

String-net condensation and topological phases in quantum spin systems

by

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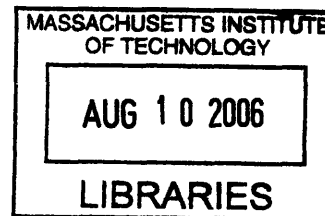
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Abstract

For many years, it was thought that Landau's theory of symmetry breaking could describe essentially all phases and phase transitions. However, in the last twenty years, it has become clear that at zero temperature, quantum mechanics allows for the possibility of new phases of matter beyond the Landau paradigm. In this thesis, we develop a general theoretical framework for these "exotic phases" analogous to Landau's framework for symmetry breaking phases. We focus on a particular type of exotic phase, known as "topological phases", and a particular physical realization of topological phases - namely frustrated quantum magnets. Our approach is based on a new physical picture for topological phases. We argue that, just as symmetry breaking phases originate from the condensation of particles, topological phases originate from the condensation of extended objects called "string-nets." Using this picture we show that, just as symmetry breaking phases can be classified using symmetry groups, topological phases can be classified using objects known as "tensor categories." In addition, just as symmetry breaking order manifests itself in local correlations in a ground state wave function, topological order manifests itself in nonlocal correlations or quantum entanglement. We introduce a new quantity - called "topological entropy" - which measures precisely this nonlocal entanglement. Many of our results are applicable to other (non-topological) exotic phases.

Thesis Supervisor: Xiao-Gang Wen

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Acknowledgments

One of the most common questions one gets as a student is: “when are you going to graduate?” Over the last five years, I’ve been asked that question many times. I usually give a rough estimate, but then I always add “But, I don’t want to!” That comment almost always elicits shock and confusion. “How could you not want to graduate?!” is the invariable reply. After all, graduate school is the butt of so many jokes - with its low pay, long hours, and stressful qualifying exams. Graduate students are supposed to want to finish as quickly as possible. Well, my time at MIT has been a wonderful experience in many ways, and even though I’m only going a few miles away, I’m certain I will miss it.

On the academic front, this is mainly due to my adviser Xiao-Gang. There are many things about Xiao-Gang that amaze me, but the first thing that comes to mind is his incredible energy and enthusiasm. Many times during the last four years, I came to his office discouraged and uncertain about my research. However, by the end of our meeting - usually an intense one or two hour discussion - I’d invariably be excited and cheered! Of course, much of that excitement originated not from Xiao-Gang’s personality, but from his brilliance, particularly his remarkable creativity. Somehow, he always managed to find new ways of looking at things, and new directions for research. It was hard to be stuck for long when working with Xiao-Gang.

Equally important was his warmth and humility. Xiao-Gang patiently answered any question, no matter how naive, and discussed any progress, no matter how minor. As a result, ideas could flow freely - fostering my own creativity and explorations. Xiao-Gang taught me more than just physics; he taught me a unique style for doing physics, where no idea was out of bounds, and everything from atomic physics to quantum gravity was a potential research topic.

I am also grateful to Xiao-Gang for his help in more practical matters. He encouraged me to attend many stimulating conferences and programs, introduced me to many potential collaborators, and nominated me for awards and fellowships. Also, he was always happy to give advice on presenting and communicating my work. No matter how busy he was, Xiao-Gang always seemed to have time to talk. His kindness and generosity made working with him a real pleasure. Xiao-Gang is truly one of a kind and I am extremely fortunate to have been his student.

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whole corridor - making the whole CMT experience more fun, relaxing, and social.

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Chapter 1

Introduction

1.1 Background and motivation

Seventy years ago, the Russian physicist Lev Landau developed a framework for understanding ordered phases of matter. [1] This framework - known as Landau theory - is based on two physical concepts: symmetry breaking and order parameters. These concepts can explain virtually all familiar phases of matter from crystals and ferromagnets to superfluids and superconductors. For a time, it seemed that Landau theory could explain all the phases and phase transitions in condensed matter physics. All that was left to do was to apply Landau theory to particular cases.

However, in the last twenty years, it has become clear that at zero temperature, quantum mechanics allows for the possibility of new phases beyond the Landau paradigm. These phases are ordered in some sense, but their order is completely different from the familiar order of crystals and ferromagnets. It cannot be thought of in terms of symmetry breaking or order parameters. Instead, it has a nonlocal character and is manifest in the nonlocal entanglement in the ground state. This new order gives rise to new low energy physics: quasiparticles typically carry fractional quantum numbers and interact via emergent gauge bosons!

Recently researchers have coined the term “exotic phases” to describe these non-Landau phases of matter. Theoretically, exotic phases can occur in a wide variety of $T = 0$ strongly correlated condensed matter systems. They are known to occur in the fractional quantum Hall liquids [2, 3, 4, 5] and are suspected of occurring in the high T_c superconductors [6, 7] as well as frustrated quantum magnets. [8, 9, 10, 11, 12] More examples will likely be found. Indeed, it is quite possible that exotic phases are commonplace. Our lack of examples may stem more from a lack of good experimental probes than from a lack of materials.

In any case, if we are optimistic and assume that these exotic phases are indeed common, then we are faced with an important theoretical problem. We have a large class of phases with a unified structure, but our current methods - based on Landau theory - are useless for understanding them. Clearly, we need to develop a new framework analogous to Landau theory for analyzing and characterizing these new exotic phases. Such a framework could unlock a whole new world of condensed

matter physics. In addition, it could have implications for quantum computation: exotic phases are ideal “hardware” from which to build a quantum computer. [13] Finally, it could even be relevant to high energy physics: there is a possibility that the vacuum we live in is such an exotic phase, and the photons and electrons we see around us are simply the low energy excitations of this medium. [14, 15, 16, 17] This hypothesis is particularly compelling, because in these phases, gauge bosons and fermions naturally emerge *together* - just as in our universe. [18]

In this thesis, we will attempt to address the theoretical problem of building a Landau-like framework for exotic phases. We will focus primarily on a class of exotic phases known as “topological phases.” We will also focus on a particular physical realization of these phases - namely frustrated quantum magnets. However, many of our results are relevant to other (non-topological) exotic phases, and to other physical realizations.

1.2 Theory of topological phases

What are topological phases? Topological phases are quantum ($T = 0$) phases of matter with a topological character. They are fundamentally different from any familiar phase - such as ferromagnets, crystals, superconductors, etc. They can occur in any spatial dimension $d > 1$, but here we will focus on the two dimensional case for simplicity. Two dimensional topological phases have several physical properties:

1. Gapped ground state
2. Degenerate ground state on a torus
3. Anyonic quasiparticle excitations
4. Gapless edge excitations (in chiral case)

Let us consider these properties in more detail. The first property of topological phases is that they are gapped. For example, suppose one had a two dimensional quantum magnet in a topological phase. If one measured the energy spectrum of the sample one would find a finite gap Δ between the ground state and the first excited state.

This, in itself, is not new or particularly interesting. What makes topological phases interesting and different is the second property. Suppose one were to take the quantum magnet and wrap it up so that it forms a torus. If one then measured the energy spectrum of this toroidal sample, one would again find a finite gap Δ between the ground state and the first excited state. However, the ground state would now be multiply degenerate! (See Fig. 1-1).

This degeneracy is completely different from the ground state degeneracy that occurs in an ordered quantum phase such as an Ising ferromagnet. Unlike an Ising ferromagnet, this degeneracy has nothing to do with symmetry. Indeed, one can perturb the system in any way, break all symmetries, and the degeneracy cannot

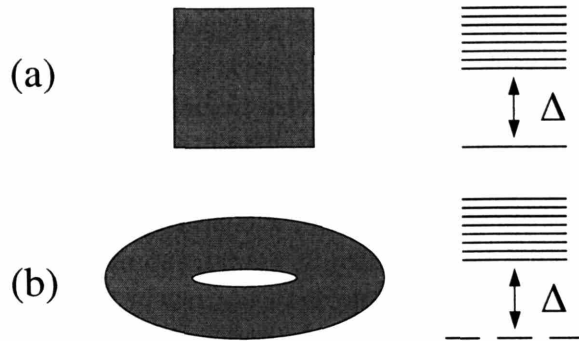


Figure 1-1: The energy spectrum of (a) a piece of topologically ordered material and (b) the same material wrapped into the shape of a torus. In both cases, the ground state is gapped, but in the case of the toroidal sample, the ground state is multiply degenerate.

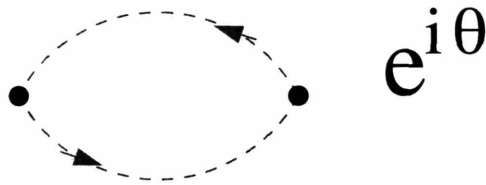


Figure 1-2: In a topological phase, quasiparticle excitations are typically anyons. If one exchanges two quasiparticles, the resulting phase, $e^{i\theta}$, is typically neither $+1$ (bosons) nor -1 (fermions), but rather something in between.

be split (in the thermodynamic limit). [19, 20, 2] This is in contrast to an Ising ferromagnet where the degeneracy splits immediately once the symmetry is broken - for example, by the application of a magnetic field.

The ground state degeneracy in a topological phase is fundamentally tied to the topology of the system. This is strange because all the correlations in a topological phase are short ranged. Yet somehow the material “feels” the global topology of the system. This sensitivity is an indication of the nonlocal correlations and entanglement in topological phases.

The third and perhaps most important property of topological phases is that they exhibit fractional statistics. Imagine exciting the ground state. The result will be a localized excitation - a quasiparticle. In a topological phase, the quasiparticles are typically anyons. If one exchanges two quasiparticles, the resulting phase, $e^{i\theta}$, is typically neither $+1$ (bosons) nor -1 (fermions), but rather something in between (see Fig. 1-2). [21, 22, 23] Moreover, if one moves one quasiparticle in a closed loop around another, it typically acquires a nontrivial phase. Again, this is something that can never occur in a phase with usual order. In an ordered spin system, the excitations are always bosonic.

The final property of topological phases (which is only true for the class of topological phases that break time reversal symmetry) is that they have gapless edge excitations. These edge excitations are described by chiral Luttinger liquids. [24]

This property is the most experimentally accessible property of topological phases.

The ground state degeneracy, fractional statistics, and gapless edge excitations in a topological phase are manifestations of a new kind of order. This order has a topological character, so it has been named “topological order.” [19] The above properties can be thought of as a physical definition of topological phases and topological order. Any phase with these properties is topological and vice versa.

A more formal and precise definition is that a topological phase is a phase whose low energy physics is described by a topological quantum field theory. [25] Topological quantum field theories (TQFTs) have been defined axiomatically by mathematicians [26], but roughly speaking, they can be thought of as quantum field theories whose action S is topologically invariant. That is, S is invariant under any continuous deformation of space and time. The simplest example are abelian Chern-Simons theories. These are topological quantum field theories in $(2 + 1)$ dimensions defined by

$$S = -\frac{k}{4\pi} \int d^3x \epsilon^{\lambda\mu\nu} a_\lambda \partial_\mu a_\nu \quad (1.1)$$

where the indices λ, μ, ν run over $0, 1, 2$, and $a_\nu(x)$ is a gauge field, and k is an integer. One can check that the Chern-Simons action is invariant under any continuous deformation of space-time: $x \rightarrow x'$. It is therefore a valid TQFT.

Another class of TQFTs are lattice gauge theories with a discrete gauge group. The simplest example is Z_2 gauge theory.

At the moment all known two dimensional TQFTs can be represented as either Chern-Simons theories or discrete gauge theories. It is an open question as to whether all TQFTs belong to these two classes (or even whether all TQFTs have a field theoretic description!).

The topological quantum field theory describing a given topological phase contains all the information about the topological content of that phase. In particular, the ground state degeneracies, quasiparticle statistics, and edge excitations can all be calculated once the TQFT is known. Therefore, classifying topological phases is equivalent to classifying TQFTs.

1.3 Experimental realizations of topological phases

The first example of a topological phase was found in 1982. In that year, Tsui, Stormer, and Gossard discovered the fractional quantum Hall effect. [27] They found that a two dimensional electron gas in a strong perpendicular magnetic field displayed unusual behavior at certain densities and magnetic fields. In particular, when the Hall resistance ρ_{xy} was plotted as a function of the magnetic field B , it contained a plateau at $\rho_{xy} = \frac{3h}{e^2}$.

This result was puzzling because it implied that the interacting electron gas formed an incompressible liquid at fractional filling fraction $\nu = \frac{nhc}{eB} = \frac{1}{3}$, even though non-interacting electrons could only form incompressible liquids at integer filling fraction $\nu = m$. The $\nu = \frac{1}{3}$ fractional quantum Hall liquid was clearly rooted in the strong interactions between the electrons.

A year after the experiment was performed, Laughlin explained the nature of the $\nu = \frac{1}{3}$ fractional quantum Hall (FQH) liquid with a simple wave function. [28] Laughlin's wave function was a highly entangled state with very unusual properties - including quasiparticles with fractional charge $e/3$. Subsequently, it was realized that these properties are manifestations of topological order. [19] The FQH liquids are actually topological phases!

Consider, for example, the $\nu = 1/3$ Laughlin state. This state is gapped. Furthermore, if one puts this state on a torus, one finds there are 3 different ways of doing this. Thus, the $\nu = 1/3$ FQH liquid has a threefold degenerate ground state on the torus. [2] Also, the quasiparticle excitations of the Laughlin state have fractional statistics. For example, if two quasiparticles are exchanged the resulting phase is $e^{\pi i/3}$. [23] Finally, the Laughlin state can be shown to have gapless edge excitations. [24] These properties imply that the $\nu = 1/3$ FQH liquid is indeed a topological phase. The topological quantum field theory that describes its low energy physics is the $k = 3$ Abelian Chern-Simons theory (1.1).

At the moment, the fractional quantum Hall liquids are the only known examples of topological phases. In fact, even for the FQH liquids, complete experimental confirmation of their topological order (in particular, their fractional statistics) has not yet been obtained and is an active area of research. [29, 30, 31, 32] Nevertheless, there is reason to believe that topological phases - or more generally exotic phases - exist in many other strongly correlated systems. In this thesis, we focus on the possibility of these phases in frustrated quantum magnets.

One reason why frustrated magnets are considered good candidates for topological phases is that we have currently have many theoretical spin models that realize topological phases. These models range from "engineered" models that are known to realize topological phases, to more realistic models that are suspected of realizing topological phases. [33, 13, 34, 35] We also have a few candidate materials: Cs_2CuCl_4 and $\kappa - (\text{BEDT-TTF})_2\text{Cu}_2(\text{CN})_3$.

The first material, Cs_2CuCl_4 , is a spin-1/2 triangular antiferromagnet with anisotropic exchange couplings. At low temperatures, $T < 0.62\text{K}$, the spins order into an incommensurate spiral state. The low energy excitations of this state are spin waves, and inelastic neutron scattering shows spin wave peaks in good agreement with theory. However, spin wave theory cannot account for the breadth of these peaks. [36]

Such broad continua naturally occur in exotic fractionalized phases where the elementary excitations are not $S = 1$ magnons, but rather $S = 1/2$ "spinons." In such phases, neutrons can excite pairs of spinons with a continua of energies. Therefore, a number of authors have suggested that, while Cs_2CuCl_4 is not itself fractionalized, it is likely proximate to a fractionalized phase (or critical point). A number of proposals have been made for the proximate fractionalized phase. Some involve topological phases, and some involve gapless exotic phases. [37, 38, 39] However, all are at least cousins of topological phases.

The second material, $\kappa - (\text{BEDT-TTF})_2\text{Cu}_2(\text{CN})_3$, is perhaps the best candidate for an exotic phase. This organic compound is just on the insulating side of a Mott transition. It can be thought of as a complicated spin-1/2 system on the triangular lattice. Recent NMR measurements show no signs of spin order down to 32 mK, which

is particularly surprising since the antiferromagnetic exchange energy is estimated to be approximately 250 K! [40]

A natural interpretation is that the ground state of this compound is a spin liquid - a spin state that doesn't break rotational symmetry. Measurements of the nuclear spin-lattice relaxation rate suggest that the excitations are gapless - so that strictly speaking, the spin liquid is not topologically ordered. However, like all spin liquids, it is a close relative of a topological phase. One intriguing proposal is that the low energy physics of this spin liquid is described by a Fermi liquid of spin-1/2 spinons interacting via an emergent $U(1)$ gauge field. [41, 42]

At the moment, these two compounds are the best candidates for exotic physics. However, the field of frustrated quantum magnets is still very young, so this is likely only the tip of the iceberg. In any case, a general framework for analyzing exotic phases would be useful for making further progress. In this thesis, we will attempt to accomplish this goal for the case of topological phases in quantum magnets. Many of our results are also applicable to non-topological exotic phases.

1.4 Landau theory

In order to understand what kind of framework we need to build for topological phases, we need to first review the framework for ordered phases. This framework - known as Landau theory - has four basic components (see Fig. 1-3).

The first component of Landau theory is a physical characterization of ordered phases of matter. Landau theory teaches us that ordered phases of matter are characterized by long range order, symmetry breaking, and order parameters. For every ordered phase there is an associated order parameter that characterizes its structure.

The second component of Landau theory is a description of the low energy physics of ordered phases. The Landau framework teaches us that the low energy physics can be described by Ginzburg-Landau field theories. [43] These theories can be derived by considering the low energy/long wavelength fluctuations of the order parameter.

The third component of Landau theory is a physical picture for ordered phases. This physical picture is particle condensation. Consider, for example, the simplest ordered phase: the superfluid. A superfluid arises when particles (in particular bosons) become highly fluctuating and condense. Thus, our picture for a superfluid involves particle condensation. In fact, one can argue that all ordered phases can be thought about in terms of particle condensation, or more generally condensation of some particle-like object (such as spin).

The final component of Landau theory is a mathematical framework for classifying and characterizing ordered phases of matter. This framework is group theory. All ordered phases can be characterized by the symmetry group G of the Hamiltonian, and H of the ground state. Thus, there is a one-to-one correspondence between orders and pairs of symmetry groups (G, H) . Using symmetry groups, one can classify ordered phases and understand their phase transitions.

Physical characterization	Symmetry breaking Long range order
Low energy effective theory	Ginzburg-Landau field theory
Physical picture	Particle condensation
Mathematical framework	Group theory: Orders \longleftrightarrow Groups G

Figure 1-3: The four basic components of Landau's theory of ordered phases.

1.5 Statement of the problem

We would like to build an framework for topological phases analogous to Landau theory. Some progress has been made toward this goal (see Fig. 1-4). Indeed, we have a physical characterization of topological phases. Just as ordered phases are characterized by long range order and symmetry breaking, topological phases are characterized by ground state degeneracy and fractional statistics. [19]

We also have an understanding of the low energy effective theories for topological phases. Just as the low energy effective theories for ordered phases are Ginzburg-Landau field theories, the low energy effective theories for topological phases are topological quantum field theories. [25]

However, we are missing the last two components of Landau theory. We have no physical picture for topological phases. How can simple spins on a lattice give rise to fractional statistics and ground state degeneracy? What kinds of interactions favor topological phases? This is mysterious.

Equally problematic is our lack of a general mathematical framework for topological phases. How do we classify and characterize topological phases? What is the mathematical structure of topological phases? Again, we are at a loss.

There is yet another missing piece in the theory of topological phases. This piece involves the physical characterization of topological phases. Our physical characterization of topological phases is not nearly as powerful as for ordered phases. In particular, while we know how to detect order in a wave function (e.g. by looking for long range correlations) we do not know how to detect topological order. Intuitively, we expect that topological order is encoded in the entanglement in the ground state wave function. But how exactly is it encoded?

Physical characterization	Degenerate gd. states Fractional statistics
Low energy effective theory	Topological quantum field theory
Physical picture	??
Mathematical framework	??

Figure 1-4: The four basic components of the theory of topological phases. While previous research has provided a physical characterization and low energy effective theory for topological phases, we are missing a physical picture and mathematical framework.

1.6 Outline of the thesis

In this thesis, we will attempt to fill in these missing pieces in the theory of topological phases, focusing on the case of frustrated quantum magnets. We will argue that the physical picture for topological phases in quantum magnets involves a concept called “string-net condensation” while the mathematical framework for topological phases is something called “tensor category theory.” Finally, we will show that topological order - like long range order - can be detected in a ground state wave function. The method involves computing a quantity called “topological entropy” which probes nonlocal quantum entanglement.

The thesis is organized as follows. In chapter 2, we explain the basic physical picture of string-net condensation. In chapter 3, we present exactly soluble spin models that demonstrate this picture. In the process, we derive the mathematical framework of tensor category theory. In chapter 4, we introduce the concept of topological entropy and show that it can be used to detect topological order in a ground state wave function. In the final chapter we summarize our results and describe several new directions for research. Many of the mathematical details can be found in the appendix.

Most of the material in this thesis has been adapted from material published elsewhere. Chapters 2 and 3 were adapted from Ref. [44, 14], while chapter 4 is adapted from Ref. [45].

Chapter 2

String-net picture of topological phases

While we understand many of the macroscopic properties of topological phases (ground state degeneracy, fractional statistics, edge excitations, etc.), we are missing a physical picture for how these macroscopic properties emerge from microscopic degrees of freedom - such as spins on a lattice. We do not know the mechanism responsible for these phases, nor do we have intuition for what kinds of energetics favor them.

In this chapter, we will attempt to remedy this problem. We will describe a physical picture for topological phases in quantum spin systems. Our physical picture is very general. As we will see in the next chapter, it can be applied to any non-chiral topological phase (It is currently unclear whether it can be extended to the chiral case - such as in FQH effect).

The physical picture we will present is based on “string-net condensation.” String-net condensation is a phenomenon that can occur in a quantum spin system. Roughly speaking, it occurs whenever local energetic constraints cause the spins to organize into effective extended objects called “string-nets”, and these string-nets become highly fluctuating and condense. We will argue that topological phases in spin systems originate from string-net condensation in the same way that traditional ordered phases originate from particle condensation. The string-net condensation picture can also be applied to other (non-topological) exotic phases whose low energy physics is described by gauge theory.

The chapter is organized as follows. In section 2.1, we describe a special case of string-net condensation, known as “string condensation.” In section 2.2, we describe the general string-net condensation picture. In section 2.3, we motivate the string-net picture using lattice gauge theory.

2.1 String condensation

It is useful to begin with a special case of string-net condensation, known as “string condensation.” What is string condensation? String condensation is a phenomenon that can occur in a quantum spin system. There are two requirements for this phe-

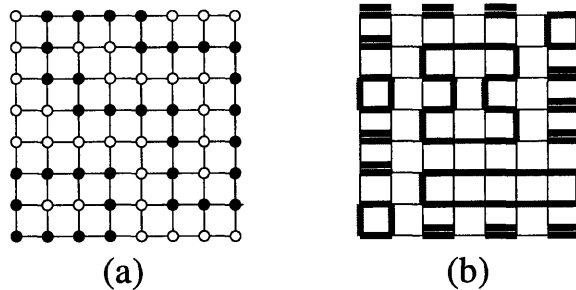


Figure 2-1: Two ways that spin models can give rise to string-like degrees of freedom. (a) Interactions can favor configurations where down spins (denoted by red dots) have an even number of down spin neighbors. (b) In the RVB picture for antiferromagnets, low energy configurations correspond to dimer coverings of the square lattice. A dimer covering (denoted by red lines) can be mapped onto a collection of closed loops by superimposing a reference dimer state (denoted by black lines).

nomenon to occur.

First, the spin interactions must favor spin configurations which are string-like. There are a number of ways that this can happen. One naive way is the following. The interactions in spin-1/2 system could potentially favor configurations where down spins have an even number of down spin neighbors (indeed, on the kagome and pyrochlore lattices, nearest neighbor spin interactions can have this effect [46, 12]). Then the low energy configurations will consist of closed loops of down spins (see Fig. 2-1a).

Another way that spin interactions could give rise to string-like configurations is the RVB picture for antiferromagnets. [47] In a spin-1/2 antiferromagnet, one could imagine that the low energy spin states are configurations where each spin forms a singlet with one of its neighbors. These valence bond states can be thought of as dimer states where every site is contained in exactly one dimer. Any dimer state can be thought of as a string state. To see this, one can simply superimpose the dimer state on a fixed reference dimer state. [48] The result is a collection of closed loops (see Fig. 2-1b). In this way, a frustrated antiferromagnet can give rise to string-like degrees of freedom.

The second requirement for string condensation is that these string-like degrees of freedom become highly fluctuating and condense. More precisely, consider any spin system whose low energy degrees of freedom are string-like. The low energy physics of such a spin system is described by some kind of effective “string model.” The Hamiltonian of an effective string model is typically a sum of potential and kinetic energy pieces:

$$H = UH_U + tH_t \quad (2.1)$$

The kinetic energy H_t gives dynamics to these low energy string states while the potential energy H_U is typically some kind of string tension. When $U \gg t$, the string tension dominates and we expect the ground state to be the vacuum with a few small strings. On the other hand, when $t \gg U$, the kinetic energy dominates, and we

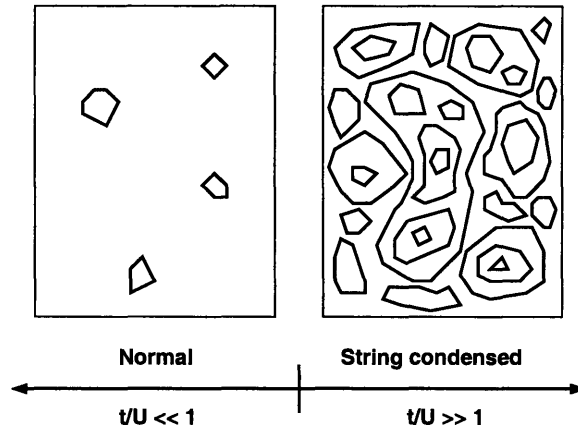


Figure 2-2: A schematic phase diagram for the generic string Hamiltonian (2.1). When t/U (the ratio of the kinetic energy to the string tension) is small the system is in the normal phase. The ground state is essentially the vacuum with a few small strings. When t/U is large the strings condense and large fluctuating string-nets fill all of space. We expect a phase transition between the two states at some t/U of order unity.

expect the ground state to consist of many fluctuating strings. Large strings with a typical length on the order of the system size fill all of space. We expect that there is a quantum phase transition between the two states at some t/U on the order of unity (see Fig. 2-2). In the latter case, we say that the spin system is “string condensed.”

What does string condensation have to do with topological phases? We will see that string condensed phases are *naturally* topological phases.

However, string condensation is not sufficiently general to account for all non-chiral topological phases. To do that, we need to consider “string-net condensation.”

2.2 String-net condensation

To explain string-net condensation in more detail, we need to define “string-nets” and “string-net models.” We begin with string-nets.¹ As the name suggests, string-nets are networks of strings. Here we will assume that they are *trivalent* networks where each node or branch point is attached to exactly 3 strings. The strings, which form the edges or links of the network, can come in different “types” and can carry a sense of orientation. Thus, string-nets can be thought of as trivalent networks or graphs with oriented, labeled edges (see Fig. 2-6). String-nets can be realized by lattice spin systems in the same way as strings (see Fig. 2-3) though they typically require more complicated interactions.

Now that we have defined string-nets, it is natural to consider the concept of “string-net models” - quantum mechanical models whose basic degrees of freedom are

¹A particular kind of string-net (where the strings are labeled by positive integers) plays an important role in the theory of loop quantum gravity, a background independent approach to quantum gravity. [49] However, in this field such string-nets are known as “spin-networks.”

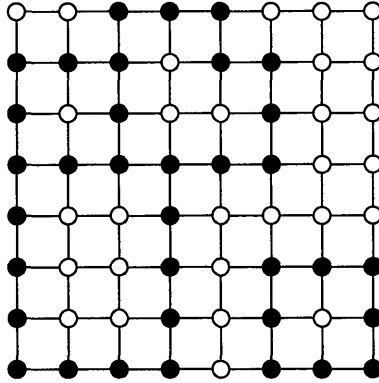


Figure 2-3: Spin models can also give rise to string-net degrees of freedom at low energies. For example, interactions can favor configurations where down spins have 2 or 3 nearest neighbors. The result is branching strings like the ones shown above. Usually, however, the formation of string-nets requires more complicated interactions than strings.

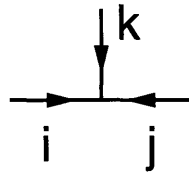


Figure 2-4: The orientation convention for the branching rules.

string-nets. String-net models are a very general: there are infinitely many different string-net models. To specify a particular string-net model one has to provide several pieces of information. First, one needs to specify the structure of the string-nets in the model. The structure of the string-nets can be characterized by the following pieces of data:

1. **String types:** The number of different string types N . (We will label the different string types with the integers $i = 1, \dots, N$).
2. **Branching rules:** The set of all triplets of string-types $\{\{i, j, k\} \dots\}$ that are allowed to meet at a point. (see Fig. 2-4).
3. **String orientations:** The dual string type i^* associated with each string type i . The duality must satisfy $(i^*)^* = i$. The type- i^* string corresponds to the type- i string with the opposite orientation. If $i = i^*$, then the string is unoriented (see Fig. 2-5).

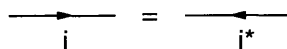


Figure 2-5: i and i^* label strings with opposite orientations.

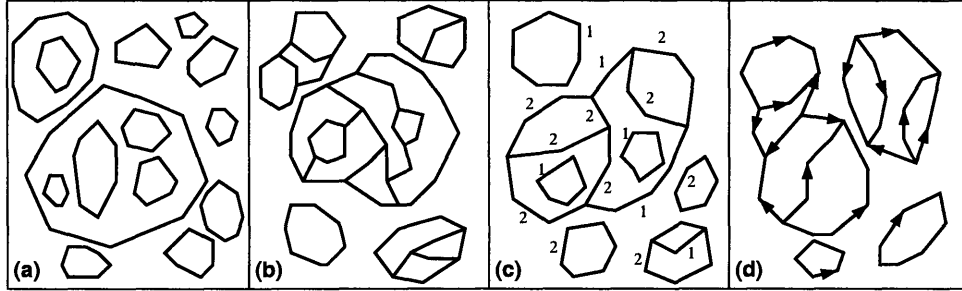


Figure 2-6: Four examples of string-net models. In (a) there is one unoriented string with no branching. In (b) there is one unoriented string type with branching. In (c) there are 2 unoriented string types with two types of branching: $\{1, 2, 2\}, \{2, 2, 2\}$. In (d) there are 2 oriented string types, $1^* = 2$, with branching $\{1, 1, 1\}, \{2, 2, 2\}$ (we've omitted labels for clarity). Each string-net model gives rise to a different kind of topological order when the string-nets condense (see section 3.3 for a detailed discussion of the four models).

This data describes the detailed structure of the string-nets (see Fig. 2-6). To complete the string-net model, one also has to specify a string-net Hamiltonian. Just like the string Hamiltonian (2.1), the typical string-net Hamiltonian is a sum of potential and kinetic energy pieces, $H = UH_U + tH_t$ where the kinetic energy H_t gives dynamics to the string-nets, while the potential energy H_U is typically some kind of string tension.

We can now explain string-net condensation in detail. Like string condensation, string-net condensation is a phenomenon that can occur in a quantum spin system. There are two requirements for string-net condensation to occur. First, the spin interactions must be such that the low energy degrees of freedom are some kind of effective string-nets. Thus the spin system must be described by some kind of effective string-net model at low energies. The second requirement is that the effective string-nets have a large kinetic energy compared with their string tension - that is $t \gg U$. If this happens, the effective string-nets will become highly fluctuating and condense. The resulting ground state will be a quantum liquid of large string-nets - a string-net condensate.

As we will see later, these string-net condensed phases are *naturally* topological phases. Moreover, depending on the structure of string-nets, and the kind of string-net condensation, many different kinds of topological orders can occur. Thus, string-net condensation provides a physical picture/mechanism for the emergence of topological phases.

But how general is this picture? Mathematical results suggest that it is very general. In $(2 + 1)$ dimensions, all “doubled” (in other words, non-chiral) topological orders can be described by string-net condensation (provided that we generalize the string-net picture as in appendix A.1). [26] Examples include all discrete gauge theories, and all doubled Chern-Simons theories. The situation for dimension $d > 2$ is less well understood. However, we know that string-net condensation quite generally describes all lattice gauge theories with or without emergent Fermi statistics (see

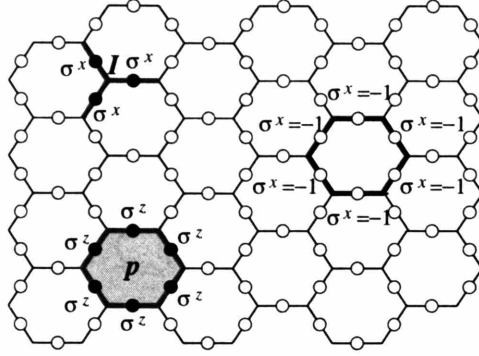


Figure 2-7: The constraint term $\prod_{\text{legs of } I} \sigma_i^x$ and magnetic term $\prod_{\text{edges of } p} \sigma_j^z$ in Z_2 lattice gauge theory. In the dual picture, we regard the links with $\sigma^x = -1$ as being occupied by a string, and the links with $\sigma^x = +1$ as being unoccupied. The constraint term then requires the strings to be closed - as shown on the right.

chapter 3).

In the following section we motivate and give intuition for the string-net picture by explaining the close connection between lattice gauge theory and string-net condensation.

2.3 Gauge theories and string-nets

In this section, we review the string-net picture of lattice gauge theory. [50, 51, 46] We point out that, quite generally, the ground states of deconfined lattice gauge theories can be understood as string-net condensates. Since deconfined lattice gauge theories form a large class of topological phases, this result provides intuition for (and motivates) the string-net picture of topological phases.

We begin with the simplest lattice gauge theory - Z_2 lattice gauge theory [52]. For simplicity we will restrict our discussion to trivalent lattices such as the honeycomb lattice (see Fig. 2-7). The Hamiltonian is

$$H_{Z_2} = -U \sum_i \sigma_i^x - t \sum_p \prod_{\text{edges of } p} \sigma_j^z \quad (2.2)$$

where $\sigma^{x,y,z}$ are the Pauli matrices, and I , i , p label the sites, links, and plaquettes of the lattice. The Hilbert space is formed by states satisfying

$$\prod_{\text{legs of } I} \sigma_i^x |\Phi\rangle = |\Phi\rangle, \quad (2.3)$$

for every site I .

It is well known that Z_2 lattice gauge theory is dual to the Ising model in $(2 + 1)$ dimensions [53]. What is less well known is that there is a more general dual description of Z_2 gauge theory that exists in any number of dimensions [54]. To obtain this dual picture, we view links with $\sigma^x = -1$ as being occupied by a string

and links with $\sigma^x = +1$ as being unoccupied. The constraint (2.3) then implies that only closed strings are allowed in the Hilbert space (Fig. 2-7).

In this way, Z_2 gauge theory can be reformulated as a closed string theory, and the Hamiltonian can be viewed as a closed string Hamiltonian. The electric and magnetic energy terms have a simple interpretation in this dual picture: the “electric energy” $-U \sum_i \sigma_i^x$ is a string tension while the “magnetic energy” $-t \sum_p \prod_{\text{edges of } p} \sigma_j^z$ is a string kinetic energy. The physical picture for the confining and deconfined phases is also clear. The confining phase corresponds to a large electric energy and hence a large string tension, $U \gg t$. The ground state is therefore the vacuum configuration with a few small strings. The deconfined phase corresponds to a large magnetic energy and hence a large kinetic energy, $t \gg U$. The ground state is thus a superposition of many large string configurations. In other words, the ground state of deconfined Z_2 gauge theory is a quantum liquid of large strings - a string condensate. (Fig. 2-8a).

A similar, but more complicated, picture exists for other deconfined gauge theories. The next layer of complexity is revealed when we consider other Abelian theories, such as $U(1)$ gauge theory. As in the case of Z_2 , $U(1)$ lattice gauge theory can be reformulated as a theory of electric flux lines. However, unlike Z_2 , there is more than one type of flux line. The electric flux on a link can take any integral value in $U(1)$ lattice gauge theory. Therefore, the electric flux lines need to be labeled with integers to indicate the amount of flux carried by the line. In addition, the flux lines need to be oriented to indicate the direction of the flux. The final point is that the flux lines don’t necessarily form closed loops. It is possible for three flux lines E_1, E_2, E_3 to meet at a point, as long as Gauss’ law is obeyed: $E_1 + E_2 + E_3 = 0$. Thus, the dual formulation of $U(1)$ gauge theory involves not strings, but more general objects: networks of strings (or string-nets). The strings in a string-net are labeled, oriented, and obey branching rules, given by Gauss’ law (Fig. 2-8b).

This string-net picture exists for general gauge theories. In the general case, the strings (electric flux lines) are labeled by representations of the gauge group. The branching rules (Gauss’ law) require that if three strings E_1, E_2, E_3 meet at a point, then the product of the representations $E_1 \otimes E_2 \otimes E_3$ must contain the trivial representation. (For example, in the case of $SU(2)$, the strings are labeled by half-integers $E = 1/2, 1, 3/2, \dots$, and the branching rules are given by the triangle inequality: $\{E_1, E_2, E_3\}$ are allowed to meet at a point if and only if $E_1 \leq E_2 + E_3$, $E_2 \leq E_3 + E_1$, $E_3 \leq E_1 + E_2$ and $E_1 + E_2 + E_3$ is an integer (Fig. 2-8c)) [50]. These string-nets provide a general dual formulation of gauge theory. As in the case of Z_2 , the deconfined phase of the gauge theory always corresponds to highly fluctuating string-nets - a string-net condensate.

In this section, we have shown that string-net condensation can give rise to general deconfined gauge theories. However, we would like to emphasize that string-net condensation is more than just a reformulation of gauge theory. Instead, it should be viewed as a generalization of gauge theory. As we will see in the next chapter, string-net condensation can give rise to other topological orders beyond discrete gauge theory - such as doubled Chern-Simons theories. In fact, it’s possible that some string-net condensates do not even have a field theoretic formulation!

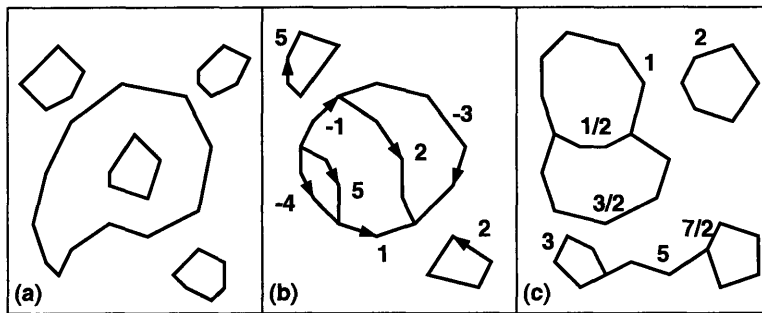


Figure 2-8: Typical string-net configurations in the dual formulation of (a) Z_2 , (b) $U(1)$, and (c) $SU(2)$ gauge theory. In the case of (a) Z_2 gauge theory, the string-net configurations consist of closed (non-intersecting) loops. In (b) $U(1)$ gauge theory, the string-nets are oriented graphs with edges labeled by integers. The string-nets obey the branching rules $E_1 + E_2 + E_3 = 0$ for any three edges meeting at a point. In the case of (c) $SU(2)$ gauge theory, the string-nets consist of (unoriented) graphs with edges labeled by half-integers $1/2, 1, 3/2, \dots$. The branching rules are given by the triangle inequality: $\{E_1, E_2, E_3\}$ are allowed to meet at a point if and only if $E_1 \leq E_2 + E_3$, $E_2 \leq E_3 + E_1$, $E_3 \leq E_1 + E_2$, and $E_1 + E_2 + E_3$ is an integer.

Chapter 3

Theory of string-net condensation

In the previous chapter, we described a physical picture for non-chiral topological phases based on the concept of string-net condensation. In this chapter, we make this picture more quantitative. In the process, we find a solution of the second major problem in the theory of topological phases - the problem of finding a mathematical framework for classifying and characterizing topological phases.

This mathematical framework is tensor category theory. We find that there is a one-to-one correspondence between non-chiral topological orders and mathematical objects known as “tensor categories.” [55] Tensor categories can be thought of as collections of numerical constants (d_i, F_{lmn}^{ijk}) which satisfy certain algebraic equations (3.5). Each tensor category determines the universal properties of the associated topological order (quasiparticle statistics, ground state degeneracy, etc.) just as symmetry groups do in Landau theory.

Our analysis is based on a constructive approach. For each two dimensional string-net condensate, we construct a string-net wave function that captures its universal properties. We show that each wave function is associated with a tensor category. This gives a classification of string-net condensates and hence non-chiral topological phases.

To complete the picture we construct exactly soluble $2D$ spin models realizing each string-net condensate. These models explicitly demonstrate the string-net condensation picture. They realize all discrete gauge theories and all doubled Chern-Simons theories (in $(2 + 1)$ dimensions). One of the Hamiltonians - a spin-1/2 model on the honeycomb lattice - is a simple theoretical realization of a universal fault tolerant quantum computer. [56]

The models can also be generalized to higher dimensions. The higher dimensional models yield an interesting result: they demonstrate that $(3 + 1)D$ string-net condensation naturally gives rise to both emerging gauge bosons and emerging fermions. Thus, string-net condensation provides a mechanism for unifying gauge bosons and fermions in $(3 + 1)$ and higher dimensions.

We feel that our constructive approach is one of the most important features of our analysis. Indeed, in the mathematical community it is well known that topological field theory, tensor category theory and knot theory are all intimately related [26, 57, 58]. Thus it is not surprising that topological phases are closely connected to tensor

categories and string-nets. The contribution of this work is the demonstration that these elegant mathematical relations have a concrete realization in condensed matter systems.

The chapter is organized as follows. In section 3.1, we consider the case of $(2 + 1)$ dimensions. In 3.1.1 and 3.1.2, we construct string-net wave functions and Hamiltonians for each $(2 + 1)D$ string-net condensed phase. Then, in 3.1.3, we use this mathematical framework to calculate the quasiparticle statistics in each phase. In section 3.2, we discuss the generalization to 3 and higher dimensions. In the last section, we present several examples of string-net condensed states - including a spin-1/2 model theoretically capable of fault tolerant quantum computation. The main mathematical calculations can be found in appendix A.

3.1 String-net condensation in $(2 + 1)$ dimensions

3.1.1 Fixed-point wave functions

In this section, we try to understand the universal features of each string-net condensed phase in $(2 + 1)$ dimensions. Our approach, inspired by Ref. [50, 57, 58, 59, 60, 61], is based on the string-net wave function. We construct a special “fixed-point” wave function for each string-net condensed phase. We believe that these “fixed-point” wave functions capture the universal properties of the corresponding phases. We show that each wave function is associated with a collection of numerical constants (d_i, F_{lmn}^{ijk}) that satisfy certain algebraic equations (3.5). In this way, we derive a one-to-one correspondence between string-net condensates (and hence non-chiral topological phases) and tensor categories (d_i, F_{lmn}^{ijk}) . We would like to mention that a related result on the classification of $(2 + 1)D$ topological quantum field theories was obtained independently in the mathematical community.[26]

To begin, imagine we have a string-net model with some number of string types N , some branching rules $\{\{i, j, k\}\}$, some orientations, and some string-net Hamiltonian H . As we vary the parameters in H , we expect that the model can realize both normal phases and string-net condensed phases.

Let us try to visualize the ground state wave function in one of the string-net condensed phase. We expect that a string-net condensed state is a superposition of many different string-net configurations. Each string-net configuration has a size typically on the same order as the system size. The large size of the string-nets implies that a string-net condensed wave function has a non-trivial long distance structure. It is this long distance structure that distinguishes the condensed state from the “normal” state.

In general, we expect that the universal features of a string-net condensed phase are contained in the long distance character of the wave functions. Imagine comparing two different string-net condensed states that belong to the same quantum phase. The two states will have different wave functions. However, by the standard RG reasoning, we expect that the two wave functions will look the same at long distances. That is, the two wave functions will only differ in short distance details - like those shown in

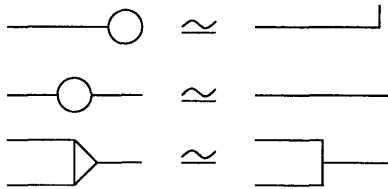


Figure 3-1: Three pairs of string-net configurations that differ only in their short distance structure. We expect string-net wave functions in the same quantum phase to only differ by these short distance details.

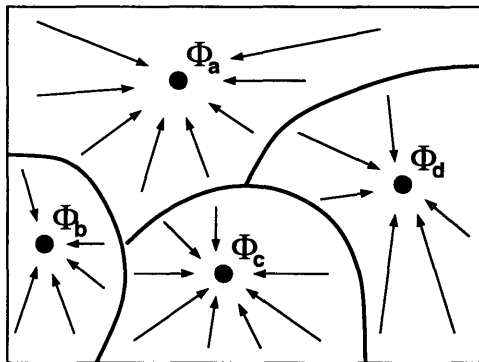


Figure 3-2: A schematic RG flow diagram for a string-net model with 4 string-net condensed phases a , b , c , and d . All the states in each phase flow to fixed-points in the long distance limit. The corresponding fixed-point wave functions Φ_a , Φ_b , Φ_c , and Φ_d capture the universal long distance features of the associated quantum phases. Our ansatz is that the fixed-point wave functions Φ are described by local constraints of the form (3.1-3.4).

Fig. 3-1.

Continuing with this line of thought, we imagine performing an RG analysis on ground state functions. All the states in a string-net condensed phase should flow to some special “fixed-point” state. We expect that the wave function of this state captures the universal long distance features of the whole quantum phase. (See Fig. 3-2). In the following, we will construct these special fixed-point wave functions.

How can we find the fixed-point wave functions Φ ? Our approach is based on the following observation: for each fixed-point wave function Φ , there is some series of local constraint equations on string-net wave functions for which it is the unique solution (an argument for this is given in appendix A.2). This means we only need to find appropriate constraint equations. Each set of constraint equations will completely specify a fixed-point wave function Φ , albeit implicitly.

How can we find the local constraint equations? We will use a heuristic approach. We will simply guess the form of the local constraints. Then, in the next section, we will verify our guess by constructing fixed-point Hamiltonians whose ground state wave functions satisfy exactly these constraints.

Consider a string-net model with some number of string types N , some branching

not complete: another constraint is needed to specify the ground state wave function uniquely. The last rule (3.4) is the simplest local constraint with this property. An alternative motivation for this rule is the fusion algebra in conformal field theory.[62]

By applying the local rules in (3.1-3.4) multiple times, one can compute the amplitude of any string-net configuration in terms of the amplitude of the no-string configuration. (For example, we can compute the amplitude

$$\begin{aligned} \Phi \left(\begin{array}{c} \textcircled{i} \\ \textcircled{m} \\ \textcircled{j} \end{array} \right) &= \sum_{n=0}^N F_{j^*i^*n}^{ijm} \Phi \left(\begin{array}{c} \textcircled{i} \\ | \\ \textcircled{n} \\ | \\ \textcircled{j} \end{array} \right) = F_{j^*i^*0}^{ijm} \Phi \left(\begin{array}{c} \textcircled{i} \\ \textcircled{j} \end{array} \right) \\ &= F_{j^*i^*0}^{ijm} d_i d_j \end{aligned}$$

by applying the fourth rule, the third rule, and the second rule in sequence). Thus the string-net wave function Φ is completely determined by (3.1-3.4). Equivalently, Φ is determined by the numerical constants (d_i, F_{kln}^{ijm}) .

However, an arbitrary choice of (d_i, F_{kln}^{ijm}) does not lead to a well defined Φ . This is because two string-net configurations may be related by more than one sequence of local rules. We need to choose the (d_i, F_{kln}^{ijm}) carefully so that different sequences of local rules produce the same results. That is, we need to choose (d_i, F_{kln}^{ijm}) so that the rules are *self-consistent*. This mathematical problem is solved in appendix A.3. We find that the only (d_i, F_{kln}^{ijm}) that give rise to self-consistent rules and a well-defined wave function Φ are those that satisfy

$$\begin{aligned} F_{j^*i^*0}^{ijk} &= \frac{v_k}{v_i v_j} \delta_{ijk} \\ F_{kln}^{ijm} = F_{jin}^{lkm^*} &= F_{lkn^*}^{jim} = F_{k^*nl}^{imj} \frac{v_m v_n}{v_j v_l} \\ \sum_{n=0}^N F_{kp^*n}^{mlq} F_{mns^*}^{jip} F_{lkr^*}^{js^*n} &= F_{q^*kr^*}^{jip} F_{m ls^*}^{riq^*} \end{aligned} \quad (3.5)$$

where $v_i = v_{i^*} = \sqrt{d_i}$ (and $v_0 = 1$). Here, we have introduced a new object δ_{ijk} defined by the branching rules:

$$\delta_{ijk} = \begin{cases} 1, & \text{if } \{i, j, k\} \text{ is allowed,} \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

There is a one-to-one correspondence between solutions of (3.5) and $(2+1)D$ string-net condensed phases: each solution corresponds to a string-net wave function Φ which in turn corresponds to a string-net condensed phase. These solutions have a rich mathematical structure, and can be viewed in a much more elegant and abstract way. Each solution is an example of a mathematical object known as a “tensor category.” [55]

According to our analysis, tensor categories give a complete classification of $(2+1)D$ string-net condensed phases (or equivalently non-chiral topological phases [26]).

We will show later that tensor categories also provide a convenient framework for deriving the physical properties of quasiparticles. Thus, tensor category theory is the fundamental mathematical framework for string-net condensed phases, just as group theory is for symmetry breaking phases.

It is highly non-trivial to find solutions of (3.5). However, it turns out each group G provides a solution. The solution is obtained by (a) letting the string-type index i run over the irreducible representations of the group, (b) letting the numbers d_i be the dimensions of the representations and (c) letting the 6 index object F_{klm}^{ijn} be the 6j symbol of the group. The low energy effective theory of the corresponding string-net condensed state turns out to be a deconfined gauge theory with gauge group G . Another class of solutions can be obtained from 6j symbols of quantum groups. In these cases, the low energy effective theories of the corresponding string-net condensed states are doubled Chern-Simons gauge theories.

These two classes of solutions are not necessarily exhaustive: Eq. (3.5) may have solutions other than gauge theories or Chern-Simons theories. Nevertheless, it is clear that gauge bosons and gauge groups emerge from string-net condensation in a very natural way.

In fact, string-net condensation provides a new perspective on gauge theory. Traditionally, we think of gauge theories geometrically. The gauge field A_μ is analogous to an affine connection, and the field strength $F_{\mu\nu}$ is essentially a curvature tensor. From this point of view, gauge theory describes the dynamics of certain geometric objects (e.g. fiber bundles). The gauge group determines the structure of these objects and is introduced by hand as part of the basic definition of the theory. In contrast, according to the string-net condensation picture, the geometrical character of gauge theory is not fundamental. Gauge theories are fundamentally theories of extended objects. The gauge group and the geometrical gauge structure emerge dynamically at low energies and long distances. A string-net system “chooses” a particular gauge group, depending on the coupling constants in the underlying Hamiltonian: these parameters determine a string-net condensed phase which in turn determines a solution to (3.5). The nature of this solution determines the gauge group.

One advantage of this alternative picture is that it unifies two seemingly unrelated phenomena: gauge interactions and Fermi statistics. Indeed, as we will show in section 3.1, string-net condensation naturally gives rise to both gauge interactions and Fermi statistics (or fractional statistics in $(2+1)D$). In addition, these structures always appear together. [18]

3.1.2 Fixed-point Hamiltonians

In this section, we construct exactly soluble lattice spin Hamiltonians that explicitly demonstrate the string-net condensation picture. These Hamiltonians provide a lattice realization of all $(2+1)D$ string-net condensates and therefore all non-chiral $(2+1)D$ topological phases (provided that we generalize these models as discussed in appendix A.1). In addition, they put the ansatz in the previous section on firm ground: their ground states are precisely the fixed point wave functions Φ . We would like to mention that a related result was obtained independently by researchers in the

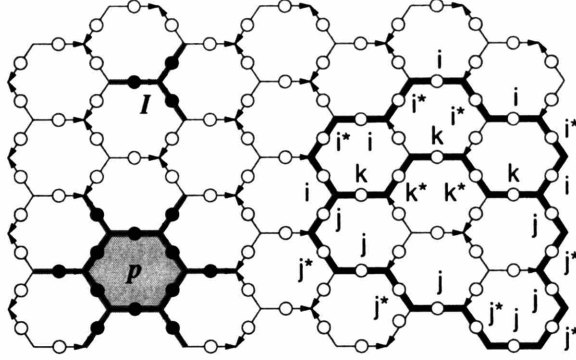


Figure 3-3: A picture of the lattice spin model (3.7). The electric charge operator Q_I acts on the three spins adjacent to the vertex I , while the magnetic energy operator B_p acts on the 12 spins adjacent to the hexagonal plaquette p . The term Q_I constrains the string-nets to obey the branching rules, while B_p provides dynamics. A typical state satisfying the low-energy constraints is shown on the right. The empty links have spins in the $i = 0$ state.

quantum computation community. [63]

For every (d_i, F_{klm}^{ijm}) satisfying the self-consistency conditions (3.5) and the unitarity condition (3.11), we can construct an exactly soluble spin Hamiltonian. The model is a spin system on a (2D) honeycomb lattice, with a spin located on each link of the lattice. However, the spins are not usual spin-1/2 spins. Each spin can be in $N + 1$ different states labeled by $i = 0, 1, \dots, N$.

It is useful to think of the spin states using the string-net language. To do that, we assign each link an arbitrary orientation. When a spin is in state i , we think of the link as being occupied by a type- i string oriented in the appropriate direction. If a spin is in state $i = 0$, then we think of the link as empty. In this way spin states correspond to string-net states.

The exactly soluble Hamiltonian for our model is given by

$$H = - \sum_I Q_I - \sum_p B_p, \quad B_p = \sum_{s=0}^N a_s B_p^s \quad (3.7)$$

where the sums run over vertices I and plaquettes p of the honeycomb lattice. The coefficients a_s satisfy $a_{s^*} = a_s^*$ but are otherwise arbitrary.

Let us explain the terms in (3.7). The first term, Q_I acts on the 3 spins adjacent to the site I :

$$Q_I \left| \begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right\rangle = \delta_{ijk} \left| \begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right\rangle \quad (3.8)$$

where δ_{ijk} is the branching rule symbol (3.6). If we think of the spin states in terms of string-nets, this term constrains the strings to obey the branching rules described by δ_{ijk} . With this constraint the low energy Hilbert space is simply the set of all allowed string-net configurations on a honeycomb lattice. (See Fig. 3-3). We think of Q_I as an electric charge operator. It measures the “electric charge” at site I , and

favors states with no charge.

We think of the second term $B_{\mathbf{p}}$ as a magnetic flux operator. It measures the “magnetic flux” through the plaquette \mathbf{p} (or more precisely, the cosine of the magnetic flux) and favors states with no flux. This term provides dynamics for the string-net configurations.

The magnetic flux operator $B_{\mathbf{p}}$ is a linear combination of $(N + 1)$ terms $B_{\mathbf{p}}^s$, $s = 0, 1, \dots, N$. Each $B_{\mathbf{p}}^s$ is an operator that acts on the 12 links that are adjacent to the hexagon \mathbf{p} . (See Fig. 3-3). Thus, the $B_{\mathbf{p}}^s$ are essentially $(N + 1)^{12} \times (N + 1)^{12}$ matrices. However, the action of $B_{\mathbf{p}}^s$ does not change the spin states on the 6 outer links of \mathbf{p} . Therefore the $B_{\mathbf{p}}^s$ can be block diagonalized into $(N + 1)^6$ blocks, each of dimension $(N + 1)^6 \times (N + 1)^6$. Let $B_{\mathbf{p},ghijkl}^{s,g'h'i'j'k'l'}(abcdef)$, with $a, b, c, \dots = 0, 1, \dots, N$, denote the matrix elements of these $(N + 1)^6$ matrices:

$$\begin{aligned}
& B_{\mathbf{p}}^s \left| \begin{array}{c} b \text{---} h \text{---} c \\ \diagup \quad \diagdown \\ a \text{---} g \text{---} i \\ \diagdown \quad \diagup \\ f \text{---} k \text{---} e \\ \diagup \quad \diagdown \\ \quad \quad \quad d \end{array} \right\rangle \\
&= \sum_{m, \dots, r} B_{\mathbf{p},ghijkl}^{s,g'h'i'j'k'l'}(abcdef) \left| \begin{array}{c} b \text{---} h \text{---} c \\ \diagup \quad \diagdown \\ a \text{---} g \text{---} i \\ \diagdown \quad \diagup \\ f \text{---} k \text{---} e \\ \diagup \quad \diagdown \\ \quad \quad \quad d \end{array} \right\rangle \quad (3.9)
\end{aligned}$$

Then the operators $B_{\mathbf{p}}^s$ are defined by

$$\begin{aligned}
& B_{\mathbf{p},ghijkl}^{s,g'h'i'j'k'l'}(abcdef) \\
&= F_{s^*g'l'^*}^{al^*g} F_{s^*h'g'^*}^{bg^*h} F_{s^*i'h'l'^*}^{ch^*i} F_{s^*j'i'^*}^{di^*j} F_{s^*k'j'l'^*}^{ej^*k} F_{s^*l'k'^*}^{fk^*l} \quad (3.10)
\end{aligned}$$

(See appendix A.4 for a graphical representation of $B_{\mathbf{p}}^s$). One can check that the Hamiltonian (3.7) is Hermitian if F satisfies

$$F_{k^*l^*n^*}^{i^*j^*m^*} = (F_{klm}^{ijn})^* \quad (3.11)$$

in addition to (3.5). Our model is only applicable to topological phases satisfying this additional constraint. We believe that this is true much more generally: only topological phases satisfying the unitarity condition (3.11) are physically realizable.

The Hamiltonian (3.7) has a number of interesting properties, provided that (d_i, F_{klm}^{ijn}) satisfy the self-consistency conditions (3.5). It turns out that:

1. The $B_{\mathbf{p}}^s$ and Q_I 's all commute with each other. Thus the Hamiltonian (3.7) is exactly soluble.
2. Depending on the choice of the coefficients a_s , the system can be in $N + 1$ different quantum phases.
3. The choice $a_s = \frac{d_s}{\sum_{i=0}^N d_i^2}$ corresponds to a topological phase with a smooth continuum limit. The ground state wave function for this parameter choice is

topologically invariant, and obeys the local rules (3.1-3.4). It is precisely the wave function Φ , defined on a honeycomb lattice. Furthermore, $Q_{\mathbf{I}}$, $B_{\mathbf{p}}$ are projection operators in this case. Thus, the ground state satisfies $Q_{\mathbf{I}} = B_{\mathbf{p}} = 1$ for all \mathbf{I} , \mathbf{p} , while the excited states violate these constraints.

Thus, the Hamiltonian (3.7) with the above choice of a_s provides an exactly soluble realization of the doubled topological phase described by (d_i, F_{klm}^{ijm}) . We can obtain some intuition for this by considering the case where d_i are the dimensions of the irreducible representations of some group G and F_{klm}^{ijm} is the $6j$ symbol. In this case, it turns out that $Q_{\mathbf{I}}$ and $B_{\mathbf{p}}$ are precisely the electric charge and magnetic flux operators in the standard lattice gauge theory with group G . Thus, (3.7) is the usual Hamiltonian of lattice gauge theory, except with no electric field term. This is exactly the Kitaev model - the zero coupling, exactly soluble Hamiltonian of lattice gauge theory. [52, 13] In this way, our construction can be viewed as a generalization of lattice gauge theory.

In this paper, we will focus on the smooth topological phase corresponding to the parameter choice $a_s = \frac{d_s}{\sum_{i=0}^N d_i^2}$ (see appendix A.4). However, we would like to mention that the other N quantum phases also have non-trivial topological (or quantum) order. However, in these phases, the ground state wave function does not have a smooth continuum limit. Thus, these are new topological phases beyond those described by continuum theories.

3.1.3 Quasiparticle excitations

In this section, we find the quasiparticle excitations of the string-net Hamiltonian (3.7), and calculate their statistics (e.g. the twists θ_α and the S matrix $s_{\alpha\beta}$). We will only consider the topological phase with smooth continuum limit. That is, we will choose $a_s = \frac{d_s}{\sum_{i=0}^N d_i^2}$ in our lattice model.

Recall that the ground state satisfies $Q_{\mathbf{I}} = B_{\mathbf{p}} = 1$ for all vertices \mathbf{I} , and all plaquettes \mathbf{p} . The quasiparticle excitations correspond to violations of these constraints for some local collection of vertices and plaquettes. We are interested in the topological properties (e.g. statistics) of these excitations.

We will focus on topologically nontrivial quasiparticles - that is, particles with nontrivial statistics or mutual statistics. By the analysis in Ref. [18], we know that these types of particles are always created in pairs, and that their pair creation operator has a string-like structure, with the newly created particles appearing at the ends. (See Fig. 3-4). The position of this string operator is unobservable in the string-net condensed state - only the endpoints of the string are observable. Thus the two ends of the string behave like independent particles.

If the two endpoints of the string coincide so that the string forms a loop, then the associated closed string operator commutes with the Hamiltonian. This follows from the fact that the string is truly unobservable; the action of an open string operator on the ground state depends only on its endpoints. Thus, *each topologically nontrivial quasiparticle is associated with a (closed) string operator that commutes*

with the Hamiltonian. To find the quasiparticles, we need to find these closed string operators. [13]

An important class of string operators are what we will call “simple” string operators. The defining property of simple string operators is their action on the vacuum state. If we apply a type- s simple string operator $W(P)$ to the vacuum state, it creates a type- s string along the path of the string, P . We already have some examples of these operators, namely the magnetic flux operators $B_{\mathbf{p}}^s$. When $B_{\mathbf{p}}^s$ acts on the vacuum configuration $|0\rangle$, it creates a type- s string along the boundary of the plaquette \mathbf{p} . Thus, we can think of $B_{\mathbf{p}}^s$ as a short type- s simple string operator, $W(\partial\mathbf{p})$.

We would like to construct simple string operators $W(P)$ for arbitrary paths $P = \mathbf{I}_1, \dots, \mathbf{I}_N$ on the honeycomb lattice. Using the definition of $B_{\mathbf{p}}^s$ as a guide, we make the following ansatz. The string operator $W(P)$ only changes the spin states along the path P . The matrix element of a general type- s simple string operator $W(P)$ between an initial spin state i_1, \dots, i_N and final spin state i'_1, \dots, i'_N is of the form

$$W_{i_1 i_2 \dots i_N}^{i'_1 i'_2 \dots i'_N}(e_1 e_2 \dots e_N) = \left(\prod_{k=1}^N F_{s_k i'_{k-1} i_k^*}^{e_k i_k^* i_{k-1}} \right) \left(\prod_{k=1}^N \omega_k \right) \quad (3.12)$$

where e_1, \dots, e_N are the spin states of the N “legs” of P (see Fig. 3-4) and

$$\omega_k = \begin{cases} \frac{v_{i_k} v_s}{v_{i'_k}} \omega_{i_k}^{i'_k}, & \text{if } P \text{ turns right, left at } \mathbf{I}_k, \mathbf{I}_{k+1} \\ \frac{v_{i_k} v_s}{v_{i'_k}} \bar{\omega}_{i_k}^{i'_k}, & \text{if } P \text{ turns left, right at } \mathbf{I}_k, \mathbf{I}_{k+1} \\ 1, & \text{otherwise} \end{cases} \quad (3.13)$$

Here, $\omega_j^i, \bar{\omega}_j^i$ are two (complex) two index objects that characterize the string W .

Note the similarity to the definition of $B_{\mathbf{p}}^s$. The major difference is the additional factor $\prod_{k=1}^N \omega_k$. We conjecture that $|\omega_i^j \frac{v_i v_s}{v_j}| = 1$ for a type- s string, so $\prod_{k=1}^N \omega_k$ is simply a phase factor that depends on the initial and final spin states $i_1, i_2, \dots, i_N, i'_1, i'_2, \dots, i'_N$. This phase vanishes for paths P that make only left or only right turns, such as plaquette boundaries $\partial\mathbf{p}$. In that case, the definition of $W(P)$ coincides with $B_{\mathbf{p}}^s$.

A straightforward calculation shows that the operator $W(P)$ defined above commutes with the Hamiltonian (3.7) if $\omega_j^i, \bar{\omega}_j^i$ satisfy

$$\begin{aligned} \bar{\omega}_j^m F_{kjm^*}^{sl^*i} \omega_i^l \frac{v_j v_s}{v_m} &= \sum_{n=0}^N F_{s^*nl^*}^{ji^*k} \omega_k^n F_{ksm^*}^{jll^*n} \\ \bar{\omega}_i^j &= \sum_{k=0}^N \omega_i^k F_{i^*sj^*}^{is^*k} \end{aligned} \quad (3.14)$$

The solutions to these equations give all the type- s simple string operators.

For example, consider the case of Abelian gauge theory. In this case, the solutions to (3.14) can be divided into three classes. The first class is given by $s \neq 0$,

$\omega_i^j \frac{v_i v_s}{v_j} = \bar{\omega}_i^j \frac{v_i v_s}{v_j} = 1$. These string operators create electric flux lines and the associated quasiparticles are electric charges. In more traditional nomenclature, these are known as (Wegner-)Wilson loop operators [52, 64]. The second class of solutions is given by $s = 0$, and $\omega_i^j \frac{v_i v_s}{v_j} = (\bar{\omega}_i^j \frac{v_i v_s}{v_j})^* \neq 1$. These string operators create magnetic flux lines and the associated quasiparticles are magnetic fluxes. The third class has $s \neq 0$ and $\omega_i^j \frac{v_i v_s}{v_j} = (\bar{\omega}_i^j \frac{v_i v_s}{v_j})^* \neq 1$. These strings create both electric and magnetic flux and the associated quasiparticles are electric charge/magnetic flux bound states. This accounts for all the quasiparticles in $(2+1)D$ Abelian gauge theory. Therefore, all the string operators are simple in this case.

However, this is not true for non-Abelian gauge theory or other $(2+1)D$ topological phases. To compute the quasiparticle spectrum of these more general theories, we need to generalize the expression (3.12) for $W(P)$ to include string operators that are not simple.

One way to guess the more general expression for $W(P)$ is to consider products of simple string operators. Clearly, if $W_1(P)$ and $W_2(P)$ commute with the Hamiltonian, then $W(P) = W_1(P) \cdot W_2(P)$ also commutes with the Hamiltonian. Thus, we can obtain other string operators by taking products of simple string operators. In general, the resulting operators are not simple. If W_1 and W_2 are type- s_1 and type- s_2 simple string operators, then the action of the product string on the vacuum state is:

$$W(P)|0\rangle = W_1(P)W_2(P)|0\rangle = W_1(P)|s_2\rangle = \sum_s \delta_{ss_1 s_2} |s\rangle$$

where $|s\rangle$ denotes the string state with a type- s string along the path P and the vacuum everywhere else. If we take products of more than two simple string operators then the action of the product string on the vacuum is of the form $W(P)|0\rangle = \sum_s n_s |s\rangle$ where n_s are some non-negative integers.

We now generalize the expression for $W(P)$ so that it includes arbitrary products of simple strings. Let W be a product of simple string operators, and let n_s be the non-negative integers characterizing the action of W on the vacuum: $W(P)|0\rangle = \sum_s n_s |s\rangle$. Then, one can show that the matrix elements of $W(P)$ are always of the form

$$W_{i_1 i_2 \dots i_N}^{i'_1 i'_2 \dots i'_N}(e_1 e_2 \dots e_N) = \sum_{\{s_k\}} \left(\prod_{k=1}^N F_{s_k^* i'_k - 1 i'_k}^{e_k i_k^* i_{k-1}} \right) \text{Tr} \left(\prod_{k=1}^N \Omega_k^{s_k} \right) \quad (3.15)$$

where

$$\Omega_k^{s_k} = \begin{cases} \frac{v_{i_k} v_{s_k}}{v_{i'_k}} \Omega_{s_k s_{k+1} i_k}^{i'_k}, & \text{if } P \text{ turns right, left at } \mathbf{I}_k, \mathbf{I}_{k+1} \\ \frac{v_{i_k} v_{s_k}}{v_{i'_k}} \bar{\Omega}_{s_k s_{k+1} i_k}^{i'_k}, & \text{if } P \text{ turns left, right at } \mathbf{I}_k, \mathbf{I}_{k+1} \\ \delta_{s_k s_{k+1}} \cdot \text{Id}, & \text{otherwise} \end{cases} \quad (3.16)$$

and $\Omega_{stj}^i, \bar{\Omega}_{stj}^i$ are two 4 index objects that characterize the string operator W . For any quadruple of string types i, j, s, t , $(\Omega_{stj}^i, \bar{\Omega}_{stj}^i)$ are (complex) rectangular matrices of dimension $n_s \times n_t$. Note that type- s_0 simple string operators correspond to the special case where $n_s = \delta_{s_0 s}$. In this case, the matrices $\Omega_{stj}^i, \bar{\Omega}_{stj}^i$ reduce to complex

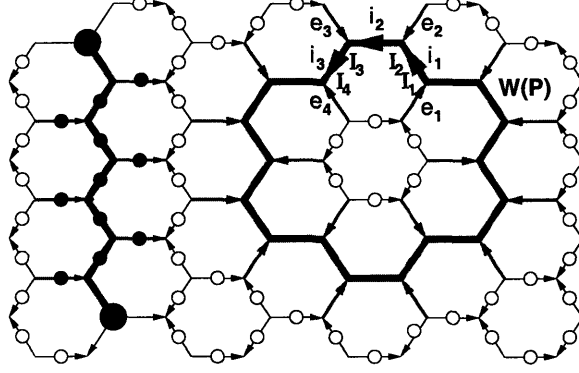


Figure 3-4: Open and closed string operators for the lattice spin model (3.7). Open string operators create quasiparticles at the two ends, as shown on the left. Closed string operators, as shown on the right, commute with the Hamiltonian. The closed string operator $W(P)$ only acts non-trivially on the spins along the path $P = \mathbf{I}_1, \mathbf{I}_2, \dots$ (thick line), but its action depends on the spin states on the legs (thin lines). The matrix element between an initial state i_1, i_2, \dots and a final state i'_1, i'_2, \dots is $W_{i_1 i_2 \dots}^{i'_1 i'_2 \dots}(e_1 e_2 \dots) = (F_{s^* i'_1 i'_2}^{e_2 i_2 i_1} F_{s^* i'_2 i'_3}^{e_3 i_3 i_2} \dots) \cdot (\frac{v_{i_1} v_{s_1}}{v_{i'_1}} \omega_{i_1}^{i'_1} \frac{v_{i_3} v_{s_3}}{v_{i'_3}} \bar{\omega}_{i_3}^{i'_3} \dots)$ for a type- s simple string and $W_{i_1 i_2 \dots}^{i'_1 i'_2 \dots}(e_1 e_2 \dots) = \sum_{\{s_k\}} (F_{s^* i'_1 i'_2}^{e_2 i_2 i_1} F_{s^* i'_2 i'_3}^{e_3 i_3 i_2} \dots) \cdot \text{Tr}(\frac{v_{i_1} v_{s_1}}{v_{i'_1}} \Omega_{s_1 s_2 i_1}^{i'_1} \delta_{s_2 s_3} \text{Id} \frac{v_{i_3} v_{s_3}}{v_{i'_3}} \bar{\Omega}_{s_3 s_4 i_3}^{i'_3} \dots)$ for a general string.

numbers, and we can identify

$$\Omega_{stj}^i = \omega_j^i \delta_{ss_0} \delta_{ts_0}, \quad \bar{\Omega}_{stj}^i = \bar{\omega}_j^i \delta_{ss_0} \delta_{ts_0}. \quad (3.17)$$

As we mentioned above, products of simple string operators are always of the form (3.15). In fact, we believe that *all string operators* are of this form. Thus, we will use (3.15) as an ansatz for general string operators in $(2+1)D$ topological phases. This ansatz is complicated algebraically, but like the definition of $B_{\mathbf{p}}^s$, it has a simple graphical interpretation (see appendix A.5).

A straightforward calculation shows that the closed string $W(P)$ commutes with the Hamiltonian (3.7) if Ω and $\bar{\Omega}$ satisfy

$$\begin{aligned} \sum_{s=0}^N \bar{\Omega}_{rsj}^m F_{kjm^*}^{sl^*i} \Omega_{sti}^l \frac{v_j v_s}{v_m} &= \sum_{n=0}^N F_{t^*nl^*}^{ji^*k} \Omega_{rtk}^n F_{krm^*}^{jl^*n} \\ \bar{\Omega}_{sti}^j &= \sum_{k=0}^N \Omega_{sti^*}^k F_{i^*sj^*}^{it^*k} \end{aligned} \quad (3.18)$$

The solutions $(\Omega_m, \bar{\Omega}_m)$ to these equations give all the different closed string operators W_m . However, not all of these solutions are really distinct. Notice that two solutions

$(\Omega_1, \bar{\Omega}_1)$, $(\Omega_2, \bar{\Omega}_2)$ can be combined to form a new solution $(\Omega', \bar{\Omega}')$:

$$\begin{aligned}\Omega'_{sti} &= \Omega_{1,sti} \oplus \Omega_{2,sti} \\ \bar{\Omega}'_{sti} &= \bar{\Omega}_{1,sti} \oplus \bar{\Omega}_{2,sti}\end{aligned}\tag{3.19}$$

This is not surprising: the string operator W' corresponding to $(\Omega', \bar{\Omega}')$ is simply the sum of the two operators corresponding to $(\Omega_{1,2}, \bar{\Omega}_{1,2})$: $W' = W_1 + W_2$.

Given this additivity property, it is natural to consider the “irreducible” solutions $(\Omega_\alpha, \bar{\Omega}_\alpha)$ that cannot be written as a sum of two other solutions. Only the “irreducible” string operators W_α create quasiparticle-pairs in the usual sense. Reducible string operators W create superpositions of different strings - which correspond to superpositions of different quasiparticles.¹

To analyze a topological phase, one only needs to find the irreducible solutions $(\Omega_\alpha, \bar{\Omega}_\alpha)$ to (3.18). The number M of such solutions is always finite. In general, each solution corresponds to an irreducible representation of an algebraic object. In the case of lattice gauge theory, there is one solution for every irreducible representation of the quantum double $D(G)$ of the gauge group G . Similarly, in the case of doubled Chern-Simons theories there is one solution for each irreducible representation of a doubled quantum group.

The structure of these irreducible string operators W_α determines all the universal features of the topological phase. The number M of irreducible string operators is the number of different kinds of quasiparticles. The fusion rules $W_\alpha W_\beta = \sum_{\gamma=1}^M h_{\alpha\beta}^\gamma W_\gamma$ determine how bound states of type- α and type- β quasiparticles can be viewed as a superposition of other types of quasiparticles.

The topological properties of the quasiparticles are also easy to compute. As an example, we now derive two particularly fundamental objects that characterize the spins and statistics of quasiparticles: the M twists θ_α and the $M \times M$ S-matrix, $s_{\alpha\beta}$ [65, 25, 57, 20].

The twists θ_α are defined to be statistical angles of the type- α quasiparticles. By the spin-statistics theorem they are closely connected to the quasiparticle spins s_α : $e^{i\theta_\alpha} = e^{2\pi i s_\alpha}$. We can calculate θ_α by comparing the quantum mechanical amplitude for the following two processes. In the first process, we create a pair of quasiparticles $\alpha, \bar{\alpha}$ (from the ground state), exchange them, and then annihilate the pair. In the second process, we create and then annihilate the pair without any exchange. The ratio of the amplitudes for these two processes is precisely $e^{i\theta_\alpha}$.

The amplitude for each process is given by the expectation value of the closed string operator W_α for a particular path P :

$$\mathcal{A}_1 = \langle \Phi | \left[\text{diagram of } W_\alpha \text{ with crossing} \right] | \Phi \rangle \tag{3.20}$$

$$\mathcal{A}_2 = \langle \Phi | \left[\text{diagram of } W_\alpha \text{ without crossing} \right] | \Phi \rangle \tag{3.21}$$

¹Note that reducible quasiparticles should not be confused with bound states. Indeed, in the case of Abelian gauge theory, most of the irreducible quasiparticles are bound states of electric charges and magnetic fluxes.

Here, $|\Phi\rangle$ denotes the ground state of the Hamiltonian (3.7).

Let $(\Omega_\alpha, \bar{\Omega}_\alpha, n_\alpha)$ be the irreducible solution corresponding to the string operator W_α . The above two amplitudes can be then be expressed in terms of $(\Omega_\alpha, \bar{\Omega}_\alpha, n_\alpha)$ (see appendix A.5):

$$\mathcal{A}_1 = \sum_s d_s^2 \cdot \text{Tr}(\Omega_{\alpha, sss}^0) \quad (3.22)$$

$$\mathcal{A}_2 = \sum_s n_{\alpha, s} d_s \quad (3.23)$$

Combining these results, we find that the twists are given by

$$e^{i\theta_\alpha} = \frac{\mathcal{A}_1}{\mathcal{A}_2} = \frac{\sum_s d_s^2 \cdot \text{Tr}(\Omega_{\alpha, sss}^0)}{\sum_s n_{\alpha, s} d_s} \quad (3.24)$$

Just as the twists θ_α are related to the spin and statistics of individual particle types α , the elements of the S-matrix, $s_{\alpha\beta}$ describe the mutual statistics of two particle types α, β . Consider the following process: We create two pairs of quasiparticles $\alpha, \bar{\alpha}, \beta, \bar{\beta}$, braid α around β , and then annihilate the two pairs. The element $s_{\alpha\beta}$ is defined to be the quantum mechanical amplitude \mathcal{A} of this process, divided by a proportionality factor D where $D^2 = \sum_\alpha (\sum_s n_{\alpha, s} d_s)^2$. The amplitude \mathcal{A} can be calculated from the expectation value of W_α, W_β for two “linked” paths P :

$$\mathcal{A} = \langle \Phi | \left\langle \begin{array}{c} \beta \\ \diamond \\ \alpha \end{array} \right| \Phi \rangle \quad (3.25)$$

Expressing \mathcal{A} in terms of $(\Omega_\alpha, \bar{\Omega}_\alpha, n_\alpha)$, we find

$$s_{\alpha\beta} = \frac{\mathcal{A}}{D} = \frac{1}{D} \sum_{ijk} \text{Tr}(\Omega_{\alpha, iij}^k) \cdot \text{Tr}(\Omega_{\beta, jjj}^{k*}) d_i d_j \quad (3.26)$$

3.2 String-net condensation in $(3 + 1)$ and higher dimensions

In this section, we generalize our results to $(3+1)$ and higher dimensions. We find that there is a one-to-one correspondence between $(3+1)$ (and higher) dimensional string-net condensates and mathematical objects known as “symmetric tensor categories.” [55] The low energy effective theories for these states are gauge theories coupled to bosonic or fermionic charges. (See Ref. [14, 15] for a simplified, alternative derivation).

Our approach is based on the exactly soluble lattice spin Hamiltonian (3.7). In section 3.1.2, we analyzed that model in the case of the honeycomb lattice. However, the choice of lattice was somewhat arbitrary: we could equally well have chosen any trivalent lattice in two dimensions.

Trivalent lattices can also be constructed in three and higher dimensions. For example, we can create a space-filling trivalent lattice in three dimensions, by “split-

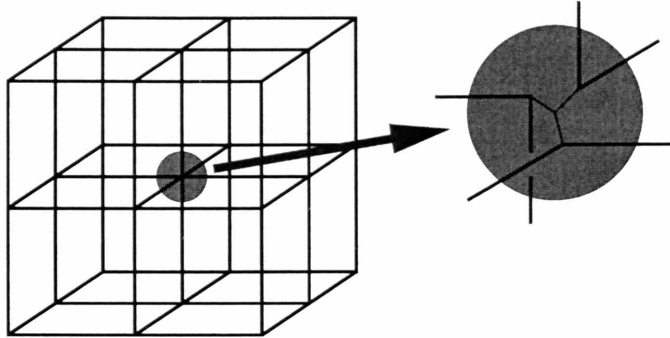


Figure 3-5: A three dimensional trivalent lattice, obtained by splitting the sites of the cubic lattice. We replace each vertex of the cubic lattice with 4 other vertices as shown above.

ting” the sites of the cubic lattice (see Fig. 3-5). Consider the spin Hamiltonian (3.7) for this lattice, where \mathbf{I} runs over all the vertices of the lattice, and \mathbf{p} runs over all the “plaquettes” (that is, the closed loops that correspond to plaquettes in the original cubic lattice).

This model is a natural candidate for string-net condensation in three dimensions. Unfortunately, it turns out that the Hamiltonian (3.7) is not exactly soluble on this lattice. The magnetic flux operators $B_{\mathbf{p}}^s$ do not commute in general.

This lack of commutativity originates from two differences between the plaquettes in the honeycomb lattice and in higher dimensional trivalent lattices. The first difference is that in the honeycomb lattice, neighboring plaquettes always share precisely two vertices, while in higher dimensions the boundary between plaquettes can contain three or more vertices (see Fig. 3-6). The existence of these interior vertices has the following consequence. Imagine we choose orientation conventions for each vertex, so that we have a notion of “left turns” and “right turns” for oriented paths on our lattice (such an orientation convention can be obtained by projecting the 3D lattice onto a 2D plane - as in Fig. 3-6). Then, no matter how we assign these orientations, some plaquette boundaries will always make both left and right turns. Thus, we cannot regard the boundaries of the 3D plaquettes as small closed strings the way we did in two dimensions (since small closed strings always make all left turns, or all right turns). On the other hand, the magnetic flux operators $B_{\mathbf{p}}^s$ only commute if their boundaries are small closed strings. It is this inconsistency between the algebraic definition of $B_{\mathbf{p}}^s$ and the topology of the plaquettes that leads to the lack of commutativity.

To resolve this problem, we need to define a Hamiltonian using the general simple string operators $W(\partial\mathbf{p})$ rather than the small closed strings $B_{\mathbf{p}}^s$. Suppose $(\omega_{sj}^i, \bar{\omega}_{sj}^i)$, $s = 0, 1, \dots, N$ are type- s solutions of (3.14). After picking some “left turn”, “right turn” orientation convention at each vertex, we can define the corresponding type- s simple string operators $W_s(P)$ as in (3.12). Suppose, in addition, that we choose $(\omega_{sj}^i, \bar{\omega}_{sj}^i)$ so that the string operators satisfy $W_r \cdot W_s = \sum_t \delta_{rst} W_t$ (this property ensures that $W_s(\partial\mathbf{p})$ are analogous to $B_{\mathbf{p}}^s$). Then, a natural higher dimensional gen-

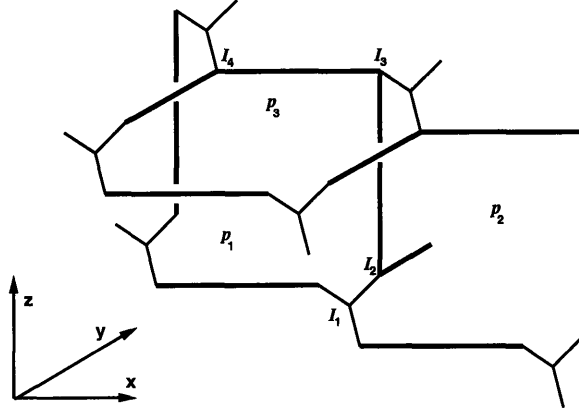


Figure 3-6: Three plaquettes demonstrating the two fundamental differences between higher dimensional trivalent lattices and the honeycomb lattice. The plaquettes p_1 , p_2 lie in the xz plane, while p_3 is oriented in the xy direction. Notice that p_1 and p_2 share three vertices, I_1, I_2, I_3 - unlike neighboring plaquettes in the honeycomb lattice, which share two vertices. Also, notice that the plaquette boundaries ∂p_1 and ∂p_3 intersect only at the line segment I_3I_4 . The boundary ∂p_1 makes a left turn at I_3 , and a right turn at I_4 . Thus, if we shrink the segment I_3I_4 to a point, these two plaquette boundaries intersect exactly once - unlike neighboring plaquettes in the honeycomb lattice, which intersect *tangentially* when their common boundary is shrunk to a point.

erealization of the Hamiltonian (3.7) is

$$H = - \sum_I Q_I - \sum_p W_p, \quad W_p = \sum_{s=0}^N a_s W_s(\partial p) \quad (3.27)$$

For a two dimensional lattice, the conditions (3.14) are sufficient to guarantee that the closed strings $W_s(\partial p)$ commute within the ground state subspace. The Hamiltonian (3.27) is then an exactly soluble realization of a doubled topological phase. However, in higher dimensions, one additional constraint is necessary.

This constraint stems from the second, and perhaps more fundamental, difference between $2D$ and higher dimensional lattices. In two dimensions, two closed curves always intersect an even number of times. For higher dimensional lattices, this is not the case. Small closed curves, in particular plaquette boundaries, can (in a sense) intersect exactly once (see Fig. 3-6). Because of this, the objects ω_{jk}^i must satisfy the additional relation:

$$\omega_{jk}^i = \bar{\omega}_{kj}^i \quad (3.28)$$

One can show that if this additional constraint is satisfied, then (a) the higher dimensional Hamiltonian (3.27) is exactly soluble, and (b) the ground state wave function Φ is defined by local topological rules analogous to (3.1-3.4). This means that (3.27) provides an exactly soluble realization of topological phases in $(3 + 1)$ and higher dimensions.

Each exactly soluble Hamiltonian is associated with a solution $(F_{kln}^{ijm}, \omega_{jk}^i, \bar{\omega}_{jk}^i)$ of (3.5), (3.14), (3.28). By analogy with the two dimensional case, we conjecture that there is a one-to-one correspondence between topological string-net condensed phases in $(3 + 1)$ or higher dimensions, and these solutions. The solutions $(F_{kln}^{ijm}, \omega_{jk}^i, \bar{\omega}_{jk}^i)$ correspond to a special class of tensor categories - symmetric tensor categories. [55] Just as tensor categories are the mathematical objects underlying string condensation in $(2 + 1)$ dimensions, symmetric tensor categories are fundamental to string condensation in higher dimensions.

There are relatively few solutions to (3.5), (3.14), (3.28). Physically, this is a consequence of the restrictions on quasiparticle statistics in 3 or higher dimensions. Unlike in two dimensions, higher dimensional quasiparticles necessarily have trivial mutual statistics, and must be either bosonic or fermionic. From a more mathematical point of view, the scarcity of solutions is a result of the symmetry condition (3.28).

Nevertheless, each group G does provide a solution. This solution is obtained by (a) letting the string-type index i run over the irreducible representations of the gauge group, (b) letting the numbers d_i be the dimensions of the representations, (c) letting the 6 index object F_{kln}^{ijm} be the $6j$ symbol of the group, and (d) setting $\omega_{jk}^i = \pm \frac{v_i}{v_j v_k}$. Here we choose the positive sign in almost all cases. The only time when the sign is negative is when $j = k$ and the unique G -invariant tensor in $k \otimes k \otimes i$ is antisymmetric in the first two indices. With this choice, one can show that (3.5), (3.14) and (3.28) are satisfied. The low energy effective theory of the corresponding string-net condensed state is a deconfined gauge theory with gauge group G .

It is not surprising that we have this class of solutions since the string-net picture of gauge theory (section 2.3) is valid in any number of dimensions. However, there is another class of solutions that is more interesting. These are obtained by “twisting” the usual gauge theory solution. We replace ω_{jk}^i by $\tilde{\omega}_{jk}^i$ where

$$\tilde{\omega}_{jk}^i = \omega_{jk}^i \cdot (-1)^{P(j)P(k)} \quad (3.29)$$

Here $P(i)$ is some assignment of parity (“even”, or “odd”) to each representation i . The assignment must be self-consistent in the sense that the tensor product of two representations with the same (different) parity decomposes into purely even (odd) representations. The low energy effective theories for these string-net condensed states are variants of gauge theories - called “twisted gauge theories.” Notice that if all the representations are assigned an even parity, we are back to the usual gauge theory solution: $\tilde{\omega}_{jk}^i = \omega_{jk}^i$.

The major physical distinction between twisted gauge theories and standard gauge theories is the quasiparticle statistics. In standard gauge theory, the fundamental quasiparticles are the $N + 1$ electric charges corresponding to the $N + 1$ string types. These quasiparticles are all bosonic. In twisted gauge theories, there are also $N + 1$ different electric charges corresponding to the $N + 1$ string types. However, in this case, the charges corresponding to “odd” representations i are fermionic.

It appears that gauge theories coupled to fermionic or bosonic charged particles are the *only* possibilities for higher dimensional string-net condensates: mathematical work on symmetric tensor categories suggests that the only solutions to (3.5), (3.14),

(3.28) are those corresponding to gauge theories and twisted gauge theories. [66]

Thus, higher dimensional string-net condensation naturally gives rise to *both* emerging gauge bosons and emerging fermions. This is interesting because it suggests that gauge interactions and Fermi statistics may be intimately connected. The string-net picture may be the bridge between these two seemingly unrelated phenomena. [18, 14, 15]

We would like to mention that (3+1) dimensional string-net condensed states also exhibit membrane condensation. These membrane operators are entirely analogous to the string operators. Just as open string operators create charges at their two ends, open membrane operators create magnetic flux loops along their boundaries. Furthermore, just as string condensation makes the string unobservable, membrane condensation leads to the unobservability of the membrane. Only the boundary of the membrane - the magnetic flux loop - is observable.

3.3 Examples

3.3.1 $N = 1$ string model

We begin with the simplest string-net model. In the notation from chapter 2, this model is given by

1. Number of string types: $N = 1$
2. Branching rules: \emptyset (no branching)
3. String orientations: $1^* = 1$.

In other words, the string-nets in this model contain one unoriented string type and have no branching. Thus, they are simply closed loops. (See Fig. 2-6a).

We would like to find the different topological phases that can emerge from these closed loops. According to the discussion in section 3.1.1, each phase is captured by a fixed-point wave function, and each fixed-point wave function is specified by local rules (3.1-3.4) that satisfy the self-consistency relations (3.5). It turns out that (3.5) have only two solutions in this case (up to rescaling):

$$\begin{aligned}
 d_0 &= 1 \\
 d_1 &= F_{110}^{110} = \pm 1 \\
 F_{000}^{000} &= F_{101}^{101} = F_{011}^{011} = 1 \\
 F_{111}^{000} &= F_{001}^{110} = F_{010}^{101} = F_{100}^{011} = 1
 \end{aligned} \tag{3.30}$$

where the other elements of F all vanish. The corresponding local rules (3.1-3.4) are:

$$\begin{aligned}
 \Phi \left(\blacksquare \circ \right) &= \pm \Phi \left(\blacksquare \right) \\
 \Phi \left(\blacktriangleright \blacktriangleleft \right) &= \pm \Phi \left(\blacksquare \square \blacksquare \right)
 \end{aligned} \tag{3.31}$$

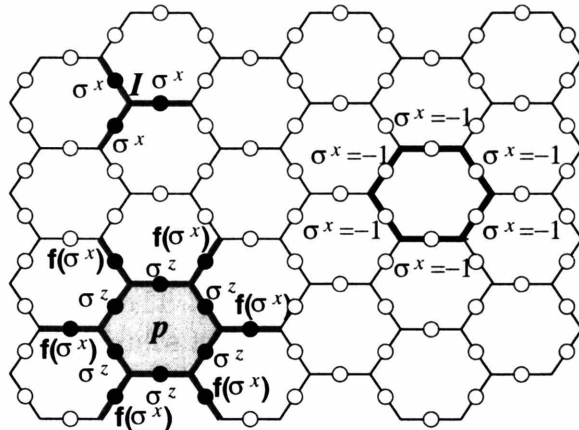


Figure 3-7: The Hamiltonians (3.35), (3.36), realizing the two $N = 1$ string condensed phases. Each circle denotes a spin-1/2 spin. The links with $\sigma^x = -1$ are thought of as being occupied by a type-1 string, while the links with $\sigma^x = +1$ are regarded as empty. The electric charge term acts on the three legs of the vertex I with σ^x . The magnetic energy term acts on the 6 edges of the plaquette p with σ^z , and acts on the 6 legs of p with an operator of the form $f(\sigma^x)$. For the Z_2 phase, $f = 1$, while for the Chern-Simons phase, $f(x) = i^{(1-x)/2}$.

We have omitted those rules that can be derived from topological invariance (3.1).

The fixed-point wave functions Φ_{\pm} satisfying these rules are given by

$$\Phi_{\pm}(X) = (\pm 1)^{X_c} \quad (3.32)$$

where X_c is the number of disconnected components in the string configuration X .

The two fixed-point wave functions Φ_{\pm} correspond to two simple topological phases. As we will see, Φ_+ corresponds to Z_2 gauge theory, while Φ_- is a $U(1) \times U(1)$ Chern-Simons theory. (Actually, other topological phases can emerge from closed loops - such as in Ref. [59, 60, 61]. However, we regard these phases as emerging from more complicated string-nets. The closed loops organize into these effective string-nets in the infrared limit).

The exactly soluble models (3.7) realizing these two phases can be written as spin 1/2 systems with one spin on each link of the honeycomb lattice (see Fig. 3-7). We regard a link with $\sigma^x = -1$ as being occupied by a type-1 string, and the state $\sigma^x = +1$ as being unoccupied (or equivalently, occupied by a type-0 or null string). The Hamiltonians for the two phases are of the form

$$H_{\pm} = - \sum_I Q_{I,\pm} - \sum_p B_{p,\pm}$$

The electric charge term is the same for both phases (since it only depends on the branching rules):

$$Q_{I,\pm} = \frac{1}{2} \left(1 + \prod_{\text{legs of } I} \sigma_i^x \right) \quad (3.33)$$

The magnetic terms for the two phases are

$$\begin{aligned}
B_{\mathbf{p},\pm} &= \frac{1}{2}(B_{\mathbf{p},\pm}^0 \pm B_{\mathbf{p},\pm}^1) \\
&= \frac{1}{2} \left(1 \pm \prod_{\text{edges of } \mathbf{p}} \sigma_j^z \cdot \prod_{\text{legs of } \mathbf{p}} (\sqrt{\pm 1})^{\frac{1-\sigma_j^x}{2}} \right) P_{\mathbf{p}}
\end{aligned} \tag{3.34}$$

where $P_{\mathbf{p}}$ is the projection operator $P_{\mathbf{p}} = \prod_{I \in \mathbf{p}} Q_I$. The projection operator $P_{\mathbf{p}}$ can be omitted without affecting the physics (or the exact solubility of the Hamiltonian). We have included it only to be consistent with (3.7). If we omit this term, the Hamiltonian for the first phase (Φ_+) reduces to the Kitaev toric code [13] - the exactly soluble zero coupling limit of Z_2 lattice gauge theory:

$$H_+ = - \sum_I \prod_{\text{legs of } I} \sigma_i^x - \sum_{\mathbf{p}} \prod_{\text{edges of } \mathbf{p}} \sigma_j^z \tag{3.35}$$

The Hamiltonian for the second phase,

$$H_- = - \sum_I \prod_{\text{legs of } I} \sigma_i^x + \sum_{\mathbf{p}} \left(\prod_{\text{edges of } \mathbf{p}} \sigma_j^z \right) \left(\prod_{\text{legs of } \mathbf{p}} i^{\frac{1-\sigma_j^x}{2}} \right) \tag{3.36}$$

is less familiar. However, one can check that in both cases, the Hamiltonians are exactly soluble and the two ground state wave functions are precisely Φ_{\pm} (in the σ^x basis).

Next we find the quasiparticle excitations for the two phases, and the corresponding S-matrix and twists θ_{α} .

In both cases, equation (3.18) has 4 irreducible solutions $(n_{\alpha,s}, \Omega_{\alpha,st}^{ij}, \bar{\Omega}_{\alpha,st}^{ij})$, $\alpha = 1, 2, 3, 4$ - corresponding to 4 quasiparticle types. For the first phase (Φ_+) these solutions are given by:

$$\begin{aligned}
n_{1,0} &= 1, & n_{1,1} &= 0, & \Omega_{1,000}^0 &= 1, & \Omega_{1,001}^1 &= 1 \\
n_{2,0} &= 0, & n_{2,1} &= 1, & \Omega_{2,110}^1 &= 1, & \Omega_{2,111}^2 &= 1 \\
n_{3,0} &= 1, & n_{3,1} &= 0, & \Omega_{3,000}^0 &= 1, & \Omega_{3,001}^1 &= -1 \\
n_{4,0} &= 0, & n_{4,1} &= 1, & \Omega_{4,110}^1 &= 1, & \Omega_{4,111}^0 &= -1
\end{aligned}$$

The other elements of Ω vanish. In all cases $\bar{\Omega} = \Omega$.

The corresponding string operators for a path P are

$$\begin{aligned}
W_1 &= \text{Id} \\
W_2 &= \prod_{\text{edges of } P} \sigma_j^z \\
W_3 &= \prod_{\text{R-legs}} \sigma_{\mathbf{k}}^x \\
W_4 &= \prod_{\text{edges of } P} \sigma_j^z \prod_{\text{R-legs}} \sigma_{\mathbf{k}}^x
\end{aligned} \tag{3.37}$$

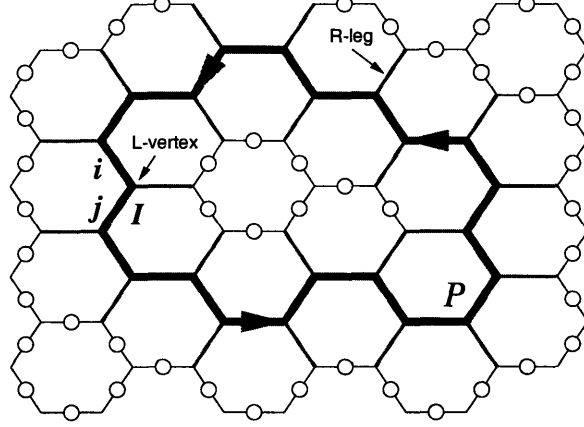


Figure 3-8: A closed string operator $W(P)$ for the two models (3.35),(3.36). The path P is drawn with a thick line, while the legs are drawn with thin lines. The action of the string operators (3.37),(3.40) on the legs is different for legs that branch to the right of P , “R-legs”, and legs that branch to the left of P , “L-legs.” Similarly, we distinguish between “R-vertices” and “L-vertices” which are ends of “R-leg” and “L-leg” respectively.

where the “R-legs” k are the legs that are to the right of P . (See Fig. 3-8). Technically, we should multiply these string operators by an additional projection operator $\prod_{I \in P} Q_I$, in order to be consistent with the general result (3.15). However, we will neglect this factor since it doesn’t affect the physics.

Once we have the string operators, we can easily calculate the twists and the S-matrix. We find:

$$e^{i\theta_1} = 1, \quad e^{i\theta_2} = 1, \quad e^{i\theta_3} = 1, \quad e^{i\theta_4} = -1 \quad (3.38)$$

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad (3.39)$$

This is in agreement with the twists and S-matrix for Z_2 gauge theory: W_1 creates trivial quasiparticles, W_2 creates magnetic fluxes, W_3 creates electric charges, W_4 creates electric/magnetic bound states.

In the second phase (Φ_-), we find

$$\begin{aligned} n_{1,0} &= 1, & n_{1,1} &= 0, & \Omega_{1,000}^0 &= 1, & \Omega_{1,001}^1 &= 1 \\ n_{2,0} &= 0, & n_{2,1} &= 1, & \Omega_{2,110}^1 &= 1, & \Omega_{2,111}^0 &= i \\ n_{3,0} &= 0, & n_{3,1} &= 1, & \Omega_{3,110}^1 &= 1, & \Omega_{3,111}^0 &= -i \\ n_{4,0} &= 1, & n_{4,1} &= 0, & \Omega_{4,000}^0 &= 1, & \Omega_{4,001}^1 &= -1 \end{aligned}$$

Once again, the other elements of Ω vanish. Also, in all cases, $\bar{\Omega} = \Omega^*$. The corresponding string operators for a path P are

$$\begin{aligned}
W_1 &= \text{Id} \\
W_2 &= \prod_{\text{edges of } P} \sigma_j^z \prod_{\text{R-legs}} i^{\frac{1-\sigma_j^x}{2}} \prod_{\text{L-vertices}} (-1)^{s_I} \\
W_3 &= \prod_{\text{edges of } P} \sigma_j^z \prod_{\text{R-legs}} (-i)^{\frac{1-\sigma_j^x}{2}} \prod_{\text{L-vertices}} (-1)^{s_I} \\
W_4 &= \prod_{\text{R-legs}} \sigma_j^x
\end{aligned} \tag{3.40}$$

where the ‘‘L-vertices’’ I are the vertices of P adjacent to legs that are to the left of P . The exponent s_I is defined by $s_I = \frac{1}{4}(1 - \sigma_i^x)(1 + \sigma_j^x)$, where i, j are the links just before and just after the vertex I , along the path P . (See Fig. 3-8).

We find the twists and S-matrix are

$$e^{i\theta_1} = 1, \quad e^{i\theta_2} = i, \quad e^{i\theta_3} = -i, \quad e^{i\theta_4} = 1 \tag{3.41}$$

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{3.42}$$

We see that W_1 creates trivial quasiparticles, W_2, W_3 create semions with opposite chiralities and trivial mutual statistics, and W_4 creates bosonic bound states of the semions. These results agree with the $U(1) \times U(1)$ Chern-Simons theory

$$L = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda}, \quad I, J = 1, 2 \tag{3.43}$$

with K -matrix

$$K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \tag{3.44}$$

Thus the above $U(1) \times U(1)$ Chern-Simons theory is the low energy effective theory of the second exactly soluble model (with $d_1 = -1$).

Note that the Z_2 gauge theory from the first exactly soluble model (with $d_1 = 1$) can also be viewed as a $U(1) \times U(1)$ Chern-Simons theory with K -matrix [67]

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{3.45}$$

3.3.2 $N = 1$ string-net model

The next simplest string-net model also contains only one oriented string type - but with branching. Simple as it is, we will see that this model contains non-Abelian anyons and is theoretically capable of universal fault tolerant quantum computation

[56]. Formally, the model is defined by

1. Number of string types: $N = 1$
2. Branching rules: $\{\{1, 1, 1\}\}$
3. String orientations: $1^* = 1$.

The string-nets are unoriented trivalent graphs (see Fig. 2-6b). To find the topological phases that can emerge from these objects, we solve the self-consistency relations (3.5). We find two sets of self-consistent rules:

$$\begin{aligned}
\Phi \left(\begin{array}{c} \blacksquare \\ \square \end{array} \right) &= \gamma_{\pm} \cdot \Phi \left(\blacksquare \right) \\
\Phi \left(\begin{array}{c} \blacktriangleright \\ \blacktriangleleft \end{array} \right) &= \gamma_{\pm}^{-1} \cdot \Phi \left(\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \right) + \gamma_{\pm}^{-1/2} \cdot \Phi \left(\begin{array}{c} \blacksquare \\ \square \end{array} \right) \\
\Phi \left(\begin{array}{c} \blacktriangleright \\ \blacktriangleleft \end{array} \right) &= \gamma_{\pm}^{-1/2} \cdot \Phi \left(\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \right) - \gamma_{\pm}^{-1} \cdot \Phi \left(\begin{array}{c} \blacksquare \\ \square \end{array} \right)
\end{aligned}$$

where $\gamma_{\pm} = \frac{1 \pm \sqrt{5}}{2}$. (Once again, we have omitted those rules that can be derived from topological invariance). Unlike the previous case, there is no closed form expression for the wave function amplitude.

Note that the second solution, $d_1 = \frac{1 - \sqrt{5}}{2}$ does not satisfy the unitarity condition (3.11). Thus, only the first solution corresponds to a physical topological phase. As we will see, this phase is described by an $SO_3(3) \times SO_3(3)$ Chern-Simons theory.

As before, the exactly soluble realization of this phase (3.7) is a spin-1/2 model with spins on the links of the honeycomb lattice. We regard a link with $\sigma^x = -1$ as being occupied by a type-1 string, and a link with $\sigma^x = 1$ as being unoccupied (or equivalently occupied by a type-0 string). However, in this case we will not explicitly rewrite (3.7) in terms of Pauli matrices, since the resulting expression is quite complicated.

We now find the quasiparticles. These correspond to irreducible solutions of (3.18). For this model, there are 4 such solutions, corresponding to 4 quasiparticles:

$$\begin{aligned}
1 : n_{1,0} &= 1, n_{1,1} = 0, \Omega_{1,000}^0 = 1, \Omega_{1,001}^1 = 1 \\
2 : n_{2,0} &= 0, n_{2,1} = 1, \Omega_{2,110}^1 = 1, \\
&\Omega_{2,111}^0 = -\gamma_+^{-1} e^{\pi i/5}, \Omega_{2,111}^1 = \gamma_+^{-1/2} e^{3\pi i/5} \\
3 : n_{3,0} &= 0, n_{3,1} = 1, \Omega_{3,110}^1 = 1, \\
&\Omega_{3,111}^0 = -\gamma_+^{-1} e^{-\pi i/5}, \Omega_{3,111}^1 = \gamma_+^{-1/2} e^{-3\pi i/5} \\
4 : n_{4,0} &= 1, n_{4,1} = 1, \Omega_{4,000}^0 = 1, \Omega_{4,110}^1 = 1, \\
&\Omega_{4,001}^1 = -\gamma_+^{-2}, \Omega_{4,111}^0 = \gamma_+^{-1}, \Omega_{4,111}^1 = \gamma_+^{-5/2}, \\
&\Omega_{4,101}^1 = (\Omega_{4,011}^1)^* = \gamma_+^{-11/4} (2 - e^{3\pi i/5} + \gamma_+ e^{-3\pi i/5}).
\end{aligned} \tag{3.46}$$

In all cases, $\bar{\Omega} = \Omega^*$.

We can calculate the twists and the S-matrix. We find:

$$e^{i\theta_1} = 1, \quad e^{i\theta_2} = e^{-4\pi i/5}, \quad e^{i\theta_3} = e^{4\pi i/5}, \quad e^{i\theta_4} = 1 \quad (3.47)$$

$$S = \frac{1}{1 + \gamma^2} \begin{pmatrix} 1 & \gamma & \gamma & \gamma^2 \\ \gamma & -1 & \gamma^2 & -\gamma \\ \gamma & \gamma^2 & -1 & -\gamma \\ \gamma^2 & -\gamma & -\gamma & 1 \end{pmatrix} \quad (3.48)$$

We conclude that W_1 creates trivial quasiparticles, W_2, W_3 create (non-Abelian) anyons with opposite chiralities, and W_4 creates bosonic bound states of the anyons. These results agree with $SO_3(3) \times SO_3(3)$ Chern-Simons theory, the so-called doubled ‘‘Yang-Lee’’ theory.

Researchers in the field of quantum computing have shown that the Yang-Lee theory can function as a universal quantum computer - via manipulation of non-Abelian anyons. [56] Therefore, the spin-1/2 Hamiltonian (3.7) associated with (3.46) is a theoretical realization of a universal quantum computer. While this Hamiltonian may be too complicated to be realized experimentally, the string-net picture suggests that this problem can be overcome. Indeed, the string-net picture suggests that *generic* spin Hamiltonians with a trivalent graph structure will exhibit a Yang-Lee phase. Thus, much simpler spin-1/2 Hamiltonians may be capable of universal fault tolerant quantum computation.

3.3.3 $N = 2$ string-net models

In this section, we discuss two $N = 2$ string-net models. The first model contains one oriented string and its dual. In the notation from section 2.2, it is given by

1. Number of string types: $N = 2$
2. Branching rules: $\{\{1, 1, 1\}, \{2, 2, 2\}\}$
3. String orientations: $1^* = 2, 2^* = 1$.

The string-nets are therefore oriented trivalent graphs with Z_3 branching rules (see Fig. 2-6d). The string-net condensed phases correspond to solutions of (3.5). Solving these equations, we find one set of self-consistent local rules:

$$\begin{aligned} \Phi \left(\begin{array}{c} \blacksquare \\ \square \end{array} \right) &= \Phi \left(\blacksquare \right) \\ \Phi \left(\begin{array}{c} \blacktriangleright \\ \blacktriangleleft \end{array} \right) &= \Phi \left(\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \right) \\ \Phi \left(\begin{array}{c} \blacktriangleright \\ \blacktriangleleft \end{array} \right) &= \Phi \left(\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \right) \end{aligned} \quad (3.49)$$

The corresponding fixed-point wave function Φ is simply the constant function: $\Phi(X) = 1$ for all X .

As before, we can construct an exactly soluble Hamiltonian, find the quasiparticles and compute the twists and S-matrices. We find that Φ is described by a Z_3 gauge

theory. There are $3^2 = 9$ elementary quasiparticles. These quasiparticles are electric charge/magnetic flux bound states formed from the 3 types of electric charges and 3 types of magnetic fluxes.

The final example we will discuss contains two unoriented strings. Formally it is given by

1. Number of string types: $N = 2$
2. Branching rules: $\{\{1, 2, 2\}, \{2, 2, 2\}\}$
3. String orientations: $1^* = 1, 2^* = 2$.

The string-nets are unoriented trivalent graphs, with edges labeled with 1 or 2 (see Fig. 2-6c). We find that there is only one set of self-consistent local rules:

$$\begin{aligned}
\Phi \left(\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \square^i \right) &= d_i \cdot \Phi \left(\begin{array}{c} \blacksquare \\ \blacksquare \end{array} \right) \\
\Phi \left(\begin{array}{c} \blacksquare \xrightarrow{1} \blacksquare \\ \blacksquare \xrightarrow{1} \blacksquare \end{array} \right) &= \Phi \left(\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \right) \\
\Phi \left(\begin{array}{c} \blacksquare \xrightarrow{1} \blacksquare \\ \blacksquare \xrightarrow{2} \blacksquare \end{array} \right) &= \Phi \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \right) \\
\Phi \left(\begin{array}{c} \blacksquare \xrightarrow{2} \blacksquare \\ \blacksquare \xrightarrow{1} \blacksquare \end{array} \right) &= \Phi \left(\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \right) \\
\Phi \left(\begin{array}{c} \blacksquare \xrightarrow{2} \blacksquare \\ \blacksquare \xrightarrow{2} \blacksquare \end{array} \right) &= - \Phi \left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \right) \\
\Phi \left(\begin{array}{c} \blacksquare \xrightarrow{2} \blacksquare \\ \blacksquare \xrightarrow{2} \blacksquare \end{array} \right) &= \sum_{n=0}^2 F_{22n}^{22m} \cdot \Phi \left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 2 & 2 \\ \hline \end{array} \right) \tag{3.50}
\end{aligned}$$

where $d_0 = d_1 = 1, d_2 = 2$, and F_{22n}^{22m} is the matrix

$$F_{22n}^{22m} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

If we construct the Hamiltonian (3.7), we find that it is equivalent to the standard exactly soluble lattice gauge theory Hamiltonian [13] with gauge group S_3 - the permutation group on 3 objects. One can show that this theory contains 8 elementary quasiparticles (corresponding to the 8 irreducible representations of the quantum double $D(S_3)$). These quasiparticles are combinations of the 3 electric charges and 3 magnetic fluxes.

Chapter 4

String-net condensation and quantum entanglement

In the previous two chapters, we described a physical picture and mathematical framework for non-chiral topological phases. It might appear that we have a complete theory of topological phases.

However, the theory of topological phases is still incomplete. There are several problems that need to be addressed. In this chapter we focus on one of these problems: the problem of physical characterizing topological phases. At the moment, our physical characterization of topological order is weak - much weaker than our characterization of symmetry-breaking order. For example, we can easily detect symmetry breaking order in a ground state wave function. To do this, one simply looks for long range correlations $\langle S_i \cdot S_j \rangle$. But we cannot detect topological order in a wave function. Indeed, the only physical characterizations of topological order [19] involve properties of the Hamiltonian - e.g. quasiparticle statistics [23], ground state degeneracy [68, 2], and edge excitations [19].

Yet we expect that topological order is a property of the ground state. Topologically ordered states are highly entangled and this entanglement is responsible for their unusual properties. If topological order is in any way analogous to symmetry-breaking order it should be a property of the ground state wave function.

This is a serious problem, not only conceptually, but also for practical reasons. In recent years, many wave functions have been proposed as examples of topological order such as Gutzwiller projected states [69], quantum loop gas wave functions [60], and resonating dimer wave functions. [48] However, we cannot make sharp statements about whether these states actually contain topological order.

In addition, this is a problem for numerical studies. For example, there is reason to believe that the ground state of the $J_1 - J_3$ model - a spin-1/2 antiferromagnet on the square lattice - is topologically ordered for some choices of parameters. But we cannot establish this definitively since current techniques only allow us to look for symmetry breaking order. [35, 34]

In this chapter, we remedy this problem. We demonstrate that topological order is manifest not only in dynamical properties but also in the basic entanglement of the ground state wave function. Furthermore, this entanglement can be detected in

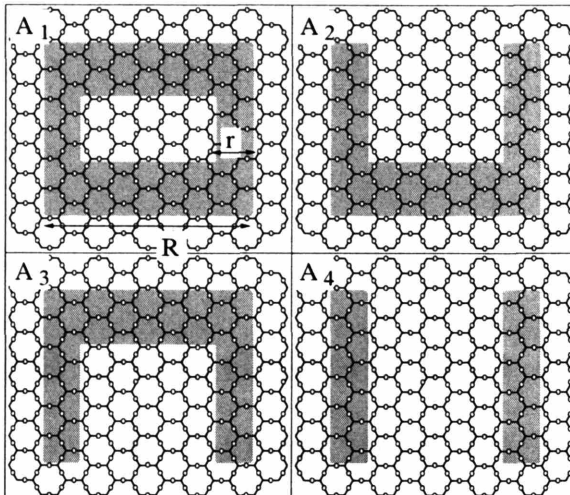


Figure 4-1: One can detect topological order in a state Ψ by computing the entanglement entropies S_1, S_2, S_3, S_4 of the above four regions, A_1, A_2, A_3, A_4 . Here the four regions are drawn in the case of the honeycomb lattice. Note that these regions have been carefully designed so that A_1 differs from A_2 in the same way that A_3 differs from A_4 .

a disk-like region: there is no need to consider non-trivial topologies such as a torus to detect topological order.

Our approach is based on the string-net condensation picture. String-net condensed states are highly entangled, and we described a method for detecting this entanglement. The method involves computing a new quantity that we call “topological entropy.” We would like to mention that a similar result was obtained independently in the recent paper, Ref. [70].

This chapter is organized as follows. In section 4.1, we describe the main result. In section 4.2, we explain the basic physical picture behind the result, and in section 4.3 we work out a simple example. In section 4.4, we derive the result for general string-net models.

4.1 Main Result

We focus on the $(2+1)$ dimensional case (though the result can be generalized to any dimension). Let Ψ be an arbitrary wave function for some two dimensional lattice model. For any subset A of the lattice, one can compute the associated quantum entanglement entropy $S_A = -\text{Tr}(\rho_A \log \rho_A)$. [71] The main result of this chapter is that one can determine the “total quantum dimension” D of Ψ by computing the entanglement entropy S_A of particular regions A in the plane. Normal states have $D = 1$ while topologically ordered states have $D > 1$. Thus, this result provides a way to distinguish topologically ordered states from normal states, *using only the wave function*.

More specifically, consider the four regions A_1, A_2, A_3, A_4 drawn in Fig. 4-1. Let

the corresponding von Neumann entanglement entropies be S_1, S_2, S_3, S_4 . Consider the linear combination $(S_1 - S_2) - (S_3 - S_4)$ computed in the limit of large, thick annuli, $R, r \rightarrow \infty$. The main result of this chapter is that

$$(S_1 - S_2) - (S_3 - S_4) = -\log(D^2) \quad (4.1)$$

where D is the total quantum dimension of the topological order associated with Ψ . Here, $D = \sum_i d_i^2$ ¹ for a topological order described by a string-net condensate $(N, d_i, F_{lmn}^{ijk}, \delta_{ijk})$. In the case of discrete gauge theories, D is simply the number of elements in the gauge group.

We call the quantity $(S_1 - S_2) - (S_3 - S_4)$ the “topological entropy”, $-S_{\text{top}}$, since it measures the entropy associated with the topological entanglement in Ψ . The above result implies that S_{top} is universal: it only depends on the type of topological order encoded in Ψ .

4.2 Physical picture

The idea behind (4.1) is that topologically ordered states contain nonlocal entanglement. Consider, for example, the Kitaev toric code [13] - the zero coupling, exactly soluble limit of Z_2 lattice gauge theory. The model is a spin-1/2 model in with spins located on the links i of the honeycomb lattice. The Hamiltonian is

$$H = -\sum_{\mathbf{I}} \prod_{\text{legs of } \mathbf{I}} \sigma_i^x - \sum_{\mathbf{p}} \prod_{\text{edges of } \mathbf{p}} \sigma_j^z \quad (4.2)$$

where $\mathbf{I}, i, \mathbf{p}$ label the vertices, links, and plaquettes of the honeycomb lattice (see Fig. 4-2).

This model is exactly soluble: all the different terms, $\prod_{\text{legs of } \mathbf{I}} \sigma_i^x$ and $\prod_{\text{edges of } \mathbf{p}} \sigma_j^z$ commute with each other. The ground state $|\Psi\rangle$ is known exactly. The easiest way to describe Ψ is in terms of strings. [72] One can think of each spin state as a string state, where a $\sigma_i^x = -1$ spin corresponds to a link occupied by a string and a $\sigma_i^x = 1$ spin corresponds to an empty link. In this language, Ψ is simple: $\Psi(X) = 1$ for string states X where the strings form closed loops, and Ψ vanishes otherwise.

All local correlations $\langle \sigma_i^x \sigma_j^x \rangle$ vanish for this state. However, Ψ contains *nonlocal* correlations. To see this, imagine drawing a curve C in the plane (see Fig. 4-3). There is a nonlocal correlation between the spins on the links crossing this curve: $\langle W(C) \rangle = \langle \prod_{i \in C} \sigma_i^x \rangle = 1$. This correlation originates from the fact that the number of strings crossing the curve is always even. Similar correlations exist for more general states that contain virtual string-breaking fluctuations. In the general case, the nonlocal correlations can be captured by “fattened string operators” $W_{\text{fat}}(C)$ that act on spins within some distance l of C where l is the length scale for string breaking.

¹The usual definition of D is $D = \sqrt{\sum_{\alpha} d_{\alpha}^2}$ where α runs over the quasiparticle types. This agrees with the formula given in the text. The reason is that i only runs over a *subset* of the quasiparticle types - roughly speaking, those corresponding to “electric charges.” This subset has the property that $\sum_i d_i^2 = \sqrt{\sum_{\alpha} d_{\alpha}^2}$.

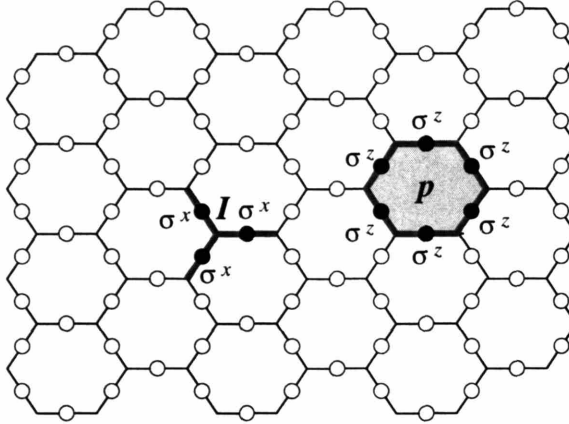


Figure 4-2: A picture of a spin-1/2 model (4.2) that realizes Z_2 topological order. The spins, denoted by open circles, are located on the links of the honeycomb lattice. The term $\prod_{\text{legs of } I} \sigma_i^x$ acts on the three spins adjacent to the vertex I , while $\prod_{\text{edges of } p} \sigma_j^z$ acts on the six spins along the boundary of the plaquette p .

To determine whether a state is topologically ordered, one has to determine whether the state contains such nonlocal correlations or entanglement. While it is difficult to find the explicit form of the fattened string operators W_{fat} , [73] one can establish their existence or non-existence using quantum information theory. The idea is that if the string operators exist, then the entropy of an annular region (such as A_1 in Fig. 4-1) will be lower than one would expect based on local correlations.

The combination $(S_1 - S_2) - (S_3 - S_4)$ measures exactly this anomalous entropy. To see this, notice that $(S_1 - S_2)$ is the amount of additional entropy associated with closing the region A_2 at the top. Similarly, $(S_3 - S_4)$ is the amount of additional entropy associated with closing the region A_4 at the top. If Ψ has only local correlations with correlation length ξ then these two quantities are the same up to corrections of order $O(e^{-R/\xi})$, since A_2, A_4 only differ by the region at the bottom. For such states, $\lim_{R \rightarrow \infty} (S_1 - S_2) - (S_3 - S_4) = 0$. Thus, a *nonzero* value for S_{top} signals the presence of nonlocal correlations and topological order.

The universality of S_{top} can also be understood from this picture. Small deformations of Ψ will typically modify the form of the string operators W_{fat} and change their width l . However, as long as l remains finite, $(S_1 - S_2) - (S_3 - S_4)$ will converge to the same value when the width r of the annular region is larger than l .

4.3 A simple example

Let us compute the topological entropy of the ground state Ψ of the Z_2 model and confirm (4.1) in this case. We will first compute the entanglement entropy S_R for an arbitrary region R . To make the boundary more symmetric, we split the sites on the boundary links into two sites (see Fig. 4-4). The wave function Ψ generalizes to the new lattice in the natural way.

We will decompose Ψ into $\Psi = \sum_l \Psi_l^{\text{in}} \Psi_l^{\text{out}}$ where Ψ_l^{in} are wave functions of spins

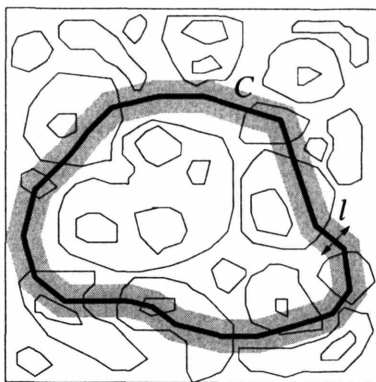


Figure 4-3: The state Ψ contains nonlocal correlations originating from the fact that strings always cross a curve C an even number of times. These correlations can be measured by a string operator $W(C)$ (blue curve). For more general states, a fattened string operator $W_{\text{fat}}(C)$ (blue region) is necessary.

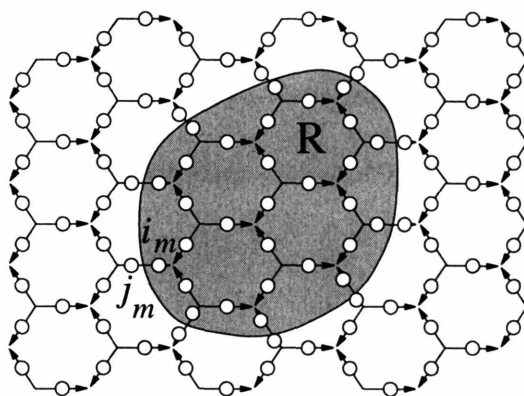


Figure 4-4: A simply connected region R in the honeycomb lattice. We split the sites on the boundary links into two sites labeled i_m and j_m , where $m = 1, \dots, n$.

inside R , Ψ_l^{out} are wave functions of spins outside R , and l is a dummy index. A simple decomposition can be obtained using the string picture. For any q_1, \dots, q_n , with $q_m = 0, 1$, and $\sum_m q_m$ even, we can define a wave function $\Psi_{q_1, \dots, q_n}^{\text{in}}$ on the spins inside of R : $\Psi_{q_1, \dots, q_n}^{\text{in}}(X) = 1$ if (a) the strings in X form closed loops and (b) X satisfies the boundary condition that there is a string on i_m if $q_m = 1$, and no string if $q_m = 0$. Similarly, we can define a set of wave functions $\Psi_{r_1, \dots, r_n}^{\text{out}}$ on the spins outside of R .

If we glue Ψ^{in} and Ψ^{out} together - setting $q_m = r_m$ for all m - the result is Ψ . Formally, this means that

$$\Psi = \sum_{q_1 + \dots + q_n \text{ even}} \Psi_{q_1, \dots, q_n}^{\text{in}} \Psi_{q_1, \dots, q_n}^{\text{out}} \quad (4.3)$$

It is not hard to see that the functions $\{\Psi_{q_1, \dots, q_n}^{\text{in}} : \sum_m q_m \text{ even}\}$, and $\{\Psi_{r_1, \dots, r_n}^{\text{out}} : \sum_m r_m \text{ even}\}$ are orthonormal. Therefore, the density matrix for the region R is an

equal weight mixture of all the $\{\Psi_{q_1, \dots, q_n}^{\text{in}} : \sum_m q_m \text{ even}\}$. There are 2^{n-1} such states. The entropy is therefore $S_R = (n - 1) \log 2$. [71]

This formula applies to simply connected regions like the one in Fig. 4-4. The same argument can be applied to general regions R and leads to $S_R = (n - j) \log 2$, where n is the number of spins along ∂R , and j is the number of disconnected boundary curves in ∂R .

We are now ready to calculate the topological entropy associated with Ψ . According to (4.1) we need to calculate the entropy of the four regions shown in Fig. 4-1. From $S_R = (n - j) \log 2$, we find

$$\begin{aligned} S_1 &= (n_1 - 2) \log 2 \\ S_2 &= (n_2 - 1) \log 2 \\ S_3 &= (n_3 - 1) \log 2 \\ S_4 &= (n_4 - 2) \log 2 \end{aligned}$$

where n_1, n_2, n_3, n_4 are the number of spins along the boundaries of the four regions. The topological entropy is therefore

$$-S_{\text{top}} = (n_1 - n_2 - n_3 + n_4 - 2) \log 2$$

But the four regions are chosen such that $(n_1 - n_2) = (n_3 - n_4)$. Thus the size dependent factor cancels out and

$$-S_{\text{top}} = -2 \log 2 = -\log(2^2)$$

This agrees with (4.1) since the total quantum dimension of Z_2 gauge theory is $D = 2$.

4.4 General string-net models

To derive (4.1) in the general (parity invariant) case, we compute the the topological entropy for the exactly soluble string-net models discussed in chapter 3. The ground states of these models describe all $(2 + 1)$ dimensional parity invariant topological orders. Recall that the models and the associated topological orders are characterized by several pieces of data: (a) An integer N - the number of string types. (b) A completely symmetric tensor δ_{ijk} where $i, j, k = 0, 1, \dots, N$ and δ_{ijk} only takes on the values 0 or 1. This tensor represents the branching rules: three string types i, j, k are allowed to meet at a point if and only if $\delta_{ijk} = 1$. (c) A dual string type i^* corresponding to each string type i . This dual string type corresponds to the same string, but with the opposite orientation. (d) A real tensor d_i and a complex tensor

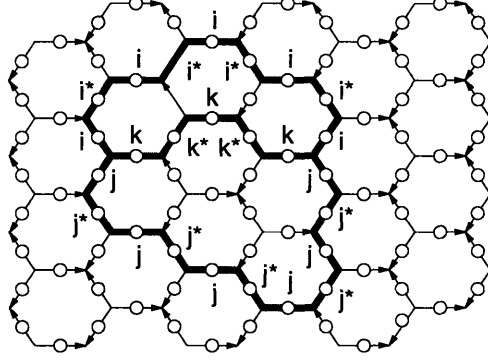


Figure 4-5: A typical string-net state on the honeycomb lattice. The empty links correspond to spins in the $i = 0$ state. The orientation conventions on the links are denoted by arrows.

F_{kln}^{ijm} satisfying the algebraic relations

$$\begin{aligned}
 F_{j^*i^*0}^{ijk} &= \frac{v_k}{v_i v_j} \delta_{ijk} \\
 F_{kln}^{ijm} &= F_{jin}^{lkm^*} = F_{lkn^*}^{jim} = F_{k^*nl}^{imj} \frac{v_m v_n}{v_j v_l} \\
 \sum_{n=0}^N F_{kp^*n}^{mlq} F_{mns^*}^{jip} F_{lkr^*}^{js^*n} &= F_{q^*kr^*}^{jip} F_{mns^*}^{riq^*} \\
 (F_{kln}^{ijm})^* &= F_{k^*l^*n^*}^{i^*j^*m^*}
 \end{aligned} \tag{4.4}$$

where v_i is defined by $v_i = v_{i^*} = \sqrt{d_i}$. For each set of $(F_{kln}^{ijm}, d_i, \delta_{ijk})$ satisfying these relations, there is a corresponding exactly soluble topologically ordered spin model.

The spins in the model are located on the links \mathbf{k} of the honeycomb lattice. However, the spins are not usual spin-1/2 spins. Each spin can be in $N + 1$ different states which we will label by $i = 0, 1, \dots, N$. The Hamiltonian of the model involves a 12 spin interaction. The model is known to be gapped and topologically ordered and all the relevant quantities - ground state degeneracies, quasiparticle statistics, etc., can be calculated explicitly (see chapter 3).

The ground state wave function Φ is also known exactly. It is easiest to describe using the string-net language. One first needs to pick an orientation for each link on the honeycomb lattice. When a spin is in state i , we think of the link as being occupied by a type- i string oriented in the appropriate direction. If a spin is in state $i = 0$, then we think of the link as empty. In this way spin states correspond to string-net states (see Fig. 4-5).

If a spin configuration $\{i_{\mathbf{k}}\}$ corresponds to an invalid string-net configuration - that is, a string-net configurations that doesn't obey the branching rules defined by δ_{ijk} - then $\Phi(\{i_{\mathbf{k}}\}) = 0$. On the other hand, if $\{i_{\mathbf{k}}\}$ corresponds to a valid string-net configuration then the amplitude is in general nonzero. What are these amplitudes?

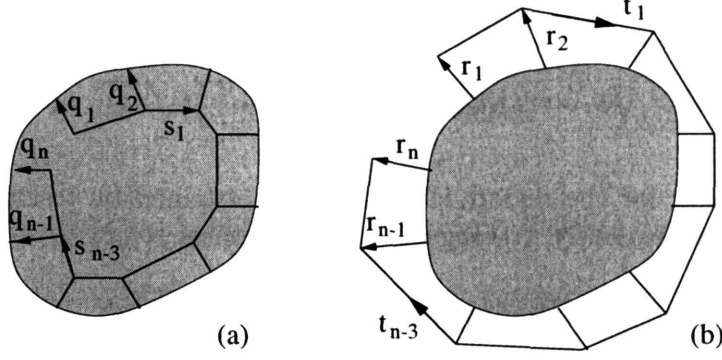


Figure 4-6: The basic string-net configurations (a) $X_{\{q,s\}}$ for inside R and (b) $Y_{\{r,t\}}$ for outside R .

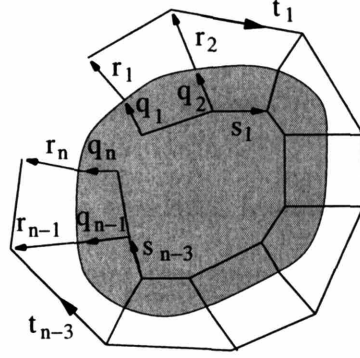


Figure 4-7: The string-net configuration $Z_{\{q,s,r,t\}}$ obtained by “gluing” the configuration $X_{\{q,s\}}$ to $Y_{\{r,t\}}$.

where the $Y_{\{r,t\}}$ are shown in Fig. 4-6(b).

Now consider the product wave functions $\Phi_{\{q,s\}}^{\text{in}} \Phi_{\{r,t\}}^{\text{out}}$. These are wave functions on the all the spins in the system - both inside and outside R . They can be generated from the amplitudes for the string-net configurations $Z_{\{q,s,r,t\}}$ in Fig. 4-7:

$$\Phi_{\{q,s\}}^{\text{in}} \Phi_{\{r,t\}}^{\text{out}}(Z_{\{q',s',r',t'\}}) = \delta_{\{q\},\{q'\}} \delta_{\{s\},\{s'\}} \delta_{\{r\},\{r'\}} \delta_{\{t\},\{t'\}}$$

On the other hand, it is not hard to show that for the ground state wave function Φ , the amplitude for $Z_{\{q,s,r,t\}}$ is

$$\Phi(Z_{\{q,s,r,t\}}) = \delta_{\{q\},\{r\}} \delta_{\{s\},\{t\}} \prod_m (\sqrt{d_{q_m}})$$

Comparing the two, we see that

$$\Phi = \sum_{\{q,s,r,t\}} \Phi_{\{q,s\}}^{\text{in}} \Phi_{\{r,t\}}^{\text{out}} \delta_{\{q\},\{r\}} \delta_{\{s\},\{t\}} \prod_m (\sqrt{d_{q_m}}) \quad (4.9)$$

It turns out that the wave functions $\{\Phi_{\{q,s\}}^{\text{in}}\}$ are orthonormal, as are the $\{\Phi_{\{r,t\}}^{\text{out}}\}$ (see appendix A.6). This means that we can use them as a basis. If we denote

$\Phi_{\{q,s\}}^{\text{in}} \Phi_{\{r,t\}}^{\text{out}}$ by $|\{q, s, r, t\}\rangle$, then in this basis, the wave function Φ is

$$\langle\{q, s, r, t\}|\Phi\rangle = \delta_{\{q\},\{r\}}\delta_{\{s\},\{t\}} \prod_m (\sqrt{d_{q_m}}) \quad (4.10)$$

The density matrix for the region R can now be obtained by tracing out the spins outside of R , or equivalently, tracing out the spin states $|\{r, t\}\rangle$:

$$\langle\{q', s'\}|\rho_R|\{q, s\}\rangle = \delta_{\{q\},\{q'\}}\delta_{\{s\},\{s'\}} \prod_m d_{q_m} \quad (4.11)$$

To normalize ρ_R , we need to compute its trace. Note that

$$\begin{aligned} \text{Tr}(\rho_R) &= \sum_{\{q,s\}} \prod_m d_{q_m} \\ &= \sum_{\{q\}} N_{\{q\}} \prod_m d_{q_m} \end{aligned}$$

where $N_{\{q\}}$ is the number of allowed labelings of $\{s\}$ for a given $\{q\}$. Using the expression

$$N_{\{q\}} = \sum_{\{s\}} \delta_{q_1 q_2 s_1^*} \delta_{s_1 q_3 s_2^*} \cdots \delta_{s_{n-3} q_{n-1} q_n}$$

and the identity $\sum_i d_i \delta_{ijk} = d_j d_k$, we find

$$\text{Tr}(\rho_R) = \sum_{\{q\}} N_{\{q\}} \prod_m d_{q_m} = D^{n-1} \quad (4.12)$$

where $D = \sum_k d_k^2$. Thus, the normalized density matrix ρ_R is

$$\langle\{q', s'\}|\rho_R|\{q, s\}\rangle = \delta_{\{q\},\{q'\}}\delta_{\{s\},\{s'\}} \frac{\prod_m d_{q_m}}{D^{n-1}} \quad (4.13)$$

Since the density matrix is diagonal, we can easily obtain the entanglement entropy for S_R . Taking $-\text{Tr} \rho_R \log \rho_R$, we find

$$\begin{aligned} S_R &= - \sum_{\{q,s\}} \frac{\prod_m d_{q_m}}{D^{n-1}} \log \left(\frac{\prod_l d_{q_l}}{D^{n-1}} \right) \\ &= - \sum_{\{q\}} \frac{N_{\{q\}} \prod_m d_{q_m}}{D^{n-1}} \log \left(\frac{\prod_l d_{q_l}}{D^{n-1}} \right) \end{aligned}$$

Expanding and rearranging, the expression simplifies to

$$S_R = (n-1) \log(D) - \sum_{\{q\}} \frac{N_{\{q\}} \prod_m d_{q_m}}{D^{n-1}} \sum_l \log(d_{q_l})$$

Reversing the order of the two summations,

$$\begin{aligned}
S_R &= (n-1) \log(D) - \sum_l \sum_{\{q\}} \log(d_{q_l}) \cdot \frac{N_{\{q\}} \prod_m d_{q_m}}{D^{n-1}} \\
&= (n-1) \log(D) - \sum_l \sum_{q_l=0}^N \frac{d_{q_l}}{D^{n-1}} \log(d_{q_l}) \\
&\quad \cdot \sum_{\{q_m: m \neq l\}} N_{\{q\}} \prod_{m \neq l} d_{q_m} \\
&= (n-1) \log(D) - \sum_l \sum_{q_l=0}^N \frac{d_{q_l}}{D^{n-1}} \log(d_{q_l}) \cdot d_{q_l} D^{n-2} \\
&= (n-1) \log(D) - \sum_l \sum_{q_l=0}^N \frac{d_{q_l}^2}{D} \log(d_{q_l}) \\
&= (n-1) \log(D) - n \sum_{k=0}^N \frac{d_k^2}{D} \log(d_k)
\end{aligned}$$

where the third equality follows from the same manipulations as (4.12). Rewriting this so that the finite size correction is manifest, we derive

$$S_R = -\log(D) - n \sum_{k=0}^N \frac{d_k^2}{D} \log\left(\frac{d_k}{D}\right) \quad (4.14)$$

This result applies to simply connected regions like the one shown in Fig. 4-1. The same argument can be applied to general regions R . In the general case, we find

$$S_R = -j \log(D) - n \sum_{k=0}^N \frac{d_k^2}{D} \log\left(\frac{d_k}{D}\right) \quad (4.15)$$

where n is the number of spins along ∂R , and j is the number of disconnected boundary curves in ∂R .

We can now calculate the topological entropy associated with Φ . Applying (4.15), we find

$$\begin{aligned}
S_1 &= -2 \log D - n_1 s_0 \\
S_2 &= -\log D - n_2 s_0 \\
S_3 &= -\log D - n_3 s_0 \\
S_4 &= -2 \log D - n_4 s_0
\end{aligned}$$

where n_1, n_2, n_3, n_4 are the numbers of spins along the boundaries of the four regions,

and

$$s_0 = \sum_{k=0}^N \frac{d_k^2}{D} \log \left(\frac{d_k}{D} \right)$$

The topological entropy is therefore

$$-S_{\text{top}} = -2 \log D + (n_1 - n_2 - n_3 + n_4) s_0 = -2 \log D$$

in agreement with (4.1). This establishes (4.1) for the exactly soluble string-net models.

To prove (4.1) more generally we appeal to universality. By the argument presented in section 4.2, the topological entropy is universal throughout each quantum phase. Therefore, the previous result implies that (4.1) holds for *general* string-net condensed states, not just the above exactly soluble points. Since string-net condensation describes all non-chiral topological phases, this establishes (4.1) for general non-chiral topological orders. We believe (4.1) also holds for chiral topological orders. This was established using different methods in the recent paper [70].

Chapter 5

Conclusion

5.1 Summary of results

In this thesis, we have shown that the string-net picture can provide many of the components of a Landau-like framework for topological phases. In particular, we have shown that it can provide a physical picture and mathematical framework for these phases (see Fig. 5-1).

The physical picture we have presented was based on the phenomenon of “string-net condensation.” If a spin model happens to have local energetics which favor string-like configurations of spins, and these string-like configurations become highly fluctuating and condense, the spin model can enter a string-net condensed phase. These string-net condensed phases are naturally non-chiral topological phases. Thus, string-net condensation provides a physical picture for non-chiral topological phases - analogous in many ways to the particle condensation picture for ordered phases. We hope that this physical picture may help direct the search for topological phases in frustrated quantum magnets. It may also be useful in the search for non-topological exotic phases.

We have also shown that string-net condensation can give a mathematical framework for topological phases. This framework is tensor category theory. We have shown that each $(2 + 1)$ dimensional non-chiral topological phase is associated with a tensor category - a 6 index object F_{kln}^{ijm} and a set of real numbers d_i satisfying the algebraic relations (3.5). All the universal properties of the topological phase are contained in the associated tensor category. In particular, the tensor category directly determines the quasiparticle statistics of the associated topological phase (3.24, 3.26). Thus tensor categories can be used to classify non-chiral topological phases in the same way that groups can be used to classify ordered phases.

We have constructed exactly soluble spin Hamiltonians (3.7) that demonstrate this physical picture and mathematical framework. These models are very general and can realize all non-chiral topological phases. Their generality comes at a price, however - the models require 12 spin interactions.

These results were derived for string-net condensation in $(2 + 1)$ dimensions. We have also investigated the consequences of string-net condensation in $(3 + 1)$ dimen-

Physical characterization	Degenerate gd. states Fractional statistics Topological entropy
Low energy effective theory	Topological quantum field theory
Physical picture	String-net condensation
Mathematical framework	Tensor category theory Orders $\longleftrightarrow (d, F_{kln}^{ijm})$

Figure 5-1: The four basic components of the theory of topological phases. In this thesis, we have filled in the bottom two boxes. We have shown that the physical picture for topological phases is string-net condensation, while the mathematical framework is tensor category theory. We have also shown that topological phases are characterized by nonzero topological entanglement entropy.

sions. In that case, we found that string-net condensation naturally gives rise to gauge theories coupled to bosonic or fermionic charges. Thus, string-net condensation provides a mechanism for unifying Fermi statistics and gauge interactions. This result may have implications for high energy physics. [14, 15, 17, 16]

Finally, we have shown that topological order can be detected in the wave function just like normal order. Instead of being encoded in local correlation functions, topological order is encoded in nonlocal correlations or entanglement. We have shown that a quantity called “topological entropy” can measure this nonlocal entanglement. In addition to its applications to numerical studies, this result is important conceptually. It gives a deeper understanding of what topological order actually *is*. Topological order is fundamentally a kind of non-local quantum entanglement. To move beyond the Landau paradigm, we need to develop new tools to handle this entanglement.

5.2 New directions

While it may seem that our theoretical framework for topological phases is complete, there are still many missing pieces.

First, and perhaps most importantly we are missing a crucial component of Landau theory: mean field theory. We do not have a good mean field approach for exotic phases (since the slave particle formalism is likely unreliable). Because of this, analytical results have been restricted to special exactly soluble points. We cannot make predictions about where topological phases and exotic phases should occur in real materials. We cannot bridge the gap between theory and experiment. Thus, a good mean field approach would represent a breakthrough in the theory of exotic phases. A natural direction for research would be to develop such a mean field theory using the

string-net picture. Such a mean field theory would be particularly well suited to spin models whose low energy physics is described by strings or dimers. More generally, if one could combine this approach with projected entangled pair states - a recent development in quantum information theory [74] - one might be able to develop a very general mean field technique for frustrated quantum magnets.

Another interested direction involves the question of phase transitions. Recall that Landau theory is not only a theory of phases, but also a theory of phase transitions. It would be useful to have a similar framework for phase transitions in topological phases. At the very least, one should be able to predict what pairs of topological phases can be separated by a second order phase transition.

Yet another direction would be to generalize the string-net picture to chiral topological phases. At the moment, the string-net picture only works for non-chiral phases. Only these phases can occur when extended objects condense. However, there must be a related picture that can explain chiral phases. This would be particularly useful given the fact that the quantum Hall states are chiral.

Also, the string-net picture is not the only picture for exotic phases in spin systems. Another way of constructing exotic states is the slave particle approach. In this approach, the spin operator is written as a fermion or boson bilinear and the spin Hamiltonian is mapped onto a fermion or boson Hamiltonian where the fermions or bosons are coupled via a gauge field. The slave particle approach and the string-net approach must be connected, but at the moment the precise connection is missing. Where are the strings in the slave particle states? This is an open question.

Finally, there are a number of new directions related to the concept of topological entropy. Topological entropy could potentially be a useful numerical tool for detecting topological order. A natural problem to consider is the $J_1 - J_3$ model on the Heisenberg lattice. This model is suspected of being topologically ordered in certain parameter regimes, but current methods can only probe symmetry-breaking order in the ground state. [35, 34] It would be interesting to compute the topological entropy for this model. This would be the first example of topological order in an $SU(2)$ invariant system. In fact, it would be the first example of a spin liquid.

A related direction is whether there exist concepts analogous to topological entropy for gapless exotic phases. Can one detect gapless quantum order in a wave function? In general, to what extent are dynamics encoded in ground state wave functions?

From a more general point of view, all of the phases described by Landau's symmetry breaking theory can be understood in terms of particle condensation. These phases are classified using group theory and lead to emergent gapless scalar bosons [75, 76], such as phonons, spin waves, *etc*. In this thesis, we have shown that there is a much richer class of phases - arising from the condensation of extended objects. These phases are classified using tensor category theory and lead to emergence of anyons, fermions, and gauge interactions. Clearly, there is whole new world beyond the paradigm of symmetry breaking and long range order. It is a virgin land waiting to be explored.

Appendix A

Mathematics of string-net condensation

A.1 General string-net models

In this section, we discuss the most general string-net models. These models can describe all doubled topological phases, including all discrete gauge theories and doubled Chern-Simons theories.

In these models, there is a “spin” degree of freedom at each branch point or node of a string-net, in addition to the usual string-net degrees of freedom. The dimension of this “spin” Hilbert space depends on the string types of the 3 strings incident on the node.

To specify a particular model one needs to provide a 3 index tensor δ_{ijk} which gives the dimension of the spin Hilbert space associated with $\{i, j, k\}$ (in addition to the usual information). The string-net models discussed in the body of the thesis correspond to the special case where $\delta_{ijk} = 0, 1$ for all i, j, k . (To get intuition about the more general δ_{ijk} note that in the case of gauge theory, δ_{ijk} is the number of copies of the trivial representation that appear in the tensor product $i \otimes j \otimes k$. Thus we need the more general string-net picture to describe gauge theories where the trivial representation appears multiple times in $i \otimes j \otimes k$).

The Hilbert space of the string-net model is defined in the natural way: the states in the string-net Hilbert space are linear superpositions of different spatial configurations of string-nets with different spin states at the nodes.

One can analyze string-net condensed phases as before. The universal properties of each phase are captured by a fixed-point ground state wave function Φ . The wave function Φ is specified by the local rules (3.1), (3.2) and simple modifications of (3.3), (3.4):

$$\begin{aligned} \Phi \left(\begin{array}{c} \sigma \quad k \quad \tau \\ \text{---} \circ \text{---} \\ i \quad | \quad j \end{array} \right) &= \delta_{ij} \delta_{\sigma\tau} \Phi \left(\begin{array}{c} \sigma \quad k \quad \sigma \\ \text{---} \circ \text{---} \\ i \quad | \quad i \end{array} \right) \\ \Phi \left(\begin{array}{c} i \quad | \quad l \\ \text{---} \times \text{---} \\ j \quad \sigma \quad \tau \quad k \end{array} \right) &= \sum_{n\mu\nu} (F_{kln}^{ijm})_{\mu\nu}^{\sigma\tau} \Phi \left(\begin{array}{c} i \quad | \quad \mu \quad l \\ \text{---} \times \text{---} \\ j \quad \sigma \quad \nu \quad k \end{array} \right) \end{aligned}$$

The complex numerical constant F_{kln}^{ijm} is now a complex tensor $(F_{kln}^{ijm})_{\mu\nu}^{\sigma\tau}$ of dimension $\delta_{ijm} \times \delta_{klm^*} \times \delta_{inl} \times \delta_{jkn^*}$.

One can proceed as before, with self-consistency conditions, fixed-point Hamiltonians, string operators, and the generalization to $(3 + 1)$ dimensions. The exactly soluble models are similar to (3.7). The main difference is the existence of an additional spin degree of freedom at each site of the honeycomb lattice. These spins account for the degrees of freedom at the nodes of the string-nets.

A.2 Argument for local constraint equations

In this section we give a heuristic argument for why fixed point wave functions can be described as the unique solution of local constraint equations on string-net wave functions.

Suppose Φ is some fixed-point wave function. We know that Φ is the ground state of some fixed-point Hamiltonian H . Based on our experience with gauge theories, we expect that H is a zero coupling fixed point. That is, H contains no string tension terms. This means that H is simply a sum of local kinetic energy terms, $H_{t,i}$:

$$H = tH_t = t \sum_i H_{t,i}$$

We expect that all the kinetic energy terms commute with each other. Thus, H is completely unfrustrated, and the ground state wave function minimizes the expectation values of all the kinetic energy terms $\{H_{t,i}\}$ simultaneously. Minimizing the expectation value of an individual kinetic energy term $H_{t,i}$ is equivalent to imposing a local constraint on the ground state wave function, namely $H_{t,i}|\Phi\rangle = E_i|\Phi\rangle$ (where E_i is the smallest eigenvalue of $H_{t,i}$). We conclude that the wave function Φ can be specified uniquely by local constraint equations. The local constraints are linear relations between several string-net amplitudes $\Phi(X_1), \Phi(X_2), \Phi(X_3)\dots$ where the configurations $X_1, X_2, X_3\dots$ only differ by local transformations (e.g. transformations generated by the local kinetic energy operators $H_{t,i}$).

A.3 Self-consistency conditions

In this section, we derive the self-consistency conditions (3.5). We begin with the last relation, the so-called ‘‘pentagon identity’’, since it is the most fundamental. To derive this condition, we use the fusion rule (3.4) to relate the amplitude $\Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagdown \\ i \quad r \quad s \quad m \end{array} \right)$

to the amplitude $\Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagdown \\ i \quad r \quad s \quad m \end{array} \right)$ in two distinct ways (see Fig. A-1). On the one hand,

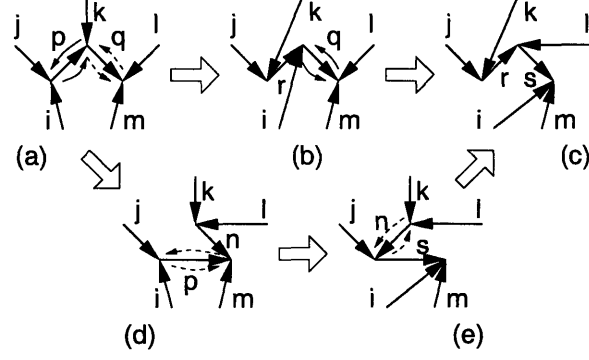


Figure A-1: The fusion rule (3.4) can be used to relate the amplitude of (a) to the amplitude of (c) in two different ways. On the one hand, we can apply the fusion rule (3.4) twice - along the links denoted by solid arrows - to relate (a) \rightarrow (b) \rightarrow (c). But we can also apply (3.4) three times - along the links denoted by dashed arrows - to relate (a) \rightarrow (d) \rightarrow (e) \rightarrow (c). Self-consistency requires that the two sequences of the operation lead to the same linear relations between the amplitudes of (a) and (c).

we can apply the fusion rule (3.4) twice to obtain the relation

$$\begin{aligned} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ p \quad q \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) &= \sum_r F_{q^*kr^*}^{jip} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ r \quad q \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) \\ &= \sum_{r,s} F_{q^*kr^*}^{jip} F_{m^*ls^*}^{riq^*} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ r \quad s \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) \end{aligned}$$

(Here, we neglected to draw a shaded region surrounding the whole diagram. Just as in the local rules (3.1-3.4) the ends of the strings i, j, k, l, m are connected to some arbitrary string-net configuration). But we can also apply the fusion rule (3.4) three times to obtain a different relation:

$$\begin{aligned} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ p \quad q \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) &= \sum_n F_{kp^*n}^{mlq} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ n \quad q \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) \\ &= \sum_{n,s} F_{kp^*n}^{mlq} F_{m^*ns^*}^{jip} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ n \quad s \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) \\ &= \sum_{n,r,s} F_{kp^*n}^{mlq} F_{m^*ns^*}^{jip} F_{lkr^*}^{js^*n} \Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ r \quad s \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right) \end{aligned}$$

If the rules are self-consistent, then these two relations must agree with each other. Thus, the two coefficients of $\Phi \left(\begin{array}{c} j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagup \\ r \quad s \\ \diagup \quad \diagdown \\ i \quad m \end{array} \right)$ must be the same. This equality implies the pentagon identity (3.5).

The first two relations in (3.5) are less fundamental. In fact, the first relation is

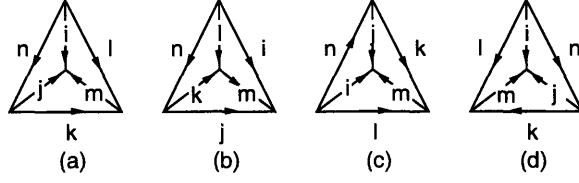


Figure A-2: Four string-net configurations related by tetrahedral symmetry. In diagram (a), we show the tetrahedron corresponding to G_{kln}^{ijm} . In diagrams (b), (c), (d), we show the tetrahedrons $G_{jin}^{lkm^*}$, $G_{lkn^*}^{jim}$, $G_{k^*nl}^{imj}$, obtained by reflecting (a) in 3 different planes: the plane joining n to the center of m , the plane joining m to the center of n , and the plane joining i to the center of k . The four tetrahedrons correspond to the four terms in the second relation of (3.5).

not required by self-consistency at all; it is simply a useful convention. To see this, consider the following rescaling transformation on wave functions $\Phi \rightarrow \tilde{\Phi}$. Given a string-net wave function Φ , we can obtain a new wave function $\tilde{\Phi}$ by multiplying the amplitude $\Phi(X)$ for a string-net configuration X by an arbitrary factor $f(i, j, k)$ for each vertex $\{i, j, k\}$ in X . As long as $f(i, j, k)$ is symmetric in i, j, k and $f(0, i, i^*) = 1$, this operation preserves the topological invariance of Φ . The rescaled wave function $\tilde{\Phi}$ satisfies the same set of local rules with rescaled F_{kln}^{ijm} :

$$F_{kln}^{ijm} \rightarrow \tilde{F}_{kln}^{ijm} = F_{kln}^{ijm} \frac{f(i, j, m)f(k, l, m^*)}{f(n, l, i)f(j, k, n^*)} \quad (\text{A.1})$$

Since Φ and $\tilde{\Phi}$ describe the same quantum phase, we regard F and \tilde{F} as equivalent local rules. Thus the first relation in (3.5) is simply a normalization convention for F or Φ (except when i, j or k vanishes; these cases require an argument similar to the derivation of the pentagon identity).

The second relation in (3.5) has more content. This relation can be derived by computing the amplitude for a tetrahedral string-net configuration. We have:

$$\begin{aligned} \Phi \left(\begin{array}{c} \triangle \\ \begin{array}{ccc} n & i & l \\ \diagdown & | & / \\ j & m & \\ \diagup & & \diagdown \\ k & & \end{array} \end{array} \right) &= F_{kln}^{ijm} \Phi \left(\begin{array}{c} \square \\ \begin{array}{ccc} n & & \\ \diagdown & | & / \\ j & m & \\ \diagup & & \diagdown \\ n & & \end{array} \end{array} \right) \\ &= F_{kln}^{ijm} F_{kn^*0}^{nk^*j^*} d_k \Phi \left(\begin{array}{c} \square \\ \begin{array}{ccc} & & \\ \diagdown & | & / \\ & i & \\ \diagup & & \diagdown \\ n & & \end{array} \end{array} \right) \\ &= F_{kln}^{ijm} F_{kn^*0}^{nk^*j^*} F_{in0}^{n^*i^*l^*} d_k d_i d_n \Phi(\emptyset) \quad (\text{A.2}) \\ &= F_{kln}^{ijm} v_i v_j v_k v_l \Phi(\emptyset) \quad (\text{A.3}) \end{aligned}$$

We define the above combination in the front of $\Phi(\emptyset)$ as:

$$G_{kln}^{ijm} \equiv F_{kln}^{ijm} v_i v_j v_k v_l \quad (\text{A.4})$$

Imagine that the above string-net configuration lies on a sphere. In that case, topological invariance (together with parity invariance) requires that G_{kln}^{ijm} be invariant

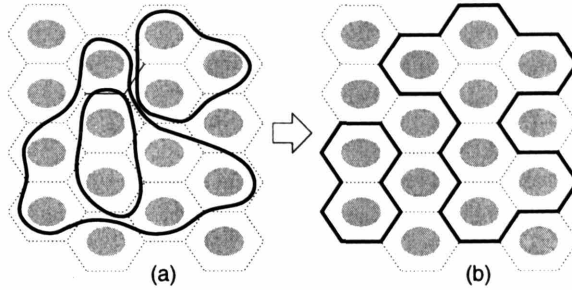


Figure A-3: The fattened honeycomb lattice. The strings are forbidden in the shaded region. A string state in the fattened honeycomb lattice (a) can be viewed as a superposition of string states on the links (b).

under all 24 symmetries of a regular tetrahedron. The second relation in (3.5) is simply a statement of this tetrahedral symmetry requirement - written in terms of F_{kln}^{ijm} . (See Fig. A-2).

In this section, we have shown that the relations (3.5) are necessary for self-consistency. It turns out that these relations are also sufficient. One way of proving this is to use the lattice model (3.7). A straightforward algebraic calculation shows that the ground state of (3.7) obeys the local rules (3.1-3.4), as long as (3.5) is satisfied. This establishes that the local rules are self-consistent.

A.4 Graphical representation of the Hamiltonian

In this section, we provide an alternative, graphical, representation of the lattice model (3.7). This graphical representation provides a simple visual technique for understanding properties (a)-(c) of the Hamiltonian (3.7).

We begin with the 2D honeycomb lattice. Imagine we fatten the links of the lattice into stripes of finite width (see Fig. A-3). Then, any string-net state in the fattened honeycomb lattice (Fig. A-3a) can be viewed as a superposition of string-net states in the original, unfattened lattice (Fig. A-3b). This mapping is obtained via the local rules (3.1-3.4). Using these rules, we can relate the amplitude $\Phi(X)$ for a string-net in the fattened lattice to a linear combination of string-net amplitudes in the original lattice: $\Phi(X) = \sum a_i \Phi(X_i)$. This provides a natural linear relation between the states in the fattened lattice and those in the unfattened lattice: $|X\rangle = \sum a_i |X_i\rangle$. This linear relation is independent of the particular way in which the local rules (3.1-3.4) are applied, as long as the rules are self-consistent.

In this way, the fattened honeycomb lattice provides an alternative notation for representing the states in the Hilbert space of (3.7). This notation is useful because the magnetic energy operators $B_{\mathbf{p}}^s$ are simple in this representation. Indeed, the action

of the operator $B_{\mathbf{p}}^s$ on the string-net state $\left| \begin{array}{c} \text{a} \quad \text{b} \quad \text{h} \quad \text{c} \\ \text{g} \quad \text{f} \quad \text{i} \quad \text{d} \\ \text{e} \quad \text{j} \quad \text{k} \quad \text{o} \end{array} \right\rangle$ is equivalent to simply adding

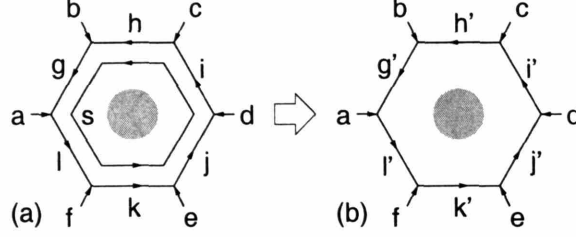


Figure A-4: The action of B_p^s is equivalent to adding a loop of type- s string. The resulting string-net state (a) is actually a linear combination of the string-net states (b). The coefficients in this linear relation can be obtained by using the local rules (3.1-3.4) to reduce (a) to (b).

a loop of type- s string:

$$B_p^s \left| \begin{array}{c} b \quad h \quad c \\ g \quad \quad i \\ a \quad \quad d \\ l \quad \quad j \\ f \quad k \quad e \end{array} \right\rangle = \left| \begin{array}{c} b \quad h \quad c \\ g' \quad \quad i' \\ a \quad \quad d \\ l' \quad \quad j' \\ f \quad k' \quad e \end{array} \right\rangle$$

As we described above, we can use the local rules (3.1-3.4) to rewrite $\left| \begin{array}{c} b \quad h \quad c \\ g \quad \quad i \\ a \quad \quad d \\ l \quad \quad j \\ f \quad k \quad e \end{array} \right\rangle$ as

a linear combination of the physical string-net states with strings only on the links, that is to reduce Fig. A-4a to Fig. A-4b. This allows us to obtain the matrix elements of B_p^s .

The following is a particular way to implement the above procedure:

$$\begin{aligned}
B_p^s \left| \begin{array}{c} b \quad h \quad c \\ g \quad \quad i \\ a \quad \quad d \\ l \quad \quad j \\ f \quad k \quad e \end{array} \right\rangle &= \left| \begin{array}{c} b \quad h \quad c \\ g \quad \quad i \\ a \quad \quad d \\ l \quad \quad j \\ f \quad k \quad e \end{array} \right\rangle = \sum_{g'h'i'j'k'l'} F_{s^*sg'^*}^{gg^*0} F_{s^*sh'^*}^{hh^*0} F_{s^*si'^*}^{ii^*0} F_{s^*sj'^*}^{jj^*0} F_{s^*sk'^*}^{kk^*0} F_{s^*sl'^*}^{ll^*0} \left| \begin{array}{c} b \quad h \quad h' \quad c \\ g \quad \quad s \quad s \quad i \\ a \quad \quad s \quad s \quad d \\ l \quad \quad s \quad s \quad j \\ f \quad k \quad k' \quad e \end{array} \right\rangle \\
&= \sum_{g'h'i'j'k'l'} F_{s^*sg'^*}^{gg^*0} F_{s^*sh'^*}^{hh^*0} F_{s^*si'^*}^{ii^*0} F_{s^*sj'^*}^{jj^*0} F_{s^*sk'^*}^{kk^*0} F_{s^*sl'^*}^{ll^*0} F_{s^*h'g'^*}^{bg^*h} F_{s^*i'h'^*}^{ch^*i} F_{s^*j'i'^*}^{di^*j} F_{s^*k'j'^*}^{ej^*k} F_{s^*l'k'^*}^{fk^*l} F_{s^*g'l'^*}^{al^*g} \left| \begin{array}{c} b \quad h' \quad h' \quad c \\ g' \quad \quad s \quad s \quad i' \\ a \quad \quad s \quad s \quad d \\ l' \quad \quad s \quad s \quad j' \\ f \quad k' \quad k' \quad e \end{array} \right\rangle \\
&= \sum_{g'h'i'j'k'l'} F_{s^*h'g'^*}^{bg^*h} F_{s^*i'h'^*}^{ch^*i} F_{s^*j'i'^*}^{di^*j} F_{s^*k'j'^*}^{ej^*k} F_{s^*l'k'^*}^{fk^*l} F_{s^*g'l'^*}^{al^*g} \left| \begin{array}{c} b \quad h' \quad c \\ g' \quad \quad i' \\ a \quad \quad d \\ l' \quad \quad j' \\ f \quad k' \quad e \end{array} \right\rangle \tag{A.5}
\end{aligned}$$

Notice that (A.5) is exactly (3.10). Thus, the graphical representation of B_p^s agrees with the original algebraic definition.

Using the graphical representation of $B_{\mathbf{p}}^s$ we can easily show that $B_{\mathbf{p}_1}^{s_1}$ and $B_{\mathbf{p}_2}^{s_2}$ commute. The derivation is much simpler than the more straightforward algebraic calculation. First note that these operators will commute if $\mathbf{p}_1, \mathbf{p}_2$ are well-separated. Thus, we only have to consider the case where \mathbf{p}_1 and \mathbf{p}_2 are adjacent, or the case where $\mathbf{p}_1, \mathbf{p}_2$ coincide. We begin with the nearest neighbor case. The action of $B_{\mathbf{p}_1}^{s_1} B_{\mathbf{p}_2}^{s_2}$ on the string-net state Fig. A-5a can be represented as Fig. A-5b. Fig. A-5b can then be related to a linear combination of the string-net states shown in Fig. A-5c. The coefficients in this relation are the matrix elements of $B_{\mathbf{p}_1}^{s_1} B_{\mathbf{p}_2}^{s_2}$. But by the same argument, the action of $B_{\mathbf{p}_2}^{s_2} B_{\mathbf{p}_1}^{s_1}$ can also be represented by Fig. A-5b. We conclude that $B_{\mathbf{p}_2}^{s_2} B_{\mathbf{p}_1}^{s_1}, B_{\mathbf{p}_1}^{s_1} B_{\mathbf{p}_2}^{s_2}$ have the same matrix elements. Thus, the two operators commute in this case.

On the other hand, when $\mathbf{p}_1 = \mathbf{p}_2$, we have

$$\begin{aligned}
B_{\mathbf{p}}^{s_2} B_{\mathbf{p}}^{s_1} \left| \text{Hexagon} \right\rangle &= \left| \text{Hexagon with } s_1, s_2 \right\rangle \\
&= \sum_k F_{s_2^* s_2 k^*}^{s_1 s_1^* 0} \left| \text{Hexagon with } s_1, s_2, k \right\rangle \\
&= \sum_k F_{s_2^* s_2 k^*}^{s_1 s_1^* 0} F_{s_2^* k 0}^{k^* s_2 s_1} d_{s_2^*} \left| \text{Hexagon with } k \right\rangle \\
&= \sum_k \delta_{k^* s_2 s_1} \left| \text{Hexagon with } k \right\rangle
\end{aligned} \tag{A.6}$$

Thus,

$$B_{\mathbf{p}}^{s_2} B_{\mathbf{p}}^{s_1} = \sum_k \delta_{k^* s_2 s_1} B_{\mathbf{p}}^k. \tag{A.7}$$

Since $\delta_{k^* s_2 s_1}$ is symmetric in s_2, s_1 , we conclude that $B_{\mathbf{p}}^{s_1} B_{\mathbf{p}}^{s_2} = B_{\mathbf{p}}^{s_2} B_{\mathbf{p}}^{s_1}$, so the operators commute in this case as well. This establishes property (a) of the Hamiltonian (3.7).

Equation (A.7) also sheds light on the spectrum of the $B_{\mathbf{p}}^s$ operators. Let the simultaneous eigenvalues of $B_{\mathbf{p}}^s$ (with \mathbf{p} fixed) be $\{b_q^s\}$. Then, by (A.7) these eigenvalues satisfy

$$\sum_k \delta_{k^* s_2 s_1} b_q^k = b_q^{s_2} b_q^{s_1}$$

We can view this as an eigenvalue equation for the $(N+1) \times (N+1)$ matrix M_{s_2} , defined by $M_{s_2, j}^i = \delta_{j^* s_2 i}$. The simultaneous eigenvalues $b_q^{s_2}$ are simply the simultaneous eigenvalues of the matrices M_{s_2} . In particular, this means that the index q ranges over a set of size $N+1$.

Each value of q corresponds to a different possible state for the plaquette \mathbf{p} . The

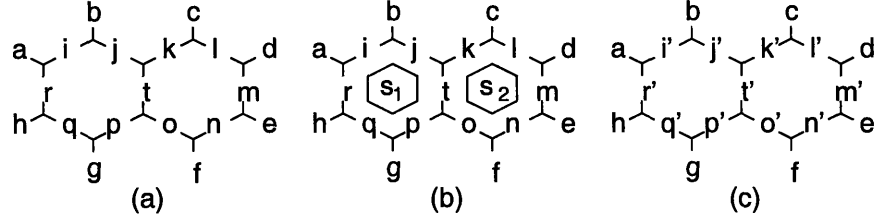


Figure A-5: The action of $B_{\mathbf{p}_1}^{s_1} B_{\mathbf{p}_2}^{s_2}$ on the string-net state (a) can be represented by adding two loops of type- s_1 and type- s_2 strings as shown in (b). The string-net state (b) is a linear combination of the string-net states (c). The coefficients are obtained by using (3.1 -3.4) to reduce (b) to (c).

magnetic energies of these $N + 1$ different states are given by: $E_q = -\sum_s a_s b_q^s$. Depending on the parameter choice a_s , all on the plaquettes \mathbf{p} will be in one of these states q . In this way, the Hamiltonian (3.7) can be in $N + 1$ different quantum phase. This establishes property (b) of the Hamiltonian (3.7).

One particular state q is particularly interesting. This state corresponds to the simultaneous eigenvalues $b^s = d_s$. It is not hard to show that the parameter choice $a_s = \frac{d_s}{\sum_k d_k^2}$ makes this state energetically favorable. In fact, using (A.7) one can show that $B_{\mathbf{p}}$ is a projector for this parameter choice, and that $B_{\mathbf{p}} = 1$ for this state.

Furthermore, the ground state wave function for this parameter choice obeys the local rules (3.1-3.4). One way to see this is to compare $B_{\mathbf{p}} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle$ with $B_{\mathbf{p}} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle$. For the first state, we find

$$\begin{aligned}
 B_{\mathbf{p}} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle &= \sum_s a_s \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle \\
 &= \sum_{j,s} \frac{d_s}{\sum_k d_k^2} F_{s^* s j^*}^{i i^* 0} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle \\
 &= \sum_{j,s} \frac{v_j v_s}{v_i \sum_k d_k^2} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle
 \end{aligned}$$

For the second state, we find the same result:

$$B_{\mathbf{p}} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle = \sum_{j,s} \frac{v_j v_s}{v_i \sum_k d_k^2} \left| i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \right\rangle$$

It follows that

$$\begin{aligned}
 0 &= \left\langle i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \left| B_{\mathbf{p}} \right| \Phi \right\rangle - \left\langle i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \left| B_{\mathbf{p}} \right| \Phi \right\rangle \\
 &= \left\langle i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \left| \Phi \right\rangle - \left\langle i \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} i \left| \Phi \right\rangle
 \end{aligned}$$

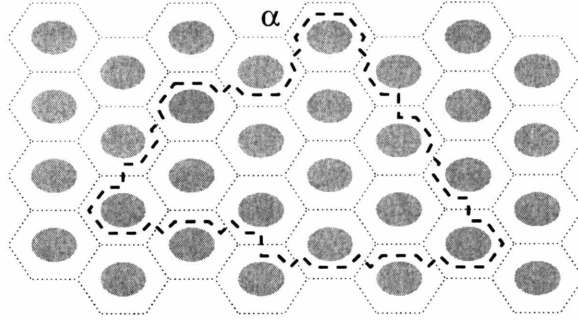


Figure A-6: The action of the string operator $W_\alpha(P)$ is equivalent to adding a type- α string along the path P . The resulting string-net state can be reduced to a linear combination of states on the honeycomb lattice, using the local rules (3.1-3.4), (A.8).

so

$$\Phi \left(i \text{---} \text{---} \text{---} i \right) = \Phi \left(i \text{---} \text{---} \text{---} i \right)$$

This result means that the strings can be moved through the forbidden regions at the center of the hexagons. Thus, the local rules which were originally restricted to the fattened honeycomb lattice can be extended throughout the entire $2D$ plane. The wave function Φ obeys these continuum local rules and has a smooth continuum limit. We call such a state smooth topological state. This establishes property (c) of the Hamiltonian (3.7).

We would like to mention that the wave functions of some smooth topological states are positive definite. So these wave functions can be viewed as the Boltzmann weights of statistical models in the same spatial dimension. What is interesting is that these statistical models are local models with short-ranged interactions [57, 58, 77].

A.5 Graphical representation of the string operators

In this section, we describe a graphical representation of the long string operators $W_\alpha(P)$. Just as in the previous section, this representation involves the fattened honeycomb lattice. The action of the string operator $W_\alpha(P)$ on a general string state X , is simply to create a string labeled α along the path P (see Fig. A-6). The resulting string-net state can then be reduced to a linear combination of string-net states on the unfattened lattice. The coefficients in this linear combination are the matrix elements of $W_\alpha(P)$.

However, none of the rules (3.1-3.4) involve strings labeled α , nor do they allow for crossings. Thus, the reduction to string-net states on the unfattened lattice requires new local rules. These new local rules are defined by the 4 index objects $\Omega_{\alpha,sti}^j$, $\bar{\Omega}_{\alpha,sti}^j$,

and the integers $n_{\alpha,i}$:

$$\begin{aligned}
|\square \circlearrowleft^\alpha\rangle &= \sum_i n_{\alpha,i} |\square \circlearrowright^i\rangle \\
|\begin{array}{c} \alpha \\ \diagup \quad \diagdown \\ \quad \quad i \end{array}\rangle &= \sum_{jst} (\Omega_{\alpha,sti}^j)_{\sigma\tau} |\begin{array}{c} j \quad s \\ \diagup \quad \diagdown \\ t \quad i \end{array}\rangle \\
|\begin{array}{c} \diagup \quad \diagdown \\ \alpha \quad \quad i \end{array}\rangle &= \sum_{jst} (\bar{\Omega}_{\alpha,sti}^j)_{\sigma\tau} |\begin{array}{c} t \quad j \\ \diagup \quad \diagdown \\ i \quad s \end{array}\rangle
\end{aligned} \tag{A.8}$$

Here, σ, τ are the two indices of the matrix Ω_{st}^{ij} . (Until now, we've neglected to write out these indices explicitly).

After applying these rules, we then need to join together the resulting string-nets. The ‘‘joining rule’’ for two string types s_1, s_2 is as follows. If $s_1 \neq s_2$, we don't join the two strings: we simply throw away the diagram. If $s_1 = s_2$, then we join the two strings and contract the two corresponding indices σ_1, σ_2 . That is, we multiply the two Ω matrices together in the usual way. Using the same approach as (A.5), one can show that the graphical definition of $W_\alpha(P)$ agrees with the algebraic definition (3.15).

In the previous section, we used the graphical representation of B_p^s to show that these operators commute. The string operators $W_\alpha(P)$ can be analyzed in the same way. With a simple graphical argument one can show that the string operators $W_\alpha(P)$ commute with the magnetic operators B_p^s provided that (3.1-3.4),(A.8) satisfy the conditions

$$|\begin{array}{c} j \\ \diagdown \quad \diagup \\ \alpha \quad \quad k \end{array}\rangle = |\begin{array}{c} j \\ \diagup \quad \diagdown \\ \quad \quad \alpha \\ \quad \quad k \end{array}\rangle \tag{A.9}$$

$$|\begin{array}{c} \diagup \quad \diagdown \\ \quad \quad i \\ \alpha \end{array}\rangle = |\begin{array}{c} \diagdown \quad \diagup \\ \quad \quad i \\ \quad \quad \alpha \end{array}\rangle \tag{A.10}$$

These relations are precisely the commutativity conditions (3.18), written in graphical form.

A.6 Orthonormality

In this section, we show that the set of wave functions $\{\Phi_{\{q,s\}}^{\text{in}}\}$ are orthonormal (the same argument applies to the wave functions $\{\Phi_{\{q,s\}}^{\text{out}}\}$).

First, we rephrase the problem in a more convenient language. Suppose X is some string-net configuration inside R . If we apply the graphical rules we can calculate the amplitude of X in terms of the basic configurations $X_{\{q,s\}}$:

$$\Phi(X) = \sum_{\{q,s\}} c_{\{q,s\}}(X) \Phi(X_{\{q,s\}})$$

Notice that the coefficient in this expansion is nothing but the wave function $\Phi_{\{q,s\}}^{\text{in}}$:

$$\Phi_{\{q,s\}}^{\text{in}}(X) = c_{\{q,s\}}(X)$$

Thus, we can establish the orthonormality of $\Phi_{\{q,s\}}^{\text{in}}$ from the orthonormality of the coefficients $c_{\{q,s\}}$. But these coefficients are simply the result of applying the graphical rules many times in succession. Therefore, it suffices to show that the coefficients in the graphical rules are orthonormal.

Only two rules are necessary to reduce an arbitrary configuration X to $X_{\{q,s\}}$. The first is (3.4):

$$\Phi \left(\begin{array}{c} i \\ \text{---} \\ j \text{---} m \text{---} k \\ \text{---} \\ j \end{array} \right) = \sum_n F_{kln}^{ijm} \Phi \left(\begin{array}{c} i \\ \text{---} \\ j \text{---} n \text{---} k \\ \text{---} \\ j \end{array} \right) \quad (\text{A.11})$$

This rule can be written as

$$\Phi(Y_m) = \sum_n F_{kln}^{ijm} \Phi(X_n)$$

where Y_m, X_n are the string-net configurations on the left and right hand sides respectively. The coefficients are therefore $c_n(Y_m) = F_{kln}^{ijm}$. We need to establish that

$$\sum_m c_n(Y_m)(c_p(Y_m))^* = \delta_{np}$$

This can be derived from the relations in (4.4). First, substitute $i = 0, p = j^*, s = m^*$ into the third relation in (4.4). This gives

$$\sum_{n=0}^N F_{kjn}^{mlq} F_{mnm}^{j0j^*} F_{lkr^*}^{jmn} = F_{q^*kr^*}^{j0j^*} F_{mlm}^{r0q^*}$$

Applying the first and second relation, we find $F_{mnm}^{j0j^*} = \delta_{jmn}$, $F_{q^*kr^*}^{j0j^*} = \delta_{jq^*k} \delta_{qr}$, $F_{mlm}^{r0q^*} = \delta_{qml} \delta_{qr}$. The identity simplifies to

$$\sum_{n=0}^N F_{kjn}^{mlq} F_{lkr^*}^{jmn} = \delta_{qr} \delta_{jq^*k} \delta_{qml}$$

Applying the second relation and the last relation, we find $F_{kjn}^{mlq} = (F_{lkq^*}^{jmn})^*$ so

$$\sum_{n=0}^N (F_{lkq^*}^{jmn})^* F_{lkr^*}^{jmn} = \delta_{qr} \delta_{jq^*k} \delta_{qml}$$

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