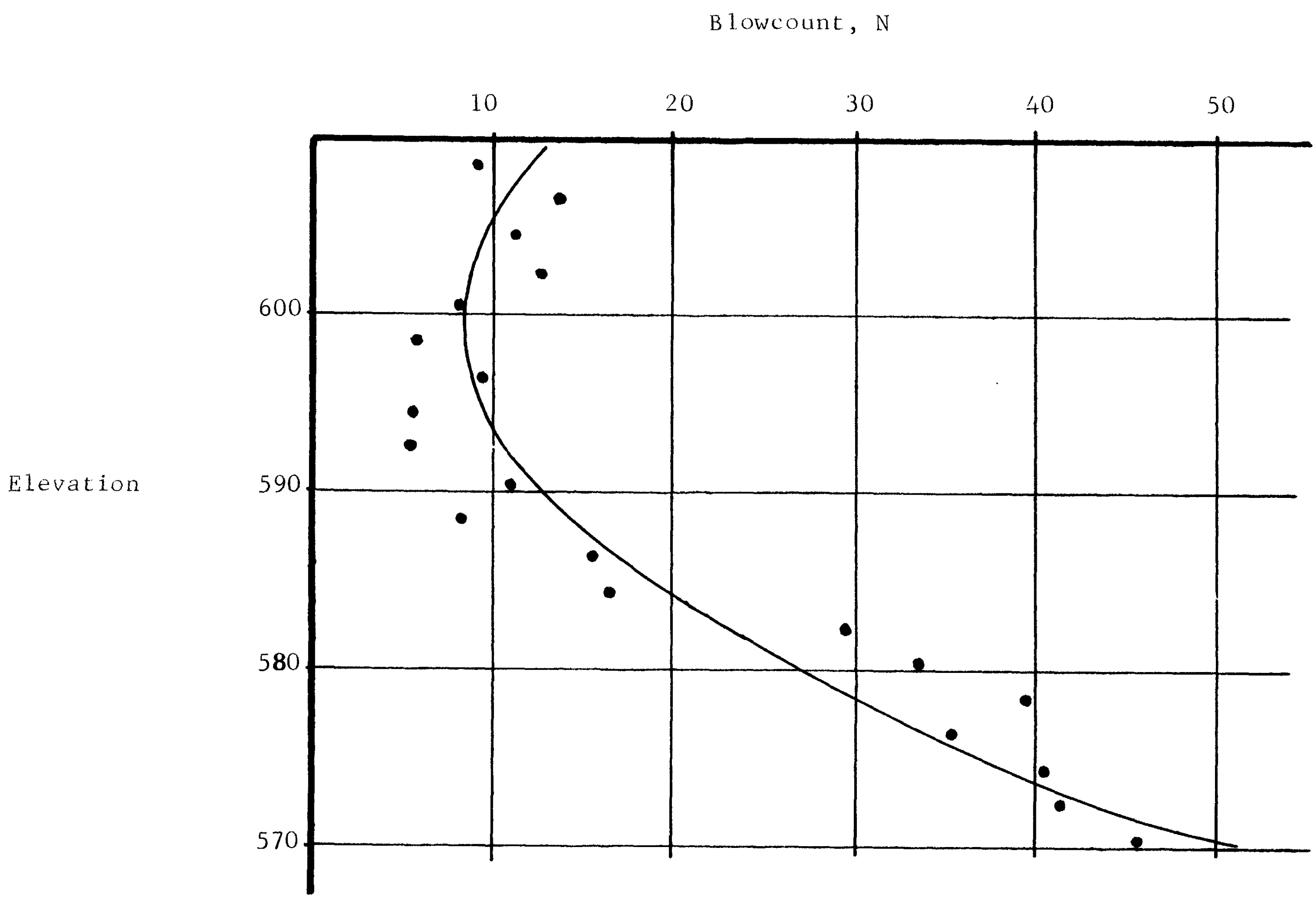


Fig. 3.3.3 Average Blowcount vs. Depth of Area B

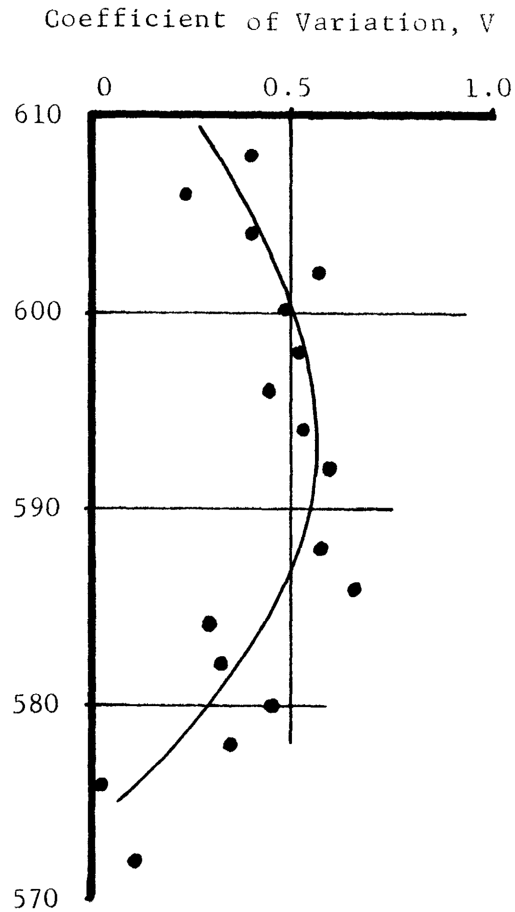


Fig. 3.3.4 Coefficient of Variation versus Depth for Area A

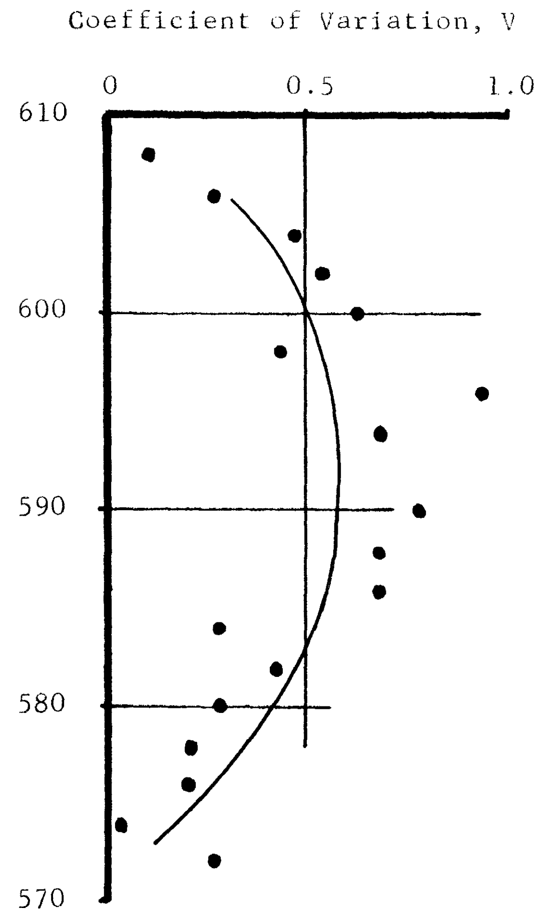


Fig. 3.3.5 Coefficent of Variation versus Depth for Area B

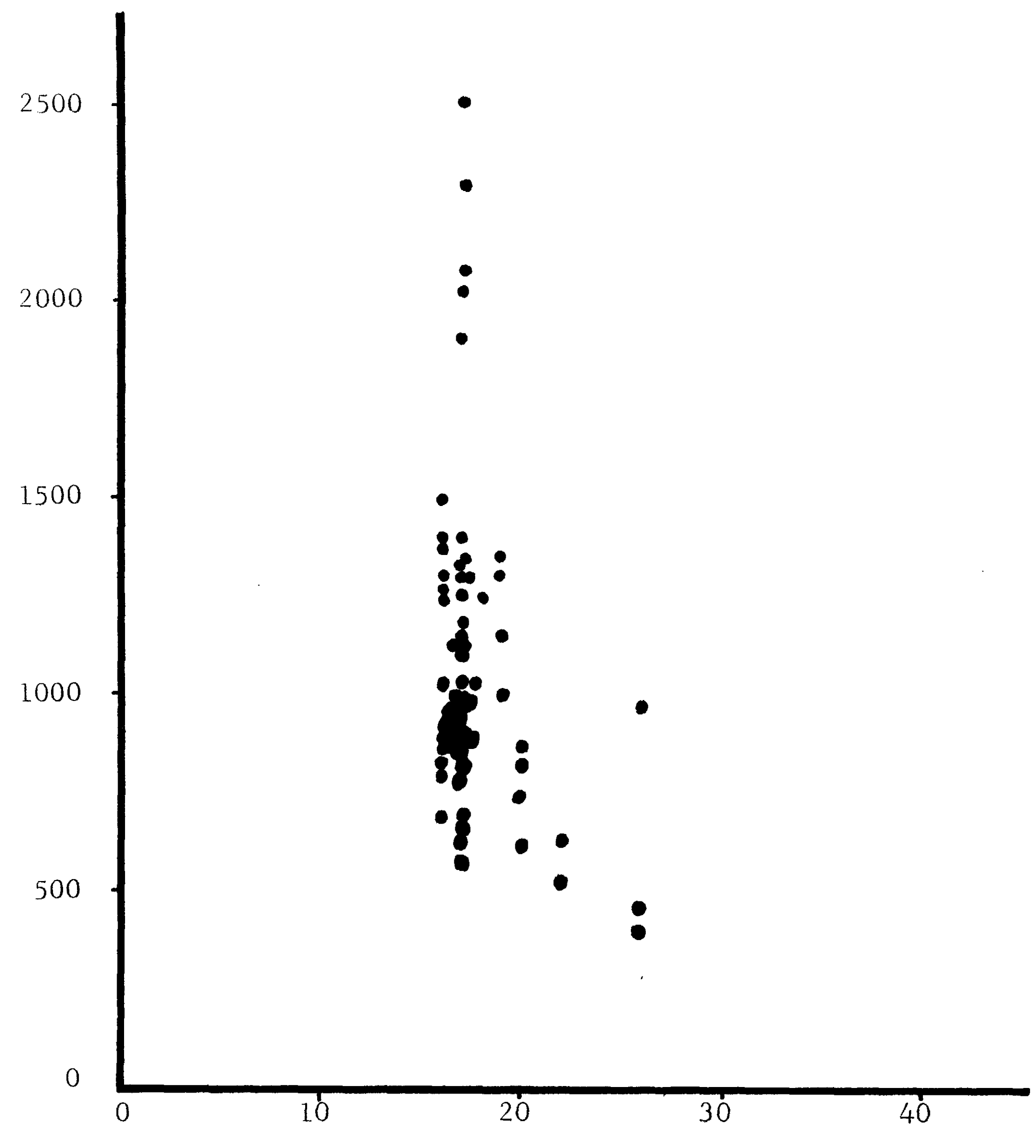


Fig. 3.3.6 Modulus - Blowcount Correlation for Area A

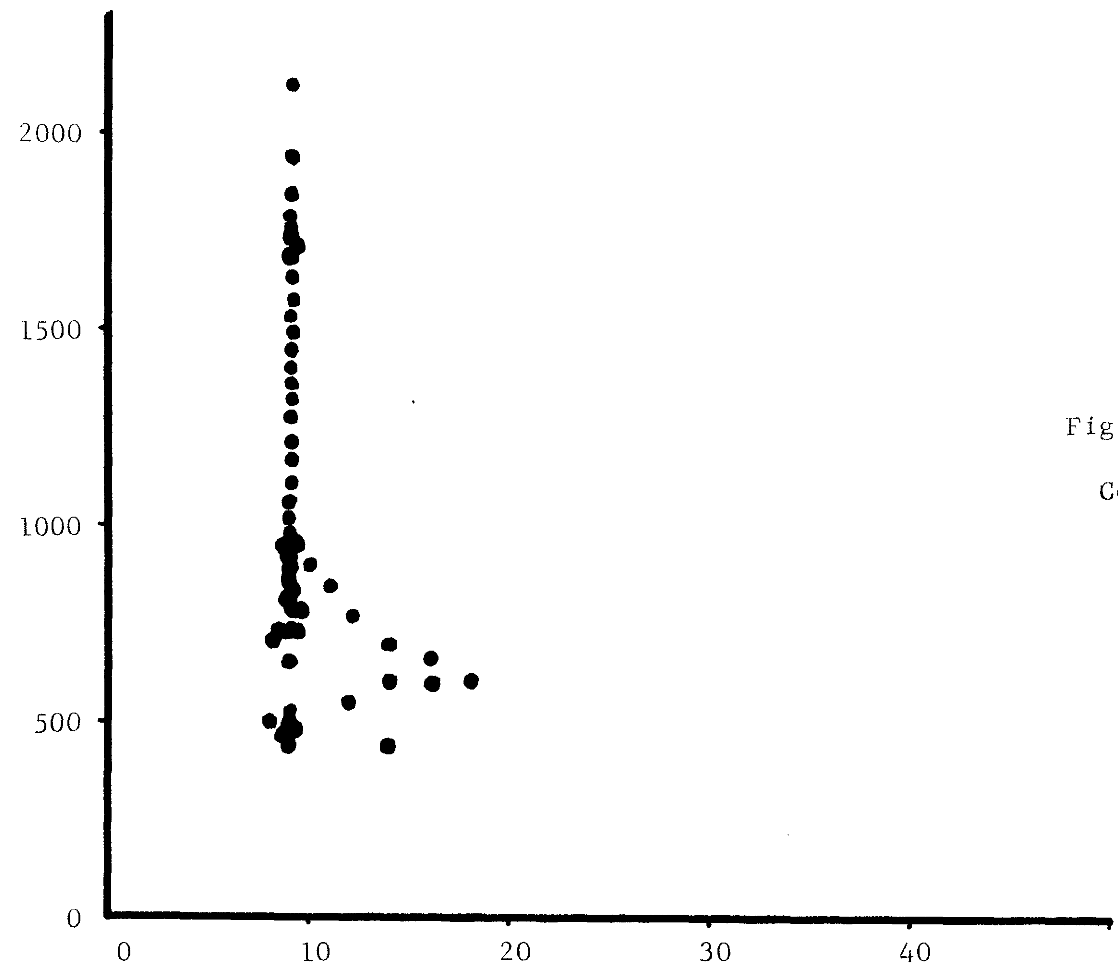


Fig. 3.3.7 Modulus - Blowcount Correlation for Area B

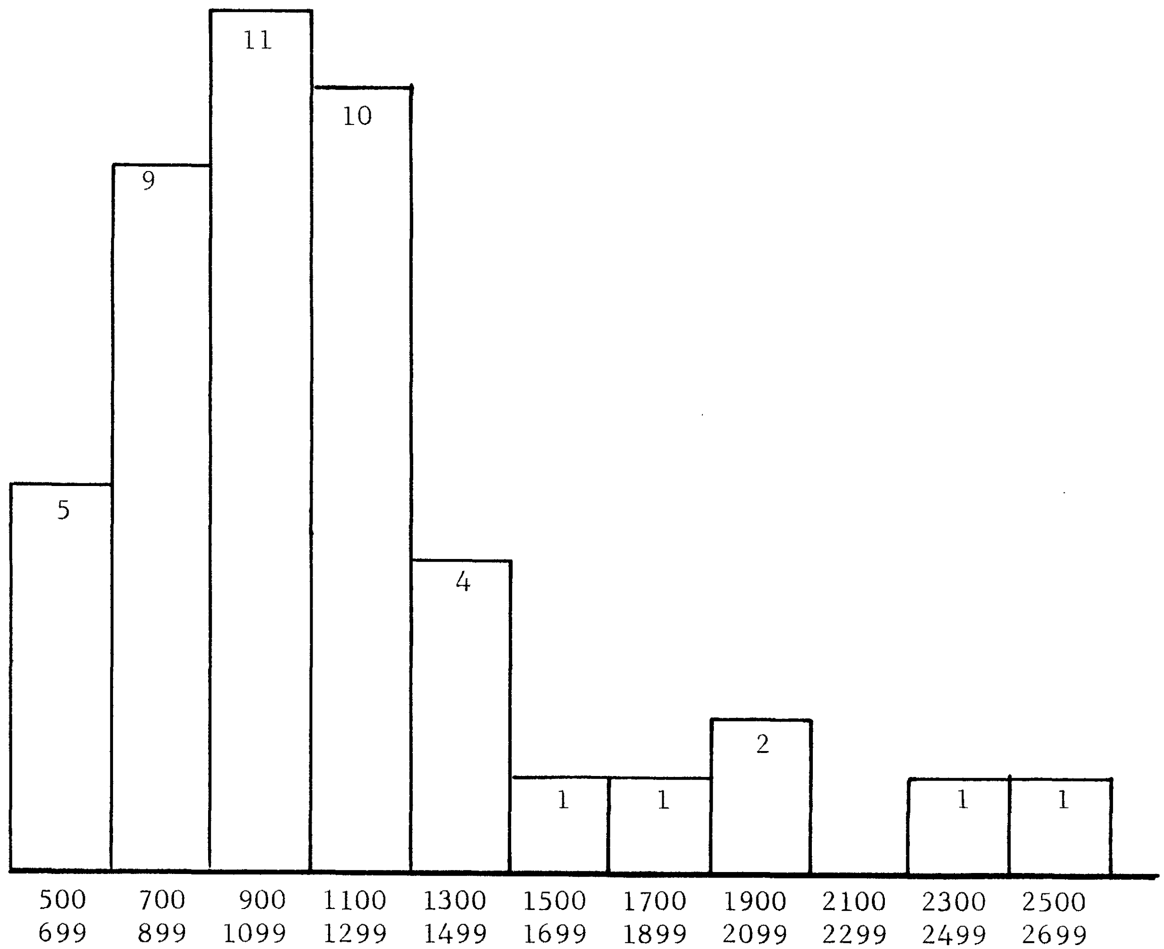


Fig. 3.3.8 Frequency of the Modulus at a Blowcount of 17, Area A

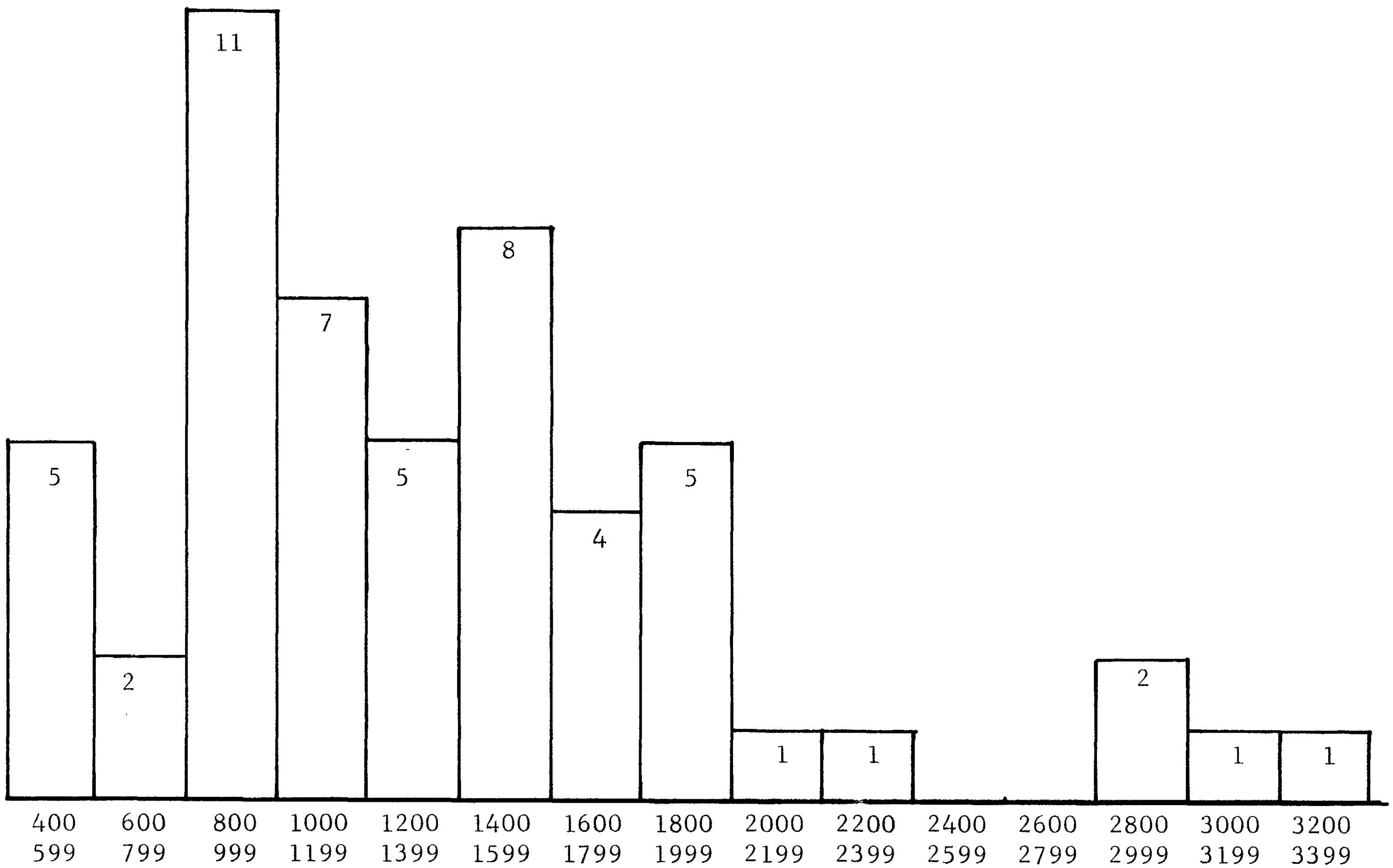


Fig. 3.3.9 Frequency of the Modulus at a Blowcount of 9, Area B

3.4 SPATIAL CORRELATION

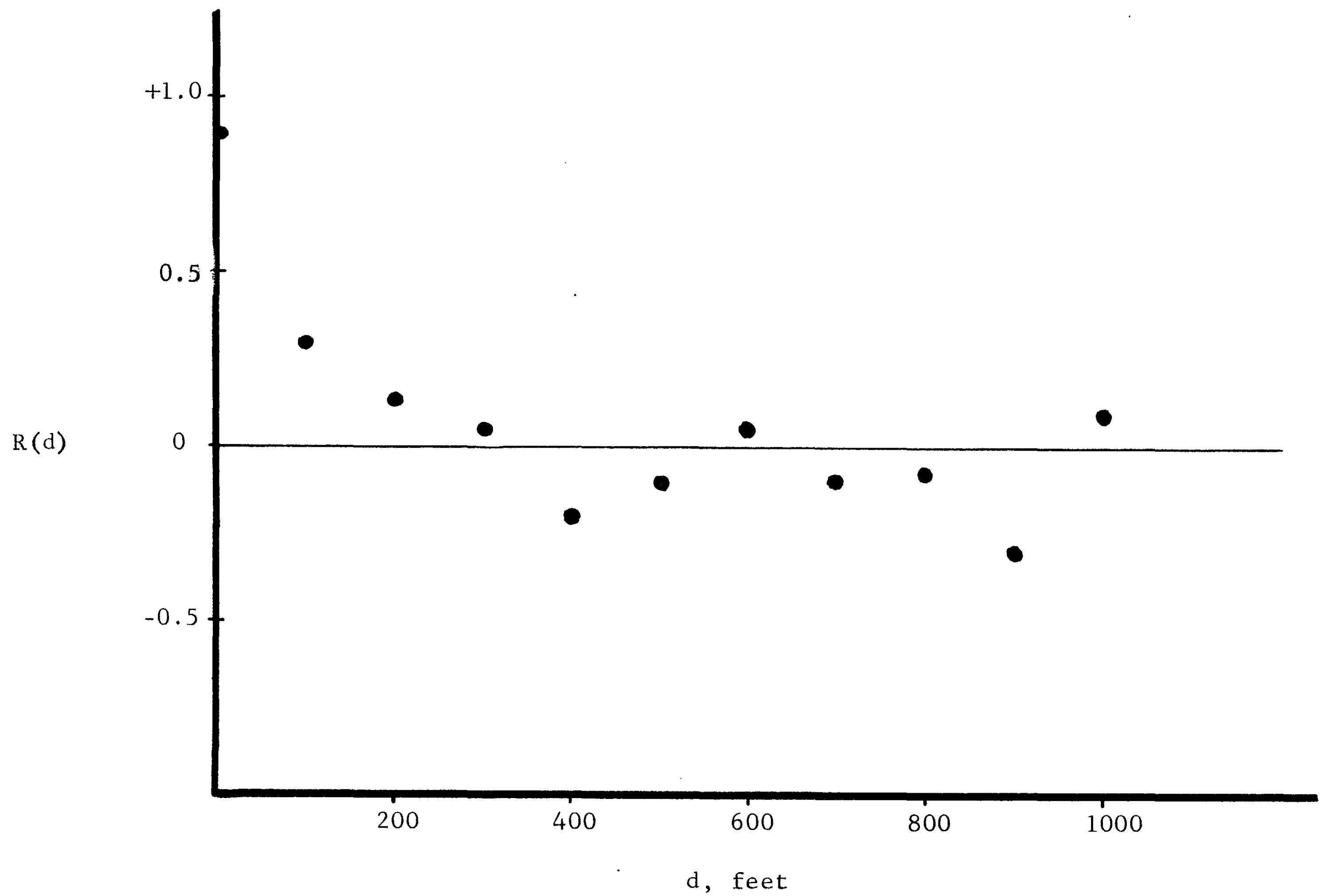
Several types of statistical tests were conducted on the blowcount data to determine if it was spatially correlated.

3.4.1 Horizontal Spatial Correlation of N

Correlation in the horizontal direction was evaluated by estimating the autocorrelation function based on the irregularly spaced borings.[12] The autocorrelation function was estimated for each 2 foot layer for data sets A and B. An incremental distance of approximately 100 feet was chosen. In all cases, the correlation coefficient measured at 100 feet was less than 0.30. Figure 3.4.1 is typical of the results of the horizontal correlation tests. On the basis of these test results it is assumed that for this site the measured blowcount is essentially uncorrelated at distances between 100 and 1000 feet. Sufficient data was not available to determine the horizontal correlation for distances less than 100 feet.

3.4.2 Vertical Spatial Correlation of N

The correlation in the vertical direction was studied using an approach based on variance functions.[11] Prior to computing the variance function all blowcount data N_{ij} were normalized by subtracting the mean value m_i for the layer i and dividing the difference $N_{ij} - m_i$ by the standard deviation s_i for the layer i. That is,



Fig, 3.4.1 Typical Autocorrelation of a Layer in Group A and Group B

(3.4.1) $$R_{ij} = \frac{N_{ij} - m_i}{s_i}$$

where $$m = 1/n \sum_{j=1}^{n} N_{ij}$$

$$s_i^2 = 1/n \sum_{j=1}^{n} (N_{ij} - m_i)^2$$

The variance function was calculated using the normalized data, the variance function being

(3.4.2) $$\sigma^2(k) = \frac{1}{t} \sum_{j=1}^{t} \left[\sum_{i=1}^{k} R_{ij} \right]^2 - \frac{1}{t^2} \left[\sum_{j=1}^{t} \sum_{i=1}^{k} R_{ij} \right]^2$$

t = total number of borings in the group

k = number of layers

Figures 3.4.2 and 3.4.3 show a comparison of the measured normalized variance for different distances and the variance function corresponding to an assumed coefficient of correlation of the form:

(3.4.3) $$\rho = e^{-d^2/b^2}$$

Comparing the observed and the computed variance functions it is found that a "best fit" is obtained by using b = 5 for group A and b = 4 for group B.

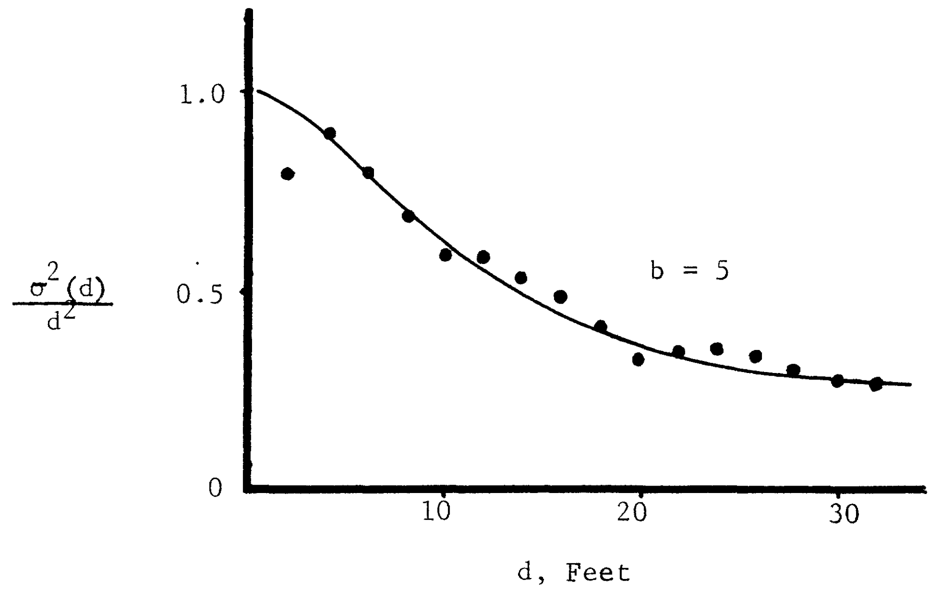


Fig. 3.4.2 Normalized Variance Function of N for Area A

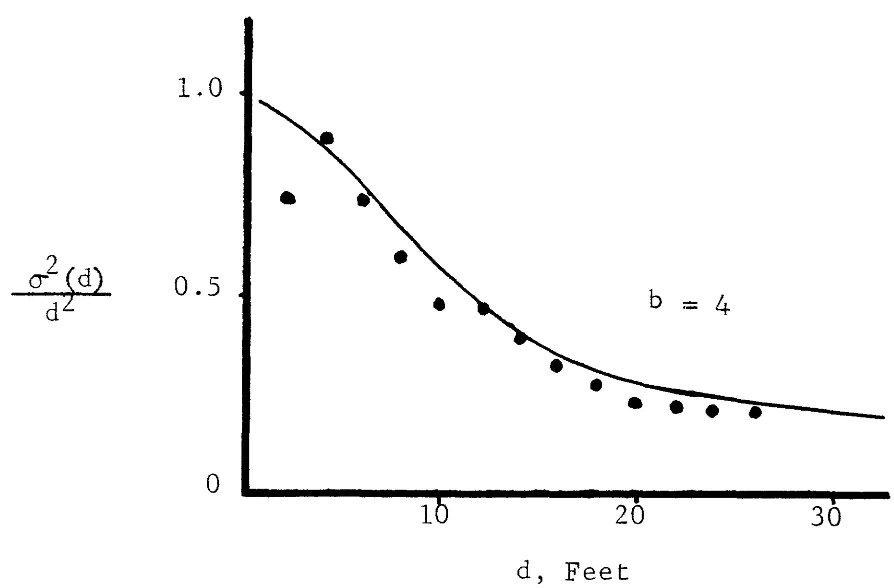


Fig. 3.4.3 Normalized Variance Function of N for Area B

3.5 ACTUAL VERSUS PREDICTED SETTLEMENT - SINGLE PARAMETER MODEL

Three groups of footings were chosen for the purpose of comparing the actual settlement with the predicted settlement. These groups were selected on the basis of similar values of K, elevation and width. Table 3.5.1 summarizes the pertinent information for each group.

Table 3.5.1

Group	Number of Footings	K	Elevation	B
A1	23	30	607	14
B1	10	26	600	10
B2	20	34	607	10

The mean and variance of the total settlement for these three groups of data was calculated based on the measured settlement. Tables 3.5.2 through 3.5.4 list the footings and the measured mean and variance of the total settlement. Figures 3.5.1 through 3.5.3 show the CDF's for the same groups.

The mean value of the blowcount needed to predict the total and differential settlement of a group of footings was determined using the layer means and variances previously calculated. That is,

(3.5.1)
$$m_N = \sum_{i=1}^{n} a_i m_{N_i}$$

where
$$\sum_{i=1}^{n} a_i = 1$$

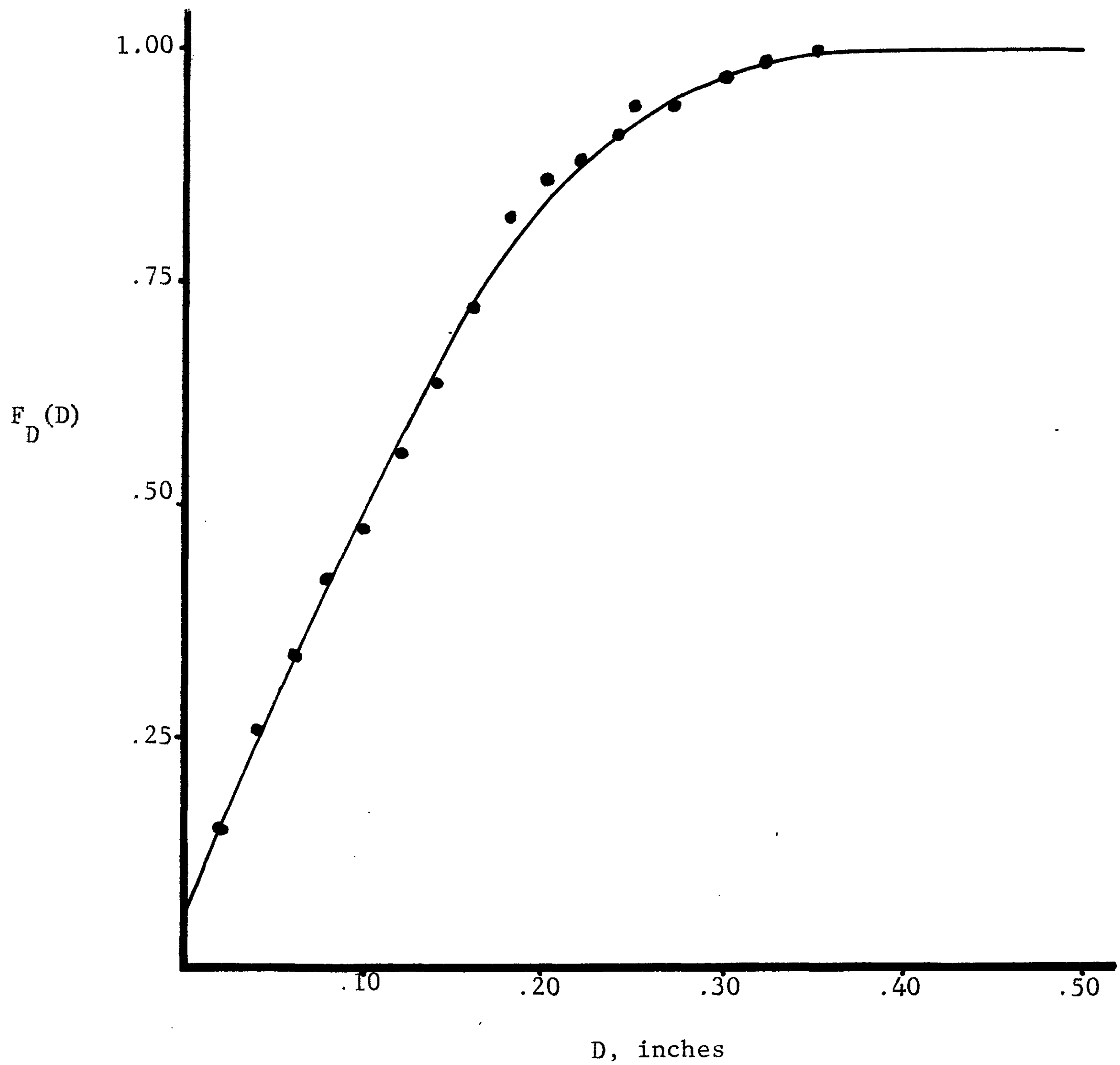


Fig. 3.5.1 CDF of the Differential Settlement for Group A1

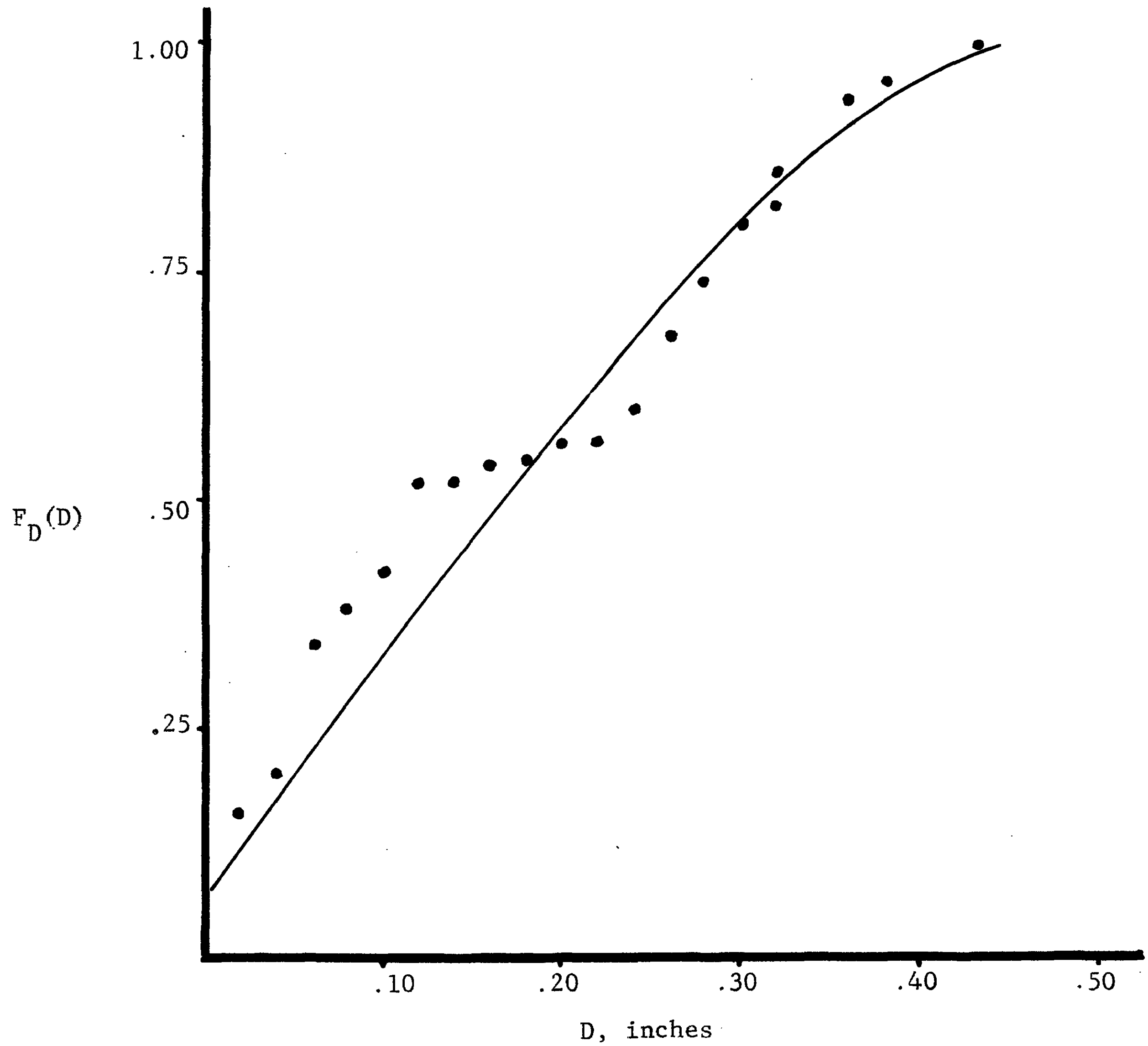


Fig. 3.5.2 CDF of the Differential Settlement for Group B2

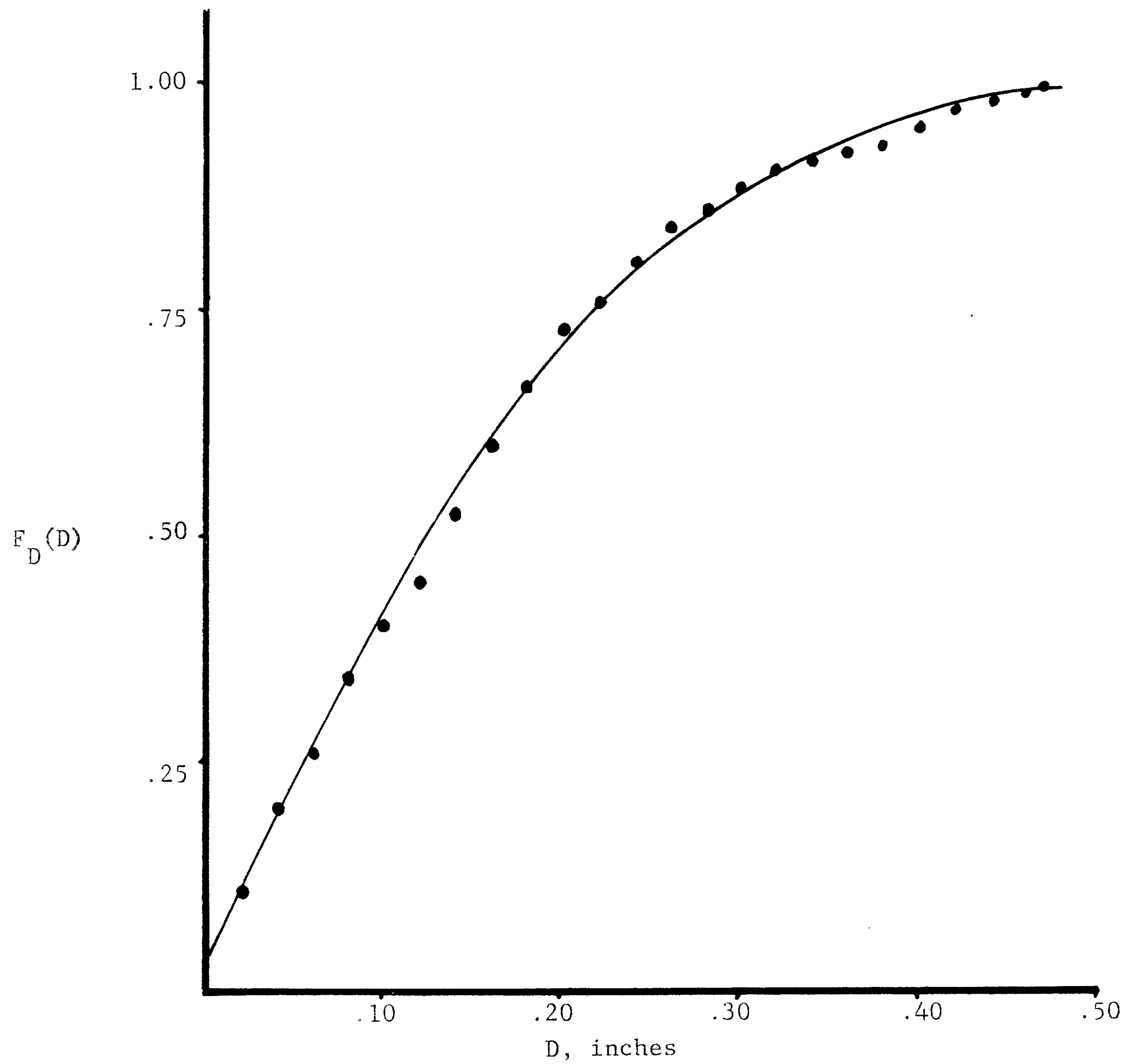


Fig. 3.5.3 CDF of the Differential Settlement for Group B2

Table 3.5.2

Group Al Footings and Measured Settlements

Footing	Settlement
A33	.26
A35	.13
A37	.12
A41	.24
A43	.28
A45	.31
A47	.30
A51	.15
A53	.12
A55	.22
A57	.16
B35	.30
B37	.30
B39	.47
B41	.43
B43	.38
B45	.34
B47	.36
B51	.32
B53	.30
B55	.29
E5	.19

$m_S = .28$ in.

$s_S = .10$ in.^{2}

Table 3.5.3

Group B1 Footings and Measured Settlements

Footing	Settlement
H71	.28
K57	.25
K59	.25
K61	.20
K63	.13
K64	.19
K66	.56
K68	.55
K70	.30
K72	.31
K73	.55

$$m_S = .325 \text{ in.}$$

$$s_S = .15 \text{ in.}^2$$

Table 3.5.4

Group B2 Footings and Measured Settlements

Footing	Settlement
E103	.28
E105	.24
E107	.29
E109	.31
K99	.24
K101	.12
K103	.10
K105	.11
K107	.26
K109	.16
H84	.45
H86	.35
H88	.11
H90	.20
H91	.12
K80	.53
K91	.31
K93	.34
K64	.28

$m_S = .27$ in.

$s_S = .13$ in.^{2}

The weighting factor a_i is the fraction of the i^{th} layer which contributes to the depth B divided by B.

The variance of the average blowcount was determined using eq. (3.5.2).

$$(3.5.2) \qquad s_N^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j s_{N_i} s_{N_j} e^{-d_{ij}^2/b^2}$$

Here d_{ij} is the distance between the center of the i^{th} layer to the center of the j^{th} layer. Note that eq. (3.5.2) includes the effect of vertical correlation. A value of b equal to 5 was used for area A while b equal to 4 was used for area B. Tables 3.5.5 through 3.5.7 summarize the weighting factors, layer means and variances for the three groups of footings.

Since it is more convenient to calculate settlement in inches, the modulus-blowcount relationship was also converted to inches. Using the modulus-blowcount relationship of the previous section, the constants

$$k_1 = 71.5$$

$$k_2 = 0.67$$

$$k_3 = 34.5$$

will give settlement in inches for the formula

$$(3.5.3) \qquad S = PBI/(k_1 + k_2 N + k_3 U)$$

where P is in kips per square feet

B is in feet

U is an independent, zero mean and unit variance random variable.

Table 3.5.5

Group A1 Weighting Factors, Layer Means, and Layer Variances

a_i	N_i	s_i
.143	14.82	3.45
.143	17.36	6.99
.143	14.25	8.14
.143	14.25	6.94
.143	21.00	10.94
.143	16.75	7.80
.143	17.55	9.23

Table 3.5.6

Group B1 Weighting Factors, Layer Means, and Layer Variances

a_i	N_i	s_i
.10	8.36	5.31
.20	5.50	2.58
.20	9.10	8.81
.20	5.20	3.62
.20	5.13	5.24
.10	11.00	9.00

Table 3.5.7

Group B2 Weighting Factors, Layer Means, and Layer Variances

a_i	N_i	s_i
.20	13.44	3.70
.20	11.00	5.20
.20	12.33	6.97
.20	8.36	5.31
.20	5.50	2.58

Table 3.5.8 gives a comparison of the actual anddthe predicted values of the mean and variance of the total and differential settlement.

3.6 COEFFICIENT OF COMPRESSIBILITY - BLOWCOUNT RELATIONSHIP

The relationship between the coefficient of compressibility and the blowcount was assumed in section 2.4 to be of the form

$$V_i = k_1' + k_2'/N_i + k_3'U$$

The control group of borings and footings was used to examine the correlation between V and N. The procedure was as follows: The assumed relationship was substituted into eq. (2.3.1) to obtain:

(3.6.1) $$S_j = \sum_{i=1}^{n} (k_1' + k_2'/N_{ij})P_{ij}Z_jq_j$$

or $$S_j = k_1'X_j + k_2'Y_j$$

where $$X_j = Z_jq_j \sum_{i=1}^{n} P_{ij}/N_{ij}$$

$$Y_j = Z\ q \sum_{i=1}^{n} P_{ij}$$

$$S_j = \text{measured settlement of the } j^{th} \text{ footing}$$

The X's and Y's were estimated from the data using six layers, each of thickness B/3. An estimate of P was obtained by taking the average stress coefficient at the midpoint of the layer for the footing geometry. The N_{ij} used was the blowcount measured in the i^{th} layer

Table 3.5.8

Predicted Settlement Using the Single Parameter Model

				Measured			Predicted		
Group	K	m_N	s_N	m_S	s_S	m_D	m_S	s_S	m_D
A1	30	16.6	6.6	.28	.10	.11	.41	.12	.17
B1	26	6.9	4.7	.32	.15	.15	.42	.14	.19
B2	34	10.1	4.3	.27	.13	.15	.50	.19	.27

$$m_M = k_1 + k_2 m_N$$

$$s_M = \sqrt{k_2^2 s_N^2 + k_N^2}$$

$$m_S = K/m_M \left[1 + s_M^2/m_M^2 \right]$$

$$s_S = K s_M / m_M^2$$

$$m_D = \sqrt{2} \quad s_S$$

of the j^{th} footing. A regression analysis[10] was then done to obtain the relationship:

$$V_i = k_1' + k_2'/N + k_3'U$$

where

$$k_1' = .011$$

$$k_2' = .021$$

$$k_3' = .124$$

If it is assumed that k_1' is equal to zero, a regression analysis determines the remaining constants to be

$$k_2' = .139 \qquad (k_1' = 0)$$

$$k_3' = .161$$

Figures 3.6.1 and 3.6.2 show the correlation between S and X and S and Y.

3.7 VERTICAL SPATIAL CORRELATION OF 1/N

Vertical spatial correlation of the coefficient of compressibility was determined using the reciprocal of the blowcount. The procedure used is that given in section 3.4.2 if N_{ij} is replaced by $1/N_{ij}$. Figure 3.7.1 and 3.7.2 show a comparison of the measured normalized variance for different distances and the variance function corresponding

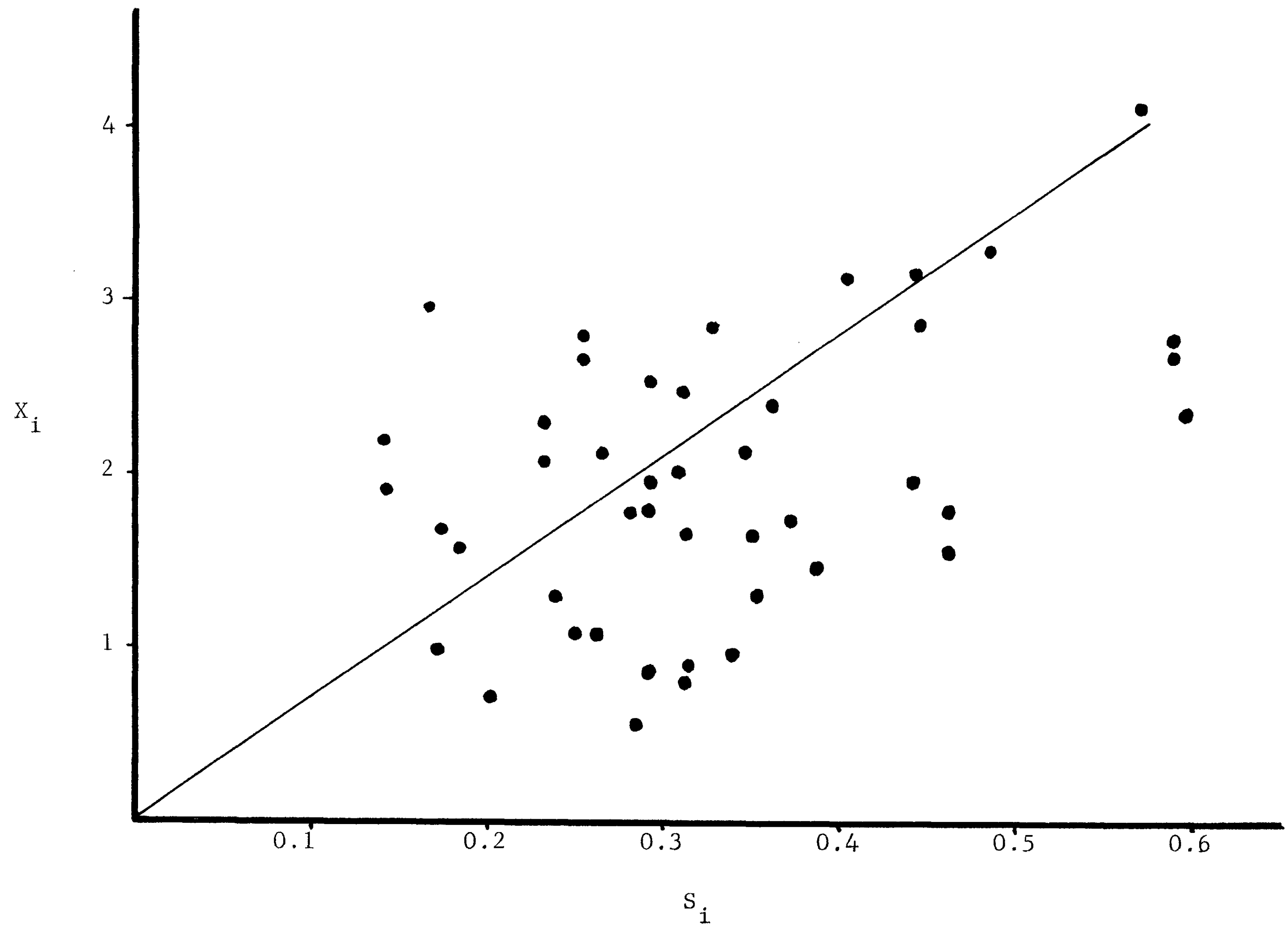


Fig. 3.6.1 S_i vs. X_i for the Control Group

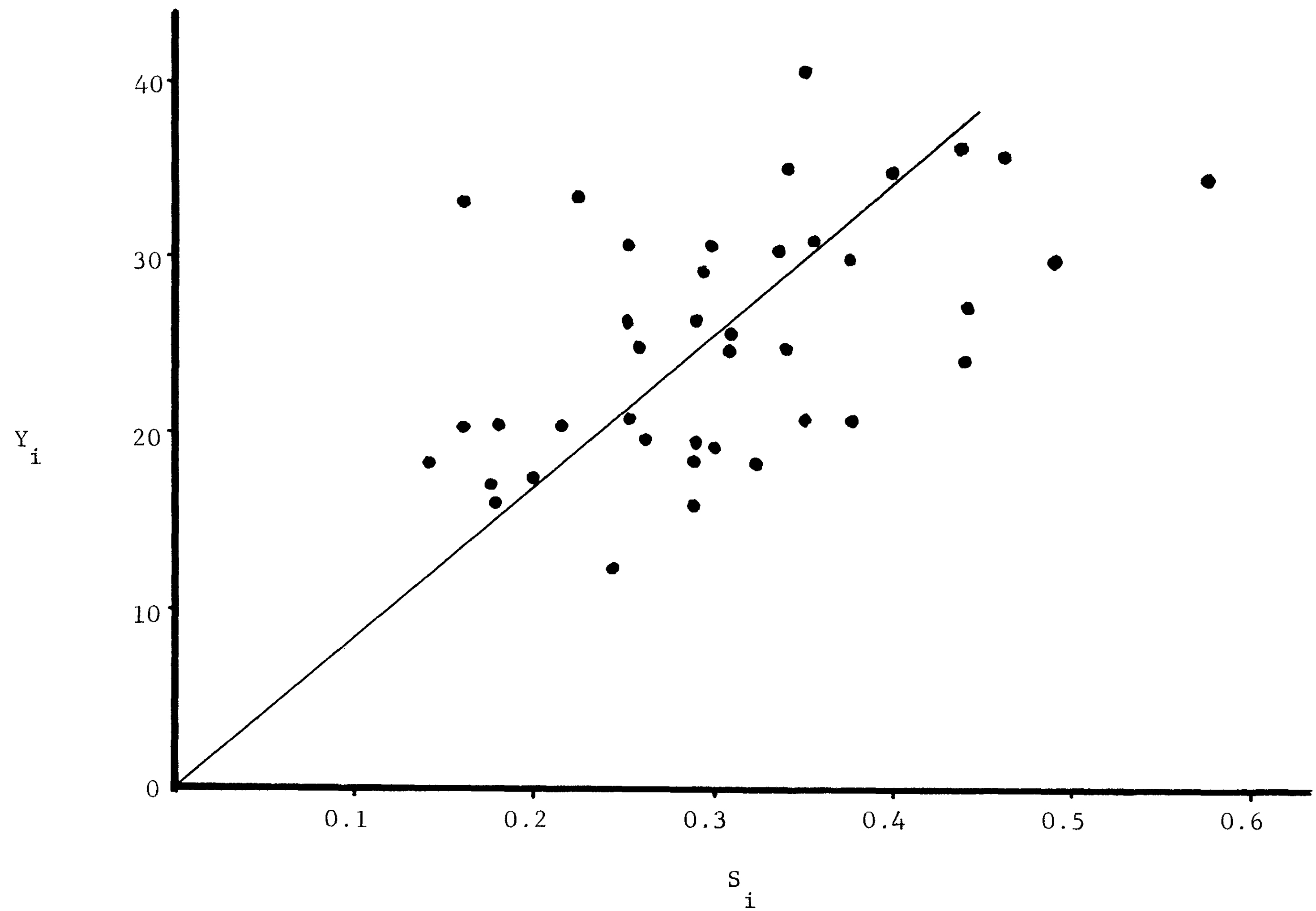


Fig. 3.6.2 S_i vs. Y_i for the Control Group

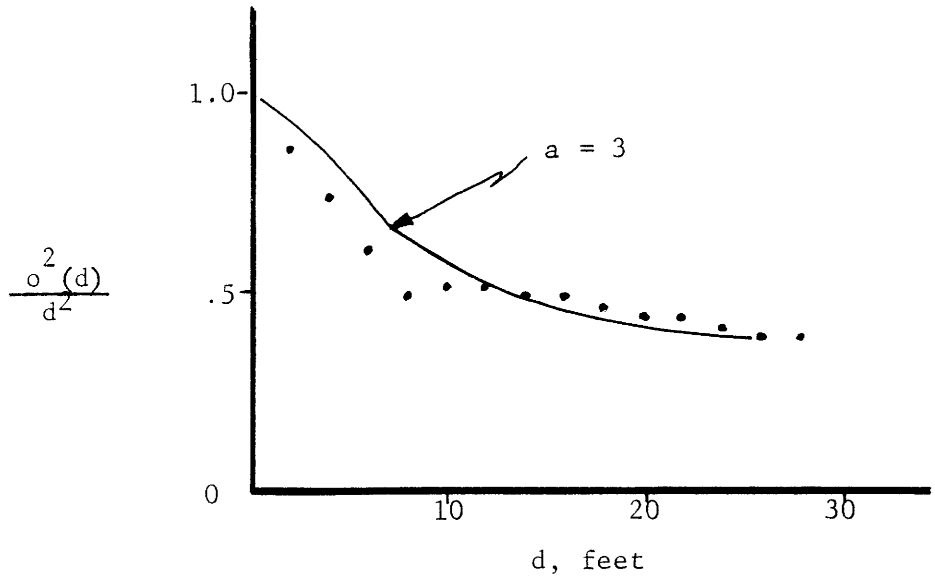


Fig. 3.7.1 Normalized Variance Function of 1/N for Area A

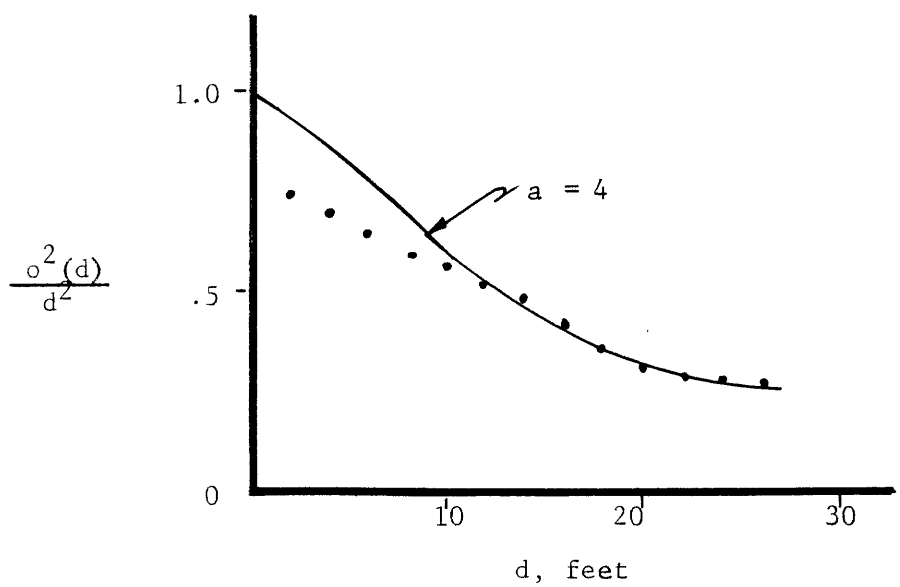


Fig. 3.7.2 Normalized Variance Function of 1/N for Area B

to an assumed coefficient of correlation of the form:

$$\rho = e^{-d^2/a^2}$$

Comparing the observed and the computed variance functions it is found that a "best fit" if obtained by using a = 3 for group A and a = 4 for group B.

3.8 ACTUAL VERSUS PREDICTED SETTLEMENT - MULTIPARAMETER MODEL

The same three subgroups of footings from groups A and B were again chosen for the purpose of comparing the actual and the predicted settlement using the multilayered model. Table 3.8.1 summarizes the pertinent information about each group needed for this comparison.

Table 3.8.1

Group Number	Number of Footings	q	Z	B	a	Elevation
A1	23	2.60	2	14	3	607
B1	10	4.33	2	10	4	600
B2	20	4.27	2	10	4	607

The measured means and variances of the settlement given in Tables 3.5.2 through 3.5.4 still apply.

Results from Chapter II show that the mean and variance of the settlement of a footing are given by:

(3.8.1) $$m_S = \sum_{i=1}^{n} qP_i ZV_i^o$$

(3.8.2) $$s_S^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} q^2 Z^2 P_i P_j s_{M_i} s_{M_j} \exp(-d^2/a^2)$$

where

Z = layer thickness

q = applied load

P_i = average stress coefficient of the i^{th} layer

V_i^o = expected value of the coefficient of compressibility

V_i' = variance of the coefficient of compressibility

n = number of layers

d = distance between the i^{th} and the j^{th} layers

Again using results of Chapter II, V_i^o and V_i' can be approximated by:

(3.8.3) $$V_i^o \approx k_1' + \frac{k_2'}{m_{N_i}} \left[1 + \left[\frac{s_{N_i}}{m_{N_i}} \right]^2 \right]$$

(3.8.4) $$V_i' = k_3'^2 + \left[\frac{k_2' \, s_{N_i}}{m_{N_i}^2} \right]^2$$

where

m_{N_i} = mean of the blowcount in the i^{th} layer

s_{N_i} = variance of the blowcount in the i^{th} layer

For the purpose of evaluating this model, a layer thickness of two feet was chosen and the average stress coefficient was evaluated at the midpoint of the layer. Tables 3.8.2 and 3.8.3 show a comparison of the actual and the predicted values of the total and the differential settlement for the blowcount - coefficient of compressibility relationships given in the previous section. These tables also include a comparison of the actual and the predicted total and differential settlement if k_3' is neglected.

Table 3.8.2

Measured Values of the Mean and Variance of the Total Settlement Compared with Predicted Values Using $k_1' = 0$ and $k_2' = .139$

Group	Measured		Predicted			
			$k_3' = .164$		$k_3' = 0$	
	m_S	s_S	m_S	s_S	m_S	s_S
A1	.28	.10	.17	2.1	.17	.05
B1	.32	.15	.66	3.0	.66	.25
B2	.27	.13	.49	3.1	.49	.16

Table 3.8.3

Measured Values of the Mean and Variance of the Total Settlement Compared with Predicted Values Using $k_1' = .011$ and $k_2' = .021$

Group	Measured		Predicted			
			$k_3' = .124$		$k_3' = 0$	
	m_S	s_S	m_S	s_S	m_S	s_S
A1	.28	.10	.21	1.56	.21	.008
B1	.32	.15	.34	2.21	.34	.037
B2	.27	.13	.31	2.18	.31	.022

CHAPTER IV

CONCLUSIONS

The mean and the variance of the total and differential settlement predicted for the three subgroups of footings using the single parameter model tend to be conservative. While the mean value of the total settlement predicted using this model were between 30% and 50% too high, the standard deviation was within 20% for all three groups. Since the differential settlement depends on the variance of the total settlement, the single parameter model also gave good estimates of the differential settlement. Thus, the single parameter model appears to be appropriate for obtaining estimates of the differential settlement if the modulus - blowcount relationship can be obtained for the area of interest. At this time it is not clear if the modulus - blowcount relationship is the same for all sands. Studies done by D'Appolonia et al.[3] indicate that this relationship is different for normally loaded sands than it is for preloaded sand.

The study of the spatial correlation in the SPT resulted in new information as previous studies done by Resendiz and Herrera[4] assumed that vertical correlation of soil properties could be neglected. However, this study shows that significant correlation does exist in the vertical direction. For the SPT, the coefficient of correlation decays from 1.0 to about 0.5 as the distance increases from zero to four feet. This study also shows that if one assumed that the correlation coefficient is a function of the square of the distance not only can solutions be found for the variance function of continuous processes but also that these solutions compare well with the measured variance functions.

The predicted mean of the total settlement using the layered model compares quite well if the blowcount - coefficient of compressibility relationship is assumed to be of the form:

$$V_i = k_1' + k_2'/N_i$$

When k_1' was assumed to be zero, the predicted means and variances were quite conservative. However, this may not always be the case, for results indicate that as the value of N becomes larger, the predicted values tend to be less conservative.

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APPENDIX A

DERIVATION OF THE FIRST AND SECOND MOMENTS OF THE TOTAL SETTLEMENT

An exact solution for the moments of the total settlement S can be obtained by deriving the distribution of S based on the distribution of the modulus M. The function $S = g(M)$ is a monotonically decreasing function $g(M) = K/M$. Therefore S is greater than or equal to some value S_o if and only if M is less than or equal to some value m_o, where $s_o = K/m_o$. (See Fig. A1). It then follows that the probability that S is greater than or equal to the probability that M is less than or equal to m_o. Or, in terms of the cumulative density function (CDF):

$$1 - F_S(s_o) = F_M(m_o) = F_M(K/s_o)$$

where $\qquad F_s(s_o)$ = CDF of S

Taking the derivitive of both sides:

$$-f_S(s_o) = d/ds_o(F_M(K/s_o))$$

But, by definition

$$F_M(K/s_o) = \int_o^{K/s_o} f_M(x)\, dx$$

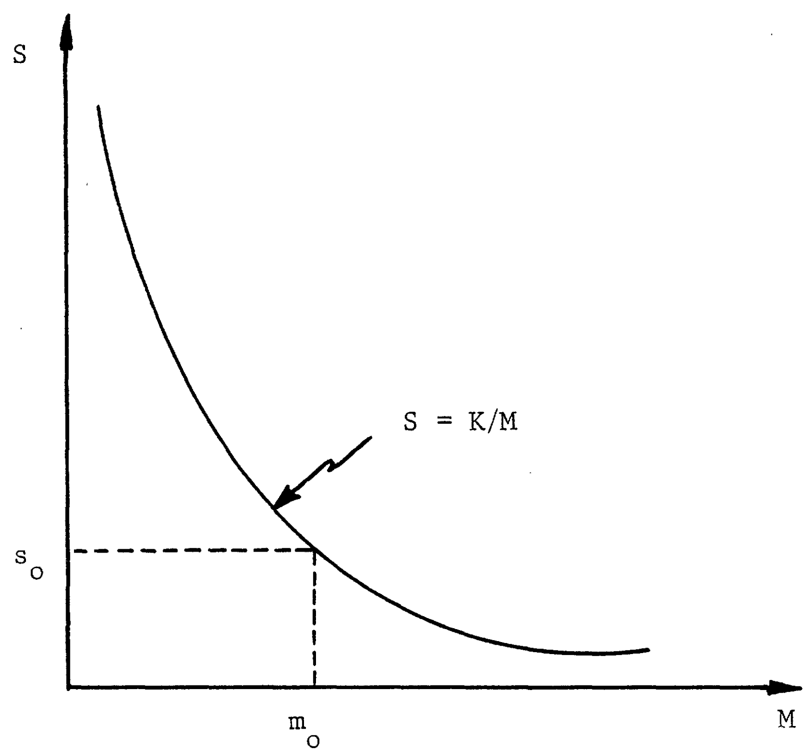

Fig. A 1   The Function S = K/M in the SM Plane and so the distribution of S can be expressed as:

$$f_S(s_o) = -d/ds_o \int_o^{K/s_o} f_M(x)dx = k/s_o^2 f_M(k/s_o).$$

Recalling that

$$E\left[S^n\right] = \int_{-\infty}^{+\infty} s^n f_S(s)ds$$

Then it follows that

$$E\left[S\right] = \int_o^{\infty} (K/s)f_M(K/s)ds$$

and

$$E\left[S^2\right] = \int_o^{\infty} Kf_M(K/s)ds$$

Thus, the variance of the settlement would be:

$$Var\left[S^2\right] = K\int_c^{\infty} f_M(K/s)ds - \left[K\int_o^{\infty}(1/s)f_M(K/s)ds\right]^2$$

APPENDIX B

DERIVATION OF THE DISTRIBUTION OF THE DIFFERENTIAL SETTLEMENT

Consider two footings which have a settlement distribution $f_{S1}(s_1)$ and $f_{S2}(s_2)$. Since the differential settlement can be defined as

$$D = |S_1 - S_2|$$

where D = differential settlement

then the CDF of D, assuming that S_1 and S_2 are independent, would be (B.1)

$$F_D[d] = P[D \leq d] = P[|S_1 - S_2| \leq d]$$

$$= \iint_{R_D} f_{S1}(s_1) f_{S2}(s_2) ds_1 ds_2$$

where R_D is the region on the $s_1 s_2$ plane where $|S_1 - S_2| \leq d$. This region is shown in Fig. (B.1)

Since the integral over the entire $S_1 S_2$ plane is equal to one, then eq. (B.1) can be written as:

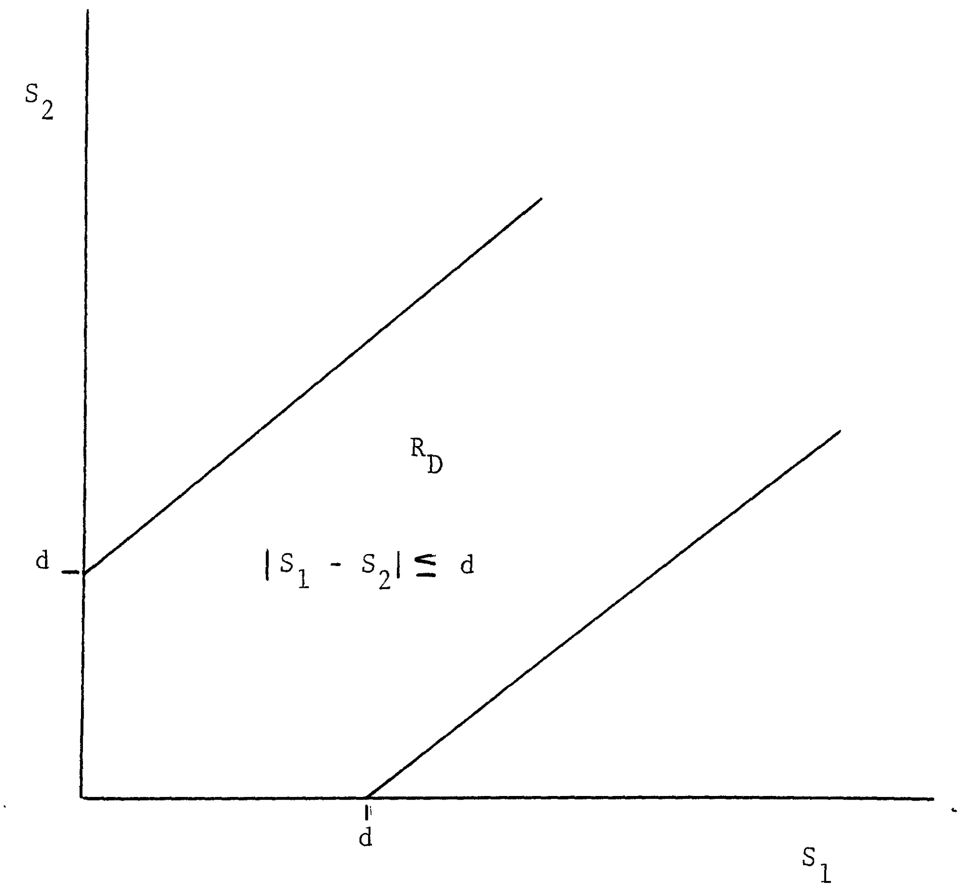

Fig. B 1   The Region R_D in the S_1S_2 Plane

(B.2)
$$F_D(d) = \int_o^{\infty}\int_{d+s_2}^{\infty} f_{S_1}(s_1)f_{S_2}(s_2)ds_1ds_2 -$$
$$- \int_o^{\infty}\int_{d+s_1}^{\infty} f_{S_1}(s_1)f_{S_2}(s_2)ds_1ds_2$$

Recalling that $f_D(d) = d/ddF_D(d)$ (5)

(B.3)
$$f_D(d) = -d/dd \left\{ \int_o^{\infty}\int_{d+s_2}^{\infty} f_{S_1}(s_1)f_{S_2}(s_2)\, ds_1ds_2 + \right.$$
$$\left. + \int_o^{\infty}\int_{d+s_1}^{\infty} f_{S_1}(s_1)f_{S_2}(s_2)\, ds_1ds_2 \right\}$$
$$= \int_o^{\infty} f_{S_2}(s_2)f_{S_1}(d+s_2)ds_2 +$$
$$+ \int_o^{\infty} f_{S_1}(s_i)f_{S_2}(d+s_1)ds_1$$

And if $f_{S_1}(s_1) = f_{S_2}(s_2)$, then

$$f_D(d) = 2\int_o^{\infty} f_S(s)f_S(d+s)ds$$

The expected value of the differential settlement can be derived using the distribution of S_1 and S_2. That is,

(B.4) $$m_D = E\left[D\right] = E\left[|S_1 - S_2|\right]$$

$$= \int_0^{\infty}\int_0^{\infty} |s_1 - s_2| \, f_{S_1}(s_1) f_{S_2}(s_2) ds_1 ds_2$$

$$= \int_0^{\infty}\int_0^{s_2} (s_1 - s_2) f_{S_1}(s_1) f_{S_2}(s_2) ds_1 ds_2 -$$

$$- \int_0^{\infty}\int_{s_2}^{\infty} (s_1 - s_2) f_{S_1}(s_1) f_{S_2}(s_2) ds_1 ds_2$$

Clearly, an exact expression for the expected value of the differential settlement in terms of the mean and variance of the total settlement cannot be found except for certain types of distributions of total settlement.

APPENDIX C

CORRELATION MODELS

Consider a discrete two dimensional model (See Fig. C1) where all the X_{ij}'s are identically distributed. If one assumes that the covariance function is of the form[11]

$$\text{(C.1)} \qquad \mathrm{Cov}\,(X_{ij},X_{kl}) = s_X^2 \exp\left(- \;(k-\;)^2 + (\;-j)^2 \;/a^2\right)$$

Then the variance on an n x m area would be

$$\text{(C.2)} \qquad \mathrm{Var}(n,m) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} s_x^2 \exp\left(- \;(k-i)^2 + (l-j)^2 \;/a^2\right)$$

$$= s_x^2 \left[\sum_{i=1}^{n}\sum_{k=1}^{n} \exp\left(-(k-i)^2/2\right)\right]\left[\sum_{l=1}^{m}\sum_{j=1}^{m} \exp\left(-(l-j)^2/2\right)\right]$$

The continuous analog to eq.(C.2)is far more useful as it can be expressed in terms of Error functions. If one assumes that the variance of an elemental volume is s^{*2}, then the variance can be obtained by integrating over the area. In terms of Error functions, the variance of an l x h area is given by eq.(C.3).[11],(12)

$$\text{(C.3)} \qquad s^2(l,h) = s^{*2}\left(\sqrt{\pi}\,a l \;\mathrm{erf}(l/a) + a^2\left[\exp(-l^2/a^2) - 1\;\right]\right) \mathrm{x}$$

$$\mathrm{x} \quad \left(\sqrt{\pi}\,a h \;\mathrm{erf}(h/a) + a^2\left[\exp(-h^2/a^2) - 1\right]\right)$$

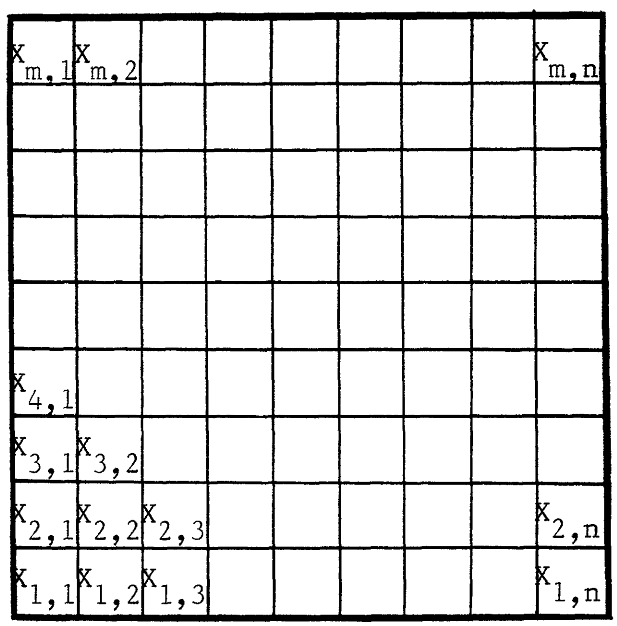

Fig. C 1 Discrete Two Dimensional Correlation Model

The variance s^{*2} can be evaluated by using the measured value of the variance and the size of the sample. That is,

$$s^{*2} = s^2(1,h) / \; g(1/a) g(h/a) \tag{C.4}$$

where $$g(x,a) = \sqrt{\pi}\, ax \, \text{erf}(x/a) + a^2\left[\exp(-x^2/a^2) - 1\right]$$

A solution for the variance of a circular area of radius R is given in Appendix E. Reference (12) discusses correlation models in general and gives solutions for several types of correlation functions which depend on distance.

APPENDIX D

AVERAGE STRESS DUE TO A UNIFORM RECTANGULAR LOAD

Experience has shown that in most cases good estimates of the vertical stresses beneath a uniformly loaded footing can be obtained by assuming that the soil is elastic, homogeneous and isotropic.[7] Based on these assumptions the vertical stress at any point under a 2a by 2b rectangular area loaded with a uniform pressure q is given by eq. (D.1).[6]

$$\sigma_z(x,y,z) = 3qz^3/(2\pi) \int_{-a}^{a} \int_{-b}^{b} \frac{du\,dv}{((x-u)^2 + (y-z)^2 + z^2)^{3/2}} \qquad \text{(D.1)}$$

The above integral can easily be evaluated for $\sigma_z(0,0,z)$ to give the vertical stress under the center of the rectangle:

$$\sigma_z(0,0,z) = 2q/\pi \left[\frac{abz(a^2 + b^2 + 2z^2)}{(a^2 + z^2)(b^2 + z^2)(a^2 + b^2 + z^2)^{1/2}} + \right.$$

$$\left. + \sin^{-1} \frac{ab}{(a^2 + b^2)^{1/2}(b^2 + z^2)^{1/2}} \right]$$

By the principle of superposition, the vertical stress under the corner of an a x b rectangular area would be one quarter of the stress under the center of a 2a x 2b area.[6]

$$\sigma_Z = 1/4(\sigma_Z(0,0,z)$$

$$= q/\pi \left[\frac{mn}{(1 + m^2 + n^2)^{1/2}} \cdot \frac{1 + m^2 + 2n^2}{(1 + n^2)(m^2 + n^2)} + \right.$$

$$\left. + \sin^{-1} \frac{m}{(m^2 + n^2)^{1/2}(1 + n^2)^{1/2}} \right]$$

where $m = a/b$
$n = z/b$

The above equation can also be written as

$$\sigma_Z = qK$$

where K = dimensionless influence coefficient

Thus, to obtain the vertical stress at any point p lying directly beneath the loaded area, one can add up the stresses at the four corners of the smaller rectangles which coincide with the point p. (See Fig. D1). Or, in terms of the influence factors of the smaller areas

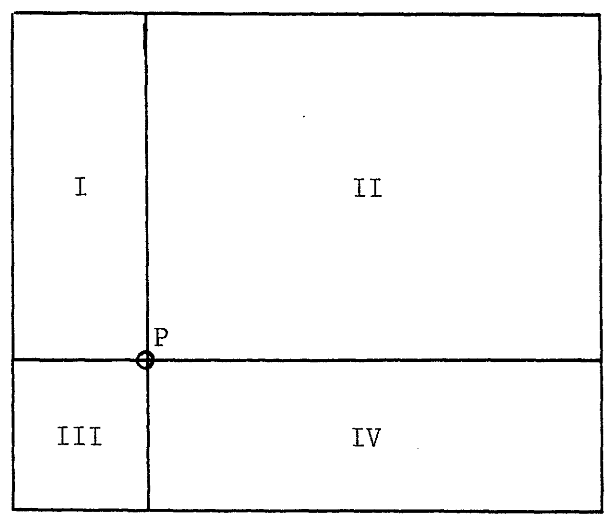

Fig. D 1 Superposition of Corner Stresses

$$\sigma_Z = q(K_I + K_{II} + K_{III} + K_{IV})$$

The average stress P of the area directly beneath the loaded area for a given depth Z can be estimated by averaging the vertical stresses of the individual points at the depth Z. Figure D2 gives a plot of P as a function of the dimensionless quantities m and n.

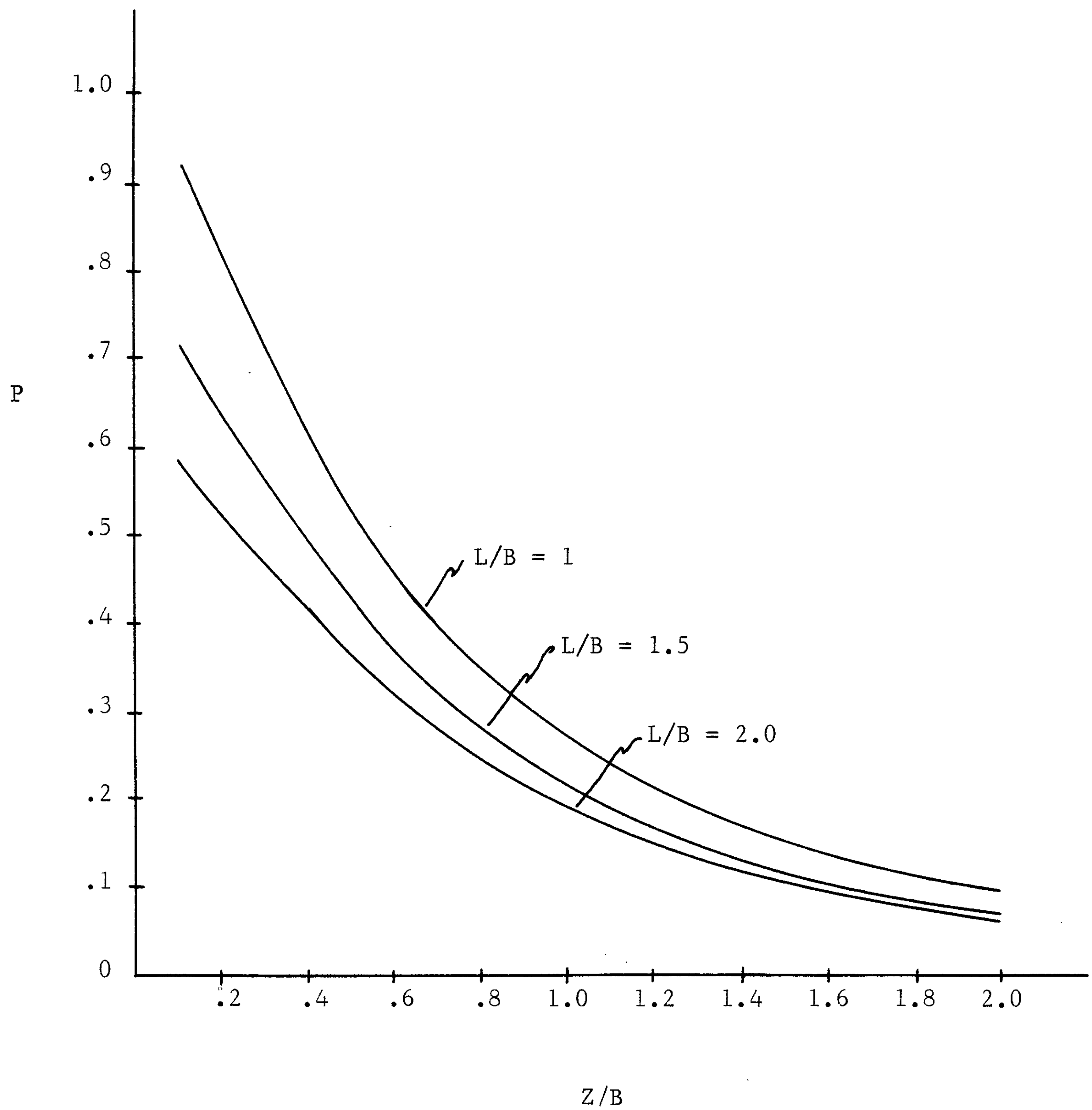

Fig. D 2 Average Stress Coefficient for Uniformly Loaded Rectangular Footings

APPENDIX E

DERIVATION OF THE VARIANCE OF A CIRCULAR AREA

Consider the case where the covariance between two points, p_1 and p_2, which are separated by a distance d is defined as

$$\mathrm{Cov}(p_1, p_2) = s^2 e^{-d^2/a^2}$$

In polar coordinates p_1 and p_2 would be of the form

$$p_1 = f(\phi_1, \bar{r}_1)$$
$$p_2 = f(\phi_2, \bar{r}_2)$$

and the square of the distance between the two points would be

$$d^2 = \bar{r}_1^{\,2} + \bar{r}_2^{\,2} - 2\bar{r}_1\bar{r}_2 \cos(\phi_1 - \phi_2)$$

Hence, the covariance function could be written

$$\mathrm{Cov}(p_1, p_2) = s^2 \exp\left[- \frac{\bar{r}_1^{\,2} + \bar{r}_2^{\,2} - 2\bar{r}_1\bar{r}_2\cos(\phi_1 - \phi_2)}{a^2} \right]$$

Suppose that p_1 and p_2 are representative of the elemental areas $r_1 d\phi_1 dr_1$ and $r_2 d\phi_2 dr_2$, respectively. Then integrating over the circular area R, one obtains the variance, s_R^2, of the area

$$s_R^2 = \int_0^R \int_0^{2\pi} \int_0^R \int_0^{2\pi} s^2 \bar{r}_1 \bar{r}_2 \exp\left[- \frac{\bar{r}_1^{\,2} + \bar{r}_2^{\,2} - 2\bar{r}_1\bar{r}_2\cos(\phi_1 - \phi_2)}{a^2} \quad d\phi_1 d\bar{r}_1 d\phi_2 d\bar{r}_2 \right]$$

Making the change of variables

$$r_1 = \bar{r}_1/a \text{ and } r_2 = \bar{r}_2/a$$

and changing the order of integration, one obtains:

$$s_R^2 = \int_0^{R/a} \int_0^{R/a} \int_0^{2\pi} \int_0^{2\pi} s^2 a^4 r_1 r_2 \exp\left[- \left[r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\phi_1 - \phi_2)\right]\right] d\phi_1 d\phi_2 dr_1 dr_2$$

The above equation can be rewritten as

$$s_R^2 = s^2 a^4 \int_0^{R/a} \int_0^{R/a} \int_0^{2\pi} r_1 r_2 \exp\left[- \quad r_1^{\,2} + r_2^{\,2} \right] \int_0^{2\pi} \exp\left[2r_1r_2\cos(\phi_1 - \phi_2)\right] d\phi_1 d\phi_2 \, x$$

$$x dr_1 dr_2$$

Consider now

$$J = \int_0^{2\pi} \exp\left[B\cos(\theta_1 - \theta) \right] d\theta_1$$

Introducing the change of variable $z = e^{i\theta_1}$ it follows that

$$J = \oint_{c_1} \frac{\exp\left[B\left(\frac{z^2+1}{2z}\right)\right]}{iz}\, dz$$

where c_1 is the positive circuit around the unit circle in the complex plane. Applying Cauchy's residue theorem, one obtains

$$J = \oint_{c_1} \frac{\exp\left[B\left(\frac{z^2+1}{2z}\right)\right]}{iz}\, dz = 2\pi i \mathrm{Res}(0)$$

To find the residue at zero, it is convenient to rewrite J as

$$J = \oint_{c_1} \frac{\exp\left[\frac{Bz}{2}\right]}{iz} \cdot \exp\left[\frac{B}{2z}\right] dz$$

Expanding the two expressions about zero, one obtains

$$\frac{\exp \frac{Bz}{2}}{iz} = \frac{1}{i}\left[\frac{1}{z} + \frac{(B/2)}{1} + \frac{(B/2)^2 z}{2!} + \frac{(B/2)^3 z^2}{3!} + \ldots + \frac{(B/2)^n z^{n-1}}{n!}\right]$$

and

$$\exp\left[\frac{B}{2z}\right] = 1 + \frac{(B/2)}{z} + \frac{(B/2)^2}{z^2 z!} + \frac{(B/2)^3}{z^3 3!} + \ldots +$$

$$+ \frac{(B/2)^n}{z^n n!}$$

Multiplying the above expansions and collecting terms of the order 1/z

$$\frac{1}{iz}\left[1 + (B/2)^2 + \frac{(B/2)^4}{2!2!} + \frac{(B/2)^6}{3!3!} + \ldots + \frac{(B/2)^{2n}}{n!n!}\right] + 0(z^o)$$

$$= \frac{1}{iz}\sum_{i=0}^{\infty}\frac{(B/2)^{2n}}{n!n!} + 0(z^o)$$

The residue at zero is

$$\text{Res}(0) = \frac{1}{i}\sum_{i=0}^{\infty}\frac{(B/2)^{2n}}{n!n!}$$

J can now be evaluated.

$$J = 2\pi i\text{Res}(0) = 2\pi\sum_{i=0}^{\infty}\frac{(B/2)^{2n}}{n!n!}$$

Recalling the identity

$$I_p(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+p}}{k!(k+p)!}$$

where I_p is the modified Bessel function of the first kind of order p, J can be expressed as:

$$J = 2\pi I_o(B)$$

Letting $B = 2r_1r_2$, s_R^2 can be expressed as

$$s_R^2 = 2\pi s^2 a^4 \int_0^{R/a}\int_0^{R/a}\int_0^{2\pi} r_1 r_2 \exp\left[-r_1^2+r_2^2\right] I_0(2r_1r_2)d\phi_2 dr_1 dr_2$$

and integrating over ϕ_2,

$$s_R^2 = (2\pi sa^2)^2 \int_0^{R/a}\int_0^{R/a} r_1 r_2 \exp\left[-r_1^2-r_2^2\right] I_0(2r_1r_2)dr_1 dr_2$$

Again expanding I_0, one obtains

$$s_R^2 = (2\pi s a^2)^2 \int_0^{R/a}\int_0^{R/a} r_1 r_2 \exp\,(-r_1^2 - r_2^2)\left[1 + \frac{(r_1r_2)^2}{1!1!} + \right.$$

$$\left. + \frac{(r_1r_2)^4}{2!2!} + \ . \ . \ . + \frac{(r_1r_2)^{2n}}{n!n!}\right] dr_1 dr_2$$

Assuming that the integral of the sum is equal to the sum of the integrals and noticing that the integrals can be separated, s_R^2 can be rewritten as

$$s_R^2 = (2\pi s a^2)^2 \left[J_0^{\,2} + J_1^{\,2} + J_2^{\,2} + \ . \ . \ . + J_n^{\,2} \right]$$

where

$$J_0 = \int_0^{R/a} r\exp(-r^2)dr$$

$$J_1 = \int_0^{R/a} r^3\exp(-r^2)dr$$

$$J_2 = \int_0^{R/a} \frac{r^5}{2!}\exp(-r^2)dr$$

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$$J_n = \int_0^{R/a} \frac{r^{2n+1}}{n!}\exp(-r^2)dr$$

Examining the integral J_0, it is found that

$$J_0 = \int_0^R r \exp(-r^2)\, dr = -\frac{1}{2} \exp(-r^2) \Bigg|_{r=0}^{R/a}$$

$$= -\frac{1}{2}\left[\exp(-R^2/a^2) - 1\right] = \frac{1}{2} \quad 1-\exp(-R^2/a^2)$$

Consider now the integral J_1. Making the change of variable $t = r^2$, J_1 becomes

$$J_1 = \frac{1}{2} \int_0^{R^2/a^2} te^{-t} dt$$

Recalling the known integral

$$\int xe^{ax} = \frac{e^{ax}}{a^2} (ax-1)$$

the integral J_1 can be evaluated as

$$J_1 = \frac{1}{2} \int_0^{R^2/a^2} te^{-t}dt = \frac{1}{2}\left[e^{-t}(-t-1)\right]\Bigg|_0^{R^2/a^2}$$

$$= \frac{1}{2}\left[e^{-R^2/a^2}(-R^2/a^2-1) + 1\right] = \frac{1}{2}\left[1-e^{-R^2/a^2}(1 + R^2/a^2)\right]$$

Again recalling a known integral

$$\int x^m e^{ax} dx = e^{ax} \sum_{r=0}^{m} \frac{(-1)^r m! x^{m-r}}{(m-r)! a^{r+1}}$$

the general term J_n can be evaluated if the change of variable $t = r^2$ is used.

$$J_n = \int_0^{R/a} \frac{r^{2n+1}}{n!} e^{-r^2} dr = \int_0^{R^2/a^2} \frac{t^n}{2(n!)} e^{-t}dt$$

$$= \frac{1}{2(n!)} e^{-t} \sum_{k=0}^{n} \frac{(-1)^k n! t^{n-k}}{(n-k)!(-1)^{k+1}} \Bigg|_0^{R^2/a^2}$$

$$= -\frac{1}{2} e^{-t} \sum_{k=0}^{n} \frac{t^{n-k}}{(n-k)!} \Bigg|_{0}^{R^2/a^2} = -\frac{1}{2} e^{-t} \sum_{k=0}^{n} \frac{t^{k}}{k!} \Bigg|_{0}^{R^2/a^2}$$

$$= -\frac{1}{2} \left[e^{-R^2/a^2} \sum_{k=0}^{n} \frac{(R/a)^{2k}}{k!} - \left[1 + \cancelto{0}{\sum_{k=1}^{n} \frac{0^k}{k!}} \right] \right]$$

$$= \frac{1}{2} \left[1 - e^{-R^2/a^2} \sum_{k=0}^{n} \frac{(R/a)^{2k}}{k!} \right]$$

And thus s_R^2 finally becomes

$$s_R^2 = (\pi s a^2)^2 \sum_{n=0}^{\infty} \left[1 - e^{-R^2/a^2} \sum_{k=0}^{n} \frac{(R/a)^{2k}}{k!} \right]^2$$