

Essays in Capital Markets

by

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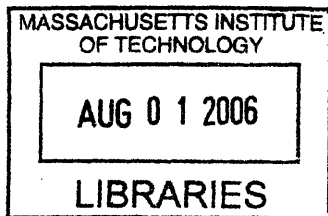
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Abstract

This thesis consists of three essays in capital markets.

The first essay presents a dynamic asset pricing model with heterogeneously informed agents. Unlike previous research, the general case where differential information leads to the problem of “forecasting the forecasts of others” and to non-trivial dynamics of higher order expectations is studied. In particular, it is proved that the model does not admit a finite number of state variables. A comparison of equilibria characterized by identical fundamentals but different information structure shows that the distribution of information has substantial impact on equilibrium prices and returns.

In the second essay we explore several sources of serial correlation in returns of hedge funds and other alternative investments. We show that the most likely explanation is illiquidity exposure, i.e., investments in securities that are not actively traded and for which market prices are not always readily available. For portfolios of illiquid securities, reported returns will tend to be smoother than true economic returns, which will understate volatility and increase risk-adjusted performance measures such as the Sharpe ratio. We propose an econometric model of illiquidity exposure and develop estimators for the smoothing profile as well as a smoothing-adjusted Sharpe ratio. For a sample of 908 hedge funds drawn from the TASS database, we show that our estimated smoothing coefficients vary considerably across hedge-fund style categories and may be a useful proxy for quantifying illiquidity exposure.

In the third essay our objective is to study analytically the effect of borrowing constraints on asset returns. We explicitly characterize the equilibrium for an exchange economy with two agents who differ in their risk aversion and are prohibited from borrowing. In a representative-agent economy with CRRA preferences, the Sharpe ratio of equity returns and the risk-free rate are linked by the risk aversion parameter. We show that allowing for preference heterogeneity and imposing borrowing constraints breaks this link. We find that an economy with borrowing constraints exhibits simultaneously a relatively high Sharpe ratio of stock returns and a relatively

low risk-free interest rate, compared to both representative-agent and unconstrained heterogeneous-agent economies.

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Chapter 1

Forecasting the Forecasts of Others: Implications for Asset Pricing

1.1 Introduction

One of the major difficulties arising in the analysis of dynamic asset pricing models with asymmetric information is the problem of “forecasting the forecasts of others.” When investors possess different information about an asset’s payoff, prices generally reflect not only investors’ expectations of the asset’s fundamental value, but also their expectations of other investors’ expectations of it. Iterating this logic forward, prices must depend on the whole hierarchy of investors’ beliefs. This problem has interested economists for decades, as evidenced by Keynes’ (1936) much-cited comparison of financial markets with beauty contests: “We devote our intelligence to anticipating what average opinion expects the average opinion to be.”

In most cases the successive forecasts of the forecasts of others differ from one another. To account for all of them one needs an infinite number of variables, making the model not only analytically involved, but also numerically challenging. Almost all existing models get around this problem by employing various assumptions that,

by restricting the possible dynamics of expectations, guarantee that all higher order beliefs can be described by a few carefully chosen state variables¹. It is not clear, however, that insights obtained from these models survive in a more general informational environment.

In this paper we do not impose standard simplifying restrictions and show that, as long as two fairly general conditions hold, the infinite regress problem cannot be avoided and an infinite number of state variables is required to describe the dynamics of prices². The two conditions are that each agent lack a component of fundamental information which is known to some other agents and that fundamentals evolve stochastically over time.

The first condition guarantees that information held by other agents is relevant to each agent's payoff and, as a result, his beliefs about other agents' beliefs affect his demand for the risky asset. We call this information setup differential and contrast it with the hierarchical setup, in which one agent is better informed than the other. In the latter case, the informed agent knows the forecasting error of the uninformed one and therefore does not need to forecast it, so higher order expectations collapse. The second condition forces agents to form new sets of higher order beliefs every period. Since no agent ever becomes fully informed, they all need to incorporate the entire history of prices into their predictions³.

There is a vast literature related to dynamic asset pricing with asymmetric information. In most papers, however, the role of higher order expectations is limited. Grundy and McNichols (1989), and Brown and Jennings (1989) study two-period models, which are very restrictive for analysis of dynamic effects of differential information. Singleton (1987), and Grundy and Kim (2002) consider models in which all private information becomes public after one or two periods. As a result, investors' learning problem in these papers becomes a static one, weakening the effects

¹See Brunnermeier (2001) for a review of asset pricing under asymmetric information.

²Surprisingly, this proof is not trivial and is not excessive. Despite the apparent simplicity of intuition leading to this conclusion, in each particular case it is not easy to show that the dynamics of the model do not admit description in terms of a finite number of state variables.

³This observation suggests that price histories may play an important role in financial markets in which asymmetric information is ubiquitous, thus lending support to technical analysis, which is often employed in practice.

of asymmetric information on expectations and prices.

Enabling private information to be long-lived allows for non-trivial interplay between expectations and fundamentals which sometimes reverts the conclusions of simplified models. For example, in contrast to Grundy and Kim (2002), we demonstrate that volatility of returns under differential information may be lower than in an otherwise identical economy with no information asymmetry.

Wang (1993, 1994) and He and Wang (1995) analyze multi-period economies with asymmetric information. To avoid the aforementioned infinite regress problem Wang (1993, 1994) considers a model with hierarchical information. In the model of He and Wang (1995) agents have differential information about the unknown value of payoff, which, however, does not evolve over time. Our setup naturally extends these models by allowing both a general information structure and stochastic evolution of fundamentals.

The above papers assume competitive markets. Admati and Pfleiderer (1988), Foster and Viswanathan (1996), Back, Cao, and Willard (2000), and Bernhardt, Seiler, and Taub (2004), among others, consider dynamic noncompetitive models under asymmetric information. The theme of our paper is also aligned with another strand of literature which explicitly analyzes higher order expectations. Allen, Morris, and Shin (2004) argue that under asymmetric information agents tend to underreact to private information, making price biased towards the public signal. Bacchetta and Wincoop (2004) show that under asymmetric information price deviations from its fundamental value can be large.

The rest of the paper is organized as follows. Section 2.4 describes the model. In Section 1.3 we solve for the equilibria in benchmark cases of full and hierarchical information dispersion setups. In Section 1.4 we consider differential information. Section 1.5 presents details of the numerical algorithm used to solve the model. In Sections 1.6 we analyze the impact of information dispersion on prices and returns. Section 1.8 concludes. Technical details are presented in Appendices A, B, and C.

1.2 The Model

In this section, we present a noisy rational expectation model. We assume that investors are fully rational and know the model.

1.2.1 Financial Assets

There are two assets. The first asset is a riskless asset in perfectly elastic supply that generates a rate of return $1 + r$. The second asset is a claim on a hypothetical firm which pays no dividends⁴ but has a chance of being liquidated every period. We assume that the probability of liquidation in period $t + 1$, given that the firm has survived until period t , is equal to λ . Upon liquidation the firm pays its equity holders a stochastic liquidation value V_t . This liquidation value can be decomposed into two components: $V_t = V_t^1 + V_t^2$, and each component evolves according to a first-order autoregressive process:

$$V_{t+1}^j = aV_t^j + b_V \epsilon_{t+1}^j, \quad j = 1, 2.$$

We assume that $\epsilon_t^j \sim \mathcal{N}(0, 1)$ are *i.i.d.* across time and components. For simplicity we take identical parameters a and b_V for the processes V_t^1 and V_t^2 . The total amount of risky equity⁵ available to rational agents is $1 + \theta_t$, where $\theta_t \equiv b_\Theta \epsilon_t^\Theta$ and $\epsilon_t^\Theta \sim \mathcal{N}(0, 1)$.

1.2.2 Preferences

There is an infinite set of competitive rational investors indexed by i and uniformly distributed on a unit interval $[0, 1]$. Each of them is endowed with some piece of information about the fundamentals V_t^1 and V_t^2 . We assume that investors are mean-variance optimizers and each investor i submits the demand X^i which is proportional

⁴We model the firm as not paying dividends for simplicity, since the current dividend would be an additional signal about future cash flows.

⁵This can be interpreted as supply of stock by noise traders. Following Grossman and Stiglitz (1980), we introduce stochastic amount of equity to prevent prices from being fully revealing.

to his expectation of excess stock return Q_{t+1} :

$$X_t^i = \frac{1}{\alpha} \frac{E[Q_{t+1}|\mathcal{F}_t^i]}{\text{Var}[Q_{t+1}|\mathcal{F}_t^i]}, \quad Q_{t+1} = \lambda V_{t+1} + (1 - \lambda)P_{t+1} - (1 + r)P_t. \quad (1.1)$$

Here \mathcal{F}_t^i is the information set of investor i at time t . All investors are assumed to have the same coefficient of risk aversion α .

1.2.3 Properties of the model

Before we turn to analysis of equilibrium, it is worthwhile to make several comments about the model. First of all, we make the model very stylized, since we want to demonstrate and analyze the “forecasting the forecasts of others” problem in the simplest setting. In particular, we assume that all shocks are normally distributed and this property is inherited by other random variables in the model, leading to the linear form of conditional expectations and, therefore, to a linear equilibrium. Next, we consider a model with an infinite horizon and focus on stationary equilibria which enables us to use powerful methods from the theory of stationary Gaussian processes. Finally, a major simplification is achieved by assigning agents’ mean-variance preferences. This assumption is similar to the assumption of logarithmic utility with lognormally distributed shocks in that hedging demand is zero. Since calculation of hedging demand in the economy with infinite number of state variables is complicated by itself⁶, sidestepping this problem allows us to preserve tractability of the model but still relate equilibrium price to agents’ higher order beliefs and characterize their dynamics.

1.2.4 The rational expectation equilibrium

We focus on a rational expectation equilibrium of this model which is defined by two conditions:

- 1) all agents rationally form their demands according to (1.1);
- 2) market clearing condition holds: $\int X_t^i di = 1 + \theta_t$.

⁶See Schroder and Skiadas (1999) for some results in this case.

In the most general case, information sets of investors \mathcal{F}_t^i are different, investors have to forecast the forecasts of others, and non-trivial higher order expectations appear. As a basis for our subsequent analysis, it is useful to represent the price in terms of fundamentals and expectations of agents. It is convenient to first define the *weighted average expectation operator* $\bar{E}_t^w[x]$ of agents as follows:

$$\bar{E}_t^w[x] = \int \frac{\omega_i}{\Omega} E[x|\mathcal{F}_t^i] di, \quad \Omega = \int \omega_i di, \quad \omega_i = \frac{1}{\alpha} \frac{1}{\text{Var}[Q_{t+1}|\mathcal{F}_t^i]}$$

Note that the weights ω_i are endogenous and determined by the conditional variances of excess returns given investors' information sets. The expectations of agents with better information get larger weights than those who are less informed. Using the market clearing condition we can derive a relation between the current price and the next period price:

$$P_t = -\frac{1 + \theta_t}{\Omega(1 + r)} + \frac{1}{1 + r} \bar{E}_t^w[\lambda V_{t+1} + (1 - \lambda)P_{t+1}].$$

Iterating this relation forward and imposing the no-bubble condition, we get

$$P_t = -\frac{1}{\Omega(r + \lambda)} - \frac{1}{\Omega(1 + r)}\theta_t + \frac{a\lambda}{1 + r} \sum_{s=0}^{\infty} \left(\frac{1 - \lambda}{1 + r}\right)^s \bar{E}_t^w \bar{E}_{t+1}^w \dots \bar{E}_{t+s}^w V_{t+s}. \quad (1.2)$$

This equation represents the price as a series over iterated weighted average expectations of future values of V_t : we have arrived at a mathematical formulation of forecasting the forecasts of others. It highlights two essential difficulties. The first is that the law of iterated expectations need not hold because agents may have different information; this point was recently emphasized by Allen, Morris and Shin (2004). The second and even more significant obstacle is that the current price also depends on agents' future expectations which, in turn, depend on future prices. Consequently, in order to compute their expectations, we have to solve for the entire sequence of prices as a fixed point. Since this problem is quite complicated, before attempting to find a solution for the general case, let us first consider some special cases in which the solution is not as involved.

1.3 Benchmarks

1.3.1 Full information

As a starting point, we consider the full information setup. Full information means that all investors $i \in [0, 1]$ observe both components V_t^1 and V_t^2 and their information sets are

$$\mathcal{F}_t^i = \{P_\tau, V_\tau^1, V_\tau^2 : \tau \leq t\}.$$

In this case we are back to the representative agent framework, and the law of iterated expectations holds: $\bar{E}_t^w \bar{E}_{t+1}^w \dots \bar{E}_{t+s}^w V_{t+s} = E_t V_{t+s} = a^s V_t$. Now observing the price is sufficient to infer the demand of noise traders θ_t . We have the following proposition:

Proposition 1 *Suppose that*

- 1) *all investors observe V_t ;*
- 2) $2\sqrt{2}b_V b_\Theta \frac{\lambda(1-\lambda)}{1+r-a(1-\lambda)} \leq \frac{1}{\alpha}$.

Then there exists a full information equilibrium in which the equilibrium price of the risky asset is given by

$$P_t = -\frac{1}{\Omega(r+\lambda)} + \frac{a\lambda}{1+r-a(1-\lambda)} V_t - \frac{1}{\Omega(1+r)} \theta_t. \quad (1.3)$$

$$\Omega = \frac{(1+r-a(1-\lambda))^2}{4b_V^2 \lambda^2 (1+r)^2} \left(\frac{1}{\alpha} + \sqrt{\frac{1}{\alpha^2} - \frac{8b_V^2 b_\Theta^2 \lambda^2 (1-\lambda)^2}{(1+r-a(1-\lambda))^2}} \right) \quad (1.4)$$

Proof. See Appendix A.

The obtained price function has a structure which is common to linear rational expectations models⁷. The first term corresponds to a risk premium for uncertain payoffs. The second term is the value of expected future payoffs discounted at the risk-free rate adjusted for the probability of liquidation. The third term compensates the investors for noise trading related risk.

⁷See Campbell and Kyle (1993), Wang (1993)

Formally, the equations determining equilibrium price admit two solutions. One of them is given in Proposition 1, and we take this solution as the full information benchmark in the future. The reason for discriminating between equilibria is that the other solution is unstable, meaning that minor errors in agents' behavior significantly impact prices and destabilize the economy. Having this in mind, we consider only the full information equilibrium which is most sensible from the economic point of view.

1.3.2 Hierarchical information

Now consider the equilibrium with hierarchical information⁸, which means that investors can be ranked according to the amount of their information: some investors are better informed than others. Formally, the information sets of investors at time t are hierarchically embedded in each other and generate a filtration: $\mathcal{F}_t^i \subseteq \mathcal{F}_t^{i'} \subseteq \dots$. We focus on the simplest case, and assume that there are only two types of investors which we denote as 1 and 2. Investors of type 1, which are indexed by $i \in [0, \gamma]$, are informed and observe both V_t^1 and V_t^2 . Investors of type 2, with $i \in (\gamma, 1]$, are partially informed and observe V_t^2 only. We can write their information sets of informed and uninformed investors as

$$\mathcal{F}_t^1 = \{P_\tau, V_\tau^1, V_\tau^2 : \tau \leq t\}, \quad \mathcal{F}_t^2 = \{P_\tau, V_\tau^2 : \tau \leq t\}.$$

There are several reasons why this informational structure is interesting. First of all, it is an intermediate setup between the full information and the differential information equilibria. Despite the investors having heterogeneous information, the infinite regress problem does not arise and we can find a closed-form solution. The intuition behind this result is simple and can be easily conveyed in terms of expectations. When trying to extract the unknown piece of information from the price, investors of type 2 form their expectations $\hat{V}_t^1 = E[V_t^1 | \mathcal{F}_t^2]$ about the current value of V_t^1 . Since all agents of type 2 make an identical estimation error, \hat{V}_t^1 is a new state

⁸The idea to analyze hierarchical information setup in order to avoid the infinite regress problem was suggested by Townsend (1983) and elaborated in the asset pricing context by Wang (1993, 1994).

variable influencing the price of the asset. In their turn, the investors of type 1 need to form their own expectations about expectations of type 2 investors. $E[\hat{V}_t^1|\mathcal{F}_t^1]$, and in the general case of differential information, it would be represented by another state variable. However, since $\mathcal{F}_t^2 \subseteq \mathcal{F}_t^1$ we get $E[\hat{V}_t^1|\mathcal{F}_t^1] = E[E[V_t^1|\mathcal{F}_t^2]|\mathcal{F}_t^1] = \hat{V}_t^1$ and the infinite regress problem does not arise. Basically, since the type 1 agents have all the information, they can, without mistake, deduce the mistake of type 2 agents, thus their prediction of the price is accurate. So the hierarchical information case illustrates how iterated expectations collapse and the state space of the model remains finite dimensional. The hierarchical information equilibrium in our model is characterized by Proposition 2.

Proposition 2 *If investors of type 1, with $i \in [0, \gamma]$, observe V_t^1 and V_t^2 and investors of type 2, with $i \in (\gamma, 1]$, observe only V_t^2 the equilibrium price of the risky asset is given by*

$$P_t = -\frac{1}{\Omega(r + \lambda)} + p_V V_t - \frac{1}{\Omega(1 + r)} \theta_t + p_\Delta (\hat{V}_t^1 - V_t^1), \quad (1.5)$$

where p_V , p_Δ and Ω are constants which solve a system of nonlinear equations given in Appendix B.

Proof. See Appendix B.

1.4 Differential information equilibrium

Now consider the informational structure in which all agents are endowed only with a piece of relevant information and the rest of the information is never revealed. Again, assume that there are two types of agents, $j = 1, 2$ with $i \in [0, \gamma]$ and $i \in (\gamma, 1]$ respectively, such that their information sets are given by

$$\mathcal{F}_t^1 = \{P_\tau, V_\tau^1 : \tau \leq t\}, \quad \mathcal{F}_t^2 = \{P_\tau, V_\tau^2 : \tau \leq t\}. \quad (1.6)$$

1.4.1 Forecasting the forecasts of others

It means that the agents of type j can observe only V^j and the history of prices. Let us show how the problem of “forecasting the forecasts of others” arises in this case. First of all, due to the presence of noise traders, the price is not fully revealing, i.e. knowing the price and their own component of information V^j , the agents cannot infer the other component V^{-j} . However, the information about V^{-j} is relevant to agent j , since it helps him predict his own future payoff and, consequently, to form his demand for the asset. Moreover, due to the market clearing condition, the information of each investor is partially incorporated in the price, each agent has an incentive to extract the missing information of the other type from the price. Therefore, an agent will form his own expectations about the unknown piece of information. For example, agent 1 forms his expectations about agent 2’s information. These expectations of agent 1 affect his demand and, subsequently, the price. So the inference problem of agent 2 is not only to extract the information of agent 1, but also the expectations of agent 1 about the information of agent 2. Agent 1, in turn, faces a similar problem; we can see how the infinite regress starts to appear.

The above reasoning might seem to be quite general, however, it does not always produce an infinite set of different higher order expectations. He and Wang (1995) provide an example how the higher order expectations can be reduced to first-order expectations even when investors have differential information. They consider a similar setup but assume that the firm is liquidated with probability one at some future time T and that the liquidation value does not evolve over time. In this situation, investors also try to predict the weighted average of investors’ expectations \widehat{V} of V . The paper demonstrates that \widehat{V} can be written as a weighted average of V conditional on public information (price) and the true value of V . Given this, investor i ’s expectation of \widehat{V} is a weighed average of his first-order expectations, conditional on price and on his private signals. Averaging them, one can show that second-order expectations of V can be again expressed as weighted average of V conditional on price and the true value of V . As will be shown later, this logic breaks down when V

evolves stochastically over time.

It is necessary to distinguish between the cases with finite vs. infinite dimensional state space because they are conceptually different and call for different solution techniques. In the former case, the major problem is to find appropriate state space variables. In the latter, the search for a finite set of state variables that can capture the dynamics is worthless by default, and the solution of such models presents a greater challenge.

1.4.2 Markovian dynamics

To provide the ground for rigorous treatment of the “forecasting the forecasts of others”, we introduce the concept of Markovian dynamics. Let $(\Omega, \mathcal{F}_t, \mu)$, $t \in \mathbb{Z}$ be a complete probability space equipped with a filtration \mathcal{F}_t . In what follows, all the processes are assumed to be defined on this space.

Definition. Let X_t be an adaptive random process. We say that X_t admits Markovian dynamics if there exists a collection of $n < \infty$ adaptive random processes $\bar{Y}_t = \{Y_t^i\}$, $i = 1..n$, such that the joint process (X_t, \bar{Y}_t) is Markov, that is

$$Prob(X_t \leq x, \bar{Y}_t \leq y | X_\tau, \bar{Y}_\tau : \tau \leq t-1) = Prob(X_t \leq x, \bar{Y}_t \leq y | X_{t-1}, \bar{Y}_{t-1}).$$

Obviously any Markov process admits Markovian dynamics. The next example will further help to clarify the ideas.

Example. Let ε_t , $t \in \mathbb{Z}$ be *i.i.d.* standard normal random variables. Define $X_t = \varepsilon_t - \theta\varepsilon_{t-1}$, an MA(1) process. X_t is not a Markov process, or even an n-Markov process: $Prob(X_t | X_\tau : \tau \leq t-1) \neq Prob(X_t | X_{t-1}, \dots, X_{t-n})$ for any n . However, X_t can be easily extended to a Markov process if one augments it with ε_t .

An important consequence of X_t admitting Markovian dynamics is that the filtered process \hat{X}_t then also admits Markovian dynamics, provided that signals obey the Markov property. As a result, all relevant information is summarized by a finite number of variables.

Applying the concept of Markovian dynamics to our model we get the following

result.

Proposition 3 *Let $\mathcal{F}_t = \sigma(V_s^1, V_s^2, \theta_s, s \leq t)$. Suppose agents' information sets are given by*

$$\mathcal{F}_t^j = \{P_\tau, V_\tau^j : \tau \leq t\}, \quad j = 1, 2.$$

Then in the linear equilibrium of the described economy the system $\{V^1, V^2, \theta, P\}$ does not admit Markovian dynamics.

Proof. See Appendix C.

Although we give a detailed proof in Appendix C, it is useful to make some comments on it here. The idea behind the proof is to use the following result from the theory of stationary Gaussian processes: if the process admits Markovian dynamics, then it is described by a rational function in the frequency domain. We start with the assumption that the price admits Markovian dynamics. The main part of the proof is to show that it is impossible to satisfy the market clearing condition and to simultaneously solve the optimal filtering problem of each agent working only with rational functions. This contradiction proves that the equilibrium price does not admit Markovian dynamics and the infinite regress problem is there.

To highlight the significance of this result from the theoretical standpoint, we refer to the paper by Townsend (1983), which inspired the study of the infinite regress problem and coined the term “forecasting the forecasts of others”. Townsend attempted to create a setup in which traders would have to estimate the beliefs of others in order to solve their own forecasting problems. However, Sargent (1991) and Kasa (2000) show how to reduce all higher order expectations in his model to just a small number of cleverly chosen low order expectations. Since then, a lot of effort has been made to state the necessary and sufficient conditions for the infinite regress problem to exist. We demonstrate that our setup is, in a sense, a minimal model where this phenomenon appears. We know from the result of He and Wang (1995) that if the value of the payoff remains constant over time, it is possible to reduce higher order expectations to first order expectations. In our model, we relax just this condition. It is still interesting to search for other cases, in which solution can take

a simple form. Our result, however, severely restricts the set of possible candidates. It suggests that the infinite regress problem is almost unavoidable if one is willing to consider a situation more general than ones previously studied.

The result also provides support for technical analysis. The simplicity of the fundamentals in our model leads to a straightforward solution in the case of complete information. However, asymmetric information results in highly non-trivial price dynamics. Now, to be as efficient as possible, agents have to use the entire price history in their predictions: as stated in Proposition 3, they cannot choose a finite number of state variables to summarize the price dynamics. This suggests that in financial markets, where fundamentals are not as simple and asymmetric information is commonplace, price history may be informative for investors.

1.5 Numerical procedure

Unfortunately, systems with an infinite number of state variables are very difficult to analyze and, in general, do not admit an analytical solution. So, in our solution we have to rely on reasonable numerical approximation. We use a variant of projection method described below. In Makarov and Rytchkov (2005) we verify that its outcome coincides with the k -lag revelation approximation, considered by Townsend (1983), in which all information is revealed to all investors after k periods.

Consider all random variables that admit the following decomposition:

$$x = b_V \sum_{k=0}^{\infty} f_k^1 \epsilon_{t-k}^1 + b_V \sum_{k=0}^{\infty} f_k^2 \epsilon_{t-k}^2 + b_{\Theta} \sum_{k=0}^{\infty} f_k^{\Theta} \epsilon_{t-k}^{\Theta},$$

where

$$\sum_{k=0}^{\infty} \left(b_V^2 (f_k^1)^2 + b_V^2 (f_k^2)^2 + b_{\Theta}^2 (f_k^{\Theta})^2 \right) < \infty$$

and denote the set of such random variables as H . From Appendix C we know that the demeaned price process \tilde{P} is in this set: $\tilde{P} \in H$. In fact, H is a Hilbert space with a scalar product defined as follows. If $x \in H$ and $y \in H$ can be decomposed over shocks with the coefficients $(f_k^1, f_k^2, f_k^{\Theta})_{k=0}^{\infty}$ and $(g_k^1, g_k^2, g_k^{\Theta})_{k=0}^{\infty}$ respectively, then

the scalar product is

$$(x, y) = E[xy] = \sum_{k=0}^{\infty} (b_V^2 f_k^1 g_k^1 + b_V^2 f_k^2 g_k^2 + b_{\Theta}^2 f_k^{\Theta} g_k^{\Theta}).$$

In what follows we describe each element $x \in H$ by its coefficients $(f_k^1, f_k^2, f_k^{\Theta})_{k=0}^{\infty}$. Note that in this representation θ, V^1 , and V^2 , are $(0; 0; b_{\Theta})$, $(b_V, b_V a, b_V a^2, \dots; 0; 0)$, and $(0; b_V, b_V a, b_V a^2, \dots; 0)$ respectively. It is convenient to introduce a shift operator L such as

$$L(f_k^1, f_k^2, f_k^{\Theta})_{k=0}^{\infty} = (\hat{f}_k^1, \hat{f}_k^2, \hat{f}_k^{\Theta})_{k=0}^{\infty},$$

where

$$\hat{f}_0^1 = 0, \hat{f}_0^2 = 0, \hat{f}_0^{\Theta} = 0, \text{ and } \hat{f}_{k+1}^1 = f_k^1, \hat{f}_{k+1}^2 = f_k^2, \hat{f}_{k+1}^{\Theta} = f_k^{\Theta} \text{ for } k = 0, 1, \dots$$

Using this operator we can represent the demeaned excess return \tilde{Q} as $\tilde{Q} = \lambda V + (1 - \lambda)\tilde{P} - (1 + r)L\tilde{P}$. Let $M \subseteq H$ be a linear subspace and define $\pi_{\{M\}}$ as a projection operator on M . Denote the projection operator of each agent on his information set as π^i : $\pi^i = \pi_{\{LV^i, L^2V^i, \dots, L\tilde{P}, L^2\tilde{P}, \dots\}}$. Then we have $\omega_i = \frac{1}{\alpha \|\tilde{Q} - \pi^i \tilde{Q}\|^2}$ and the equilibrium price \tilde{P} is such that the following equation is satisfied:

$$(\gamma \omega_1 \pi^1 + (1 - \gamma) \omega_2 \pi^2) \tilde{Q} = \theta. \quad (1.7)$$

While this equilibrium condition might appear simple it should be understood as an infinite dimensional system of non-linear equations in the Hilbert space H which determines the coefficients $(p_k^1, p_k^2, p_k^{\Theta})_{k=0}^{\infty}$ of the price process \tilde{P} . To tackle this problem we consider a sequence of finite dimensional approximations. Instead of an infinite dimensional system (1.7) we consider a finite dimensional one:

$$(\gamma \omega_1^N \pi_N^1 + (1 - \gamma) \omega_2^N \pi_N^2) \tilde{Q} = \theta. \quad (1.8)$$

Here $\pi_N^i = \pi_{\{\pi_N LV^i, \pi_N L^2 V^i, \dots, \pi_N L^N V^i, \pi_N L \tilde{P}, \pi_N L^2 \tilde{P}, \dots, \pi_N L^N \tilde{P}\}}$ is a finite dimensional analog of π^i and it projects the elements of H on a finite dimensional subspace generated

by vectors $\{\pi_N L V^i, \pi_N L^2 V^i, \dots, \pi_N L^N V^i, \pi_N L \tilde{P}, \pi_N L^2 \tilde{P}, \dots, \pi_N L^N \tilde{P}\}$. π_N is a projection operator on the space spanned by the first N basis vectors, which correspond to N most recent shocks in the initial space: $\pi_N (f_k^1, f_k^2, f_k^\Theta)_{k=0}^\infty = (f_k^1, f_k^2, f_k^\Theta)_{k=0}^N$. Correspondingly, $\omega_i^N = 1/(\alpha \|\tilde{Q} - \pi_N^i \tilde{Q}\|^2)$. Thus, instead of an infinite dimensional problem (1.7) we solve (1.8). We demonstrate that as N tends to infinity the approximations are more and more closer to each other. This fact indicates that as $N \rightarrow \infty$ the approximate solution converges to the real one.

1.6 Implications for asset pricing

In this section we analyze how the underlying information structure affects stock prices, returns, and their basic statistical properties. Most of the comparative static analysis is concerned with the effect of changing the information dispersion setup. Namely, we consider economies with full, hierarchical, and differential information.

1.6.1 Stock prices

Propositions 1, 2, and 3 describe the structure of the equilibrium price in economies with full, hierarchical, and differential information, respectively. Figure (1-1) shows the impulse response of the equilibrium price to the underlying shocks in these three economies.

Panel (a) shows the decomposition of the equilibrium price with respect to fundamental shocks ε_t^1 . We can notice that, as we move from full to hierarchical and then to differential information, it takes longer for fundamental shocks to be impounded into the price. The quantitative effect is much more pronounced in the case of differential information. The reason for this is that under hierarchical information, fully informed investors know perfectly well the states of the economy: mistakes of the uninformed and demand of the liquidity traders. Competition makes them arbitrage the mistakes of the uninformed quite aggressively, and by the second lag the price reflects the underlying value almost perfectly. When investors are differentially informed they all make errors in valuations. Moreover, the errors made by one type depend not only

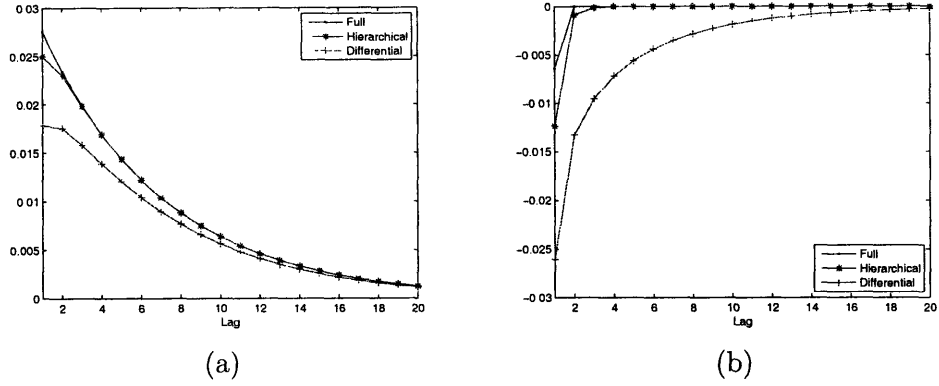


Figure 1-1: Impulse response of the price to underlying shocks

Panel (a) plots impulse response of the price to shocks ε_t^1 for economies with full, hierarchical, and differential information respectively. Panel (b) plots impulse response of the price to shocks ε_t^Θ for economies with full, hierarchical, and differential information respectively. The following parameter values are used: $\lambda = \frac{0.05}{12}$, $r = \frac{0.01}{12}$, $a = 0.85$, $\alpha = 3$, $\gamma = 1/2$, $b_V = 1.2$, $b_\Theta = 1$.

on fundamentals, but also the errors made by the other type of investors. Without fully informed arbitrageurs, it takes much longer to correct them: in the figure we see that it takes up to 20 lags for the price to reveal the true value.

Panel (b) shows the decomposition of the price with respect to supply shocks ε_t^Θ . Here we observe the opposite effect: as we move from full to hierarchical and then to differential information the equilibrium price becomes more and more sensitive to noise trading. Investors with perfect information trade against liquidity traders. On the other hand, investors who do not have full information confuse supply and fundamental shocks and therefore require higher compensation to absorb supply shocks. The price is much more affected by supply shocks under differential information, since in this case there is much more uncertainty about the true value of the firm.

A few comments are in order on the choice of parameter values in Figure 1-1 and the figures to follow. The parameters are chosen somewhat arbitrarily. Our purpose is to show the qualitative relationship between the behavior of stock prices and the underlying information structure. Since the model is too simplified we make no attempt to match parameters with historic data. We also vary the parameter values within a wide range and find that most of the figures are robust in their qualitative features.

1.6.2 Return volatility

Let us consider how the information dispersion setup affects the volatility of returns. In the Figure (1-2) we plot the ratio of volatilities of both Q and dP in economies with hierarchical and full information (Panels (a) and (b)) and differential and full information (Panels (c) and (d)). We see that volatility is lowest under differential information. This observation contradicts the conclusion of Grundy and Kim (2002), who assert that differential information causes returns to be more volatile than in the benchmark case with no information asymmetry. The cause for this discrepancy is that in Grundy and Kim's model private information is short lived, so investors can only trade on their information for one period, and therefore trade more aggressively. If, on the other hand, information is not revealed every period, as in our model, investors have plenty of time to trade on their information. As a result, it takes a long time for shocks to be impounded into prices, making returns less volatile.

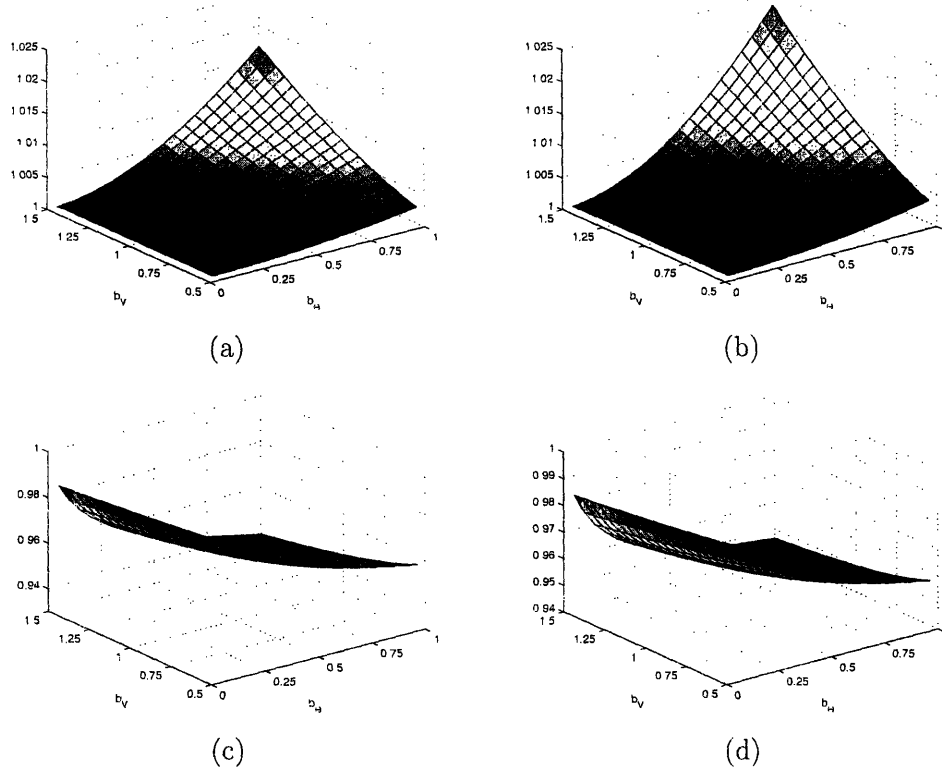


Figure 1-2: Effect of asymmetric information on volatility of returns

Panel (a) plots the ratio of volatility of dP in the economy with hierarchical information and full information. Panel (b) plots the ratio of volatility of dP in the economy with differential information and full information. Panel (c) plots the ratio of volatility of returns Q in the economy with hierarchical information and full information. Panel (d) plots the ratio of volatility of returns Q in the economy with differential information and full information. The following parameter values are used: $\lambda = \frac{0.05}{12}$, $r = \frac{0.01}{12}$, $a = 0.85$, $\alpha = 3$, $\gamma = 1/2$, $b_V \in [0.5, 1.5]$, $b_\Theta \in [0, 1]$.

On the other hand, we can see that in the hierarchical information case volatility is slightly larger than under full information. This result is consistent with findings in Wang (1993) who considers a similar model. In this case it also takes longer for shocks to fundamentals to be impounded into price compared to the full information setup. However, another effect is also at work: the uninformed investors face the risk of being taken advantage of by the informed investors. As a result, they are afraid of trading and taking large positions against liquidity traders, which causes the returns to be more volatile⁹. The overall result depends on the interaction of these two effects. In our simulations we could not find a region where the first effect is stronger than the second one. It is interesting to notice that under the differential information the opposite is true: the first effect dominates the second. These results provide another example in which introduction of fully informed arbitragers makes returns more volatile¹⁰.

1.6.3 Risk-premium

Because we assigned our agents a mean-variance demand over a one period horizon, the volatility of one period returns has a direct effect on their perception of risk, producing an inverse relation between expected returns and volatility. This is a result of our simplifying assumptions, and a more thorough modeling of agents' preferences, for example as in Wang (1993) would be required if one is interested in rigorous analysis of the effect of asymmetric information on risk premium.

1.7 Serial correlation in returns

In this section we consider the correlation of Q_{t+1} and the realized return is $\Delta P_t^e = P_t - P_{t-1}$. We use ΔP_t^e instead of Q_t because in the current model investors do not observe Q_t , but rather the history of prices. So, for example, in order for the model to generate momentum it is this return that we would need to see positively correlated

⁹For more results see Makarov and Rytchkov (2005).

¹⁰See also Stein (1987) and Wang (1993).

with Q_{t+1} . As we show next most models with asymmetric information put severe restrictions on the possible sign of this correlations.

Let \tilde{X}_t^i be the demeaned demand of the investor i . The market clearing condition then implies

$$\int \tilde{X}_t^i di = \theta_t. \quad (1.9)$$

Recall that $X_t^i = \omega_i E[Q_{t+1} | \mathcal{F}_t^i]$. If we multiply both sides of equation (1.9) by ΔP_t^e and take the unconditional expectation, then by the law of iterated expectations, we arrive at the following equation:

$$\text{Cov}(Q_{t+1}, \Delta P_t^e) = \frac{1}{\Omega} E(\theta_t \Delta P_t^e). \quad (1.10)$$

If correlation Q_{t+1} with ΔP_t^e is positive, then when agents see the price increase they have higher expected returns, and therefore should hold a larger number of shares. If θ_t are *i.i.d.*, however, this is highly unlikely, since $E(\theta_t \Delta P_t^e) = E(\theta_t P_t)$. This quantity is negative in most models, because a positive supply shock normally leads to lower price.

In deriving this result, we use the fact that agents have myopic preferences. In general, there will also be a hedging demand. Note, however, that if the hedging demand results solely from information asymmetry, then it is a linear combination of agents' forecasting mistakes, and therefore is orthogonal to the public information set. Since everyone observes the price, the covariance of the hedging demand with ΔP_t^e is zero, which leaves the left hand side of equation (1.10) unchanged. As a result, the distribution of information between agents can change the magnitude of the correlation but not the sign. In fact, one can prove a similar result for a case in which θ_t has a more general dynamics:

Proposition 4 *Let $a_{t-s} = E(\varepsilon_s \theta_t)$ is non-increasing sequence and $p_{t-s}^\theta = E(\varepsilon_s P_t) < 0$ for all $s, s \leq t$. Then $\text{Cov}(Q_{t+1}, \Delta P_t^e) < 0$.*

Proof. We have to show that $E(\theta_t \Delta P_t^e) < 0$.

$$E(\theta_t \Delta P_t^e) = a_0 p_0^\theta + \sum_{k=1}^{\infty} a_k (p_k^\theta - p_{k-1}^\theta) = \sum_{k=0}^{\infty} p_k^\theta (a_k - a_{k+1}) < 0. \quad (1.11)$$

It is worthwhile to compare this result with that of Brown and Jennings (1989), who are able to generate positive autocorrelation for a wider range of parameters in a two period, but otherwise similar model. This difference underscores the importance of considering a stationary economy where the initial conditions have little effect on properties of equilibrium.

1.8 Concluding remarks

This paper presents a dynamic equilibrium model of asset pricing under different information dispersion setups. The model allows us to clarify the mechanics behind the infinite regress problem and explicitly demonstrate the effect of information distribution. By analyzing differential information coupled with time evolving fundamentals we are able to provide new insights about the behavior of prices and returns.

Due to the complexity of the problem, we made a number of simplifying assumptions. It is reasonable to believe that the intuition we gain from our analysis can be applied to more realistic models as well. There are several directions in which our paper can be developed. First, it would be interesting to consider a setup with multiple stocks and analyze the effect of information distribution on cross-correlations of prices and returns¹¹. Next, we consider myopic investors who do not have hedging demand, significantly simplifying the model, since otherwise we would have to solve a dynamic program with an infinite dimensional space of state variables. The impact of hedging could be non-trivial and needs further research.

In our model the agents are exogenously endowed with their information and can neither buy new information, nor release their own information if they find this exchange profitable. It might be interesting to relax this assumption and to introduce

¹¹See Admati (1985), Easley and O'Hara (2004), and Hughes, Liu, and Liu (2005), among others, for a static analysis.

the market for information. This direction was explored in a static setting by Verrecchia (1982), Admati and Pfleiderer (1986), and others but dynamic properties of the market for information are not thoroughly explored¹².

Although our analysis pertains mostly to asset pricing, the insights about various aspects of the “forecasting the forecasts of others” problem and iterated expectations, as well as the intuition behind our results, are much more general and also relevant for other fields. For example, higher order expectations naturally arise in different macroeconomic settings (Woodford (2002)), in the analysis of exchange rate dynamics (Bacchetta and Wincoop (2003)), in models of industrial organization where, for example, firms have to extract information about unknown cost structure of competitors (Vives (1988)). The application of our approach and analysis of higher order expectations in these fields might be fruitful and need further research.

¹²See Naik (1997b) for analysis of monopolistic information market in a dynamic framework.

1.9 Appendix A

Proof of Proposition 1.

Our starting point is a representation of equilibrium price (1.2). If all investors know V_t^1 and V_t^2 then the infinite sum can be computed explicitly and we get

$$P_t = -\frac{1}{\Omega(r + \lambda)} + \frac{a\lambda}{1 + r - a(1 - \lambda)}V_t - \frac{1}{\Omega(1 + r)}\theta_t.$$

So the only remaining problem is to calculate Ω which is endogenous and is determined by conditional variance of Q_{t+1} . A simple calculation yields

$$\text{Var}(Q_{t+1}|\mathcal{F}_t^i) = \frac{2\lambda^2(1 + r)^2b_V^2}{(1 + r - a(1 - \lambda))^2} + \frac{(1 - \lambda)^2b_\Theta^2}{\Omega^2(1 + r)^2}.$$

By definition of Ω

$$\Omega = \int \frac{1}{\alpha} \frac{di}{\text{Var}[Q_{t+1}|\mathcal{F}_t^i]} = \frac{1}{\alpha} \frac{1}{\text{Var}[Q_{t+1}|\mathcal{F}_t^i]},$$

which gives the following equation for Ω

$$\frac{1}{\alpha} \frac{1}{\Omega} = \frac{2\lambda^2(1 + r)^2b_V^2}{(1 + r - a(1 - \lambda))^2} + \frac{(1 - \lambda)^2b_\Theta^2}{\Omega^2(1 + r)^2}$$

or, equivalently,

$$\frac{2\lambda^2(1 + r)^2b_V^2}{(1 + r - a(1 - \lambda))^2}\Omega^2 - \frac{\Omega}{\alpha} + \frac{(1 - \lambda)^2b_\Theta^2}{(1 + r)^2} = 0.$$

This is a quadratic equation which has real solutions only if its discriminant is non-negative. or

$$2\sqrt{2}b_Vb_\Theta \frac{\lambda(1 - \lambda)}{1 + r - a(1 - \lambda)} \leq \frac{1}{\alpha}.$$

Under this condition there is a full information solution with Ω as given in Proposition 1.

1.10 Appendix B

Proof of Proposition 2.

If investors are hierarchically informed the infinite sequence of iterated expectations collapses to one term $\hat{V}_t^1 = E[V_t^1 | \mathcal{F}_t^2]$, which is a new state variable of the economy. So we conjecture that the price is a linear function of state variables:

$$P_t = p_0 + p_{V^1} V_t^1 + p_{V^2} V_t^2 + p_\Theta \theta_t + p_\Delta (\hat{V}_t^1 - V_t^1), \quad (\text{B1})$$

where $p_0, p_{V^1}, p_{V^2}, p_\Theta$ and p_Δ are constants. The dynamics of \hat{V}_t^1 can be found from the filtering problem of uninformed agents. To solve this problem we use the following theorem¹³.

Theorem 1 (*Kalman - Bucy filter*)

Consider a discrete linear system of the form

$$\begin{aligned} x_t &= \Phi x_{t-1} + \Gamma \epsilon_{x,t}, \\ y_t &= M x_t + \epsilon_{y,t}, \end{aligned}$$

where x_t is an n -vector of unobservable state variables at t , y_t is an m -vector of observations at t . Φ , Γ and M are $(n \times n)$, $(n \times r)$, and $(m \times n)$ constant matrices respectively. $\epsilon_{x,t}$ and $\epsilon_{y,t}$ are r -vector and m -vector white Gaussian sequences: $\epsilon_{x,t} \sim \mathcal{N}(0, Q)$, $\epsilon_{y,t} \sim \mathcal{N}(0, R)$, $\epsilon_{x,t}$ and $\epsilon_{y,t}$ are independent. Denote the optimal estimation of x_t at time t as \hat{x}_t :

$$\hat{x}_t = E[x_t | y_\tau : \tau \leq t]$$

and define

$$\Sigma = E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)' | y_\tau : \tau \leq t].$$

¹³See Jazwinski (1970) for textbook discussion of linear filtering theory.

Then

$$\hat{x}_t = (I_n - KM)\Phi\hat{x}_{t-1} + Ky_t, \quad (\text{B2})$$

$$\Sigma = (I_n - KM)(\Phi\Sigma\Phi' + \Gamma Q\Gamma'), \quad (\text{B3})$$

$$K = (\Phi\Sigma\Phi' + \Gamma Q\Gamma')M'[M(\Phi\Sigma\Phi' + \Gamma Q\Gamma')M' + R]^{-1}, \quad (\text{B4})$$

where I_n is the $(n \times n)$ identity matrix.

In our case the system of unobservable state variables is $V_{t+1}^1 = aV_t^1 + b_V\epsilon_{t+1}^1$. The partially informed investors effectively observe $Z_t = (p_{V^1} - p_\Delta)V_t^1 + p_\Theta\theta_t$. We have the following mapping:

$$\begin{aligned} x_t &= V_t^1, \quad y_t = Z_t, \quad \Phi = a, \quad \Gamma = b_V, \\ M &= p_{V^1} - p_\Delta, \quad R = (p_\Theta b_\Theta)^2, \quad Q = 1. \end{aligned}$$

Applying the Kalman-Bucy filter we arrive at

$$\hat{V}_t^1 = a(1 - k(p_{V^1} - p_\Delta))\hat{V}_{t-1}^1 + k(p_{V^1} - p_\Delta)V_t^1 + kp_\Theta\theta_t, \quad (\text{B5})$$

where k solves the quadratic equation

$$p_\Theta^2 b_\Theta^2 a^2 (p_{V^1} - p_\Delta) k^2 + (p_\Theta^2 b_\Theta^2 (1 - a^2) + b_V^2 (p_{V^1} - p_\Delta)^2) k - b_V^2 (p_{V^1} - p_\Delta) = 0. \quad (\text{B6})$$

Equation (B5) implies AR(1) dynamics of the estimation error:

$$\hat{V}_t^1 - V_t^1 = ac(\hat{V}_{t-1}^1 - V_{t-1}^1) - b_V c \epsilon_t^1 + kb_\Theta p_\Theta \epsilon_t^\Theta, \quad c = 1 - k(p_{V^1} - p_\Delta). \quad (\text{B7})$$

Consider now the demand functions of investors and the market clearing condition.

The aggregate demand of partially informed investors is

$$X_t^2 = (1 - \gamma) \frac{E[Q_{t+1} | \mathcal{F}_t^2]}{\alpha \text{Var}[Q_{t+1} | \mathcal{F}_t^2]}$$

Using our conjecture for the price function we can rewrite it as

$$\begin{aligned} X_t^2 &= \omega_2 \left((1-\lambda)p_0 + a(\lambda + (1-\lambda)p_{V^2})V_t^2 + a(\lambda + (1-\lambda)p_{V^1})\hat{V}_t^1 - (1+r)P_t \right) \\ &= \omega_2 \left((1-\lambda)p_0 + a(\lambda + (1-\lambda)p_{V^1})V_t^1 + a(\lambda + (1-\lambda)p_{V^2})V_t^2 + \right. \\ &\quad \left. + a(\lambda + (1-\lambda)p_{V^1})(\hat{V}_t^1 - V_t^1) - (1+r)P_t \right), \end{aligned}$$

where, by definition, $\omega_2 = (1-\gamma)/(\alpha \text{Var}[Q_{t+1}|\mathcal{F}_t^2])$. Similarly, the aggregate demand of informed investors is:

$$\begin{aligned} X_t^1 &= \gamma \frac{E[Q_{t+1}|\mathcal{F}_t^1]}{\alpha \text{Var}[Q_{t+1}|\mathcal{F}_t^1]} = \omega_1 \left(a\lambda V_t + (1-\lambda)E[P_{t+1}|\mathcal{F}_t^1] - (1+r)P_t \right), \\ \omega_1 &= \gamma/(\alpha \text{Var}[Q_{t+1}|\mathcal{F}_t^1]). \end{aligned}$$

Using (B7) we can rewrite it as

$$\omega_1 \left((1-\lambda)p_0 + a(\lambda + (1-\lambda)p_{V^1})V_t^1 + a(\lambda + (1-\lambda)p_{V^2})V_t^2 + a c p_\Delta (1-\lambda)(\hat{V}_t^1 - V_t^1) - (1+r)P_t \right).$$

The market clearing condition $X_t^1 + X_t^2 = 1 + \theta_t$ gives

$$\begin{aligned} P_t &= \frac{\Omega(1-\lambda)p_0 - 1}{\Omega(1+r)} + \frac{a(\lambda + (1-\lambda)p_{V^1})}{1+r} V_t^1 + \frac{a(\lambda + (1-\lambda)p_{V^2})}{1+r} V_t^2 \\ &\quad - \frac{1}{\Omega(1+r)} \theta_t + \frac{a(\omega_2\lambda + (1-\lambda)(\omega_2 p_{V^1} + \omega_1 c p_\Delta))}{\Omega(1+r)} (\hat{V}_t^1 - V_t^1), \quad (\text{B8}) \end{aligned}$$

where $\Omega = \omega_1 + \omega_2$. Comparing (B8) with the conjectured expression for price we get a set of equations for the coefficients p_0 , p_{V^1} , p_{V^2} , p_Θ and p_Δ :

$$\begin{aligned} p_0 &= \frac{\Omega(1-\lambda)p_0 - 1}{\Omega(1+r)}, & p_{V^1} &= \frac{a(\lambda + (1-\lambda)p_{V^1})}{1+r}, \\ p_{V^2} &= \frac{a(\lambda + (1-\lambda)p_{V^2})}{1+r}, & p_\Theta &= -\frac{1}{\Omega(1+r)}, \\ p_\Delta &= \frac{a(\omega_2\lambda + (1-\lambda)(\omega_2 p_{V^1} + \omega_1 c p_\Delta))}{\Omega(1+r)}. \end{aligned}$$

Solving these equations we obtain:

$$p_0 = -\frac{1}{\Omega(r + \lambda)}, \quad p_{V^1} = p_{V^2} = \frac{a\lambda}{1 + r - a(1 - \lambda)},$$

$$p_\Theta = -\frac{1}{\Omega(1 + r)}, \quad p_\Delta = \frac{\omega_2 \lambda a(1 + r)}{(1 + r - a(1 - \lambda))(\Omega(1 + r) - \omega_1 a c(1 - \lambda))}.$$

Coefficients p_{V^1} and p_{V^2} are expressed in terms of exogenous parameters of the model.

In order to get p_0 , p_Θ , and p_Δ we have to compute $\text{Var}[Q_{t+1}|\mathcal{F}_t^1]$ and $\text{Var}[Q_{t+1}|\mathcal{F}_t^2]$.

We have:

$$\begin{aligned} \text{Var}[Q_{t+1}|\mathcal{F}_t^1] &= \\ &= b_V^2 [(\lambda + (1 - \lambda)(p_{V^1} - cp_\Delta))^2 + (\lambda + (1 - \lambda)p_{V^2})^2] + b_\Theta^2(1 - \lambda)^2 p_\Theta^2(1 + kp_\Delta)^2, \end{aligned}$$

$$\begin{aligned} \text{Var}[Q_{t+1}|\mathcal{F}_t^2] &= \text{Var}[Q_{t+1}|\mathcal{F}_t^1] + a^2(\lambda + (1 - \lambda)(p_{V^1} - cp_\Delta))^2 \text{Var}[\hat{V}_t^1 - V_t^1|\mathcal{F}_t^2] = \\ &= \text{Var}[Q_{t+1}|\mathcal{F}_t^1] + (\lambda + (1 - \lambda)(p_{V^1} - cp_\Delta))^2 \frac{a^2 c}{1 - a^2 c} b_V^2. \end{aligned}$$

As a result, we have the following system of nonlinear equations for p_Θ , p_Δ , c , ω_1 , ω_2 and Ω :

$$p_\Theta = -\frac{1}{\Omega(1 + r)},$$

$$p_\Delta = \frac{\omega_2 \lambda a(1 + r)}{(1 + r - a(1 - \lambda))(\Omega(1 + r) - \omega_1 a c(1 - \lambda))},$$

$$\omega_1 (b_V^2 [(\lambda + (1 - \lambda)(p_{V^1} - cp_\Delta))^2 + (\lambda + (1 - \lambda)p_{V^2})^2] + b_\Theta^2(1 - \lambda)^2 p_\Theta^2(1 + kp_\Delta)^2) = \gamma,$$

$$\omega_2 \left(b_V^2 \left[\frac{1}{1 - a^2 c} (\lambda + (1 - \lambda)(p_{V^1} - cp_\Delta))^2 + (\lambda + (1 - \lambda)p_{V^2})^2 \right] + \right.$$

$$\left. + b_\Theta^2(1 - \lambda)^2 p_\Theta^2(1 + kp_\Delta)^2 \right) = 1 - \gamma,$$

$$p_\Theta^2 b_\Theta^2 a^2 (1 - c)^2 + (p_\Theta^2 b_\Theta^2 (1 - a^2) + b_V^2 (p_{V^1} - p_\Delta)^2) (1 - c) - b_V^2 (p_{V^1} - p_\Delta)^2 = 0,$$

$$\Omega = \omega_1 + \omega_2.$$

The solution to the above system then can be obtained numerically.

1.11 Appendix C

Proof of Proposition 3.

To save space we give the proof for $\alpha = 1$ and $\gamma = 1/2$, and the components V_t^1 and V_t^2 are treated symmetrically. The proof for the general case follows the same logic but is more involved. Denote demeaned price by \tilde{P}_t . We assume that the model has a stationary linear equilibrium, i.e. \tilde{P}_t is a stationary regular Gaussian process¹⁴ which admits the following decomposition:

$$\tilde{P}_t = b_V \sum_{k=0}^{\infty} f_k \varepsilon_{t-k}^i + b_V \sum_{k=0}^{\infty} f_k \varepsilon_{t-k}^{-i} + b_{\Theta} \sum_{k=0}^{\infty} f_k^{\Theta} \varepsilon_{t-k}^{\Theta}, \quad (\text{C1})$$

where

$$\sum_{k=0}^{\infty} \left(b_V^2 f_k^2 + b_V^2 f_k^2 + b_{\Theta}^2 (f_k^{\Theta})^2 \right) < \infty. \quad (\text{C2})$$

Instead of working with an infinite number of coefficients it is convenient to put the series in z -representation¹⁵, i.e. introduce functions $f(z)$ and $f_{\Theta}(z)$ such that

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad f_{\Theta}(z) = \sum_{k=0}^{\infty} f_k^{\Theta} z^k. \quad (\text{C3})$$

Due to (C2) f and f_{Θ} are well-defined analytical functions in the unit disk $D_0 = \{z : |z| < 1\}$ in the complex plane \mathbb{C} . Let L be a shift operator defined as $L\varepsilon_t = \varepsilon_{t-1}$. Then using z -representation we can put the conjectured price function into the following form:

$$\tilde{P}_t = b_V f(L) \varepsilon_t^i + b_V f(L) \varepsilon_t^{-i} + b_{\Theta} f^{\Theta}(L) \varepsilon_t^{\Theta}. \quad (\text{C4})$$

One can verify that if two random processes x_t and y_t are

$$\begin{aligned} x_t &= b_V f_x^1(L) \varepsilon_t^i + b_V f_x^2(L) \varepsilon_t^{-i} + b_{\Theta} f_x^{\Theta}(L) \varepsilon_t^{\Theta} \\ y_t &= b_V f_y^1(L) \varepsilon_t^i + b_V f_y^2(L) \varepsilon_t^{-i} + b_{\Theta} f_y^{\Theta}(L) \varepsilon_t^{\Theta} \end{aligned}$$

¹⁴See all relevant definitions in Ibragimov and Rozanov (1978).

¹⁵For other applications of z -representation to analysis of rational expectation equilibrium see Futia (1981), Kasa (2000), Kasa, Walker, and Whiteman (2004).

then

$$E[x_t y_t] = \frac{1}{2\pi i} \oint \left\{ b_V^2 f_x^1(z) f_y^1\left(\frac{1}{z}\right) + b_V^2 f_x^2(z) f_y^2\left(\frac{1}{z}\right) + b_\Theta^2 f_x^\Theta(z) f_y^\Theta\left(\frac{1}{z}\right) \right\} \frac{dz}{z}.$$

It turns out that the notion of Markovian dynamics has a nice counterpart in the frequency domain. We will use extensively the following result from the theory of Gaussian stationary processes (see Doob (1944) for original results and Ibragimov and Rozanov (1978) for textbook treatment).

Theorem 2 *Let X_t be a regular Gaussian stationary process with discrete time defined on a complete probability space $(\Omega, \mathcal{F}, \mu)$. Let \mathcal{F}_t be a natural filtration generated by X_t . The process X_t admits Markovian dynamics with a finite number of Gaussian state variables if and only if its spectral density is a rational function $e^{i\lambda}$.*

Remark. It is a well-known result then that a Gaussian process X_t with a rational spectral density is an ARMA(p,q) process, that is, it can be represented as

$$X_t - \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q} \quad (\text{C5})$$

for some $\phi_i, i = 1..p$, $\theta_i, i = 1..q$, and $\varepsilon_t, t \in \mathbb{Z}$.

Let us reformulate the equilibrium conditions in terms of functions $f(z)$ and $f_\Theta(z)$. It is convenient to start from the filtering problem of each agent. When forming his demand each agent has to find the best estimate of $\lambda V_{t+1}^{-i} + (1 - \lambda)P_{t+1}^i$ given his information set $\mathcal{F}_t^i = \{V_s^i, P_s\}_{-\infty}^t$. Since some components of P_t are known to agent i , observation of $\mathcal{F}_t^i = \{V_s^i, P_s\}_{-\infty}^t$ is equivalent to observation of $\mathcal{F}_t^i = \{V_s^i, Z_s^i\}_{-\infty}^t$, where

$$Z_t^i = b_V f(L) \varepsilon_t^{-i} + b_\Theta f^\Theta(L) \varepsilon_t^\Theta. \quad (\text{C6})$$

The filtering problem is equivalent to finding a projector G such that:

$$E[\lambda V_{t+1}^{-i} + (1 - \lambda)Z_{t+1}^i | \mathcal{F}_t^i] = G(L)Z_t^i. \quad (\text{C7})$$

By definition, $\lambda V_{t+1}^{-i} + (1 - \lambda)Z_{t+1}^i - G(L)Z_t^i$ is orthogonal to all Z_s^i , $s \leq t$:

$$E[(\lambda V_{t+1}^{-i} + (1 - \lambda)Z_{t+1}^i - G(L)Z_t^i) Z_s^i] = 0. \quad (\text{C8})$$

Calculating expectations we get

$$\begin{aligned} E[V_{t+1}^{-i} Z_s^i] &= \frac{1}{2\pi i} \oint \frac{1}{z} \frac{a}{1 - az} \frac{1}{z^{t-s}} f\left(\frac{1}{z}\right) dz, \\ E[Z_{t+1}^i Z_s^i] &= \frac{1}{2\pi i} \oint \left\{ b_V^2 \frac{1}{z^2} f(z) \frac{1}{z^{t-s}} f\left(\frac{1}{z}\right) + b_\Theta^2 \frac{1}{z^2} f^\Theta(z) \frac{1}{z^{t-s}} f^\Theta\left(\frac{1}{z}\right) \right\}, \\ E[G(L)Z_t^i Z_s^i] &= \frac{1}{2\pi i} \oint \left\{ b_V^2 \frac{1}{z} G(z) f(z) \frac{1}{z^{t-s}} f\left(\frac{1}{z}\right) + b_\Theta^2 \frac{1}{z} G(z) f^\Theta(z) \frac{1}{z^{t-s}} f^\Theta\left(\frac{1}{z}\right) \right\} dz. \end{aligned} \quad (\text{C9})$$

Collecting all terms the orthogonality condition (C8) takes the form

$$\frac{1}{2\pi i} \oint \frac{1}{z^k} U(z) = 0, \quad k = 1, 2, \dots \quad (\text{C10})$$

where the function $U(z)$ is

$$\begin{aligned} U(z) &= b_V^2 \frac{a\lambda}{1 - az} f\left(\frac{1}{z}\right) + (1 - \lambda) \left(b_V^2 \frac{1}{z} f(z) f\left(\frac{1}{z}\right) + b_\Theta^2 \frac{1}{z} f_\Theta(z) f_\Theta\left(\frac{1}{z}\right) \right) - \\ &\quad - G(z) \left(b_V^2 f(z) f\left(\frac{1}{z}\right) + b_\Theta^2 f_\Theta(z) f_\Theta\left(\frac{1}{z}\right) \right). \end{aligned} \quad (\text{C11})$$

This means that $U(z)$ is analytic in $D_\infty = \{z : |z| > 1\}$ and $U(\infty) = 0$. In other words, the series expansion of $U(z)$ at $z = \infty$ doesn't have the terms z^s , $s \geq 0$. The demand function of i' agent in z -representation can be written as

$$\begin{aligned} X_t^i &= -(r + \lambda)p_0 + b_V \left(\frac{a\lambda}{1 - aL} - (1 + r)f(L) + (1 - \lambda) \frac{f(L) - f(0)}{L} \right) \varepsilon_t^i \\ &\quad + b_V \left(-(1 + r) + G(L) \right) f(L) \varepsilon_t^{-i} + b_\Theta \left(-(1 + r) + G(L) \right) f_\Theta(L) \varepsilon_t^\Theta. \end{aligned} \quad (\text{C12})$$

The market clearing condition $\omega_1 X_t^1 + \omega_2 X_t^2 = 1 + \theta_t$, where $\omega_1 = \omega_2 = \Omega/2$ should

be valid for all realizations of shocks, which yields the following set of equations:

$$\frac{a\lambda}{1-az} - 2(1+r)f(z) + (1-\lambda)\frac{f(z)-f(0)}{z} + G(z)f(z) = 0, \quad (\text{C13})$$

$$-\Omega(1+r)f_{\Theta}(z) + \Omega G(z)f_{\Theta}(z) = 1. \quad (\text{C14})$$

Given these equations $U(z)$ can be rewritten as

$$U(z) = 2b_V^2(1+r)f(z)f\left(\frac{1}{z}\right) + b_V^2(1-\lambda)\frac{f(0)}{z}f\left(\frac{1}{z}\right) - 2b_V^2G(z)f(z)f\left(\frac{1}{z}\right) + b_{\Theta}^2(1-\lambda)\frac{1}{z}f_{\Theta}(z)f_{\Theta}\left(\frac{1}{z}\right) - b_{\Theta}^2\left(\frac{1}{2} + (1+r)f_{\Theta}(z)\right)f_{\Theta}\left(\frac{1}{z}\right). \quad (\text{C15})$$

Note that the term $b_V^2(1-\lambda)\frac{f(0)}{z}f\left(\frac{1}{z}\right)$ does not have terms with non-negative powers of z , so it can be discarded. Similarly, the term $-\frac{1}{2}b_{\Theta}^2f_{\Theta}\left(\frac{1}{z}\right)$ contributes only the constant $-\frac{1}{2}b_{\Theta}^2f_{\Theta}(0)$. So $U(z)$ takes an equivalent form:

$$U(z) = 2b_V^2((1+r) - G(z))f(z)f\left(\frac{1}{z}\right) + b_{\Theta}^2\left((1-\lambda)\frac{1}{z} - (1+r)\right) \times f_{\Theta}(z)f_{\Theta}\left(\frac{1}{z}\right) - \frac{1}{2}b_{\Theta}^2f_{\Theta}(0). \quad (\text{C16})$$

Let us introduce a function $g(z)$ such that $g(z) = G(z) - (1+r)$. Then equations (C13), (C14), and (C16) take the following forms:

$$f(z) = -\frac{a(\lambda + (1-\lambda)f(0))z - (1-\lambda)f(0)}{(1-az)(1-\lambda - (1+r)z + zg(z))}, \quad (\text{C17})$$

$$f_{\Theta}(z) = \frac{1}{\Omega g(z)}, \quad (\text{C18})$$

$$U(z) = -2b_V^2g(z)f(z)f\left(\frac{1}{z}\right) + b_{\Theta}^2(1-\lambda - (1+r)z + zg(z))\frac{1}{z} \times f_{\Theta}(z)f_{\Theta}\left(\frac{1}{z}\right) - b_{\Theta}^2\left(\frac{1}{2} + \frac{1}{\Omega}\right)f_{\Theta}(0). \quad (\text{C19})$$

So the rational expectation equilibrium in our model is characterized by functions $f(z)$, $f_{\Theta}(z)$, $g(z)$ and $U(z)$ such that $f(z)$, $f_{\Theta}(z)$ and $g(z)$ are analytic inside the unit

circle, $U(z)$ is analytic outside the unit circle, $U(\infty) = 0$ and equations (C17), (C18), and (C19) hold.

Now let us turn to the main part of the proof. By Theorem 2, if the system $\{V^1, V^2, \theta, P\}$ admits Markovian dynamics, then its joint spectral density should be rational, which, in turn, implies that function $g(z)$ has to be rational as well. Given relationships (C17) and (C18), functions $f(z)$ and $f_\Theta(z)$ should also be rational. So to prove that our model has non-Markovian dynamics we have to show that there do not exist rational functions $f(z)$ and $f_\Theta(z)$ solving equations (C17), (C18), (C19) and satisfying all conditions specified above.

We construct the proof by contradiction. Suppose that function $g(z)$ is rational. For further convenience we introduce the function $H(z)$ such that

$$g(z) = (zg(z) + 1 - \lambda - (1+r)z)H(z) \quad (\text{C20})$$

Consequently, in terms of $H(z)$, the function $g(z)$ is

$$g(z) = (1+r) \frac{z_0 - z}{H^{-1}(z) - z}, \quad z_0 = \frac{1-\lambda}{1+r} \quad (\text{C21})$$

The following lemmas describe the properties of $H(z)$.

Lemma 1 $H(z)$ is rational, $H(z) \neq 0$ for $z \in D_0$, and $H(z_0) = \frac{1}{z_0}$.

Proof. Since $f_\Theta(z) = 1/(\Omega g(z))$, we have

$$f_\Theta(z) = \frac{1}{\Omega(1+r)} \frac{1 - zH(z)}{(z_0 - z)H(z)}. \quad (\text{C22})$$

Statements of the lemma now follow from the fact that $f_\Theta(z)$ is rational and analytic in D_0 .

Lemma 2 $(z - z_1)H(z)$, where $z_1 = \frac{(1-\lambda)f(0)}{a(\lambda+(1-\lambda)f(0))}$ is analytic in D_0 .

Proof. Substituting (C21) into (C17) gives

$$f(z) = \frac{a(\lambda + (1-\lambda)f(0))}{(1+r)} \frac{z_1 - z}{(1-az)(z_0 - z)} (1 - zH(z)). \quad (\text{C23})$$

The lemma now follows from analyticity of $f(z)$ in D_0 .

Substitution of (C22) and (C23) into $U(z)$ results in

$$U(z) = -2b_V^2 a^2 (\lambda + (1 - \lambda)f(0))^2 \frac{(z - z_1)(\frac{1}{z} - z_1)}{(1 - az)(1 - \frac{a}{z})g(\frac{1}{z})} \times \\ \times H(z)H\left(\frac{1}{z}\right) + b_\Theta^2 \frac{1}{\Omega^2 z H(z)g(\frac{1}{z})} - b_\Theta^2 \left(\frac{1}{2} + \frac{\Omega}{2}\right) f_\Theta(0). \quad (\text{C24})$$

Also from (C22),

$$f_\Theta(0) = \frac{1}{\Omega(1 - \lambda)H(0)}. \quad (\text{C25})$$

Since $g(z)$ does not have poles in D_0 (and consequently $g(\frac{1}{z})$ does not have poles in D_∞), analyticity of $U(z)$ in D_∞ implies analyticity of $U^g(z) = U(z)g(\frac{1}{z})$ in D_∞ .

Using (C25) we see that

$$U^g(z) = -2b_V^2 a^2 (\lambda + (1 - \lambda)f(0))^2 \frac{(z - z_1)(\frac{1}{z} - z_1)}{(1 - az)(1 - \frac{a}{z})} \times \\ \times H(z)H\left(\frac{1}{z}\right) + b_\Theta^2 \frac{1}{\Omega^2 z H(z)} - \left(\frac{1}{2} + \frac{1}{2\Omega}\right) b_\Theta^2 \quad (\text{C26})$$

must be analytical in D_∞ . This means that the pole $1/a$ in (C26) must be canceled.

It might happen only due to one of the following reasons:

1. $H(1/a) = 0$,
2. $H(a) = 0$,
3. $z_1 = a$, or, equivalently, $f(0) = \frac{a^2}{1 - a^2} \frac{\lambda}{1 - \lambda}$
4. $z_1 = 1/a$
5. The pole in the first term is canceled by a pole in the second term.

It is easy to notice that the first reason does not work since in this case a pole in the second term appears. Similarly, the fifth possibility cannot realize. The equation $z_1 = 1/a$ is inconsistent unless $\lambda = 0$. The second option contradicts the condition that $H(z)$ does not have zeros inside the unit circle. This leaves only the third

possibility should realize and we can fix the value of $f(0)$. Consequently, we rewrite $U^g(z)$ as

$$U^g(z) = -2b_V^2 \frac{a^2 \lambda^2}{(1-a^2)^2} H(z) H\left(\frac{1}{z}\right) + b_\Theta^2 \frac{1}{\Omega^2 z H(z)} - \left(\frac{1}{2} + \frac{1}{2\Omega}\right) b_\Theta^2 \quad (\text{C27})$$

with the condition

$$H\left(\frac{1-\lambda}{1+r}\right) = \frac{1+r}{1-\lambda} \quad \text{and} \quad U^g(\infty) = 0. \quad (\text{C28})$$

Now we will show that there is no such rational function $H(z)$. Assume for now that $H(z)$ has a pole z_h . From Lemma 2, $z_h = a$ or $z_h \in D_\infty$. If $z_h \in D_\infty$ and $z_h \neq \infty$, then, for analyticity of $U(z)$ in D_∞ , we have to have $H(1/z_h) = 0$, but it contradicts Lemma 1. If $z_h = a$ then $U(z)$ has a pole at $1/a$. Indeed, if a is a pole of $H(z)$ then $1/a$ is a pole of $H(1/z)$. The only possibility to cancel it in the first term of $U(z)$ is to have $H(1/a) = 0$. But in this case a pole in the second term arises. So $H(z)$ does not have poles in \mathbb{C} . As a result, the only possibility is $z_h = \infty$. This means that $H(z)$ is a polynomial. Let $w_0 \in \mathbb{C}$ be a zero of $H(z)$. Because of Lemma 1, w_0 can be only in D_∞ . However, this means that, unless $H(1/z)$ or $H(z)$ have a pole at w_0 , $U(z)$ is not analytic in D_∞ . We know that $H(z)$ (and consequently $H(1/z)$) do not have poles in \mathbb{C} . Thus we can conclude that $H(z)$ does not have zeros. Hence by Liouville's theorem $H(z) = H = \text{const}$. We have two equations that this constant has to satisfy:

$$H = \frac{1+r}{1-\lambda}, \quad -2b_V^2 \frac{a^2 \lambda^2}{(1-a^2)^2} H^2 - \frac{1}{\Omega} b_\Theta^2 = 0.$$

Obviously, these conditions are inconsistent and this concludes the proof.

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Chapter 2

An Econometric Model of Serial Correlation and Illiquidity In Hedge Fund Returns (joint with Mila Getmansky and Andrew W. Lo)

2.1 Introduction

One of the fastest growing sectors of the financial services industry is the hedge-fund or “alternative investments” sector. Long the province of foundations, family offices, and high-net-worth investors, hedge funds are now attracting major institutional investors such as large state and corporate pension funds and university endowments, and efforts are underway to make hedge-fund investments available to individual investors through more traditional mutual-fund investment vehicles. One of the main reasons for such interest is the performance characteristics of hedge funds—often known as “high-octane” investments, many hedge funds have yielded double-digit returns to their investors and, in some cases, in a fashion that seems uncorrelated with general market swings and with relatively low volatility. Most hedge funds accomplish this by maintaining both long and short positions in securities—hence

the term “hedge” fund—which, in principle, gives investors an opportunity to profit from both positive and negative information while, at the same time, providing some degree of “market neutrality” because of the simultaneous long and short positions.

However, several recent empirical studies have challenged these characterizations of hedge-fund returns, arguing that the standard methods of assessing their risks and rewards may be misleading. For example, Asness, Krail and Liew (2001) show in some cases where hedge funds purport to be market neutral, i.e., funds with relatively small market betas, including both contemporaneous and lagged market returns as regressors and summing the coefficients yields significantly higher market exposure. Moreover, in deriving statistical estimators for Sharpe ratios of a sample of mutual and hedge funds, Lo (2002) shows that the correct method for computing annual Sharpe ratios based on monthly means and standard deviations can yield point estimates that differ from the naive Sharpe ratio estimator by as much as 70%.

These empirical properties may have potentially significant implications for assessing the risks and expected returns of hedge-fund investments, and can be traced to a single common source: significant serial correlation in their returns.

This may come as some surprise because serial correlation is often (though incorrectly) associated with market inefficiencies, implying a violation of the Random Walk Hypothesis and the presence of predictability in returns. This seems inconsistent with the popular belief that the hedge-fund industry attracts the best and the brightest fund managers in the financial services sector. In particular, if a fund manager’s returns are predictable, the implication is that the manager’s investment policy is not optimal; if his returns next month can be reliably forecasted to be positive, he should increase his positions this month to take advantage of this forecast, and vice versa for the opposite forecast. By taking advantage of such predictability the fund manager will eventually eliminate it, along the lines of Samuelson’s (1965) original “proof that properly anticipated prices fluctuate randomly”. Given the outsize financial incentives of hedge-fund managers to produce profitable investment strategies, the existence of significant unexploited sources of predictability seems unlikely.

In this paper, we argue that in most cases, serial correlation in hedge-fund re-

turns is not due to unexploited profit opportunities, but is more likely the result of illiquid securities that are contained in the fund, i.e., securities that are not actively traded and for which market prices are not always readily available. In such cases, the reported returns of funds containing illiquid securities will appear to be smoother than “true” economic returns—returns that fully reflect all available market information concerning those securities—and this, in turn, will impart a downward bias on the estimated return variance and yield positive serial return correlation. The prospect of spurious serial correlation and biased sample moments in reported returns is not new. Such effects have been derived and empirically documented extensively in the literature on “nonsynchronous trading”, which refers to security prices recorded at different times but which are erroneously treated as if they were recorded simultaneously.¹ However, this literature has focused exclusively on equity market-microstructure effects as the sources of nonsynchronicity—closing prices that are set at different times, or prices that are “stale”—where the temporal displacement is on the order of minutes, hours, or, in extreme cases, several days.² In the context of hedge funds, we argue in this paper that serial correlation is the outcome of illiquidity exposure, and while nonsynchronous trading may be one symptom or by-product of illiquidity, it is not the only aspect of illiquidity that affects hedge-fund returns. Even if prices were sampled synchronously, they may still yield highly serially correlated returns if the securities are not actively traded.³ Therefore, although our formal

¹ For example, the daily prices of financial securities quoted in the *Wall Street Journal* are usually “closing” prices, prices at which the last transaction in each of those securities occurred on the previous business day. If the last transaction in security A occurs at 2:00pm and the last transaction in security B occurs at 4:00pm, then included in B’s closing price is information not available when A’s closing price was set. This can create spurious serial correlation in asset returns since economy-wide shocks will be reflected first in the prices of the most frequently traded securities, with less frequently traded stocks responding with a lag. Even when there is no statistical relation between securities A and B, their reported returns will appear to be serially correlated and cross-correlated simply because we have mistakenly assumed that they are measured simultaneously. One of the first to recognize the potential impact of nonsynchronous price quotes was Fisher (1966). Since then more explicit models of non-trading have been developed by Atchison, Butler, and Simonds (1987), Dimson (1979), Cohen, Hawawini, et al. (1983a,b), Shanken (1987), Cohen, Maier, et al. (1978, 1979, 1986), Kadlec and Patterson (1999), Lo and MacKinlay (1988, 1990), and Scholes and Williams (1977). See Campbell, Lo, and MacKinlay (1997, Chapter 3) for a more detailed review of this literature.

²For such application, Lo and MacKinlay (1988, 1990) and Kadlec and Patterson (1999) show that nonsynchronous trading cannot explain all of the serial correlation in weekly returns of equal- and value-weighted portfolios of US equities during the past three decades.

³In fact, for most hedge funds, returns computed on a monthly basis, hence the pricing or “mark-

econometric model of illiquidity is similar to those in the nonsynchronous trading literature, the motivation is considerably broader—linear extrapolation of prices for thinly traded securities, the use of smoothed broker-dealer quotes, trading restrictions arising from control positions and other regulatory requirements, and, in some cases, deliberate performance-smoothing behavior—and the corresponding interpretations of the parameter estimates must be modified accordingly.

Regardless of the particular mechanism by which hedge-fund returns are smoothed and serial correlation is induced, the common theme and underlying driver is illiquidity exposure, and although we argue that the sources of serial correlation are spurious for most hedge funds, nevertheless, the economic impact of serial correlation can be quite real. For example, spurious serial correlation yields misleading performance statistics such as volatility, Sharpe ratio, correlation, and market beta estimates, statistics commonly used by investors to determine whether or not they will invest in a fund, how much capital to allocate to a fund, what kinds of risk exposures they are bearing, and when to redeem their investments. Moreover, spurious serial correlation can lead to wealth transfers between new, existing, and departing investors, in much the same way that using stale prices for individual securities to compute mutual-fund net-asset-values can lead to wealth transfers between buy-and-hold investors and day-traders (see, for example, Boudoukh et al., 2002).

In this paper, we develop an explicit econometric model of smoothed returns and derive its implications for common performance statistics such as the mean, standard deviation, and Sharpe ratio. We find that the induced serial correlation and impact on the Sharpe ratio can be quite significant even for mild forms of smoothing. We estimate the model using historical hedge-fund returns from the TASS Database, and show how to infer the true risk exposures of a smoothed fund for a given smoothing profile. Our empirical findings are quite intuitive: funds with the highest serial correlation tend to be the more illiquid funds, e.g., emerging market debt, fixed income arbitrage, etc., and after correcting for the effects of smoothed returns, some of the most successful types of funds tend to have considerably less attractive performance

to-market” of a fund’s securities typically occurs synchronously on the last day of the month.

characteristics.

Before describing our econometric model of smoothed returns, we provide a brief literature review in Section 2.2 and then consider other potential sources of serial correlation in hedge-fund returns in Section 2.3. We show that these other alternatives—time-varying expected returns, time-varying leverage, and incentive fees with high-water marks—are unlikely to be able to generate the magnitudes of serial correlation observed in the data. We develop a model of smoothed returns in Section 2.4 and derive its implications for serial correlation in observed returns, and we propose several methods for estimating the smoothing profile and smoothing-adjusted Sharpe ratios in Section 2.5. We apply these methods to a dataset of 909 hedge funds spanning the period from November 1977 to January 2001 and summarize our findings in Section 2.6, and conclude in Section 2.7.

2.2 Literature Review

Thanks to the availability of hedge-fund returns data from sources such as AltVest, Hedge Fund Research (HFR), Managed Account Reports (MAR), and TASS, a number of empirical studies of hedge funds have been published recently. For example, Ackermann, McEnally, and Ravenscraft (1999), Agarwal and Naik (2000b, 2000c), Edwards and Caglayan (2001), Fung and Hsieh (1999, 2000, 2001), Kao (2002), and Liang (1999, 2000, 2001) provide comprehensive empirical studies of historical hedge-fund performance using various hedge-fund databases. Agarwal and Naik (2000a), Brown and Goetzmann (2001), Brown, Goetzmann, and Ibbotson (1999), Brown, Goetzmann, and Park (1997, 2000, 2001), Fung and Hsieh (1997a, 1997b), and Lofchoff (2002) present more detailed performance attribution and “style” analysis for hedge funds. None of these empirical studies focus directly on the serial correlation in hedge-fund returns or the sources of such correlation.

However, several authors have examined the persistence of hedge-fund performance over various time intervals, and such persistence may be indirectly linked to serial correlation, e.g., persistence in performance usually implies positively autocor-

related returns. Agarwal and Naik (2000c) examine the persistence of hedge-fund performance over quarterly, half-yearly, and yearly intervals by examining the series of wins and losses for two, three, and more consecutive time periods. Using net-of-fee returns, they find that persistence is highest at the quarterly horizon and decreases when moving to the yearly horizon. The authors also find that performance persistence, whenever present, is unrelated to the type of a hedge fund strategy. Brown, Goetzmann, Ibbotson, and Ross (1992) show that survivorship gives rise to biases in the first and second moments and cross-moments of returns, and apparent persistence in performance where there is dispersion of risk among the population of managers. However, using annual returns of both defunct and currently operating offshore hedge funds between 1989 and 1995, Brown, Goetzmann, and Ibbotson (1999) find virtually no evidence of performance persistence in raw returns or risk-adjusted returns, even after breaking funds down according to their returns-based style classifications. None of these studies considers illiquidity and smoothed returns as a source of serial correlation in hedge-fund returns.

The findings by Asness, Kraib, and Liew (2001)—that lagged market returns are often significant explanatory variables for the returns of supposedly market-neutral hedge funds—is closely related to serial correlation and smoothed returns, as we shall demonstrate in Section 2.4. In particular, we show that even simple models of smoothed returns can explain both serial correlation in hedge-fund returns and correlation between hedge-fund returns and lagged index returns, and our empirically estimated smoothing profiles imply lagged beta coefficients that are consistent with the lagged beta estimates reported in Asness, Kraib, and Liew (2001). Their framework is derived from the nonsynchronous trading literature, specifically the estimators for market beta for infrequently traded securities proposed by Dimson (1977), Scholes and Williams (1977), and Schwert (1977) (see footnote 1 for additional references to this literature). A similar set of issues affects real-estate prices and price indexes, and Ross and Zisler (1991), Gyourko and Keim (1992). Fisher, Geltner, and Webb (1994), and Fisher et al. (2003), have proposed various econometric estimators that have much in common with those in the nonsynchronous trading literature.

An economic implication of nonsynchronous trading that is closely related to the hedge-fund context is the impact of stale prices on the computation of daily net-asset-values (NAVs) of certain open-end mutual funds, e.g., Bhargava, Bose and Dubofsky (1998), Chalmers, Edelen, and Kadlec (2001), Goetzmann, Ivkovic, and Rouwenhorst (2001), Boudoukh et al. (2002), Greene and Hodges (2002), and Zitzewitz (2002). In these studies, serially correlated mutual fund returns are traced to nonsynchronous trading effects in the prices of the securities contained in the funds, and although the correlation is spurious, it has real effects in the form of wealth transfers from a fund's buy-and-hold shareholders to those engaged in opportunistic buying and selling of shares based on forecasts of the fund's daily NAVs. Although few hedge funds compute daily NAVs or provide daily liquidity, the predictability in some hedge-fund return series far exceeds levels found among mutual funds, hence the magnitude of wealth transfers attributable to hedge-fund NAV-timing may still be significant.

With respect to the deliberate smoothing of performance by managers, a recent study of closed-end funds by Chandar and Bricker (2002) concludes that managers seem to use accounting discretion in valuing restricted securities so as to optimize fund returns with respect to a passive benchmark. Because mutual funds are highly regulated entities that are required to disclose considerably more information about their holdings than hedge funds, Chandar and Bricker (2002) were able to perform a detailed analysis of the periodic adjustments—both discretionary and non-discretionary—that fund managers made to the valuation of their restricted securities. Their findings suggest that performance smoothing may be even more relevant in the hedge-fund industry which is not nearly as transparent, and that econometric models of smoothed returns may be an important tool for detecting such behavior and unraveling its effects on true economic returns.

2.3 Other Sources of Serial Correlation

Before turning to our econometric model of smoothed returns in Section 2.4, we first consider four other potential sources of serial correlation in asset returns: (1) market

inefficiencies; (2) time-varying expected returns; (3) time-varying leverage; and (4) incentive fees with high water marks.

Perhaps the most common explanation (at least among industry professionals and certain academics) for the presence of serial correlation in asset returns is a violation of the Efficient Markets Hypothesis, one of the central pillars of modern finance theory. As with so many of the ideas of modern economics, the origins of the Efficient Markets Hypothesis can be traced back to Paul Samuelson (1965), whose contribution is neatly summarized by the title of his article: “Proof that Properly Anticipated Prices Fluctuate Randomly”. In an informationally efficient market, price changes must be unforecastable if they are properly anticipated, i.e., if they fully incorporate the expectations and information of all market participants. Fama (1970) operationalizes this hypothesis, which he summarizes in the well-known epithet “prices fully reflect all available information”, by placing structure on various information sets available to market participants. This concept of informational efficiency has a wonderfully counter-intuitive and seemingly contradictory flavor to it: the more efficient the market, the more random the sequence of price changes generated by such a market, and the most efficient market of all is one in which price changes are completely random and unpredictable. This, of course, is not an accident of Nature but is the direct result of many active participants attempting to profit from their information. Unable to curtail their greed, an army of investors aggressively pounce on even the smallest informational advantages at their disposal, and in doing so, they incorporate their information into market prices and quickly eliminate the profit opportunities that gave rise to their aggression. If this occurs instantaneously, which it must in an idealized world of “frictionless” markets and costless trading, then prices must always fully reflect all available information and no profits can be garnered from information-based trading (because such profits have already been captured).

In the context of hedge-fund returns, one interpretation of the presence of serial correlation is that the hedge-fund manager is not taking full advantage of the information or “alpha” contained in his strategy. For example, if a manager’s returns are highly positively autocorrelated, then it should be possible for him to improve

his performance by exploiting this fact—in months where his performance is good, he should increase his bets in anticipation of continued good performance (due to positive serial correlation), and in months where his performance is poor, he should reduce his bets accordingly. The reverse argument can be made for the case of negative serial correlation. By taking advantage of serial correlation of either sign in his returns, the hedge-fund manager will eventually eliminate it along the lines of Samuelson (1965). i.e., properly anticipated hedge-fund returns should fluctuate randomly. And if this self-correcting mechanism of the Efficient Markets Hypothesis is at work among any group of investors in the financial community, it surely must be at work among hedge-fund managers, which consists of a highly trained, highly motivated, and highly competitive group of sophisticated investment professionals.

Of course, the natural counter-argument to this somewhat naive application of the Efficient Markets Hypothesis is that hedge-fund managers cannot fully exploit such serial correlation because of transactions costs and liquidity constraints. But once again, this leads to the main thesis of this paper: serial correlation is a proxy for illiquidity and smoothed returns.

There are, however, at least three additional explanations for the presence of serial correlation. One of the central tenets of modern financial economics is the necessity of some trade-off between risk and expected return, hence serial correlation may not be exploitable in the sense that an attempt to take advantage of predictabilities in fund returns might be offset by corresponding changes in risk, leaving the fund manager indifferent at the margin between his current investment policy and other alternatives. Specifically, LeRoy (1973), Rubinstein (1976), and Lucas (1978) have demonstrated conclusively that serial correlation in asset returns need not be the result of market inefficiencies, but may be the result of time-varying expected returns, which is perfectly consistent with the Efficient Markets Hypothesis.⁴ If an investment

⁴Grossman (1976) and Grossman and Stiglitz (1980) go even further. They argue that perfectly informationally efficient markets are an *impossibility*, for if markets are perfectly efficient, the return to gathering information is nil, in which case there would be little reason to trade and markets would eventually collapse. Alternatively, the degree of market *inefficiency* determines the effort investors are willing to expend to gather and trade on information, hence a non-degenerate market equilibrium will arise only when there are sufficient profit opportunities, i.e., inefficiencies, to compensate investors for the costs of trading and information-gathering. The profits earned by these

strategy’s required expected return varies through time—because of changes in its risk exposures, for example—then serial correlation may be induced in realized returns without implying any violation of market efficiency (see Figure 2-1). We examine this possibility more formally in Section 2.3.1.

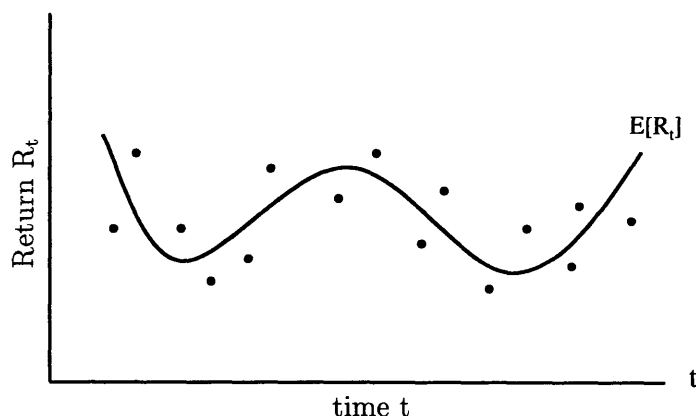


Figure 2-1: Time-varying expected returns can induce serial correlation in asset returns.

Another possible source of serial correlation in hedge-fund returns is time-varying leverage. If managers change the degree to which they leverage their investment strategies, and if these changes occur in response to lagged market conditions, this is tantamount to time-varying expected returns. We consider this case in Section 2.3.2.

Finally, we investigate one more potential explanation for serial correlation: the compensation structure of the typical hedge fund. Because most hedge funds charge an incentive fee coupled with a “high water mark” that must be surpassed before incentive fees are paid, this path dependence in the computation for net-of-fee returns may induce serial correlation. We develop a formal model of this phenomenon in Section 2.3.3.

The analysis of Sections 2.3.1–2.3.3 show that time-varying expected returns, time-varying leverage, and incentive fees with high water marks can all generate serial correlation in hedge-fund returns, but none of these effects can plausibly generate attentive investors may be viewed as economic rents that accrue to those willing to engage in such activities. Who are the providers of these rents? Black (1986) gives a provocative answer: noise traders, individuals who trade on what they think is information but is in fact merely noise.

serial correlation to the degree observed in the data, e.g., 30% to 50% for monthly returns. Therefore, illiquidity and smoothed returns are more likely sources of serial correlation in hedge-fund returns.

2.3.1 Time-Varying Expected Returns

Let R_t denote a hedge fund's return in month t , and suppose that its dynamics are given by the following time-series process:

$$R_t = \mu_1 I_t + \mu_0 (1 - I_t) + \epsilon_t \quad (1)$$

where ϵ_t is assumed to be independently and identically distributed (IID) with mean 0 and variance σ_ϵ^2 , and I_t is a two-state Markov process with transition matrix:

$$P \equiv \begin{matrix} & \begin{matrix} I_{t+1}=1 & I_{t+1}=0 \end{matrix} \\ \begin{matrix} I_t=1 \\ I_t=0 \end{matrix} & \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \end{matrix} \quad (2)$$

and μ_0 and μ_1 are the equilibrium expected returns of fund i in states 0 and 1, respectively. This is a particularly simple model of time-varying expected returns in which we abstract from the underlying structure of the economy that gives rise to (1), but focus instead on the serial correlation induced by the Markov regime-switching process (2).⁵ In particular, observe that

$$P^k = \frac{1}{2-p-q} \begin{pmatrix} 1-q & 1-p \\ 1-q & 1-p \end{pmatrix} + \frac{(p+q-1)^k}{2-p-q} \begin{pmatrix} 1-p & -(1-p) \\ -(1-q) & 1-q \end{pmatrix} \quad (3)$$

⁵For examples of dynamic general equilibrium models that yield a Markov-switching process for asset prices, and econometric methods to estimate such processes, see Cecchetti and Mark (1990). Goodwin (1993). Hamilton (1989, 1990, 1996). Kandel and Stambaugh (1991), and Turner, Startz, and Nelson (1989).

assuming that $|p + q - 1| < 1$, hence the steady-state probabilities and moments for the regime-switching process I_t are:

$$P^\infty = \begin{pmatrix} \pi_1 \\ \pi_0 \end{pmatrix} = \begin{pmatrix} \frac{1-q}{2-p-q} \\ \frac{1-p}{2-p-q} \end{pmatrix} \quad (4)$$

$$E[I_t] = \frac{1-q}{2-p-q} \quad (5)$$

$$\text{Var}[I_t] = \frac{(1-p)(1-q)}{(2-p-q)^2} \quad (6)$$

These, in turn, imply the following moments for R_t :

$$E[R_t] = \mu_1 \frac{1-q}{2-p-q} + \mu_0 \frac{1-p}{2-p-q} \quad (7)$$

$$\text{Var}[R_t] = (\mu_1 - \mu_0)^2 \frac{(1-p)(1-q)}{(2-p-q)^2} + \sigma_{\epsilon_t}^2 \quad (8)$$

$$\rho_k \equiv \text{Corr}[R_{t-k}, R_t] = \frac{(p+q-1)^k}{1 + \sigma_{\epsilon_t}^2 / [(\mu_1 - \mu_0)^2 \frac{(1-p)(1-q)}{(2-p-q)^2}]} \quad (9)$$

By calibrating the parameters μ_1 , μ_0 , p , q , and $\sigma_{\epsilon_t}^2$ to empirically plausible values, we can compute the serial correlation induced by time-varying expected returns using (9).

Observe from (9) that the serial correlation of returns depends on the squared difference of expected returns, $(\mu_1 - \mu_0)^2$, not on the particular values in either regime. Moreover, the absolute magnitudes of the autocorrelation coefficients ρ_k are monotonically increasing in $(\mu_1 - \mu_0)^2$ —the larger the difference in expected returns between the two states, the more serial correlation is induced. Therefore, we begin our calibration exercise by considering an extreme case where $|\mu_1 - \mu_0|$ is 5% per month, or 60% per year, which yields rather dramatic shifts in regimes. To complete the calibration exercise, we fix the unconditional variance of returns at a particular value, say $(20\%)^2/12$ (which is comparable to the volatility of the S&P 500 over the past 30 years), vary p and q , and solve for the values of $\sigma_{\epsilon_t}^2$ that are consistent with the values of p , q , $(\mu_1 - \mu_0)^2$, and the unconditional variance of returns.

The top panel of Table 2.1 reports the first-order autocorrelation coefficients for

various values of p and q under these assumptions, and we see that even in this most extreme case, the largest absolute magnitude of serial correlation is only 15%. The second panel of Table 2.1 shows that when the unconditional variance of returns is increased from 20% to 50% per year, the correlations decline in magnitude with the largest absolute correlation of 2.4%. And the bottom panel illustrates the kind of extreme parameter values needed to obtain autocorrelations that are empirically relevant for hedge-fund returns—a difference in expected returns of 20% per month or 240% per year, and probabilities p and q that are either both 80% or higher, or both 20% or lower. Given the implausibility of these parameter values, we conclude that time-varying expected returns—at least of this form—may not be the most likely explanation for serial correlation in hedge-fund returns.

2.3.2 Time-Varying Leverage

Another possible source of serial correlation in hedge-fund returns is time-varying leverage. Since leverage directly affects the expected return of any investment strategy, this can be considered a special case of the time-varying expected returns model of Section 2.3.1. Specifically, if L_t denotes a hedge fund's leverage ratio, then the actual return R_t^o of the fund at date t is given by:

$$R_t^o = L_t R_t \tag{10}$$

where R_t is the fund's unlevered return.⁶ For example if a fund's unlevered strategy yields a 2% return in a given month, but 50% of the funds are borrowed from various counterparties at fixed borrowing rates, the return to the fund's investors is approximately 4%,⁷ hence the leverage ratio is 2.

The specific mechanisms by which a hedge fund determines its leverage can be quite complex and often depend on a number of factors including market volatility,

⁶For simplicity, and with little loss in generality, we have ignored the borrowing costs associated with leverage in our specification (10). Although including such costs will obviously reduce the net return, the serial correlation properties will be largely unaffected because the time variation in borrowing rates is not significant relative to R_t and L_t .

⁷Less the borrowing rate, of course, which we assume is 0 for simplicity.

ρ_1 (%)		q (%)								
		10	20	30	40	50	60	70	80	90
$ \mu_1 - \mu_0 = 5\%$, $\text{Var}[R_t] = (20\%)^2/12$										
p (%)	10	-15.0	-13.1	-11.1	-9.0	-6.9	-4.8	-2.8	-1.1	0.0
	20	-13.1	-11.3	-9.3	-7.3	-5.3	-3.3	-1.5	0.0	0.7
	30	-11.1	-9.3	-7.5	-5.6	-3.6	-1.7	0.0	1.3	1.6
	40	-9.0	-7.3	-5.6	-3.8	-1.9	0.0	1.7	2.8	2.8
	50	-6.9	-5.3	-3.6	-1.9	0.0	1.9	3.5	4.6	4.2
	60	-4.8	-3.3	-1.7	0.0	1.9	3.8	5.5	6.7	6.0
	70	-2.8	-1.5	0.0	1.7	3.5	5.5	7.5	9.0	8.4
	80	-1.1	0.0	1.3	2.8	4.6	6.7	9.0	11.3	11.7
	90	0.0	0.7	1.6	2.8	4.2	6.0	8.4	11.7	15.0
$ \mu_1 - \mu_0 = 5\%$, $\text{Var}[R_t] = (50\%)^2/12$										
p (%)	10	-2.4	-2.1	-1.8	-1.4	-1.1	-0.8	-0.5	-0.2	0.0
	20	-2.1	-1.8	-1.5	-1.2	-0.9	-0.5	-0.2	0.0	0.1
	30	-1.8	-1.5	-1.2	-0.9	-0.6	-0.3	0.0	0.2	0.3
	40	-1.4	-1.2	-0.9	-0.6	-0.3	0.0	0.3	0.5	0.4
	50	-1.1	-0.9	-0.6	-0.3	0.0	0.3	0.6	0.7	0.7
	60	-0.8	-0.5	-0.3	0.0	0.3	0.6	0.9	1.1	1.0
	70	-0.5	-0.2	0.0	0.3	0.6	0.9	1.2	1.4	1.4
	80	-0.2	0.0	0.2	0.5	0.7	1.1	1.4	1.8	1.9
	90	0.0	0.1	0.3	0.4	0.7	1.0	1.4	1.9	2.4
$ \mu_1 - \mu_0 = 20\%$, $\text{Var}[R_t] = (50\%)^2/12$										
p (%)	10	-38.4	-33.5	-28.4	-23.0	-17.6	-12.3	-7.2	-2.9	0.0
	20	-33.5	-28.8	-23.9	-18.8	-13.6	-8.5	-3.8	0.0	1.9
	30	-28.4	-23.9	-19.2	-14.3	-9.3	-4.4	0.0	3.3	4.2
	40	-23.0	-18.8	-14.3	-9.6	-4.8	0.0	4.3	7.2	7.1
	50	-17.6	-13.6	-9.3	-4.8	0.0	4.7	9.0	11.8	10.7
	60	-12.3	-8.5	-4.4	0.0	4.7	9.6	14.1	17.1	15.4
	70	-7.2	-3.8	0.0	4.3	9.0	14.1	19.2	23.0	21.6
	80	-2.9	0.0	3.3	7.2	11.8	17.1	23.0	28.8	29.9
	90	0.0	1.9	4.2	7.1	10.7	15.4	21.6	29.9	38.4

Table 2.1: First-order autocorrelation coefficients of returns from a two-state Markov model of time-varying expected returns, $R_t = \mu_1 I_t + \mu_0(1 - I_t) + \epsilon_t$, where $p \equiv \text{Prob}(I_{t+1} = 1|I_t = 1)$, $q \equiv \text{Prob}(I_{t+1} = 0|I_t = 0)$, μ_1 and μ_0 are the monthly expected returns in states 1 and 0, respectively, and $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and σ_ϵ^2 is calibrated to fix the unconditional variance $\text{Var}[R_t]$ of returns at a prespecified level.

credit risk, and various constraints imposed by investors, regulatory bodies, banks, brokers, and other counterparties. But the basic motivation for typical leverage dynamics is the well-known trade-off between risk and expected return: by increasing its leverage ratio, a hedge fund boosts its expected returns proportionally, but also increases its return volatility and, eventually, its credit risk or risk of default. Therefore, counterparties providing credit facilities for hedge funds will impose some ceiling on the degree of leverage they are willing to provide. More importantly, as market prices move against a hedge fund's portfolio, thereby reducing the value of the fund's collateral and increasing its leverage ratio, or as markets become more volatile and the fund's risk exposure increases significantly, creditors (and, in some cases, securities regulations) will require the fund to either post additional collateral or liquidate a portion of its portfolio to bring the leverage ratio back down to an acceptable level. As a result, the leverage ratio of a typical hedge fund varies through time in a specific manner, usually as a function of market prices and market volatility. Therefore we propose a simple data-dependent mechanism through which a hedge fund determines its ideal leverage ratio.

Denote by R_t the return of a fund in the absence of any leverage, and to focus squarely on the ability of leverage to generate serial correlation, let R_t be IID through time, hence:

$$R_t = \mu + \epsilon_t, \quad \epsilon_t \text{ IID } \mathcal{N}(0, \sigma_\epsilon^2) \quad (11)$$

where we have assumed that ϵ_t is normally distributed only for expositional convenience.⁸ Given (10), the k -th order autocorrelation coefficient of leveraged returns R_t^o is:

$$\rho_k = \frac{1}{\text{Var}[R_t^o]} \left[\mu^2 \text{Cov}[L_t, L_{t+k}] + \mu \text{Cov}[L_t, L_{t+k}\epsilon_{t+k}] + \mu \text{Cov}[L_{t+k}, L_t\epsilon_t] + \text{Cov}[L_t\epsilon_t, L_{t+k}\epsilon_{t+k}] \right]. \quad (12)$$

Now suppose that the leverage process L_t is independently distributed through time

⁸Other distributions can easily be used instead of the normal in the Monte Carlo simulation experiment described below.

and also independent of ϵ_{t+k} for all k . Then (12) implies that $\rho_k = 0$ for all $k \neq 0$, hence time-varying leverage of this sort will not induce any serial correlation in returns R_t^o .

However, as discussed above, leverage is typically a function of market conditions, which can induce serial dependence in L_t and dependence between L_{t+k} and ϵ_t for $k \geq 0$, yielding serially correlated observed returns R_t^o .

To see how, we propose a simple but realistic mechanism by which a hedge fund might manage its leverage ratio. Suppose that, as part of its enterprise-wide risk management protocol, a fund has adopted a policy of limiting the 95% Value-at-Risk of its portfolio to no worse than δ —for example, if $\delta = -10\%$, this policy requires managing the portfolio so that the probability of a loss greater than or equal to 10% of the fund's assets is at most 5%. If we assume that the only control variable available to the manager is the leverage ratio L_t and that unleveraged returns R_t are given by (11), this implies the following constraint on leverage:

$$\text{Prob}(R_t^o \leq \delta) \leq 5\% \quad , \quad \delta \leq 0$$

$$\text{Prob}(L_t R_t \leq \delta) \leq 5\%$$

$$\text{Prob}\left(\frac{R_t - \mu}{\sigma} \leq \frac{\delta/L_t - \mu}{\sigma}\right) \leq 5\%$$

$$\Phi\left(\frac{\delta/L_t - \mu}{\sigma}\right) \leq 5\% \tag{13}$$

$$\delta/L_t \leq \sigma\Phi^{-1}(5\%) \tag{14}$$

$$\Rightarrow L_t \leq \frac{\delta}{\sigma\Phi^{-1}(5\%)} \tag{15}$$

where, following common industry practice, we have set $\mu = 0$ in (13) to arrive at (14).⁹ Now in implementing the constraint (15), the manager must estimate the portfolio volatility σ , which is typically estimated using some rolling window of historical data, hence the manager's estimate is likely to be time-varying but persistent

⁹Setting the expected return of a portfolio equal to 0 for purposes of risk management is often motivated by a desire to be conservative. Most portfolios will tend to have positive expected return, hence setting μ equal to 0 will generally yield larger values for VaR. However, for actively managed portfolios that contain both long and short positions, Lo (2002) shows that the practice of setting expected returns equal to 0 need not be conservative, but in some cases, can yield severely downward-biased estimates of VaR.

to some degree. This persistence, and the dependence of the volatility estimate on past returns, will both induce serial correlation in observed returns R_t^o . Specifically, let:

$$\hat{\sigma}_t^2 \equiv \frac{1}{n} \sum_{k=1}^n (R_{t-k} - \hat{\mu})^2 \quad , \quad \hat{\mu}_t \equiv \frac{1}{n} \sum_{k=1}^n R_{t-k} \quad (16)$$

$$L_t = \frac{\delta}{\hat{\sigma}_t \Phi^{-1}(5\%)} \quad (17)$$

where we have assumed that the manager sets his leverage ratio L_t to the maximum allowable level subject to the VaR constraint (15).

To derive the impact of this heuristic risk management policy on the serial correlation of observed returns, we perform a Monte Carlo simulation experiment where we simulate a time series of 100,000 returns $\{R_t\}$ and implement the leverage policy (17) to obtain a time series of observed returns $\{R_t^o\}$, from which we compute its autocorrelation coefficients $\{\rho_k\}$. Given the large sample size, our estimate should yield an excellent approximation to the population values of the autocorrelation coefficients. This procedure is performed for the following combinations of parameter values:

$$\begin{aligned} n &= 3, 6, 9, 12, 24, 36, 48, 60 \\ 12\mu &= 5\% \\ \sqrt{12}\sigma &= 10\%, 20\%, 50\% \\ \delta &= -25\% \end{aligned}$$

and the results are summarized in Table 2.2. Note that the autocorrelation of observed returns (12) is homogeneous of degree 0 in δ , hence we need only simulate our return process for one value of δ without loss of generality as far as ρ_k is concerned. Of course, the mean and standard of observed returns and leverage will be affected by our choice of δ . but because these variables are homogeneous of degree 1, we can obtain results for any arbitrary δ simply by rescaling our results for $\delta = -25\%$.

n	Return R_t^o		Leverage L_t		Return R_t^o			Leverage L_t		
	12 Mean (%)	$\sqrt{12}$ SD (%)	Mean	SD	ρ_1 (%)	ρ_2 (%)	ρ_3 (%)	ρ_1 (%)	ρ_2 (%)	ρ_3 (%)
$12\mu = 5\%$, $\sqrt{12}\sigma = 10\%$, $\delta = -25\%$										
3	50.53	191.76	9.52	15.14	0.7	0.3	0.4	17.5	2.9	0.0
6	29.71	62.61	5.73	2.45	0.1	0.4	0.3	70.6	48.5	32.1
12	24.34	51.07	4.96	1.19	0.1	0.4	-0.3	88.9	78.6	68.8
24	24.29	47.27	4.66	0.71	0.3	0.1	-0.2	95.0	90.0	85.1
36	21.46	46.20	4.57	0.57	-0.2	0.0	0.1	96.9	93.9	90.9
48	22.67	45.61	4.54	0.46	0.3	-0.5	0.3	97.6	95.3	92.9
60	22.22	45.38	4.51	0.43	-0.2	0.0	0.2	98.2	96.5	94.7
$12\mu = 5\%$, $\sqrt{12}\sigma = 20\%$, $\delta = -25\%$										
3	26.13	183.78	4.80	8.02	0.0	-0.1	0.0	13.4	1.9	-0.6
6	14.26	62.55	2.87	1.19	0.2	0.1	0.4	70.7	48.6	32.0
12	12.95	50.99	2.48	0.59	0.2	-0.1	0.1	89.1	79.0	69.4
24	11.58	47.22	2.33	0.36	0.2	0.0	0.1	95.2	90.4	85.8
36	11.23	46.14	2.29	0.28	-0.1	0.3	-0.3	97.0	94.0	90.9
48	11.00	45.63	2.27	0.24	0.2	-0.5	-0.1	97.8	95.5	93.3
60	12.18	45.37	2.26	0.21	0.1	-0.1	0.4	98.3	96.5	94.8
$12\mu = 5\%$, $\sqrt{12}\sigma = 50\%$, $\delta = -25\%$										
3	9.68	186.59	1.93	3.42	-1.1	0.0	-0.5	14.7	1.8	-0.1
6	6.25	62.43	1.16	0.48	-0.2	0.3	-0.2	70.9	49.4	32.9
12	5.90	50.94	0.99	0.23	-0.1	0.1	0.0	89.0	78.6	69.0
24	5.30	47.29	0.93	0.15	0.2	0.3	0.4	95.2	90.5	85.7
36	5.59	46.14	0.92	0.12	-0.1	0.3	-0.2	97.0	94.1	91.1
48	4.07	45.64	0.91	0.10	-0.4	-0.6	0.1	97.8	95.7	93.5
60	5.11	45.34	0.90	0.08	0.4	0.3	-0.3	98.2	96.5	94.7

Table 2.2: Monte Carlo simulation results for time-varying leverage model with a VaR constraint. Each row corresponds to a separate and independent simulation of 100,000 observations of independently and identically distributed $\mathcal{N}(\mu, \sigma^2)$ returns R_t which are multiplied by a time-varying leverage factor L_t to generated observed returns $R_t^o \equiv L_t R_t$.

For a VaR constraint of -25% and an annual standard deviation of unlevered returns of 10% , the mean leverage ratio ranges from 9.52 when $n = 3$ to 4.51 when $n = 60$. For small n , there is considerably more sampling variation in the estimated standard deviation of returns, hence the leverage ratio—which is proportional to the reciprocal of $\hat{\sigma}_t$ —takes on more extreme values as well and has a higher expectation in this case.

As n increases, the volatility estimator becomes more stable over time since each month’s estimator has more data in common with the previous month’s estimator, leading to more persistence in L_t as expected. For example, when $n = 3$, the average first-order autocorrelation coefficient of L_t is 43.2% , but increases to 98.2% when $n = 60$. However, even with such extreme levels of persistence in L_t , the autocorrelation induced in observed returns R_t^o is still only -0.2% . In fact, the largest absolute return-autocorrelation reported in Table 2.2 is only 0.7% , despite the fact that leverage ratios are sometimes nearly perfectly autocorrelated from month to month. This suggests that time-varying leverage, at least of the form described by the VaR constraint (15), cannot fully account for the magnitudes of serial correlation in historical hedge-fund returns.

2.3.3 Incentive Fees with High-Water Marks

Yet another source of serial correlation in hedge-fund returns is an aspect of the fee structure that is commonly used in the hedge-fund industry: an incentive fee—typically 20% of excess returns above a benchmark—which is subject to a “high-water mark”, meaning that incentive fees are paid only if the cumulative returns of the fund are “above water”, i.e., if they exceed the *cumulative* return of the benchmark since inception.¹⁰ This type of nonlinearity can induce serial correlation in net-of-fee returns because of the path dependence inherent in the definition of the high-water mark—when the fund is “below water” the incentive fee is not charged, but over time, as

¹⁰For more detailed analyses of high water marks and other incentive-fee arrangements in the context of delegated portfolio management, see Bhattacharya and Pfleiderer (1985), Brown, Goetzmann, and Liang (2002), Carpenter (2000), Carpenter, Dybvig, and Farnsworth (2001), Elton and Gruber (2002), and Goetzmann, Ingersoll, and Ross (1997).

the fund’s cumulative performance rises “above water”, the incentive fee is reinstated and the net-of-fee returns is reduced accordingly.

Specifically, denote by F_t the incentive fee paid to the manager in period t and for simplicity, set the benchmark to 0. Then:

$$F_t \equiv \text{Max} [0 , \gamma(X_{t-1} + R_t)] , \quad \gamma > 0 \quad (18a)$$

$$X_t \equiv \text{Min} [0 , X_{t-1} + R_t] \quad (18b)$$

where X_t is a state variable that is non-zero only when the manager is “under water”, in which case it measures the cumulative losses that must be recovered before an incentive fee is paid. The net-of-fee returns R_t^o are then given by:

$$R_t^o = R_t - F_t = (1-\gamma)R_t + \gamma(X_t - X_{t-1}) \quad (19)$$

which is clearly serially correlated due to the presence of the lagged state variable X_{t-1} .¹¹

Because the high-water mark variable X_t is a nonlinear recursive function of X_{t-1} and R_t , its statistical properties are quite complex and difficult to derive in closed form. Therefore, we perform a Monte Carlo simulation experiment in which we simulate a time series of returns $\{R_t\}$ of length $T=100,000$ where R_t is given by (11), compute the net-of-fee returns $\{R_t^o\}$, and estimate the first-order autocorrelation coefficient ρ_1 . We follow this procedure for each of the combinations of the following parameter values:

$$\begin{aligned} 12\mu &= 5\%, 10\%, 15\%, \dots, 50\% \\ \sqrt{12}\sigma &= 10\%, 20\%, \dots, 50\% \\ \gamma &= 20\% . \end{aligned}$$

¹¹This is a simplified model of how a typical hedge fund’s incentive fee is structured. In particular, (18) ignores the fact that incentive fees are usually paid on an annual or quarterly basis whereas high-water marks are tracked on a monthly basis. Using the more realistic fee cycle did not have significant impact on our simulation results. hence we use (18) for expositional simplicity. Also, some funds do pay their employees and partners monthly incentive compensation, in which case (18) is the exact specification of their fee structure.

Table 2.3 summarizes the results of the simulations which show that although incentive fees with high-water marks do induce some serial correlation in net-of-fee returns, they are generally quite small in absolute value. For example, the largest absolute value of all the entries in Table 2.3 is only 4.4%. Moreover, all of the averages are negative, a result of the fact that all of the serial correlation in R_t^o is due to the first difference of X_t in (19). This implies that incentive fees with high-water marks are even less likely to be able to explain the large positive serial correlation in historical hedge-fund returns.

ρ_1 (%)		12μ (%)									
		5	10	15	20	25	30	35	40	45	50
$\sigma \times \sqrt{12}$ (%)	10	-1.4	-2.5	-3.2	-3.4	-3.4	-3.2	-2.9	-2.4	-2.0	-1.5
	20	-1.6	-2.3	-2.9	-3.4	-3.8	-4.1	-4.3	-4.4	-4.4	-4.3
	30	-0.6	-1.1	-1.6	-2.1	-2.4	-2.8	-3.0	-3.3	-3.5	-3.6
	40	-0.2	-0.7	-1.1	-1.4	-1.8	-2.1	-2.3	-2.6	-2.8	-3.0
	50	0.0	-0.3	-0.6	-0.9	-1.2	-1.5	-1.7	-1.9	-2.1	-2.3

Table 2.3: First-order autocorrelation coefficients for Monte Carlo simulation of net-of-fee returns under an incentive fee with a high-water mark. Each entry corresponds to a separate and independent simulation of 100,000 observations of independently and identically distributed $\mathcal{N}(\mu, \sigma^2)$ returns R_t , from which a 20% incentive fee $F_t \equiv \text{Max}[0, 0.2 \times (X_{t-1} + R_t)]$ is subtracted each period to yield net-of-fee returns $R_t^o \equiv R_t - F_t$, where $X_t \equiv \text{Min}[0, X_{t-1} + R_t]$ is a state variable that is non-zero only when the fund is “under water”, in which case it measures the cumulative losses that must be recovered before an incentive fee is paid.

2.4 An Econometric Model of Smoothed Returns

Having shown in Section 2.3 that other possible sources of serial correlation in hedge-fund returns are hard-pressed to yield empirically plausible levels of autocorrelation, we now turn to the main focus of this study: illiquidity and smoothed returns. Although illiquidity and smoothed returns are two distinct phenomena, it is important to consider them in tandem because one facilitates the other –for actively traded securities, both theory and empirical evidence suggest that in the absence of transactions costs and other market frictions, returns are unlikely to be very smooth.

As we argued in Section 2.1, nonsynchronous trading is a plausible source of serial correlation in hedge-fund returns. In contrast to the studies by Lo and MacKinlay (1988, 1990) and Kadlec and Patterson (1999) in which they conclude that it is difficult to generate serial correlations in weekly US equity portfolio returns much greater than 10% to 15% through nonsynchronous trading effects alone, we argue that in the context of hedge funds, significantly higher levels of serial correlation can be explained by the combination of illiquidity and smoothed returns, of which nonsynchronous trading is a special case. To see why, note that the empirical analysis in the nonsynchronous-trading literature is devoted exclusively to exchange-traded equity returns, not hedge-fund returns, hence their conclusions may not be relevant in our context. For example, Lo and MacKinlay (1990) argue that securities would have to go without trading for several days on average to induce serial correlations of 30%, and they dismiss such nontrading intervals as unrealistic for most exchange-traded US equity issues. However, such nontrading intervals are considerably more realistic for the types of securities held by many hedge funds, e.g., emerging-market debt, real estate, restricted securities, control positions in publicly traded companies, asset-backed securities, and other exotic OTC derivatives. Therefore, nonsynchronous trading of this magnitude is likely to be an explanation for the serial correlation observed in hedge-fund returns.

But even when prices are synchronously measured—as they are for many funds that mark their portfolios to market at the end of the month to strike a net-asset-value at which investors can buy into or cash out of the fund—there are several other channels by which illiquidity exposure can induce serial correlation in the reported returns of hedge funds. Apart from the nonsynchronous-trading effect, naive methods for determining the fair market value or “marks” for illiquid securities can yield serially correlated returns. For example, one approach to valuing illiquid securities is to extrapolate linearly from the most recent transaction price (which, in the case of emerging-market debt, might be several months ago), which yields a price path that is a straight line, or at best a series of straight lines. Returns computed from such marks will be smoother, exhibiting lower volatility and higher serial correlation

than true economic returns, i.e., returns computed from mark-to-market prices where the market is sufficiently active to allow all available information to be impounded in the price of the security. Of course, for securities that are more easily traded and with deeper markets, mark-to-market prices are more readily available, extrapolated marks are not necessary, and serial correlation is therefore less of an issue. But for securities that are thinly traded, or not traded at all for extended periods of time, marking them to market is often an expensive and time-consuming procedure that cannot easily be performed frequently. Therefore, we argue in this paper that serial correlation may serve as a proxy for a fund's liquidity exposure.

Even if a hedge-fund manager does not make use of any form of linear extrapolation to mark the securities in his portfolio, he may still be subject to smoothed returns if he obtains marks from broker-dealers that engage in such extrapolation. For example, consider the case of a conscientious hedge-fund manager attempting to obtain the most accurate mark for his portfolio at month end by getting bid/offer quotes from three independent broker-dealers for every security in his portfolio, and then marking each security at the average of the three quote midpoints. By averaging the quote midpoints, the manager is inadvertently downward-biasing price volatility, and if any of the broker-dealers employ linear extrapolation in formulating their quotes (and many do, through sheer necessity because they have little else to go on for the most illiquid securities), or if they fail to update their quotes because of light volume, serial correlation will also be induced in reported returns.

Finally, a more prosaic channel by which serial correlation may arise in the reported returns of hedge funds is through "performance smoothing", the unsavory practice of reporting only part of the gains in months when a fund has positive returns so as to partially offset potential future losses and thereby reduce volatility and improve risk-adjusted performance measures such as the Sharpe ratio. For funds containing liquid securities that can be easily marked to market, performance smoothing is more difficult and, as a result, less of a concern. Indeed, it is only for portfolios of illiquid securities that managers and brokers have any discretion in marking their positions. Such practices are generally prohibited by various securities laws and

accounting principles, and great care must be exercised in interpreting smoothed returns as deliberate attempts to manipulate performance statistics. After all, as we have discussed above, there are many other sources of serial correlation in the presence of illiquidity, none of which is motivated by deceit. Nevertheless, managers do have certain degrees of freedom in valuing illiquid securities—for example, discretionary accruals for unregistered private placements and venture capital investments—and Chandar and Bricker (2002) conclude that managers of certain closed-end mutual funds do use accounting discretion to manage fund returns around a passive benchmark. Therefore, the possibility of deliberate performance smoothing in the less regulated hedge-fund industry must be kept in mind in interpreting our empirical analysis of smoothed returns.

To quantify the impact of all of these possible sources of serial correlation, denote by R_t the true economic return of a hedge fund in period t , and let R_t satisfy the following linear single-factor model:

$$R_t = \mu + \beta\Lambda_t + \epsilon_t, \quad E[\Lambda_t] = E[\epsilon_t] = 0, \quad \epsilon_t, \Lambda_t \sim \text{IID} \quad (20a)$$

$$\text{Var}[R_t] \equiv \sigma^2. \quad (20b)$$

True returns represent the flow of information that would determine the equilibrium value of the fund's securities in a frictionless market. However, true economic returns are not observed. Instead, R_t^o denotes the reported or observed return in period t , and let

$$R_t^o = \theta_0 R_t + \theta_1 R_{t-1} + \cdots + \theta_k R_{t-k} \quad (21)$$

$$\theta_j \in [0, 1], \quad j = 0, \dots, k \quad (22)$$

$$1 = \theta_0 + \theta_1 + \cdots + \theta_k \quad (23)$$

which is a weighted average of the fund's true returns over the most recent $k+1$ periods, including the current period.

This averaging process captures the essence of smoothed returns in several re-

spects. From the perspective of illiquidity-driven smoothing, (21) is consistent with several models in the nonsynchronous trading literature. For example, Cohen, Maier et al. (1986, Chapter 6.1) propose a similar weighted-average model for observed returns.¹² Alternatively, (21) can be viewed as the outcome of marking portfolios to simple linear extrapolations of acquisition prices when market prices are unavailable, or “mark-to-model” returns where the pricing model is slowly varying through time. And of course, (21) also captures the intentional smoothing of performance.

The constraint (23) that the weights sum to 1 implies that the information driving the fund’s performance in period t will eventually be fully reflected in observed returns, but this process could take up to $k+1$ periods from the time the information is generated.¹³ This is a sensible restriction in the current context of hedge funds for several reasons. Even the most illiquid securities will trade eventually, and when that occurs, all of the cumulative information affecting that security will be fully impounded into its transaction price. Therefore the parameter k should be selected to match the kind of illiquidity of the fund—a fund comprised mostly of exchange-traded US equities would require a much lower value of k than a private equity fund. Alternatively, in the case of intentional smoothing of performance, the necessity of periodic external audits of fund performance imposes a finite limit on the extent to which deliberate smoothing can persist.¹⁴

¹²In particular, their specification for observed returns is:

$$r_{j,t}^o = \sum_{l=0}^N (\gamma_{j,t-l,l} r_{j,t-l} + \theta_{j,t-l})$$

where $r_{j,t-l}$ is the true but unobserved return for security j in period $t-l$, the coefficients $\{\gamma_{j,t-l,l}\}$ are assumed to sum to 1, and $\theta_{j,t-l}$ are random variables meant to capture “bid/ask bounce”. The authors motivate their specification of nonsynchronous trading in the following way (p. 116): “Alternatively stated, the $\gamma_{j,t,0}, \gamma_{j,t,1}, \dots, \gamma_{j,t,N}$ comprise a delay distribution that shows how the true return generated in period t impacts on the returns actually observed during t and the next N periods”. In other words, the essential feature of nonsynchronous trading is the fact that information generated at date t may not be fully impounded into prices until several periods later.

¹³In Lo and MacKinlay’s (1990) model of nonsynchronous trading, they propose a stochastic non-trading horizon so that observed returns are an infinite-order moving average of past true returns, where the coefficients are stochastic. In that framework, the waiting time for information to become fully impounded into future returns may be arbitrarily long (but with increasingly remote probability).

¹⁴In fact, if a fund allows investors to invest and withdraw capital only at pre-specified intervals, imposing lock-ups in between, and external audits are conducted at these same pre-specified intervals.

2.4.1 Implications For Performance Statistics

Given the smoothing mechanism outlined above, we have the following implications for the statistical properties of observed returns:

Proposition 1 *Under (21)–(23), the statistical properties of observed returns are characterized by:*

$$E[R_t^o] = \mu \quad (24)$$

$$\text{Var}[R_t^o] = c_\sigma^2 \sigma^2 \leq \sigma^2 \quad (25)$$

$$\text{SR}^o \equiv \frac{E[R_t^o]}{\sqrt{\text{Var}[R_t^o]}} = c_s \text{SR} \geq \text{SR} \equiv \frac{E[R_t]}{\sqrt{\text{Var}[R_t]}} \quad (26)$$

$$\beta_m^o \equiv \frac{\text{Cov}[R_t^o, \Lambda_{t-m}]}{\text{Var}[\Lambda_{t-m}]} = \begin{cases} c_{\beta,m} \beta & \text{if } 0 \leq m \leq k \\ 0 & \text{if } m > k \end{cases} \quad (27)$$

$$\text{Cov}[R_t^o, R_{t-m}^o] = \begin{cases} \left(\sum_{j=0}^{k-m} \theta_j \theta_{j+m} \right) \sigma^2 & \text{if } 0 \leq m \leq k \\ 0 & \text{if } m > k \end{cases} \quad (28)$$

$$\text{Corr}[R_t^o, R_{t-m}^o] = \frac{\text{Cov}[R_t^o, R_{t-m}^o]}{\text{Var}[R_t^o]} = \begin{cases} \frac{\sum_{j=0}^{k-m} \theta_j \theta_{j+m}}{\sum_{j=0}^k \theta_j^2} & \text{if } 0 \leq m \leq k \\ 0 & \text{if } m > k \end{cases} \quad (29)$$

then it may be argued that performance smoothing is irrelevant. For example, no investor should be disadvantaged by investing in a fund that offers annual liquidity and engages in annual external audits with which the fund's net-asset-value is determined by a disinterested third party for purposes of redemptions and new investments. However, there are at least two additional concerns that remain—historical track records are still affected by smoothed returns, and estimates of a fund's liquidity exposure are also affected—both of which are important factors in the typical hedge-fund investor's overall investment process. Moreover, given the apparently unscrupulous role that the auditors at Arthur Andersen played in the Enron affair, there is the further concern of whether third-party auditors are truly objective and free of all conflicts of interest.

where:

$$c_\mu \equiv \theta_0 + \theta_1 + \dots + \theta_k \quad (30)$$

$$c_\sigma^2 \equiv \theta_0^2 + \theta_1^2 + \dots + \theta_k^2 \quad (31)$$

$$c_s \equiv 1/\sqrt{\theta_0^2 + \dots + \theta_k^2} \quad (32)$$

$$c_{\beta,m} \equiv \theta_m \quad (33)$$

Proposition 1 shows that smoothed returns of the form (21)–(23) do not affect the expected value of R_t^o but reduce its variance, hence boosting the Sharpe ratio of observed returns by a factor of c_s . From (27), we see that smoothing also affects β_0^o , the contemporaneous market beta of observed returns, biasing it towards 0 or “market neutrality”, and induces correlation between current observed returns and lagged market returns up to lag k . This provides a formal interpretation of the empirical analysis of Asness, Krail, and Liew (2001) in which many hedge funds were found to have significant lagged market exposure despite relatively low contemporaneous market betas.

Smoothed returns also exhibit positive serial correlation up to order k according to (29), and the magnitude of the effect is determined by the pattern of weights $\{\theta_j\}$. If, for example, the weights are disproportionately centered on a small number of lags, relatively little serial correlation will be induced. However, if the weights are evenly distributed among many lags, this will result in higher serial correlation. A useful summary statistic for measuring the concentration of weights is

$$\xi \equiv \sum_{j=0}^k \theta_j^2 \in [0, 1] \quad (34)$$

which is simply the denominator of (29). This measure is well known in the industrial organization literature as the *Herfindahl index*, a measure of the concentration of firms in a given industry where θ_j represents the market share of firm j . Because $\theta_j \in [0, 1]$, ξ is also confined to the unit interval, and is minimized when all the θ_j 's are identical, which implies a value of $1/(k+1)$ for ξ , and is maximized when one coefficient is 1

and the rest are 0, in which case $\xi = 1$. In the context of smoothed returns, a lower value of ξ implies more smoothing, and the upper bound of 1 implies no smoothing, hence we shall refer to ξ as a “smoothing index”.

In the special case of equal weights, $\theta_j = 1/(k+1)$ for $j = 0, \dots, k$, the serial correlation of observed returns takes on a particularly simple form:

$$\text{Corr}[R_t^o, R_{t-m}^o] = 1 - \frac{m}{k+1}, \quad 1 \leq m \leq k \quad (35)$$

which declines linearly in the lag m . This can yield substantial correlations even when k is small—for example, if $k = 2$ so that smoothing takes place only over a current quarter (i.e. this month and the previous two months), the first-order autocorrelation of monthly observed returns is 66.7%.

To develop a sense for just how much observed returns can differ from true returns under the smoothed-return mechanism (21)–(23), denote by $\Delta(T)$ the difference between the cumulative observed and true returns over T holding periods, where we assume that $T > k$:

$$\Delta(T) \equiv (R_1^o + R_2^o + \dots + R_T^o) - (R_1 + R_2 + \dots + R_T) \quad (36)$$

$$= \sum_{j=0}^{k-1} (R_{-j} - R_{T-j}) \left(1 - \sum_{i=0}^j \theta_i\right) \quad (37)$$

Then we have:

Proposition 2 *Under (21)–(23) and for $T > k$,*

$$E[\Delta(T)] = 0 \quad (38)$$

$$\text{Var}[\Delta(T)] = 2\sigma^2 \sum_{j=0}^{k-1} \left(1 - \sum_{l=0}^j \theta_l\right)^2 = 2\sigma^2 \zeta \quad (39)$$

$$\zeta \equiv \sum_{j=0}^{k-1} \left(1 - \sum_{l=0}^j \theta_l\right)^2 \leq k \quad (40)$$

Proposition 2 shows that the cumulative difference between observed and true returns has 0 expected value, and its variance is bounded above by $2k\sigma^2$.

2.4.2 Examples of Smoothing Profiles

To develop further intuition for the impact of smoothed returns on observed returns, we consider the following three specific sets of weights $\{\theta_j\}$ or “smoothing profiles”:¹⁵

$$\theta_j = \frac{1}{k+1} \quad (\text{Straightline}) \quad (41)$$

$$\theta_j = \frac{k+1-j}{(k+1)(k+2)/2} \quad (\text{Sum-of-Years}) \quad (42)$$

$$\theta_j = \frac{\delta^j(1-\delta)}{1-\delta^{k+1}}, \quad \delta \in (0,1) \quad (\text{Geometric}) . \quad (43)$$

The straightline profile weights each return equally. In contrast, the sum-of-years and geometric profiles weight the current return the most heavily, and then has monotonically declining weights for lagged returns, with the sum-of-years weights declining linearly and the geometric weights declining more rapidly (see Figure 2-2).

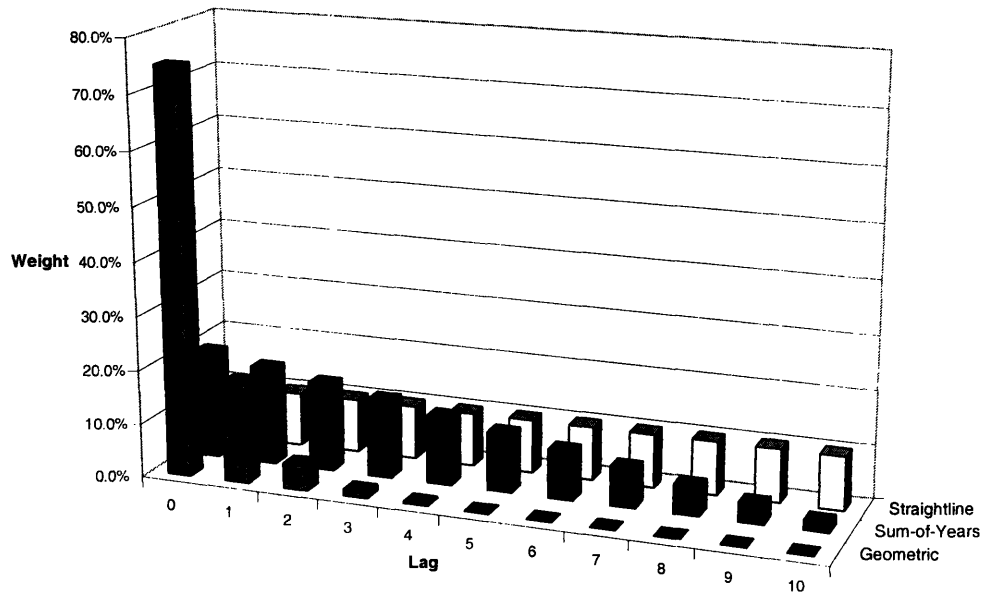


Figure 2-2: Straightline, sum-of-years, and geometric smoothing profiles for $k=10$.

More detailed information about the three smoothing profiles is contained in Table

¹⁵Students of accounting will recognize these profiles as commonly used methods for computing depreciation. The motivation for these depreciation schedules is not entirely without relevance in the smoothed-return context.

2.4. The first panel reports the smoothing coefficients $\{\theta_j\}$, constants $c_{\beta,0}$, c_σ , c_s , ζ , and the first three autocorrelations of observed returns for the straightline profile for $k = 0, 1, \dots, 5$. Consider the case where $k=2$. Despite the relatively short smoothing period of three months, the effects are dramatic: smoothing reduces the market beta by 67%, increases the Sharpe ratio by 73%, and induces first- and second-order serial correlation of 67% and 33%, respectively, in observed returns. Moreover, the variance of the cumulative discrepancy between observed and true returns, $2\sigma^2\zeta$, is only slightly larger than the variance of monthly true returns σ^2 , suggesting that it may be difficult to detect this type of smoothed returns even over time.

As k increases, the effects become more pronounced—for $k=5$, the market beta is reduced by 83%, the Sharpe ratio is increased by 145%, and first three autocorrelation coefficients are 83%, 67%, and 50%, respectively. However, in this extreme case, the variance of the discrepancy between true and observed returns is approximately three times the monthly variance of true returns, in which case it may be easier to identify smoothing from realized returns.

The sum-of-years profile is similar to, although somewhat less extreme than, the straightline profile for the same values of k because more weight is being placed on the current return. For example, even in the extreme case of $k=5$, the sum-of-years profile reduces the market beta by 71%, increases the Sharpe ratio by 120%, induces autocorrelations of 77%, 55%, and 35%, respectively, in the first three lags, and has a discrepancy variance that is approximately 1.6 times the monthly variance of true returns.

The last two panels of Table 2.4 contain results for the geometric smoothing profile for two values of δ , 0.25 and 0.50. For $\delta = 0.25$, the geometric profile places more weight on the current return than the other two smoothing profiles for all values of k , hence the effects tend to be less dramatic. Even in the extreme case of $k = 5$, 75% of current true returns are incorporated into observed returns, the market beta is reduced by only 25%, the Sharpe ratio is increased by only 29%, the first three autocorrelations are 25%, 6%, and 1% respectively, and the discrepancy variance is approximately 13% of the monthly variance of true returns. As δ increases, less weight

k	θ_0 (%)	θ_1 (%)	θ_2 (%)	θ_3 (%)	θ_4 (%)	θ_5 (%)	c_β	c_σ	c_s	ρ_1^o (%)	ρ_2^o (%)	ρ_3^o (%)	ρ_4^o (%)	ρ_5^o (%)	ζ (%)
<i>Straightline Smoothing</i>															
0	100.0	—	—	—	—	—	1.00	1.00	1.00	0.0	0.0	0.0	0.0	0.0	—
1	50.0	50.0	—	—	—	—	0.50	0.71	1.41	50.0	0.0	0.0	0.0	0.0	25.0
2	33.3	33.3	33.3	—	—	—	0.33	0.58	1.73	66.7	33.3	0.0	0.0	0.0	55.6
3	25.0	25.0	25.0	25.0	—	—	0.25	0.50	2.00	75.0	50.0	25.0	0.0	0.0	87.5
4	20.0	20.0	20.0	20.0	20.0	—	0.20	0.45	2.24	80.0	60.0	40.0	20.0	0.0	120.0
5	16.7	16.7	16.7	16.7	16.7	16.7	0.17	0.41	2.45	83.3	66.7	50.0	33.3	16.7	152.8
<i>Sum-of-Years Smoothing</i>															
0	100.0	—	—	—	—	—	1.00	1.00	1.00	0.0	0.0	0.0	0.0	0.0	—
1	66.7	33.3	—	—	—	—	0.67	0.75	1.34	40.0	0.0	0.0	0.0	0.0	11.1
2	50.0	33.3	16.7	—	—	—	0.50	0.62	1.60	57.1	21.4	0.0	0.0	0.0	27.8
3	40.0	30.0	20.0	10.0	—	—	0.40	0.55	1.83	66.7	36.7	13.3	0.0	0.0	46.0
4	33.3	26.7	20.0	13.3	6.7	—	0.33	0.49	2.02	72.7	47.3	25.5	9.1	0.0	64.9
5	28.6	23.8	19.0	14.3	9.5	4.8	0.29	0.45	2.20	76.9	54.9	35.2	18.7	6.6	84.1
<i>Geometric Smoothing ($\delta = 0.25$)</i>															
0	100.0	—	—	—	—	—	1.00	1.00	1.00	0.0	0.0	0.0	0.0	0.0	—
1	80.0	20.0	—	—	—	—	0.80	0.82	1.21	23.5	0.0	0.0	0.0	0.0	4.0
2	76.2	19.0	4.8	—	—	—	0.76	0.79	1.27	24.9	5.9	0.0	0.0	0.0	5.9
3	75.3	18.8	4.7	1.2	—	—	0.75	0.78	1.29	25.0	6.2	1.5	0.0	0.0	6.5
4	75.1	18.8	4.7	1.2	0.3	—	0.75	0.78	1.29	25.0	6.2	1.6	0.4	0.0	6.6
5	75.0	18.8	4.7	1.2	0.3	0.1	0.75	0.77	1.29	25.0	6.2	1.6	0.4	0.1	6.7
<i>Geometric Smoothing ($\delta = 0.50$)</i>															
0	100.0	—	—	—	—	—	1.00	1.00	1.00	0.0	0.0	0.0	0.0	0.0	—
1	66.7	33.3	—	—	—	—	0.67	0.75	1.34	40.0	0.0	0.0	0.0	0.0	11.1
2	57.1	28.6	14.3	—	—	—	0.57	0.65	1.53	47.6	19.0	0.0	0.0	0.0	20.4
3	53.3	26.7	13.3	6.7	—	—	0.53	0.61	1.63	49.4	23.5	9.4	0.0	0.0	26.2
4	51.6	25.8	12.9	6.5	3.2	—	0.52	0.60	1.68	49.9	24.6	11.7	4.7	0.0	29.6
5	50.8	25.4	12.7	6.3	3.2	1.6	0.51	0.59	1.71	50.0	24.9	12.3	5.9	2.3	31.4

Table 2.4: Implications of three different smoothing profiles for observed betas, standard deviations, Sharpe ratios, and serial correlation coefficients for a fund with IID true returns. Straightline smoothing is given by $\theta_j = 1/(k+1)$; sum-of-years smoothing is given by $\theta_j = (k+1-j)/[(k+1)(k+2)/2]$; geometric smoothing is given by $\theta_j = \delta^j(1-\delta)/(1-\delta^{k+1})$. c_β , c_σ , and c_s denote multipliers associated with the beta, standard deviation, and Sharpe ratio of observed returns, respectively, ρ_j^o denotes the j -th autocorrelation coefficient of observed returns, and ζ is proportional to the variance of the discrepancy between true and observed multi-period returns.

is placed on the current observation and the effects on performance statistics become more significant. When $\delta = 0.50$ and $k = 5$, geometric smoothing reduces the market beta by 49%, increases the Sharpe ratio by 71%, induces autocorrelations of 50%, 25%, and 12%, respectively, for the first three lags, and yields a discrepancy variance that is approximately 63% of the monthly variance of true returns.

The three smoothing profiles have very different values for ζ in (40):

$$\zeta = \frac{k(2k+1)}{6(k+1)} \quad (44)$$

$$\zeta = \frac{k(3k^2+6k+1)}{15(k+1)(k+2)} \quad (45)$$

$$\zeta = \frac{\delta^2(-1 + \delta^k(2 + 2\delta + \delta^k(-1 - 2\delta + k(\delta^2 - 1))))}{(\delta^2 - 1)(\delta^{k+1} - 1)^2} \quad (46)$$

with the straightline and sum-of-years profiles implying variances for $\Delta(T)$ that grow approximately linearly in k , and the geometric profile implying a variance for $\Delta(T)$ that asymptotes to a finite limit (see Figure 2-3).

The results in Table 2.4 and Figure 2-3 show that a rich set of biases can be generated by even simple smoothing profiles, and even the most casual empirical observation suggests that smoothed returns may be an important source of serial correlation in hedge-fund returns. To address this issue directly, we propose methods for estimating the smoothing profile in Section 2.5 and apply these methods to the data in Section 2.6.

2.5 Estimation of Smoothing Profiles and Sharpe Ratios

Although the smoothing profiles described in Section 2.4.2 can all be easily estimated from the sample moments of fund returns, e.g., means, variances, and autocorrelations, we wish to be able to estimate more general forms of smoothing. Therefore, in this section we propose two estimation procedures—maximum likelihood and linear regression—that place fewer restrictions on a fund’s smoothing profile than the three

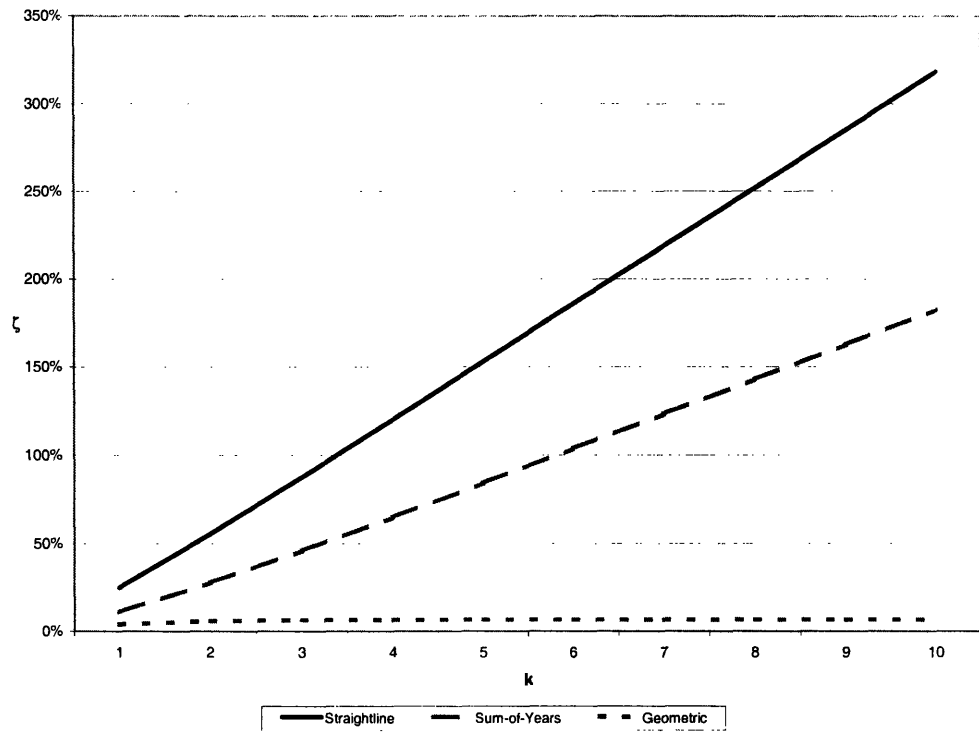


Figure 2-3: Straightline, sum-of-years, and geometric smoothing profiles for $k=10$.

examples in Section 2.4.2. In Section 2.5.1 we review the steps for maximum likelihood estimation of an MA(k) process, slightly modified to accommodate our context and constraints, and in Section 2.5.2 we consider a simpler alternative based on linear regression under the assumption that true returns are generated by the linear single-factor model (20). We propose several specification checks to evaluate the robustness of our smoothing model in Section 2.5.3, and in Section 2.5.4 we show how to adjust Sharpe ratios to take smoothed returns into account.

2.5.1 Maximum Likelihood Estimation

Given the specification of the smoothing process in (21)–(23), we can estimate the smoothing profile using maximum likelihood estimation in a fashion similar to the estimation of standard moving-average time series models (see, for example, Brockwell and Davis, 1991, Chapter 8). We begin by defining the de-measured observed returns process X_t :

$$X_t = R_t^o - \mu \quad (47)$$

and observing that (21)–(23) implies the following properties for X_t :

$$X_t = \theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k} \quad (48)$$

$$1 = \theta_0 + \theta_1 + \cdots + \theta_k \quad (49)$$

$$\eta_k \sim \mathcal{N}(0, \sigma_\eta^2) \quad (50)$$

where, for purposes of estimation, we have added the parametric assumption (50) that η_k is normally distributed. From (48), it is apparent that X_t is a moving-average process of order k , or an “MA(k)”. For a given set of observations $\mathbf{X} \equiv [X_1 \cdots X_T]'$, the likelihood function is well known to be:

$$\mathcal{L}(\boldsymbol{\theta}, \sigma_\eta) = (2\pi)^{-T/2} (\det \boldsymbol{\Gamma})^{-1/2} \exp(-\frac{1}{2} \mathbf{X}' \boldsymbol{\Gamma}^{-1} \mathbf{X}) \quad , \quad \boldsymbol{\Gamma} \equiv \text{E}[\mathbf{X}\mathbf{X}'] \quad (51)$$

where $\boldsymbol{\theta} \equiv [\theta_0 \ \cdots \ \theta_k]'$ and the covariance matrix $\boldsymbol{\Gamma}$ is a function of the parameters $\boldsymbol{\theta}$ and σ_η . It can be shown that for any constant κ ,

$$\mathcal{L}(\kappa\boldsymbol{\theta}, \sigma_\eta/\kappa) = \mathcal{L}(\boldsymbol{\theta}, \sigma_\eta), \quad (52)$$

therefore, an additional identification condition is required. The most common identification condition imposed in the time-series literature is the normalization $\theta_0 \equiv 1$. However, in our context, we impose the condition (49) that the MA coefficients sum to 1—an economic restriction that smoothing takes place over only the most recent $k+1$ periods—and this is sufficient to identify the parameters $\boldsymbol{\theta}$ and σ_η . The likelihood function (51) may be then evaluated and maximized via the “innovations algorithm” of Brockwell and Davis (1991, Chapter 8.3),¹⁶ and the properties of the estimator are given by:

Proposition 3 *Under the specification (48)–(50), X_t is invertible on the set $\{\boldsymbol{\theta} : \theta_0 + \theta_1 + \theta_2 = 1, \theta_1 < 1/2, \theta_1 < 1 - 2\theta_2\}$, and the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ satisfies the following properties:*

¹⁶Specifically, let $\hat{\mathbf{X}} = [\hat{X}_1 \ \cdots \ \hat{X}_T]'$ where $\hat{X}_1 = 0$ and $\hat{X}_j = E[X_j|X_1, \dots, X_{j-1}]$, $j \geq 2$. Let $r_t = E[(X_{t+1} - \hat{X}_{t+1})^2]/\sigma_\eta^2$. Brockwell and Davis (1991) show that (51) can be rewritten as:

$$\mathcal{L}(\boldsymbol{\theta}, \sigma_\eta^2) = (2\pi\sigma_\eta^2)^{-T/2} (r_0 \cdots r_{T-1})^{-1/2} \exp \left[-\frac{1}{2}\sigma_\eta^2 \sum_{t=1}^T (X_t - \hat{X}_t)^2 / r_{t-1} \right] \quad (53)$$

where the one-step-ahead predictors \hat{X}_t and their normalized mean-squared errors r_{t-1} , $t = 1, \dots, T$ are calculated recursively according to the formulas given in Brockwell and Davis (1991, Proposition 5.2.2). Taking the derivative of (53) with respect to σ_η^2 , we see that the maximum likelihood estimator $\hat{\sigma}_\eta^2$ is given by:

$$\hat{\sigma}_\eta^2 = S(\boldsymbol{\theta}) = T^{-1} \sum_{t=1}^T (X_t - \hat{X}_t)^2 / r_{t-1} \quad (54)$$

hence we can “concentrate” the likelihood function by substituting (54) into (53) to obtain:

$$\mathcal{L}_o(\boldsymbol{\theta}) = \log S(\boldsymbol{\theta}) + T^{-1} \sum_{t=1}^T \log r_{t-1} \quad (55)$$

which can be minimized in $\boldsymbol{\theta}$ subject to the constraint (49) using standard numerical optimization packages (we use Matlab’s Optimization Toolbox in our empirical analysis). Maximum likelihood estimates obtained in this fashion need not yield an invertible MA(k) process, but it is well known that any non-invertible process can always be transformed into an invertible one simply by adjusting the parameters σ_η^2 and $\boldsymbol{\theta}$. To address this identification problem, we impose the additional restriction that the estimated MA(k) process be invertible.

$$1 = \hat{\theta}_0 + \hat{\theta}_1 + \hat{\theta}_2 \quad (56)$$

$$\sqrt{T} \left(\begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} - \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \right) \stackrel{a}{\sim} \mathcal{N}(0, \mathbf{V}_\theta) \quad (57)$$

$$\begin{bmatrix} -(-1 + \theta_1)(-1 + 2\theta_1)(-1 + \theta_1 + 2\theta_2) & \mathbf{V}_\theta - \theta_2(-1 + 2\theta_1)(-1 + \theta_1 + 2\theta_2) \\ -\theta_2(-1 + 2\theta_1)(-1 + \theta_1 + 2\theta_2) & (-1 + \theta_1 - 2(-1 + \theta_2)\theta_2)(-1 + \theta_1 + 2\theta_2) \end{bmatrix} \quad (58)$$

By applying the above procedure to observed de-meaned returns, we may obtain estimates of the smoothing profile $\hat{\theta}$ for each fund.¹⁷ Because of the scaling property (52) of the MA(k) likelihood function, a simple procedure for obtaining estimates of our smoothing model with the normalization (49) is to transform estimates $(\check{\theta}, \check{\sigma})$ from standard MA(k) estimation packages such as SAS or RATS by dividing each $\check{\theta}_i$ by $1 + \check{\theta}_1 + \dots + \check{\theta}_k$ and multiplying $\check{\sigma}$ by the same factor. The likelihood function remains unchanged but the transformed smoothing coefficients will now satisfy (49).

2.5.2 Linear Regression Analysis

Although we proposed a linear single-factor model (20) in Section 2.4 for true returns so as to derive the implications of smoothed returns for the market beta of observed returns, the maximum likelihood procedure outlined in Section 2.5.1 is designed to estimate the more general specification of IID Gaussian returns, regardless of any factor structure. However, if we are willing to impose (20), a simpler method for estimating the smoothing profile is available. By substituting (20) into (21), we can re-express observed returns as:

$$R_t^o = \mu + \beta(\theta_0\Lambda_t + \theta_1\Lambda_{t-1} + \dots + \theta_k\Lambda_{t-k}) + u_t \quad (59)$$

$$u_t = \theta_0\epsilon_t + \theta_1\epsilon_{t-1} + \dots + \theta_k\epsilon_{t-k} . \quad (60)$$

¹⁷Recall from Proposition 1 that the smoothing process (21)–(23) does not affect the expected return, i.e., the sample mean of observed returns is a consistent estimator of the true expected return. Therefore, we may use $R_t^o - \hat{\mu}$ in place of X_t in the estimation process without altering any of the asymptotic properties of the maximum likelihood estimator.

Suppose we estimate the following linear regression of observed returns on contemporaneous and lagged market returns:

$$R_t^o = \mu + \gamma_0 \Lambda_t + \gamma_1 \Lambda_{t-1} + \cdots + \gamma_k \Lambda_{t-k} + u_t \quad (61)$$

as in Asness, Krail and Liew (2001). Using the normalization (23) from our smoothing model, we can obtain estimators for β and $\{\theta_j\}$ readily:

$$\hat{\beta} = \hat{\gamma}_0 + \hat{\gamma}_1 + \cdots + \hat{\gamma}_k \quad , \quad \hat{\theta}_j = \hat{\gamma}_j / \hat{\beta} . \quad (62)$$

Moreover, a specification check for (59)–(60) can be performed by testing the following set of equalities:

$$\beta = \frac{\gamma_0}{\theta_0} = \frac{\gamma_1}{\theta_1} = \cdots = \frac{\gamma_k}{\theta_k} . \quad (63)$$

Because of serial correlation in u_t , ordinary least squares estimates (62) will not be efficient and the usual standard errors are incorrect, but the estimates are still consistent and may be a useful first approximation for identifying smoothing in hedge-fund returns.¹⁸

There is yet another variation of the linear single-factor model that may help to disentangle the effects of illiquidity from return smoothing.¹⁹ Suppose that a fund's true economic returns R_t satisfies:

$$R_t = \mu + \beta \Lambda_t + \epsilon_t \quad , \quad \epsilon_t \sim \text{IID}(0, \sigma_\epsilon^2) \quad (64)$$

but instead of assuming that the common factor Λ_t is IID as in (20), let Λ_t be serially correlated. While this alternative may seem to be a minor variation of the smoothing model (21)–(23), the difference in interpretation is significant. A serially correlated Λ_t captures the fact that a fund's returns may be autocorrelated because of an illiquid common factor, even in the absence of any smoothing process such as (21)–(23). Of

¹⁸To obtain efficient estimates of the smoothing coefficients, a procedure like the maximum likelihood estimator of Section 2.5.1 must be used.

¹⁹We thank the referee for encouraging us to explore this alternative.

course, this still begs the question of what the ultimate source of serial correlation in the common factor might be, but by combining (64) with the smoothing process (21)–(23), it may be possible to distinguish between “systematic” versus “idiosyncratic” smoothing, the former attributable to the asset class and the latter resulting from fund-specific characteristics.

To see why the combination of (64) and (21)–(23) may have different implications for observed returns, suppose for the moment that there is no smoothing, i.e., $\theta_0 = 1$ and $\theta_k = 0$ for $k > 0$ in (21)–(23). Then observed returns are simply given by:

$$R_t^o = \mu + \beta\Lambda_t + \epsilon_t, \quad \epsilon_t \sim \text{IID}(0, \sigma_\epsilon^2) \quad (65)$$

where R_t^o is now serially correlated solely through Λ_t . This specification implies that the ratios of observed-return autocovariances will be identical across all funds with the same common factor:

$$\frac{\text{Cov}[R_t^o, R_{t-k}^o]}{\text{Cov}[R_t^o, R_{t-l}^o]} = \frac{\beta \text{Cov}[\Lambda_t, \Lambda_{t-k}]}{\beta \text{Cov}[\Lambda_t, \Lambda_{t-l}]} = \frac{\text{Cov}[\Lambda_t, \Lambda_{t-k}]}{\text{Cov}[\Lambda_t, \Lambda_{t-l}]} \quad (66)$$

Moreover, (64) implies that in the regression equation (61), the coefficients of the lagged factor returns are zero and the error term is not serially correlated.

More generally, consider the combination of a serially correlated common factor (64) and smoothed returns (21)–(23). This more general econometric model of observed returns implies that the appropriate specification of the regression equation is:

$$R_t^o = \mu + \gamma_0\Lambda_t + \gamma_1\Lambda_{t-1} + \cdots + \gamma_k\Lambda_{t-k} + u_t \quad (67)$$

$$u_t = \theta_0\epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_k\epsilon_{t-k}, \quad \epsilon_t \sim \text{IID}(0, \sigma_\epsilon^2) \quad (68)$$

$$1 = \theta_0 + \theta_1 + \cdots + \theta_k. \quad (69)$$

To the extent that serial correlation in R_t^o can be explained mainly by the common factor, the lagged coefficient estimates of (67) will be statistically insignificant, the residuals will be serially uncorrelated, and the ratios of autocovariance coefficients

will be roughly constant across funds with the same common factor. To the extent that the smoothing process (21)–(23) is responsible for serial correlation in R_t^o , the lagged coefficient estimates of (67) will be significant, the residuals will be serially correlated, and the ratios $\hat{\gamma}_j/\hat{\theta}_j$ will be roughly the same for all $j \geq 0$ and will be a consistent estimate of the factor loading or beta of the fund’s true economic returns with respect to the factor Λ_t .

Perhaps the most difficult challenge in estimating (67)–(69) is to correctly identify the common factor Λ_t . Unlike a simple market-model regression that is meant to estimate the sensitivity of a fund’s returns to a broad-based market index, the ability to distinguish between the effects of systematic illiquidity and idiosyncratic return smoothing via (67) relies heavily on the correct specification of the common factor. Using a common factor in (67) that is highly serially correlated but not exactly the right factor for a given fund may yield misleading estimates for the degree of smoothing in that fund’s observed returns. Therefore, the common factor Λ_t must be selected or constructed carefully to match the specific risk exposures of the fund, and the parameter estimates of (67) must be interpreted cautiously and with several specific alternative hypotheses at hand. In Section 2.6.4, we provide an empirical example that highlights the pitfalls and opportunities of the common factor specification (67)–(69).

2.5.3 Specification Checks

Although the maximum likelihood estimator proposed in Section 2.5.1 has some attractive properties—it is consistent and asymptotically efficient under certain regularity conditions—it may not perform well in small samples or when the underlying distribution of true returns is not normal as hypothesized.²⁰ Moreover, even if normality is satisfied and a sufficient sample size is available, our proposed smoothing model (21)–(23) may simply not apply to some of the funds in our sample. There-

²⁰There is substantial evidence that financial asset returns are not normally distributed, but characterized by skewness, leptokurtosis, and other non-gaussian properties (see, for example, Lo and MacKinlay, 1999). Given the dynamic nature of hedge-fund strategies, it would be even less plausible for their returns to be normally distributed.

fore, it is important to have certain specification checks in mind when interpreting the empirical results.

The most obvious specification check is whether or not the maximum likelihood estimation procedure, which involves numerical optimization, converges. If not, this is one sign that our model is misspecified, either because of non-normality or because the smoothing process is inappropriate.

A second specification check is whether or not the estimated smoothing coefficients are all positive in sign (we do not impose non-negative restrictions in our estimation procedure, despite the fact that the specification does assume non-negativity). Estimated coefficients that are negative and significant may be a sign that the constraint (49) is violated, which suggests that a somewhat different smoothing model may apply.

A third specification check is to compare the smoothing-parameter estimates from the maximum likelihood approach of Section 2.5.1 with the linear regression approach of Section 2.5.2. If the linear single-factor model (20) holds, the two sets of smoothing-parameter estimates should be close. Of course, omitted factors could be a source of discrepancies between the two sets of estimates, so this specification check must be interpreted cautiously and with some auxiliary information about the economic motivation for the common factor Λ_t .

Finally, a more direct approach to testing the specification of (21)–(23) is to impose a different identification condition than (49). Suppose that the standard deviation σ_η of true returns was observable; then the smoothing parameters θ are identified, and a simple check of the specification (21)–(23) is to see whether the estimated parameters $\hat{\theta}$ sum to 1. Of course, σ_η is not observable, but if we had an alternative estimator $\tilde{\sigma}_\eta$ for σ_η , we can achieve identification of the MA(k) process by imposing the restriction:

$$\sigma_\eta = \tilde{\sigma}_\eta \tag{70}$$

instead of (49). If, under this normalization, the smoothing parameter estimates are significantly different, this may be a sign of misspecification.

The efficacy of this specification check depends on the quality of $\tilde{\sigma}_\eta$. We propose to construct such an estimator by exploiting the fact that the discrepancy between observed and true returns becomes “small” for multiperiod returns as the number of periods grows. Specifically, recall from (37) that:

$$\begin{aligned} (R_1^o + R_2^o + \cdots + R_T^o) &= (R_1 + R_2 + \cdots + R_T) + \sum_{j=0}^{k-1} (R_{-j} - R_{T-j}) \left(1 - \sum_{i=0}^j \theta_i\right) \\ \frac{1}{T} \text{Var} \left[\sum_{t=1}^T R_t^o \right] &= \sigma_\eta^2 + \frac{\sigma_\eta^2}{T} \sum_{j=0}^{k-1} \left(1 - \sum_{i=0}^j \theta_i\right) \left(1 - 2 \sum_{i=0}^j \theta_i\right) \end{aligned} \quad (72)$$

and under the specification (21)–(23), it is easy to show that the second term on the right side of (72) vanishes as T increases without bound, hence:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{Var} \left[\sum_{t=1}^T R_t^o \right] = \sigma_\eta^2. \quad (73)$$

To estimate this normalized variance of multiperiod observed returns, we can apply Newey and West’s (1987) estimator:

$$\tilde{\sigma}_\eta^2 \equiv \frac{1}{T} \sum_1^T (R_t^o - \hat{\mu})^2 + \frac{2}{T} \sum_{j=1}^m \left(1 - \frac{j}{m+1}\right) \left(\sum_{t=j+1}^T (R_t^o - \hat{\mu})(R_{t-j}^o - \hat{\mu}) \right) \quad (74)$$

where $\hat{\mu}$ is the sample mean of $\{R_t^o\}$ and m is a truncation lag that must increase with T but at a slower rate to ensure consistency and asymptotic normality of the estimator. By imposing the identification restriction

$$\sigma_\eta = \tilde{\sigma}_\eta \quad (75)$$

in estimating the smoothing profile of observed returns, we obtain another estimator of θ which can be compared against the first. As in the case of the normalization (49), the alternate normalization (75) can be imposed by rescaling estimates $(\check{\theta}, \check{\sigma})$ from standard MA(k) estimation packages, in this case by dividing each $\check{\theta}_i$ by $\tilde{\sigma}_\eta/\check{\sigma}$ and multiplying $\check{\sigma}$ by the same factor.

2.5.4 Smoothing-Adjusted Sharpe Ratios

One of the main implications of smoothed returns is that Sharpe ratios are biased upward, in some cases substantially (see Proposition 1).²¹ The mechanism by which this bias occurs is through the reduction in volatility because of the smoothing, but there is an additional bias that occurs when monthly Sharpe ratios are annualized by multiplying by $\sqrt{12}$. If monthly returns are independently and identically distributed, this is the correct procedure, but Lo (2002) shows that for non-IID returns, an alternative procedure must be used, one that accounts for serial correlation in returns in a very specific manner.²² Specifically, denote by $R_t(q)$ the following q -period return:

$$R_t(q) \equiv R_t + R_{t-1} + \cdots + R_{t-q+1} \quad (76)$$

where we ignore the effects of compounding for computational convenience.²³ For IID returns, the variance of $R_t(q)$ is directly proportional to q , hence the Sharpe ratio satisfies the simple relation:

$$\text{SR}(q) = \frac{E[R_t(q)] - R_f(q)}{\sqrt{\text{Var}[R_t(q)]}} = \frac{q(\mu - R_f)}{\sqrt{q} \sigma} = \sqrt{q} \text{SR} . \quad (77)$$

²¹There are a number of other concerns regarding the use and interpretation of Sharpe ratios in the context of hedge funds. See Agarwal and Naik (2000a, 2002), Goetzmann et al. (2002), Lo (2001), Sharpe (1994), Spurgin (2001), and Weisman (2002) for examples where Sharpe ratios can be misleading indicators of the true risk-adjusted performance of hedge-fund strategies, and for alternate methods of constructing optimal portfolios of hedge funds.

²²See also Jobson and Korkie (1981), who were perhaps the first to derive rigorous statistical properties of performance measures such as the Sharpe ratio and the Treynor measure.

²³The exact expression is, of course:

$$R_t(q) \equiv \prod_{j=0}^{q-1} (1 + R_{t-j}) - 1 .$$

For most (but not all) applications, (76) is an excellent approximation. Alternatively, if R_t is defined to be the continuously compounded return, i.e., $R_t \equiv \log(P_t/P_{t-1})$ where P_t is the price or net asset value at time t , then (76) is exact.

Using Hansen's (1982) GMM estimator, Lo (2002) derives the asymptotic distribution of $\widehat{\text{SR}}(q)$ as:

$$\sqrt{T}(\widehat{\text{SR}}(q) - \sqrt{q}\text{SR}) \stackrel{a}{\sim} \mathcal{N}\left(0, V_{\text{IID}}(q)\right), \quad V_{\text{IID}}(q) = q V_{\text{IID}} = q\left(1 + \frac{1}{2}\text{SR}^2\right). \quad (78)$$

For non-IID returns, the relation between SR and $\text{SR}(q)$ is somewhat more involved because the variance of $R_t(q)$ is not just the sum of the variances of component returns, but also includes all the covariances. Specifically, under the assumption that returns $\{R_t\}$ are stationary,

$$\text{Var}[R_t(q)] = \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} \text{Cov}[R_{t-i}, R_{t-j}] = q\sigma^2 + 2\sigma^2 \sum_{k=1}^{q-1} (q-k)\rho_k \quad (79)$$

where $\rho_k \equiv \text{Cov}[R_t, R_{t-k}]/\text{Var}[R_t]$. This yields the following relation between SR and $\text{SR}(q)$:

$$\text{SR}(q) = \eta(q)\text{SR}, \quad \eta(q) \equiv \frac{q}{\sqrt{q + 2 \sum_{k=1}^{q-1} (q-k)\rho_k}}. \quad (80)$$

Note that (80) reduces to (77) if the autocorrelations $\{\rho_k\}$ are zero, as in the case of IID returns. However, for non-IID returns, the adjustment factor for time-aggregated Sharpe ratios is generally not \sqrt{q} but a function of the first $q-1$ autocorrelations of returns, which is readily estimated from the sample autocorrelations of returns, hence:

$$\widehat{\text{SR}}(q) = \hat{\eta}(q)\widehat{\text{SR}}, \quad \hat{\eta}(q) \equiv \frac{q}{\sqrt{q + 2 \sum_{k=1}^{q-1} (q-k)\hat{\rho}_k}} \quad (81)$$

where $\hat{\rho}_k$ is the sample k -th order autocorrelation coefficient.

Lo (2002) also derives the asymptotic distribution of (81) under fairly general assumptions for the returns process (stationarity and ergodicity) using generalized method of moments. However, in the context of hedge-fund returns, the usual asymptotic approximations may not be satisfactory because of the small sample sizes that characterize hedge-fund data—a five-year track record, which amounts to only 60 monthly observations, is considered quite a long history in this fast-paced industry.

Therefore, we derive an alternate asymptotic distribution using the continuous-record asymptotics of Richardson and Stock (1989). Specifically, as the sample size T increases without bound, let q grow as well so that the ratio converges to some finite limit between 0 and 1:

$$\lim_{q, T \rightarrow \infty} q/T = \tau \in (0, 1). \quad (82)$$

This condition is meant to provide an asymptotic approximation that may be more accurate for small-sample situations, i.e., situations where q is a significant fraction of T . For example, in the case of a fund with a five-year track record, computing an annual Sharpe ratio with monthly data corresponds to a value of 0.20 for the ratio q/T .

Now as q increases without bound, $\text{SR}(q)$ also tends to infinity, hence we must renormalize it to obtain a well-defined asymptotic sampling theory. In particular, observe that:

$$\text{SR}(q) = \frac{E[R_t(q)] - R_f(q)}{\sqrt{\text{Var}[R_t(q)]}} = \frac{q(\mu - R_f)}{\sqrt{\text{Var}[R_t(q)]}} \quad (83)$$

$$\text{SR}(q)/\sqrt{q} = \frac{\mu - R_f}{\sqrt{\text{Var}[R_t(q)]/q}} \quad (84)$$

$$\lim_{q \rightarrow \infty} \text{SR}(q)/\sqrt{q} = \frac{\mu - R_f}{\bar{\sigma}} \quad (85)$$

where $\bar{\sigma}$ can be viewed as a kind of long-run average return standard deviation, which is generally not identical to the unconditional standard deviation σ of monthly returns except in the IID case. To estimate $\bar{\sigma}$, we can either follow Lo (2002) and use sample autocorrelations as in (81), or estimate $\bar{\sigma}$ directly accordingly to Newey and West (1987):

$$\hat{\sigma}_{\text{NW}}^2 \equiv \frac{1}{T} \sum_1^T (R_t - \hat{\mu})^2 + \frac{2}{T} \sum_{j=1}^m \left(1 - \frac{j}{m+1}\right) \sum_{t=j+1}^T (R_t - \hat{\mu})(R_{t-j} - \hat{\mu}) \quad (86)$$

where $\hat{\mu}$ is the sample mean of $\{R_t\}$. For this estimator of $\bar{\sigma}$, we have the following asymptotic result:

Proposition 4 *As m and T increase without bound so that $m/T \rightarrow \lambda \in (0, 1)$, $\hat{\sigma}_{\text{NW}}^2$ converges weakly to the following functional $f(W)$ of standard Brownian motion on $[0, 1]$.²⁴*

$$f(W) \equiv \frac{2\bar{\sigma}^2}{\lambda} \left(\int_0^1 W(r)[W(r) - W(\min(r + \lambda, 1))]dr - W(1) \int_0^\lambda (\lambda - r)(W(1 - r) - W(r))dr + \frac{\lambda(1 - \frac{\lambda^2}{3})}{2} W^2(1) \right) \quad (87)$$

From (87), a straightforward computation yields the following expectations:

$$E[\hat{\sigma}_{\text{NW}}^2] = 1 - \lambda + \frac{\lambda^2}{3}, \quad E[1/\hat{\sigma}_{\text{NW}}] \approx \sqrt{\frac{1 + \lambda}{1 - \lambda + \lambda^2/3}} \quad (88)$$

hence we propose the following bias-corrected estimator for the Sharpe ratio for small samples:

$$\widehat{\text{SR}}(q) = \frac{\sqrt{q}(\hat{\mu} - R_f)}{\hat{\sigma}_{\text{NW}}} \sqrt{\frac{1 - \lambda + \lambda^3/2}{1 + \lambda}} \quad (89)$$

and its asymptotic distribution is given by the following proposition:

Proposition 5 *As m , q , and T increase without bound so that $m/T \rightarrow \lambda \in (0, 1)$ and $q/T \rightarrow \tau \in (0, 1)$, the Sharpe ratio estimator $\widehat{\text{SR}}(q)$ converges weakly to the following random variable:*

$$\widehat{\text{SR}}(q) \Rightarrow \left(\frac{\text{SR}(q)}{f(W)} + \frac{\sqrt{\tau}W(1)}{f(W)} \right) \sqrt{\frac{1 - \lambda + \lambda^3/2}{1 + \lambda}} \quad (90)$$

where $f(W)$ is given by (87), $\text{SR}(q)$ is given by (89) and $W(\cdot)$ is standard Brownian motion defined on $[0, 1]$.

Monte Carlo simulations show that the second term of (90) does not account for much bias when $\tau \in (0, \frac{1}{2}]$, and that (90) is an excellent approximation to the small-sample distributions of Sharpe ratios for non-IID returns.²⁵

²⁴See Billingsley (1968) for the definition of weak convergence and related results.

²⁵We have tabulated the percentiles of the distribution of (90) by Monte Carlo simulation for an extensive combination of values of q , τ , and λ and would be happy to provide them to interested readers upon request.

2.6 Empirical Analysis

For our empirical analysis, we use the TASS database of hedge funds which consists of monthly returns and accompanying information for 2,439 hedge funds (as of January 2001) from November 1977 to January 2001.²⁶ The database is divided into two parts: “Live” and “Graveyard” funds. Hedge funds that belong to the Live database are considered to be active as of January 1, 2001; once a hedge fund decides not to report its performance, is liquidated, restructured, or merged with other hedge funds, the fund is transferred into the Graveyard database. A hedge fund can only be listed in the Graveyard database after being listed in the Live database, but the TASS database is subject to backfill bias—when a fund decides to be included in the database, TASS adds the fund to the Live database and includes available prior performance of the fund (hedge funds do not need to meet any specific requirements to be included in the TASS database). Due to reporting delays and time lags in contacting hedge funds, some Graveyard funds can be incorrectly listed in the Live database for a period of time. However, TASS has adopted a policy of transferring funds from the Live to the Graveyard database if they do not report over a 6–8 month period.

As of January 1, 2001, the combined data set of both live and dead hedge funds contained 2,439 funds with at least one monthly net return observation. Out of these 2,439 funds, 1,512 are in the Live database and 927 are in the Graveyard database. The earliest data available for a fund in either database is November 1, 1977. The Graveyard database became active only in 1994, i.e., funds that were dropped from the Live database prior to 1994 are not included in the Graveyard database, which may yield a certain degree of survivorship bias.²⁷

A majority of the 2,439 funds reported returns net of various fees on a monthly

²⁶For further information about the database and TASS, see <http://www.tassresearch.com>.

²⁷For studies attempting to quantify the degree and impact of survivorship bias, see Baquero, Horst, and Verbeek (2002), Brown, Goetzmann, Ibbotson, and Ross (1992), Brown, Goetzmann, and Ibbotson (1999), Brown, Goetzmann, and Park (1997), Carpenter and Lynch (1999), Fung and Hsieh (1997b, 2000), Horst, Nijman, T. and M. Verbeek (2001), Hendricks, Patel, and Zeckhauser (1997), and Schneeweis and Spurgin (1996).

basis.²⁸ We eliminated 30 funds that reported only gross returns and/or quarterly returns (15 from each of the Live and Graveyard databases, respectively), leaving 2,409 funds in our sample. We imposed an additional filter of including only those funds with at least five years of data, leaving 651 funds in the Live database and 258 in the Graveyard database for a combined total of 909 funds. This obviously creates additional survivorship bias in our sample, but since our main objective is to estimate smoothing profiles and not to make inferences about overall performance, our filter may not be as problematic.²⁹

TASS also attempts to classify funds according to one of 17 different investment styles, listed in Table 2.5 and described in Appendix 2.8.4; funds that TASS are not able to categorize are assigned a category code of '0'.³⁰ Table 2.5 also reports the number of funds in each category for the Live, Graveyard, and Combined databases, and it is apparent from these figures that the representation of investment styles is not evenly distributed, but is concentrated among six categories: US Equity Hedge (162), Event Driven (109), Non-Directional/Relative Value (85), Pure Managed Futures (93), Pure Emerging Market (72), and Fund of Funds (132). Together, these six categories account for 72% of the funds in the Combined database.

To develop a sense of the dynamics of the TASS database and the impact of our minimum return-history filter, in Table 2.6 we report annual frequency counts of the funds in the database at the start of each year, funds entering during the year, funds exiting during the year, and funds entering and exiting within the year. The left panel contains counts for the entire TASS database, and the right panel contains counts for our sample of 909 funds with at least five years of returns. The left panel shows that despite the start date of November 1977, the database is relatively sparsely populated

²⁸TASS defines returns as the change in net asset value during the month (assuming the reinvestment of any distributions on the reinvestment date used by the fund) divided by the net asset value at the beginning of the month, net of management fees, incentive fees, and other fund expenses. Therefore, these reported returns should approximate the returns realized by investors. TASS also converts all foreign-currency denominated returns to US-dollar returns using the appropriate exchange rates.

²⁹See the references in footnote 27.

³⁰A hedge fund can have at most 2 different categories (CAT1 and CAT2) in the TASS database. For all hedge funds in the TASS database, the second category (CAT2) is always 17, 'Fund of Funds'.

Code	Category	Number of Funds In:		
		Combined	Live	Graveyard
0	Not Categorized	111	44	67
1	US Equity Hedge	162	139	23
2	European Equity Hedge	22	19	3
3	Asian Equity Hedge	5	5	0
4	Global Equity Hedge	27	24	3
5	Dedicated Shortseller	7	6	1
6	Fixed-Income Directional	13	12	1
7	Convertible Fund (Long Only)	15	12	3
8	Event Driven	109	97	12
9	Non-Directional/Relative Value	85	63	22
10	Global Macro	25	15	10
11	Global Opportunity	1	1	0
12	Natural Resources	3	1	2
13	Pure Leveraged Currency	26	15	11
14	Pure Managed Futures	93	28	65
15	Pure Emerging Market	72	54	18
16	Pure Property	1	1	0
17	Fund of Funds	132	115	17
	All	909	651	258

Table 2.5: Number of funds in the TASS Hedge Fund Live and Graveyard databases with at least five years of returns history during the period from November 1977 to January 2001.

until the 1990's, with the largest increase in new funds in 1998 and, in the aftermath of the collapse of LTCM, the largest number of funds exiting the database in 1999 and 2000. The right panel of Table 2.6 illustrates the impact of our five-year filter—the number of funds is considerably smaller, and although the impact of survivorship bias can be ameliorated by the use of Live and Graveyard funds, our sample of 909 funds will not include any of the funds started in 1997 and later which is a substantial proportion of the TASS database.

The attrition rates reported in Table 2.6 are defined as the ratio of funds exiting in a given year to the number of existing funds at the start of the year. TASS began tracking the exits of funds starting only in 1994 hence attrition rates could not be computed in prior years. For the unfiltered sample of all funds, the average attrition rate from 1994–1999 is 9.11%, which is very similar to the 8.54% attrition rate obtained by Liang (2001) for the same period. As observed above, the attrition rate skyrocketed in 2000 in the wake of LTCM's demise. In the right panel of Table 2.6, we see smaller attrition rates—the average over the 1994–1999 period is only 3.81%—because of our five-year minimum return history filter; since many hedge funds fail in their first three years, our filtered sample is likely to have a much lower attrition rate by construction.

Figure 2-4 contains a visual depiction of the variation in sample sizes of our 909 funds. The start and end dates of the return history for each fund are connected by a vertical line and plotted in Figure 2-4 according to the primary category of the fund—Categories 0–7 in the top panel and Categories 8–17 in the bottom panel. It is apparent from the increasing density of the graphs as we move from the bottom to the top that the majority of funds in our sample are relatively new.

In Section 2.6.1 we present summary statistics for the sample of hedge funds included in our analysis. We implement the smoothing profile estimation procedures outlined in Section 2.5 for each of the funds, and in Section 2.6.2 we summarize the results of the maximum likelihood and linear regression estimation procedures for the entire sample of funds and for each style category. Section 2.6.3 reports the results of cross-sectional regressions of the estimated smoothing coefficients, and in Section

Year	All Funds							Funds with At Least 5 Years' History						
	Existing Funds	New Entries	New Exits	Intrayear Entry/Exit	Total Funds	Attrition Rate (%)	Existing Funds	New Entries	New Exits	Intrayear Entry/Exit	Total Funds	Attrition Rate (%)		
1977	0	1	0	0	1	—	0	1	0	0	1	—		
1978	1	2	0	0	3	0.0	1	2	0	0	3	0.0		
1979	3	1	0	0	4	0.0	3	1	0	0	4	0.0		
1980	4	1	0	0	5	0.0	4	1	0	0	5	0.0		
1981	5	3	0	0	8	0.0	5	3	0	0	8	0.0		
1982	8	4	0	0	12	0.0	8	4	0	0	12	0.0		
1983	12	6	0	0	18	0.0	12	6	0	0	18	0.0		
1984	18	14	0	0	32	0.0	18	14	0	0	32	0.0		
1985	32	8	0	0	40	0.0	32	8	0	0	40	0.0		
1986	40	19	0	0	59	0.0	40	19	0	0	59	0.0		
1987	59	32	0	0	91	0.0	59	32	0	0	91	0.0		
1988	91	28	0	0	119	0.0	91	28	0	0	119	0.0		
1989	119	39	0	0	158	0.0	119	39	0	0	158	0.0		
1990	158	99	0	0	257	0.0	158	93	0	0	251	0.0		
1991	257	87	0	0	344	0.0	251	78	0	0	329	0.0		
1992	344	156	0	0	500	0.0	329	118	0	0	447	0.0		
1993	500	233	0	0	733	0.0	447	135	0	0	582	0.0		
1994	733	245	28	0	950	3.8	582	153	3	0	732	0.5		
1995	950	256	71	1	1135	7.5	732	142	15	0	859	2.0		
1996	1135	263	127	9	1271	11.2	859	32	30	0	861	3.5		
1997	1271	290	105	7	1456	8.3	861	0	14	0	847	1.6		
1998	1456	255	166	10	1545	11.4	847	0	51	0	796	6.0		
1999	1545	222	193	7	1574	12.5	796	0	73	0	723	9.2		
2000	1574	90	421	19	1243	26.7	723	0	182	0	541	25.2		

Table 2.6: Annual frequency counts of entries into and exits out of the TASS Hedge Fund Database from November 1977 to January 2001.

2.6.4 we consider idiosyncratic and systematic effects of illiquidity and smoothing by estimating a linear factor model with smoothing for all funds in one particular style category using a common factor appropriate for that style. And in Section 2.6.5 we summarize the properties of smoothing-adjusted Sharpe ratios for all the funds in our sample and compare them to their unadjusted counterparts.

2.6.1 Summary Statistics

Table 2.7 contains basic summary statistics for the 909 funds in our combined extract from the TASS Live and Graveyard databases. Not surprisingly, there is a great deal of variation in mean returns and volatilities both across and within categories. For example, the 162 US Equity Hedge funds in our sample exhibited a mean return of 22.53%, but with a standard deviation of 10.80% in the cross section, and a mean volatility of 21.69% with a cross-sectional standard deviation of 11.63%. Average serial correlations also vary considerably across categories, but five categories stand out as having the highest averages:³¹ Fixed Income Directional (21.6%), Convertible Fund (Long Only) (22.5%), Event Driven (20.8%), Non-Directional/Relative Value (18.2%), and Pure Emerging Market (18.8%). Given the descriptions of these categories provided by TASS (see Appendix 2.8.4) and common wisdom about the nature of the strategies involved—these categories include some of the most illiquid securities traded—serial correlation seems to be a reasonable proxy for illiquidity and smoothed returns. Alternatively, equities and futures are among the most liquid securities in which hedge funds invest, and not surprising, the average first-order serial correlation for US Equity Hedge funds and Pure Managed Futures is 7.8% and -0.1% , respectively. In fact, all of the equity funds have average serial correlations that are much smaller than those of the top five categories. Dedicated Shortseller funds also have a low average first-order autocorrelation, 4.4%, which is consistent with the high degree of liquidity that often characterize shortsellers (since, by definition, the ability

³¹At 23.1% and -23.1% , respectively. Global Opportunity and Pure Property have higher first-order autocorrelation coefficients in absolute value than the other categories, but since these two categories contain only a single fund each, we set them aside in our discussions.

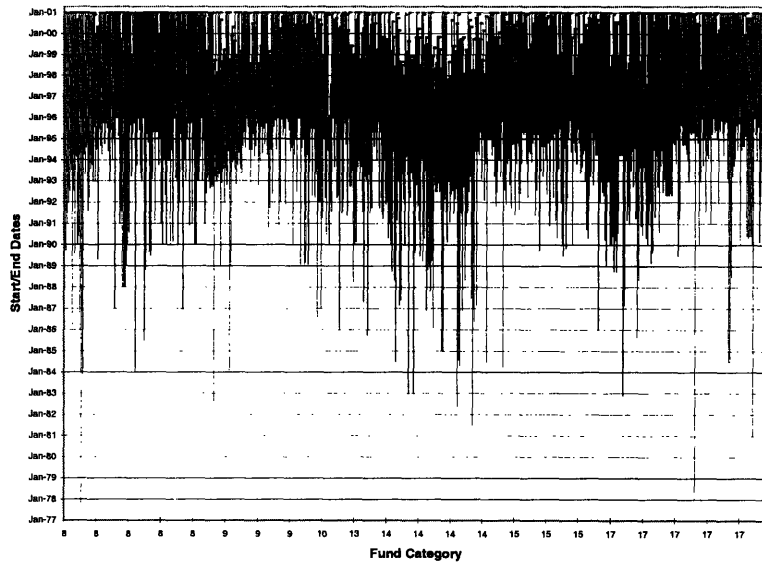
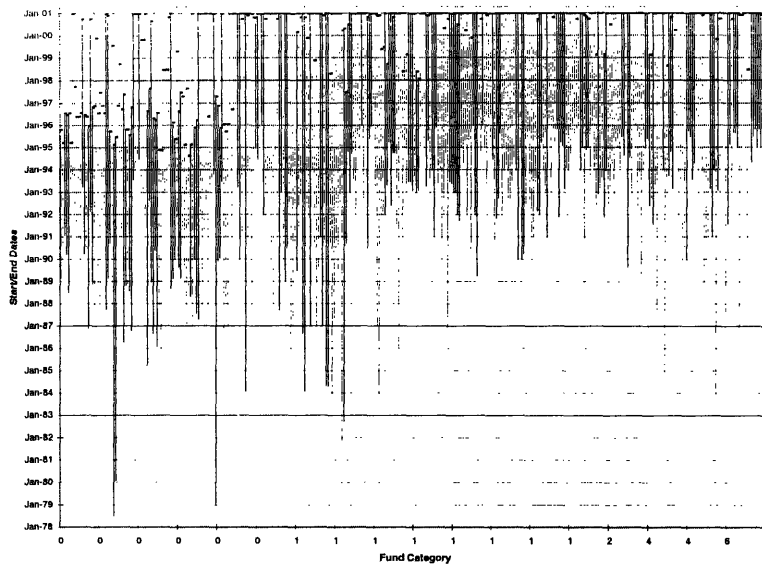


Figure 2-4: Length of return histories, depicted by vertical solid lines, for all funds in the TASS Hedge Fund database with at least five years of returns during the period from November 1977 to January 2001, ordered by categories 0 to 7 in the top panel and categories 8 to 17 in the bottom panel. Each fund is represented by a single solid vertical line that spans the start and end dates of the fund's return history.

to short a security implies a certain degree of liquidity).

These summary statistics suggest that illiquidity and smoothed returns may be important attributes for hedge-fund returns which can be captured to some degree by serial correlation and our time-series model of smoothing.

2.6.2 Smoothing Profile Estimates

Using the methods outlined in Section 2.5, we estimate the smoothing model (21)–(23) and summarize the results in Tables 2.8–2.9. Our maximum likelihood procedure—programmed in Matlab using the *Optimization Toolbox* and replicated in Stata using its MA(k) estimation routine—converged without difficulty for all but one of the 909 funds:³² fund 1055, a Global Macro fund with returns from June 1994 to January 2001 for which the maximum likelihood estimation procedure yielded the following parameter estimates:

$$\hat{\theta}_0 = 490.47 \quad , \quad \hat{\theta}_1 = -352.63 \quad , \quad \hat{\theta}_2 = -136.83$$

which suggests that our MA(2) model is severely misspecified for this fund. Therefore, we drop this fund from our sample and for the remainder of our analysis, we focus on the smoothing profile estimates for the remaining 908 funds in our sample.³³

Table 2.8 contains summary statistics for maximum likelihood estimate of the smoothing parameters $(\theta_0, \theta_1, \theta_2)$ and smoothing index ξ , Table 2.12 reports comparable statistics for the regression estimates of the smoothing parameters under the assumption of a linear one-factor model for true returns, and Table 2.9 presents the

³²We also constrain our maximum likelihood estimators to yield invertible MA(2) processes, and this constraint was binding for only two funds: 1711 and 4298.

³³The apparent source of the problem in this case is two consecutive outliers, 39.4% in December 1999 followed by -27.6% in January 2000 (these are *monthly* returns, not annualized). The effect of two outliers on the parameter estimates of the MA(2) model (21)–(23) is to pull the values of the coefficients in opposite directions so as to fit the extreme reversals. We contacted TASS to investigate these outliers and were informed that they were data errors. We also checked the remaining 908 funds in our sample for similar outliers, i.e., consecutive extreme returns of opposite sign, and found none. We also computed the maximum and minimum monthly returns for each fund in our sample, ranked the 908 funds according to these maxima and minima, and checked the parameter estimates of the top and bottom 10 funds, and none exhibited the extreme behavior of fund 1055's parameter estimates.

Category	N	Annual Mean		Annual SD		Skewness		Kurtosis		$\hat{\rho}_1(\%)$		$\hat{\rho}_2(\%)$		$\hat{\rho}_3(\%)$		p-Value(Q)	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Not Categorized	111	10.78	6.99	15.60	11.25	0.12	1.06	3.67	4.94	5.5	17.0	0.9	14.7	0.5	11.4	36.1	30.4
US Equity Hedge	162	22.53	10.80	21.69	11.63	0.17	0.96	3.82	5.02	7.8	13.5	0.6	11.2	-4.2	11.0	33.9	28.8
European Equity Hedge	22	18.01	7.15	14.71	6.41	0.39	0.81	2.97	2.73	12.7	10.9	11.2	11.3	-2.0	8.9	21.2	23.0
Asian Equity Hedge	5	9.16	7.96	21.13	6.64	0.50	0.85	2.32	0.73	11.7	13.0	5.0	3.6	-5.4	13.4	40.2	21.1
Global Equity Hedge	27	13.71	8.41	17.41	6.61	-0.24	1.06	3.96	5.60	12.4	11.3	-0.1	9.6	-2.7	7.2	41.9	27.2
Dedicated Shortseller	7	0.33	10.26	21.55	13.80	0.49	0.20	2.28	2.40	4.4	8.4	-2.7	9.5	-6.5	6.9	31.8	27.7
Fixed-Income Directional	13	9.82	3.49	9.55	8.74	-0.94	1.29	5.37	7.32	21.6	15.7	14.1	13.2	1.9	13.6	18.1	21.0
Convertible Fund (Long Only)	15	14.95	5.33	11.03	6.76	-0.19	1.68	6.14	9.97	22.5	12.3	6.2	13.4	-2.1	12.5	17.2	25.2
Event Driven	109	15.35	6.70	9.51	9.65	-0.58	1.82	8.26	11.68	20.8	16.8	6.4	13.4	-0.1	12.3	25.7	29.6
Non-Directional/Relative Value	85	12.54	6.65	8.19	5.05	-0.76	1.71	6.44	9.50	18.2	23.5	12.2	15.4	5.0	14.5	17.1	25.0
Global Macro	25	16.75	7.94	18.92	9.53	0.53	1.03	4.66	6.82	7.2	15.0	-0.1	10.8	-0.9	8.1	33.1	22.9
Global Opportunity	1	-17.39	—	31.03	—	-0.48	—	2.80	—	23.1	—	14.1	—	0.0	—	20.8	—
Natural Resources	3	11.39	4.89	18.91	1.06	0.48	0.44	1.25	1.49	5.0	14.5	8.5	12.1	1.9	7.0	47.6	43.7
Pure Leveraged Currency	26	9.29	7.11	17.18	8.93	0.44	1.08	3.50	4.92	5.2	8.7	-6.2	9.5	-4.4	8.1	35.5	26.6
Pure Managed Futures	93	9.60	9.28	21.74	17.06	0.22	1.35	4.93	8.31	-0.1	12.9	-3.8	10.0	-3.5	10.7	35.4	29.1
Pure Emerging Market	72	9.57	10.94	26.60	14.44	-0.69	1.49	6.17	8.05	18.8	11.8	4.4	11.6	-1.4	9.0	32.6	30.6
Pure Property	1	3.96	—	9.41	—	-1.33	—	5.43	—	-23.1	—	2.3	—	7.3	—	18.9	—
Fund of Funds	132	11.04	5.16	10.07	5.63	-0.26	1.30	4.78	6.16	17.6	14.1	5.7	11.5	-0.7	9.3	26.3	27.5
All	909	13.72	9.46	16.06	12.21	-0.15	1.38	5.05	7.56	12.1	16.8	3.4	13.1	-1.2	11.2	30.1	28.7

Table 2.7: Means and standard deviations of basic summary statistics for 909 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001. The columns ' p -Value(Q)' contain means and standard deviations of p -values for the Box-Pierce Q -statistic for each fund using the first 6 autocorrelations of returns.

maximum likelihood estimates of the smoothing model for the 50 most illiquid funds of the 908 funds, as ranked by $\hat{\theta}_0$.

The left panel of Table 2.8 reports summary statistics for the maximum likelihood estimates under the normalization (49) where the smoothing coefficients sum to 1, and the right panel reports the same statistics for the maximum likelihood estimates under the normalization (75) where the variance σ_η^2 is set equal to the nonparametric estimate $\tilde{\sigma}_\eta^2$ given by (74). A comparison of the right and left panels reveals roughly similar characteristics, indicating the general equivalence of these two normalization methods and the fact that the smoothing model (21)–(23) may be a reasonable specification for hedge-fund returns.³⁴

Table 2.8 shows that seven categories seem to exhibit smaller average values of $\hat{\theta}_0$ than the rest—European Equity Hedge (0.82), Fixed-Income Directional (0.76), Convertible Fund (Long Only) (0.84), Event Driven (0.81), Non-Directional/Relative Value (0.82), Pure Emerging Market (0.83), and Fund of Funds (0.85).³⁵ Consider, in particular, the Fixed-Income Directional category, which has a mean of 0.76 for $\hat{\theta}_0$. This is, of course, the average across all 13 funds in this category, but if it were the point estimate of a given fund, it would imply that only 76% of that fund’s true current monthly return would be reported, with the remaining 24% distributed over the next two months (recall the constraint that $\hat{\theta}_0 + \hat{\theta}_1 + \hat{\theta}_2 = 1$). The estimates 0.15 and 0.08 for $\hat{\theta}_1$ and $\hat{\theta}_2$ imply that on average, the current reported return also includes 15% of last month’s true return and 8% of the true return two months ago.³⁶ These averages suggest a significant amount of smoothing and illiquidity in this category, and are approximated by the geometric smoothing model of Section 2.4.2 with $\delta = 0.25$ (see Table 2.4). Recall from Table 2.4 that in this case, with $k = 2$, the impact of geometric smoothing was a 24% decrease in the market beta and a 27% increase in

³⁴However, Table 2.8 contains only summary statistics, not the maximum likelihood estimators of individual funds, hence differences in the two normalizations for given funds may not be apparent from this table. In particular, side-by-side comparisons of maximum likelihood estimates for an individual fund under these two normalizations may still be a useful specification check despite the broad similarities that these two approaches seem to exhibit in Table 2.8.

³⁵We omit the Global Opportunity category from our discussions because it consists of only a single fund.

³⁶The averages do not sum to 1 exactly because of rounding errors.

Category	N	MA(2) with Constrained Sum						MA(2) with Constrained σ_{η}									
		$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$		ξ		$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$			
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD		
Not Categorized	111	1.04	0.33	0.01	0.18	-0.05	0.23	1.27	1.06	0.97	0.27	0.01	0.16	-0.04	0.19	1.08	0.60
US Equity Hedge	162	0.95	0.21	0.06	0.15	-0.01	0.15	1.00	0.54	0.98	0.20	0.07	0.14	-0.01	0.15	1.04	0.42
European Equity Hedge	22	0.82	0.15	0.08	0.09	0.11	0.11	0.72	0.28	0.90	0.17	0.09	0.10	0.13	0.12	0.88	0.32
Asian Equity Hedge	5	0.94	0.30	0.05	0.19	0.01	0.11	1.00	0.71	0.90	0.20	0.06	0.17	0.01	0.10	0.88	0.40
Global Equity Hedge	27	0.91	0.14	0.11	0.10	-0.02	0.09	0.88	0.27	0.96	0.20	0.11	0.10	-0.02	0.09	0.99	0.37
Dedicated Shortseller	7	1.03	0.20	0.03	0.12	-0.06	0.13	1.12	0.48	1.13	0.14	0.05	0.10	-0.05	0.14	1.33	0.34
Fixed-Income Directional	13	0.76	0.17	0.15	0.10	0.08	0.12	0.67	0.27	0.75	0.21	0.15	0.10	0.09	0.12	0.66	0.30
Convertible Fund (Long Only)	15	0.84	0.35	0.17	0.15	-0.02	0.35	0.98	1.36	0.89	0.27	0.20	0.10	0.00	0.24	0.97	0.58
Event Driven	109	0.81	0.17	0.15	0.13	0.04	0.11	0.74	0.28	0.84	0.24	0.16	0.12	0.04	0.11	0.82	0.44
Non-Directional/Relative Value	85	0.82	0.24	0.09	0.25	0.09	0.14	0.82	0.62	0.77	0.27	0.09	0.24	0.08	0.13	0.75	0.70
Global Macro	24	0.95	0.18	0.08	0.11	-0.03	0.15	0.99	0.40	0.97	0.23	0.08	0.10	-0.03	0.17	1.03	0.51
Global Opportunity	1	0.74	—	0.16	—	0.10	—	0.58	—	0.63	—	0.14	—	0.09	—	0.43	—
Natural Resources	3	0.91	0.19	0.02	0.11	0.07	0.09	0.87	0.34	0.85	0.04	0.03	0.10	0.08	0.10	0.74	0.04
Pure Leveraged Currency	26	1.10	0.34	0.03	0.13	-0.13	0.25	1.41	1.38	1.00	0.16	0.04	0.10	-0.10	0.16	1.07	0.37
Pure Managed Futures	93	1.13	0.32	-0.05	0.22	-0.08	0.18	1.47	1.19	1.10	0.25	-0.03	0.18	-0.07	0.16	1.33	0.63
Pure Emerging Market	72	0.83	0.16	0.15	0.09	0.02	0.13	0.76	0.31	0.79	0.17	0.14	0.08	0.02	0.13	0.70	0.30
Pure Property	1	1.23	—	-0.31	—	0.07	—	1.62	—	0.95	—	-0.24	—	0.06	—	0.96	—
Fund of Funds	132	0.85	0.21	0.13	0.13	0.03	0.13	0.81	0.50	0.85	0.23	0.13	0.12	0.03	0.13	0.83	0.45
All	908	0.92	0.26	0.08	0.17	0.00	0.17	0.98	0.76	0.91	0.25	0.08	0.16	0.01	0.15	0.95	0.53

Table 2.8: Means and standard deviations of maximum likelihood estimates of MA(2) smoothing process $R_t^* = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$. $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$, for 908 hedge funds in the TASS combined database with at least five years of returns history during the period from November 1977 to January 2001.

the Sharpe ratio of observed returns. Overall, the summary statistics in Table 2.8 are broadly consistent with common intuition about the nature of the strategies and securities involved in these fund categories, which contain the most illiquid securities and, therefore, have the most potential for smoothed returns.

Code	Category	Period	T	Status	$\hat{\theta}_0$	SE($\hat{\theta}_0$)	$\hat{\theta}_1$	SE($\hat{\theta}_1$)	$\hat{\theta}_2$	SE($\hat{\theta}_2$)	$\hat{\xi}$
1201	Non-Directional/Relative Value	199409-200101	77	1	0.464	0.040	0.365	0.025	0.171	0.047	0.378
4346	Event Driven	199412-200011	71	1	0.471	0.041	0.335	0.029	0.195	0.048	0.371
180	Not Categorized	198906-199608	87	0	0.485	0.041	0.342	0.027	0.173	0.046	0.382
1204	Non-Directional/Relative Value	199510-200101	64	1	0.492	0.049	0.339	0.033	0.169	0.055	0.386
1584	Fund of Funds	199601-200101	61	1	0.504	0.046	0.245	0.040	0.251	0.051	0.377
518	Non-Directional/Relative Value	199312-200005	78	0	0.514	0.034	0.142	0.037	0.343	0.037	0.403
971	Non-Directional/Relative Value	199409-200012	76	1	0.515	0.038	0.176	0.038	0.309	0.041	0.392
1234	Fund of Funds	199410-200012	75	1	0.527	0.061	0.446	0.020	0.027	0.061	0.477
2185	Fixed Income Directional	199108-200101	114	1	0.532	0.040	0.265	0.032	0.202	0.043	0.395
26	Non-Directional/Relative Value	199303-200101	95	1	0.533	0.047	0.305	0.033	0.162	0.050	0.403
171	Non-Directional/Relative Value	199304-200101	94	1	0.536	0.040	0.193	0.037	0.271	0.043	0.398
1696	Fund of Funds	199501-200001	61	0	0.536	0.067	0.406	0.030	0.058	0.068	0.456
120	Non-Directional/Relative Value	198207-199810	196	0	0.546	0.032	0.238	0.026	0.215	0.034	0.402
1396	Non-Directional/Relative Value	199507-200101	67	1	0.548	0.056	0.255	0.044	0.197	0.059	0.404
2774	Non-Directional/Relative Value	199501-200006	66	0	0.553	0.058	0.262	0.044	0.185	0.061	0.409
1397	Non-Directional/Relative Value	199410-200101	76	1	0.556	0.055	0.260	0.042	0.184	0.057	0.410
57	Non-Directional/Relative Value	199210-200101	100	1	0.561	0.052	0.311	0.034	0.127	0.053	0.428
4158	Event Driven	199510-200101	64	1	0.565	0.062	0.261	0.046	0.174	0.064	0.417
1773	Fund of Funds	199506-200101	68	1	0.569	0.056	0.194	0.049	0.238	0.058	0.417
415	Not Categorized	198807-199608	98	0	0.570	0.054	0.306	0.035	0.124	0.055	0.434
1713	Fixed Income Directional	199601-200101	61	1	0.573	0.067	0.269	0.048	0.158	0.068	0.426
1576	Fund of Funds	199504-200010	67	1	0.575	0.066	0.293	0.044	0.132	0.067	0.434
1633	Event Driven	199304-199901	70	0	0.576	0.066	0.309	0.042	0.115	0.066	0.440
1883	Event Driven	199306-200101	92	1	0.578	0.053	0.236	0.041	0.187	0.054	0.424
1779	Non-Directional/Relative Value	199512-200101	62	1	0.584	0.067	0.241	0.051	0.175	0.068	0.429
3860	Fund of Funds	199511-200101	63	1	0.584	0.064	0.207	0.053	0.210	0.065	0.428
482	Non-Directional/Relative Value	199406-200101	80	1	0.584	0.060	0.253	0.044	0.163	0.060	0.432
2755	US Equity Hedge	199203-200010	104	1	0.584	0.057	0.321	0.034	0.095	0.056	0.453
1968	Fund of Funds	199402-200012	83	1	0.587	0.060	0.253	0.044	0.160	0.060	0.434
1240	Fund of Funds	199405-200101	81	1	0.592	0.055	0.168	0.049	0.240	0.056	0.436
1884	Event Driven	199301-200101	97	1	0.592	0.059	0.298	0.038	0.110	0.058	0.451
2864	Non-Directional/Relative Value	199408-200101	78	1	0.598	0.064	0.255	0.046	0.147	0.064	0.444
1	Non-Directional/Relative Value	199101-200101	121	1	0.600	0.051	0.241	0.038	0.159	0.051	0.444
412	Pure Emerging Market	199211-200101	99	1	0.600	0.061	0.318	0.036	0.082	0.059	0.468
1046	Pure Emerging Market	199406-200101	80	1	0.603	0.058	0.169	0.051	0.228	0.058	0.444
2570	Event Driven	199001-200101	133	1	0.604	0.046	0.171	0.039	0.225	0.046	0.445
945	Fund of Funds	199210-200101	100	1	0.605	0.060	0.273	0.040	0.122	0.058	0.456
1994	Non-Directional/Relative Value	199507-200101	67	1	0.606	0.069	0.217	0.053	0.177	0.068	0.446
1691	Event Driven	199310-200101	88	0	0.610	0.062	0.232	0.046	0.158	0.061	0.451
34	Non-Directional/Relative Value	199209-200101	101	1	0.612	0.052	0.143	0.047	0.245	0.052	0.455
2630	Non-Directional/Relative Value	199001-200101	133	1	0.612	0.052	0.255	0.036	0.133	0.051	0.457
2685	US Equity Hedge	199207-200101	103	1	0.613	0.060	0.271	0.040	0.116	0.058	0.463
2549	Non-Directional/Relative Value	199408-200101	78	1	0.614	0.079	0.412	0.029	-0.026	0.069	0.547
490	Event Driven	199404-199908	65	0	0.615	0.074	0.237	0.054	0.148	0.072	0.456
3099	Convertible Fund (Long Only)	199501-200101	73	1	0.615	0.060	0.121	0.057	0.263	0.060	0.463
1326	US Equity Hedge	199601-200101	61	1	0.619	0.069	0.145	0.062	0.236	0.068	0.460
1418	Fund of Funds	199301-199804	64	0	0.621	0.085	0.363	0.041	0.016	0.078	0.518
3712	Not Categorized	199304-200101	94	1	0.625	0.062	0.211	0.047	0.164	0.060	0.462
1534	Fund of Funds	199402-200101	84	1	0.625	0.068	0.246	0.048	0.129	0.065	0.468
167	Event Driven	199207-200101	103	1	0.627	0.061	0.239	0.044	0.134	0.059	0.468

Table 2.9: First 50 funds of ranked list of 908 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001, ranked in increasing order of the estimated smoothing parameter $\hat{\theta}_0$ of the MA(2) smoothing process $R_t^o = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, subject to the normalization $1 = \theta_0 + \theta_1 + \theta_2$, and estimated via maximum likelihood.

Table 2.9 contains the smoothing parameter estimates for the top 50 funds ranked by $\hat{\theta}_0$, which provides a more direct view of illiquidity and smoothed returns. In

contrast to the averages of Table 2.8, the parameter estimates of θ_0 among these 50 funds range from 0.464 to 0.627, implying that only half to two-thirds of the current month's true returns are reflected in observed returns. The asymptotic standard errors are generally quite small, ranging from 0.032 to 0.085, hence the smoothing parameters seem to be estimated reasonably precisely.

The funds in Table 2.9 fall mainly into three categories: Non-Directional/Relative Value, Event Driven, and Fund of Funds. Together, these three categories account for 40 of the 50 funds in Table 2.9. A more complete summary of the distribution of smoothing parameter estimates across the different fund categories is provided in Figures 2-5 and 2-6. Figure 2-5 contains a graph of the smoothing coefficients $\hat{\theta}_0$ for all 908 funds by category, and Figure 2-6 contains a similar graph for the smoothing index $\hat{\xi}$. These figures show that although there is considerable variation within each category, nevertheless, some differences emerge between categories. For example, categories 6–9, 15, and 17 (Fixed-Income Directional, Convertible Fund (Long Only), Event Driven Non-Directional/Relative Value, Pure Emerging Market, and Fund of Funds, respectively), have clearly discernible concentrations that are lower than the other categories, suggesting more illiquid funds and more smoothed returns. On the other hand, categories 1, 4, and 14 (US Equity Hedge, Global Equity Hedge, and Pure Managed Futures, respectively) have concentrations that are at the upper end, suggesting just the opposite—more liquidity and less return-smoothing. The smoothing index estimates $\hat{\xi}$ plotted in Figure 2-6 show similar patterns of concentration and dispersion within and between the categories.

To develop further intuition for the smoothing model (21)–(23) and the possible interpretations of the smoothing parameter estimates, we apply the same estimation procedure to the returns of the Ibbotson stock and bond indexes, the Merrill Lynch Convertible Securities Index,³⁷ the CSFB/Tremont hedge-fund indexes, and two mutual funds, the highly liquid Vanguard 500 Index Fund, and the considerably less

³⁷ This is described by Merrill Lynch as a “market value-weighted index that tracks the daily price only, income and total return performance of corporate convertible securities, including US domestic bonds, Eurobonds, preferred stocks and Liquid Yield Option Notes”.

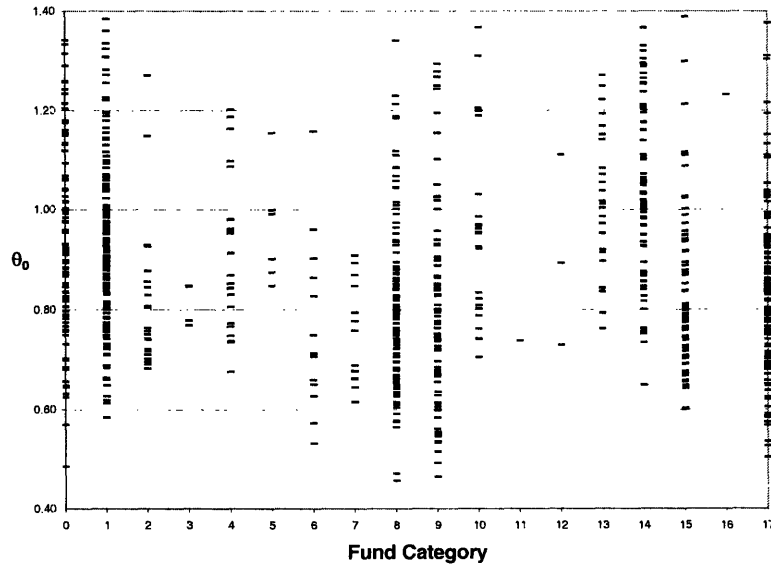
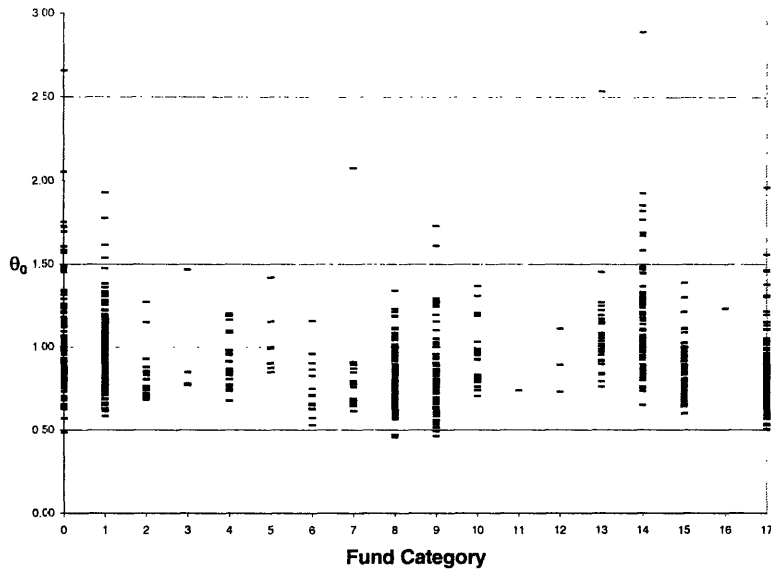


Figure 2-5: Estimated smoothing coefficients $\hat{\theta}_0$ for all funds in the TASS Hedge Fund database with at least five years of returns during the period from November 1977 to January 2001, ordered by categories 0 to 17. The two panels differ only in the range of the vertical axis, which is smaller for the lower panel so as to provide finer visual resolution of the distribution of estimated coefficients in the sample.

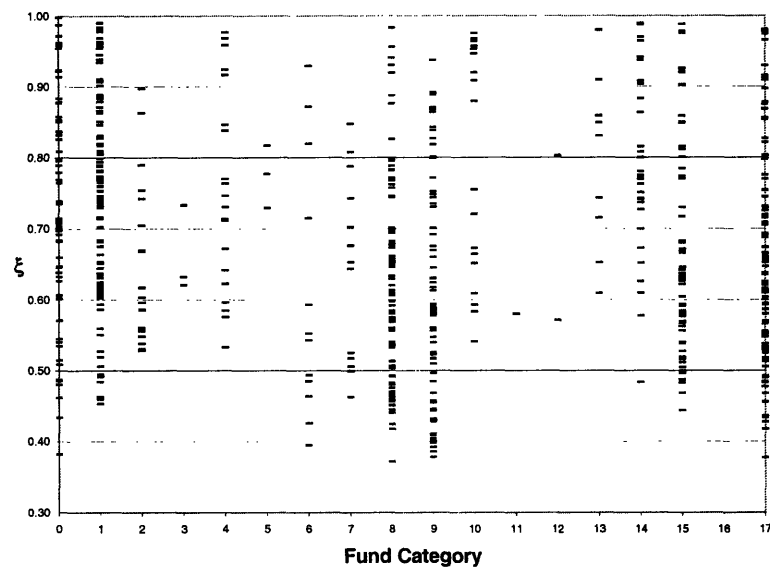
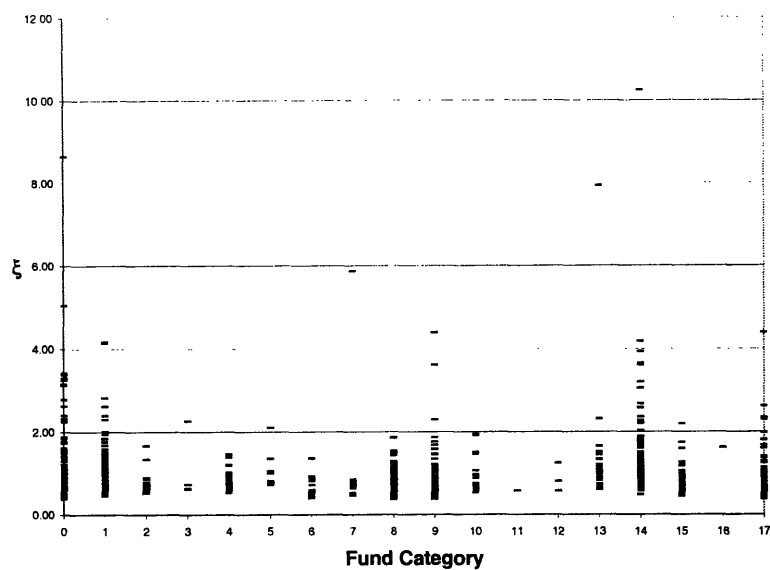


Figure 2-6: Estimated smoothing index $\hat{\xi}$ for all funds in the TASS Hedge Fund database with at least five years of returns during the period from November 1977 to January 2001, ordered by categories 0 to 17. The two panels differ only in the range of the vertical axis, which is smaller for the lower panel so as to provide finer visual resolution of the distribution of estimated smoothing indexes in the sample.

liquid American Express Extra Income Fund.³⁸ Table 2.10 contains summary statistics, market betas, contemporaneous and lagged market betas as in Asness, Kraib and Liew (2001), and smoothing-coefficient estimates for these index and mutual-fund returns.³⁹

Consistent with our interpretation of $\hat{\theta}_0$ as an indicator of liquidity, the returns of the most liquid portfolios in the first panel of Table 2.10—the Ibbotson Large Company Index, the Vanguard 500 Index Fund (which is virtually identical to the Ibbotson Large Company Index, except for sample period and tracking error), and the Ibbotson Long-Term Government Bond Index—have smoothing parameter estimates near unity: 0.92 for the Ibbotson Large Company Index, 1.12 for the Vanguard 500 Index Fund, and 0.92 for the Ibbotson Long-Term Government Bond Index. The first-order autocorrelation coefficients and lagged market betas also confirm their lack of serial correlation; 9.8% first-order autocorrelation for the Ibbotson Large Company Index, −2.3% for the Vanguard 500 Index Fund, and 6.7% for the Ibbotson Long-Term Government Bond Index, and lagged market betas that are statistically indistinguishable from 0. However, the values of $\hat{\theta}_0$ of the less liquid portfolios are less than 1.00: 0.82 for the Ibbotson Small Company Index, 0.84 for the Ibbotson Long-Term Corporate Bond Index, 0.82 for the Merrill Lynch Convertible Securities Index, and 0.67 for the American Express Extra Income Fund, and their first-order serial correlation coefficients are 15.6%, 15.6%, 6.4% and 35.4%, respectively, which, with the exception of the Merrill Lynch Convertible Securities Index, are considerably higher than those

³⁸As of January 31, 2003, the net assets of the Vanguard 500 Index Fund (ticker symbol: VFINX) and the AXP Extra Income Fund (ticker symbol: INEAX) are given by <http://finance.yahoo.com/> as \$59.7 billion and \$1.5 billion, respectively, and the descriptions of the two funds are as follows:

“The Vanguard 500 Index Fund seeks investment results that correspond with the price and yield performance of the S&P 500 Index. The fund employs a passive management strategy designed to track the performance of the S&P 500 Index, which is dominated by the stocks of large U.S. companies. It attempts to replicate the target index by investing all or substantially all of its assets in the stocks that make up the index.”

“AXP Extra Income Fund seeks high current income; capital appreciation is secondary. The fund ordinarily invests in long-term high-yielding, lower-rated corporate bonds. These bonds may be issued by U.S. and foreign companies and governments. The fund may invest in other instruments such as: money market securities, convertible securities, preferred stocks, derivatives (such as futures, options and forward contracts), and common stocks.”

³⁹Market betas were obtained by regressing returns on a constant and the total return of the S&P 500, and contemporaneous and lagged market betas were obtained by regressing returns on a constant, the contemporaneous total return of the S&P 500, and the first two lags.

of the more liquid portfolios.⁴⁰ Also, the lagged market betas are statistically significant at the 5% level for the Ibbotson Small Company Index (a t -statistic for $\hat{\beta}_1$: 5.41), the Ibbotson Long-Term Government Bond Index (t -statistic for $\hat{\beta}_1$: -2.30), the Merrill Lynch Convertible Securities Index (t -statistic for $\hat{\beta}_1$: 3.33), and the AXP Extra Income Fund (t -statistic for $\hat{\beta}_1$: 4.64).

The results for the CSFB Hedge Fund Indexes in the second panel of Table 2.10 are also consistent with the empirical results in Tables 2.8 and 2.9—indexes corresponding to hedge-fund strategies involving less liquid securities tend to have lower $\hat{\theta}_0$'s. For example, the smoothing-parameter estimates $\hat{\theta}_0$ of the Convertible Arbitrage, Emerging Markets, and Fixed-Income Arbitrage Indexes are 0.49, 0.75, and 0.63, respectively, and first-order serial correlation coefficients of 56.6%, 29.4%, and 39.6%, respectively. In contrast, the smoothing-parameter estimates of the more liquid hedge-fund strategies such as Dedicated Short Bias and Managed Futures are 0.99 and 1.04, respectively, with first-order serial correlation coefficients of 7.8% and 3.2%, respectively. While these findings are generally consistent with the results in Tables 2.8 and 2.9, it should be noted that the process of aggregation can change the statistical behavior of any time series. For example, Granger (1980, 1988) observes that the aggregation of a large number of stationary autoregressive processes can yield a time series that exhibits long-term memory, characterized by serial correlation coefficients that decay very slowly (hyperbolically, as opposed to geometrically as in the case of a stationary ARMA process). Therefore, while it is true that the aggregation of a collection of illiquid funds will generally yield an index with smoothed returns,⁴¹ the reverse need not be true—smoothed index returns need not imply that all of the funds comprising the index are illiquid. The latter inference can only be made with the benefit of additional information—essentially identification restrictions—about

⁴⁰However, note that the second-order autocorrelation of the Merrill Lynch Convertible Securities Index is 12.0% which is second only to the AXP Extra Income Fund in absolute magnitude, two orders of magnitude larger than the second-order autocorrelation of the Ibbotson bond indexes, and one order of magnitude larger than the Ibbotson stock indexes.

⁴¹It is, of course, possible that the smoothing coefficients of some funds may exactly offset those of other funds so as to reduce the degree of smoothing in an aggregate index. However, such a possibility is extremely remote and pathological if each of the component funds exhibits a high degree of smoothing.

the statistical relations among the funds in the index, i.e., covariances and possibly other higher-order co-moments, or the existence of common factors driving fund returns.

It is interesting to note that the first lagged market beta, $\hat{\beta}_1$, for the CSFB/Tremont Indexes is statistically significant at the 5% level in only three cases (Convertible Arbitrage, Event Driven, and Managed Futures), but the second lagged beta, $\hat{\beta}_2$, is significant in five cases (the overall index, Convertible Arbitrage, Fixed Income Arbitrage, Global Macro, and Long/Short). Obviously, the S&P 500 Index is likely to be inappropriate for certain styles, e.g., Emerging Markets, and these somewhat inconsistent results suggest that using a lagged market-beta adjustment may not completely account for the impact of illiquidity and smoothed returns.

Overall, the patterns in Table 2.10 confirm our interpretation of smoothing coefficients and serial correlation as proxies for liquidity, and suggest that there may be broader applications of our model of smoothed returns to other investment strategies and asset classes.

2.6.3 Cross-Sectional Regressions

A more quantitative summary of the cross-sectional properties of the smoothing parameter estimates for the 908 funds is given in Table 2.11, which contains the results of cross-sectional regressions of the smoothing parameter $\hat{\theta}_0$ and the smoothing index $\hat{\xi}$ on a number of 0/1 indicator variables.⁴² In the first two regressions, $\hat{\theta}_0$ and $\hat{\xi}$ are the dependent variables, respectively, and the regressors include a constant term, 17 indicator variables corresponding to the 17 hedge-fund categories defined by TASS (see Appendix 2.8.4), and an indicator variable that takes on the value 1 if the fund is open and 0 if it is closed to new investors. The third and fourth regressions have the same dependent variables— $\hat{\theta}_0$ and $\hat{\xi}$, respectively—and include the same regressors as the first two regressions but also include 0/1 indicator variables that indicate

⁴²To conserve space, we report regression results only for the maximum likelihood estimates under the constraint (49). Table A.8 of the Appendix reports corresponding results for the maximum likelihood estimates under the alternate constraint (75).

Series	Period	T	Mean (%)	SD (%)	$\hat{\beta}_1$ (%)	$\hat{\beta}_2$ (%)	$\hat{\beta}_3$ (%)	Market Model		Contemporaneous and Lagged Market Model							
								$\hat{\beta}$	SE($\hat{\beta}$)	R^2 (%)	$\hat{\beta}_0$	SE($\hat{\beta}_0$)	$\hat{\beta}_1$	SE($\hat{\beta}_1$)	$\hat{\beta}_2$	SE($\hat{\beta}_2$)	R^2 (%)
Ibbotson Small Company	192601-200112	912	1.35	8.63	15.6	1.7	-10.6	1.27	0.03	66.9	1.25	0.03	0.16	0.03	0.03	0.03	68.0
Ibbotson Long-Term Government Bonds	192601-200112	912	0.46	2.22	6.7	0.3	-8.3	0.07	0.01	2.8	0.07	0.01	-0.03	0.01	-0.02	0.01	3.6
Ibbotson Long-Term Corporate Bonds	192601-200112	912	0.49	1.96	15.6	0.3	-6.0	0.08	0.01	5.2	0.08	0.01	-0.01	0.01	-0.01	0.01	5.3
Ibbotson Large Company	192601-200112	912	1.03	5.57	9.8	-3.2	-10.7	1.00	0.00	100.0	1.00	0.00	0.00	0.00	0.00	0.00	100.0
Merrill Lynch Convertibles Index	199401-200210	168	0.99	3.43	6.4	12.0	5.1	0.59	0.05	48.6	0.60	0.05	0.15	0.05	0.07	0.04	52.2
AXP Extra Income Fund (INEAX)	199401-200112	210	0.67	2.04	35.4	13.1	2.5	0.21	0.03	20.7	0.21	0.03	0.12	0.03	0.04	0.03	28.7
Vanguard 500 Index Trust (VFINX)	197609-200112	304	1.16	4.36	-2.3	-6.8	-3.2	1.00	0.00	100.0	1.00	0.00	0.00	0.00	0.00	0.00	100.0
CSFB/Tremont Indices:																	
Aggregate Hedge Fund Index	199401-200210	106	0.87	2.58	11.2	4.1	-0.4	0.31	0.05	24.9	0.32	0.05	0.06	0.05	0.16	0.05	32.1
Convertible Arbitrage	199401-200210	106	0.81	1.40	56.6	42.6	15.6	0.03	0.03	1.1	0.04	0.03	0.09	0.03	0.06	0.03	12.0
Dedicated Short Bias	199401-200210	106	0.22	3.29	7.8	-6.3	-5.0	-0.94	0.08	58.6	-0.93	0.08	-0.06	0.08	0.08	0.08	59.3
Emerging Markets	199401-200210	106	0.34	3.38	29.4	1.2	-2.1	0.62	0.11	24.0	0.63	0.11	0.19	0.11	0.03	0.12	26.2
Equity Market Neutral	199401-200210	106	0.89	0.92	29.4	18.1	8.4	0.10	0.02	21.1	0.10	0.02	0.02	0.02	0.00	0.02	22.1
Event Driven	199401-200210	106	0.83	1.81	39.6	14.7	5.4	0.23	0.04	30.2	0.23	0.03	0.11	0.03	0.04	0.03	38.2
Fixed Income Arbitrage	199401-200210	106	0.55	1.18	34.8	10.8	4.6	0.24	0.09	7.5	0.26	0.09	-0.01	0.09	0.09	0.03	12.9
Global Macro	199401-200210	106	1.17	3.69	5.6	4.6	8.3	0.24	0.06	36.7	0.49	0.06	-0.06	0.06	0.23	0.15	14.1
Long/Short	199401-200210	106	0.98	3.34	15.9	5.9	-4.6	0.48	0.06	36.7	-0.13	0.06	0.06	0.06	0.15	0.06	14.1
Managed Futures	199401-200210	106	0.55	3.44	3.2	-6.3	0.7	-0.12	0.08	2.5	-0.13	0.08	-0.17	0.08	0.02	0.06	7.8

Series	Period	T	$\hat{\theta}_0$	SE($\hat{\theta}_0$)	$\hat{\theta}_1$	SE($\hat{\theta}_1$)	$\hat{\theta}_2$	SE($\hat{\theta}_2$)	ξ
Ibbotson Small Company	192601-200112	912	0.82	0.03	0.13	0.02	0.04	0.03	0.69
Ibbotson Long-Term Government Bonds	192601-200112	912	0.92	0.05	0.06	0.03	0.01	0.03	0.86
Ibbotson Long-Term Corporate Bonds	192601-200112	912	0.84	0.04	0.14	0.02	0.02	0.03	0.73
Ibbotson Large Company	192601-200112	912	0.92	0.05	0.09	0.03	0.03	0.03	0.85
Merrill Lynch Convertibles Index	199401-200112	168	0.82	0.07	0.02	0.06	0.16	0.05	0.70
AXP Extra Income Fund (INEAX)	199401-200112	210	0.67	0.03	0.24	0.03	0.09	0.04	0.51
Vanguard 500 Index Trust (VFINX)	197609-200112	304	1.12	0.17	-0.03	0.07	-0.09	0.07	1.26
CSFB/Tremont Indices:									
Aggregate Hedge Fund Index	199401-200210	106	0.86	0.12	0.09	0.08	0.04	0.08	0.76
Convertible Arbitrage	199401-200210	106	0.49	0.01	0.26	0.03	0.25	0.03	0.37
Dedicated Short Bias	199401-200210	106	0.39	0.20	0.08	0.09	-0.07	0.10	0.99
Emerging Markets	199401-200210	106	0.75	0.08	0.24	0.05	0.07	0.06	0.62
Equity Market Neutral	199401-200210	106	0.71	0.06	0.18	0.03	0.12	0.06	0.54
Event Driven	199401-200210	106	0.68	0.05	0.23	0.05	0.09	0.06	0.52
Fixed Income Arbitrage	199401-200210	106	0.63	0.04	0.28	0.04	0.08	0.05	0.49
Global Macro	199401-200210	106	0.91	0.14	0.04	0.08	0.05	0.08	0.84
Long/Short	199401-200210	106	0.82	0.10	0.13	0.07	0.06	0.07	0.58
Managed Futures	199401-200210	106	1.04	0.23	0.04	0.10	-0.08	0.11	1.08

Table 2.10: Summary statistics and maximum likelihood estimates of MA(2) smoothing process $R_t^e = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$. $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$, subject to the normalization $1 = \theta_0 + \theta_1 + \theta_2$, for the returns of various indexes and two mutual funds, the Vanguard 500 Index Trust (which tracks the S&P 500 index), and the AXP Extra Income Fund (which focuses on high current income and invests in long-term high-yielding lower-rated corporate bonds). Total returns of the S&P 500 index are used for both market models.

whether the fund is US-based (USBASED), and whether the geographical focus of the fund is global (GF-GLB), US (GF-USA), Asia/Pacific (GF-APC), Western Europe (GF-WEU), Eastern Europe (GF-EEU), and Africa (GF-AFR).

The results of the first regression are consistent with the general intuition gleaned from Figures 2-5 and 2-6. The category indicator variables with the most negative coefficients that are statistically significant at the 5% level are European Equity Hedge (-0.212), Fixed-Income Directional (-0.262), Event Driven (-0.218), Non-Directional/Relative Value (-0.211), Pure Emerging Market (-0.195), Fund of Funds (-0.178), implying that on average, funds in these categories have smaller smoothing coefficients $\hat{\theta}_0$, i.e., less liquidity or smoother returns. These point estimates can be used to approximate the marginal impact that a given investment style has on the smoothing profile of the fund's monthly returns. For example, from a no-smoothing baseline of 1, conditioning on belonging to the Fixed-Income Directional category yields an expected smoothing parameter $\hat{\theta}_0$ of $1 - 0.262 = 0.738$ and an expected smoothing index of $\hat{\xi}$ of $1 - 0.583 = 0.417$, other things equal (and assuming that the remaining indicator variables in the two regression equations are 0).

In contrast, the coefficients for Dedicated Shortseller and Pure Leveraged Currency indicators— -0.001 and 0.069 , respectively, with t -statistics of 0.01 and 0.11 , respectively—are positive and statistically insignificant at the 5% level, which is consistent with common intuition about the liquidity of these types of funds. Moreover, the coefficient for the Pure Managed Futures indicator is both positive and significant at the 5% level— -0.101 with a t -statistic of 3.00 —which is also consistent with the intuition that managed futures involve relatively liquid securities with well established marks that cannot easily be manipulated.

The last indicator variable included in the first two regressions takes on the value 1 if the fund is open to new investors and 0 if closed. If return-smoothing is actively being pursued, we might expect it to be more important for funds that are open since such funds are still attempting to attract new investors. This implies that the coefficient for this indicator variable should be negative—open funds should be more prone to smoothing than closed funds. Table 2.11 confirms this hypothesis: the

estimated coefficient for OPEN is -0.040 with a t -statistic of 2.03, implying that funds open to new investors have a smoothing coefficient $\hat{\theta}_0$ that is lower by 0.040 on average than funds that are closed. An alternate interpretation is that funds that are still open to new investors are typically those with smaller assets under management, and as a result, are less likely to be able to afford costly third-party valuations of illiquid securities in their portfolios. Unfortunately, many funds in the TASS database do not report assets under management so we were unable to investigate this alternative.

The third and fourth regressions in Table 2.11 include additional indicator variables that capture the fund's geographical base as well as the geographical focus of its investments, and we see that being in the US has a positive marginal impact on the conditional mean of $\hat{\theta}_0$, but being US-focused in its investments has a negative marginal impact. The latter result is somewhat counterintuitive but becomes less puzzling in light of the fact that approximately 46% of the funds are US-focused, hence many of the most illiquid funds are included in this category. Apart from this indicator, the geographical aspects of our sample of funds seem to have little impact on the cross-sectional variability in smoothing parameter estimates.

With R^2 's ranging from 9.0% to 17.7%, the regressions in Table 2.11 leave considerable cross-sectional variation unexplained, but this is no surprise given the noise inherent in the category assignments and the heterogeneity of investment styles even within each category. However, the general pattern of coefficients and t -statistics do suggest that the smoothing coefficients are capturing significant features of the cross section of hedge fund returns in our sample.

The final set of empirical estimates of the smoothing process (21)–(23) is for the linear regression model of Section 2.5.2, and is summarized in Table 2.12. Recall from Section 2.5.2 that the linear-regression estimates of $(\theta_0, \theta_1, \theta_2)$ are based on the assumption that true returns are given by the linear single-factor model (20) where the factor is the return on the S&P 500 index. To the extent that this assumption is a poor approximation to the true return-generating process, the corresponding smoothing parameter estimates will be flawed as well.

Table 2.12 reports the means and standard deviations of the estimates $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2)$

Regressor	$\hat{\theta}_0$	$\hat{\xi}$	$\hat{\theta}_0$	$\hat{\xi}$
Constant	1.059 (42.05)	1.314 (17.26)	1.086 (35.62)	1.407 (15.21)
US Equity Hedge	-0.077 (2.56)	-0.255 (2.82)	-0.076 (2.36)	-0.257 (2.62)
European Equity Hedge	-0.212 (3.79)	-0.531 (3.14)	-0.260 (4.06)	-0.676 (3.48)
Asian Equity Hedge	-0.086 (0.78)	-0.254 (0.77)	-0.076 (0.65)	-0.248 (0.71)
Global Equity Hedge	-0.113 (2.19)	-0.366 (2.35)	-0.084 (1.55)	-0.294 (1.80)
Dedicated Shortseller	0.001 (0.01)	-0.128 (0.45)	0.001 (0.01)	-0.130 (0.46)
Fixed-Income Directional	-0.262 (3.73)	-0.583 (2.75)	-0.258 (3.64)	-0.575 (2.67)
Convertible Fund (Long Only)	-0.180 (2.73)	-0.264 (1.32)	-0.188 (2.83)	-0.289 (1.44)
Event Driven	-0.218 (6.69)	-0.514 (5.23)	-0.223 (6.57)	-0.527 (5.12)
Non-Directional/Relative Value	-0.211 (6.07)	-0.432 (4.12)	-0.194 (5.37)	-0.399 (3.64)
Global Macro	-0.075 (1.38)	-0.271 (1.66)	-0.049 (0.86)	-0.230 (1.33)
Global Opportunity	-0.282 (1.17)	-0.656 (0.9)	-0.275 (1.14)	-0.647 (0.89)
Natural Resources	-0.109 (0.78)	-0.363 (0.86)	-0.112 (0.8)	-0.372 (0.87)
Pure Leveraged Currency	0.069 (1.32)	0.152 (0.96)	0.073 (1.37)	0.158 (0.98)
Pure Managed Futures	0.101 (3.00)	0.210 (2.06)	0.116 (3.29)	0.244 (2.29)
Pure Emerging Market	-0.195 (5.34)	-0.486 (4.40)	-0.189 (4.33)	-0.481 (3.63)
Pure Property	0.173 (0.72)	0.304 (0.42)	0.195 (0.80)	0.339 (0.46)
Fund of Funds	-0.178 (5.66)	-0.433 (4.57)	-0.170 (5.28)	-0.419 (4.29)
OPEN	-0.040 (2.03)	-0.077 (1.30)	-0.024 (1.17)	-0.041 (0.65)
USBASED			0.018 (1.00)	0.030 (0.54)
GF-GLB			-0.050 (2.37)	-0.139 (2.19)
GF-USA			-0.053 (2.45)	-0.144 (2.19)
GF-APC			-0.049 (1.59)	-0.128 (1.37)
GF-WEU			0.018 (0.61)	0.052 (0.58)
GF-EEU			-0.011 (0.26)	-0.045 (0.35)
GF-AFR			0.016 (0.28)	0.054 (0.31)
Sample Size	908	908	891	891
Adjusted R^2 (%)	16.5	9.0	17.7	9.9

Table 2.11: Regressions of maximum likelihood estimated smoothing coefficient $\hat{\theta}_0$ and smoothing index $\hat{\xi}$ on indicator variables for 908 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001, where the maximum likelihood estimators of the MA coefficients $(\theta_0, \theta_1, \theta_2)$ are constrained to sum to 1. Absolute values of t -statistics are given in parentheses. The indicator variables are OPEN (1 if the fund is open, 0 otherwise); the fund categories (1 if the fund belongs to the category, 0 otherwise); USBASED (1 if the fund is based in the US, 0 otherwise); and geographical focus categories (1 if the geographical focus of the fund is in a given region, 0 otherwise, where the regions are USA, Asia Pacific, Western Europe, Eastern Europe, and Africa, respectively).

and $\hat{\xi}$ for each of the categories, as well as the Durbin-Watson statistic and the regression R^2 . In contrast to the maximum likelihood estimates of Table 2.8, the regression estimates are considerably more noisy, with cross-sectional standard deviations for the coefficients that are often an order of magnitude larger than the means, and in almost every case larger than the standard deviations of Table 2.8. For example, the average $\hat{\theta}_0$ for the Not Categorized category is 0.659, but the standard deviation is 8.696. The mean of $\hat{\theta}_0$ for Fixed-Income Directional funds is -1.437 and the standard deviation is 6.398. These results are not unexpected given the role that the linear single-factor model plays in the estimation process—if true returns contain additional common factors, then the linear-regression approach (62) will yield biased and inconsistent estimators for the smoothing parameters in (21)–(23).

The R^2 statistics in Table 2.12 yield some indication of the likelihood of omitted factors among the different categories. The highest mean R^2 's are for the US Equity Hedge, Dedicated Shortseller, and Convertible Fund (Long Only) categories, with values of 26.1%, 43.0%, and 25.0%, respectively, which is consistent with the fact that our single factor is the S&P 500.⁴³ However, several categories have mean R^2 's below 10%, implying relatively poor explanatory power for the single-factor model and, therefore, noisy and unreliable estimates of the smoothing process.

Overall, the results in Table 2.12 suggest that the linear regression approach is dominated by the maximum likelihood procedure, and that while the regression coefficients of lagged market returns may provide some insight into the net market exposure of some funds, they are considerably less useful for making inferences about illiquidity and smoothed returns.

2.6.4 Illiquidity Vs. Smoothing

To address the issue of systematic versus idiosyncratic effects of illiquidity and return-smoothing, we estimate the more general linear factor model of smoothing (67)–(69) with $k = 3$ for the subset of Convertible Funds (Long Only), which consists of 15

⁴³We have omitted the Global Opportunity category from this comparison despite its R^2 of 30.9% because it contains only a single fund.

Category	N	$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\xi}$		D.W.		$R^2(\%)$	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Not Categorized	111	0.659	8.696	0.459	9.363	-0.118	2.260	167.540	1432.164	1.86	0.33	13.6	16.8
US Equity Hedge	162	0.695	1.502	0.094	0.415	0.211	1.220	4.427	38.503	1.80	0.28	26.1	17.5
European Equity Hedge	22	0.561	0.155	0.340	0.683	0.099	0.708	1.386	4.287	1.76	0.19	18.0	10.8
Asian Equity Hedge	5	0.405	0.252	0.323	0.179	0.272	0.088	0.425	0.086	1.82	0.24	7.4	4.5
Global Equity Hedge	27	0.846	0.400	0.057	0.331	0.097	0.364	1.115	0.913	1.65	0.22	22.1	19.8
Dedicated Shortseller	7	0.908	0.182	0.115	0.101	-0.023	0.203	0.911	0.360	1.94	0.16	43.0	14.7
Fixed-Income Directional	13	-1.437	6.398	4.385	14.552	-1.948	8.179	320.099	1151.235	1.62	0.29	13.1	9.9
Convertible Fund (Long Only)	15	0.538	0.262	0.343	0.152	0.120	0.178	0.536	0.219	1.53	0.24	25.0	15.0
Event Driven	109	0.582	0.479	0.293	0.493	0.125	0.225	0.958	3.568	1.63	0.36	18.0	11.3
Non-Directional/Relative Value	85	0.643	4.057	0.895	2.321	-0.538	4.761	45.493	244.844	1.62	0.46	12.2	9.2
Global Macro	24	0.558	0.349	0.194	0.341	0.247	0.190	0.674	0.409	1.76	0.28	13.5	8.0
Global Opportunity	1	0.619	—	0.191	—	0.191	—	0.455	—	1.55	—	30.9	—
Natural Resources	3	0.691	0.138	0.106	0.087	0.203	0.068	0.551	0.163	1.88	0.31	2.6	0.9
Pure Leveraged Currency	26	-0.080	1.376	0.360	1.066	0.720	1.987	7.363	24.457	1.89	0.16	4.4	6.0
Pure Managed Futures	93	-0.032	1.588	0.696	1.649	0.336	1.611	8.350	26.079	1.99	0.26	5.9	5.9
Pure Emerging Market	72	0.899	0.801	0.157	0.465	-0.056	0.452	1.883	6.486	1.51	0.29	22.8	10.6
Pure Property	1	1.319	—	-0.024	—	-0.295	—	1.828	—	2.31	—	15.4	—
Fund of Funds	132	0.674	1.516	-0.049	2.566	0.375	3.059	18.694	171.624	1.63	0.30	21.6	15.0
All	908	0.554	3.534	0.366	3.950	0.081	2.401	34.268	529.561	1.73	0.34	17.8	15.1

Table 2.12: Means and standard deviations of linear regression estimates of MA(2) smoothing process $R_t^o = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$. $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$ under the assumption of a linear single-factor model for R_t where the factor is the total return of the S&P 500 Index. for 908 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001.

funds in our sample of 908 funds. We take as our common factor Λ_t the Merrill Lynch Convertible Securities Index (see footnote 37 for a description), and estimate the linear regression equation via maximum likelihood and then renormalize the MA coefficients according to (69) and recompute the standard errors accordingly. Table 2.13 contains the regression coefficients as well as the smoothing coefficients, and t -statistics are reported instead of standard errors because we have specific null hypotheses to test as described in Section 2.5.2.

The estimates in Table 2.13 show that including a common factor can have a significant impact on the smoothing parameter estimates. For example, the value of $\hat{\theta}_0$ for Fund 1146 under the smoothing process (21)–(23) is 0.689, and its t -statistic under the null hypothesis that $\theta_0 = 1$ is -6.01 . However, under the linear factor specification (67)–(69), the smoothing coefficient estimate becomes 1.172 with a t -statistic of 0.96. Nevertheless, for other funds in our sample of 15, the smoothing parameter estimates are virtually unchanged by including the contemporaneous and lagged common factors. For example, the value of $\hat{\theta}_0$ for Fund 4243 under the smoothing process (21)–(23) and the linear factor model (67)–(69) is 0.645 and 0.665, respectively, with t -statistics of -8.18 and -5.36 , respectively.

We see from Table 2.13 that the Convertible Securities Index is statistically significant at the 5% level for most, but not all, of the 15 funds, and that its first lag is significant for only four funds (818, 2245, 4204, and 4326), and its second lag is significant for only two funds (1908 and 4216). For five of these six funds, the lagged-index coefficients are positive in sign, which is consistent with the smoothing model (68)–(69) (assuming that the funds' contemporaneous factor loadings and smoothing parameters are positive). For Fund 4216, the smoothing parameter estimate $\hat{\theta}_0$ is still significantly different from 1 even after accounting for the common factor, but for Fund 4204, it is not.

It is tempting to conclude from these results that the linear factor model (67)–(69) is capable of distinguishing between systematic illiquidity and idiosyncratic return-smoothing behavior. For example, we might argue that those funds which continue to exhibit significant smoothing parameters $\hat{\theta}_0$ even after accounting for common factors

must be engaged in return-smoothing behavior. However, several caveats must be kept in mind before reaching such conclusions. First, we cannot be certain that the Merrill Lynch Convertible Securities Index is the appropriate common factor for these funds, despite the TASS classification—some funds may be involved in complex trading strategies involving convertible securities while others are engaged in simpler buy-and-hold strategies.⁴⁴ Regressing a fund’s returns on a highly serially correlated common factor that is not directly relevant to that fund’s investment process will, nevertheless, have an effect on the smoothing-parameter estimates $\hat{\theta}_k$, and the effect may be in either direction depending on the relation between the common factor and the fund’s observed returns. Second, even if a common factor can account for much of the serial correlation in a fund’s observed returns, an explanation for the source of the factor’s serial correlation is still required—if the fund is a buy-and-hold version of the common factor, e.g., a fund-of-funds designed to replicate the CSFB/Tremont Convertible Arbitrage Index, then it is of small comfort to investors in such a fund-of-funds that there is not much smoothing in observed returns beyond what is already present in the common factor. And finally, no econometric model can fully capture the many qualitative and often subjective characteristics of a fund’s investment process, and such information is likely to be of particular relevance in distinguishing between illiquidity and smoothed returns at the fund level.

These caveats suggest that a more comprehensive econometric analysis of hedge-fund returns may be worthwhile, with particular emphasis on constructing common factors for hedge funds with similar investment mandates and processes. By developing a better understanding for the common risk exposures that certain hedge funds share, it may be possible to differentiate between systematic and idiosyncratic illiquidity and provide investors and managers with a more refined set of tools with which to optimize their investment plans.

⁴⁴ In particular, of the 15 “Convertible Fund (Long Only)” funds in our sample, 12 funds involve long-only positions in convertible bonds, possibly with a limited degree of leverage, but with no equity or credit protection, and the remaining 3 funds (4204, 4145 and 4326) are convertible arbitrage funds that involve long positions in convertible bonds and short positions in the corresponding stocks. We thank Stephen Jupp of TASS for providing us with this information.

Code	T	Period	$\hat{\mu}$ (t-stat)	$\hat{\gamma}_0$ (t-stat)	$\hat{\gamma}_1$ (t-stat)	$\hat{\gamma}_2$ (t-stat)	Maximum Likelihood Smoothing Parameter Estimates						
							$\hat{\theta}_0$	$t(\hat{\theta}_0)$	$\hat{\theta}_1$	$t(\hat{\theta}_1)$	$\hat{\theta}_2$	$t(\hat{\theta}_2)$	$\hat{\xi}$
31	90	199307-200012	-1.159 (-2.28)	1.171 (11.51)	0.036 (0.36)	-0.110 (-0.92)	0.903 -0.55	-0.236 -0.24	-2.39 -3.24	0.393 0.123	6.42 1.94	1.06 0.65	
818	72	199502-200101	-0.754 (-1.66)	0.387 (5.84)	0.168 (1.98)	0.014 (0.15)	0.986 -3.46	0.139 0.261	1.31 4.71	-0.124 -0.019	-0.96 -0.28	1.01 0.64	
843	79	199407-200101	-0.716 (-1.86)	0.340 (4.50)	0.144 (1.61)	0.106 (0.97)	0.971 -0.18	0.094 0.159	0.53 2.34	-0.065 -0.006	-0.51 -0.08	0.96 0.74	
1146	94	199010-199807	-1.503 (-7.59)	1.043 (13.15)	0.138 (1.75)	-0.029 (-0.39)	1.172 0.689	-0.044 -6.01	-0.30 0.187	-0.128 3.79	-0.98 0.124	1.39 0.53	
1148	73	199501-200101	-0.966 (-3.07)	0.617 (10.49)	0.106 (1.25)	-0.009 (-0.11)	1.125 0.795	0.51 -2.81	0.312 0.211	3.16 3.39	-0.437 -0.005	-1.72 -0.08	1.55 0.68
1908	81	199405-200101	-1.191 (-1.37)	1.325 (10.51)	0.121 (0.82)	-0.425 (-2.09)	0.989 0.909	-0.06 -1.12	0.052 0.140	0.42 1.93	-0.041 -0.049	-0.37 -0.60	0.98 0.85
2245	67	199507-200101	-0.607 (-1.27)	0.287 (3.48)	0.174 (2.37)	0.034 (0.31)	0.938 -0.36	0.087 0.173	0.70 2.34	-0.025 -0.043	-0.22 -0.50	0.89 0.79	
2564	63	199511-200101	-0.621 (-1.10)	0.291 (3.22)	0.108 (1.39)	0.123 (0.39)	0.866 0.778	-0.67 -2.86	0.279 0.303	2.17 5.45	-0.145 -0.080	-0.90 -1.03	0.85 0.70
3099	73	199501-200101	0.093 (0.41)	-0.040 (-0.96)	-0.047 (-1.32)	0.018 (0.39)	0.632 -4.10	0.104 0.121	1.71 2.37	0.264 0.263	5.49 5.68	0.48 0.46	
4145	63	199511-200101	-0.487 (-0.73)	0.368 (2.14)	-0.094 (-0.61)	-0.031 (-0.14)	2.042 2.077	4.82 4.41	0.015 0.161	0.03 1.16	-1.058 -5.07	-1.08 5.87	5.29 0.52
4204	72	199501-200012	-0.876 (-2.88)	0.303 (4.52)	0.304 (6.37)	0.069 (0.95)	0.769 -1.26	0.122 0.217	0.85 4.08	0.109 0.105	1.46 1.80	0.62 0.50	
4216	104	199206-200101	-0.115 (-1.07)	0.012 (0.59)	0.012 (0.50)	0.079 (2.09)	0.650 -4.60	0.151 -7.48	3.47 2.84	0.300 0.302	4.60 4.49	0.48 0.50	
4243	118	199101-200010	-0.322 (-1.07)	0.036 (0.59)	0.110 (1.70)	0.081 (1.15)	0.665 -5.36	0.270 0.279	6.24 7.76	0.066 0.076	0.89 1.76	0.52 0.50	
4295	82	199403-200012	-0.014 (-0.07)	0.039 (0.51)	-0.032 (-0.72)	0.014 (0.21)	0.894 -0.61	-0.031 0.009	-6.22 0.10	0.136 0.098	1.63 1.26	0.82 0.81	
4326	149	198807-200101	-0.154 (-2.10)	0.082 (6.02)	0.039 (2.60)	0.021 (1.24)	0.700 -8.62	-4.71 0.243	0.229 7.02	0.071 0.095	1.40 2.43	0.55 0.51	

Table 2.13: Maximum likelihood estimates of linear regression model with MA(2) errors, $R_t^q = \mu + \gamma_0 A_t + \gamma_1 A_{t-1} + \gamma_2 A_{t-2} + u_t$, $u_t = \theta_0 \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$, subject to the normalization $1 = \theta_0 + \theta_1 + \theta_2$, for all 15 Convertible Funds (Long Only) among the sample of 908 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001. The factor A_t is taken to be the Merrill Lynch Convertible Securities Index. For comparison, maximum likelihood estimates of θ_0 , θ_1 , θ_2 , and t -statistics from the MA(2) smoothing process $R_t^q = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, subject to the normalization $1 = \theta_0 + \theta_1 + \theta_2$, are provided below each of the main rows. t -statistics are computed with respect to the null hypothesis that the coefficient is 0, except for $\hat{\theta}_0$, for which the null hypothesis $\theta_0 = 1$ is used, hence the t -statistic is computed as $(\hat{\theta}_0 - 1)/SE(\hat{\theta}_0)$.

2.6.5 Smoothing-Adjusted Sharpe Ratio Estimates

For each of the 908 funds in our sample, we compute annual Sharpe ratios in three ways relative to a benchmark return of 0: the standard method ($\sqrt{12}$ times the ratio of the mean monthly return to the monthly return standard deviation), the serial-correlation-adjusted method in Lo (2002), and the small-sample method described in Section 2.5.4. The results are summarized in Table 2.14.

The largest differences between standard and smoothing-adjusted Sharpe ratios are found in the same categories that the smoothing-process estimates of Section 2.6.2 identified as the most illiquid: Fixed-Income Directional (20.3% higher average Sharpe ratio relative to SR**), Convertible Fund (Long Only) (17.8%), Non-Directional/Relative Value (16.0%), Pure Emerging Market (16.3%), and Fund of Funds (17.8%). For two categories—Dedicated Shortseller and Managed Futures—the bias is reversed, a result of negative serial correlation in their returns. For the other categories, Table 2.14 shows that the smoothing-adjusted Sharpe ratios are similar in magnitude to the usual estimates. These differences across categories suggest the importance of taking illiquidity and smoothed returns into account in evaluating the performance of hedge funds.

Category	Sharpe Ratios For Combined Sample						
	<i>N</i>	SR		SR*		SR**	
		Mean	SD	Mean	SD	Mean	SD
Not Categorized	111	1.12	1.09	1.06	0.87	1.00	0.82
US Equity Hedge	162	1.26	0.75	1.31	0.75	1.23	0.69
European Equity Hedge	22	1.43	0.74	1.43	0.80	1.33	0.74
Asian Equity Hedge	5	0.50	0.39	0.52	0.39	0.49	0.37
Global Equity Hedge	27	0.90	0.61	0.95	0.66	0.89	0.61
Dedicated Shortseller	7	0.28	0.59	0.32	0.64	0.30	0.61
Fixed-Income Directional	13	2.02	2.35	1.80	2.23	1.68	2.06
Convertible Fund (Long Only)	15	1.83	1.20	1.66	0.85	1.55	0.80
Event Driven	109	2.36	1.45	2.21	1.57	2.08	1.47
Non-Directional/Relative Value	85	2.20	1.86	2.03	2.39	1.89	2.22
Global Macro	24	1.08	0.67	1.14	0.73	1.07	0.70
Global Opportunity	1	-0.56	—	-0.39	—	-0.37	—
Natural Resources	3	0.60	0.25	0.56	0.23	0.52	0.21
Pure Leveraged Currency	26	0.63	0.49	0.65	0.50	0.61	0.47
Pure Managed Futures	93	0.54	0.55	0.63	0.60	0.60	0.56
Pure Emerging Market	72	0.39	0.45	0.36	0.44	0.34	0.40
Pure Property	1	0.42	—	0.45	—	0.41	—
Fund of Funds	132	1.44	1.01	1.30	0.88	1.22	0.82
All	908	1.32	1.24	1.27	1.27	1.19	1.18

Table 2.14: Means and standard deviations of Sharpe ratios of 908 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001. SR is the standard Sharpe ratio, SR* is the smoothing-adjusted Sharpe ratio of Lo (2002), and SR** is the smoothing-adjusted Sharpe ratio using $\hat{\sigma}_{NW}$. All Sharpe ratios are computed with respect to a 0 benchmark.

2.7 Conclusions

Although there are several potential explanations for serial correlation in asset returns, we have argued in this paper that the serial correlation present in the returns of hedge funds is due primarily to illiquidity and smoothed returns. Using a simple econometric model in which observed returns are a finite moving-average of unobserved economic returns, we are able to generate empirically realistic levels of serial correlation for historical hedge-fund returns while, at the same time, explaining the findings of Asness, Krail and Liew (2001) regarding the significance of lagged market returns in market-model regressions for hedge funds. Although our moving-average specification is similar to some of the early models of nonsynchronous trading, our motivation is quite different and is meant to cover a broader set of factors that give rise to serial correlation and smoothed returns, even in the presence of synchronously recorded prices.

Maximum likelihood estimates of our smoothing model for the returns of 908 hedge funds in the TASS Hedge Fund database yield empirically plausible estimates of smoothing coefficients and suggest that simple time-series measures such as our smoothing index may serve as useful proxies for a hedge fund's illiquidity risk exposure. In some cases, our econometric model may also be useful for flagging possible cases of deliberate performance-smoothing behavior, although additional information will need to be gathered before any firm conclusions regarding such behavior can be made. Regardless of the sources of serial correlation, illiquidity exposure is the main implication and this has potentially important consequences for both managers and investors. Therefore, we also develop a set of tools for quantifying the degree of smoothing in the data and adjusting for smoothed returns in computing performance statistics such as means, variances, market betas, and Sharpe ratios, and derive their asymptotic distributions using continuous-record asymptotics that can better accommodate the small sample sizes of most hedge-fund datasets.

Our empirical results suggest several applications for our econometric model of illiquidity and smoothed returns. Despite the general consistency of our empirical

results with common intuition regarding the levels of illiquidity among the various hedge-fund investment styles, the variation in estimated smoothing coefficients within each category indicates that there may be better ways of categorizing hedge funds. Given the importance of liquidity for the typical hedge-fund investor, it may be useful to subdivide each style category into “liquidity tranches” defined by our smoothing index. This may prove to be especially useful in identifying and avoiding the potential wealth transfers between new and existing investors that can occur from the opportunistic timing of hedge-fund investments and redemptions. Alternatively, our smoothing parameter estimates may be used to compute illiquidity exposure measures for portfolios of hedge funds or fund of funds, which may serve as the basis for a more systematic approach to managing portfolios that include alternative investments.

Although we have focused on hedge funds in this paper, our analysis may be applied to other investments and asset classes, e.g., real estate, venture capital, private equity, art and other collectibles, and other assets for which illiquidity and smoothed returns are even more problematic, and where the estimation of smoothing profiles can be particularly useful for providing investors with risk transparency. More generally, our econometric model may be applied to a number of other contexts in which there may be a gap between reported results and economic realities. For example, recent events surrounding the collapse of Enron and other cases of corporate accounting irregularities have created renewed concerns about “earnings management” in which certain corporations are alleged to have abused accounting conventions so as to smooth earnings, presumably to give the appearance of stability and consistent growth.⁴⁵ The impact of such smoothing can sometimes be “undone” using an econometric model such as ours.

There are a number of outstanding issues regarding our analysis of illiquidity and smoothed returns that warrant further study. Perhaps the most pressing issue is whether the proximate source of smoothing is inadvertent or deliberate. Our linear regression model with contemporaneous and lagged common factors may serve as the

⁴⁵See Beneish (2001) and Healy and Wahlen (1999) for reviews of the extensive literature on earnings management.

starting point for distinguishing between systematic illiquidity versus idiosyncratic smoothing behavior. However, this issue is likely to require additional information about each fund along the lines of Chandar and Bricker's (2002) study, e.g., the size of the fund, the types of the securities in which the fund invests, the accounting conventions used to mark the portfolio, the organization's compensation structure, and other operational aspects of the fund. With these additional pieces of information, we may construct more relevant common factors for our linear-regression framework, or relate the cross-sectional variation in smoothing coefficients to assets under management, security type, fee structure, and other characteristics, yielding a more complete picture of the sources of smoothed returns.

It may also be fruitful to view the hedge-fund industry from a broader perspective, one that acknowledges the inherent capacity constraints of certain types of strategies as well as the time lags involved in shifting assets from one type to another. Because of the inevitable cross-sectional differences in the performance of hedge-fund styles, assets often flow in loosely coordinated fashion from one style to another, albeit under various institutional restrictions such as calendar-specific periods of liquidity, tiered redemption schedules, redemption fees, and other frictions. The interactions between asset flows and institutional rigidities—especially over time—may sometimes cause statistical side-effects that include periodicities in performance and volatility, time-varying correlations, structural breaks, and under certain conditions, serial correlation. The dynamics of the hedge-fund industry are likely to be quite different than that of more traditional investment products, hence the “ecological” framework of Niederhoffer (1998, Chapter 15), Farmer (1998), and Farmer and Lo (1999), or the system dynamics approach of Getmansky and Lo (2003) might be more conducive paradigms for addressing these issues.

Finally, from the hedge-fund investor's perspective, a natural extension of our analysis is to model illiquidity directly and quantify the illiquidity premium associated with each hedge-fund investment style, perhaps in a linear-factor framework such as Chordia, Roll and Subrahmanyam (2000) and Pastor and Stambaugh (2002). Whether such factor models can forecast liquidity crises like August 1998, and whether

there are “systematic” illiquidity factors that are common to categories of hedge funds are open questions that are particularly important in the context of hedge-fund investments. We plan to address these and other related questions in our ongoing research.

2.8 Appendix

Proofs of Propositions 3, 4, and 5 are provided in Sections 2.8.1, 2.8.2, and 2.8.3. respectively. Section 2.8.4 provides definitions of the 17 categories from TASS, and Section 2.8.5 contains additional empirical results.

2.8.1 Proof of Proposition 3

The constraint (49) may be used to substitute out θ_0 , hence we need only consider (θ_1, θ_2) . Now it is well known that in the standard MA(2) specification where the usual identification condition is used in place of (49), i.e.,

$$X_t = \varepsilon_t + a\varepsilon_{t-1} + b\varepsilon_{t-2},$$

the asymptotic distribution of the maximum likelihood estimators (\hat{a}, \hat{b}) is given by:

$$\sqrt{T} \left(\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) \stackrel{a}{\sim} \mathcal{N}(0, \mathbf{V}) \quad (\text{A.1})$$

where

$$\mathbf{V} \equiv \begin{bmatrix} 1 - b^2 & a(1 - b) \\ a(1 - b) & 1 - b^2 \end{bmatrix}. \quad (\text{A.2})$$

But under our normalization (49), there is a simple functional relation between (\hat{a}, \hat{b}) and $(\hat{\theta}_1, \hat{\theta}_2)$:

$$\hat{\theta}_1 = \frac{\hat{a}}{1 + \hat{a} + \hat{b}}, \quad \hat{\theta}_2 = \frac{\hat{b}}{1 + \hat{a} + \hat{b}}. \quad (\text{A.3})$$

Therefore, we can apply the delta method to obtain the asymptotic distribution of $(\hat{\theta}_1, \hat{\theta}_2)$ as:

$$\sqrt{T} \left(\begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} - \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \right) \stackrel{a}{\sim} \mathcal{N}(0, \mathbf{J}\mathbf{V}\mathbf{J}') \quad (\text{A.4})$$

where the matrix \mathbf{J} is Jacobian associated with (A.3):

$$\mathbf{J} = \frac{1}{(1+a+b)^2} \begin{bmatrix} 1+b & -a \\ -b & 1+a \end{bmatrix}. \quad (\text{A.5})$$

Then we have:

$$\mathbf{J}\mathbf{V}\mathbf{J}' = \frac{1}{(1+a+b)^3} \begin{bmatrix} -(1+b)(-1+a-ab+b^2) & b(-1+a-ab+b^2) \\ b(-1+a-ab+b^2) & -(-1+b)(-1+a-ab+b^2) \end{bmatrix} \quad (\text{A.6})$$

and solving for a and b as a function of θ_1 and θ_2 using (A.3) and substituting these expressions into A.6 yields the desired result.

The process X_t is invertible if and only if the roots of characteristic polynomial

$$f(x) = \theta_0 x^2 + \theta_1 x + \theta_2 \quad (\text{A.7})$$

lie inside the unit circle in the complex plane. It is easy to see that this is equivalent to the condition that the roots of

$$f(x) = f\left(\frac{z+1}{z-1}\right) = \frac{z^2 + 2(1-\theta_1-2\theta_2)z + 1 - 2\theta_1}{(z-1)^2} \quad (\text{A.8})$$

lie in the left half-plane (Samuelson, 1941, was perhaps the first to state this result). Applying the Routh-Hurwitz necessary and sufficient conditions to (A.8) then yields the desired result.

2.8.2 Proof of Proposition 4

Theorem 3 [Herrndorf (1984)] *If $\{\varepsilon_t\}$ satisfies the following assumptions:*

(A1) $E[\varepsilon_t] = \mu$ for all t .

(A2) $\sup_t E|\varepsilon_t - \mu|^\beta < \infty$ for some $\beta > 2$.

$$(A3) \quad 0 < \sigma_o^2 = \lim_{T \rightarrow \infty} E \left[\frac{1}{T} \sum_{j=1}^n (\varepsilon_j - \mu)^2 \right] < \infty .$$

(A4) $\{\varepsilon_t\}$ is strong-mixing with mixing coefficients α_k that satisfy:

$$\sum_{j=1}^{\infty} \alpha_j^{1-\frac{2}{\beta}} < \infty . \quad (A.9)$$

then as n increases without bound,

$$W_n(s) \equiv \frac{1}{\sigma_o \sqrt{n}} \sum_{j=1}^{[ns]} (\varepsilon_j - \mu) \Rightarrow W(s) , \quad s \in [0, 1] \quad (A.10)$$

where $[ns]$ denotes the greater integer less than or equal to ns and ' \Rightarrow ' denotes weak convergence.

With these results in hand, we are ready to prove Proposition 4. Let returns R_t be given by

$$R_t = \varepsilon_t, \quad (A.11)$$

where ε_t satisfies assumptions (A1)–(A4) of Theorem 3, and recall that

$$\hat{\mu} = \frac{1}{T} \sum_1^T \varepsilon_t \quad (A.12)$$

$$\hat{\sigma}_{NW}^2 = \frac{1}{T} \sum_1^T (\varepsilon_t - \hat{\mu})^2 + \frac{2}{T} \sum_{j=1}^{\theta} \left(1 - \frac{j}{\theta+1}\right) \sum_{t=j+1}^T (\varepsilon_t - \hat{\mu})(\varepsilon_{t-j} - \hat{\mu}). \quad (A.13)$$

Let us define

$$I_1 = \frac{1}{T} \sum_1^T (\varepsilon_t - \hat{\mu})^2, \quad (A.14)$$

$$I_2 = \frac{2}{T} \sum_{j=1}^{\theta} \left(1 - \frac{j}{\theta+1}\right) \sum_{t=j+1}^T (\varepsilon_t - \hat{\mu})(\varepsilon_{t-j} - \hat{\mu}). \quad (A.15)$$

Observe that

$$I_1 = \frac{1}{T} \sum_1^T (\varepsilon_t - \hat{\mu})^2 = \frac{1}{T} \sum_1^T \varepsilon_t^2 - \hat{\mu}^2 \xrightarrow{p} \sigma_\varepsilon^2, \quad (\text{A.16})$$

where

$$\sigma_\varepsilon^2 \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \sum_1^T \varepsilon_t^2 = \mathbb{E}[\varepsilon_t^2]. \quad (\text{A.17})$$

The second term I_2 can be written as

$$I_2 \frac{T(m+1)}{2} = \sum_{j=0}^m (m+1-j) \sum_{t=j+1}^T (\varepsilon_t - \hat{\mu})(\varepsilon_{t-j} - \hat{\mu}) = J_1 + J_2 + J_3, \quad (\text{A.18})$$

where

$$J_1 = \sum_{j=0}^m (m+1-j) \sum_{t=j+1}^T \varepsilon_t \varepsilon_{t-j}, \quad (\text{A.19})$$

$$J_2 = \sum_{j=0}^m (m+1-j) \sum_{t=j+1}^T \hat{\mu} (\varepsilon_t + \varepsilon_{t-j}), \quad (\text{A.20})$$

$$J_3 = \sum_{j=0}^m (m+1-j) \sum_{t=j+1}^T \hat{\mu}^2. \quad (\text{A.21})$$

One can show that

$$J_1 = \sum_{t=1}^{T-1} S_t^2 - (m-1) \sum_{t=1}^{T-1} \varepsilon_t S_t - \sum_{t=1}^{T-(m+1)} S_t S_{t+m+1} + S_T \left(m S_{T-1} - \sum_{t=T-m}^{T-1} S_t \right). \quad (\text{A.22})$$

Applying the Functional Central Limit Theorem to (A.22) we have

$$J_1/T^2 \Rightarrow \sigma^2 \int_0^1 W(r)(W(r) - W(\min(r+\lambda, 1))) dr - \frac{(m-1)}{2T} (\sigma^2 W^2(1) + \sigma_\varepsilon^2) + \frac{m}{T} \sigma^2 W^2(1). \quad (\text{A.23})$$

Similar, one can show that

$$J_2 = \frac{m(m+1)}{2} \hat{\mu} S_T - \hat{\mu} \sum_{j=1}^m (m+1-j)(S_{T-j} - S_j). \quad (\text{A.24})$$

Applying the Functional Central Limit Theorem to (A.24) yields

$$J_2/T^2 \Rightarrow -\frac{m(m+1)}{2T^2} \sigma^2 W^2(1) - \sigma^2 W(1) \int_0^\lambda (\lambda-r)(W(1-r) - W(r)) dr. \quad (\text{A.25})$$

The third term of (A.18) can be rewritten as

$$J_3 = \hat{\mu}^2 \frac{m(m+1)(3T-m-5)}{6}. \quad (\text{A.26})$$

By the Functional Central Limit Theorem

$$J_3/T^2 \Rightarrow \sigma^2 W^2(1) \frac{(1+m)m(3T-m-5)}{6T^3}. \quad (\text{A.27})$$

Combining (A.23), (A.25) and (A.27) we have

$$I_2 \Rightarrow \frac{2}{\lambda} \sigma^2 \left(\int_0^1 W(r)[W(r) - W(\min(r+\lambda, 1))] dr - W(1) \int_0^\lambda (\lambda-r)(W(1-r) - W(r)) dr \right) + \sigma^2 W^2(1) \left(1 - \frac{\lambda^2}{3}\right) - \sigma_\epsilon^2. \quad (\text{A.28})$$

Summing (A.16) and (A.28) we finish the proof.

2.8.3 Proof of Proposition 5

Given the weak convergence of $\hat{\sigma}_{\text{NW}}^2$ to the functional $f(W)$ in (87), Proposition 5 is an almost trivial consequence of the following well-known result:

Theorem 4 [*Extended Continuous Mapping Theorem*]⁴⁶ Let h_n and h be measurable mappings from $D[0, 1]$ -the space of all functions on $[0, 1]$ that are right continuous with left-hand limits—to itself and denote by E the set of $x \in D[0, 1]$ such that

⁴⁶See Billingsley (1968) for a proof.

$h_n(x_n) \rightarrow h(x)$ fails to hold for some sequence x_n converging to x . If $W_n(s) \Rightarrow W(s)$ and E is of Wiener-measure zero, i.e. $P(W \in E) = 0$, then $h_n(W_n) \Rightarrow h(W)$.

2.8.4 TASS Fund Category Definitions

The following is a list of category descriptions, taken directly from TASS documentation, that define the criteria used by TASS in assigning funds in their database to one of 17 possible categories:

Equity Hedge This directional strategy involves equity-oriented investing on both the long and short sides of the market. The objective is not to be market neutral. Managers have the ability to shift from value to growth, from small to medium to large capitalization stocks, and from a net long position to a net short position. Managers may use futures and options to hedge. The focus may be regional, such as long/short US or European equity, or sector specific, such as long and short technology or healthcare stocks. Long/short equity funds tend to build and hold portfolios that are substantially more concentrated than those of traditional stock funds. US equity Hedge, European equity Hedge, Asian equity Hedge and Global equity Hedge is the regional Focus.

Dedicated Short Seller Short biased managers take short positions in mostly equities and derivatives. The short bias of a manager's portfolio must be constantly greater than zero to be classified in this category.

Fixed Income Directional This directional strategy involves investing in Fixed Income markets only on a directional basis.

Convertible Arbitrage This strategy is identified by hedge investing in the convertible securities of a company. A typical investment is to be long the convertible bond and short the common stock of the same company. Positions are designed to generate profits from the fixed income security as well as the short sale of stock, while protecting principal from market moves.⁴⁷

Event Driven This strategy is defined as 'special situations' investing designed to capture price movement generated by a significant pending corporate event such as a merger, corporate restructuring, liquidation, bankruptcy or reorganization. There are three popular sub-categories in event-driven strategies: risk (merger) arbitrage, distressed/high yield securities, and Regulation D.

Non Directional/Relative Value This investment strategy is designed to exploit equity and/or fixed income market inefficiencies and usually involves being simultaneously long and short matched market portfolios of the same size within a country. Market neutral portfolios are designed to be either beta or currency neutral, or both.

⁴⁷Note that the category closest to this definition in the TASS database is "Convertible Fund (Long Only)", which is related to but different from convertible arbitrage—see footnote 44 for details. TASS recently changed their style classifications, and now defines "Convertible Arbitrage" as a distinct category.

Global Macro Global macro managers carry long and short positions in any of the world's major capital or derivative markets. These positions reflect their views on overall market direction as influenced by major economic trends and or events. The portfolios of these funds can include stocks, bonds, currencies, and commodities in the form of cash or derivatives instruments. Most funds invest globally in both developed and emerging markets.

Global Opportunity Global macro managers carry long and short positions in any of the world's major capital or derivative markets on an opportunistic basis. These positions reflect their views on overall market direction as influenced by major economic trends and or events. The portfolios of these funds can include stocks, bonds, currencies, and commodities in the form of cash or derivatives instruments. Most funds invest globally in both developed and emerging markets.

Natural Resources This trading strategy has a focus for the natural resources around the world.

Leveraged Currency This strategy invests in currency markets around the world.

Managed Futures This strategy invests in listed financial and commodity futures markets and currency markets around the world. The managers are usually referred to as Commodity Trading Advisors, or CTAs. Trading disciplines are generally systematic or discretionary. Systematic traders tend to use price and market specific information (often technical) to make trading decisions, while discretionary managers use a judgmental approach.

Emerging Markets This strategy involves equity or fixed income investing in emerging markets around the world.

Property The main focus of the investments are property.

Fund of Funds A 'Multi Manager' fund will employ the services of two or more trading advisors or Hedge Funds who will be allocated cash by the Trading Manager to trade on behalf of the fund.

2.8.5 Supplementary Empirical Results

In Tables A.1–A.7, we provide corresponding empirical results to Tables 2.7–2.14 but with Live and Graveyard funds separated so that the effects of survivorship bias can be seen. Of course, since we still apply our five-year minimum returns history filter to both groups, there is still some remaining survivorship bias. Tables A.1 and A.2 contain summary statistics for the two groups of funds, Tables A.3 and A.4 report summary statistics for the maximum likelihood estimates of the smoothing model (21)–(23). Tables A.5 and A.6 report similar statistics for the regression model estimates (62) of the smoothing model, and Table A.7 contains smoothing-adjusted Sharpe ratios for the two groups of funds. Finally, Table A.8 corresponds to the regressions of Table 2.11 but with dependent variables $\hat{\theta}_0$ and $\hat{\xi}$ estimated by maximum likelihood under the alternate constraint (75).

Category	N	Annual Mean		Annual SD		Skewness		Kurtosis		$\beta_1(\%)$		$\beta_2(\%)$		$\beta_3(\%)$		p -Value(Q)	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Not Categorized	44	12.08	6.04	13.08	10.44	0.05	0.73	2.37	2.54	8.8	15.9	2.3	14.5	1.4	9.8	31.8	32.0
US Equity Hedge	139	23.53	10.92	21.86	11.95	0.20	0.98	3.79	4.89	8.0	13.7	0.7	11.4	-4.1	11.2	33.0	29.0
European Equity Hedge	19	18.75	7.33	15.36	6.53	0.51	0.73	3.10	2.86	13.0	11.1	13.5	9.0	-2.9	8.9	16.4	20.7
Asian Equity Hedge	5	9.16	7.96	21.13	6.64	0.50	0.85	2.32	0.73	11.7	13.0	5.0	5.6	-5.4	13.4	40.2	21.1
Global Equity Hedge	24	15.04	7.67	18.28	6.31	-0.16	1.06	4.17	5.83	10.5	9.9	-0.6	8.8	-4.1	5.4	41.8	26.9
Dedicated Shortseller	6	-0.17	11.15	23.01	14.51	0.44	0.17	2.24	2.63	2.6	7.4	-0.9	9.0	-5.6	7.0	28.3	28.7
Fixed-Income Directional	12	10.17	3.39	9.80	9.08	-1.00	1.33	5.78	7.49	22.8	15.8	14.9	13.4	1.7	14.1	17.6	21.8
Convertible Fund (Long Only)	12	14.05	5.54	11.67	7.39	-0.26	1.88	7.16	10.93	22.9	11.9	5.3	14.9	-1.0	11.9	15.2	22.1
Event Driven	97	15.61	6.42	8.96	6.66	-0.58	1.74	8.10	11.71	20.0	16.6	5.8	13.2	-0.9	12.1	25.2	28.6
Non-Directional/Relative Value	63	13.23	5.91	7.89	4.84	-0.61	1.74	6.08	9.52	18.4	26.0	15.3	14.0	5.4	15.8	14.6	24.0
Global Macro	15	14.92	7.82	17.34	10.46	0.79	1.07	4.67	4.79	5.3	11.8	-1.3	12.2	-0.9	8.4	41.9	24.2
Global Opportunity	1	-17.39	—	31.03	—	-0.48	—	2.80	—	23.1	—	14.1	—	0.0	—	20.8	—
Natural Resources	1	14.49	—	18.42	—	0.30	—	-0.14	—	-10.1	—	-1.6	—	7.8	—	40.6	—
Pure Leveraged Currency	15	10.36	5.23	15.27	6.25	0.81	0.75	3.22	4.04	7.4	8.7	-3.3	9.9	-3.1	7.1	31.2	18.7
Pure Managed Futures	28	10.54	6.19	16.46	7.62	0.42	1.55	5.92	12.20	0.9	13.2	-4.1	11.4	-2.5	8.6	32.4	29.9
Pure Emerging Market	54	9.08	10.21	26.12	11.66	-0.56	1.21	4.97	5.86	18.3	12.0	4.1	11.9	-1.7	9.2	30.7	29.4
Pure Property	1	3.96	—	9.41	—	-1.33	—	5.43	—	-23.1	—	2.3	—	7.3	—	18.9	—
Pund of Funds	115	11.45	4.95	10.16	5.75	-0.29	1.27	4.83	6.17	17.3	14.0	6.2	11.7	-1.2	9.1	26.5	27.9
All Live	651	14.88	9.33	15.05	10.65	-0.15	1.36	5.00	7.63	13.7	16.5	4.6	13.1	-1.2	11.1	28.0	28.3

Table A.1: Means and standard deviations of basic summary statistics for 651 hedge funds in the TASS Hedge Fund Live database with at least five years of returns history during the period from November 1977 to January 2001. The columns ' p -Value(Q)' contain means and standard deviations of p -values for the Box-Pierce Q -statistic for each fund using the first 6 autocorrelations of returns.

Category	N	Annual Mean		Annual SD		Skewness		Kurtosis		$\hat{\rho}_1(\%)$		$\hat{\rho}_2(\%)$		$\hat{\rho}_3(\%)$		$p\text{-Value}(Q)$			
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
		Not Categorized	67	9.92	7.48	17.26	11.52	0.17	1.23	4.53	5.88	3.4	17.5	-0.1	14.9	-0.1	12.4	39.0	29.2
US Equity Hedge	23	16.47	7.82	20.68	9.67	0.03	0.83	3.97	5.90	7.2	12.5	0.4	9.7	-5.3	10.1	38.9	27.6		
European Equity Hedge	3	13.33	3.85	10.58	4.17	-0.38	1.06	2.15	1.89	10.5	11.7	-3.7	14.8	3.8	7.6	51.4	9.1		
Asian Equity Hedge	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—		
Global Equity Hedge	3	3.06	7.16	10.41	5.29	-0.88	0.97	2.29	3.48	27.5	12.9	3.2	16.9	7.8	12.4	42.2	36.6		
Dedicated Shortseller	1	3.32	—	12.78	—	0.78	—	2.46	—	15.5	—	-13.5	—	-12.0	—	52.4	—		
Fixed-Income Directional	1	5.58	—	6.49	—	-0.15	—	0.50	—	7.7	—	4.2	—	4.1	—	24.0	—		
Convertible Fund (Long Only)	3	18.55	2.34	8.48	2.77	0.09	0.31	2.05	2.71	20.8	16.5	9.7	4.5	-6.5	16.7	25.2	40.7		
Event Driven	12	13.22	8.67	13.98	22.43	-0.66	2.45	9.55	11.85	26.6	17.8	11.4	14.8	5.9	13.1	29.7	37.5		
Non-Directional/Relative Value	22	10.58	8.26	9.02	5.67	-1.18	1.61	7.47	9.56	17.8	14.5	3.5	16.3	3.9	10.4	24.4	26.8		
Global Macro	10	19.49	7.70	21.29	7.83	0.14	0.86	4.64	9.40	10.1	19.2	1.6	8.7	-0.8	8.0	20.6	13.3		
Global Opportunity	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—		
Natural Resources	2	9.84	5.78	19.15	1.38	0.56	0.59	1.95	1.24	12.5	9.0	13.6	11.9	-1.0	6.7	51.1	61.2		
Pure Leveraged Currency	11	7.83	9.18	19.78	11.48	-0.07	1.28	3.90	6.13	2.2	8.2	-10.1	7.8	-6.1	9.3	41.4	34.8		
Pure Managed Futures	65	9.19	10.35	24.02	19.41	0.13	1.25	4.51	5.98	-0.6	12.9	-3.7	9.4	-3.9	11.5	36.7	28.9		
Pure Emerging Market	18	11.06	13.11	28.05	21.07	-1.08	2.12	9.79	12.04	20.2	11.6	5.1	10.9	-0.7	8.4	38.1	34.0		
Pure Property	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—		
Fund of Funds	17	8.24	5.82	9.50	4.87	-0.04	1.53	4.39	6.25	19.2	14.6	2.2	9.6	2.9	10.2	24.6	24.9		
All Graveyard	258	10.80	9.17	18.61	15.21	-0.14	1.45	5.17	7.40	8.2	16.8	0.2	12.7	-1.0	11.4	35.2	29.3		

Table A.2: Means and standard deviations of basic summary statistics for 258 hedge funds in the TASS Hedge Fund Graveyard database with at least five years of returns history during the period from November 1977 to January 2001. The columns ' $p\text{-Value}(Q)$ ' contain means and standard deviations of p -values for the Box-Pierce Q -statistic for each fund using the first 6 autocorrelations of returns.

Category	N	MA(2) with Constrained Sum										MA(2) with Constrained σ_η									
		$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$		ξ		$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$		ξ					
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD				
Not Categorized	44	0.973	0.273	0.049	0.149	-0.023	0.174	1.075	0.704	0.942	0.269	0.046	0.145	-0.015	0.160	1.006	0.584				
US Equity Hedge	139	0.951	0.213	0.065	0.159	-0.016	0.156	1.004	0.562	0.974	0.194	0.071	0.147	-0.008	0.151	1.035	0.416				
European Equity Hedge	19	0.790	0.115	0.077	0.088	0.133	0.073	0.673	0.196	0.902	0.170	0.096	0.101	0.156	0.095	0.893	0.317				
Asian Equity Hedge	5	0.942	0.296	0.053	0.188	0.005	0.109	0.998	0.712	0.904	0.198	0.064	0.167	0.013	0.096	0.883	0.404				
Global Equity Hedge	24	0.926	0.143	0.095	0.091	-0.020	0.099	0.903	0.271	0.994	0.173	0.100	0.098	-0.018	0.100	1.045	0.346				
Dedicated Shortseller	6	1.031	0.220	0.008	0.109	-0.039	0.135	1.131	0.528	1.149	0.152	0.024	0.115	-0.031	0.142	1.368	0.363				
Fixed-Income Directional	12	0.752	0.177	0.160	0.105	0.088	0.126	0.652	0.280	0.733	0.203	0.157	0.104	0.093	0.128	0.633	0.303				
Convertible Fund (Long Only)	12	0.868	0.392	0.189	0.069	-0.057	0.387	1.074	1.516	0.896	0.300	0.202	0.086	-0.027	0.263	0.997	0.630				
Event Driven	97	0.816	0.170	0.148	0.128	0.036	0.115	0.747	0.285	0.856	0.240	0.154	0.126	0.038	0.113	0.842	0.439				
Non-Directional/Relative Value	63	0.809	0.253	0.077	0.279	0.114	0.133	0.831	0.697	0.764	0.296	0.070	0.275	0.110	0.127	0.777	0.793				
Global Macro	15	1.006	0.186	0.046	0.115	-0.053	0.168	1.088	0.422	1.054	0.214	0.052	0.120	-0.049	0.188	1.205	0.525				
Natural Opportunity	1	0.737	—	0.162	—	0.100	—	0.580	—	0.632	—	0.139	—	0.086	—	0.426	—				
Natural Resources	1	1.111	—	-0.113	—	0.003	—	1.246	—	0.861	—	-0.088	—	0.002	—	0.748	—				
Pure Leveraged Currency	15	0.970	0.132	0.074	0.091	-0.044	0.121	0.986	0.271	0.926	0.099	0.076	0.084	-0.035	0.115	0.892	0.186				
Pure Managed Futures	28	1.138	0.305	-0.034	0.171	-0.104	0.219	1.470	0.904	1.129	0.253	-0.030	0.165	-0.090	0.190	1.406	0.621				
Pure Emerging Market	54	0.836	0.165	0.148	0.087	0.016	0.139	0.774	0.328	0.797	0.162	0.141	0.082	0.021	0.136	0.706	0.296				
Pure Property	1	1.232	—	-0.306	—	0.074	—	1.618	—	0.951	—	-0.236	—	0.057	—	0.963	—				
Fund of Funds	115	0.848	0.220	0.119	0.132	0.034	0.136	0.817	0.522	0.859	0.232	0.122	0.126	0.035	0.133	0.841	0.470				
All	651	0.891	0.231	0.093	0.160	0.016	0.158	0.906	0.578	0.899	0.242	0.096	0.155	0.021	0.148	0.923	0.512				

Table A.3: Means and standard deviations of maximum likelihood estimates of MA(2) smoothing process $R_t^c = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$, for 651 funds in the TASS Hedge Fund Live database with at least five years of returns history during the period from November 1977 to January 2001.

Category	N	MA(2) with Constrained Sum						MA(2) with Constrained σ_η									
		θ_0		θ_1		θ_2		ξ		θ_0		θ_1		θ_2		ξ	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Not Categorized	67	1.079	0.353	-0.009	0.194	-0.070	0.267	1.399	1.226	0.993	0.268	-0.007	0.165	-0.049	0.214	1.130	0.613
US Equity Hedge	23	0.945	0.179	0.053	0.113	0.002	0.115	0.951	0.363	0.993	0.212	0.065	0.126	0.000	0.123	1.063	0.471
European Equity Hedge	3	0.984	0.251	0.073	0.107	-0.057	0.147	1.041	0.544	0.859	0.201	0.065	0.094	-0.049	0.127	0.788	0.378
Asian Equity Hedge	0	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
Global Equity Hedge	3	0.790	0.099	0.220	0.043	-0.011	0.057	0.683	0.131	0.670	0.196	0.179	0.003	-0.014	0.044	0.508	0.242
Dedicated Shortseller	1	0.999	---	0.166	---	-0.165	---	1.052	---	1.035	---	0.172	---	-0.171	---	1.131	---
Fixed-Income Directional	1	0.902	---	0.064	---	0.033	---	0.820	---	0.977	---	0.069	---	0.036	---	0.961	---
Convertible Fund (Long Only)	3	0.753	0.121	0.138	0.113	0.109	0.013	0.617	0.166	0.885	0.062	0.182	0.155	0.131	0.033	0.854	0.158
Event Driven	12	0.761	0.174	0.184	0.135	0.055	0.101	0.670	0.286	0.740	0.270	0.166	0.115	0.058	0.099	0.666	0.438
Non-Directional/Relative Value	22	0.844	0.196	0.142	0.099	0.014	0.137	0.796	0.317	0.768	0.203	0.135	0.101	0.014	0.127	0.673	0.332
Global Macro	9	0.869	0.155	0.135	0.061	-0.005	0.126	0.813	0.302	0.818	0.188	0.123	0.050	-0.002	0.124	0.732	0.351
Global Opportunity	0	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
Natural Resources	2	0.811	0.115	0.079	0.030	0.110	0.085	0.687	0.164	0.845	0.050	0.084	0.039	0.119	0.099	0.741	0.055
Pure Leveraged Currency	11	1.272	0.446	-0.024	0.147	-0.248	0.331	1.980	2.016	1.107	0.163	0.000	0.107	-0.183	0.169	1.319	0.424
Pure Managed Futures	65	1.129	0.325	-0.053	0.233	-0.076	0.159	1.467	1.295	1.082	0.244	-0.037	0.193	-0.066	0.144	1.292	0.642
Pure Emerging Market	18	0.817	0.148	0.154	0.085	0.029	0.109	0.730	0.254	0.789	0.181	0.145	0.078	0.028	0.107	0.692	0.313
Pure Property	0	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
Fund of Funds	17	0.845	0.165	0.168	0.103	-0.013	0.107	0.789	0.250	0.823	0.208	0.158	0.090	-0.012	0.106	0.761	0.324
All	257	1.001	0.314	0.042	0.190	-0.042	0.196	1.177	1.064	0.950	0.259	0.046	0.164	-0.031	0.162	1.025	0.569

Table A.4: Means and standard deviations of maximum likelihood estimates of MA(2) smoothing process $R_t^o = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$, for 257 funds in the TASS Hedge Fund Graveyard database with at least five years of returns history during the period from November 1977 to January 2001.

Category	N	$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\xi}$		D.W.		$R^2(\%)$	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Not Categorized	44	1.598	4.141	-0.613	3.212	0.015	2.294	34.919	108.677	1.81	0.32	15.0	13.9
US Equity Hedge	139	0.713	1.557	0.080	0.433	0.207	1.268	4.747	41.449	1.80	0.28	24.8	17.4
European Equity Hedge	19	0.581	0.113	0.344	0.735	0.075	0.756	1.527	4.614	1.76	0.19	18.0	9.3
Asian Equity Hedge	5	0.405	0.252	0.323	0.179	0.272	0.088	0.425	0.086	1.82	0.24	7.4	4.5
Global Equity Hedge	24	0.913	0.323	0.028	0.331	0.059	0.354	1.163	0.956	1.69	0.20	24.3	19.9
Dedicated Shortseller	6	0.886	0.189	0.104	0.106	0.010	0.202	0.869	0.376	1.96	0.17	41.1	15.0
Fixed-Income Directional	12	-1.495	6.678	4.689	15.156	-2.193	8.493	346.600	1198.277	1.61	0.30	14.1	9.7
Convertible Fund (Long Only)	12	0.513	0.239	0.354	0.160	0.133	0.162	0.506	0.152	1.51	0.22	23.8	15.2
Event Driven	97	0.550	0.279	0.330	0.288	0.120	0.232	0.638	0.661	1.64	0.36	18.0	11.4
Non-Directional/Relative Value	63	0.607	2.010	0.762	1.808	-0.369	3.342	19.268	80.833	1.63	0.51	12.7	9.5
Global Macro	15	0.541	0.428	0.182	0.424	0.277	0.161	0.765	0.491	1.84	0.29	12.0	7.0
Global Opportunity	1	0.619	—	0.191	—	0.191	—	0.455	—	1.55	—	30.9	—
Natural Resources	1	0.640	—	0.175	—	0.186	—	0.474	—	2.20	—	2.3	—
Pure Leveraged Currency	15	0.049	1.238	0.596	0.971	0.354	1.241	4.234	10.045	1.85	0.15	4.7	7.1
Pure Managed Futures	28	-0.290	1.930	0.776	2.494	0.514	2.027	14.502	41.277	1.96	0.26	3.8	3.1
Pure Emerging Market	54	0.928	0.916	0.125	0.527	-0.053	0.506	2.226	7.470	1.52	0.28	22.0	10.8
Pure Property	1	1.319	—	-0.024	—	-0.295	—	1.828	—	2.31	—	15.1	—
Fund of Funds	115	0.616	1.287	-0.070	2.694	0.454	3.231	19.776	183.336	1.63	0.30	22.0	14.8
All	651	0.636	1.888	0.256	2.649	0.109	2.284	16.252	184.926	1.71	0.34	19.1	11.7

Table A.5: Means and standard deviations of linear regression estimates of MA(2) smoothing process $R_t^* = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$. $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$ under the assumption of a linear single-factor model for R_t where the factor is the total return of the S&P 500 Index, for 651 hedge funds in the TASS Hedge Fund Live database with at least five years of returns history during the period from November 1977 to January 2001.

Category	N	$\hat{\theta}_0$		$\hat{\theta}_1$		$\hat{\theta}_2$		ξ		D.W.		$R^2(\%)$	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Not Categorized	67	0.042	10.672	1.163	11.752	-0.206	2.251	254.635	1841.566	1.89	0.33	12.6	18.5
US Equity Hedge	23	0.586	1.129	0.182	0.266	0.232	0.900	2.492	8.220	1.81	0.26	33.8	16.6
European Equity Hedge	3	0.434	0.329	0.312	0.173	0.254	0.275	0.493	0.173	1.78	0.17	18.3	21.0
Asian Equity Hedge	—	—	—	—	—	—	—	—	—	—	—	—	—
Global Equity Hedge	3	0.308	0.626	0.294	0.271	0.397	0.361	0.737	0.293	1.40	0.26	4.1	1.3
Dedicated Shortseller	1	1.040	—	0.176	—	-0.216	—	1.159	—	1.84	—	54.7	—
Fixed-Income Directional	1	-0.738	—	0.748	—	0.990	—	2.084	—	1.83	—	1.6	—
Convertible Fund (Long Only)	3	0.636	0.383	0.295	0.127	0.068	0.272	0.655	0.428	1.63	0.32	30.1	16.3
Event Driven	12	0.837	1.223	-0.008	1.244	0.171	0.159	3.543	10.629	1.50	0.38	18.1	11.0
Non-Directional/Relative Value	22	0.748	7.341	1.275	3.420	-1.023	7.574	120.592	461.012	1.59	0.28	10.5	8.2
Global Macro	9	0.587	0.173	0.214	0.139	0.198	0.234	0.522	0.138	1.63	0.22	16.0	9.3
Global Opportunity	—	—	—	—	—	—	—	—	—	—	—	—	—
Natural Resources	2	0.717	0.184	0.072	0.089	0.211	0.095	0.589	0.211	1.71	0.19	2.7	1.2
Pure Leveraged Currency	11	-0.255	1.590	0.037	1.150	1.218	2.690	11.631	36.324	1.95	0.16	4.0	3.5
Pure Managed Futures	65	0.079	1.418	0.662	1.132	0.260	1.406	5.700	15.336	2.00	0.26	6.8	6.5
Pure Emerging Market	18	0.811	0.231	0.256	0.151	-0.067	0.237	0.852	0.460	1.46	0.30	24.9	9.8
Pure Property	—	—	—	—	—	—	—	—	—	—	—	—	—
Fund of Funds	17	1.064	2.613	0.093	1.469	-0.158	1.366	11.379	40.134	1.59	0.30	19.1	17.0
All	257	0.346	5.928	0.643	6.112	0.010	2.680	79.904	950.698	1.80	0.34	14.4	15.6

Table A.6: Means and standard deviations of linear regression estimates of MA(2) smoothing process $R_t^o = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, $\xi \equiv \theta_0^2 + \theta_1^2 + \theta_2^2$ under the assumption of a linear single-factor model for R_t where the factor is the total return of the S&P 500 Index, for 257 hedge funds in the TASS Hedge Fund Graveyard database with at least five years of returns history during the period from November 1977 to January 2001.

Category	Sharpe Ratios For Live Funds						Sharpe Ratios For Graveyard Funds							
	SR		SR*		SR**		SR		SR*		SR**			
	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD		
Not Categorized	44	1.55	1.34	1.41	1.00	1.33	0.94	67	0.84	0.79	0.83	0.70	0.77	0.65
US Equity Hedge	139	1.31	0.77	1.36	0.76	1.27	0.70	23	0.99	0.54	1.06	0.60	1.00	0.57
European Equity Hedge	19	1.44	0.78	1.46	0.85	1.36	0.78	3	1.36	0.57	1.26	0.49	1.18	0.46
Asian Equity Hedge	5	0.50	0.39	0.52	0.39	0.49	0.37	0	—	—	—	—	—	—
Global Equity Hedge	24	0.95	0.60	1.00	0.66	0.94	0.61	3	0.55	0.75	0.51	0.59	0.48	0.55
Dedicated Shortseller	6	0.28	0.65	0.33	0.70	0.31	0.67	1	0.26	—	0.30	—	0.28	—
Fixed-Income Directional	12	2.11	2.42	1.87	0.70	2.31	2.14	1	0.86	—	0.87	—	0.84	—
Convertible Fund (Long Only)	12	1.67	1.18	1.48	0.72	1.38	0.69	3	2.45	1.26	2.39	1.07	2.23	1.00
Event Driven	97	2.45	1.46	2.31	1.58	2.17	1.47	12	1.61	1.16	1.47	1.39	1.37	1.29
Non-Directional/Relative Value	63	2.36	1.82	2.21	2.52	2.06	2.35	22	1.73	1.93	1.53	1.91	1.42	1.76
Global Macro	15	1.12	0.77	1.30	0.86	1.22	0.82	9	1.02	0.51	0.87	0.34	0.82	0.32
Global Opportunity	1	-0.56	—	-0.39	—	-0.37	—	0	—	—	—	—	—	—
Natural Resources	1	0.79	—	0.75	—	0.69	—	2	0.50	0.27	0.46	0.23	0.43	0.22
Pure Leveraged Currency	15	0.83	0.50	0.83	0.50	0.78	0.47	11	0.35	0.32	0.41	0.39	0.38	0.37
Pure Managed Futures	28	0.81	0.71	0.92	0.66	0.87	0.63	65	0.43	0.43	0.51	0.52	0.48	0.49
Pure Emerging Market	54	0.39	0.48	0.37	0.48	0.34	0.44	18	0.39	0.36	0.34	0.30	0.32	0.27
Pure Property	1	0.42	—	0.45	—	0.41	—	0	—	—	—	—	—	—
Fund of Funds	115	1.48	1.02	1.35	0.90	1.27	0.83	17	1.15	0.98	0.95	0.70	0.89	0.65
All	651	1.50	1.29	1.45	1.35	1.36	1.26	257	0.85	0.95	0.83	0.90	0.78	0.84

Table A.7: Means and standard deviations of Sharpe ratios of 651 funds in the TASS Hedge Funds Live database and 257 funds in the TASS Hedge Fund Graveyard database, both with at least five years of returns history during the period from November 1977 to January 2001. SR is the standard Sharpe ratio, SR* is the smoothing-adjusted Sharpe ratio of Lo (2002), and SR** is the smoothing-adjusted Sharpe ratio using $\hat{\sigma}_{nw}$. All Sharpe ratios are computed with respect to a 0 benchmark.

Regressor	$\hat{\theta}_0$	$\hat{\xi}$	$\hat{\theta}_0$	$\hat{\xi}$
Constant	1.007 (42.02)	1.152 (21.91)	1.029 (35.56)	1.212 (19.04)
US Equity Hedge	0.020 (0.69)	-0.009 (0.15)	0.026 (0.85)	-0.004 (0.07)
European Equity Hedge	-0.063 (1.19)	-0.175 (1.50)	-0.120 (1.97)	-0.281 (2.10)
Asian Equity Hedge	-0.054 (0.52)	-0.167 (0.73)	-0.030 (0.28)	-0.131 (0.54)
Global Equity Hedge	0.005 (0.11)	-0.053 (0.49)	0.032 (0.64)	0.006 (0.05)
Dedicated Shortseller	0.178 (2.01)	0.291 (1.50)	0.184 (2.06)	0.293 (1.49)
Fixed-Income Directional	-0.203 (3.04)	-0.386 (2.63)	-0.196 (2.92)	-0.375 (2.54)
Convertible Fund (Long Only)	-0.060 (0.95)	-0.073 (0.53)	-0.064 (1.01)	-0.086 (0.62)
Event Driven	-0.116 (3.74)	-0.228 (3.36)	-0.124 (3.85)	-0.247 (3.49)
Non-Directional/Relative Value	-0.194 (5.88)	-0.303 (4.18)	-0.180 (5.26)	-0.281 (3.73)
Global Macro	0.004 (0.09)	-0.029 (0.26)	0.034 (0.63)	0.026 (0.22)
Global Opportunity	-0.314 (1.37)	-0.598 (1.19)	-0.309 (1.35)	-0.59 (1.18)
Natural Resources	-0.096 (0.72)	-0.281 (0.96)	-0.095 (0.72)	-0.284 (0.97)
Pure Leveraged Currency	0.042 (0.85)	0.019 (0.17)	0.045 (0.89)	0.022 (0.20)
Pure Managed Futures	0.133 (4.16)	0.267 (3.78)	0.142 (4.25)	0.284 (3.87)
Pure Emerging Market	-0.161 (4.63)	-0.344 (4.50)	-0.146 (3.53)	-0.311 (3.42)
Pure Property	-0.056 (0.25)	-0.190 (0.38)	-0.028 (0.12)	-0.146 (0.29)
Fund of Funds	-0.099 (3.32)	-0.211 (3.21)	-0.092 (3.03)	-0.199 (2.96)
OPEN	-0.061 (3.27)	-0.128 (3.13)	-0.055 (2.78)	-0.118 (2.73)
USBASED			0.012 (0.68)	0.022 (0.58)
GF-GLB			-0.034 (1.69)	-0.078 (1.78)
GF-USA			-0.043 (2.09)	-0.087 (1.92)
GF-APC			-0.051 (1.73)	-0.104 (1.61)
GF-WEU			0.048 (1.70)	0.079 (1.28)
GF-EEU			-0.058 (1.46)	-0.133 (1.51)
GF-AFR			0.071 (1.27)	0.137 (1.12)
Sample Size	908	908	891	891
Adjusted R^2 (%)	15.7	11.4	16.9	12.6

Table A.8: Regressions of maximum likelihood estimated smoothing coefficient $\hat{\theta}_0$ and smoothing index $\hat{\xi}$ on indicator variables for 908 hedge funds in the TASS Hedge Fund Combined (Live and Graveyard) database with at least five years of returns history during the period from November 1977 to January 2001, where the maximum likelihood estimator $\hat{\sigma}_\eta$ is constrained to equal a nonparametric estimator $\tilde{\sigma}_\eta$ of the innovation standard deviation. Absolute values of t -statistics are given in parentheses. The indicator variables are OPEN (1 if the fund is open, 0 otherwise); the fund categories (1 if the fund belongs to the category, 0 otherwise); USBASED (1 if the fund is based in the US, 0 otherwise); and geographical focus categories (1 if the geographical focus of the fund is in a given region, 0 otherwise, where the regions are USA, Asia Pacific, Western Europe, Eastern Europe, and Africa, respectively).

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Chapter 3

The Equity Risk Premium and the Riskfree Rate in an Economy with Borrowing Constraints (joint with Leonid Kogan and Raman Uppal)

3.1 Introduction

A feature of standard representative agent models with constant relative risk aversion (CRRA) preferences is that the Sharpe ratio of stock returns and the risk-free rate are linked to one another. This is a major limitation. For instance, attempts to resolve the finding in Mehra and Prescott (1985) that the risk premium is too small and the risk-free rate is too high in such a model relative to the data, run into the problem that an increase in the Sharpe ratio of stock returns is associated with an increase in the risk-free rate. known as the “interest rate puzzle” (Weil 1989).

Our objective in this article is to study analytically the effect of borrowing constraints on the link between the Sharpe ratio and the risk-free rate. We do this by considering a model that is a straightforward extension of the homogeneous agent economy of Mehra and Prescott where financial markets are effectively complete.

The extension is to introduce a borrowing constraint in a general equilibrium exchange economy with two agents who have CRRA preferences, and to give the borrowing constraint a meaningful role we assume that the two agents differ in their risk aversion. We characterize exactly in closed form the equilibrium of this economy. General-equilibrium economies with borrowing constraints are typically not amenable to explicit analysis and are studied using numerical simulation methods.¹ Our model is extremely tractable and amenable to rigorous theoretical analysis.

Our main result is that, unlike in a representative agent model, in an economy with borrowing constraints the Sharpe ratio of stock returns can be relatively high, while the risk-free interest rate remains relatively low. In particular, we show that the Sharpe ratio of stock returns in the constrained heterogeneous-agent economy is the same as in the representative-agent economy populated only by the more risk averse of the two agents, while the risk-free rate in the constrained heterogeneous-agent economy may be even lower than in the representative-agent economy populated by the less risk averse of the two agents. And, comparing the constrained heterogeneous-agent economy to one where agents are heterogeneous but unconstrained, we find that imposing a borrowing constraint increases the Sharpe ratio of stock returns and lowers the risk-free interest rate. Moreover, we show that the unconstrained economy with heterogeneous agents suffer from the same limitations as the representative-agent economy with CRRA preferences, namely the tight link between the Sharpe ratio of stock returns and the level of the risk-free rate (we establish this new analytical result for the unconstrained economy), which is not the case in an economy with borrowing constraints.

Borrowing constraints are an important feature of the real economy and as argued by Constantinides (2002) it is important to consider these constraints when studying the implications of asset pricing models. However, taking into account borrowing constraints is a challenging task since even in models without borrowing constraints but with heterogeneous risk aversion (Dumas 1989, Wang 1996, Chan and Kogan 2002)

¹A notable exception is a model of Detemple and Murthy (1997), in which explicit results can be obtained when all agents have logarithmic preferences, but differ in their beliefs about the aggregate endowment process.

most of the asset-pricing results are obtained using numerical analysis.² In models *with* borrowing constraint, for instance Heaton and Lucas (1996) and Constantinides, Donaldson and Mehra (2002), the analysis is undertaken using numerical methods, while in Kogan and Uppal (2002) the analysis is undertaken using approximation methods that apply in the neighborhood of log utility which then limits the range of the risk aversion parameter for which the effect of borrowing constraints can be analyzed. In contrast, we characterize exactly in closed form the equilibrium in an economy with borrowing constraints.

There is another important difference between our model and the models of Heaton and Lucas (1996) and Constantinides, Donaldson and Mehra (2002), which are the two papers closest to our work. In both these models, the source of heterogeneity across agents is idiosyncratic endowment shocks and therefore the mechanism through which the borrowing constraint works is different. In Heaton and Lucas, the constraint on borrowing and a cost for trading stocks and bonds raises individual consumption variability, and hence, lowers the risk-free rate of return due to the demand for precautionary savings. Constantinides, Donaldson and Mehra model do not have trading costs; instead, they consider an overlapping generations model. In their model, the young would like to invest in equity by collateralizing future wages but are prevented from doing so because of the constraint on borrowing. On the other hand, for the middle-aged wage uncertainty has largely been resolved and so most of variation in their consumption occur from variation in financial wealth; thus, stock returns are highly correlated with consumption. Hence, this age cohort requires a higher rate of return for holding equity. Thus, in their model “the *deus ex machina* is the stage in the life cycle of the marginal investor.”

In contrast to these two papers, in our model the source of heterogeneity is risk aversion, and therefore no additional source of risk is introduced relative to the standard representative-agent framework considered in Mehra and Prescott (1985). Moreover, because we solve for the equilibrium in closed-form, the economic forces driving

²Wang can solve for only some of the quantities of the model in closed form but even this is possible only for particular combinations of the number of agents and the degree of risk aversion for each of these agents.

the results in our paper are transparent.

Our work is also related to the paper by Basak and Cuoco (1998) who characterize the equilibrium in a model where agents differ with respect to their risk aversion and, instead of a constraint on borrowing, face a constraint on participating in the stock market. In contrast to our model where all agents face the same constraint on borrowing, in their setup the constraint is applied asymmetrically across agents; in particular, they assume that it's the less risk averse agent who is excluded from the stock market, which is counter to what one would expect.

The rest of the paper is arranged as follows. In Section 3.2, we describe an exchange economy with heterogeneous agents who face borrowing constraints. In Section 3.3, we characterize analytically the equilibrium in this economy. In Section 3.4, we consider the robustness of our results to more general forms of the borrowing constraint. We conclude in Section 3.5. Our main results are highlighted in propositions and the proofs for all the propositions are collected in the appendix.

3.2 A model of an exchange economy with heterogeneous agents

In this section, we study a general-equilibrium exchange (endowment) economy with multiple agents who differ in their level of risk aversion. Wang (1996) analyzes this economy for the case where there are two agents who do not face any portfolio constraints.

3.2.1 The aggregate endowment process

The infinite-horizon exchange economy has an aggregate endowment, D_t , that evolves according to

$$dD_t = \mu D_t dt + \sigma D_t dZ_t,$$

where μ and σ are constant parameters. We assume that the growth rate of the endowment is positive, $\mu - \sigma^2/2 > 0$. Without much loss of generality we also assume

that $D_0 = 1$.

3.2.2 Financial assets

We assume that there are two assets available for trading in the economy. The first asset is a short-term risk-free bond, available in zero net supply, which pays the interest rate r_t that will be determined in equilibrium. The second asset is a stock that is a claim on the aggregate endowment. The price of the stock is denoted by S_t . The cumulative stock return process is given by

$$\frac{dS_t + D_t dt}{S_t} = \mu_{S_t} dt + \sigma_{S_t} dZ_t, \quad (1)$$

with μ_{S_t} and σ_{S_t} to be determined in equilibrium.

3.2.3 Preferences

There are two competitive agents in the economy. The utility function of both agents is time-separable and is given by

$$E_0 \left[\int_0^\infty e^{-\rho t} \frac{1}{1-\gamma} (C_{\gamma,t}^{1-\gamma} - 1) dt \right],$$

where ρ is the constant subjective time discount rate, and C_t is the flow of consumption. The agent's relative risk aversion equals γ , and for agents with unit risk aversion ($\gamma = 1$), the utility function is logarithmic:

$$E_0 \left[\int_0^\infty e^{-\rho t} \ln C_{1,t} dt \right].$$

We assume that the first agent has risk aversion greater than one, while the second agent has unit risk aversion. Most of our results can be easily generalized to an arbitrary combination of risk aversion coefficients³.

³See Appendix B.

3.2.4 Individual endowments

We assume that both agents are initially endowed with shares of the stock. We will let $\omega_{\alpha,0}$, $\alpha \in \{1, \gamma\}$, denote the initial share of the aggregate endowment owned by the agent with relative risk aversion equal to α .

3.2.5 The constraint on borrowing

We consider a leverage constraint that restricts the proportion of individual wealth that can be invested in the risky asset. The base case of our model assumes that borrowing is prohibited. We establish our analytical results under this assumption. As an extension, we analyze numerically a more general case, when the proportion of individual wealth invested in the risky asset is bounded from above, $\pi \leq \bar{\pi} > 1$.

3.2.6 The competitive equilibrium

The equilibrium in this economy is defined by the stock price process, P_t , the interest rate process r_t , and the portfolio and consumption policies, such that (i) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents, (ii) the goods market and the markets for the stock and the bond clear.

3.3 The equilibrium and asset prices

In this section, we characterize an equilibrium in the economy described above. We compare the equilibrium in this economy with homogeneous representative-agent economies. We conclude by comparing the equilibrium in the economy with constraints to the one that is unconstrained.

3.3.1 Equilibrium in the economy with borrowing constraints

We look for an equilibrium in which the policy of the less risk averse agent is affected by the borrowing constraint, while the more risk averse agent is effectively uncou-

strained. Clearly, one could construct other equilibria by lowering the risk-free rate relative to the values that we identify. The equilibrium we identify has an intuitive appeal, since it can be also interpreted as approximating an economy in which a small amount of borrowing is allowed. In such an economy, while portfolio holdings of both agents would consist almost entirely of the risky asset, the more risk averse agent would be unconstrained. Our numerical results in Section 3.4 further illustrate this point.

The following proposition characterizes equilibrium prices and allocations in the constrained economy. To simplify notation, we let $R_t = \int_0^t r_s ds$ denote the cumulative return on the risk-free asset and define $\bar{R}_t = R_t - (\rho + \mu - \gamma\sigma^2)t$. The short-term interest rate can then be recovered from the process R_t by differentiation.

Proposition 1 *Let $\rho > \max[(1 - \gamma)\mu + \frac{\gamma(\gamma-1)}{2}\sigma^2, 0]$. There exists a competitive equilibrium in which*

(i) *The consumption processes of the two agents are given by*

$$C_{\gamma,t} = (1 - A) \exp\left(\frac{\bar{R}_t + (1 - \gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) D_t, \quad (2)$$

$$C_{1,t} = A \exp(\bar{R}_t) D_t, \quad (3)$$

where the constant $A = C_{1,0}/D_0 \in [0, 1]$ and the deterministic process \bar{R}_t are determined as a unique solution of the following system of equations:

$$(1 - A) \exp\left(\frac{\bar{R}_t + (1 - \gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) + A \exp(\bar{R}_t) = 1, \quad (4)$$

$$A - \rho(1 - \omega_{\gamma,0}) \int_0^\infty \exp(-\bar{R}_t - \rho t) dt = 0. \quad (5)$$

(ii) *The instantaneous Sharpe ratio of stock returns equals*

$$\frac{\mu_{St} - r_t}{\sigma_{St}} = \gamma\sigma; \quad (6)$$

(iii) *The instantaneous volatility of stock returns equals*

$$\sigma_{St} = \sigma; \tag{7}$$

(iv) *The risk-free interest rate process r_t is deterministic and is given by*

$$r_t = \frac{d\bar{R}_t}{dt} + (\rho + \mu - \gamma\sigma^2).$$

Proposition 1 states that in equilibrium the moments of asset returns are deterministic. Moreover, the instantaneous Sharpe ratio and volatility of stock returns are constant. The reason for why the moments of returns are not affected by shocks to the aggregate endowment, which is the only source of uncertainty in this economy, is very intuitive. Since the growth rate of the aggregate endowment process is independent of its past history, the moments of asset returns may depend only on the distribution of wealth in the economy. Because the agents cannot borrow, in equilibrium they both invest all of their wealth in the stock, and therefore their wealth processes are instantaneously perfectly correlated. Thus, the cross-sectional wealth distribution in the economy evolves in a locally predictable manner. Moreover, since both agents have CRRA preferences, their consumption policies (consumption rate as a fraction of individual wealth) depend only on the contemporaneous investment opportunity set in the economy, that is, on the wealth distribution. Thus, we conclude that the instantaneously riskless rate of change of the wealth distribution in the economy must be a function of the wealth distribution itself, implying that the latter evolves deterministically over time, and hence all moments of asset returns are also deterministic functions of time.

The fact that the cross-sectional wealth distribution evolves deterministically has another important implication. The consumption policies of both agents (consumption as a share of individual wealth) then must be deterministic as well, therefore the volatility of consumption growth of each agent coincides with the volatility of the growth rate of aggregate endowment. The standard CCAPM relation then implies

that the maximum Sharpe ratio is given by the product of the volatility of aggregate endowment growth and the relative risk aversion coefficient of the effectively unconstrained agent, i.e., of the more risk averse agent. Because there is only one source of risk in this economy, the aggregate stock returns are instantaneously perfectly correlated with shocks to the aggregate endowment and therefore the Sharpe ratio of stock returns coincides with the maximum achievable Sharpe ratio, thus the Sharpe ratio of stock returns is effectively set by the more risk averse of the two agents.

3.3.2 Comparison with representative agent economies

Having characterized the competitive equilibrium, we are now in a position to identify the impact of heterogeneity on the properties of asset returns. We compare our heterogeneous economy to a representative agent economy populated by identical agents with a relative risk aversion of γ^* . By the same logic as above, we look for an equilibrium which is supposed to approximate an economy in which a small amount of borrowing is allowed, that is, we are looking for an equilibrium in which the representative agent is unconstrained. The solution to this problem is well-known. The moments of asset returns in this economy are given by

$$\sigma_S = \sigma, \quad \frac{\mu_S - r}{\sigma_S} = \gamma^* \sigma, \quad r = \rho + \gamma^* \mu - \frac{\gamma^*(1 + \gamma^*)}{2} \sigma^2. \quad (8)$$

Both the Sharpe ratio of stock returns and the risk-free rate depend on the same preference parameter. This gives rise to the well-known interest rate puzzle (Weil 1989): in a representative agent model with CRRA preferences, realistic values of the Sharpe ratio of stock returns are associated with unrealistically high levels of the risk-free rate.

The economy with borrowing constraints has properties that are markedly different from those in the representative-agent economy. Comparing the results in Proposition 1 with equation (8), we see that the Sharpe ratio of stock returns in the constrained heterogeneous-agent economy is the same as in the representative-agent economy populated only by the second, more risk averse, of the two agents, i.e., the

economy with $\gamma^* = \gamma$. At the same time, the risk-free rate in the constrained heterogeneous economy is lower than the corresponding value suggested by (8), which we will henceforth denote by $r(\gamma^*)$. The following proposition summarizes the properties of the risk-free rate.

Proposition 2 *The risk-free interest rate in the economy with the borrowing constraint is a monotonically decreasing function of time. At time 0, the initial value of the interest rate is given by*

$$r_0 = z[r(1) + (1 - \gamma)\sigma^2] + (1 - z)r(\gamma), \quad (9)$$

where $r(\gamma^*)$ denotes the risk-free rate in a representative-agent economy with risk aversion equal to γ^* , as given in (8), and $z = \gamma A/[1 + (\gamma - 1)A] \in [0, 1]$, where $A = C_{1,0}/D_0$ is the time-zero consumption share of the log-utility agent (see Proposition 1). The initial value of the interest rate is a convex combination of $r(\gamma)$ and $r(1) + (1 - \gamma)\sigma^2$ and the weight, z , is a decreasing function of the wealth distribution $\omega_{\gamma,0}$. In the long run, as time approaches infinity,

- (i) If $\mu - \gamma\sigma^2/2 > 0$, then $r(\gamma) > r(1)$ and $\lim_{t \rightarrow \infty} r_t = \rho + \mu - \gamma\sigma^2 = r(1) + (1 - \gamma)\sigma^2$;
- (ii) If $\mu - \gamma\sigma^2/2 < 0$, then $r(\gamma) < r(1)$ and $\lim_{t \rightarrow \infty} r_t = \rho + \gamma\mu - \gamma(1 + \gamma)\sigma^2/2 = r(\gamma)$;
- (iii) If $\mu - \gamma\sigma^2/2 = 0$, then $r_t = r(\gamma) = r(1)$.

Case (i) of Proposition 2 is the one in which the “interest rate puzzle” can arise in a representative agent economy: a relatively high Sharpe ratio in the economy with $\gamma^* = \gamma$ is also associated with a relatively high risk-free interest rate, i.e. $r(\gamma) > r(1)$. It is also the case that is relevant for empirical analysis, since most reasonable parameter choices would satisfy the condition $\mu - \gamma\sigma^2/2 > 0$, which says that the risk-adjusted growth rate of the economy is positive. The proposition shows that in this case, in the heterogeneous economy, the risk-free rate is always lower than in the representative-agent economy with risk aversion equal to γ , i.e., $r_t < r(\gamma)$. Moreover, for sufficiently large values of t , or equivalently for an initial wealth distribution with

enough wealth controlled by the log-agent, the risk-free rate in the heterogeneous economy is even lower than in the log-agent economy, that is, $r_t < r(1)$. Thus, in contrast to the representative agent economies, in our heterogeneous economy the risk-free rate is almost entirely divorced from the Sharpe ratio of stock returns. In fact, in an economy with only a small fraction of wealth controlled by the more risk averse type of agents, the Sharpe ratio of stock returns is the same as in a homogeneous economy with risk aversion of γ , while the risk-free rate is even *lower* than in a homogeneous economy with risk aversion of one.

The intuition underlying the result in Proposition 2 is the following. When most of the wealth in the economy is controlled by the log investor, the level of expected stock returns is close to that in a homogeneous economy with a log-utility representative agent, that is, $\rho + \mu$. This is because the consumption rate of the log-utility agent is a constant fraction of his/her wealth, given by the time preference parameter ρ (e.g., Merton, 1969). Market clearing requires that the wealth of the log agent is approximately equal to the stock price, while his/her consumption approximately equals the aggregate endowment, from which the result on the price level and expected stock returns follows immediately. However, following Proposition 1, we argued that it is quite intuitive why the Sharpe ratio of stock returns is determined by risk aversion of the effectively unconstrained, more risk averse investor. Thus, the presence of the borrowing constraint drives a wedge between the risk-free rate and the Sharpe ratio of stock returns.

3.3.3 Comparison with an economy without borrowing constraints

To further isolate the effect of the borrowing constraint, we consider the benchmark economy where agents are heterogeneous and there is no constraint on borrowing. This is precisely the setting studied by Wang (1996). We assume that the agent's preferences in the unconstrained economy are identical to those in the constrained economy. Unfortunately, the asset prices in the unconstrained economy cannot be

computed in closed form, which limits the scope of our analysis. Nevertheless, some comparative results can be established.

Proposition 3 *The instantaneous Sharpe ratio of stock returns in the unconstrained economy falls between σ and $\gamma\sigma$ and hence is lower or equal to that in the economy with borrowing constraints, regardless of the cross-sectional distribution of wealth in each of the economies.*

Proposition 3 establishes that imposing the borrowing constraint raises the Sharpe ratio of stock returns. In the unconstrained economy, there is no well-defined marginal investor, risk aversion of both agents affects the Sharpe ratio of stock returns. As we argued above, imposing a borrowing constraint effectively makes the logarithmic agent infra-marginal and the Sharpe ratio of stock returns is now set by the more risk averse agents. Thus, it is not surprising that in the constrained economy the Sharpe ratio is higher than in its unconstrained counterpart.

Intuitively, one could also conjecture that imposing the borrowing constraint lowers the risk-free interest rate. This is because imposing the constraint reduces the demand for borrowing on behalf of the less risk averse investors, so for the bond market to clear, the more risk averse investors should not be willing to lend, and therefore the risk-free rate must fall. This argument is heuristic, since it ignores the general-equilibrium effects that the borrowing constraint has on the dynamic properties of stock returns and the risk-free rate. Nevertheless, this intuition is appealing and is formalized in Proposition 4 below.

Because the wealth distribution in the unconstrained economy cannot be derived explicitly, it is difficult to compare interest rates in the constrained and the unconstrained economies while controlling for the wealth distribution. In the following proposition, we take a different approach, by assuming that the consumption distribution in the two economies is identical and comparing the corresponding interest rates. This is not a standard comparative statics experiment, since the consumption distribution in the two economies being same is not equivalent to the wealth distribution being the same. However, together with Proposition 3 this establishes the

following important result. For an unconstrained economy with any wealth distribution one can always find a wealth distribution in a constrained economy with the same preferences to simultaneously achieve a lower value of the risk-free rate and a higher value of the Sharpe ratio of stock returns.

Proposition 4 *Given the same cross-sectional distribution of consumption in the unconstrained and the constrained economies, the risk-free interest rate in the constrained economy is lower than or equal to that in the unconstrained economy.*

Propositions 3 and 4 show that, *holding the agents' preferences fixed*, imposing a borrowing constraint increases the Sharpe ratio of stock returns and lowers the risk-free interest rate. One could, however, argue that since the distribution of risk-aversion coefficients is not directly observable, one would often treat it as a free parameter in calibration, and therefore an unconstrained economy could potentially have properties similar to a constrained economy, albeit with a different choice of risk aversion parameters. The following proposition demonstrates that this is the case. In fact, an unconstrained heterogeneous economy exhibits a tradeoff between the Sharpe ratio of stock returns and the risk-free rate that is very similar to the one in representative-agent economies with CRRA preferences. In the latter case, the Sharpe ratio of stock returns, denoted by $SR(\gamma)$, and the risk-free rate are related by

$$r(\gamma) = \rho + \frac{\mu}{\sigma}SR(\gamma) - \frac{1 + \gamma^{-1}}{2}SR(\gamma)^2.$$

For realistic choices of model parameters, a high Sharpe ratio of returns implies a relatively high risk-free rate. As the following proposition shows, the situation is not very different in an unconstrained heterogeneous economy.

Proposition 5 *Let SR_t^{unc} denote the instantaneous Sharpe ratio of stock returns in an unconstrained heterogeneous-agent economy. Then the risk-free interest rate and the Sharpe ratio of stock returns satisfy*

$$r_t^{unc} > \rho + \frac{\mu}{\sigma}SR_t^{unc} - (SR_t^{unc})^2. \quad (10)$$

The inequality (10) does not explicitly depend on the preference parameter γ or the distribution of wealth in a heterogeneous economy, that is, it applies for any wealth distribution within a particular economy and also across various economies, differing in agents' risk aversion. Figure 3-1 below illustrates the implication of Proposition 5. Note that, as the wealth distribution shifts from the log-utility agent to the more risk averse agent (as ω_γ increases), both the interest rate and the Sharpe ratio rise in the unconstrained economy. However, to achieve a high value of the Sharpe ratio, the risk-free rate must be unrealistically high. On the other hand, a constrained economy with the same parameter values can generate a high Sharpe ratio while the risk-free rate remains relatively low.

Finally, we find that the borrowing constraint does not have an obvious systematic effect on the volatility of stock returns. In the constrained economy, the instantaneous return volatility equals the volatility of the endowment process σ , while in the unconstrained economy it can be either higher or lower, depending on the choice of model parameters.

3.4 Numerical analysis: More general borrowing constraints

In this section, we analyze a more general case, when the proportion of individual wealth invested in the risky asset is bounded from above, $\pi \leq \bar{\pi} > 1$. An explicit solution in this case is not available and we have to resort to numerical simulations. Our objective is to illustrate that the explicit solution in the economy without borrowing is qualitatively similar to the behavior of an economy with a relatively tight restriction on leverage.

We consider an economy in which the moments of the aggregate endowment growth are given by $\mu = 0.018$ and $\sigma = 0.033$ (these correspond to the unconditional moments of the century-long U.S. aggregate consumption series). We set the subjective time discount rate to $\rho = 0.02$. We assume that the more risk averse agent

in the economy has the relative risk aversion parameter $\gamma = 10$. We set $\bar{\pi} = 1.05$, i.e., the agents cannot borrow more than 5% of their individual wealth. We solve for equilibrium prices and strategies using the same iterative procedure as in Kogan and Uppal (2002).

Our results are shown in Figure 3-1. In the region where the borrowing constraint is binding, which corresponds approximately to $\omega_\gamma > 0.05$, the Sharpe ratio of stock returns is close to the value in the economy without borrowing, which is the same as in the representative-agent economy with risk aversion of γ . The risk-free rate is monotonically increasing in ω_γ , as in the economy analyzed above. Note that if most of wealth in the economy is controlled by the less risk-averse, log-utility, agent, the interest rate can be lower than in a representative-agent log-utility economy, as predicted by our analytical solution above (the results of Proposition 2, obtained in the limit of large values of time t are equivalent to the limit of the wealth distribution ω_γ approaching zero, since the wealth distribution in the economy without borrowing is a deterministic monotone function of time and $\lim_{t \rightarrow \infty} \omega_{\gamma,t} = 0$).

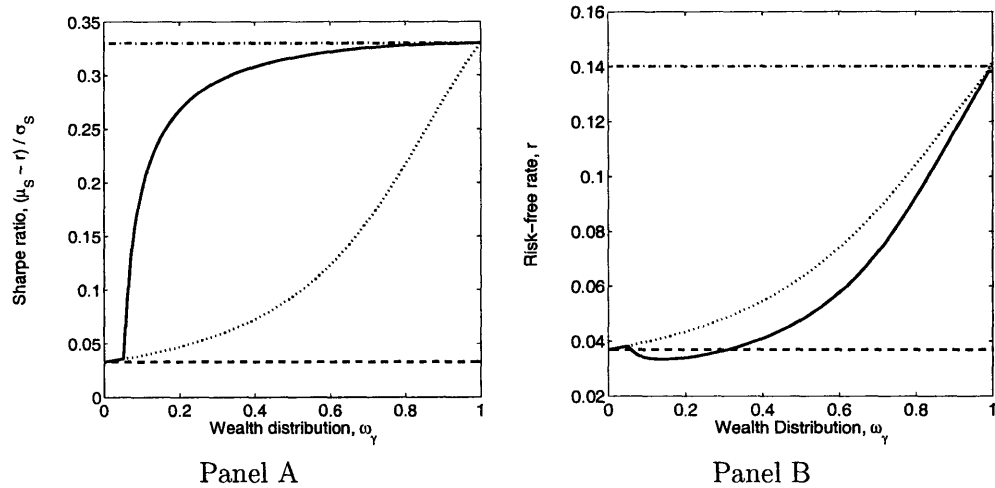


Figure 3-1: Effect of borrowing constraint on Sharpe ratio and risk-free rate

Panel (a) plots the instantaneous Sharpe ratio of stock returns in the constrained economy (solid line) and in the unconstrained economy (dotted line) as a function of the wealth distribution, ω_γ . Panel (b) gives the corresponding plots of the risk-free interest rate. The following parameter values are used: $\mu = 0.018$, $\sigma = 0.033$, $\rho = 0.02$. The more risk averse agent in the economy has $\gamma = 10$. The constraint on borrowing is given by $\pi \leq 1.05$, i.e., the agents cannot borrow more than 5% of their individual wealth. The dashed and dashed-dotted lines correspond to the representative-agent economies with risk aversion of $\gamma = 1$ and $\gamma = 10$ respectively.

3.5 Conclusion

In this article, we study a general equilibrium exchange economy with multiple agents who differ in their degree of risk aversion and face borrowing constraints. We show that, unlike in a representative agent model, in an economy with borrowing constraints the Sharpe ratio of stock returns can be relatively high, while the risk-free interest rate remains relatively low. In particular, the Sharpe ratio of stock returns in the constrained heterogeneous-agent economy is the same as in the representative-agent economy populated only by the more risk averse of the two agents, while the risk-free rate in the constrained heterogeneous-agent economy may be even lower than in the representative-agent economy populated by the less risk averse of the two agents. And, comparing the constrained heterogeneous-agent economy to one where agents are heterogeneous but unconstrained, we find that imposing a borrowing constraint increases the Sharpe ratio of stock returns and lowers the risk-free interest rate. Moreover, we show that the heterogeneous-agent unconstrained economies suffer from the same limitations as the representative-agent economies with CRRA preferences, namely the tight link between the Sharpe ratio of stock returns and the level of the risk-free rate, which is not the case in economies with borrowing constraints.

3.6 Appendix A: Proofs and technical results

Proof of Proposition 1

We first examine the decision problem of individual agents, subject to the market prices given in parts (ii–iv) of the Proposition. We then show that markets clear as long as the system of equations (4,5) has a solution. Finally, we prove that such a solution exists and is unique.

Individual agents' consumption/portfolio choice

Since the first, more risk averse, agent is unconstrained in equilibrium, his problem can be formulated in an equivalent static form (see Cox and Huang, 1989)

$$\max_{C_{\gamma,t}} E_0 \left[\int_0^\infty e^{-\rho t} \frac{C_{\gamma,t}^{1-\gamma}}{1-\gamma} dt \right], \quad (\text{A1})$$

subject to the budget constraint

$$E_0 \left[\int_0^\infty e^{-Rt} \xi_t C_{\gamma,t} dt \right] = \omega_\gamma E_0 \left[\int_0^\infty e^{-Rt} \xi_t D_t dt \right] = \omega_{\gamma,0} S_0. \quad (\text{A2})$$

where ξ_t is the density of the equivalent martingale measure (EMM density). Given (ii) ξ_t takes the following form

$$\xi_t = e^{-\frac{1}{2}\gamma^2\sigma^2 t - \gamma\sigma W_t}. \quad (\text{A3})$$

The optimal consumption of the first agent then satisfies

$$e^{-\rho t} C_{\gamma,t}^{-\gamma} = \lambda_1 e^{-Rt} \xi_t, \quad (\text{A4})$$

where λ_1 is the Lagrange multiplier on his budget constraint. Thus,

$$C_{\gamma,t} = \lambda_1^{-\frac{1}{\gamma}} e^{\frac{Rt}{\gamma}} e^{\frac{-\rho - \gamma\mu + \gamma(1+\gamma)\sigma^2/2}{\gamma} t} D_t. \quad (\text{A5})$$

To solve the problem of the log-utility agent, we use the technique developed in Cvitanic and Karatzas (1992) for portfolio optimization with constraints. Specifically, we introduce a fictitious market in which the diffusion component of stock returns is the same as in the original market, but the EMM density is now given by

$$\tilde{\xi}_t = e^{-\frac{1}{2}\sigma^2 t - \sigma W_t} \quad (\text{A6})$$

and the interest rate by

$$\tilde{r}_t = r_t - (1 - \gamma) \sigma^2. \quad (\text{A7})$$

It is easy to check that the expected stock return in the fictitious market is the same as in the original market. If it turns out that the optimal portfolio strategy for the agent in the fictitious market satisfies the original constraints, this strategy would also be optimal in the original market (see Cvitanic and Karatzas, 1992).

The log-utility agent's problem in fictitious market is

$$\max_{C_{1,t}} E_0 \left[\int_0^\infty e^{-\rho t} \ln C_{1,t} ds \right], \quad (\text{A8})$$

subject to

$$E_0 \left[\int_0^\infty e^{-R_t + (1-\gamma)\sigma^2 t} \tilde{\xi}_t C_{1,t} dt \right] = (1 - \omega_{\gamma,0}) S_0. \quad (\text{A9})$$

The optimality condition takes form

$$e^{-\rho t} C_{1,t}^{-1} = \lambda_2 e^{-R_t + (1-\gamma)\sigma^2 t} \tilde{\xi}_t, \quad (\text{A10})$$

and therefore

$$C_{1,t} = \lambda_2^{-1} e^{R_t} e^{(-\rho - \mu + \gamma\sigma^2)t} D_t. \quad (\text{A11})$$

Market clearing conditions

Let us define

$$\bar{R}_t = R_t - (\rho + \mu - \gamma\sigma^2)t. \quad (\text{A12})$$

Then, equations (A5) and (A11) take form

$$C_{\gamma,t} = \lambda_1^{-\frac{1}{\gamma}} e^{\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}} D_t, \quad (\text{A13})$$

$$C_{1,t} = \lambda_2^{-1} e^{\bar{R}_t} D_t. \quad (\text{A14})$$

The market clearing condition in the consumption market is then given by

$$\lambda_1^{-\frac{1}{\gamma}} e^{\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}} + \lambda_2^{-1} e^{\bar{R}_t} = 1, \quad (\text{A15})$$

which should hold for every $t \in [0, \infty)$. The condition (A15) at time $t = 0$ implies that $\lambda_2^{-1} = 1 - \lambda_1^{-\frac{1}{\gamma}} = C_{1,0}/D_0$. Let us denote $C_{1,0}/D_0$ as A and express (A15) as

$$(1 - A) \exp\left(\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) + A \exp(\bar{R}_t) = 1. \quad (\text{A16})$$

Consider now the budget constraint of the log-utility agent:

$$(1 - \omega_{\gamma,0})S_0 = E_0 \left[\int_0^\infty e^{-R_t + (1-\gamma)\sigma^2 t} \tilde{\xi}_t C_{1,t} dt \right] = A \int_0^\infty e^{-\rho t} dt = \frac{A}{\rho}. \quad (\text{A17})$$

Since

$$S_0 = E_0 \left[\int_0^\infty e^{-R_t} \xi_t D_t dt \right] = \int_0^\infty e^{-R_t} e^{(\mu - \gamma\sigma^2)t} dt = \int_0^\infty e^{-\bar{R}_t - \rho t} dt, \quad (\text{A18})$$

equation (A17) is equivalent to

$$A = \rho(1 - \omega_{\gamma,0}) \int_0^\infty e^{-\bar{R}_t - \rho t} dt. \quad (\text{A19})$$

As long as the budget constraint of the log-utility agent is satisfied, so is the budget constraint of the non-log agent, which follows from equations (A13), (A16), and (A18). Finally, note also that according to (A18), the ratio of the stock price to the aggregate endowment is a deterministic function of time, and hence the instantaneous volatility of stock returns equals σ . Similarly, the volatility of wealth of the log-utility agent (computed using the EMM density $\tilde{\xi}_t$) is equal to σ . Hence, the log-utility agent

invests all of his/her wealth in the stock market, and therefore, the no-borrowing constraint is satisfied. The same is true for the non-log agent. Thus, we conclude that the equilibrium postulated in Proposition 1 exists as long as the system of two equations (A16, A19) (equations (4) and (5) in Proposition 1) has a solution.

Existence and uniqueness of solution to equations (4) and (5)

Differentiating (4) with respect to A we have

$$(\partial_A \bar{R}_t) \frac{1 - (1 - \gamma)A \exp(\bar{R}_t)}{\gamma} = \exp\left(\frac{\bar{R}_t + (1 - \gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) - \exp(\bar{R}_t). \quad (\text{A20})$$

Consider two cases:

Case: $\mu - \gamma\sigma^2/2 > 0$

Equation (4) implies that in this case $\bar{R}_t \geq 0$. From equation (A20) then it follows that $\partial_A \bar{R}_t < 0$. Let us define a mapping

$$I(A) = \rho(1 - \omega_{\gamma,0}) \int_0^\infty e^{-\bar{R}_t - \rho t} dt. \quad (\text{A21})$$

Differentiating (A21) with respect to A we have

$$I'(A) = -\rho(1 - \omega_{\gamma,0}) \int_0^\infty \partial_A \bar{R}_t e^{-\bar{R}_t - \rho t} dt \geq 0. \quad (\text{A22})$$

From (4) we have

$$\begin{aligned} 0 < I(0) &= (1 - \omega_{\gamma,0})\rho / [\rho - (1 - \gamma)(\mu - \gamma\sigma^2/2)] < 1 \\ 0 < I(1) &= (1 - \omega_{\gamma,0}) < 1. \end{aligned}$$

Therefore, by the Brouwer Fixed-Point theorem, the system of equations (4), (5) has a solution. To show that the solution is unique, we compute the second derivative of $I(A)$:

$$I''(A) = \rho(1 - \omega_{\gamma,0}) \int_0^\infty [(\partial_A \bar{R}_t)^2 - \partial_{AA} \bar{R}_t] e^{-\bar{R}_t - \rho t} dt. \quad (\text{A23})$$

Differentiating (A20) with respect to A we have

$$\begin{aligned} & \partial_{AA} \bar{R}_t (1 - (1 - \gamma)A \exp(\bar{R}_t)) - (1 - \gamma) \partial_A \bar{R}_t (1 + A \partial_A \bar{R}_t) \exp(\bar{R}_t) = \\ & = \partial_A \bar{R}_t \left(\exp\left(\frac{\bar{R}_t + (1 - \gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) - \exp(\bar{R}_t) \right) + (1 - \gamma) \partial_A \bar{R}_t \exp(\bar{R}_t). \end{aligned}$$

Therefore,

$$\begin{aligned} & [(\partial_A \bar{R})_{A,t}^2 - \partial_{AA} \bar{R}_t] (1 - (1 - \gamma)A \exp(\bar{R}_t)) = \\ & \partial_A \bar{R}_t (\gamma - 1) \frac{[1 + (2\gamma - 1)A e^{\bar{R}_t}] \exp\left(\frac{\bar{R}_t + (1 - \gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) + \exp(\bar{R}_t) (1 - A \exp(\bar{R}_t))}{1 - (1 - \gamma)A \exp(\bar{R}_t)}. \end{aligned}$$

Since $\partial_A \bar{R}_t < 0$ and $\gamma > 1$, we conclude that $I''(A) < 0$ and the uniqueness of the solution follows.

Case: $\mu - \gamma\sigma^2/2 \leq 0$

Equation (4) implies that in this case $\bar{R}_t \leq 0$. From equation (A20) then it follows that $\partial_A \bar{R}_t > 0$ and, therefore, $I'(A) \leq 0$. This implies that the equation $I(A) = A$ has a unique solution.

Proof of Proposition 2

Differentiating equation (4) with respect to t at $t = 0$ proves (9). To show that z is a decreasing function of the $\omega_{\gamma,0}$, it is enough to prove that A is a decreasing function of $\omega_{\gamma,0}$, since z is monotonically increasing in A . $A = I(A, \omega_{\gamma,0})$ holds for every $\omega_{\gamma,0} \in [0, 1]$. Differentiating this equality in $\omega_{\gamma,0}$, we find that

$$[1 - \partial_A I(A, \omega_{\gamma,0})] \partial_{\omega_{\gamma,0}} A = \partial_{\omega_{\gamma,0}} I < 0.$$

At the fixed point of the mapping $I(A)$, it must be that $\partial I(A, \omega_{\gamma,0}) < 1$, and hence $\partial_{\omega_{\gamma,0}} A < 0$.

To establish the asymptotic properties of the risk-free rate, we examine (4) as t approaches infinity.

(i) Case $\mu - \gamma\sigma^2/2 > 0$: $\lim_{t \rightarrow \infty} \bar{R}_t = -\ln A$, and therefore, $\lim_{t \rightarrow \infty} r_t = \rho + \mu - \gamma\sigma^2$.

(ii) Case $\mu - \gamma\sigma^2/2 < 0$: $\lim_{t \rightarrow \infty} \bar{R}_t + (1 - \gamma)(\mu - \gamma\sigma^2/2)t = \text{const}$, and therefore,
 $\lim_{t \rightarrow \infty} r_t = \rho + \gamma\mu - \gamma(1 + \gamma)\sigma^2/2$.

(iii) Case $\mu - \gamma\sigma^2/2 = 0$: $\bar{R}_t = 0$ and $r_t = \rho + \mu - \gamma\sigma^2$.

Proof of Propositions 3–5

First, we establish some properties of the unconstrained economy. The equilibrium allocation of consumption in such economy is Pareto-optimal and can be recovered as a solution of the central planner's problem (see Wang, 1996)

$$\max_{C_\gamma + C_1 = D} \frac{C_\gamma^{1-\gamma}}{1-\gamma} + \lambda \ln C_1 \quad (\text{A24})$$

for a suitable choice of the utility weight λ . Let $u(D; \lambda)$ denote the solution of (A24), which can be interpreted as a utility function of the representative agent (social planner). Using the optimality conditions, it is easy to show that (A24) implies

$$\partial_D C_\gamma = \frac{C_\gamma}{D - (1 - \gamma)C_1} \quad (\text{A25})$$

$$\partial_{DD} C_\gamma = \frac{(1 - \gamma)\gamma C_\gamma C_1}{(D - (1 - \gamma)C_1)^3}, \quad (\text{A26})$$

and therefore,

$$\partial_D u(D; \lambda) = \lambda \frac{1}{C_1}, \quad (\text{A27})$$

and

$$\partial_{DD} u(D; \lambda) = -\lambda \frac{1}{C_1} \frac{\gamma}{D - (1 - \gamma)C_1}. \quad (\text{A28})$$

According to the consumption CAPM, the instantaneous Sharpe ratio of stock returns is given by

$$SR_0^{unc} = \sigma \frac{-D_0 \partial_{DD} u(D; \lambda)}{\partial_D u(D; \lambda)} = \sigma \frac{\gamma}{(1 - A) + \gamma A} \in [\sigma, \gamma\sigma], \quad (\text{A29})$$

where $A = C_{1,0}/D_0$ denotes the consumption share of the log-utility agent at time zero, as in the constrained economy characterized in Proposition 1. This proves Proposition 3.

The risk-free rate in the unconstrained economy can be computed using the derived utility function of the representative agent. Specifically,

$$r_t^{unc} = \rho - \frac{D_t \partial_{DD} u(D_t; \lambda)}{\partial_D u(D_t; \lambda)} \mu - \frac{1}{2} \frac{D_t^2 \partial_{DDD} u(D_t; \lambda)}{\partial_D u(D_t; \lambda)} \sigma^2. \quad (\text{A30})$$

It then follows that

$$r_0^{unc} = \rho + \frac{\gamma}{1 + (\gamma - 1)A} \left[\mu - \frac{1 + \gamma - \frac{\gamma(1-\gamma)A}{1+(\gamma-1)A} \sigma^2}{(1 + (\gamma - 1)A)} \right]. \quad (\text{A31})$$

Now compare r_0^{unc} to the interest rate in the constrained economy with the same initial distribution of consumption, as given by Proposition 1. We find that

$$r_0^{unc} - r_0 = \frac{\gamma A \sigma^2}{2(1 - A + \gamma A)^3} [(\gamma - 1)^3 A^2 + A(\gamma^3 + \gamma^2 - 5\gamma + 3) + 2(\gamma - 1)].$$

It is easy to see that since $A \in [0, 1]$ and $\gamma \geq 1$,

$$r_0^{unc} - r_0 \geq 0.$$

This proves Proposition 4. The result of Proposition 5 follows from (A29) and (A31).

3.7 Appendix B: General case

In this section we consider a general case of two agents with CRRA utility functions.

$$U_t^1(c_1) = E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{c_{\gamma_1,s}^{1-\gamma_1}}{1-\gamma_1} ds \right] \quad \text{and} \quad U_t^2(c_2) = E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{c_{\gamma_2,s}^{1-\gamma_2}}{1-\gamma_2} ds \right]. \quad (\text{B1})$$

Without loss of generality we assume that $\gamma_1 > \gamma_2$. We prove the following analog of Proposition 1:

Proposition 6 *Let $\rho > \max[(1 - \gamma_1)\mu + \frac{\gamma_1(\gamma_1-1)}{2}\sigma^2, 0]$. There exists a competitive equilibrium in which*

(i) *The consumption processes of the two agents are given by*

$$C_{\gamma_1,t} = (1 - A) \exp \left(\frac{\bar{R}_t + (\gamma_2 - \gamma_1)(\mu + (1 - \gamma_1 - \gamma_2)\sigma^2/2)t}{\gamma_1} \right) D_t, \quad (\text{B2})$$

$$C_{\gamma_2,t} = A \exp(\bar{R}_t) D_t, \quad (\text{B3})$$

where the constant $A = C_{1,0}/D_0 \in [0, 1]$ and the deterministic process \bar{R}_t are determined as a solution of the following system of equations:

$$(1 - A) \exp \left(\frac{\bar{R}_t + (\gamma_2 - \gamma_1)(\mu + (1 - \gamma_1 - \gamma_2)\sigma^2/2)t}{\gamma_1} \right) + A \exp \left(\frac{\bar{R}_t}{\gamma_2} \right) = 1, \quad (\text{B4})$$

$$A \int_0^\infty e^{-\frac{\gamma_2-1}{\gamma_2}\bar{R}_t - \psi t} dt - (1 - \omega_{\gamma_1,0}) \int_0^\infty e^{-\bar{R}_t - \psi t} dt = 0, \quad (\text{B5})$$

where

$$\psi = \rho + (\gamma_2 - 1)(\mu - \gamma_2\sigma^2/2). \quad (\text{B6})$$

(ii) *The instantaneous Sharpe ratio of stock returns equals*

$$\frac{\mu_{St} - r_t}{\sigma_{St}} = \gamma_1 \sigma; \quad (\text{B7})$$

(iii) The instantaneous volatility of stock returns equals

$$\sigma_{St} = \sigma; \quad (\text{B8})$$

(iv) The risk-free interest rate process r_t is deterministic and is given by

$$r_t = \frac{d\bar{R}_t}{dt} + (\rho + \gamma_2\mu - (\gamma_1 + \gamma_2(\gamma_2 - 1)/2)\sigma^2).$$

Proof. Following the same line of arguments as before we find that the first agent's solution to the unconstrained problem with EMM density

$$\xi_t = e^{-\frac{1}{2}\gamma_1^2\sigma^2t - \gamma_1\sigma W_t} \quad (\text{B9})$$

is

$$C_{\gamma_1,t} = \lambda_1^{-\frac{1}{\gamma_1}} e^{\frac{R_t}{\gamma_1}} e^{\frac{-\rho - \gamma_1\mu + \gamma_1(1+\gamma_1)\sigma^2/2}{\gamma_1} t} D_t. \quad (\text{B10})$$

where λ_1 is the Lagrange multiplier on his budget constraint. To solve the problem of the less risk-averse second agent, we use the technique developed in Cvitanic and Karatzas (1992) for portfolio optimization with constraints. Specifically, we introduce a fictitious market in which the diffusion component of stock returns is the same as in the original market, but the EMM density is now given by

$$\xi_t = e^{-\frac{1}{2}\gamma_2^2\sigma^2t - \gamma_2\sigma W_t} \quad (\text{B11})$$

and the interest rate by

$$\tilde{r}_t = r_t - (\gamma_2 - \gamma_1)\sigma^2. \quad (\text{B12})$$

Thus, the second agent solves

$$\max_{C_{\gamma_2,t}} E_0 \left[\int_0^\infty e^{-\rho s} \frac{C_{\gamma_2,s}^{1-\gamma_2}}{1-\gamma_2} ds \right], \quad (\text{B13})$$

subject to

$$E_0 \left[\int_0^\infty e^{-R_t + (\gamma_2 - \gamma_1)\sigma^2 t} \tilde{\xi}_t C_{\gamma_2, t} dt \right] = (1 - \omega_{\gamma_1, 0}) S_0. \quad (\text{B14})$$

As a result, his consumption is given by

$$C_{\gamma_2, t} = \lambda_2^{-\frac{1}{\gamma_2}} e^{\frac{R_t}{\gamma_2}} e^{\frac{-\rho - \gamma_2 \mu + (\gamma_1 + \gamma_2(\gamma_2 - 1)/2)\sigma^2}{\gamma_2} t} D_t. \quad (\text{B15})$$

Market clearing conditions

Let us define

$$\bar{R}_t = R_t - (\rho + \gamma_2 \mu - (\gamma_1 + \gamma_2(\gamma_2 - 1)/2)\sigma^2)t. \quad (\text{B16})$$

The market clearing condition in the consumption market is then given by

$$(1 - A) \exp \left(\frac{\bar{R}_t + (\gamma_2 - \gamma_1)(\mu + (1 - \gamma_1 - \gamma_2)\sigma^2/2)t}{\gamma_1} \right) + A \exp \left(\frac{\bar{R}_t}{\gamma_2} \right) = 1, \quad (\text{B17})$$

where, as before, $A = C_{1,0}/D_0 = \lambda_2^{-\frac{1}{\gamma_2}} = 1 - \lambda_1^{-\frac{1}{\gamma_1}} = C_{1,0}/D_0$. Consider now the budget constraint of the less risk-averse agent:

$$(1 - \omega_{\gamma, 0}) S_0 = E_0 \left[\int_0^\infty e^{-R_t + (\gamma_2 - \gamma_1)\sigma^2 t} \tilde{\xi}_t C_{\gamma_2, t} dt \right] = A \int_0^\infty e^{-\frac{\rho}{\gamma_2} t} e^{\frac{\gamma_2 - 1}{\gamma_2} [-R_t + (\frac{\gamma_2}{2} - \gamma_1)\sigma^2 t]}. \quad (\text{B18})$$

Since

$$S_0 = E_0 \left[\int_0^\infty e^{-R_t} \xi_t D_t dt \right] = \int_0^\infty e^{-R_t} e^{(\mu - \gamma_1 \sigma^2)t} dt = \int_0^\infty e^{-\bar{R}_t - (\rho + (\gamma_2 - 1)(\mu - \gamma_2 \sigma^2/2))t} dt, \quad (\text{B19})$$

equation (B18) is equivalent to

$$A \int_0^\infty e^{-\frac{\rho}{\gamma_2} t} e^{\frac{\gamma_2 - 1}{\gamma_2} [-R_t + (\frac{\gamma_2}{2} - \gamma_1)\sigma^2 t]} = (1 - \omega_{\gamma, 0}) \int_0^\infty e^{-\bar{R}_t - (\rho + (\gamma_2 - 1)(\mu - \gamma_2 \sigma^2/2))t} dt, \quad (\text{B20})$$

or

$$A \int_0^\infty e^{-\frac{\gamma_2 - 1}{\gamma_2} \bar{R}_t - \psi t} dt = (1 - \omega_{\gamma_1, 0}) \int_0^\infty e^{-\bar{R}_t - \psi t} dt. \quad (\text{B21})$$

As long as the budget constraint of the second agent is satisfied, so is the budget constraint of the first agent, which follows from equations (B17) and (B19). Finally, note also that according to (B19), the ratio of the stock price to the aggregate endowment is a deterministic function of time, and hence the instantaneous volatility of stock returns equals σ . Similarly, the volatility of wealth of the second agent (computed using the EMM density $\tilde{\xi}_t$) is equal to σ . Hence, the second agent invests all of his wealth in the stock market, and therefore, the no-borrowing constraint is satisfied. The same is true for the first agent. Thus, we conclude that the equilibrium postulated in Proposition 1 exists as long as the system of two equations (B17, B21) has a solution.

Existence of solution to equations (B17) and (B21)

Let us define a mapping

$$I(A) = A \int_0^\infty e^{-\frac{\gamma_2-1}{\gamma_2}\bar{R}_t-\psi t} dt - (1 - \omega_{\gamma_1,0}) \int_0^\infty e^{-\bar{R}_t-\psi t} dt. \quad (\text{B22})$$

From (B17) and (B22) we have

$$I(0) < 0 < I(1).$$

Therefore, the system of equations (B17), (B21) has a solution.

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