Robust Stability and Contraction Analysis of Nonlinear Systems via Semidefinite Optimization

by

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Abstract

A wide variety of stability and performance problems for linear and certain classes of nonlinear dynamical systems can be formulated as convex optimization problems involving linear matrix inequalities (LMIs). These formulations can be solved numerically with computationally-efficient interior-point methods.

Many of the first LMI-based stability formulations applied to linear systems and the class of nonlinear systems representable as an interconnection of a linear system with bounded uncertainty blocks. Recently, stability and performance analyses of more general nonlinear deterministic systems, namely those with polynomial or rational dynamics, have been converted into an LMI framework using sum of squares (SOS) programming. SOS programming combines elements of computational algebra and convex optimization to provide efficient convex relaxations for various computationally-hard problems.

In this thesis we extend the class of systems that can be analyzed with LMI-based methods. We show how to analyze the robust stability properties of *uncertain* nonlinear systems with polynomial or rational dynamics, as well as a class of systems with external inputs, via contraction analysis and SOS programming. Specifically, we show how contraction analysis, a stability theory for nonlinear dynamical systems in which stability is defined incrementally between two arbitrary trajectories via a contraction metric, provides a useful framework for analyzing the stability of uncertain systems. Then, using SOS programming we develop an algorithmic method to search for contraction metrics for these systems. The search process is made computationally tractable by relaxing matrix definiteness constraints, the feasibility of which indicates the existence of a contraction metric, to SOS constraints on polynomial matrices. We illustrate our results through examples from the literature and show how our contraction-based approach offers advantages when compared with traditional Lyapunov analysis.

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Chapter 1

Introduction

NONLINEAR DYNAMICAL SYSTEMS arise in a variety of engineering and science contexts including mechanical and electrical engineering, physics, mathematics and biology. Since nonlinear systems are often impossible to solve analytically, it is necessary to develop theoretical and computational tools for analyzing the stability and behavior of these systems.

Computational methods have become increasingly important in nonlinear (and linear) system analysis techniques. With the introduction of the high-speed computer in the 1950s, the behavior of a system could be simulated computationally, allowing one to develop better intuition about the behavior of a nonlinear system [41]. More recently, computationally efficient interior-point methods for solving convex optimization problems [11, 22] have been developed and allow efficient computation of solutions of linear matrix inequalities (LMIs) arising from various theoretical stability and analysis approaches.

Many dynamical system analysis and design problems can be formulated as LMIs. Linear matrix inequalities first arose in systems theory in the 1890s with Lyapunov's stability theory, but only within the last 25 years have they become computationally tractable.[1] In this thesis, we focus on stability analysis problems, and refer the reader to [2] for other control and systems theory problems with LMI formulations. Originally, LMI-based methods were applied to linear systems and a specific class of nonlinear systems called Lur'e systems, which involve a feedback interconnection of a linear system with bounded uncertainty blocks.

More recently, stability and performance methods for more general nonlinear sys-

[1]Before the development of interior-point methods, a limited number of families of LMIs could be solved analytically, or by reducing the solution of LMIs to simple graphical criteria.

tems with polynomial or rational dynamics have been converted into an LMI framework using sum of squares (SOS) programming. Recent literature has shown a sum of squares program is equivalent to a semidefinite program, which is the minimization of a linear function subject to an LMI constraint [27]. Together, sum of squares and semidefinite programming provide a powerful framework for developing algorithmic methods for stability and performance analysis of linear systems, as well as various classes of nonlinear systems.

This powerful framework motivates a search for other system analysis and performance questions that can be answered algorithmically with LMI-based techniques. In this thesis we show how to analyze stability of *uncertain* nonlinear systems with polynomial or rational dynamics, as well as a class of systems with external inputs, via contraction theory (contraction analysis) and SOS programming. In Chapter 4 we introduce contraction theory, a stability theory for nonlinear systems where stability is defined incrementally between two arbitrary trajectories, and show how it provides a useful framework for analyzing the stability of these systems. Then, using SOS programming, we develop an algorithmic method to search for contraction metrics for deterministic nonlinear systems with polynomial or rational dynamics. Next, we show how, with some slight variations, this algorithmic search can be applied to systems with parametric uncertainty. The search process is made computationally tractable by relaxing matrix definiteness constraints, the feasibility of which indicates the existence of a contraction metric, to SOS constraints on polynomial matrices.

1.1 Outline and Contributions

The main contributions of this thesis are as follows:

- Using SOS programming, we develop an algorithm to search for polynomial contraction metrics for systems with polynomial dynamics.

- We discuss why contraction theory provides a useful complementary framework to traditional Lyapunov theory for studying nonlinear systems with parametric uncertainty. Specifically, we show that if a nominal system is contracting with respect to some metric, it is often the case that within a given range of uncertainty a perturbed version of that system will also be contracting with respect to the same metric, even if the perturbation changes the equilibrium of the system. Thus, the contraction theory approach can be advantageous for robust stability analysis of nonlinear systems when compared to traditional linearization or Lyapunov techniques.

- We develop an algorithm to find bounds on the uncertainty range for which a system retains the property of contractiveness with respect to the polynomial metric of the nominal system. We also develop an algorithm to obtain a polynomial metric that provides the largest uncertainty interval for which we can prove the system is contracting.

- We prove that the existence of a contraction metric with a specific structure is sufficient to guarantee contraction of a class of systems with external inputs. This feature, which is central to using contraction theory to prove synchronization of coupled nonlinear oscillators, demonstrates the flexibility of the framework in incorporating inputs and outputs.

- We show that our contraction-based methods for robust stability of nonlinear systems, whose equilibrium may shift with the uncertainty, are quantitatively better than the few techniques currently available for analyzing such systems.

Before presenting these contributions in Chapter 4, we provide a context for the work. In Chapter 2 we introduce semidefinite programming, linear matrix inequalities, and stability of dynamical systems. We review how various stability techniques can be reduced to LMIs and summarize the relevant history of LMIs in system and control theory. In addition to giving the reader background on stability analysis methods, this material provides perspective for comparing our contraction-based methods with more traditional stability analysis methods. This chapter can be skimmed by readers familiar with these traditional methods. In Chapter 3 we discuss sum of squares (SOS) polynomials, programs and matrices. We present our main contributions concerning stability and robustness analysis of nonlinear systems via contraction theory and SOS programming in Chapter 4, and finally give our conclusions and possible directions for future work in Chapter 5.

1.2 Mathematical Notation and Conventions

We denote scalars by lower case letters, matrices by capital letters, and most vectors by bold lowercase letters. However, some vector functions are denoted by scalar Greek letters (ψ). The set of real symmetric $n \times n$ matrices is denoted by S^n. A matrix $A \in S^n$ is positive semidefinite (positive definite) if $\mathbf{x}^T A\mathbf{x} \geq 0$ ($\mathbf{x}^T A\mathbf{x} > 0$) for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \neq 0$. We use the standard notation and denote this by $A \succeq 0$ ($A \succ 0$). The set of positive semidefinite matrices is denoted S^n_+ and the set of positive definite matrices is denoted S^n_{++}. Without loss of generality, only symmetric matrices need to be considered in regards to positive definiteness as $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T A_s\mathbf{x}$ where A_s denotes the symmetric part of A. A matrix A can be uniquely decomposed into its symmetric and antisymmetric parts $A = A_s + A_{as}$ where A_s denotes the symmetric part and A_{as} denotes the antisymmetric part. Finally, note $\mathbf{x}^T A_{as}\mathbf{x} = 0$ because $A_{as}^T = -A_{as}$.

Chapter 2

Semidefinite Programming and Analysis of Dynamical Systems

IN THIS CHAPTER we discuss semidefinite programming, linear matrix inequalities, and dynamical systems. We present a variety of well-known stability techniques for linear systems and systems which can be represented as a feedback connection of a linear system with nonlinear elements. Various stability properties of such systems can be determined by formulating and solving linear matrix inequalities (LMIs). In Chapter 4, we will compare some of the traditional techniques described in this chapter with our contraction-based methods.

2.1 Semidefinite Programming (SDP) and Linear Matrix Inequalities (LMIs)

Semidefinite programming (SDP) is the optimization of a linear function subject to linear matrix inequality constraints. It is a specific kind of convex optimization problem that generalizes several standard problems such as linear and quadratic programming. Even though semidefinite programs (SDPs) are more general than linear programs, they are not much more difficult to solve as many interior point methods for linear programming have been extended to semidefinite programming [22, 11]. These interior point methods have polynomial worst-case complexity and perform well in practice [44].

Because SDPs can be solved efficiently in theory and practice, semidefinite programming finds applications in various engineering disciplines as well as in combinatorial optimization. The versatility of the problem formulation and also the availability of software to solve SDPs such as SeDuMi and SDPT3 [42, 43] has resulted in extensive recent research in the areas of semidefinite programming algorithms and applications [28]. A survey of the theory and applications of semidefinite programming is given in [44].

A semidefinite program minimizes a linear function subject to the constraint that a particular affine combination of symmetric matrices is positive semidefinite (or definite). Mathematically,

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & F(\mathbf{x}) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0 \end{array} \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^m$ is the optimization variable, $\mathbf{c} \in \mathbb{R}^m$ is given, and $F_i \in S^n$ are given. The constraint in (2.1) is a linear matrix inequality and is a convex constraint on $\mathbf{x}$, i.e., the set $S = \{\mathbf{x}|F(\mathbf{x}) \succeq 0\}$ is convex. We remark that any nonstrict LMI $F(\mathbf{x}) \succeq 0$ can in principle be reduced to an equivalent strictly feasible LMI $\hat{F}(\mathbf{x}) \succ 0$ by eliminating implicit equality constraints and removing any constant nullspace [2]. In this work, we deal with both strict and nonstrict forms.

The dual problem associated with the primal problem (2.1) is

$$\begin{array}{ll} \max_{Z} & -\mathrm{Tr}[F_0 Z] \\ \text{subject to} & \mathrm{Tr}[F_i Z] = c_i, \quad i = 1, ..., m \\ & Z \succeq 0, \end{array} \tag{2.2}$$

where $\mathrm{Tr}[\cdot]$ denotes the trace operator. In the dual problem the optimization variable is the matrix $Z \in S^n$. The scalars c_i, $i = 1, ..., m$ are given and $F_i \in S^n$, $i = 0, ..., m$ are also given. It should be noted that the objective function is a linear function of Z. The dual program is also a semidefinite program and can be put in the same form as the primal problem (2.1) [44].

An important connection between the primal and dual problems is that feasible solutions of one bound the optimal solutions of the other. If $\mathbf{x}$ and Z are feasible solutions of the primal and dual problems, then

$$\mathbf{c}^T\mathbf{x} - (-\mathrm{Tr}[F_0 Z]) = \mathbf{c}^T\mathbf{x} + \mathrm{Tr}[F_0 Z] = \sum_i c_i x_i + \mathrm{Tr}[F_0 Z].$$

Since Z is feasible, we have

$$\sum_i c_i x_i + \mathrm{Tr}[F_0 Z] = \sum_i \mathrm{Tr}[F_i Z]\ x_i + \mathrm{Tr}[F_0 Z] = \mathrm{Tr}[(\sum_i x_i F_i + F_0) Z].$$

Because Z and $(\sum_i x_i F_i + F_0)$ are both positive semidefinite, we have

$$\mathrm{Tr}[(\sum_i x_i F_i + F_0) Z] \geq 0.$$ [1]

Thus,

$$\mathbf{c}^T\mathbf{x} - (-\mathrm{Tr}[F_0 Z]) \geq 0 \tag{2.3}$$

and we can conclude feasible solutions of the primal bound optimal solutions of the dual and vice versa.

This property of the primal objective function for any feasible $\mathbf{x}$ always being greater than or equal to the dual objective function for any feasible Z called *weak duality*. It allows the use of any feasible $\mathbf{x}$ to compute an upper bound for the optimum of the dual objective and any feasible Z to compute a lower bound for the optimum of the primal objective. In the case of feasibility problems where there is no objective function, the dual problem can certify the nonexistence of solutions of the primal problem and vice versa. When either the dual problem is strictly feasible, i.e., there exists and $Z = Z^T \succ 0$ such that $\mathrm{Tr}[F_i Z] = c_i\ i = 1, ..., m$, or the primal problem is strictly feasible, i.e., there exists a $\mathbf{x}$ such that $F(\mathbf{x}) \succ 0$, the inequality (2.3) holds with equality [44]. This condition is called *strong duality.*

[1] One way to get the final inequality is as follows: Let $P = (\sum_i x_i F_i + F_0)$. Since P is positive semidefinite (psd) it has the decomposition $P = LL^T$ for some L. Then we have $\mathrm{Tr}[PZ] = \mathrm{Tr}[LL^T Z] = \mathrm{Trace}[L^T Z L]$. $Z \in S_+^n \Rightarrow L^T Z L \in S_+^n$ as the definiteness of a matrix is invariant under a congruence transformation. Then, because the trace of a matrix is equal to the sum of its eigenvalues, $L^T Z L \succeq 0 \Rightarrow \mathrm{Tr}[L^T Z L] \geq 0$.

2.1.1 SDP Formulations - Conversion of Constraints to LMIs

In addition to its computational efficiency, the semidefinite programming framework is also useful because a wide variety of constraints can be converted into semidefinite programming form. Specifically, the linear matrix inequality constraint

$$F(\mathbf{x}) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i \succ 0 \tag{2.4}$$

can represent a wide variety of constraints on $\mathbf{x}$. Linear inequalities, matrix norm inequalities, Lyapunov inequalities, convex quadratic matrix inequalities, and many other constraints in system and control theory can all be converted into an LMIs [2]. Next, we present some of these conversions with strict inequalities keeping in mind that a non-strict inequality can always in principle be converted into a strict inequality [2]. One trivial conversion is turning multiple LMIs into a single LMI. Multiple LMIs $F^1(\mathbf{x}) \succ 0, ..., F^p(\mathbf{x}) \succ 0$ can be expressed as the single LMI $\mathbf{diag}(F^1(\mathbf{x}), ..., F^p(\mathbf{x})) \succ 0$ where the **diag** operator puts its arguments on the diagonal of a matrix.

Matrices as Variables

LMIs with matrices as variables arise often in stability analysis problems; for example the Lyapunov equation:

$$A^T P + PA \prec 0, \tag{2.5}$$

where $A \in \mathbb{R}^{n\times n}$ is given and $P = P^T \in \mathbb{R}^{n\times n}$ is a matrix variable, can be put in the form of the constraint (2.4) according to the following method presented in [2]: Let $P_1, ..., P_m$ be a basis for symmetric matrices ($m = n(n+1)/2$). Then let $F_0 = 0$ and $F_i = -A^T P_i + P_i A$. The x_i's in constraint (2.4) are now the decision variables which are related to the decision variable matrix P in the original problem via the basis of matrices $P_1, ...P_m$. In this work, we will leave matrix variables in our LMIs instead of explicitly putting them in the form (2.4).

Robust Constraints

Sometimes we may have constraints that depend on an uncertain parameter, and we want to impose the constraint for all values of the parameter within a certain range. If the uncertain parameter δ enters an LMI $F(\mathbf{x}, \delta) \succeq 0$ affinely and we find that the LMIs given by $F(\mathbf{x}, -\gamma) \succeq 0$ and $F(\mathbf{x}, \gamma) \succeq 0$ are satisfied then the LMI $F(\mathbf{x}, \delta) \succeq 0$ is satisfied for all δ such that $-\gamma \leq \delta \leq \gamma$. This idea can easily be extended to higher-dimensional uncertainty. Assume we have an uncertain vector $\delta \in \mathbb{R}^n$, each entry δ_i enters the dynamics affinely, and we want to see for what uncertainty values $F(\mathbf{x}, \delta) \succeq 0$. Assume we can find m points $[\gamma_{1i} \ \cdots \ \gamma_{ni}]^T$ for $i = 1, ..., m$, such that $F(\mathbf{x}, [\gamma_{1i} \ \cdots \ \gamma_{ni}]^T) \succeq 0$ for $i = 1, .., m$. Then $F(\mathbf{x}, \delta) \succeq 0$ for all δ within the polytope defined by convex hull of the points $[\gamma_{1i} \ \cdots \ \gamma_{ni}]^T$ for $i = 1, ..., m$. Analysis with robust constraints arises in Section 4.3.

Schur Complements

Schur complements are used to convert nonlinear (convex) inequalities to an equivalent LMI form.

Lemma 1. *Let $B \succ 0$ and without loss of generality assume it is symmetric. Then*

$$A = \begin{bmatrix} B & C^T \\ C & D \end{bmatrix} \succeq 0 \tag{2.6}$$

if and only if

$$D - CB^{-1}C^T \succeq 0. \tag{2.7}$$

Proof.

$$A \succeq 0 \quad \Leftrightarrow \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^T \begin{bmatrix} B & C^T \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq 0 \text{ for all } \mathbf{x}, \mathbf{y}$$

$$\Leftrightarrow \quad \mathbf{x}^T B\mathbf{x} + 2\mathbf{x}^T C^T \mathbf{y} + \mathbf{y}^T D\mathbf{y} \geq 0 \text{ for all } \mathbf{x}, \mathbf{y} \tag{2.8}$$

$$\Leftrightarrow \quad \min_{\mathbf{x}} \{\mathbf{x}^T B\mathbf{x} + 2\mathbf{x}^T C^T \mathbf{y} + \mathbf{y}^T D\mathbf{y}\} \geq 0 \text{ for all } \mathbf{y} \tag{2.9}$$

Since $B \succ 0$, the minimization problem (2.9) is strictly convex. The global minimum $\mathbf{x}^*$ is $\mathbf{x}^* = B^{-1}C^T\mathbf{y}$. Thus, (2.9) holds if and only if $\mathbf{y}^T(D - CB^{-1}C^T)\mathbf{y} \geq 0$ for all $\mathbf{y}$ which holds if and only if $D - CB^{-1}C^T \succeq 0$. □

The equivalence between (2.6) and (2.7) allows us to convert the set of nonlinear inequalities (2.7) to the LMI (2.6). The following example uses Schur complements to rewrite a quadratic matrix inequality as an LMI.

Example 2 ([2])**.** *The* quadratic *matrix inequality with decision variable P given by*

$$A^T P + PA + PBR^{-1}B^T P + Q \prec 0 \tag{2.10}$$

where A, B, $Q = Q^T$, $R = R^T \prec 0$ are given matrices of appropriate sizes, can be converted into the linear *matrix inequality in variable P*

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} \succ 0 \tag{2.11}$$

via Schur complements. The latter representation (2.11), where P enters the matrix constraint linearly, shows the quadratic matrix inequality (2.10) is convex (in P) even though it is nonlinear.

The S-Procedure

The S-procedure is one way to prove that a certain quadratic function is nonnegative whenever some other specified quadratic functions are all nonnegative.

Let $\sigma_0, ...\sigma_N$ be quadratic functions of the variable $\mathbf{z} \in \mathbb{R}^n$; that is

$$\sigma_i(\mathbf{z}) \triangleq \mathbf{z}^T T_i \mathbf{z} + 2\mathbf{u}_i^T \mathbf{z} + v_i, \qquad i = 0, ..., N$$

where $T_i^T = T_i \in \mathbb{R}^{n \times n}$, $u_i \in \mathbb{R}^n$, and $v_i \in \mathbb{R}$. We consider the following condition on $\sigma_0, ..., \sigma_N$:

$$\sigma_0(\mathbf{z}) \geq 0 \text{ for all } \mathbf{z} \text{ that satisfy } \sigma_i(\mathbf{z}) \geq 0 \quad i = 1, ..., N. \tag{2.12}$$

Clearly, if there exist $d_1 \geq 0, ..., d_N \geq 0$ such that for all $\mathbf{z}$

$$\sigma_0(\mathbf{z}) - \sum_{i=1}^{N} d_i \sigma_i(\mathbf{z}) \geq 0,$$

then (2.12) holds. When $p = 1$, the converse holds, provided that there is some $\mathbf{z}_0$ such that $\sigma_1(\mathbf{z}_0) \geq 0$ However, this converse statement is not immediate [2].

Rewriting equation (2.12) as

$$\begin{bmatrix} T_0 & \mathbf{u}_0 \\ \mathbf{u}_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{N} d_i \begin{bmatrix} T_i & \mathbf{u}_i \\ \mathbf{u}_i^T & v_i \end{bmatrix} \succeq 0$$

makes it clear that finding $d_1 \geq 0, ..., d_N \geq 0$, such that the condition (2.12) holds, is a linear matrix inequality in the variables $d_1 \geq 0, ..., d_N \geq 0$.

These techniques for reformulating problems as LMIs indicate that many different types of problems can be solved via semidefinite programming. In the following sections of this chapter we show how a variety of stability problems can be reduced to semidefinite programming feasibility problems, i.e., LMIs. Such a discussion first requires a basic understanding of dynamical systems theory.

2.2 Dynamical Systems

Design and analysis techniques for dynamical systems are critical for understanding linear and nonlinear circuits, mechanical systems, control systems, chemical kinetics, civil engineering structures, condensed matter physics, and biological systems such as predator-prey cycles [41].

A large class of dynamical systems can be modeled as coupled ordinary differential equations (ODEs). These equations take the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, ..., x_n, u_1, ..., u_p, t) \\ \dot{x}_2 &= f_2(x_1, ..., x_n, u_1, ..., u_p, t) \\ \vdots &= \vdots \\ \dot{x}_n &= f_n(x_1, ..., x_n, u_1, ..., u_p, t) \end{aligned} \tag{2.13}$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t, $u_1, ..., u_p$ are specified input (control) variables, and $x_1, ..., x_n$ are state variables. The state variables model the "memory" of the dynamical system, i.e., how the evolution of the dynamical system depends on its past. State variables might represent flows and temperatures in a jet engine, concentrations of chemicals in a reactor, or positions and velocities in a mechanical system. The functions $f_1, ..., f_n$ are determined by the system or problem under consideration and can be nonlinear. To determine a state trajectory, i.e., solution curve $\mathbf{x}(t)$ for such a system, an initial condition $\mathbf{x}(0) = \mathbf{x}_0$ must also be specified.

In vector notation the equations can be expressed compactly as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t). \tag{2.14}$$

where we have written $\mathbf{x}(t)$ as $\mathbf{x}$ etc. for notational convenience. There is also often an associated output equation

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{2.15}$$

relating the input $\mathbf{u}$ and state $\mathbf{x}$ to an the output $\mathbf{y}$. If the inputs are zero or specified as a function of the state $\mathbf{u} = \mathbf{k}(\mathbf{x})$ or a function of time $\mathbf{u} = \mathbf{k}(t)$, then the dynamics have the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t). \tag{2.16}$$

In this case, the system is called a closed-loop system. If $\mathbf{f}$ does not depend on t, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.17}$$

the system is called autonomous or time-invariant. Systems whose dynamics change slowly with respect to the time interval over which the analysis of the system occurs, can in practice be represented by time-invariant models.

An autonomous system is linear and time-invariant (LTI) if the dynamics can be written in the form

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{2.18}$$

where A is an $n \times n$ matrix. In a linear system the time derivative of each variable depends linearly on the other state variables. Linear systems are much easier to analyze than nonlinear systems because they can be broken down into parts that are solved separately, and then the parts can be recombined to get the answer. Readers unfamiliar with linear systems theory can refer to [35].

The concept of an equilibrium point is important for the analysis of autonomous systems (2.17)). An equilibrium point $\mathbf{x}^*$ is a point in the state-space such that if the initial state is $\mathbf{x}^*$, the trajectory will remain at $\mathbf{x}^*$ for all future time. The equilibrium points of (2.17) are the solutions of $\mathbf{f}(\mathbf{x}) = 0$. An equilibrium point can be isolated or there can be a continuum of equilibrium points. A linear system can only have an isolated equilibrium point at the origin or a continuum of equilibrium points that constitute a subspace through the origin. A nonlinear system can have more than one isolated equilibrium point and depending on the initial state of the system the state trajectory may converge to one of several steady-state operating behaviors.

It is often useful to transform the equation (2.17) so that the equilibrium point of interest is at the origin. This can be done by a simple translation. Assume the equilibrium point of interest is $\mathbf{x}^*$. Then, introduce a new variable $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$ and substitute $\tilde{\mathbf{x}} + \mathbf{x}^*$ for $\mathbf{x}$ in the system equation (2.17). With this substitution we obtain a new set of equations in the variable $\tilde{\mathbf{x}}$.

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^*). \tag{2.19}$$

There is a one-to-one correspondence between the trajectories of (2.17) and (2.19) and the equilibrium point of (2.19) corresponding to the equilibrium point $\mathbf{x} = \mathbf{x}^*$ of 2.4 is $\tilde{\mathbf{x}} = 0$.

Nonlinear systems are more complex and can exhibit (among others) the following phenomena not occurring in linear systems:

- Finite escape time: The state of a nonlinear system can go to infinity in finite time, while the state of an unstable linear system is always bounded for finite time.

- Limit cycles: In real systems, oscillations must be produced by a nonlinear system. There are nonlinear systems which can go into an oscillation of fixed amplitude and frequency regardless of the initial state. Such an oscillation is called a limit cycle.

- Chaos: Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions [41].

- Nonlinear systems can also exhibit quasi-periodic oscillations.

This complexity often makes it difficult to explicitly determine the state trajectory $\mathbf{x}(t)$ given the differential equations (2.14) that describe the nonlinear system. In the next section, we describe techniques which allow us to determine stability behavior of nonlinear systems even if we cannot explicitly determine the state trajectories.

A variety of notions of stability exist for dynamical systems, including internal stability, input-output stability, and absolute stability. Internal stability specifies how a closed-loop system qualitatively behaves regardless of its initial condition. Input-output stability characterizes the input-output ($\mathbf{u}$ to $\mathbf{y}$) behavior of the system. Absolute stability is a stability concept for nonlinear systems representable as a feedback connection of a linear dynamical system and a nonlinear element. As we will see, many techniques for proving these various forms of stability involve formulating and solving LMIs.

2.3 Internal (Lyapunov) Stability

Internal stability is most often defined with respect to equilibrium points.[2] Stability with respect to equilibrium points is often called 'stability in the sense of Lyapunov'. Lyapunov was a Russian mathematician and engineer who formulated the foundations of internal stability theory, now called Lyapunov theory, in 1890. We only consider autonomous systems here but there are advanced internal stability techniques for dealing with non-autonomous systems [12, 37, 40].

The questions Lyapunov theory attempts to answer are: does the state return to an equilibrium after a small perturbation away from it? Does it remain close to it in some sense? The following well-known definitions and theorems formalize these notions. Without loss of generality we assume the equilibrium is at the origin ($\mathbf{x} = \mathbf{0}$).

Definition 3 ([12]). *The equilibrium point* $\mathbf{x} = \mathbf{0}$ *of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *is*

- *stable in the sense of Lyapunov, if for each* $\epsilon > 0$, *there exists a* $\delta = \delta(\epsilon) > 0$ *such that*

$$||\mathbf{x}(0)|| < \delta \Rightarrow ||\mathbf{x}(t)|| < \epsilon, \quad \textit{for all}\, t \geq 0$$

- *unstable, if not stable*

- *asymptotically stable if it is stable and* δ *can be chosen such that*

$$||\mathbf{x}(0)|| < \delta \Rightarrow \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0}$$

[2] Occasionally, internal stability is defined with respect to a nominal motion trajectory. See Chapter 4 for a description of contraction theory, one method for analyzing stability with respect to a trajectory.

Lyapunov's Stability Theorem allows us to approach the question of whether or not a system is internally stable. This theorem is easy to understand in terms of energy principles. The basic idea is that if the "energy" of a system decreases along trajectories of the system, then the system is stable. We can formalize this idea as follows:

Definition 4 ([45])**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous map. $V(\mathbf{x})$ is called a* locally positive definite (lpd) function around $\mathbf{x} = \mathbf{0}$ *if*

- $V(\mathbf{0}) = 0$,
- $V(\mathbf{x}) > 0, \quad 0 < ||\mathbf{x}|| < r$ *for some r.*

The function is called locally positive semidefinite (lpsd) if $V(\mathbf{x}) > 0$ is replaced by $V(\mathbf{x}) \geq 0$. A function $h(\mathbf{x})$ is locally negative definite (locally negative semidefinite) if $-h(\mathbf{x})$ is locally positive definite (locally positive semidefinite).

Definition 5 ([45])**.** *A* candidate Lyapunov function *is a locally positive definite function.*

Definition 6 ([45])**.** *Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and let $V(\mathbf{x})$ be a candidate Lyapunov function. Let $\dot{V}(\mathbf{x})$ be its derivative along trajectories, i.e., $\dot{V}(\mathbf{x}) = \frac{dV(x)}{dx}\frac{dx}{dt} = \frac{dV(x)}{dx}\mathbf{f}(x)$. If $\dot{V}(\mathbf{x})$ is locally negative semidefinite, then $V(\mathbf{x})$ is a* Lyapunov function *of the system (2.17).*

With these definitions, we can now state Lyapunov's theorem for local stability.

Theorem 7 ([45])**. Lyapunov's Stability Theorem** *If there exists a Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then $\mathbf{x} = \mathbf{0}$ is a stable equilibrium in the sense of Lyapunov. If, in addition, $\dot{V}(\mathbf{x}) < 0, \quad 0 < ||\mathbf{x}|| < r_1$, for some r_1, then $\mathbf{x} = \mathbf{0}$ is an asymptotically stable equilibrium point.*

The proof of this theorem is given in many nonlinear systems analysis references including [45, 12, 40, 37].

For many physical systems, one associated Lyapunov function can often be constructed from energy considerations. The damped pendulum is a typical example of such a system. If the energy of the pendulum E is defined as the sum of its potential and kinetic energies, and the reference of the potential energy is chosen so that $E(\mathbf{0}) = 0$, then $\frac{dE}{dt} \leq 0$ (i.e. the energy is always decreasing or constant) along the trajectories of the system. Friction prohibits the energy E from staying constant while the system is in motion. Hence the energy keeps decreasing, ensuring the trajectory $\mathbf{x}(t)$, where $x_1(t)$ is the position of the pendulum and $x_2(t)$ the velocity of the pendulum, tends to $\mathbf{x} = \mathbf{0}$ as $t \to \infty$.

Lyapunov's stability theorem is powerful because in order to apply it, we do not need to solve the differential equation (2.17). It is difficult to apply generally, however, because we must search over all functions to determine whether a Lyapunov function can be found. In other words, for most systems we cannot prove instability by showing a Lyapunov function of a certain type cannot be found. However, this difficulty does not occur for all systems. As we just saw there are natural Lyapunov functions related to the energy quantities in some physical systems. In Chapter 3 we see that for systems with polynomial dynamics we can use sum of squares programming techniques to search over a large class of possible Lyapunov functions. Next, we show that in the case of linear time invariant (LTI) systems, it is sufficient to search over all quadratic functions.

Consider the LTI system

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{2.20}$$

This system is stable if and only if all of the eigenvalues of A are in the open left half of the complex plane. We can also determine stability from Lyapunov techniques. For the LTI system (2.20) all candidate Lyapunov functions can be written as

$$V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}, \quad P \succ 0, \tag{2.21}$$

where P is symmetric.[3] For this $V(\mathbf{x})$, the derivative along trajectories can be com-

[3] Symmetric P can be considered without loss of generality. See Section 1.2.

puted as follows:

$$
\begin{aligned}
\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} \\
&= \mathbf{x}^T A^T P\mathbf{x} + \mathbf{x}^T PA\mathbf{x} \\
&= \mathbf{x}^T (A^T P + PA)\mathbf{x} \qquad (2.22)\\
&= -\mathbf{x}^T Q\mathbf{x} \qquad (2.23)
\end{aligned}
$$

where $Q = -(A^T P + PA)$. According to Theorem 7, $V(\mathbf{x})$ is a Lyapunov function if $Q \succeq 0$. If $Q \succeq 0$, the equilibrium is stable in the sense of Lyapunov, and if $Q \succ 0$, the equilibrium is globally asymptotically stable.

From (2.22) and (2.23) we see that to find a quadratic Lyapunov function for system (2.20), we can pick Q and then solve the equation

$$A^T P + PA = -Q \qquad (2.24)$$

for P. This equation is linear in the entries of P. If this equation has a positive definite solution for the chosen Q, then $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function and the system (2.20) is asymptotically stable. For LTI systems the converse is also true. If the system is asymptotically stable then the equation (2.24) has a positive definite solution.

Theorem 8 ([45]). *Given the LTI dynamic system $\dot{\mathbf{x}} = A\mathbf{x}$ and any $Q \succ 0$, there exists a unique positive definite solution of the Lyapunov equation*

$$A^T P + PA = -Q$$

if and only if all the eigenvalues of A are in the open left half plane (OLHP).

Proof. See Appendix A. □

The theorem above can be written in terms of a linear matrix inequality. Lyapunov showed in his 1890 paper that the system $\dot{\mathbf{x}} = A\mathbf{x}$ was stable if and only if there exists a matrix $P \prec 0$ such that $A^T P + PA \succ 0$ [2]. This was the first LMI used in the stability analysis or performance analysis of a dynamical systems.

In the next section we see how Lyapunov techniques can be generalized to analyze input-output stability properties of systems and how these techniques also result in LMIs.

2.4 Passivity Approach

One approach to input-output stability analysis based on Lyapunov methods is the passivity approach. A system is *passive* if it does not generate energy. A system is *dissipative* if it dissipates energy. As time evolves in a dissipative system, it absorbs a fraction of its supplied energy and turns it into losses such as heat, mass, or an increase in entropy [38].

LMIs occur in a very natural way in the mathematical formulation of dissipativity, where they have the interpretation of a storage function. Storage functions can be thought of as Lyapunov-like functions for systems with inputs and outputs.

In this section we describe the basic notions and theory of dissipativity and the relation between storage functions and Lyapunov functions. Our discussion is based on [38] and [12]. We also show how LMIs naturally appear from the application of dissipativity theory to linear input-output systems described by state-space equations.

Consider a time-invariant nonlinear system described by equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{aligned} \tag{2.25}$$

where $\mathbf{x}$ takes values in a state space X, $\mathbf{u}$ takes values in an input space U, $\mathbf{y}$ takes values in the output space Y, and the system is causal. Let

$$s : U \times Y \to \mathbb{R}$$

be a mapping such that for all $t_0, t_1 \in \mathbb{R}$ and for all $\mathbf{u}, \mathbf{y}$ satisfying (2.25), the composite function

$$s(t) = s(\mathbf{u}(t), \mathbf{y}(t))$$

is locally integrable, i.e., $\int_{t_0}^{t_1} |s(t)|dt < \infty$. The mapping s is called the *supply function* or *supply rate*.

There are natural supply functions for physical systems. An example of a supply function for an RLC circuit is $s(t) = u(t)y(t)$ where $u(t)$ is the voltage input to the system, the output $y(t)$ is the current flowing from the voltage source, and $s(t)$ is the power flow into the passive network. This supply function is related to the energy in the system: If we assume there is only one inductor and one capacitor in the circuit, and we take the current through the inductor as the state variable x_1, and the voltage across the capacitor as the state variable x_2, then the energy stored in the system is given by $V(\mathbf{x}) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$. Since an RLC network is passive, the energy absorbed by the network over any time period must be greater than or equal to the increase in the energy stored in the network over the same period; mathematically

$$V(\mathbf{x}(t)) - V(\mathbf{x}(0)) \leq \int_0^t u(\tau)y(\tau)d\tau. \tag{2.26}$$

If (2.26) holds with strict inequality, then the difference between the absorbed energy and the increase in the stored energy must be the energy dissipated in the resistive components of the network. Since (2.26) holds for all t, we also have

$$\dot{V}(\mathbf{x}(t)) \leq u(t)y(t) \quad \text{for all} \;\; t. \tag{2.27}$$

This inequality indicates the power flow into the network must be greater than or equal to the rate of change of the energy stored in the network. In other words, the system is dissipative. The following definition generalizes the circuit example and the notion of dissipativity.

Definition 9 ([38]). *The system (2.25) with a supply rate s is said to be* dissipative *if there exists a non-negative function $V : X \to \mathbb{R}$ such that*

$$V(\mathbf{x}(t_0)) + \int_{t_0}^{t_1} s(t)dt \geq V(\mathbf{x}(t_1)) \tag{2.28}$$

for all $t_0 \leq t_1$ and trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ which satisfy the system equations (2.25).

The supply function should be interpreted as the work delivered to the system [38]. Work has been done on the system in the time interval $[0, T]$ if $\int_0^T s(t)dt$ is positive, while work has been done by the system if that integral is negative. $V(\mathbf{x})$ is called a storage function and generalizes the notion of an energy function for a dissipative system. The dissipation inequality (2.28) formalizes the idea that in a dissipative system, the change of internal storage $V(\mathbf{x}(t_1)) - V(\mathbf{x}(t_0))$ in the time interval $[t_0, t_1]$ will never be greater than the amount of supply that flows into the system. Whenever the composite function $V(\mathbf{x}(\cdot))$ is differentiable as a function of time then (2.28) can be written as

$$\dot{V}(\mathbf{x}(t)) \leq s(\mathbf{u}(t), \mathbf{y}(t)). \tag{2.29}$$

We have just seen that in the analysis of electrical networks, the product of voltages and currents at the external branches of a network (the power) is a supply function. In mechanical systems, the product of forces and velocities is often a supply function. Typical examples of supply functions, which arise in network theory, H_∞ theory, game theory, LQ-optimal control, and H_2 optimal control theory are [38]

$$\begin{aligned} s(\mathbf{u}, \mathbf{y}) &= \mathbf{u}^T \mathbf{y} \\ s(\mathbf{u}, \mathbf{y}) &= ||\mathbf{y}||^2 - ||\mathbf{u}||^2 \\ s(\mathbf{u}, \mathbf{y}) &= ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \\ s(\mathbf{u}, \mathbf{y}) &= ||\mathbf{y}||^2. \end{aligned}$$

The fact that the storage function in the RLC circuit is an energy function suggests that storage functions are related to Lyapunov functions. This is indeed the case.

2.4.1 Storage Functions and Lyapunov Functions

Storage functions and Lyapunov functions are closely related [38]. If $\mathbf{u}(t) = \mathbf{u}^*$ with $\mathbf{u}^* \in U$ a constant input then (2.25) becomes the autonomous system defined by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}^*) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{u}^*). \end{aligned} \tag{2.30}$$

Let $\mathbf{x}^*$ be an equilibrium point and also let (2.30) be dissipative with supply

$$s(\mathbf{u}^*, \mathbf{y}) = s(\mathbf{u}^*, \mathbf{g}(\mathbf{x}, \mathbf{u}^*)) \leq 0 \text{ for all } \mathbf{x} \text{ in a neighborhood of } \mathbf{x}^*.$$

Then from Definition 9 and inequality (2.29) we can conclude any storage function V of this system is non-negative and monotone non-increasing along solutions $\mathbf{x}(t)$ in a neighborhood of $\mathbf{x}^*$. By Lyapunov's stability theorem, we know $\mathbf{x}^*$ is a stable equilibrium if the storage function $V(\mathbf{x})$ is locally positive definite. If this is the case, the storage function V is a Lyapunov function defined in a neighborhood of $\mathbf{x}^*$.

2.4.2 Quadratic Supply Rates for Linear Dissipative Systems

Dissipativity theory provides a computationally tractable framework for studying linear input-output systems. We consider a input-output system described by the equations

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u} \end{aligned} \tag{2.31}$$

with state space $X = \mathbb{R}^n$, input space $U = \mathbb{R}^m$, output space $Y = \mathbb{R}^p$, and a quadratic supply function $s : U \times Y \to \mathbb{R}$ given by

$$s(\mathbf{u}, \mathbf{y}) = \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}^T \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}^T Q \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \tag{2.32}$$

If we substitute the output equation $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$ into the supply function (2.32), we see (2.32) can also be viewed as a quadratic function of the variables $\mathbf{x}$, and $\mathbf{u}$; that is

$$s(\mathbf{u}, \mathbf{y}) = s(\mathbf{u}, C\mathbf{x} + D\mathbf{u}) = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}^T \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}. \tag{2.33}$$

The equivalence between (2.32) and (2.33) is used in the theorem below, which provides necessary and sufficient conditions for dissipativity of system (2.31) with supply function (2.32).

Theorem 10 ([38]). *Suppose the system described by (2.31) is controllable and let the supply function s be defined by (2.32). Then the following statements are equivalent*

1. *The system (2.31) with supply function (2.32) is dissipative.*
2. *There exists a quadratic storage function $V(\mathbf{x}) \triangleq \mathbf{x}^T P \mathbf{x}$ with $P = P^T \succeq 0$ such that (2.28) holds for all $t_0 \leq t_1$ and $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ satisfying (2.31).*
3. *There exists $P = P^T \succeq 0$ such that*

$$F(P) \triangleq - \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \succeq 0 \tag{2.34}$$

The full proof is given in [38]. For an outline, see Appendix B.

We next consider a type of input-output stability for nonlinear systems which can be represented as a linear system with nonlinear bounded feedback.

2.5 Absolute Stability - The Lur'e Problem

A wide variety of nonlinear physical systems can be represented as a feedback connection of a linear system and a nonlinear element as shown in Figure 2-1. This

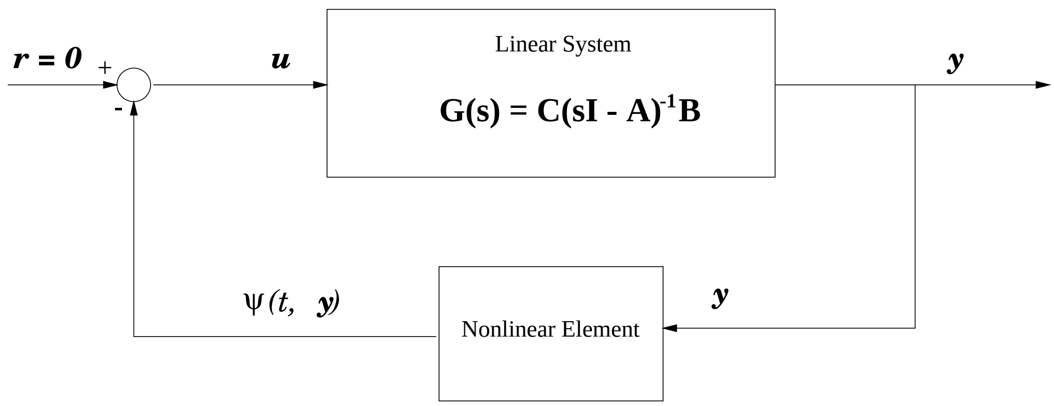


Figure 2-1: Lur'e system - Feedback interconnection of linear system with nonlinear element.

formulation was originally studied by Lur'e, Postnikov, and other researchers in the Soviet Union in the 1940s. In their work, the nonlinearity was an actuator/sensor nonlinearity [2]. A large body of relevant results were developed in the 1960-70s and now the stability of these systems, called absolute stability, is fundamental in nonlinear systems theory [20].

In this section we show how absolute stability criteria for such a system can be reduced to an LMI, closely following [12]. We consider the system in Figure 2-1 described by the equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.35}$$

$$\mathbf{y} = C\mathbf{x} \tag{2.36}$$

$$\mathbf{u} = -\psi(t, \mathbf{y}) \tag{2.37}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^p$, (A, B) is controllable, (A, C) is observable, and $\psi : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^p$ is a memoryless piecewise continuous function in t and locally Lipschitz in $\mathbf{y}$. Lur'e studied the system where $p = 1$, i.e., there is only one nonlinearity in the feedback loop.

The transfer matrix of the linear system (2.35) - (2.37) is a square strictly proper transfer function

$$G(s) = C(sI - A)^{-1}B. \tag{2.38}$$

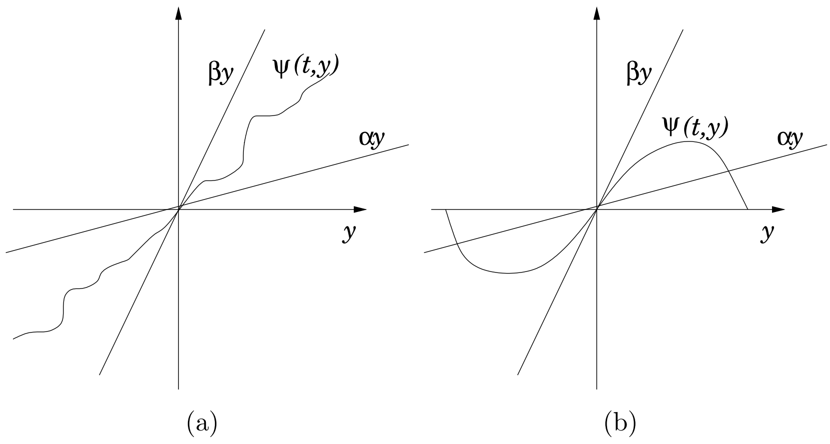


Figure 2-2: (a) Global sector nonlinearity; (b) local sector nonlinearity [12].

Since (A, B) is controllable and (A, C) is observable, (A, B, C) is a minimal realization of $G(s)$ [35].[4] In order to prove stability of the system, the nonlinearity $\psi(\cdot, \cdot)$ must satisfy a sector condition defined below. We first describe a sector condition for the nonlinear element ψ when $p = 1$, i.e., $G(s)$ is a single-input, single-output transfer function. In this case, $\psi : [0, \infty)$ satisfies a sector condition if there are constants α, β, a, b such that $\beta > \alpha$, $a < 0 < b$, and

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2 \text{ for all } t \geq 0,\ y \in [a, b]. \tag{2.39}$$

The sector condition is said to hold globally if (2.39) holds for all $y \in (-\infty, \infty)$. Figure 2-2 depicts a sector condition that holds locally and also one that holds globally. The sector condition (2.39) implies

$$[\psi(t, y) - \alpha y][\psi(t, y) - \beta y] \leq 0 \text{ for all } t \geq 0,\ y \in [a, b]. \tag{2.40}$$

This generalizes to the following multivariable sector condition.

Definition 11 ([12]). *A memoryless nonlinearity $\psi : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^p$ is said to*

[4] We remark that many of the results in this section can be extended to the more general case of non-strictly proper transfer functions, i.e., when $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$.

satisfy a sector condition *if*

$$[\psi(t,\mathbf{y}) - K_{min}\mathbf{y}]^T[\psi(t,\mathbf{y}) - K_{max}\mathbf{y}]^T \leq 0, \quad \textit{for all}\, t \geq 0,\ \mathbf{y} \in \Gamma \subset \mathbb{R}^p \qquad (2.41)$$

for some real matrices K_{min} and K_{max}, where $K = K_{max} - K_{min}$ is a positive definite symmetric matrix and the interior of Γ is connected and contains the origin. If $\Gamma = \mathbb{R}^p$, then $\psi(\cdot,\cdot)$ satisfies the sector condition globally, in which case it is said that $\psi(\cdot,\cdot)$ belongs to a sector $[K_{min}, K_{max}]$. If (2.41) holds with strict inequality, then $\psi(\cdot,\cdot)$ is said to belong to a sector (K_{min}, K_{max}).

The origin is an equilibrium point of the system (2.35) - (2.37) for all nonlinearities satisfying the sector condition (2.41). If it can be shown that the origin is asymptotically stable for all nonlinearities in the sector, the system is said to be *absolutely stable.*

Asymptotic stability of the origin can be examined with various Lyapunov function candidates. The simple quadratic Lyapunov function candidate

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}; \quad P = P^T \succ 0$$

leads to the multivariable circle criterion which we define below, while analysis with a candidate function of the form

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x} + \eta \int_0^{\mathbf{y}} \psi^T(\mathbf{r})K d\mathbf{r}; \quad P = P^T \succ 0; \quad \eta \geq 0,$$

called a Lur'e-type Lyapunov function, leads to the multivariable Popov criterion. In the latter case, the nonlinearity is assumed to be time invariant and satisfy conditions which ensure that the integral is well-defined and nonnegative. In both cases the goal, like that of Lyapunov analysis, is to determine the conditions under which there exists a P such that $P = P^T \succ 0$ and such that the derivative of $V(\mathbf{x})$ is negative definite along trajectories for all nonlinearities which satisfy the sector condition. We see in the following sections that, in both cases, finding such a P reduces to solving an LMI. Before the development of interior-point methods in 1984, there were no efficient computational methods for solving LMIs. However, in the scalar case, i.e., when the transfer function $G(s)$ was single-input single-output, the LMIs resulting from the absolute stability problem could be solved graphically via frequency domain criteria called the circle criterion and Popov criterion. These names now refer to the more general multivariable cases which we describe next.

2.5.1 Circle Criterion

Consider the system (2.35) - (2.37) and assume A is Hurwitz, i.e. has all its eigenvalues in the open left half plane, and the nonlinearity satisfies the sector condition (2.41) with $K_{min} = 0$; that is

$$\psi^T(t, \mathbf{y})[\psi(t, \mathbf{y}) - K\mathbf{y}] \leq 0, \quad \text{for all } t \geq 0,\ \mathbf{y} \in \Gamma \subset \mathbb{R}^p, \tag{2.42}$$

where K is a positive definite symmetric matrix. Take as a Lyapunov function candidate $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ where P is a positive definite symmetric matrix whose entries we want to determine in order to satisfy certain conditions. The derivative of $V(\mathbf{x})$ along trajectories of the system (2.35) - (2.37) is

$$\dot{V}(\mathbf{x}, t) = \mathbf{x}^T(PA + A^T P)\mathbf{x} - 2\mathbf{x}^T PB\psi(t, \mathbf{y}) \tag{2.43}$$

If $-2\psi^T(\psi - K\mathbf{y})$, which is always nonnegative by the sector condition (2.42), is added to the right-hand side of (2.43), we have an upper bound for $\dot{V}(\mathbf{x}, t)$:

$$\begin{aligned}
\dot{V}(\mathbf{x}, t) &\leq \mathbf{x}^T(PA + A^T P)\mathbf{x} - 2\mathbf{x}^T PB\psi(t, \mathbf{y}) - 2\psi^T(t, \mathbf{y})[\psi(t, \mathbf{y}) - K\mathbf{y}] \\
&= \mathbf{x}^T(PA + A^T P)\mathbf{x} - 2\mathbf{x}^T(C^T K - PB)\psi(t, \mathbf{y}) - 2\psi^T(t, \mathbf{y})\psi(t, \mathbf{y}) \\
&= \begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}^T \begin{bmatrix} PA + A^T P & C^T K - PB \\ KC - B^T P & -2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}
\end{aligned} \tag{2.44}$$

If we can find P such that bound (2.44) is always less than zero, then $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function for the system defined by (2.35) - (2.37). Thus, if we can solve the LMI for the matrix variable P given by

$$P \succ 0, \qquad \begin{bmatrix} PA + A^T P & C^T K - PB \\ KC - B^T P & -2 \end{bmatrix} \preceq 0 \tag{2.45}$$

P defines a Lyapunov function for the system and proves absolute stability of the system. The Kalman-Yakubovich-Popov (KYP) lemma, also called the positive real (PR) lemma, proves equivalence between feasibility of this LMI and the positive realness of the transfer matrix $Z(s) = I + KC(sI - A)^{-1}B = I + KG(s)$, where positive realness is defined as follows:

Definition 12. *The transfer matrix $Z(s)$ is* positive real *if*

$$Z(s) + Z(s)^* \succeq 0$$

for all s with $\mathbf{Re}\ s \geq 0$.

The condition $Z(s) = I + KG(s)$ be positive real is called the circle criterion because if $G(s)$ is a single-input single-output rational transfer function, i.e., p = 1, the criterion requires that the Nyquist plot of the open loop transfer function $G(j\omega)$ lie in a disk whose radius is defined by the sector bounds on the nonlinearity. A simplified presentation of the KYP lemma is give in Appendix C.

Theorem 13 ([12]). **Multivariable Circle Criterion** *Consider the system (2.35) - (2.37), where A is Hurwitz, (A, B) is controllable, (A, C) is observable, and $\psi(\cdot, \cdot)$ satisfies the sector condition (2.42) globally. Then, the system is absolutely stable if $Z(s) = I + KG(s)$ is strictly positive real. If (2.42) is satisfied only on a set $\Gamma \subset \mathbb{R}^p$, then the same condition on $Z(s)$ ensures that the system is absolutely stable with a finite domain.*

As previously mentioned, if the transfer function $G(s)$ is scalar, i.e., $p = 1$, the conditions of Theorem 13 can be verified graphically by examining the Nyquist plot of $G(j\omega)$ [12]. This is important because the Nyquist plot can be determined directly from experimental data. In addition, given the Nyquist plot of $G(j\omega)$, one can determine allowable sectors for which the system is absolutely stable.

Another graphical criterion for determining absolute stability also related to an LMI and based on Lyapunov techniques is the Popov criterion.

2.5.2 Popov Criterion

We again consider the absolute stability of system (2.35) - (2.37) where A is again assumed to be Hurwitz. $\psi(\cdot)$ is now a time-invariant sector nonlinearity satisfying

$$\psi^T(\mathbf{y})[\psi(\mathbf{y}) - K\mathbf{y}], \quad \text{for all } \mathbf{y} \in \Gamma \subset \mathbf{R}^p \tag{2.46}$$

where K is a positive definite symmetric matrix.[5] We also assume

$$\int_0^{\mathbf{y}} \psi^T(\mathbf{r})K d\mathbf{r} \geq 0, \quad \text{for all } \mathbf{y} \in \Gamma \subset \mathbf{R}^p. \tag{2.47}$$

See [12] for further details. In formulating the Popov criterion for stability, we consider the problem of finding a Lur'e type Lyapunov function of the form

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x} + 2\eta \int_0^{\mathbf{y}} \psi^T(\mathbf{r})K d\mathbf{r} \tag{2.48}$$

where $\eta \geq 0$ is to be chosen. The derivative of $V(\mathbf{x})$ along system trajectories is

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T(PA + A^TP)\mathbf{x} - 2\mathbf{x}^TPB\psi(\mathbf{y}) + 2\eta\psi^T(\mathbf{y})KC[A\mathbf{x} - B\psi(\mathbf{y})]. \tag{2.49}$$

We again can add $-2\psi^T(\psi - K\mathbf{y}) \geq 0$, to the right-hand side of (2.49) in order to get an upper bound on $\dot{V}(\mathbf{x})$. Then we have

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \mathbf{x}^T(PA + A^TP)\mathbf{x} - 2\mathbf{x}^TPB\psi(\mathbf{y}) + 2\eta\psi^T(\mathbf{y})KC[A\mathbf{x} - B\psi(\mathbf{y})] - 2\psi(\mathbf{y})[\psi(\mathbf{y}) - K\mathbf{y}] \\
&= \mathbf{x}^T(PA + A^TP)\mathbf{x} - 2\mathbf{x}^T(PB - \eta A^TC^TK - C^TK)\psi(\mathbf{y}) \\
&\qquad - \psi(\mathbf{y})(2I + \eta KCB + \eta B^TC^TK)\psi(\mathbf{y}) \\
&= \begin{bmatrix} \mathbf{x} \\ \psi(\mathbf{y}) \end{bmatrix}^T \begin{bmatrix} PA + A^TP & -PB + \eta A^TC^TK + C^TK \\ -B^TP + \eta KCA + KC & -2I - \eta(KCB + B^TC^TK) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \psi(\mathbf{y}) \end{bmatrix}.
\end{aligned}$$

[5]This is the sector condition (2.41) with $K_{min} = 0$.

By Lyapunov's theorem, solving the following LMI is enough to prove absolute stability:

$$\eta \geq 0, \quad P \succ 0, \quad \begin{bmatrix} PA + A^T P & -PB + \eta A^T C^T K + C^T K) \\ -B^T P + \eta KCA + KC & -2I - \eta(KCB + B^T C^T K) \end{bmatrix} \preceq 0 \tag{2.50}$$

where η and P are decision variables. The KYP lemma proves the equivalence between strict feasibility of this LMI and the strict positive realness of the transfer matrix $Z(s) = I + (1 + \eta s)KC(sI - A)^{-1}B$. This equivalence results in the *multivariable Popov criterion.*

Theorem 14 ([12])**. Multivariable Popov Criterion** *Consider the system (2.35) - (2.37) where A is Hurwitz, (A, B) is controllable, (A, C) is observable, and $\psi(\cdot)$ is a time-invariant nonlinearity that satisfies the sector condition (2.46) globally with a positive definite symmetric K. Suppose that $K\psi(\mathbf{y})$ is the gradient of a scalar function and (2.47) is satisfied globally. Then, the system is absolutely stable if there is $\eta \geq 0$, with $-1/\eta$ not an eigenvalue of A such that*

$$Z(s) = I + (1 + \eta s)KG(s)$$

is strictly positive real. If (2.46) and (2.47) are satisfied on a set $\Gamma \subset \mathbb{R}^p$, then the same condition on $Z(s)$ ensures that the system is absolutely with a finite domain.

If $G(s)$ is a single-input single-output system, the positive realness of $Z(s)$ can be tested graphically by verifying that the Nyquist plot of the linear part, i.e., $G(j\omega)$, is confined to a specific region of the complex plane. This important frequency domain criteria for $p = 1$ was given by Popov in 1961 [29]. Later, Yakubovich [50, 49] and Kalman [9, 10] established the connection between this frequency domain criterion and the feasibility of a set of linear matrix inequalities [2].

Next, we present yet another type of input-output stability.

2.6 Finite-Gain Input-Output Stability

Input-output stability of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{2.51}$$

characterizes the relationship between the input $\mathbf{u}$ and the output $\mathbf{y}$, and depends on how the inputs and outputs of the system are measured. When analyzing input-output properties of a nonlinear system it is best to think of the system as a map or nonlinear operator

$$\mathbf{y} = H\mathbf{u} \tag{2.52}$$

which specifies the output $\mathbf{y}$ in terms of the input $\mathbf{u}$. H must be a map from a suitably defined input space to output space [37]. A useful and common example of such suitably defined input and output spaces are normed linear spaces called L_p spaces. A norm gives a notion of lengths to vectors, and norm for a vector space is a function that maps vectors $\mathbf{g}$ to nonnegative real numbers $||\mathbf{g}||$ that satisfies the following properties

1. Positivity: $||\mathbf{g}|| > 0$ for $\mathbf{g} \neq 0$.
2. Homogeneity: $||a\mathbf{g}|| = |a|||\mathbf{g}||$.
3. Triangle inequality: $||\mathbf{g} + \mathbf{f}|| \leq ||\mathbf{g}|| + ||\mathbf{f}||$ for all $\mathbf{g}$ and $\mathbf{f}$ in the vector space.

A p-norm of a vector $\mathbf{g}$ is defined as

$$||\mathbf{g}||_p = (|g_1|^p + ... + |g_n|^p)^{1/p}$$

and

$$||\mathbf{g}||_\infty = \max_i |g_i|.$$

p-norms can be extended to signals and we can then define L_p spaces. $L_p^l[0, \infty)$ is the space of $\mathbb{R}^l$ valued functions with finite p-norm. In other words, an l-dimensional function $\mathbf{g}(\cdot)$ belongs to $L_p^l[0,\infty)$ if

$$||\mathbf{g}(\cdot)||_p = \left(\int_0^\infty ||\mathbf{g}(t)||_p^p dt\right)^{\frac{1}{p}} < \infty.$$

The L_∞ norm is defined as

$$||\mathbf{g}(\cdot)||_\infty = \sup_t \; ||\mathbf{g}(t)||_\infty,$$

where for a fixed value of t, $\mathbf{g}(t)$ is a vector.

A system is finite-gain L_p input-output stable if the output will be in L_p^q whenever the input is in L_p^m where m is the dimension of the input and q is the dimension of the output.

The mapping H cannot be defined as a mapping from L_p^m to L_p^q because some systems will be unstable and thus their output will not be in L_p^q even when the input is in L_p^m. Therefore, the mapping H is defined as a mapping from an extended space L_{pe}^m to an extended space L_{pe}^q. A function $\mathbf{g}(\cdot)$ is said to belong to the *extended L_p^n space* denoted $L_{pe}^n[0,\infty)$, if for every truncation $\mathbf{g}_T(\cdot)$, where the truncation is defined to be

$$\mathbf{g}_T(t) = \begin{cases} \mathbf{g}(t) & \text{for } 0 \leq t \leq T \\ \mathbf{0} & \text{for } t > T \end{cases},$$

we have $\mathbf{g}_T(\cdot) \in L_p^n$. For example, the signal $u(t) = t$ does not belong to the space L_∞^1. However, its truncation $u_T(t)$ belongs to L_∞^1 for every finite T and thus $u(t) = t$ belongs to the extended space $L_{\infty e}^1$.

An important notion in finite-gain input-output stability analysis is that of causality. A mapping $H : L_{pe}^n \to L_{pe}^m$ is said to be *causal* if for every input $u(\cdot) \in L_{pe}^n$ and every $T > 0$ it follows that $(H(u))_T = (H(u)_T)_T$. In other words, a system is causal if the value of the output $(H\mathbf{u})(t)$ at any time t depends only on the input up to time t. We use this the notion of causality in the following definition of finite-gain L_p stability.

Definition 15 ([12])**.** *A causal mapping H is* finite-gain L_p stable *if there exist*

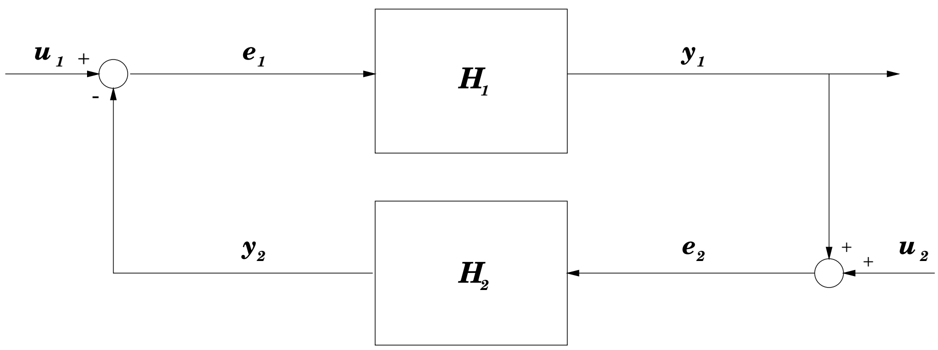


Figure 2-3: Feedback interconnection of two systems.

nonnegative constants γ *and* β *such that*

$$||(H\mathbf{u})_T||_p \leq \gamma||\mathbf{u}_T||_p + \beta \tag{2.53}$$

for all $\mathbf{u} \in L_{pe}^m$ *and* $T \in [0, \infty)$.

The constant β is included for systems where $H\mathbf{u}$ does not vanish at $\mathbf{u} = \mathbf{0}$. When γ is well defined and also the smallest possible constant for which (2.53) holds for some β, it is called the *gain* of the system. Finite-gain L_p stability with $p = \infty$ becomes the well-known notion of bounded-input bounded-output stability.

2.6.1 Small Gain Theorem

The notion of finite-gain L_p stability is useful for studying the stability of interconnected systems because gain values allow tracking of how a signal norm changes as it goes through a system. It is particularly useful for studying the feedback interconnection in Figure 2-3. Figure 2-3 suggests the question of whether the feedback connection is an L_p stable mapping from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$ when both systems H_1 and H_2 are finite-gain L_p stable with gains γ_1 and γ_2. It is assumed the interconnection is well-posed in the sense that for every pair of inputs $\mathbf{u}_1$ and $\mathbf{u}_2$ there exist unique outputs $\mathbf{e}_1$, $\mathbf{y}_1$, $\mathbf{e}_2$, and $\mathbf{y}_2$. Sandberg [36] and Zames [52, 53] answered this question in the 1960s by small gain arguments, which state the feedback interconnection of two input-output stable systems is input-output stable provided the closed loop gain is less than unity.

Theorem 16 ([12, 37]). *Assume both systems $H_1 : L_{pe}^m \to L_{pe}^q$ and $H_2 : L_{pe}^q \to L_{pe}^m$ in Figure 2-3 are both finite gain L_p stable. Assume the gain of H_1 is γ_1 and the gain of H_2 is γ_2. Also assume that the system is well defined. Then if*

$$\gamma_1\gamma_2 < 1,$$

the system viewed as a mapping from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$ is L_p stable.

A proof of this theorem is given in [12] and [37].

This theorem has several applications. One is its use in proving robust stability. Many uncertain dynamical systems can be represented in the form of a feedback connection with H_1 as a stable nominal system and H_2 as a stable perturbation [12]. In Section 2.7.1 we show how to prove a version of the small gain theorem by solving an LMI.

We have seen how we can conclude various stability properties of dynamical systems if a we can find solutions to related LMIs. In the next section, we discuss a generalization of many of these stability techniques called the integral quadratic constraint (IQC) framework developed by Alexandre Megretski and Anders Rantzer in the mid-1990s.

2.7 Integral Quadratic Constraints

In the mid-1990s, Megretski and Rantzer introduced a "unified approach to robustness analysis with respect to nonlinearities, time-variations, and uncertain parameters" [20] with the notion of *integral quadratic constraints* (IQCs). In [20, 34, 19] Megretski and Rantzer discuss how IQCs can be used to describe and analyze complex systems.

The signals considered in the IQC approach are those in $L_2^l[0, \infty)$, i.e., those that are square integrable:

$$||\mathbf{f}||^2 = \int_0^\infty |\mathbf{f}(t)|^2 dt.$$

The frequency domain representation of these signals can be found via the Fourier transform

$$\hat{\mathbf{f}}(j\omega) = \int_0^\infty e^{-j\omega t}\mathbf{f}(t)dt.$$

Operators on such signals are functions $F : L^a_{2e}[0,\infty) \to L^b_{2e}[0,\infty)$ where $L_{2e}[0,\infty)$ is a superset of $L_2[0,\infty)$. The elements of $L_{2e}[0,\infty)$ need to be square integrable only on finite intervals. The gain of the operator is given by

$$||F|| = \sup\{||F(\mathbf{f})||/||\mathbf{f}|| : \ \mathbf{f} \in L^a_{2e}[0,\infty), \ \mathbf{f} \neq 0\}.$$

Signals $\mathbf{w}(t) \in L^m_2[0,\infty)$ and $\mathbf{v}(t) \in L^l_2[0,\infty)$ are said to satisfy the (frequency domain) IQC defined by an appropriately dimensioned matrix $\Pi(j\omega)$, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{\mathbf{v}}(j\omega) \\ \hat{\mathbf{w}}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{\mathbf{v}}(j\omega) \\ \hat{\mathbf{w}}(j\omega) \end{bmatrix} d\omega > 0 \tag{2.54}$$

where absolute integrability is assumed. In the IQC (2.54) the Fourier transforms $\hat{\mathbf{v}}(j\omega)$ and $\hat{\mathbf{w}}(j\omega)$ represent the harmonic spectrum of the signals $\mathbf{v}(t)$ and $\mathbf{w}(t)$ and the IQC describes the energy distribution in the spectrum of $[\mathbf{v}\ \mathbf{w}]$. In most IQCs, $\Pi : j\mathbb{R} \to \mathbb{C}^{(l\times m)\times(l\times m)}$ is a rational function bounded on the imaginary axis, though in general it can be any measurable Hermitian-valued function.[6]

Frequency domain IQCs can describe energy relationships between signals. For example, assume we know that

$$\int_{-\infty}^{\infty} a(j\omega)|\hat{v}(j\omega)|^2 d\omega \geq \int_{-\infty}^{\infty} a(j\omega)|\hat{w}(j\omega)|^2 d\omega, \tag{2.55}$$

where $a(j\omega)$ is as given in Figure 2-4. Then if most of the energy of v is concentrated at high frequencies, the left hand side of (2.55) is small and we then know $\int_{-\infty}^{\infty} a(j\omega)|\hat{w}(j\omega)|^2$ must be small.

[6]Hermitian-valued means $\Pi(j\omega)$ is Hermitian for every ω.

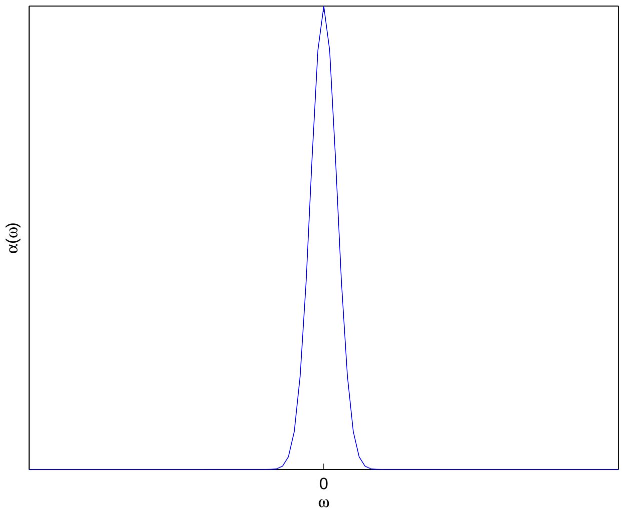


Figure 2-4: Weighting function - $\alpha(\omega)$.

A time domain form of (2.54) is

$$\int_0^\infty \sigma(\mathbf{x}_\pi(t), \mathbf{u}(t), \mathbf{v}(t))dt \geq 0, \tag{2.56}$$

where σ is a quadratic form, and $\mathbf{x}_\pi$ is defined by

$$\dot{\mathbf{x}}_\pi(t) = A_\pi \mathbf{x}_\pi + B_w \mathbf{w}(t) + B_v \mathbf{v}(t), \quad \mathbf{x}_\pi(0) = 0 \tag{2.57}$$

where A_π is a Hurwitz matrix. The IQC (2.54) can be expressed as (2.56), (2.57) by factorizing Π as $\Pi(j\omega) = \Psi(j\omega) M \Psi(j\omega)$ with $\Psi(j\omega) = F_\psi(j\omega I - A_\pi)^{-1}[B_w \ B_v] + G_\psi$ for some F_ψ, G_ψ, and then defining σ from F_ψ, G_ψ, and M [20].

Example 17. *Consider the LTI system described by the equations*

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.58}$$

where $\mathbf{x}(0) = 0$*. An IQC which describes (2.58) is*

$$\sigma_P(\mathbf{x}, \mathbf{u}) = 2\mathbf{x}^T P(A\mathbf{x} + B\mathbf{u}), \quad P \succeq 0. \tag{2.59}$$

If $P \succeq 0$,

$$\int_0^\infty \sigma_P(\mathbf{x}(t), \mathbf{u}(t))dt = \int_0^\infty \frac{d(\mathbf{x}(t)^T P \mathbf{x}(t))}{dt} = \lim_{T\to\infty} \mathbf{x}^T(T)P\mathbf{x}(T) - \mathbf{x}^T(0)P\mathbf{x}(0) \geq \mathbf{x}^T(0)P\mathbf{x}(0).$$

Since $\mathbf{x}^T(0)P\mathbf{x}(0) = 0$, *we have*

$$\int_0^\infty \sigma_P(\mathbf{x}(t), \mathbf{u}(t))dt \geq 0$$

and thus system (2.58) satisfies the IQC defined by (2.59).

Time domain and frequency domain IQCs are useful in characterizing system components. A bounded operator $\Delta : L_{2e}^l[0,\infty) \to L_{2e}^m[0,\infty)$ is said to satisfy the IQC defined by Π if (2.54) holds for all $\mathbf{w} = \Delta(\mathbf{v})$ where $\mathbf{v} \in L_{2e}^l[0,\infty)$. In [20] Megretski and Rantzer give a list of IQCs that characterize the following bounded operators Δ: uncertain linear time-invariant dynamics, a constant real scalar, a time-varying real scalar, coefficients from a polytope, a periodic real scalar, multiplication by a harmonic oscillation, a slowly time-varying real scalar, delay, a memoryless nonlinearity in a sector, and a monotonic odd nonlinearity. They also discuss the use of IQCs in specifying bounds on the autocorrelation or frequency distribution of signals.

Example 18. *An IQC characterizing passivity of a system component* $\Delta : \mathbf{v} \to \mathbf{w}$ *is*

$$\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

The IQC defined by Π *is*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{\mathbf{v}}(j\omega) \\ \hat{\mathbf{w}}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}(j\omega) \\ \hat{\mathbf{w}}(j\omega) \end{bmatrix} d\omega \geq 0. \tag{2.60}$$

which reduces to

$$\int_{-\infty}^{\infty} 2\hat{\mathbf{v}}^*(j\omega)\hat{\mathbf{w}}(j\omega)d\omega \geq 0. \tag{2.61}$$

By Parseval's relation, (2.61) is equivalent to

$$\int_0^\infty 2\mathbf{v}^T(t)\mathbf{w}(t) \geq 0$$

and thus,

$$\int_0^\infty \mathbf{v}^T(t)\mathbf{w}(t) \geq 0. \tag{2.62}$$

A system $\Delta : \mathbf{v} \to \mathbf{w}$ *is passive if (2.62) holds. In conclusion, if a system component satisfies (2.60), it is passive.*

2.7.1 System Analysis via IQCs and LMIs

We now discuss a general framework for system analysis via IQCs, originally developed by Megretski and Rantzer. We present the framework with time-domain IQCs, but note the same procedure can be carried out with frequency-domain IQCs. First, we represent the system we want to analyze in a general behavioral model form. A behavioral model lists a set $S = \{\mathbf{z}\}$ of variables of interest and the constraints that they must satisfy. Any combination of variables that satisfies the constraints is termed a *behavior* of the model. For example, in Example (17) the signals of interest are $\mathbf{x}$ and $\mathbf{u}$. Any combination of $\mathbf{x}$ and $\mathbf{u}$ that satisfy (2.58) is termed a behavior of the model. Analysis of a behavioral system model $S = \{\mathbf{z}\} \subset L_{2e}^k$, via IQCs is performed in three steps [18]:

1. Describe the system S by a sufficiently rich set of quadratic forms $\{\sigma_k(\mathbf{z})\}, \quad k = 1, ..., N$, each of which is satisfied by S. These IQCs characterize the system S.

2. Describe the analysis objective, i.e., the objective to be satisfied by S, as another quadratic constraint $\sigma_0(\mathbf{z})$. Examples of objectives are stability or performance objectives.

3. Use LMI optimization to search for multipliers (scalings) that ensure that if the system satisfies the IQCs defined by $\sigma_1, ...\sigma_N$, then it also satisfies the analysis objective IQC defined by σ_0.

Step 3 is based on the idea that if a system is described by the IQCs

$$\begin{aligned}\int_0^\infty \sigma_1(\mathbf{z}(t))dt &> 0\\ \int_0^\infty \sigma_2(\mathbf{z}(t))dt &> 0\\ &\vdots\\ \int_0^\infty \sigma_N(\mathbf{z}(t))dt &> 0,\end{aligned}$$

it is sufficient to find constants $d_i \geq 0$ such that

$$\sigma_0(\mathbf{z}) \geq \sum_{i=1}^{N} d_i \sigma_i(\mathbf{z}) \quad \text{for all} \quad \mathbf{z} \in \mathbb{R}^k,$$

to prove the system satisfies the analysis objective

$$\int_0^\infty \sigma_0(\mathbf{z}(t))dt > 0.$$

The following theorem formalizes this idea.

Theorem 19 ([18]). *Let $S = \{\mathbf{z}\} \subset L_{2e}^k$ be a behavioral system model satisfying the IQC defined by quadratic forms $\sigma_i : \mathbb{R}^k \to R$ for $k = 1, ..., N$. Let $\sigma_0 : \mathbb{R}^k \to \mathbb{R}$ be another quadratic form. If there exist constants $d_i \geq 0$ such that*

$$\sigma_0(\mathbf{z}) \geq \sum_{i=1}^{N} d_i \sigma_i(\mathbf{z}) \quad \textit{for all} \quad \mathbf{z} \in \mathbb{R}^k, \tag{2.63}$$

then S satisfies the IQC defined by σ_0.

Proof. By assumption we have that for every $\mathbf{z} \in S$

$$\int_0^\infty \sigma_i(\mathbf{z}(t))dt \geq 0 \quad \text{for all} \ \ i \in 1, ..., N.$$

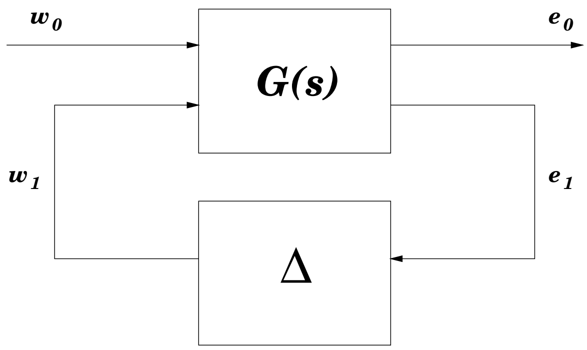


Figure 2-5: Standard description for uncertainty.

Multiplying each inequality by $d_i \geq 0$ and adding them together yields

$$\int_0^\infty \left\{ \sum_{i=1}^{N} d_i \sigma_i(\mathbf{z}(t)) \right\} dt \geq 0.$$

Hence, by (2.63)

$$\int_0^\infty \sigma_o(\mathbf{z}(t)) dt \geq 0.$$

$\square$

This theorem is exactly the S-Procedure discussed Section 2.1.1.

Because inequality (2.63) is comprised of quadratic forms, once σ_i for $i = 1, ...N$ and σ_0 are given, the search for coefficients $d_i \geq 0$ reduces to solving an LMI. If the LMI (2.63) is infeasible, i.e., no set of $d_i \geq 0 \quad i = 1, ..., N$ can be found, the IQC analysis fails because either the set of IQCs used is not rich enough, or because the conclusion to be reached is not true [18].

We can use these ideas to show a version of the small gain theorem holds if we can find a solution to a particular LMI. To do this, we consider the interconnection shown in Figure 2-5 which is a standard model for uncertainty. $G(s)$ is a linear time-invariant system and Δ is a bounded uncertain operator. One version small gain theorem states if the L_2 gain from $\mathbf{e}_1$ to $\mathbf{w}_1$ is less than 1, i.e., $||\Delta||_\infty \leq 1$, and if the L_2 gain of $G(s)$ is less than 1, i.e., $||G||_\infty < 1$, then the L_2 gain from $\mathbf{w}_0$ to $\mathbf{e}_0$ is less than 1.

The linear system $G(s)$ can be put into state-space form, i.e.,

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B_0\mathbf{w}_0 + B_1\mathbf{w}_1\\ \mathbf{e}_0 &= C_0\mathbf{x} + D_{00}\mathbf{w}_0 + D_{01}\mathbf{w}_1\\ \mathbf{e}_1 &= C_0\mathbf{x} + D_{10}\mathbf{w}_0 + D_{11}\mathbf{w}_1.\end{aligned}$$

For simplicity, let $D_{ij} = 0$. We saw in Example 17 that the $G(s)$ component of the system satisfies the IQC given by

$$\sigma(\mathbf{x}, \mathbf{w}_0, \mathbf{w}_1) = 2\mathbf{x}^T P(A\mathbf{x} + B_0\mathbf{w}_0 + B_1\mathbf{w}_1), \quad P \succeq 0.$$

Since $||\Delta||_\infty < 1$, the Δ component satisfies an IQC defined by

$$\sigma_1(\mathbf{x}, \mathbf{w}_0, \mathbf{w}_1) = -|\mathbf{w}_1|^2 + |C_1\mathbf{x}|^2,$$

where $C_1\mathbf{x} = \mathbf{e}_1$. Our analysis objective is making the L_2 gain from $\mathbf{w}_0$ to $\mathbf{e}_0$ less than one. This is described by the IQC given by

$$\sigma_0(\mathbf{x}, \mathbf{w}_0, \mathbf{w}_1) = |\mathbf{w}_0|^2 - |C_0\mathbf{x}|^2,$$

where $C_0\mathbf{x} = \mathbf{e_0}$. In accordance with the method described above, if we can find scalars $d_1 \geq 0$ and $d_p \geq 0$ such that

$$\sigma_0(\mathbf{x}, \mathbf{w}_0, \mathbf{w}_1) \geq d_1\sigma_1(\mathbf{x}, \mathbf{w}_0, \mathbf{w}_1) + d_p\sigma_p(\mathbf{x}, \mathbf{w}_0, \mathbf{w}_1) \quad \text{for all } \mathbf{x}, \mathbf{w}_0, \mathbf{w}_1 \tag{2.64}$$

then the L_2 gain from $\mathbf{w}_0$ to $\mathbf{e}_0$ is less than one. Since $P \succeq 0$ is a search parameter we can let $d_p = 1$. Then we can rewrite inequality (2.64) as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w}_0 \\ \mathbf{w}_1 \end{bmatrix}^T \begin{bmatrix} -A^TP - PA - C_0^TC_0 - d_1C_1^TC_1 & -PB_0 & -PB_1 \\ -B_0^TP & I_l & 0 \\ -B_1^TP & 0 & d_1I_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w}_0 \\ \mathbf{w}_1 \end{bmatrix} \geq 0 \quad \text{for all } \mathbf{x}, \mathbf{w}_0, \mathbf{w}_1,$$

where l is the dimension of $\mathbf{w}_0$ and m is the dimension of $\mathbf{w}_1$. Thus, finding a $P \succeq 0$ and $d_1 \geq 0$ such that

$$\begin{bmatrix} -A^T P - PA - C_0^T C_0 - d_1 C_1^T C_1 & -PB_0 & -PB_1 \\ -B_0^T P & I_l & 0 \\ -B_1^T P & 0 & d_1 I_m \end{bmatrix} \succeq 0,$$

certifies that the L_2 gain from $\mathbf{w}_0$ to $\mathbf{e}_0$ is less than 1.

We have thus far shown how a variety of stability problems can be reformulated as LMIs. Other problems in system and control theory can also be reduced to LMIs and thus solved numerically. Many of these problems are discussed in *Linear Matrix Inequalities in System and Control Theory* by Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan [2]. In this monograph, the authors also discuss the history of LMIs in system and control theory. To provide our reader with the highlights of this history, we summarize their account in the following section.

2.8 A History of LMIs in System and Control Theory

After Lyapunov's work in 1890, the next major development occurred in the 1940s when Lur'e, Postnikov and other researchers in the Soviet Union used Lyapunov theory to solve some practical control problems. They developed stability criteria for a control system with a nonlinearity in the actuator which also had the form of LMIs, though they did not explicitly form matrix inequalities. Instead, they reduced their LMIs to polynomial inequalities which they checked by hand. Their research generated "excitement by the idea that Lyapunov's theory could be applied to important (and difficult) practical problems in control engineering" [2].

In the early 1960s, Yakubovich, Popov, Kalman, and other researchers reduced the LMIs in Lur'e's problem to simple graphical criteria. In 1961, Popov's reduction gave rise to his famous frequency-domain stability criterion for the absolute stability problem in 1961 [29]. Popov's criterion could be checked via graphical means by verifying the Nyquist plot of the linear part of the nonlinear Lur'e system in Figure 2-1 was confined to a specific region in the complex plane. Then, Yakubovich and Kalman connected the Popov criterion to existence of a positive definite matrix satisfying some given matrix inequalities. The lemma resulting their work is called the Kalman-Yakubovich-Popov (KYP) lemma or the Positive Real (PR) Lemma and is given in Appendix C. When the KYP lemma and its extensions were studied in the second half of the 1960s, it was realized that they were related to the ideas of passivity, the small-gain criteria introduced by Zames [52, 53] and Sandberg [36], and quadratic optimal control [2].

By the 1970s it was known that the LMI appearing in the Positive Real Lemma was equivalent to solving a certain algebraic Ricatti equation (ARE). A 1971 paper by J.C. Willems [47] states the LMI

$$\begin{bmatrix} A^T P + PA + Q & PB + C^T \\ B^T P + C & R \end{bmatrix} \succeq 0 \tag{2.65}$$

resulting from an optimal control problem can be solved by studying the symmetric solutions of the ARE

$$A^T P + PA - (PB + C^T)R^{-1}(B^T P + C) + Q = 0. \tag{2.66}$$

The term "linear matrix inequality" was coined by Willems, and in the 1970's was used in several papers to refer to the specific LMI given in (2.65).

By 1971, researchers knew of several "closed-form" or "analytic" solutions for special forms of LMIs. As mentioned, these were graphical methods, ARE methods, and direct methods for small systems.[7] The authors of [2] state that the next major advance of LMIs in control theory was the observation that "the LMIs that arise in system and control theory can be formulated as *convex optimization problems* that are amenable to computer solution." They note that one very important consequence of this observation is that it allows a solution to LMIs "for which no "analytic" solution

[7]The ARE equation is considered by control researchers to have an "analytic" solution since standard algorithms that solve it require very predictable computational effort [2].

has ever been found or none can be found." The authors suggest that Pyatnitskii and Skorodinskii were the first to "clearly and completely" formulate LMIs as convex optimization problems in their paper [33] in 1982. Then a few years later powerful and efficient algorithms for solving such convex optimization problems, namely, interior-point methods were introduced and developed. Boyd et al. describe this development:

> In 1984, N. Karmarkar introduced a new linear programming algorithm that solves linear programs in polynomial time, like the ellipsoid method, but in contrast to the ellipsoid method, is also very efficient in practice. Karmarkar's work spurred an enormous amount of work in the area of interior-point methods for linear programming (including the rediscovery of efficient methods that were developed in the 1960's but ignored). Essentially all of this research activity concentrated on algorithms for linear and (convex) quadratic programs. Then in 1988, Nesterov and Nemirovskii developed interior-point methods that apply directly to convex problems involving LMIs, and in particular, to the problems we encounter in this book. Although there remains much to be done in this area, several interior-point algorithms for LMI problems have been implemented and tested on specific families of LMIs that arise in control theory and found to be extremely efficient.[8]

Before the development of interior-point methods, a limited number of LMIs in system and control theory could be solved analytically or graphically. Now, however, interior point methods provide efficient computational solutions to the problems described in this chapter as well as many other system and control theory problems.

2.9 Summary

In this chapter we have seen how a variety of stability analysis problems for linear systems, and also systems which can be modeled as the feedback combination of a linear system with nonlinear elements, can be reduced to LMIs. We have also seen that LMI-based methods are powerful analytic tools for systems which can be represented in these forms.

It is desirable and necessary to analyze more general forms of nonlinear systems. In the next chapter, we'll see how to can extend the class of systems we can treat

[8]Karmarkar's paper is reference [11], and Nesterov's and Nemirovskii's book as reference [22].

by using sum of squares (SOS) methods, which enable application of computational LMI-based analysis to nonlinear systems with polynomial or rational dynamics.

Then in Chapter 4, we present the main contributions of the thesis, which demonstrate how to analyze *uncertain* systems with polynomial or rational dynamics with LMI-based techniques. Specifically, we show why contraction theory, a stability theory in which stability is defined incrementally between trajectories, provides a tractable framework for the difficult problem of analyzing stability of uncertain nonlinear systems. Uncertain nonlinear systems are difficult to analyze because uncertainty in a nonlinear system may change the positions of the equilibria of the system. In the linear systems Lur'e systems described in this chapter, the nonlinearities did not affect the equilibrium point, which always remained at the origin.

Chapter 3

Sum of Squares Techniques

SUM OF SQUARES (SOS) PROGRAMMING, a method combining elements of computational algebra and convex optimization, has recently been used to provide efficient convex relaxations for several computationally-hard problems [31]. It addresses the question of whether there exists a sum of squares decomposition for a given multivariate polynomial. Such a decomposition is a sufficient condition for the polynomial's global nonnegativity ($p(\mathbf{x}) \geq 0$ for all $\mathbf{x}$). In the univariate case, an SOS decomposition is both necessary and sufficient for global nonnegativity.

If a dynamical system has polynomial or rational dynamics, SOS programming provides a framework to analyze stability properties of the system. For example, a relaxation of Lyapunov's method allows for an algorithmic search of polynomial Lyapunov functions using SOS programming [25].

In the next chapter we develop an algorithm with SOS techniques to search for contraction metrics for arbitrary nonlinear systems with polynomial or rational dynamics. Existence of a contraction metric is a necessary and sufficient condition for global exponential convergence of system trajectories. In this chapter we describe SOS polynomials, matrices, and programs, and show how an SOS program can be solved via semidefinite optimization.

3.1 Sum of Squares (SOS) Polynomials, Matrices, and Programs

The problem of checking global nonnegativity of a polynomial arises in many areas of mathematics and engineering, particularly in system and control theory. Testing global nonnegativity of a polynomial function, however, is NP-hard. Thus, as Parrilo states in [27], "(unless P = NP) *any method* guaranteed to obtain the right answer *in every possible instance* will have unacceptable behavior for a problem with a large number of variables." This computational complexity can be avoided by using the sum of squares decomposition as a sufficient condition for global nonnegativity. The question of whether or not an SOS decomposition exists can be solved in polynomial time because, as we will see below, this existence question is equivalent to a semidefinite programming problem [27].

3.1.1 Sum of Squares Polynomials

Let $\mathbf{x} = [x_1 \ \cdots \ x_n]$ and let $\mathbb{R}[\mathbf{x}]$ denote the set of all multivariate polynomials in $x_1, ..., x_n$. A multivariate polynomial $p(x_1, x_2, ..., x_n) = p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is a sum of squares if there exist polynomials $f_1(\mathbf{x}), ..., f_m(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ such that

$$p(\mathbf{x}) = \sum_{i=1}^{m} f_i^2(\mathbf{x}). \tag{3.1}$$

Existence of an SOS decomposition implies global nonnegativity, i.e., $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. The SOS condition (3.1) is equivalent to the existence of a positive semidefinite matrix Q such that

$$p(\mathbf{x}) = \mathbf{z}^T(\mathbf{x})Q\mathbf{z}(\mathbf{x}) \tag{3.2}$$

where $\mathbf{z}(\mathbf{x})$ is a vector of monomials of degree less than or equal to half the degree of the polynomial p. For example, if p has degree $2d$, then $\mathbf{z} = [1, x_1, x_2, ..., x_n, x_1x_2, ..., x_n^d]$ If the polynomial is homogenous of degree $2d$, then it is sufficient to restrict the components of $\mathbf{z}$ to the monomials of degree d [28].[1] If the matrix Q in the representation above is positive semidefinite, then $p(\mathbf{x})$ is clearly nonnegative. The matrix Q, however is not unique as the variables in $\mathbf{z}$ are not algebraically independent. It may be

[1] A homogeneous polynomial is a multivariate polynomial with all terms having the same degree. For example, $x^3 + xyz + y^2z + z^3$ is a homogeneous polynomial of degree three.

positive semidefinite for some representations but not other representations. However, if the affine subspace of matrices Q satisfying $p(\mathbf{x}) = \mathbf{z}(\mathbf{x})^T Q \mathbf{z}(\mathbf{x})$ intersects the positive semidefinite matrix cone, then $p(\mathbf{x})$ is guaranteed to be a sum of squares [28].

Example 20 ([28])**.** *Consider the quartic form in two variables*

$$\begin{aligned}
p(x,y) &= 2x^4 + 2x^3y - x^2y^2 + 5y^4 \\
&= \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix} = \mathbf{z}(x,y)^T Q \mathbf{z}(x,y) \\
&= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3. \qquad (3.3)
\end{aligned}$$

The following linear inequalities must hold to have $p(x,y) = \mathbf{z}(x,y)^T Q \mathbf{z}(x,y)$:

$$q_{11} = 2, \quad q_{22} = 5, \quad q_{33} + 2q_{12} = -1, \quad 2q_{13} = 2, \quad 2q_{23} = 0. \qquad (3.4)$$

A positive semidefinite Q satisfying these linear inequalities can be found using semidefinite programming. A particular solution is given by

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

We therefore have the sum of squares decomposition

$$p(x,y) = \mathbf{z}^T L^T L \mathbf{z} = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2.$$

Determining whether a polynomial has an SOS decomposition can be generalized as follows:

1. Express the given polynomial $p(\mathbf{x})$ as a quadratic form in new variables $\{z_i\}$. The new variables are the original variables x_i plus all the monomials of degree less than or equal to $\deg(p(\mathbf{x}))/2$. Let $\mathbf{z} = [z_1(\mathbf{x}) \;\; z_2(\mathbf{x}) \;\; \ldots \;\; z_n(\mathbf{x})]^T$ where n is the number of z_i variables needed to list the monomials degree less than or equal to $\deg(p(\mathbf{x}))/2$. Then $p(\mathbf{x})$ can be represented as

$$p(\mathbf{x}) = \mathbf{z}^T Q \mathbf{z} \qquad (3.5)$$

where Q is a constant matrix.

2. Use semidefinite programming to search for a positive semidefinite matrix Q satisfying the equality constraints given by (3.5).

3. If a positive semidefinite Q is found, we can factorize Q as $Q = G^T G$. Ways to find such a factorization include eigenvalue decomposition and Cholesky factorization.

4. The decomposition $Q = G^T G$ implies $p(\mathbf{x})$ is an SOS as $p(\mathbf{x}) = \mathbf{z}^T Q \mathbf{z} = \mathbf{z}^T G^T G \mathbf{z} = ||G\mathbf{z}||^2 = \sum_{i=1}^n \left(\sum_{j=1}^n G_{ij} z_j \right)^2$. Conversely, if $p(\mathbf{x})$ can be written as an SOS, then expanding in monomials will produce the representation $p(\mathbf{x}) = \mathbf{z}^T Q \mathbf{z}$ [27].

3.1.2 Sum of Squares Matrices

An SOS decomposition certifies nonnegativity of a scalar polynomial for all values of the indeterminates. In order to design an algorithm to search for contraction metrics in the following chapter, we need to introduce a similar idea to ensure that a polynomial matrix is positive definite for every value of the indeterminates. A natural definition is as follows:

Definition 21 ([5]). *Consider a symmetric matrix with polynomial entries* $\mathbf{S}(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]^{m \times m}$, *and let* $\mathbf{y} = [y_1, \ldots, y_m]^T$ *be a vector of new indeterminates. Then* $\mathbf{S}(\mathbf{x})$ *is a* sum of squares matrix *if the scalar polynomial* $\mathbf{y}^T \mathbf{S}(\mathbf{x}) \mathbf{y}$ *is a sum of squares in* $\mathbb{R}[\mathbf{x}, \mathbf{y}]$.

For notational convenience, we also define a stricter notion:

Definition 22. *A matrix* $\mathbf{S}(\mathbf{x})$ *is a* strict SOS matrix *if* $\mathbf{S}(\mathbf{x}) - \epsilon \mathbf{I}$ *is an SOS matrix for some* $\epsilon > 0$.

Thus, a strict SOS matrix is a matrix with polynomial entries that is positive definite for every value of the indeterminates.

An equivalent definition of an SOS matrix can be given in terms of the existence of a polynomial factorization: $\mathbf{S}(\mathbf{x})$ is an SOS matrix if and only if it can be decomposed as $\mathbf{S}(\mathbf{x}) = \mathbf{T}(\mathbf{x})^T\mathbf{T}(\mathbf{x})$ where $\mathbf{T}(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]^{p\times m}$. For example,

$$\mathbf{M}(\mathbf{x}) = \begin{bmatrix} \omega^2 + \alpha^2(x^2+k)^2 & \alpha(x^2+k) \\ \alpha(x^2+k) & 1 \end{bmatrix}$$

is an SOS matrix for all values of α and k. Indeed, this follows from the decomposition $\mathbf{M}(\mathbf{x}) = \mathbf{T}(\mathbf{x})^T\mathbf{T}(\mathbf{x})$, where

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} \omega & 0 \\ \alpha(x^2+k) & 1 \end{bmatrix}.$$

SOS matrices have been recently used by Hol and Scherer [8] and Kojima [13] to produce relaxations of polynomial optimization problems with matrix definiteness constraints.

3.1.3 Sum of Squares Programs

Using the notion of an SOS polynomial as a primitive, we can introduce a convenient class of optimization problems. A *sum of squares program* is a convex optimization problem of the form:

$$\begin{aligned} \min_{\mathbf{y}} \quad & c_1y_1 + \cdots + c_my_m \\ \text{such that} \quad & P_i(\mathbf{x},\mathbf{y}) \text{ are SOS}, \quad i = 1, ..., p \end{aligned} \tag{3.6}$$

where each y_i is a scalar real decision variable, each c_i is given, $\mathbf{x} = [x_1, ..., x_n]^T$, $\mathbf{y} = [y_1, ..., y_m]^T$, and $P_i(\mathbf{x},\mathbf{y}) \triangleq C_i(\mathbf{x}) + y_1A_{i1}(\mathbf{x}) + \cdots + y_mA_{im}(\mathbf{x})$, where C_i, A_{ij} are given polynomials in the variables x_i. There has recently been much interest in SOS programming and SOS optimization as these techniques provide convex relaxations for various computationally hard optimization and control problems; see e.g. [27, 28, 15, 31] and the volume [7].

3.2 Sum of Squares Applications

3.2.1 Global Optimization of a Polynomial

One application of SOS programming is computing global lower bounds for polynomial functions. A number γ is a global lower bound for a polynomial $p(\mathbf{x})$ if and only if the polynomial $p(\mathbf{x}) - \gamma$ is nonnegative $(p(\mathbf{x}) - \gamma) \geq 0$. Because the polynomial $p(\mathbf{x}) - \gamma$ has coefficients that depend affinely on γ, we can formulate the optimization problem

$$\max \gamma \quad \text{such that} \quad p(\mathbf{x}) - \gamma \quad \text{is an SOS} \tag{3.7}$$

as a semidefinite program. In some cases, the solution to (3.7) yields an optimal bound. However, because the existence of an SOS decomposition is sufficient but not necessary for global nonnegativity, it is also possible to obtain conservative results.

Lyapunov Functions The sum of squares approach has been applied to the problem of finding Lyapunov functions for nonlinear systems [27, 25]. SOS techniques enable a search over affinely-parameterized polynomial or rational Lyapunov functions for systems with polynomial or rational dynamics. In the search, the condition that a Lyapunov function be positive definite and that its derivative along trajectories be negative definite are imposed as SOS constraints that depend on the coefficients of the Lyapunov function.

3.2.2 Nonnegativity on Regions

Nonnegativity of a given polynomial $p(\mathbf{x})$ on regions defined by constraints $g_i(\mathbf{x}) \geq 0$, and $h_j(\mathbf{x}) = 0$ can be proved via SOS techniques. Positivstellensatz-based relaxations can be used to show that if there exists a set of sum of squares polynomials $\{\sigma_i(\mathbf{x})\}$ and a set of polynomials $\{\lambda_j(\mathbf{x})\}$ such that

$$p(\mathbf{x}) = \sigma_0(\mathbf{x}) + \sum_j \lambda_j(\mathbf{x}) h_j(\mathbf{x}) + \sum_i \sigma_i(\mathbf{x}) g_i(\mathbf{x}), \tag{3.8}$$

then $p(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ that satisfy $g_i(\mathbf{x}) \geq 0$, and $h_j(\mathbf{x}) = 0$. If we cannot find $\{\sigma_i(\mathbf{x})\}$ and $\{\lambda_j(\mathbf{x})\}$ to satisfy the equality (3.8) we can extend the sum as follows

$$p(\mathbf{x}) = \sigma_0(\mathbf{x}) + \sum_j \lambda_j(\mathbf{x})h_j(\mathbf{x}) + \sum_i \sigma_i(\mathbf{x})g_i(\mathbf{x}) + \sum_{i_1,i_2} \sigma_{i_1,i_2}(\mathbf{x})g_{i_1}(\mathbf{x})g_{i_2}(\mathbf{x}) + \cdots \tag{3.9}$$

and do another search. The following result by Schmüdgen ensures that if the set defined by $g_i(\mathbf{x}) \geq 0$ and $h_i(\mathbf{x}) = 0$ is compact, and we extend the sum in (3.9) far enough, then we are guaranteed to be able to find a representation of the form (3.9) if $p(\mathbf{x})$ is positive on this set.

Theorem 23. *[39]* ***Schmüdgen*** *Let F be a semi-algebraic set of the form*

$$F = \{\mathbf{x} \in \mathbb{R}^n | g_1(\mathbf{x}) \geq 0, ...g_k(\mathbf{x}) \geq 0\}, \quad \textit{where } \ g_1, ..., g_k \in \mathbb{R}[\mathbf{x}].$$

If F is compact, then every polynomial which is strictly positive on F belongs to

$$\sum_{I \subseteq \{1,...,k\}} (\prod_{i \in I} g_i)\Sigma, \tag{3.10}$$

where Σ denotes an element of the set of SOS polynomials.

3.2.3 Constrained Optimization

Positivstellensatz-based relaxations can be used to compute a lower bound on constrained optimization problems of the form

$$\begin{aligned} \min p(\mathbf{x}) & & & \\ \text{subject to } g_i(\mathbf{x}) & \geq 0, & & i = 1, ..., M \\ h_j(\mathbf{x}) & = 0, & & j = 1, ..., N. \end{aligned}$$

Specifically, if there exists a set of sum of squares polynomials $\{\sigma_i(\mathbf{x})\}$, and a set of polynomials $\{\lambda_j(\mathbf{x})\}$ such that

$$f(\mathbf{x}) - \gamma = \sigma_0(\mathbf{x}) + \sum_j \lambda_j(\mathbf{x}) h_j(\mathbf{x}) + \sum_i \sigma_i(\mathbf{x}) g_i(\mathbf{x}) + \sum_{i_1,i_2} \sigma_{i_1,i_2}(\mathbf{x}) g_{i_1}(\mathbf{x}) g_{i_2}(\mathbf{x}) + \cdots \tag{3.11}$$

then γ is a lower bound for the constrained optimization problem stated above. For a given degree of the expression (3.11) we can maximize γ to get the best lower bound on $p(\mathbf{x})$. This bound becomes increasingly tight as the degree of the expression (3.11) is increased [32]. In the case where only the terms linear in $g_i(\mathbf{x})$ are used, i.e.,

$$f(\mathbf{x}) - \gamma = \sigma_0(\mathbf{x}) + \sum_j \lambda_j(\mathbf{x}) h_j(\mathbf{x}) + \sum_i \sigma_i(\mathbf{x}) g_i(\mathbf{x}), \tag{3.12}$$

has been used by Lasserre [15] to give relaxations of polynomial optimization problems.

3.2.4 Other Applications

In addition to these applications, SOS programming has been used for constrained optimization problems, matrix copositivity problems, the problem of finding an upper bound for the structured singular value μ (an important object in robust control theory), the max cut problem, efforts to obtain bounds on the worst-case probability of an event [27, 30], analysis of nonlinear time delay systems [23], safety verification of linear systems [51], stability of systems with constraints, input-output and dissipativity analysis, and analysis of hybrid systems [26].

3.3 Computational Tools - SOSTOOLS and SeDuMi

Several software packages exist for solving SOS programs [30, 16] and semidefinite programs [42, 43]. We use SOSTOOLS and SeDuMi for the examples presented in Chapter 4. SOSTOOLS [30] is a free third-party MATLAB toolbox for solving sum of squares programs. We saw in Section 3.1.1 how SOS programs can be solved by reformulating them as SDPs. SOSTOOLS automates the conversion from a sum of squares program (SOSP) to a semidefinite program, runs the SDP solver, and converts the resulting SDP solution back to the solution of the original SOSP [30]. SOSTOOLS currently can use either SeDuMi [42] or SDPT3 [43] as the SDP solver. SeDuMi is a free third-party MATLAB toolbox for efficiently solving large optimization problems with linear, quadratic, and semidefiniteness constraints [42]. SDPT3 is also a MATLAB software package for semidefinite programming. In the next chapter we use SOSTOOLS with SeDuMi for finding contraction metrics for nonlinear systems.

Chapter 4

Stability and Robustness Analysis of Nonlinear Systems via Contraction Metrics and SOS Programming

IN THIS CHAPTER we present the main contributions of this thesis. First, we'll demonstrate how contraction analysis (contraction theory), a theory in which stability is defined incrementally between trajectories, provides a tractable framework for the difficult problem of analyzing the stability of uncertain nonlinear systems. In applying contraction analysis, one searches for an object called a contraction metric, which ensures that a suitably-defined distance between nearby trajectories is always decreasing. If a global contraction metric exists, the system is said to be contracting and all trajectories converge exponentially fast to a single trajectory as the system evolves in time. If an autonomous system is globally contracting, it has a single equilibrium point and all trajectories will converge exponentially to this point. Additional details of contraction analysis are given in Section 4.1. A second contribution of this thesis is using SOS programming to develop an algorithm to computationally search for contraction metrics for nonlinear systems with polynomial or rational dynamics. A third is using SOS-based techniques to develop an algorithm that finds bounds on the uncertain parameters under which a uncertain system remains stable.

Contraction theory nicely complements Lyapunov theory, a standard nonlinear stability analysis technique described in Chapter 2, by providing an alternative framework in which to study convergence and robustness properties of nonlinear systems.

For autonomous systems, one can interpret the search for a contraction metric as the search for a Lyapunov function with a certain structure. This statement is explained further in Section 4.3. There are, however, advantages to searching for a contraction metric instead of searching explicitly for a Lyapunov function. In particular, we show that contraction metrics are useful for analyzing uncertain nonlinear systems. In general, nonlinear systems with uncertain parameters can prove quite troublesome for standard Lyapunov methods, since the uncertainty can change the equilibrium point of the system in very complicated ways. This forces the use of parameter-dependent Lyapunov functions to prove stability for a range of the uncertain parameter values. In general, it may be impossible to obtain any kind of closed form expression of the equilibria in terms of the parameters, thus complicating the direct parametrization of possible Lyapunov functions.

Much of the literature on parameter-dependent Lyapunov functions focuses on linear systems with parametric uncertainty [6, 4, 3, 1]. However, if a linear model is being used to study a nonlinear system around an equilibrium point, changing the equilibrium of the nonlinear system necessitates relinearization around the new equilibrium. If the actual position of the equilibrium, in addition to the stability properties of the equilibrium, depends on the uncertainty, it may be impossible to obtain any kind of closed-form expression for the equilibrium in terms of the uncertain parameters. Thus, parameterizing the linearization in terms of the uncertainty may not be an option.

A well-studied method of dealing with specific forms of nonlinearities is to model the nonlinear system as a feedback connection of a linear system and bounded uncertainty blocks. We saw in Section 2.5 that this technique is useful since computing stability for this type of system reduces to a computationally-tractable semidefinite optimization problem. Though this method works for various kinds of uncertainty, it is also desirable to find methods to study the stability of more general uncertain nonlinear systems that do not easily admit linear approximations with the nonlinearities covered with bounded uncertainty blocks.

Contraction theory provides a framework in which to study the stability behavior of more general uncertain nonlinear systems. This framework eliminates many of the restrictions and problems that may be encountered when trying to analyze uncertain nonlinear systems with traditional linearization techniques or Lyapunov methods. If a nominal system is contracting with respect to a contraction metric, it is often the case that the uncertain system with additive or multiplicative uncertainty within a certain range is still contracting with respect to the same metric, even if the perturbation has changed the position of the equilibrium. Thus, it is possible to determine a range of values of the uncertain parameter for which the system is stable without explicitly tracking how the uncertainty changes the equilibrium location. These ideas are discussed further in Section 4.3.

Another interesting feature of the contraction framework is its relative flexibility in incorporating inputs and outputs. For instance, to prove contraction of a class of systems with external inputs, it is sufficient to show the existence of a contraction metric with a certain structure. This feature, which we discuss in Section 4.4, is central in using contraction theory determine when a system of coupled nonlinear oscillators will synchronize. These oscillators occur in a variety of research fields such as mathematics, biology, neuroscience, electronics, and robotics. A summary of the application of contraction theory to the analysis of networks of nonlinear oscillators can be found in [46] and the references listed therein.

To translate the theoretical discussion above into effective practical tools, it is necessary to have efficient computational methods to obtain contraction metrics numerically. Sum of squares (SOS) programming provides one such method. As shown in Chapter 3, if we can formulate the search for contraction metrics as a SOS program, this program can be converted to a computationally-efficient semidefinite program.

In this chapter we use SOS programming to develop an algorithmic search for contraction metrics for the class of nonlinear systems with polynomial dynamics. We also show how to use SOS methods to find bounds on the maximum amount of uncertainty allowed in a system in order for the system to retain the property of being contracting with respect to the contraction metric of the unperturbed system. Additionally, we use SOS methods to optimize the contraction metric search in order to obtain a metric that provides the largest symmetric allowable uncertainty interval for which we can prove the system is contracting. For a globally contracting autonomous system, the system remains globally stable for parameters in this allowable uncertainty range.

4.1 Contraction Analysis

Contraction analysis is a relatively new stability theory for nonlinear systems analysis [17]. The theory attempts to answer the question of whether the limiting behavior of a given dynamical system is independent of its initial conditions and is a theory in which stability is defined incrementally between two arbitrary trajectories. It is used to determine whether nearby trajectories converge to one another and can also be used to determine if trajectories converge globally. This section summarizes the main elements of contraction analysis; a much more detailed account can be found in [17].

We consider dynamical systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), t), \tag{4.1}$$

where $\mathbf{f}$ is a nonlinear vector field and $\mathbf{x}(t)$ is an n-dimensional state vector. For this analysis, it is assumed that all quantities are real and smooth and thus that all required derivatives or partial derivatives exist and are continuous. This existence and continuity assumption clearly holds for polynomial vector fields.

Under the assumption that all quantities are real and smooth, we can obtain the following differential relation from equation (4.1):

$$\delta\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}(t)}(\mathbf{x}(t), t)\delta\mathbf{x}(t), \tag{4.2}$$

where $\delta\mathbf{x}(t)$ is an infinitesimal displacement at a fixed time. For notational convenience we often write $\mathbf{x}$ for $\mathbf{x}(t)$, but in all calculations it should be noted that $\mathbf{x}$ is a function of time.

The infinitesimal squared distance between two trajectories is $\delta\mathbf{x}^T\delta\mathbf{x}$. Using (4.2), the following equation for the rate of change of the squared distance between two trajectories is obtained:

$$\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) = 2\delta\mathbf{x}^T\delta\dot{\mathbf{x}} = 2\delta\mathbf{x}^T\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\delta\mathbf{x}. \tag{4.3}$$

If $\lambda_1(\mathbf{x}, t)$ is the largest eigenvalue of the symmetric part of the Jacobian $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}$ (i.e. the largest eigenvalue of $\frac{1}{2}(\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T)$), then it follows from (4.3) that

$$\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) \leq 2\lambda_1(\mathbf{x}, t)\delta\mathbf{x}^T\delta\mathbf{x}. \tag{4.4}$$

Integrating both sides gives

$$||\delta\mathbf{x}|| \leq ||\delta\mathbf{x}_o||\, e^{\int_0^t \lambda_1(\mathbf{x},t)dt}. \tag{4.5}$$

If $\lambda_1(\mathbf{x}, t)$ is uniformly strictly negative, i.e., $(\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T) \prec 0$ for all $\mathbf{x}$ and t, it follows from (4.5) that any infinitesimal length $||\delta\mathbf{x}(t)||$ converges exponentially to zero as time goes to infinity. We can calculate a finite distance between points P_1 and P_2 on different trajectories for a fixed time with the integral $\int_{P_1}^{P_2}\sqrt{\delta\mathbf{x}(t)^T\delta\mathbf{x}(t)}$. If $(\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T) \prec 0$ for all $\mathbf{x}$ and t, this finite distance between also converges exponentially to zero as time goes to infinity.

A more general definition of length can be given by

$$\delta\mathbf{x}^T\mathbf{M}(\mathbf{x}, t)\delta\mathbf{x} \tag{4.6}$$

where $\mathbf{M}(\mathbf{x}, t)$ is a symmetric, uniformly positive definite and continuously differentiable metric (formally, this defines a Riemannian manifold). This notion of infinitesimal distance with respect to a metric can be use to define a finite distance measure between two trajectories with respect to this metric. Specifically, the distance between two points P_1 and P_2 at a fixed time with respect to the metric $\mathbf{M}(\mathbf{x}, t)$ is defined as the shortest path length, i.e., the smallest path integral $\int_{P_1}^{P_2}\sqrt{\delta\mathbf{x}^T\mathbf{M}(\mathbf{x}, t)\delta\mathbf{x}}$ between these two points. Accordingly, a ball of center $\mathbf{c}$ with radius R is defined as the set of all points whose distance to $\mathbf{c}$ with respect to $\mathbf{M}(\mathbf{x}, t)$ is strictly less than R.

From the definition of infinitesimal length given in (4.6), the equation for the rate of change of the infinitesimal length becomes

$$\frac{d}{dt}(\delta\mathbf{x}^T \mathbf{M} \delta\mathbf{x}) = \delta\mathbf{x}^T (\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})\delta\mathbf{x} \tag{4.7}$$

where $\mathbf{M}$ is shorthand notation for $\mathbf{M}(\mathbf{x}, t)$. Convergence to a single trajectory occurs in regions where $(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})$ is uniformly negative definite. It should be noted that $\dot{\mathbf{M}} = \dot{\mathbf{M}}(\mathbf{x}, t) = \frac{\partial \mathbf{M}(\mathbf{x},t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial \mathbf{M}(\mathbf{x},t)}{\partial t}$. The above analysis leads to the following definition and theorem:

Definition 24 ([17])**.** *Given the system equations* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$*, a region of the state space is called a* contraction region *with respect to a uniformly positive definite metric* $\mathbf{M}(\mathbf{x}, t)$ *if* $(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})$ *is uniformly negative definite in that region.*

Theorem 25 ([17])**.** *Consider the system equations* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$*. Assume a trajectory starts in a ball of constant radius that is defined with respect to the metric* $\mathbf{M}(\mathbf{x}, t)$*, that is centered at a another given trajectory, and that is contained at all times in a contraction region with respect to the metric* $\mathbf{M}(\mathbf{x}, t)$*. Then the first trajectory will remain in that ball and converge exponentially to the center trajectory. Furthermore, global exponential convergence to the center trajectory is guaranteed if the whole state space is a contraction region with respect to the metric* $\mathbf{M}(\mathbf{x}, t)$*.*

Definition 24 provides sufficient conditions for a system to be globally contracting. By globally contracting we mean all trajectories exponentially converge to a single trajectory as time goes to infinity. Namely, the following criteria should be satisfied:

1. The matrix $\mathbf{M}(\mathbf{x}, t)$ must be a uniformly positive definite matrix, i.e.,

$$\mathbf{M}(\mathbf{x}, t) \succeq \epsilon \mathbf{I} \succ 0 \quad \text{for all } \mathbf{x}, t \text{ and for } \epsilon > 0. \tag{4.8}$$

2. The metric variation matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}}$ must be a uniformly negative

definite matrix, i.e.,

$$\mathbf{R}(\mathbf{x},t) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} \preceq -\epsilon \mathbf{I} \prec 0 \quad \text{for all } \mathbf{x}, t, \text{ and for } \epsilon > 0. \tag{4.9}$$

An explicit rate of convergence (β) of trajectories can be found by finding a $\mathbf{M}(\mathbf{x},t)$ that satisfies (4.8) and

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} \preceq -\beta \mathbf{M}. \tag{4.10}$$

If the system dynamics are linear and $\mathbf{M}(\mathbf{x},t)$ is constant, i.e., $\mathbf{M}(\mathbf{x},t) = M$, the conditions above reduce to those in standard Lyapunov analysis techniques. As stated in Chapter 2, Lyapunov theory shows that the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ is stable, i.e., all trajectories converge to 0, if and only if there exists a positive definite matrix M such that $A^T M + MA \prec 0$. Based on our discussion in Chapter 2, we see the search for a constant contraction metric for a linear system trivially reduces to an LMI.

If a global contraction metric exists for an autonomous system, all trajectories converge to a unique equilibrium point. To prove this for a system with dynamics $\mathbf{f}(\mathbf{x})$ and an associated time-invariant contraction metric $\mathbf{M}(\mathbf{x})$, we can use $V(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{M}(\mathbf{x})\mathbf{f}(\mathbf{x})$ as a Lyapunov function to show that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ tends to $\mathbf{0}$ exponentially, and thus that $\mathbf{x}$ tends towards a finite equilibrium point [17]. For a constant metric $\mathbf{M}(\mathbf{x},t) = M$, this reduces to Krasovskii's Method [12]. We note that for systems with uncertainty there are good reasons to search for a contraction metric to create a Lyapunov function with this structure instead of searching for a Lyapunov function directly. We discuss these reasons in Section 4.3.

The problem of searching for a contraction metric thus reduces to finding a matrix function $\mathbf{M}(\mathbf{x},t)$ that satisfies the conditions (4.8) and (4.9). As we will see, SOS methods provide a computationally convenient approach to this problem.

4.2 Computational Search for Contraction Metrics via SOS Programming

As explained in Section 4.1, given a dynamical system, the conditions for a contraction metric to exist in regions of the state-space are given by a pair of matrix inequalities. In the case of metrics $\mathbf{M}(\mathbf{x})$ that do not depend explicitly on time, relaxing the matrix definiteness conditions in (4.8) and (4.9) to SOS matrix-based tests makes the search for contraction metrics a computationally tractable procedure. More specifically, the matrix definiteness constraints on $\mathbf{M}(\mathbf{x})$ (and $\mathbf{R}(\mathbf{x})$) can be relaxed to SOS matrix constraints by changing the inequality $\mathbf{M}(\mathbf{x}) - \epsilon \mathbf{I} \succeq 0$ in (4.8) (where ϵ is an arbitrarily small constant) to the weaker condition that $\mathbf{M}(\mathbf{x})$ be a strict SOS matrix. With these manipulations we see that existence of strict SOS matrices $\mathbf{M}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ is a sufficient condition for contraction. The definition of a strict SOS matrix was given in Section 3.1.2.

Lemma 26. *Existence of a strict SOS matrix* $\mathbf{M}(\mathbf{x})$ *and a strict SOS matrix* $-\mathbf{R}(\mathbf{x}) \triangleq -(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})$ *is a sufficient condition for global contraction of an autonomous system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *with polynomial dynamics.*

Proof. By Theorem 25, a sufficient condition for contraction of any nonlinear system is the existence of uniformly positive definite $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x})$. A sufficient condition for uniform positive definiteness of $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x})$ is the existence of strict SOS matrices $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x})$. □

This lemma can easily be extended to existence of certain SOS matrices implying contraction with a convergence rate β by redefining $\mathbf{R}(\mathbf{x})$ as $\mathbf{R}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} + \beta \mathbf{M}$. At this point, we do not know if the full converse of Lemma 26 holds. If a system is exponentially contracting, it is known that a contraction metric always exists [17]. Nevertheless, a system with polynomial dynamics may certainly be contracting under non-polynomial metrics. Furthermore, even if a positive definite contraction metric with polynomial entries $\mathbf{M}$ exists, it may not be the case that it is an SOS matrix. We notice, however, that some of these issues, such as the gap between "true" contraction metrics and SOS-based ones, can be bridged by using the more advanced techniques explained in [28].

4.2.1 Search Algorithm

One main contribution of this work is to show how SOS techniques can be used to algorithmically search for time-invariant contraction metrics for nonlinear systems with polynomial dynamics. For systems with polynomial dynamics, we can obtain a computationally tractable search procedure by restricting ourselves to a large class of SOS-based metrics.

As suggested by Lemma 26, the main idea is to relax the search for matrices that satisfy strict matrix definiteness constraints $\mathbf{M}(\mathbf{x}) \succ 0$ and $-\mathbf{R}(\mathbf{x}) \succ 0$ into strict SOS matrix conditions that are sufficient. Equivalently, we want to find a polynomial matrix $\mathbf{M}(\mathbf{x})$ that satisfies strict SOS matrix constraints on $\mathbf{M}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$. The SOS feasibility problem can then be formulated as finding $\mathbf{M}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ such that $\mathbf{y}^T\mathbf{M}(\mathbf{x})\mathbf{y}$ is a strict SOS and $-\mathbf{y}^T\mathbf{R}(\mathbf{x})\mathbf{y}$ is a strict SOS.

More specifically, the detailed steps in the algorithmic search of contraction metrics for systems with polynomial dynamics are as follows:

1. Choose the degree of the polynomials in the contraction metric, and write an affine parametrization of symmetric matrices of that degree. For instance, if the degree is two and the dynamical system has two dimensions, the general form of $\mathbf{M}(\mathbf{x})$ is

$$\begin{bmatrix} a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + a_4x_1 + a_5x_2 + a_6 & b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + b_4x_1 + b_5x_2 + b_6 \\ b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + b_4x_1 + b_5x_2 + b_6 & c_1x_1^2 + c_2x_1x_2 + c_3x_2^2 + c_4x_1 + c_5x_2 + c_6 \end{bmatrix}$$

   where a_i, b_i, and c_i are unknown coefficients.

2. Calculate $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ and define $\mathbf{R}(\mathbf{x}) := \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T\mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}}$. Thus, $\mathbf{R}(\mathbf{x})$ will also be a symmetric matrix with entries that depend affinely on the same unknown coefficients a_i, b_i, and c_i.

3. Change matrix constraints

$$\mathbf{M}(\mathbf{x}) \succ 0 \text{ for all } \mathbf{x},$$

and

$$\mathbf{R}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} \prec 0 \text{ for all } \mathbf{x}$$

into the following scalar constraints on polynomial functions:

$$p(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T \mathbf{M}(\mathbf{x})\mathbf{y} > 0 \text{ for all } \mathbf{x} \text{ and } \mathbf{y},$$

and

$$r(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T \mathbf{R}(\mathbf{x})\mathbf{y} = \mathbf{y}^T (\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})\mathbf{y} < 0 \text{ for all } \mathbf{x} \text{ and } \mathbf{y},$$

where $\mathbf{y}$ is an $n \times 1$ vector of new indeterminates.

4. Impose strict SOS constraints on $p(\mathbf{x}, \mathbf{y})$ and $-r(\mathbf{x}, \mathbf{y})$, and solve the associated SOS feasibility problem. If a solution exists, the SOS solver will find values for the unknown coefficients such that the constraints are satisfied. We remark again that the SOS feasibility problem is equivalent to a semidefinite feasibility problem, i.e., a linear matrix inequality (See Chapter 3.1.1).

5. Use the obtained coefficients a_i, b_i, c_i to construct the contraction metric $\mathbf{M}(\mathbf{x})$ and the corresponding $\mathbf{R}(\mathbf{x})$.

6. Optionally, for graphical presentation, independent verification, or if the convex optimization procedure runs into numerical error, further testing can be done to verify the validity of the computed solution. To do this, we can check if the matrix definiteness constraints $\mathbf{M}(\mathbf{x}) \succ 0$, and $\mathbf{R}(\mathbf{x}) \prec 0$ hold over a range of the state space by finding and plotting the eigenvalues over this range. If a true feasible solution does not exist, the minimum eigenvalue of $\mathbf{M}(\mathbf{x})$ will be negative or the maximum eigenvalue of $\mathbf{R}(\mathbf{x})$ will be positive. Either one

of these cases violates the matrix constraints which certify contraction. In most semidefinite programming solvers, the matrix Q in (3.2) is computed with floating point arithmetic. If Q is near the boundary of the set of positive semidefinite matrices, it is possible for the sign of eigenvalues that are zero or close to zero to be computed incorrectly from numerical roundoff and for the semidefinite program solver to encounter numerical difficulties. Numerical issues are further discussed at the end of the chapter.

7. An explicit lower bound on the rate of convergence can be found by using bisection to compute the largest β for which there exist matrices $\mathbf{M}(\mathbf{x}) \succ 0$ and $\mathbf{R}_\beta(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} + \beta \mathbf{M} \prec 0$.

For the specific examples presented later in the paper, we have used SOSTOOLS, an SOS toolbox for MATLAB developed for the specification and solution of SOS programs [30]. The specific structure of SOS matrices, or equivalently, the bipartite form of the polynomials $p(\mathbf{x}, \mathbf{y})$ and $r(\mathbf{x}, \mathbf{y})$ is computationally exploited through the option `sparsemultipartite` of the command `sosineq` that defines the SOS inequalities. Future versions of SOSTOOLS will allow for the direct specification of matrix SOS constraints.

Next we present two examples of using this procedure to search for contraction metrics for nonlinear systems with polynomial dynamics. The systems studied are a model of a jet engine with controller and a Van der Pol oscillator.

4.2.2 Example: Moore-Greitzer Jet Engine Model

The algorithm described was tested on the following dynamics, corresponding to the Moore-Greitzer model of a jet engine with stabilizing feedback operating in the no-stall mode [14]. In this model, a desired no-stall equilibrium is translated to the origin. The state variables correspond to $\phi = \Phi - 1$, $\psi = \Psi - \Psi_{co} - 2$, where Φ is the mass flow, Ψ is the pressure rise and Ψ_{co} is a constant [14]. The dynamic equations take the form:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ 3\phi - \psi \end{bmatrix} \tag{4.11}$$

The only real-valued equilibrium of the system is $\phi = 0$, $\psi = 0$. This equilibrium is stable.

The results of the algorithmic search for strict SOS matrices $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x})$ of various orders are given in Table 4.1. Values in the table, except the final row, are output values from SeDuMi [42], the semidefinite program solver used as the optimization engine in solving the SOS program. `CPU time` is the number of seconds it took for SeDuMi's interior point algorithm to find a solution. As expected, the computation time increases with the degree of the polynomial entries of $\mathbf{M}(\mathbf{x})$. `Feasibility ratio` is the final value of the feasibility indicator. This indicator converges to 1 for problems with a complementary solution, and to -1 for strongly infeasible problems. If the feasibility ratio is somewhere in between, this is usually an indication of numerical problems. The values `pinf` and `dinf` detect the feasibility of the problem. If `pinf` $= 1$, then the primal problem is infeasible. If `dinf` $= 1$, the dual problem is infeasible. If `numerr` is positive, the optimization algorithm (i.e., the semidefinite program solver) terminated without achieving the desired accuracy. The value `numerr` $= 1$ gives a warning of numerical problems, while `numerr` $= 2$ indicates a complete failure due to numerical problems.

As shown in Table 4.1, for this system no contraction metric with polynomial entries of degree zero or two could be found. This can be certified from the solution of the dual optimization problem. Since SeDuMi is a primal-dual solver, infeasibility certificates are computed as a byproduct of the search for contraction metrics.

Explicit lower bounds for the rate of convergence of the trajectories of the jet engine model, i.e., the largest value β for which $\mathbf{M}(\mathbf{x}) \succ 0$ and $\mathbf{R}_\beta(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} + \beta \mathbf{M} \prec 0$ were $\beta = 0.78$ for the 4^{th} degree metric and $\beta = 1.45$ for the 6^{th} degree metric.

We remark that, for this system, it is also possible to prove stability using standard Lyapunov analysis techniques. However, we illustrate stability of this example from

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	0	2	4	6
CPU time (sec)	0.140	0.230	0.481	0.671
Feasibility ratio	-1.000	-0.979	1.003	0.990
`pinf`	1	1	0	0
`dinf`	0	0	0	0
`numerr`	0	1	0	0
$M \succ 0$, $R \prec 0$ conditions met?	no	no	yes	yes
Convergence rate lower bound (β)	n/a	n/a	0.78	1.45

Table 4.1: Contraction metric search results for closed-loop jet engine dynamics.

a contraction viewpoint because contraction theory offers a good approach to study this system when there is parametric uncertainty in the plant dynamics or feedback equations. For example, the open loop jet dynamics equations are

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ -u \end{bmatrix} \tag{4.12}$$

where u is a control variable. If a nominal stabilizing feedback control u can be found (e.g., using backstepping [14] or some other design method), the SOS techniques described in Section 4.3 provide a way to find other stabilizing feedback controls which are centered around the nominal control. Specifically, if a stabilizing linear feedback control $u = k_1\phi + k_2\psi$ can be found, we can interpret k_1 and k_2 as uncertain parameters and use the methods described in Section 4.3 to search for ranges of gain values centered around the nominal values k_1 and k_2 that will also stabilize the system. We discuss the uncertain case in more detail in Section 4.3, but first we give another example of applying our search algorithm.

4.2.3 Example: Van der Pol Oscillator

The Van der Pol equation

$$\ddot{x} + \alpha(x^2 + k)\dot{x} + \omega^2 x = 0, \tag{4.13}$$

with $\alpha \geq 0$, k, and ω as parameters, played a central role in the development of nonlinear dynamics. Historically the equation arose from studying nonlinear electric circuits used in the first radios [41]. It can be regarded as an RLC electrical circuit or, equivalently, a mass-spring-damper system with a position-dependent damping coefficient. When $k < 0$, the solutions of (4.13) behave like a harmonic oscillator with a nonlinear damping term $\alpha(x^2 + k)\dot{x}$. In this case, the term provides positive damping when $x^2 > -k$ and negative damping when $x^2 < -k$. Thus, large amplitude oscillations will decay, but if they become too small they will grow larger again [41]. If $k > 0$ all trajectories converge to the origin. See Figure 4-1.

In Table 4.2 we present the results of running our contraction metric search algorithm for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\alpha(x_1^2 + k)x_2 - \omega^2 x_1 \end{bmatrix}, \tag{4.14}$$

with $\alpha = 1, \omega = 1$, which is the state-space version of the Van der Pol oscillator (4.13). We present solutions for various values of k. For each k the table displays the result of searching for a contraction metric containing quartic polynomials in each entry.

As a natural first step we searched for a constant contraction metric for each k. None could be found algorithmically. This was expected as it is easily shown analytically that a constant contraction metric for this system does not exist. Specifically, if $\mathbf{M}$ is constant, then

$$\mathbf{M} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & -x_1^2 - k \end{bmatrix},$$

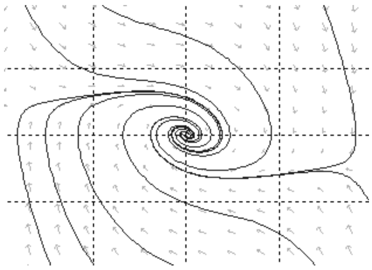
$$\mathbf{R} = \begin{bmatrix} -2b - 4bx_1x_2 & a - bx_1^2 - kb - c - 2cx_1x_2 \\ a - bx_1^2 - kb - c - 2cx_1x_2 & 2b - 2cx_1^2 - 2kc \end{bmatrix}. \tag{4.15}$$

For $\mathbf{R}$ to be negative definite $\mathbf{R}_{11}$ must be negative for all values of x_1, x_2. In other words, $-2b - 4bx_1x_2 \leq 0$ or $-1 \leq 2x_1x_2$. This inequality clearly does not hold for all values of x_1, and x_2. A more complicated analysis (or a duality argument) also shows why there is no contraction metric with quadratic entries for this system.

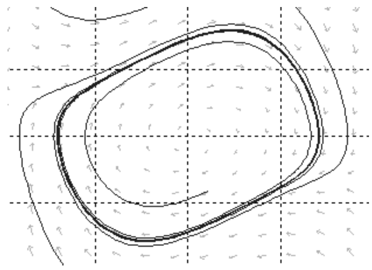
The algorithm finds a contraction metric for the system $\ddot{x} + (x^2 + k)\dot{x} + x = 0$

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	4	4	4	4	4	4	4	4	4
k	-10	-1	-0.1	-0.01	-0.001	0.001	0.01	0.1	1	10
pinf	1	1	1	0	0	0	0	0	0	0
dinf	0	0	0	0	0	0	0	0	0	0
numerr	1	1	1	1	1	0	0	0	0	0
$M \succ 0$, $R \prec 0$ conditions met?	no	no	no	no	no	yes	yes	yes	yes	yes

Table 4.2: Contraction metric search results for oscillator dynamics.



(a) $k = 0.5$



(b) $k = -0.5$

Figure 4-1: Phase portraits of Van der Pol Oscillator. The plots are vector fields of the two-dimensional nonlinear system (4.14) for $k = 0.5$ and $k = -0.5$. The vector $[x_1, x_2]^T$ represents a position point in the phase plane and $[\dot{x_1}, \dot{x_2}]^T$ is the velocity vector at that point. By flowing along the vector field we trace out a solution, i.e., a trajectory of the system. In Figure 4-1 the horizontal axis corresponds to x_1 and the vertical axis corresponds to x_2. The origin is at the center of each figure.

when $k > 0$ but not when $k < 0$. As shown in Figure 4-1, the trajectories of the oscillator converge to zero when $k > 0$, and converge to a limit-cycle when $k < 0$. Since the system is not contracting when $k < 0$, we should not be able to find a contraction metric. We note, however, that the converse does not hold. The fact that we cannot find a contraction function does not necessarily mean that the system is not contracting. Finding an SOS representation of the constrained quadratic functions is a sufficient condition for their positivity, not a necessary one.

For this example we can prove stability through Lyapunov analysis, and SOS programming can also be used to find Lyapunov functions [25]. However, we present this example in order to introduce the Van der Pol oscillator. We show in Section 4.4 how contraction theory provides a nice way to prove synchronization of coupled Van der Pol oscillators. This synchronization property is much more difficult to prove with standard Lyapunov methods.

4.3 Uncertainty Analysis with Contraction Metrics and SOS Programming

From the robust control perspective, one of the most appealing features of contraction theory is that it provides a nice framework in which to study uncertain nonlinear systems where the parametric uncertainty changes the location of the equilibrium points. In general, standard Lyapunov analysis does not handle this situation particularly well, since the Lyapunov function must track the changes in the location of the steady-state solutions. This forces the use of parameter-dependent Lyapunov functions. However, in general it may be impossible to obtain any kind of closed form expression of the equilibria in terms of the parameters, thus complicating the direct parametrization of possible Lyapunov functions.

Much attention has been given to robust stability analysis of linear systems (e.g., [6, 4, 3, 2, 54]). Less attention, however, has been paid to nonlinear systems with moving equilibria. Two papers addressing this issue are [21] and [1]. The approach in [21] is to consider systems described by the equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{h}(\mathbf{x}), \tag{4.16}$$

where $\mathbf{x}$ is a real n-vector, $\mathbf{f}$ and $\mathbf{h}$ are continuously differentiable functions, and $\mathbf{h}(\mathbf{x})$ represents the uncertainties or perturbation terms. Given an exponentially stable equilibrium $\mathbf{x}_e$, the authors of [21] establishes sufficient conditions to prove existence and local exponential stability of an equilibrium $\tilde{\mathbf{x}}_e$ for (4.16) with the property $|\mathbf{x}_e - \tilde{\mathbf{x}}_e| < \varepsilon$ where ε is sufficiently small. They do this by using the linearization of the system to produce Lyapunov functions.

Since the approach in [21] is essentially based on a fixed Lyapunov function, it is more limited than our approach using contraction theory and SOS programming, and can prove stability only under quite conservative ranges of allowable uncertainty. A quantitative measure of this conservativeness will be given in Section 4.3.3 where we discuss the results of applying the method in [21] to an uncertain model of a Moore-

Greitzer jet engine and then compare these results to our approach via contraction theory and SOS programming.

The approach in [1] is to first linearize the dynamics around an equilibrium $\mathbf{x_0}$ that is a function of the uncertain parameter, i.e., $\mathbf{x_0} = \mathbf{g}(\delta),\ \delta \in \Omega$ where Ω is the uncertainty set, and to use structured singular values to determine the eigenvalues of the linearized system $\frac{d\mathbf{z}}{dt} = A(\delta)\mathbf{z}$. In this approach the Jacobian $A(\delta)$ must be rational in δ. If $A(\delta)$ is marginally stable, no conclusions can be made about the stability of the nonlinear system. The contraction theory framework eliminates the need for linearization, and even the need to know the exact position of the equilibrium, in order to analyze stability robustness in uncertain nonlinear systems.

As noted in Section 4.1, if a global time-invariant contraction metric exists for an autonomous system, all trajectories converge to a unique equilibrium point, and we can always produce a Lyapunov function of the form $V(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{M}(\mathbf{x})\mathbf{f}(\mathbf{x})$. When a system contains parametric uncertainty, this formula yields the parameter-dependent Lyapunov function $V(\mathbf{x},\delta) = \mathbf{f}(\mathbf{x},\delta)^T\mathbf{M}(\mathbf{x})\mathbf{f}(\mathbf{x},\delta)$ for ranges of the parametric uncertainty δ where the contraction metric for the nominal system is still a contraction metric for the system with perturbed dynamics. Thus, if a contraction metric can be found for the system under a range of uncertainty, we can easily construct a Lyapunov function which tracks the uncertainty for that range.

4.3.1 Case 1: Uncertainty ranges for which the system remains contractive with respect to the nominal metric.

If the uncertainty enters the dynamics affinely, in other words if $\mathbf{f}(\mathbf{x}) = \mathbf{f_1}(\mathbf{x}) + \delta\,\mathbf{f_2}(\mathbf{x})$, we can use SOS programming to estimate the range of uncertainty under which the contraction metric for the nominal system is still a contraction metric for the perturbed system. To do this we use the ideas discussed in the paragraph on robust constraints in Section 2.1.1. Specifically, to calculate this range, we can write an SOS program to minimize or maximize the amount of uncertainty allowed subject to the constraint $\mathbf{R}_\delta(\mathbf{x}) = \frac{\partial \mathbf{f}_\delta}{\partial \mathbf{x}}^T\mathbf{M}_{nom} + \mathbf{M}_{nom}\frac{\partial \mathbf{f}_\delta}{\partial \mathbf{x}} + \dot{\mathbf{M}}_{nom}(\mathbf{f}_\delta(\mathbf{x})) \prec 0$, where $\mathbf{f}_\delta(\mathbf{x})$ are the dynamics for the system with parametric uncertainty and $\mathbf{M}_{nom}$ is the contraction metric for the nominal system. The uncertainty bound is a decision variable in the SOS program and affinely enters the constraint above in the $\frac{\partial \mathbf{f}_\delta}{\partial \mathbf{x}}$ and $\mathbf{f}_\delta(\mathbf{x})$ terms.

If we have more than one uncertain parameter in the system, we can find a polytopic inner approximation of the set of allowable uncertainties with SOS Programming. For example, if we have two uncertain parameters, we can algorithmically find a polytope in parameter space for which the original metric is still a contraction metric. The convex hull of four points, each which can be found by entering one of the four combinations, $(\delta_1, \delta_2) = (\gamma, \gamma)$, $(\delta_1, \delta_2) = (\gamma, -\gamma)$, $(\delta_1, \delta_2) = (-\gamma, \gamma)$, or $(\delta_1, \delta_2) = (-\gamma, -\gamma)$, into the uncertainty values in $\mathbf{f}_{\delta=[\delta_1,\delta_2]^T}(\mathbf{x})$ and then maximizing γ subject to the constraint $\mathbf{R}_\gamma(\mathbf{x}) = \frac{\partial \mathbf{f}_\gamma}{\partial \mathbf{x}}^T \mathbf{M}_{nom} + \mathbf{M}_{nom} \frac{\partial \mathbf{f}_\gamma}{\partial \mathbf{x}} + \dot{\mathbf{M}}_{nom}(\mathbf{f}_\gamma(\mathbf{x})) \prec 0$, defines a polytope over which stability is guaranteed. This type of formulation was previously discussed in Section 2.1.1.

4.3.2 Case 2: Search for the largest symmetric uncertainty interval for which the system is contracting.

Alternatively, we can optimize the search for a metric $\mathbf{M}(\mathbf{x})$ that provides the largest symmetric uncertainty interval for which we can prove the system is contracting. If the scalar uncertainty δ enters the system dynamics affinely $(\mathbf{f}(\mathbf{x}) = \mathbf{f_1}(\mathbf{x}) + \delta\, \mathbf{f_2}(\mathbf{x}))$, we can perform this optimization as follows. First write $\mathbf{R}(\mathbf{x}, \delta) = \mathbf{R_0}(\mathbf{x}) + \delta \mathbf{R_1}(\mathbf{x})$. To find the largest interval $(-\gamma, \gamma)$ such that for all δ that satisfy $-\gamma < \delta < \gamma$ the system is contracting, introduce the following constraints into an SOS program:

$$\mathbf{M}(\mathbf{x}) \succ 0, \quad \mathbf{R_0}(\mathbf{x}) + \gamma \mathbf{R_1}(\mathbf{x}) \prec 0, \quad \mathbf{R_0}(\mathbf{x}) - \gamma \mathbf{R_1}(\mathbf{x}) \prec 0.$$

We note that γ multiplies the scalar decision coefficients a_i, b_i, and c_i in $\mathbf{R}_1(\mathbf{x})$ and thus we must use a bisection procedure and SOS programming to find the maximum value of γ for which there exists matrices $\mathbf{M}(\mathbf{x})$, $\mathbf{R_0}(\mathbf{x})$ and $\mathbf{R_1}(\mathbf{x})$ that satisfy the constraints above.

If two uncertain parameters enter the system dynamics affinely, we can extend the procedure above as follows. To find the largest uncertainty square with width and height γ such that for all δ_1 and δ_2 that satisfy $-\gamma < \delta_1 < \gamma$ and $-\gamma < \delta_2 < \gamma$ the system is contracting, first write $\mathbf{R}(\mathbf{x}, \delta_1, \delta_2) = \mathbf{R_0}(\mathbf{x}) + \delta_1\mathbf{R_1}(\mathbf{x}) + \delta_2\mathbf{R_2}(\mathbf{x})$ Then, introduce the following constraints into and SOS program:

$$\mathbf{M}(\mathbf{x}) \succ 0, \quad \mathbf{R_0}(\mathbf{x}) + \gamma\mathbf{R_1}(\mathbf{x}) + \gamma\mathbf{R_2}(\mathbf{x}) \prec 0, \quad \mathbf{R_0}(\mathbf{x}) + \gamma\mathbf{R_1}(\mathbf{x}) - \gamma\mathbf{R_2}(\mathbf{x}) \prec 0$$
$$\mathbf{R_0}(\mathbf{x}) - \gamma\mathbf{R_1}(\mathbf{x}) + \gamma\mathbf{R_2}(\mathbf{x}) \prec 0, \quad \mathbf{R_0}(\mathbf{x}) - \gamma\mathbf{R_1}(\mathbf{x}) - \gamma\mathbf{R_2}(\mathbf{x}) \prec 0. \quad (4.17)$$

As in the scalar uncertainty case, we can use a bisection procedure and SOS Programming to find the maximum value of γ for which there exists matrices $\mathbf{M}(\mathbf{x})$, $\mathbf{R_0}(\mathbf{x})$, $\mathbf{R_1}(\mathbf{x})$ and $\mathbf{R_2}(\mathbf{x})$ that satisfy the constraints above. In the case of a large number of uncertain parameters, standard relaxation and robust control techniques can be used to avoid an exponential number of constraints.

4.3.3 Example: Moore-Greitzer Jet Engine with Uncertainty

Scalar Additive Uncertainty

As described above, SOS programming can be used to find ranges of uncertainty under which a system with uncertain perturbations is still contracting with respect to the original contraction metric. The contraction metric found for the deterministic system continues to be a metric for the perturbed system over a range of uncertainty even if the uncertainty shifts the equilibrium point and trajectories of the system. For the Moore-Greitzer jet engine model, the dynamics in (4.11) were perturbed by adding a constant term δ to the first equation:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 + \delta \\ 3\phi - \psi \end{bmatrix} \quad (4.18)$$

In Table 4.3 we display the ranges of δ where the system was still contracting with the original contraction metric for 4^{th} and 6^{th} degree contraction metrics. Note that the range of allowable uncertainty is not symmetric.

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	6
δ range	(-0.126,0.630)	(-0.070, 0.635)

Table 4.3: Range of perturbation where closed-loop uncertain jet engine system given in (4.18) is contracting with respect to the nominal metric.

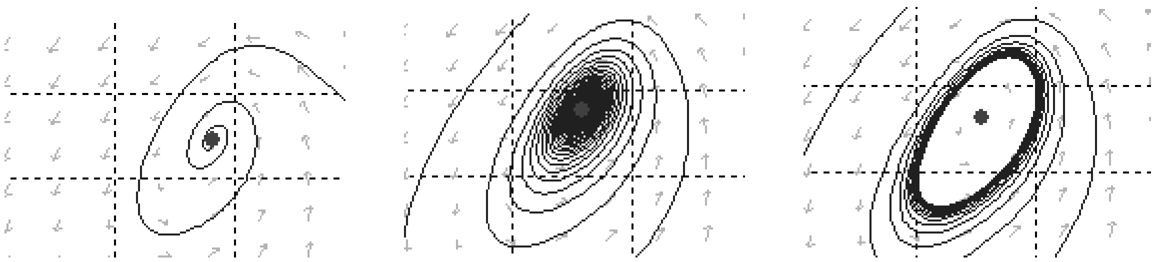
Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	6	8
δ range	$\|\delta\| \leq 0.938$	$\|\delta\| \leq 1.023$	$\|\delta\| \leq 1.023$

Table 4.4: Symmetric range of perturbation over which the uncertain closed-loop jet engine system given in (4.18) is contracting.

When we instead optimized the contraction metric search to get the largest symmetric δ interval we obtained the results listed in Table 4.4. A 6^{th} degree contraction metric finds the uncertainty range $|\delta| \leq 1.023$. Because a Hopf bifurcation occurs in this system at $\delta \approx 1.023$, making the system unstable for $\delta > 1.023$, we can conclude that the 6th degree contraction metric is the highest degree necessary to find the maximum range of uncertainty for which the system is contracting. The Hopf bifurcation is shown in Figure 4-2.

Using the techniques in [21] we computed the allowable uncertainty range for system (4.18) as $|\delta| \leq 5.1 \times 10^{-3}$. The allowable range $|\delta| \leq 1.023$ computed via contraction theory and SOS programming is much larger than the allowable uncertainty range $|\delta| \leq 5.1 \times 10^{-3}$ computed with the techniques in [21].[1]

[1] In the notation of [21], we calculated the other parameters in Assumption 1 of [21] as: $\mathbf{h} = [\delta, \ 0]^T$, $|A^{-1}|_\infty = 1$, $|D\mathbf{h}(\mathbf{x}_e)|_\infty = 0$, $a = \frac{1}{30}$, and $|\mathbf{h}(\mathbf{x}_e)|_\infty = |\delta|$, where δ is the perturbation term in (4.18).



(a) $\delta = -0.5$ (b) $\delta = -1.01$ (c) $\delta = -1.1$

Figure 4-2: Hopf bifurcation in uncertain jet dynamics.

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	6
δ range	(0.9767, 5.8686)	(0.9796, 3.9738)

Table 4.5: Range of perturbation for which the uncertain closed-loop jet engine system given in (4.19) is contracting with respect to the nominal metric.

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	6	8
δ range	$(1-0.247, 1+0.247)$	$(1-0.356, 1+0.356)$	$(1-0.364, 1+0.364)$

Table 4.6: Symmetric range of perturbation over which the uncertain closed-loop jet engine system given in (4.19) is contracting.

Scalar Multiplicative Uncertainty

The approaches in Section 4.3 also apply to multiplicative uncertainty, since multiplicative coefficients affinely enter the constraints in the SOS program. Tables 4.5 and 4.6 present the results of the described uncertainty analysis on the following system, which is equation (4.11) with multiplicative uncertainty.

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \delta(\frac{1}{2}\phi^3) \\ 3\phi - \psi \end{bmatrix}. \tag{4.19}$$

Multiple Uncertainties

We consider next the system that results from introducing two additive uncertainties to the jet dynamics in equation (4.11). We computed an uncertainty polytope (shown in Figure 4-3) for which the system

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 + \delta_1 \\ 3\phi - \psi + \delta_2 \end{bmatrix} \tag{4.20}$$

is guaranteed to be contracting with respect to the original metric. Alternatively, Table 4.7 shows the results of optimizing the contraction metric to find the largest uncertainty square with width and height γ such that for all δ_1 and δ_2 that satisfy $-\gamma < \delta_1 < \gamma$ and $-\gamma < \delta_2 < \gamma$, the system is contracting.

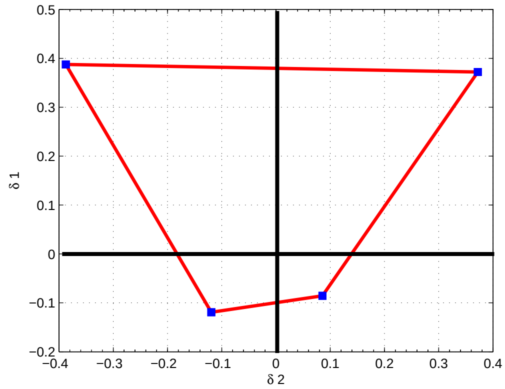


Figure 4-3: Polytopic region of uncertainty over which the closed-loop jet engine system given in (4.20) is contracting with respect to nominal metric.

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	6	8
Height and width of allowed uncertainty box	0.7093	0.7321	0.7346

Table 4.7: Symmetric range of perturbation over which the uncertain closed-loop system given in (4.20) is contracting.

4.4 Contraction and Systems with External Inputs

Another interesting feature of the contraction framework is its relative flexibility in incorporating inputs and outputs. For instance, to prove contraction of a class of systems with external inputs, it is sufficient to show the existence of a polynomial contraction metric with a certain structure. This is described in the following theorem.

Theorem 27. *Let*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\ \vdots &= \vdots \\ \dot{x}_k &= f_k(x_1, x_2, \ldots, x_n) \\ \dot{x}_{k+1} &= f_{k+1}(x_1, x_2, \ldots, x_n) + v_{k+1}(u) \\ \vdots &= \vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + v_n(u) \end{aligned} \tag{4.21}$$

be a set of nonlinear coupled differential equations where only the last $n-k$ depend explicitly on $u(t)$. If there exists an $n \times n$ matrix $\mathbf{M}(x_1, \ldots, x_k)$ such that $\mathbf{M} \succ 0$ and $\dot{\mathbf{M}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \prec 0$ then the system is contracting for all possible choices of $u(t)$.

Proof. For notational convenience, let

$$\begin{aligned} \dot{\mathbf{x}}_1 = [\dot{x}_1, \ \ldots \ \dot{x}_k]^T &= \mathbf{f_1}(\mathbf{x_1}, \mathbf{x_2}) \\ \dot{\mathbf{x}}_2 = [\dot{x}_{k+1} \ \ldots \ \dot{x}_n]^T &= \mathbf{f_2}(\mathbf{x_1}, \mathbf{x_2}, u). \end{aligned}$$

The metric $\mathbf{M}(x_1, x_2 ..., x_k) = \mathbf{M}(\mathbf{x_1})$ is independent of $\mathbf{x_2}$, and thus $\frac{\partial \mathbf{M}_{ij}}{\partial \mathbf{x_2}} = \mathbf{0}$ for all i, j. Since $\frac{\partial \mathbf{M}_{ij}}{\partial t}$ also vanishes, it follows that

$$\dot{\mathbf{M}}_{ij} = \frac{\partial \mathbf{M}_{ij}}{\partial \mathbf{x_1}} \frac{d\mathbf{x_1}}{dt} + \frac{\partial \mathbf{M}_{ij}}{\partial \mathbf{x_2}} \frac{d\mathbf{x_2}}{dt} + \frac{\partial \mathbf{M}}{\partial t} = \frac{\partial \mathbf{M}_{ij}}{\partial \mathbf{x_1}} \frac{d\mathbf{x_1}}{dt} \quad \text{for all } i, j.$$

Thus $\dot{\mathbf{M}}(\mathbf{x_1})$ is not a function of $u(t)$. In addition, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ has no dependence on $u(t)$ because $\mathbf{f}(\mathbf{x}, u) = \mathbf{h}(\mathbf{x}) + \mathbf{v}(u)$. Thus, if there exists a $n \times n$ matrix $\mathbf{M}(x_1, ..., x_k)$ such that

$$\mathbf{M} \succ 0 \text{ and } \dot{\mathbf{M}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \prec 0,$$

then the system in (4.21) is contracting for any $u(t)$. $\square$

In the following section, we illustrate how Theorem 27 is particularly useful in proving synchronization of nonlinear oscillators, an issue explored in more detail in [46]. Theorem 27 can be easily extended to the case where $u(t)$ is a vector, i.e. $\mathbf{u}(t) = [u_1(t), ..., u_m(t)]^T$.

4.4.1 Coupled Oscillators

Contraction theory is a useful tool to study synchronization behaviors of various configurations of coupled oscillators. For simplicity, we only consider here the case a pair of unidirectionally coupled oscillators (only one oscillator influences the other); more complicated and general couplings are discussed in [46].

A state-space model of two unidirectionally coupled oscillators is

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t) \\
\dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, t) - \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{x}),
\end{aligned}
\tag{4.22}
$$

where $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^m$, are the state vectors, $\mathbf{f}(\mathbf{x}, t)$ and $\mathbf{f}(\mathbf{y}, t)$ are the dynamics of the uncoupled oscillators, and $\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})$ is the coupling force. The following theorem is a modified version of Theorem 2 in [46].

Theorem 28. *If $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) - \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{x})$ in (4.22) is contracting with respect to $\mathbf{y}$ over the entire state space for arbitrary $\mathbf{u}(\mathbf{x})$,[2] the two systems will reach synchrony, i.e., $\mathbf{y}(t)$ and $\mathbf{x}(t)$ will tend toward the same trajectory, regardless of initial conditions.*

Proof. The system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) - \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{x})$ with input $\mathbf{u}(\mathbf{x})$ is contracting with respect to $\mathbf{y}$ over the entire state space and $\mathbf{y}(t) = \mathbf{x}(t)$ is a particular solution. Thus, by the properties of contraction, all solutions converge exponentially to $\mathbf{y}(t) = \mathbf{x}(t)$. □

Theorem 27 becomes especially powerful when the vector field appearing in the second subsystem of (4.22) has the structure described in equation (4.21).[3] We illustrate this in the next example.

4.4.2 Example: Coupled Van der Pol Oscillators

Consider two identical Van der Pol oscillators coupled as

$$
\begin{cases}
\ddot{x} + \alpha(x^2 + k)\dot{x} + \omega^2 x &= 0 \\
\ddot{y} + \alpha(y^2 + k)\dot{y} + \omega^2 y &= \alpha\eta(\dot{x} - \dot{y})
\end{cases}
\tag{4.23}
$$

[2] By contracting with respect to $\mathbf{y}$ for arbitrary $\mathbf{u}(\mathbf{x})$ we mean that the system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) - \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{x})$, where $\mathbf{y}$ is the state vector and $\mathbf{u}(\mathbf{x})$ is an arbitrary driving function, is contracting for all inputs $\mathbf{u}(\mathbf{x})$.

[3] If it does not have such a structure and $\mathbf{u}(\mathbf{x})$ drives each component of $\mathbf{y}$, we lose degrees of freedom in the possible forms of our contraction metric. If $\mathbf{u}(\mathbf{x})$ drives each component of $\mathbf{y}$, the only possible contraction metric is a constant.

where $\alpha > 0$, $\omega > 0$, k are arbitrary constants. We note that if $k < 0$, trajectories of the individual oscillator dynamics converge to a limit cycle. See Figure 4-1(b). We first write these coupled systems in state-space form to get the equations in the form of (4.22) and obtain

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\alpha(x_1^2 + k)x_2 - \omega^2 x_1 \end{bmatrix} \\ \\ \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\alpha(y_1^2 + k + \eta)y_2 - \omega^2 y_1 + \alpha\eta x_2 \end{bmatrix}. \end{cases} \tag{4.24}$$

By Theorem 27, this pair of unidirectional oscillators will reach synchrony regardless of initial conditions if

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) - \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{x}) = \begin{bmatrix} y_2 \\ -\alpha(y_1^2 + k + \eta)y_2 - \omega^2 y_1 + \alpha\eta x_2 \end{bmatrix} \tag{4.25}$$

is contracting with respect to $\mathbf{y}$ for arbitrary values of $\mathbf{u}(\mathbf{x}) = x_2$. We see by Theorem 27 that for this to occur, we must find a contraction metric $\mathbf{M}(\mathbf{y})$ that is only a function of y_1, i.e., $\mathbf{M}(\mathbf{y}) = \mathbf{M}(y_1)$.

When the search algorithm described in Section 4.2.1, was applied to find a metric that satisfied $\mathbf{M}(\mathbf{y}) = \mathbf{M}(y_1)$ as well as $\mathbf{M}(\mathbf{y}) \succ 0$ and $\mathbf{R}(\mathbf{y}) \prec 0$, none were found. However, it is shown in Appendix D, which is a modified version of the appendix of [46], that a metric satisfying $\mathbf{M}(\mathbf{y}) \succ 0$ and $\mathbf{R}(\mathbf{y}) \preceq 0$ implies asymptotic convergence of trajectories. A system with this metric satisfying $\mathbf{M}(\mathbf{y}) \succ 0$ and $\mathbf{R}(\mathbf{y}) \preceq 0$ is called *semi-contracting* [17, 46].

The metric

$$\mathbf{M}(\mathbf{y}) = \begin{bmatrix} \omega^2 + \alpha^2(y_1^2 + k + \eta)^2 & \alpha(y_1^2 + k + \eta) \\ \alpha(y_1^2 + k + \eta) & 1 \end{bmatrix} \tag{4.26}$$

given in [46] is only a function of y_1 and satisfies $\mathbf{M}(\mathbf{y}) \succ 0$ and $\mathbf{R}(\mathbf{y}) \preceq 0$ for the system dynamics (4.25) if $\alpha > 0$ and $(k + \eta) \geq 0$. For this $\mathbf{M}$ and the system (4.25), we have

$$\mathbf{R} = \dot{\mathbf{M}} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{y}} = \begin{bmatrix} -2\alpha\omega^2 y_1^2 - 2\alpha\omega^2(k+\eta) & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.27}$$

For $\alpha > 0$, $(k+\eta) > 0$, $\mathbf{M}(\mathbf{y}) \succ 0$ and $\mathbf{R}(\mathbf{y}) \preceq 0$. Since (4.26) and (4.27) show analytically that the system (4.25) is semi-contracting we used our search algorithm to search for a metric with $\mathbf{M}(\mathbf{y}) \succ 0$ and $\mathbf{R}(\mathbf{y}) \preceq 0$. This search initially failed due to numerical problems associated with the non-strict feasibility constraint on $\mathbf{R}(\mathbf{y})$. However, once we introduced a presolving stage that removed the nonstrict feasibility constraint by eliminating redundant variables, a feasible metric was found. Specifically, based on knowledge of the analytic solution (4.27), we constrained $R_{22} = 0$ and $R_{12} = 0$, eliminated redundant variables, and then searched for a solution in the resulting lower dimensional space.[4] With these constraints in place, a solution was found with the search algorithm.

4.5 Summary

In this chapter we developed algorithmic methods for analyzing stability and robustness of uncertain systems with polynomial dynamics via contraction analysis and SOS techniques. We illustrated through examples the advantages that contraction analysis offers when compared with traditional Lyapunov analysis and linearization techniques. Specifically, contraction analysis provides flexibility in incorporating inputs and outputs and is particularly useful in the analysis of nonlinear systems with uncertain parameters where the uncertainty shifts the equilibrium points of the system.

In the next and final chapter further summarize our conclusions and outline directions for future work.

[4]Setting $R_{22} = 0$, and $R_{12} = 0$ leads to redundant decision coefficients in the polynomial entries of $\mathbf{M}$ and $\mathbf{R}$. If these redundant variables are eliminated through a presolving stage, the search algorithm finds $\mathbf{M} \succ 0$ and $\mathbf{R} \preceq 0$.

Chapter 5

Conclusions and Future Work

IN THIS THESIS we showed how to use contraction theory and SOS programming to analyze *uncertain* systems with polynomial (or rational) dynamics. Specifically, we showed how contraction theory provides a tractable framework for the difficult problem of analyzing stability of uncertain nonlinear systems. We also developed SOS programming algorithms to find contraction metrics for uncertain systems with polynomial (or rational) dynamics. Additionally, we used SOS techniques to find bounds on the uncertainty range in which a system retains the property of contractiveness with respect to the polynomial contraction metric of the nominal system, and also to find a polynomial contraction metric that provides the largest uncertainty interval for which we can prove the system is contracting. These algorithms allow us to find ranges of values of uncertain parameters for which we can prove a nonlinear system is stable without explicitly tracking how the uncertainty changes the location of the equilibrium.

In Chapter 4 we also proved why the existence of a contraction metric with a specific structure is sufficient to guarantee contraction of a class of systems with external inputs and demonstrated how this feature can be used to prove synchronization of nonlinear oscillators.

There are several possible directions for future research. First we should carefully evaluate of how the computational resources needed by our SOS algorithms scale with system size. Second, we could explore additional benefits and limitations of the contraction-based approach to stability analysis in the context of other nonlinear techniques. For example, one could try to extend contraction-based methods to input-output stability properties, in a manner similar to the way that dissipation functions and inequalities extend Lyapunov analysis to systems with inputs and outputs.

Additionally, contraction-based methods for stability analysis of uncertain systems might be broadly applied to systems with multiple equilibria. If a contraction metric exists on closed regions of the state space, the SOS programming techniques described in Section 3.2 can be used to find these regions. However, it may be difficult to determine which trajectories will remain in a contraction region at all times. As stated in Theorem 25, this condition is necessary for proving local convergence of system trajectories when a contraction metric only exists over a region of the state space. One approach to ensuring this condition is satisfied is by calculating invariant sets in the state space. For uncertain systems, however, these sets may be difficult to characterize as they may be a function of the uncertain parameters.

This thesis could be further extended by developing a contraction and SOS-based methodology for analyzing stability of the more general class of uncertain systems with non-polynomial and non-rational dynamics. A good starting point may be [24] as it presents a systematic methodology for analyzing the stability of a more general class of deterministic non-polynomial systems by recasting them into systems with rational dynamics. Specifically, in [24] SOS decomposition techniques are used in conjunction with an extension of the Lyapunov stability theorem to investigate the stability and other properties of the recasted rational systems. Then, properties of the original, non-polynomial systems can be inferred from properties of the rational systems.

Finally, our contraction-based methods may be useful in finding performance bounds on uncertain systems, similar to the way that Lyapunov methods, which were traditionally applied to stability analysis, can also be used to find bounds on system performance [2].

Overall, SOS and LMI-based methods provide a powerful and computational framework for studying system performance and stability. In this thesis we have shown that these methods are useful and beneficial in a wide variety of uncertain and nonlinear applications, and have suggested several additional directions and related problems to explore.

Appendix A

Proof of Theorem 8

Theorem *Given the LTI dynamic system $\dot{\mathbf{x}} = A\mathbf{x}$ and any $Q \succ 0$, there exists a unique positive definite solution of the Lyapunov equation*

$$A^T P + PA = -Q$$

if and only if all the eigenvalues of A are in the open left half plane (OLHP).

Proof. (*[45]*). If $P \succ 0$ is a solution of $A^T P + PA = -Q$, then $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function for the system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \tag{A.1}$$

with $\dot{V}(\mathbf{x}) \leq 0$ for any $\mathbf{x} \neq 0$. Hence, system (A.1) is (globally) asymptotically stable, and thus the eigenvalues of A are in the open left half plane, i.e., have negative real parts.

To prove the converse, suppose A has all its eigenvalues in the open left half plane and $Q \succ 0$ is given. Define the symmetric matrix P by

$$P = \int_0^\infty e^{tA^T} Q e^{tA} dt.$$

This integral is well defined because the integrand decays exponentially to the origin since the real part of all the eigenvalues of A are negative. Then we have

$$\begin{aligned} A^T P + PA &= \int_0^\infty A^T e^{tA^T} Q e^{tA} dt + \int_0^\infty e^{tA^T} Q e^{tA} A dt \\ &= \int_0^\infty \frac{d}{dt}[e^{tA^T} Q e^{tA}] dt \\ &= -Q \end{aligned}$$

so P satisfies the Lyapunov equation. To prove P is positive definite, note

$$\mathbf{x}^T P \mathbf{x} = \int_0^\infty \mathbf{x}^T e^{tA^T} Q e^{tA} \mathbf{x} dt = \int_0^\infty ||Q^{1/2} e^{tA} \mathbf{x}||^2 dt \geq 0$$

and

$$\mathbf{x}^T P \mathbf{x} = 0 \Rightarrow Q^{1/2} e^{tA} \mathbf{x} = 0 \Rightarrow \mathbf{x} = 0,$$

where $Q^{1/2}$ is the square root of Q. To prove P is unique, suppose there is another solution P_2.

$$\begin{aligned} P_2 &= -\int_0^\infty \frac{d}{dt}[e^{tA^T} P_2 e^{tA}] dt \\ &= \int_0^\infty e^{tA^T}(A^T P_2 + P_2 A) e^{tA} dt \\ &= \int_0^\infty e^{tA^T} Q e^{tA} dt = P. \end{aligned}$$

□

Appendix B

Proof Outline for Theorem 10

Theorem *Suppose the system described by (2.31); that is*

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}\end{aligned}$$

is controllable and let the supply function s be defined by (2.32); that is

$$s(\mathbf{u},\mathbf{y}) = \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix}^T \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu}\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix}^T Q \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix}.$$

Then the following statements are equivalent

1. *The system (2.31) with supply function (2.32) is dissipative.*

2. *There exists a quadratic storage function $V(\mathbf{x}) \triangleq \mathbf{x}^T P\mathbf{x}$ with $P = P^T \succeq 0$ such that (2.28) holds for all $t_0 \le t_1$ and $(\mathbf{u},\mathbf{x},\mathbf{y})$ satisfying (2.31).*

3. *There exists $P = P^T \succeq 0$ such that*

$$F(P) \triangleq -\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0\end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I\end{bmatrix}^T \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu}\end{bmatrix}\begin{bmatrix} C & D \\ 0 & I\end{bmatrix} \succeq 0 \quad \text{(B.1)}$$

Proof. (*[38]*). $(2 \Rightarrow 1)$. Definition of dissipativity.

$(1 \Rightarrow 2)$. By dissipativity, the available storage $V_{av}(\mathbf{x}_0)$ defined as

$$V_{av}(\mathbf{x}_0) \triangleq \sup_{t_1,\,\mathbf{u}}\{-\int_0^{t_1} s(t)dt \;\mid\; t_1 \ge 0;\; (\mathbf{u},\mathbf{x},\mathbf{y}) \text{ satisfies (2.31) with } \mathbf{x}(0) = \mathbf{x}_0\}$$

is a storage function for the system and is finite for all $\mathbf{x}_0 \in \mathbb{R}^n$. The equivalence between dissipativity and $V_{av}(\mathbf{x}_0)$ being finite for all $\mathbf{x}_0$ was first given by Willems [48]. We see $V_{av}(\mathbf{x}_0)$ is a quadratic function of $\mathbf{x}$ as the supply function defined by (2.32) is quadratic and

$$V_{av}(\mathbf{x}_0) = \sup_{\mathbf{x}_0} \; - \int_{t_0}^{t_1} s(t)dt = -\inf_{\mathbf{x}_0} \int_{t_0}^{t_1} s(t)dt$$

denotes the cost of linear quadratic optimization (LQR) problem.

($2 \Rightarrow 3$). If $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ with $P \succeq 0$ is a storage function, then the dissipation inequality can be written as

$$\int_{t_0}^{t_1} \left(-\frac{d}{dt}\mathbf{x}(t)^T P \mathbf{x}(t) + s(\mathbf{u}(t), \mathbf{y}(t)) \right) dt \geq 0. \tag{B.2}$$

Substituting the system equations (2.31) in (B.2), we have

$$\int_{t_0}^{t_1} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \left(\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \geq 0 \tag{B.3}$$

Since (B.3) holds for all $t_0 \leq t_1$ and all inputs $\mathbf{u}$ this reduces to the requirement $P \succeq 0$ satisfies the LMI $F(P) \succeq 0$.

($3 \Rightarrow 2$). If there exists $P \succeq 0$ such that $F(P) \succeq 0$, then (B.3) holds and it follows that $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ is a storage function which satisfies the dissipation inequality. □

Appendix C

Kalman - Yakubovich - Popov Lemma (Positive Real Lemma)

The Kalman-Yakubovich-Popov (KYP) lemma (Positive Real lemma) proves the equivalence of the passivity of an LTI system, the feasibility of an related LMI, the positive realness of an associated transfer matrix, the existence of a solution to an algebraic Riccati equation, and a condition on the purely imaginary eigenvalue blocks of a related Hamiltonian matrix. Specifically, all of the following are equivalent conditions:

1. The system

   $$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{y} = C\mathbf{x} + D\mathbf{u}, \quad \mathbf{x}(0) = 0$$

   is passive; that is

   $$\int_0^T \mathbf{u}(t)^\top \mathbf{y}(t) dt \geq 0 \quad \text{for all } \mathbf{u}(t), \;\; T > 0.$$

2. The transfer matrix $G(s) = C(sI - A)^{-1}B + D$ is positive real; that is

   $$G(s) + G(s)^* \succeq 0 \quad \text{for all } \; s \; \text{ with } \; \mathbf{Re}\, s \geq 0$$

3. The LMI

   $$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \preceq 0,$$

in the variable $P = P^T$, is feasible.

4. There exists $P = P^T$ satisfying the algebraic Riccati equation

$$A^T P + PA + (PB - C^T)(D + D^T)^{-1}(PB - C)^T = 0.$$

5. The sizes of the Jordan blocks of the purely imaginary eigenvalues of the Hamiltonian matrix

$$M = \begin{bmatrix} A - B(D + D^T)^{-1}C & B(D + D^T)^{-1}B^T \\ C^T(D + D^T)^{-1}C & -A^T + C^T(D + D^T)^{-1}B^T \end{bmatrix}$$

are all even.

The original form of the lemma states that the transfer function $c(sI - A)^{-1}b$ of the single-input single-output minimal system (A, b, c) is positive real, i.e.,

$$\mathbf{Re}\ \ c(sI - A)^{-1}b \geq 0 \quad \text{for all} \ \ \mathbf{Re}\ \ s > 0$$

if and only if there exists $P \succ 0$ such that $A^T P + PA \leq 0$ and $Pb = c^T$ [2]. A brief history of the development of the lemma given in [2] includes Anderson's extension of the lemma to the multi-input, multi-output case and his derivation of similar results for nonexpansive systems, as well as Willems' connections between the lemma, certain quadratic optimal control problems, and the existence of symmetric solutions to an algebraic Riccati equation. Proofs of the equivalence statements above as well as Anderson's and Willems's work extending the theorem can be found in the references given at the end of Chapter 2 of [2].

Appendix D

Proving Asymptotic Convergence of Coupled Van der Pol Oscillators With a Negative Semidefinite R Matrix

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This appendix is a modified version of the appendix in [46]. Consider the system given in (4.25). Consider a 2×2 matrix $\mathbf{M}(\mathbf{y})$ that is uniformly positive definite, and a corresponding $\mathbf{R}(\mathbf{y})$ matrix that is uniformly negative semidefinite, but not uniformly negative definite. Since (4.25) is a two-dimensional system, we can assume without loss of generality that $\mathbf{R}(\mathbf{y})$ is of the form

$$\mathbf{R}(\mathbf{y}) = \begin{bmatrix} -k(\mathbf{y}) & 0 \\ 0 & 0 \end{bmatrix}$$

where $k(\mathbf{y}) > 0$ for all $\mathbf{y}$. Let $\delta\mathbf{y} = [\delta y_1 \;\; \delta y_2]^T = [\delta y \;\; \delta\dot{y}]^T$, where y_1 and y_2 are the variables in equation (4.25). With these $\mathbf{R}(\mathbf{y})$ and $\mathbf{M}(\mathbf{y})$ matrices, the general definition of differential length given by (4.6) and associated equation for the rate of change of differential length (4.7) are

$$\delta\mathbf{z}^T\delta\mathbf{z} = \delta\mathbf{y}^T\mathbf{M}(\mathbf{y})\delta\mathbf{y},$$

and

$$\begin{aligned} \frac{d}{dt}\delta \mathbf{z}^T \delta \mathbf{z} &= \frac{d}{dt}(\delta \mathbf{y}^T \mathbf{M}(\mathbf{y}) \delta \mathbf{y}) \\ &= \delta \mathbf{y}^T \mathbf{R}(\mathbf{y}) \delta \mathbf{y} \\ &= -k(\mathbf{y}) \delta y_1^2. \end{aligned} \tag{D.1}$$

Since $\frac{d}{dt}(\delta \mathbf{y}^T \mathbf{M}(\mathbf{y}) \delta \mathbf{y}) \leq 0$ and $\delta \mathbf{z}^T \delta \mathbf{z} \geq 0$, $\delta \mathbf{z}^T \delta \mathbf{z}$ has a limit as t goes to infinity.

We will prove through a Taylor series argument that all trajectories of this system converge asymptotically. If $\delta y = \delta y_1 \neq 0$, then

$$\delta \mathbf{z}^T \delta \mathbf{z}(t + dt) - \delta \mathbf{z}^T \delta \mathbf{z}(t) = -k(\mathbf{y})(\delta y_1)^2 dt + O((dt)^2)$$

while if $\delta y_1 = 0$,

$$\delta \mathbf{z}^T \delta \mathbf{z}(t + dt) - \delta \mathbf{z}^T \delta \mathbf{z}(t) = -2k(\mathbf{y})(\delta y_2)^2 \frac{dt^3}{3!} + O((dt)^4).$$

Since $\delta \mathbf{z}^T \delta \mathbf{z}$ converges, $\delta \mathbf{z}^T \delta \mathbf{z}(t + dt) - \delta \mathbf{z}^T \delta \mathbf{z}(t)$ approaches zero asymptotically and hence δy_1 and δy_2 or equivalently δy and $\delta \dot{y}$ both tend to zero. Thus, for any input $\mathbf{u}(\mathbf{x})$ all solutions of system (4.25) converge asymptotically to a single trajectory independent of initial conditions, and the unidirectional oscillators given in (4.24) will reach synchrony asymptotically regardless of initial conditions.

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