Cycle Types of Permutations with Restricted Positions and a Characterization of a New Class of Interval Orders

by

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ABSTRACT

This thesis deals with two unrelated problems. In Chapter 1, we examine permutations with restricted positions according to cycle type. R. Stanley and J. Stembridge developed this theory and defined "cycle indicators" to attack some conjectures about immanants of the Jacobi-Trudi matrix. They also proved a general formula for computing these cycle indicators. We calculate some cycle indicators using a different method for particular restrictions and prove some nice identities along the way. We also generalize the theory of classifying permutations in S_n with restricted positions by cycle type to wreath products of finite abelian groups with the symmetric group. We prove a general formula for the generalized cycle indicator which involves the forgotten symmetric function and a generalization of the power-sum symmetric function.

We study unlabelled interval orders in Chapter 2. Interval orders which correspond to interval representations in which each interval has the same length are called *semiorders* and have been studied in great detail. We classify $(1, 1, \ldots, 1, 1 + \alpha)$-interval orders where $0 < \alpha < 1$.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Applied Mathematics

Dedication:

To my parents

Kaichi and Masako Okazaki

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Contents

Chapter 1

Cycle Indicators of Permutations with Restricted Positions

1.1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$. Consider a board $B \subseteq [n]^2 = [n] \times [n]$. The classical problem of permutations with restricted positions involves counting the number of placements of n non-attacking rooks on B. Rooks are said to be non-attacking if no two occupy the same row or column. Thus, each placement of n non-attacking rooks corresponds to a permutation $\pi \in S_n$ (if a rook is in position (i, j), then $\pi(i) = j$). It is also an interesting problem to see how many placements of n non-attacking rooks correspond to a particular conjugacy class of S_n. The theory of cycle indicators for boards was developed by R. Stanley and J. Stembridge [SS93] to examine this problem. We prove some nice identities of inverse Kostka numbers, and we use them to compute the cycle indicators for particular boards, thus computing the number of placements of n non-attacking rooks according to conjugacy class for those boards. We also generalize these ideas for the symmetric group to wreath products of finite abelian groups with the symmetric group and compute a general formula for the generalized cycle indicator for these wreath products analogous to the one for the symmetric group calculated by R. Stanley and J. Stembridge.

1.2 Background

In this section, we will explain the background of the theory of cycle indicators; consequently, we will follow the notation of [SS93]. Fix $\lambda \vdash n$ (λ is a partition of n). Let χ^λ denote the irreducible character of S_n indexed by λ. Let us define the *immanant* of an $n \times n$ matrix A with respect to λ as $\mathrm{Imm}_\lambda(A) := \sum_{w \in S_n} \chi^\lambda(w) \prod_{i=1}^n a_{i,w(i)}$. Note that when $\lambda = 1^n$, χ^λ is the trivial character, and $\mathrm{Imm}_\lambda(A) = per(A)$, the permanant of A. When $\lambda = (n)$, χ^λ is the sign character, and $\mathrm{Imm}_\lambda(A) = \det(A)$, the determinant of A. Immanants were first studied long ago (see, *e.g.*, [Lit50, Sch18]).

Given partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ with $\nu \leq \mu$ (*i.e.*, $0 \leq \nu_i \leq \mu_i$ for all i), define the $n \times n$ *Jacobi-Trudi* matrix $H_{\mu/\nu} := \{h_{\mu_i - i - (\nu_j - j)}\}_{i,j}$, where h_k is the k-th homogeneous symmetric function. (Set $h_0 = 1$ and $h_k = 0$ for $k < 0$.) It is a well-known result that

$$\det H_{\mu/\nu} = s_{\mu/\nu},$$

where $s_{\mu/\nu}$ is the skew Schur function. We can ask what properties $\mathrm{Imm}_\lambda(H_{\mu/\nu})$ has for any partition λ. In the past few years, many conjectures have been made by I. Goulden and D. Jackson [GJ92] and J. Stembridge [Ste92], and many theorems have been proven about these immanants. C. Greene [Gre92] proved that they are monomial-positive (*i.e.*, nonnegative linear combinations of monomial symmetric functions), and M. Haiman [Hai93] proved that they are Schur-positive.

There is a conjecture of J. Stembridge [Ste92] that remains open that is stronger than the Schur-positivity result of M. Haiman. To state this conjecture, we must make a few definitions. Let

$$F_{\mu/\nu}(x, y) := \sum_{\lambda \vdash n} s_\lambda(y) \mathrm{Imm}_\lambda(H_{\mu/\nu})(x).$$

We can rewrite this so that

$$F_{\mu/\nu}(x, y) = \sum_{\gamma \vdash |\mu/\nu|} E^\gamma_{\mu/\nu}(y) s_\gamma(x).$$

Conjecture 1.1 *[SS93] $E^\gamma_{\mu/\nu}(y)$ is a nonnegative linear combination of complete homogeneous symmetric functions h_α of degree n.*

It turns out that this conjecture is equivalent another conjecture which we will state after we give some background. Let ch denote the characteristic map, which is an isometry between the space of class functions on

S_n and the ring of symmetric functions of degree n (see, *e.g.*, [Mac95, p. 113] and [Sag91, p. 163]) such that given an irreducible character χ^λ of S_n, we have $\mathrm{ch}(\chi^\lambda) = s_\lambda$. Let ϕ^λ denote the class function on S_n such that $\mathrm{ch}(\phi^\lambda) = m_\lambda$. J. Stembridge [Ste92] defines the *monomial immanant* to be $\sum_{w \in S_n} \phi^\lambda(w)(H_{\mu/\nu})_{1,w(1)} \cdots (H_{\mu/\nu})_{n,w(n)}$ for obvious reasons.

Let $\Gamma^\gamma_{\mu/\nu}$ denote the class function on S_n such that $\mathrm{ch}(\Gamma^\gamma_{\mu/\nu}) = E^\gamma_{\mu/\nu}(y)$. We define the usual inner product $\langle , \rangle$ on Λ, the ring of symmetric functions, such that $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$, where h_λ and m_μ denote the homogeneous and monomial symmetric functions, respectively. J. Stembridge [Ste92] explains that the coefficient of s^γ in the monomial immanant is $\langle \phi^\lambda, \Gamma^\gamma_{\mu/\nu} \rangle$. Now we can restate Conjecture 1.1 as follows:

Conjecture 1.2 *[Ste92] Monomial immanants of Jacobi-Trudi matrices are Schur-positive; i.e.,* $\langle \phi^\lambda, \Gamma^\gamma_{\mu/\nu} \rangle \geq 0$ *for* $\lambda \vdash n$ *and* $\gamma \vdash |\mu/\nu|$.

Note that if Conjecture 1.1 is true, then

$$F_{\mu/\nu}(x,y) = \sum_{\gamma \vdash |\mu/\nu|} \sum_{\alpha \vdash n} c_\alpha h_\alpha(y) s_\gamma(x),$$

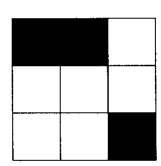
where $c_\alpha \geq 0$. Since $h_\alpha(y) = \sum_{\lambda \vdash n} K_{\lambda,\alpha} s_\lambda(y)$ [Mac95, Sag91], where $K_{\lambda,\alpha}$ is the Kostka number (which is nonnegative), we see that this conjecture would be stronger than the Schur-positivity of the immanant of the Jacobi-Trudi matrix. Now we will explain how this relates to non-attacking rooks.

Given some board $B \subseteq [n]^2$, a permutation $\pi \in S_n$ is said to be *B-compatible* if $(i,j) \in B$ for $\pi(i) = j$. R. Stanley and J. Stembridge define the *cycle indicator* of B to be

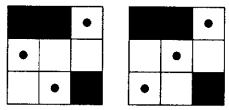
$$Z[B] := \sum_{\substack{\pi \in S_n \\ B-\text{compatible}}} p_{\rho(\pi)},$$

where $\rho(\pi)$ is the partition of n which gives the cycle type of π, and p_i is the i-th power-sum symmetric function. Thus $p_{\rho(\pi)} = p_1^{m_1(\rho(\pi))} p_2^{m_2(\rho(\pi))} \cdots$, where $m_i(\rho(\pi))$ denotes the number of parts of $\rho(\pi)$ of size i.

Example 1.1 Let B be the following board:



Here, we can see that



are the only possible placements of 3 non-attacking rooks. Hence,

$$\begin{aligned} Z[B] &= p_{\rho((132))} + p_{\rho((13)(2))} \\ &= p_3 + p_{21}. \end{aligned}$$

Let us define the partition $\delta = (n-1, n-2, \ldots, 0)$. Given the definition of $F_{\mu/\nu}(x,y)$ and the fact that $p_\alpha = \sum_\lambda \chi^\lambda_\alpha s_\lambda$ [Mac95, Sag91], we can see that

$$F_{\mu/\nu}(x,y) = \sum_{\pi \in S_n} p_{\rho(\pi)}(y) h_{\mu+\delta-\pi(\nu+\delta)}(x).$$

Again, since $h_\alpha(y) = \sum_{\lambda \vdash n} K_{\lambda,\alpha} s_\lambda(y)$, we get

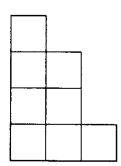
$$F_{\mu/\nu}(x,y) = \sum_{\pi \in S_n} p_{\rho(\pi)}(y) \sum_{\gamma \vdash |\mu/\nu|} K_{\gamma,\mu+\delta-\pi(\nu+\delta)} s_\gamma(x).$$

So

$$E^\gamma_{\mu/\nu}(y) = \sum_{\pi \in S_n} K_{\gamma,\mu+\delta-\pi(\nu+\delta)} p_{\rho(\pi)}(y).$$

Given a partition $\beta = (\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l > 0)$, the Young diagram of β is defined to be an array of left-justified boxes, with β_i boxes in row $n-i+1$.

Example 1.2 Let $\beta = (3221)$. Then the Young diagram of β is:



A Young tableau of shape β is a filling of the Young diagram of β with positive integers, where repetitions are allowed (see, *e.g.*,[Sag91, p. 78]). The *type* of a Young tableau T is a partition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of times that i appears in T.

Example 1.3 Say the Young tableau T is of shape $\beta = (3221)$ and looks like:

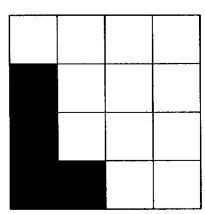
5		
2	2	
3	1	
1	7	1

Then the type of T is $(3,2,1,0,1,0,1)$.

In the case that $\gamma = (|\mu/\nu|)$, $K_{\gamma,\mu+\delta-\pi(\nu+\delta)}$ is the number of semi-standard (weakly increasing rows from left to right and strictly increasing columns from bottom to top) Young tableaux of shape $(|\mu/\nu|)$ and type $\mu+\delta-\pi(\nu+\delta)$, and hence is equal to 1. Note that $K_{\gamma,\mu+\delta-\pi(\nu+\delta)} = 0$ if $\mu+\delta-\pi(\nu+\delta) < 0$.

Let $\sigma := \sigma(\mu/\nu)$ be the partition whose Young diagram consists of all positions (i,j) such that $h_{\mu_i-i-(\nu_j-j)} = 0$. Thus, it sits in the lower-left corner of the matrix $H_{\mu/\nu}$. Define B^σ to be the board which has the diagram of σ removed from the lower left corner of $[n]^2$.

Example 1.4 If we let $\mu = (4331)$ and $\nu = (32)$, then $(2,1),(3,1),(4,1),(4,2)$ are the positions such that $h_{\mu_i-i-(\nu_j-j)} = 0$. Therefore, $\sigma = (211)$ and B^σ looks like:



When $\gamma = (|\mu/\nu|)$, we see that $E^\gamma_{\mu/\nu}(y) = \sum_{\pi\in S_n} p_{\rho(\pi)}(y)$, where the sum is taken over B^σ-compatible permutations, so $E^\gamma_{\mu/\nu}(y)$ is a cycle indicator. This motivated R. Stanley and J. Stembridge to study these cycle indicators. Now that we have given some motivation, we will explain our results.

1.3 The Symmetric Group

In this section, we will follow the notation of I. G. Macdonald [Mac95] and B. Sagan [Sag91].

We will need to define some notation. If $\mu \vdash n$, let C_μ denote the conjugacy class of S_n of cycle type μ. We know that $|C_\mu| = \frac{n!}{z_\mu}$, where

$z_\mu = 1^{m_1(\mu)}2^{m_2(\mu)}\cdots m_1(\mu)!m_2(\mu)!\cdots$. Let R^γ denote an irreducible matrix representation of S_n indexed by $\gamma \vdash n$, and let χ^γ denote the corresponding irreducible character of S_n. Then $\mathrm{tr}(R^\gamma(\pi)) = \chi^\gamma(\pi)$ for any $\pi \in S_n$. Let f^γ denote the dimension of R^γ. First, we will need the following theorem.

Theorem 1.1 (Burnside) *Take $\gamma \vdash n$. Then*

$$R^\gamma\left(\sum_{\pi\in C_\mu}\pi\right) = \frac{\frac{n!}{z_\mu}\chi^\gamma_\mu}{f^\gamma} I_{f^\gamma},$$

where I_{f^γ} is the $f^\gamma \times f^\gamma$ identity matrix. Furthermore, $\frac{\frac{n!}{z_\mu}\chi^\gamma_\mu}{f^\gamma} \in \mathbf{Z}$.

Proof [FH91]

$$\begin{aligned}\mathrm{tr}\left(\sum_{\pi\in C_\mu} R^\gamma(\pi)\right) &= \sum_{\pi\in C_\mu}\mathrm{tr}(R^\gamma(\pi)) \\ &= \sum_{\pi\in C_\mu}\chi^\gamma(\pi) \\ &= \frac{n!}{z_\mu}\chi^\gamma_\mu,\end{aligned}$$

where χ^γ_μ is the value of χ^γ on C_μ.

On the other hand, by Schur's Lemma,

$$R^\gamma\left(\sum_{\pi\in C_\mu}\pi\right) = \lambda_{\gamma,\mu} I_{f^\gamma}, \tag{1.1}$$

for some $\lambda_{\gamma,\mu} \in \mathbf{C}$. Taking the trace of both sides of Equation (1.1), we obtain

$$\frac{n!}{z_\mu}\chi^\gamma_\mu = \lambda_{\gamma,\mu} f^\gamma.$$

We see that $\lambda_{\gamma,\mu}$ is a rational number. To show that it is an integer, we need the following lemma:

Lemma 1.1 *[Ser77] Fix $\mu \vdash n$. Let*

$$u(\pi) = \begin{cases} 1 & \text{if } \pi \in C_\mu \\ 0 & \text{otherwise}\end{cases}$$

Let

$$u = \sum_{\pi \in S_n} u(\pi)\pi = \sum_{\pi \in C_\mu} \pi \in \text{center of } \mathbf{C}[S_n].$$

Then $\frac{1}{f^\gamma} \sum_{\pi \in S_n} u(\pi)\chi^\gamma(\pi) = \frac{1}{f^\gamma} \frac{n!}{z_\mu} \chi^\gamma_\mu$ *is an algebraic integer.*

Since $\lambda_{\gamma,\mu}$ is a rational number and an algebraic integer, it is an integer. ■

1.3.1 Boards Avoiding a Fixed Permutation

Fix some $\tau \in S_n$. Let B_τ denote the $n \times n$ chessboard with a permutation τ removed (*i.e.*, $B_\tau = \{(i,j) \in [n]^2 \,|\, \tau(i) \neq j\}$). We will give an expression for $Z[B_\tau]$. Before we do this, however, we need the following facts. Let $\mathcal{D}_n = \{\pi \in S_n \,|\, \pi(i) \neq i \ \forall i \in [n]\}$, the set of derangements of $[n]$. ($\mathcal{D}_n$ is the set of B_{id}-compatible permutations.) Let D_n denote $|\mathcal{D}_n|$. It is well-known (*e.g.*, [Sta86, p.67]) that $D_n = \sum_{i=0}^n (-1)^i \frac{n!}{i!}$. Take $\pi \in \mathcal{D}_n$. Note that $\tau\pi(i) \neq \tau(i)$, so $\tau\pi$ is B_τ-compatible.

Claim 1.1 *Fix* $\tau \in S_n$. *Then* $\{\sigma \in S_n \,|\, \sigma \text{ is } B_\tau - \text{compatible}\} = \{\pi\tau \in S_n \,|\, \pi \in \mathcal{D}_n\}$.

Theorem 1.2 *Fix* $\tau \in S_n$. *Then*

$$Z[B_\tau] = \sum_{\gamma \vdash n} \left[\left(\frac{\sum_{\pi \in \mathcal{D}_n} \chi^\gamma(\pi)}{f^\gamma} \right) \chi^\gamma(\tau) \right] s_\gamma.$$

Proof By definition, $Z[B_\tau] = \sum_\sigma p_{\rho(\sigma)}$, where the sum is taken over permutations σ in S_n that are B_τ-compatible. Since $p_\alpha = \sum_\lambda \chi^\lambda_\alpha s_\lambda$,

$$Z[B_\tau] = \sum_\gamma s_\gamma \sum_{\substack{\sigma \in S_n \\ B_\tau - \text{compatible}}} \chi^\gamma(\sigma).$$

So if we denote the coefficient of s_γ in $Z[B_\tau]$ by $[s_\gamma]Z[B_\tau]$,

$$\begin{aligned} [s_\gamma]Z[B_\tau] &= \sum_{\substack{\sigma \in S_n \\ B_\tau - \text{compatible}}} \chi^\gamma(\sigma) \\ &= \sum_{\pi \in \mathcal{D}_n} \chi^\gamma(\tau\pi) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{tr}\left(R^\gamma(\tau)R^\gamma\left(\sum_{\pi\in\mathcal{D}_n}\pi\right)\right)\\
&= \operatorname{tr}\left(\sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} R^\gamma(\tau)R^\gamma\left(\sum_{\pi\in C_\mu}\pi\right)\right).
\end{aligned}
$$

By Theorem 1.1,

$$
[s_\gamma]Z[B_\tau] = \operatorname{tr}\left(\sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} R^\gamma(\tau)\left(\frac{\frac{n!\chi^\gamma_\mu}{z_\mu}}{f^\gamma}I_{f^\gamma}\right)\right), \text{ so}
$$

$$
Z[B_\tau] = \sum_\gamma s_\gamma \frac{\sum_{\pi\in\mathcal{D}_n}\chi^\gamma(\pi)}{f^\gamma}\chi^\gamma(\tau).
$$

■

Note that if $\rho(\sigma)=\rho(\tau)$, then $Z[B_\sigma]=Z[B_\tau]$. Hence, if $\rho(\tau)=\lambda$, we will use the notation $Z[B_\tau]$ and $Z[B_\lambda]$ interchangeably. Before we proceed, let us state a couple of properties of the inner product $\langle,\rangle$ we defined earlier. We know (*e.g.*, [Mac95, p. 64 and p. 114]) that $\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu}z_\lambda$ and $\langle s_\lambda, p_\mu\rangle = \chi^\lambda_\mu$.

Corollary 1.1 *Fix $\tau\in S_n$ and $\lambda\vdash n$. Let $g_\lambda(\tau) = |\{\sigma\in S_n \,|\, \rho(\sigma)=\lambda,$ and σ is $B_\tau-$compatible$\}|$, the number of B_τ-compatible permutations in S_n of cycle type λ. Then*

$$
g_\lambda(\tau) = \frac{1}{z_\lambda}\sum_{\gamma\vdash n}\frac{\sum_{\pi\in\mathcal{D}_n}\chi^\gamma(\pi)}{f^\gamma}\chi^\gamma(\tau)\chi^\gamma_\lambda.
$$

Proof By definition, $Z[B_\tau]=\sum_{\mu\vdash n} g_\mu(\tau)p_\mu$, so

$$
\begin{aligned}
\langle Z[B_\tau], p_\lambda\rangle &= \left\langle \sum_{\mu\vdash n} g_\mu(\tau)p_\mu, p_\lambda\right\rangle\\
&= z_\lambda g_\lambda(\tau).
\end{aligned}
$$

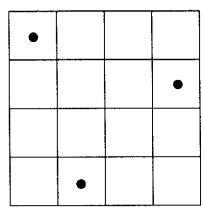
On the other hand, from Theorem 1.2,

$$
\begin{aligned}
\langle Z[B_\tau], p_\lambda\rangle &= \left\langle \sum_{\gamma\vdash n}\frac{\sum_{\pi\in\mathcal{D}_n}\chi^\gamma(\pi)}{f^\gamma}\chi^\gamma(\tau)s_\gamma, p_\lambda\right\rangle\\
&= \sum_{\gamma\vdash n}\frac{\sum_{\pi\in\mathcal{D}_n}\chi^\gamma(\pi)}{f^\gamma}\chi^\gamma(\tau)\chi^\gamma_\lambda.
\end{aligned}
$$

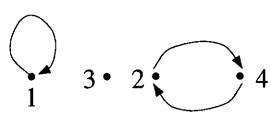
■

R. Stanley and J. Stembridge [SS93] calculated a more general formula for computing cycle indicators for any board that involves the forgotten symmetric functions f_α (see [Mac95, p. 22]). Given the involution ω on the ring of symmetric functions which is defined such that $\omega(e_i) = h_i$ (e_i is the i-th elementary symmetric function), f_α is defined to be $\omega(m_\alpha)$. Before we can state the aforementioned gneral formula, we must make some definitions. We can think of any subset B of $[n]^2$ as a directed graph, *viz.*, $i \to j$ if $(i,j) \in B$. Consider some placement S of at most n non-attacking rooks on B. The graph of S consists of disjoint cycles and directed paths. We say that the graph of S is of *type* $(\alpha;\beta)$ where $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\alpha_1, \alpha_2, \ldots$ are the number of vertices in the directed paths of S, and where $\beta = (\beta_1, \beta_2, \ldots)$ and $\beta_1, \beta_2, \ldots$ are the lengths of the cycles of S. Note that $|\alpha| + |\beta| = n$.

Example 1.5 The following is a placement S of 3 non-attacking rooks on $[4]^2$:



and the graph of S looks like:



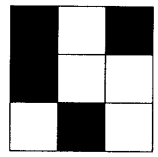
The type of the graph of S is $(1;21)$.

Theorem 1.3 *[SS93] Take any $B \subseteq [n]^2$. Then*

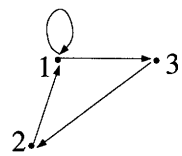
$$Z[B] = \sum_{\alpha,\beta} (-1)^{|\beta|} \prod_i m_i(\alpha)! r_{\alpha,\beta}(\overline{B}) f_\alpha p_\beta,$$

where $|\alpha| + |\beta| = n$ and $r_{\alpha,\beta}(\overline{B})$ denotes the number of subgraphs of $\overline{B}$, the complement graph to B, of type $(\alpha;\beta)$.

Example 1.6 Let B be the following board:



Then the graph of $\overline{B}$ looks like:



Here, we can see by inspection that $Z[B] = p_{\rho((123))} = p_3$. Let us verify Theorem 1.3. The following is a table of terms with non-zero $r_{\alpha;\beta}(\overline{B})$ values:

α	β	$r_{\alpha;\beta}(\overline{B})$
3	$\emptyset$	3
21	$\emptyset$	3
1^3	$\emptyset$	1
2	1	1
1^2	1	1
$\emptyset$	3	1

From this, we can compute $Z[B] = -p_3 - 2f_{1^2}p_1 - f_2p_1 + 6f_{1^3} + 3f_{21} + 3f_3$, which simplifies to $Z[B] = p_3$, using J. Stembridge's symmetric function package "SF" for *Maple*.

Here, it turns out that calculating $Z[B]$ by inspection is much simpler than using the general formula of Theorem 1.3. However, Theorem 1.3 is very useful, as we will see later.

The following symmetric function base transformations are well-known (see, *e.g.*, [Bec93, Mac95]). Let the n-th *Kostka matrix* be the matrix whose rows and columns are indexed by partitions of n such that the (λ, μ) entry is the Kostka number $K_{\lambda,\mu}$. Let $K^{-1}_{\lambda\mu}$ denote the λ, μ entry of the inverse of the Kostka matrix. Then $f_\lambda = \sum_\mu s_\mu K^{-1}_{\lambda,\mu'}$. Furthermore, since $s_\lambda = \sum_\mu p_\mu \frac{\chi^\lambda_\mu}{z_\mu}$, we have $f_\lambda = \sum_\mu p_\mu \sum_\nu K^{-1}_{\lambda\nu'} \frac{1}{z_\mu} \chi^\nu_\mu$. Given this, it seems that it might be useful to compute some values $K^{-1}_{\lambda\mu}$ in order to compute $Z[B]$ of Theorem 1.3.

Theorem 1.4 *Take $\alpha, \gamma \vdash n$, and fix $1 \leq l \leq n$. Then*

$$\sum_{\substack{\alpha \vdash n \\ l(\alpha)=l}} K^{-1}_{\alpha,\gamma'} = \begin{cases} (-1)^{n-l-l(\gamma)+1}\binom{n-l(\gamma)}{l-1} & \text{if } \gamma \text{ is a hook} \\ 0 & \text{otherwise.} \end{cases}$$

The following proof is due to R. Stanley.

Proof [Sta] Before we begin, we will need the following claim.

Claim 1.2

$$\sum_{\alpha} m_\alpha(x) t^{l(\alpha)} = \prod_i \left(1 + \frac{tx_i}{1-x_i}\right),$$

where the sum is taken over all partitions α.

Proof [Claim 1.2] The right-hand side of Claim 1.2 looks like:

$$\prod_i \left(1 + \frac{tx_i}{1-x_i}\right) = \prod_i (1 + tx_i + tx_i^2 + tx_i^3 + \cdots).$$

Take any term on the right-hand side of this equation. It will look like $t^k x_1^{a_1} x_2^{a_2} \cdots$, where $\#\{a_j \mid a_j \neq 0\} = k$. ■

Let $S = \sum_{\substack{\alpha \vdash n \\ l(\alpha)=l}} K^{-1}_{\alpha,\gamma'}$. We know (*e.g.*, [Bec93, Mac95]) that $K^{-1}_{\alpha,\gamma'} = \langle s_{\gamma'}, m_\alpha \rangle$. So

$$S = [t^l] \left\langle s_{\gamma'}, \prod_i \left(1 + \frac{tx_i}{1-x_i}\right) \right\rangle.$$

Well, we can rewrite $\prod_i \left(1 + \frac{tx_i}{1-x_i}\right)$ as

$$\left(\prod_i (1 + (t-1)x_i)\right) \left(\prod_j \frac{1}{1-x_j}\right) = \sum_{k \geq 0} (t-1)^k e_k \sum_{m \geq 0} h_m.$$

So

$$\begin{aligned} S &= \left\langle s_{\gamma'}, [t^l] \sum_{k \geq 0} (t-1)^k e_k \sum_{m \geq 0} h_m \right\rangle \\ &= \left\langle s_{\gamma'}, \sum_{k \geq 0} (-1)^{k+l} e_k \binom{k}{l} \sum_{m \geq 0} h_m \right\rangle \\ &= \sum_{k,m} (-1)^{k+l} \binom{k}{l} \langle s_{\gamma'}, e_k h_m \rangle \\ &= \sum_{k,m} (-1)^{k+l} \binom{k}{l} \langle s_{\gamma'}, s_{1^k} s_{(m)} \rangle \end{aligned}$$

Using the Littlewood-Richardson rule (see, *e.g.*, [Sag91, p. 173]), we know that $s_{1^k}s_{(m)} = s_{(m,1^k)} + s_{(m+1,1^{k-1})}$. Therefore,

$$S = \sum_{k,m}(-1)^{k+l}\binom{k}{l}\left[\delta_{\gamma',(m,1^k)} + \delta_{\gamma',(m+1,1^{k-1})}\right].$$

Clearly, if γ' is not a hook, $S = 0$. If γ' *is* a hook, say

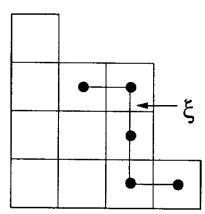
$$\gamma' = (l(\gamma), 1^{n-l(\gamma)}),$$

then

$$\begin{aligned} S &= (-1)^{n-l(\gamma)+l}\left[\binom{n-l(\gamma)}{l} - \binom{n-l(\gamma)+1}{l}\right] \\ &= (-1)^{n-l(\gamma)+l+1}\binom{n-l(\gamma)}{l-1}. \end{aligned}$$

■

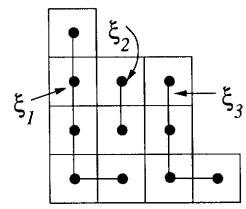
We also have a combinatorial proof of Theorem 1.4. For this, we will need some background, and we follow the notation of D. Beck [Bec93]. Let F_λ denote a Young diagram of shape λ. We say that ξ is a *rim hook* of F_λ if it is edgewise connected and lies along the north-east boundary of F_λ such that ξ contains no 2×2 subset of cells, *e.g.*,



Fix partitions μ and λ. A *rim hook tabloid of shape λ and type μ* is defined to be a sequence of partitions:

$$\emptyset = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^k = \lambda$$

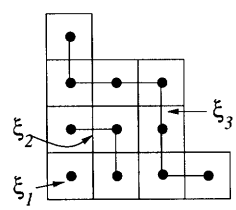
such that $\xi_i := \lambda^i - \lambda^{i-1}$ is a rim hook of size μ_i of the diagram of λ^i for all $1 \leq i \leq k$. The following is a rim hook tabloid of shape (4331) and type (524):



We say that ξ_i is a *special rim hook* if it is a rim hook of λ^i and it contains a cell from the first column of λ. We then define a *special rim hook tabloid* (SRHT) T of shape λ and type μ to be a sequence of partitions:

$$\emptyset = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^k = \lambda$$

such that ξ_i is a special rim hook of λ^i of size μ_i. The following is a special rim hook tabloid T of shape (4331) and type (137):



We define a weight function on the tabloid as follows: $wt(\xi_i) := (-1)^{ll(\xi_i)}$, where the *leg length* of ξ_i is $ll(\xi_i) =$ (the number of rows of ξ_i) -1. We define $wt(T) := \prod_{i=1}^{k} wt(\xi_i)$. In our example above, $wt(T) = (-1)^{1-1}(-1)^{2-1}(-1)^{4-1} = 1$.

Theorem 1.5 ([Bec93]) *Take $\lambda, \mu \vdash n$. Then*

$$K^{-1}_{\lambda,\mu} = \sum_T wt(T),$$

where the sum is taken over all special rim hook tabloids T of shape μ and type λ.

Claim 1.3 *Take $\alpha \vdash n$. Then*

$$K^{-1}_{\alpha,1^n} = \frac{l(\alpha)!}{\prod_i m_i(\alpha)!}(-1)^{n-l(\alpha)}.$$

Proof By definition, $K^{-1}_{\alpha,1^n} = \sum_T wt(T)$ where the sum is taken over all SRHT T of shape 1^n and type α. Take any SRHT T in this sum. We know that $wt(T) = \prod_j wt(\alpha_j)$.

$$\begin{aligned} K^{-1}_{\alpha,1^n} &= (\text{number of permutations of parts of } \alpha) \cdot \prod_j wt(\alpha_j) \\ &= \frac{l(\alpha)!}{\prod_i m_i(\alpha)!} \prod_j wt(\alpha_j). \end{aligned}$$

It is easy to see that

$$\begin{aligned} \prod_j wt(\alpha_j) &= \prod_j (-1)^{\alpha_j - 1} \\ &= (-1)^{n-l(\alpha)}. \end{aligned}$$

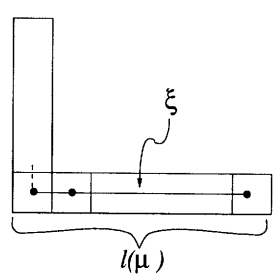
■

Claim 1.4 *Take $\alpha, \mu \vdash n$ such that μ is a hook. Then*

$$K^{-1}_{\alpha,\mu'} = \sum_{j \geq l(\mu)} (-1)^{n-l(\alpha)-l(\mu)+1} \frac{(l(\alpha)-1)!}{\prod_i m_i(\alpha)!} m_j(\alpha),$$

where μ' denotes the conjugate of μ.

Proof Since μ is a hook, μ' is also a hook. Then $K^{-1}_{\alpha,\mu'} = \sum_T wt(T)$, where the sum is taken over all special rim hook tabloids T with shape μ' and type α. In particular, any such tabloid T must contain a special rim hook ξ of length $j \geq l(\mu)$ since all special rim hooks by definition must contain a cell from the first column:



This hook contributes weight $(-1)^{j-l(\mu)}$, so

$$K^{-1}_{\alpha,\mu'} = \sum_{j \geq l(\mu)} K^{-1}_{\alpha\backslash(j),1^{n-j}}(-1)^{j-l(\mu)}.$$

By Claim 1.3, we know that

$$K^{-1}_{\alpha\backslash(j),1^{n-j}} = \frac{(l(\alpha)-1)!}{\prod_i m_i(\alpha)!} m_j(\alpha)(-1)^{n-j-(l(\alpha)-1)},$$

so

$$K^{-1}_{\alpha,\mu'} = \sum_{j \geq l(\mu)} \frac{(l(\alpha)-1)!}{\prod_i m_i(\alpha)!} m_j(\alpha)(-1)^{n-j-(l(\alpha)-1)}(-1)^{j-l(\mu)}.$$

■

Claim 1.5 *Fix $1 \leq l \leq n$. Then*

$$\sum_{\substack{\alpha \vdash n \\ l(\alpha)=l}} \frac{l!}{\prod_i m_i(\alpha)!} = \binom{n-1}{l-1}.$$

Proof Take $\alpha \vdash n$ such that $l(\alpha) = l$. We know that $\frac{l!}{\prod_i m_i(\alpha)!}$ = the number of permutations of parts of α. Then

$$\begin{aligned} \sum_{\substack{\alpha \vdash n \\ l(\alpha)=l}} \frac{l!}{\prod_i m_i(\alpha)!} &= \#\{a_1, a_2, \ldots, a_l \,|\, a_1 + a_2 + \cdots + a_l = n\} \\ &= \binom{n-1}{l-1}. \end{aligned}$$

■

Corollary 1.2 *Fix $1 \leq k \leq n$ and $1 \leq l \leq n$. Then*

$$\sum_{\substack{\alpha \vdash n \\ l(\alpha)=l}} \frac{(l-1)!}{\prod_i m_i(\alpha)!} m_k(\alpha) = \binom{n-k-1}{l-2}.$$

Proof

$$\begin{aligned}\sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} \frac{(l-1)!}{\prod_i m_i(\alpha)!} m_k(\alpha) &= \sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} \frac{(l-1)!}{m_1(\alpha)!m_2(\alpha)!\cdots(m_k(\alpha)-1)!\cdots} \\ &= \sum_{\substack{\alpha\vdash n-k\\ l(\alpha)=l-1}} \frac{(l-1)!}{\prod_i m_i(\alpha)!}.\end{aligned}$$

By Claim 1.5,

$$\sum_{\substack{\alpha\vdash n-k\\ l(\alpha)=l-1}} \frac{(l-1)!}{\prod_i m_i(\alpha)!} = \binom{n-k-1}{l-2}.$$

■

Now we will prove Theorem 1.4 combinatorially.

Proof [Theorem 1.4] First, assume that γ is a hook. From Claim 1.4, we know

$$\sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} K^{-1}_{\alpha,\gamma'} = \sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} \sum_{j\geq l(\gamma)} (-1)^{n-l-l(\gamma)+1} \frac{(l-1)!}{\prod_i m_i(\alpha)!} m_j(\alpha).$$

By Corollary 1.2, we have

$$\sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} \frac{(l-1)!}{\prod_i m_i(\alpha)!} m_j(\alpha) = \binom{n-j-1}{l-2},$$

so

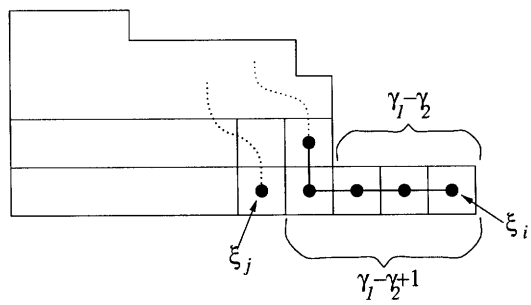
$$\begin{aligned}\sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} K^{-1}_{\alpha,\gamma'} &= (-1)^{n-l-l(\gamma)+1} \sum_{j\geq l(\gamma)} \binom{n-j-1}{l-2} \\ &= (-1)^{n-l-l(\gamma)+1} \binom{n-l(\gamma)}{l-1}.\end{aligned}$$

Now assume that γ is not a hook. Take $\gamma = (\gamma_1 \geq \gamma_2 \geq \cdots)$. Fix $1 \leq l \leq n$. Note that since γ is not a hook, $\gamma_2 > 1$. If T is a special rim hook tabloid, then let $sh(T)$ denote the shape of T and $ty(T)$ denote the type of T. Given a SRHT T such that $sh(T) = \gamma$, we say that T is in *class 1* ($cl(T) = 1$) if there is a special rim hook occupying both the γ_1 position of the bottom row and the γ_2 position of the second row from the bottom. We say that T is in *class 2* ($cl(T) = 2$) if there is a special rim hook occupying

the γ_1 and $\gamma_2 - 1$ positions of the bottom row. Note that these are mutually exclusive conditions, and any SRHT T of a fixed shape γ is either in class 1 or class 2.

Let $\mathcal{S}^1_{l,\gamma,n} := \{\text{SRHT } T : sh(T) = \gamma, l(ty(T)) = l, ty(T) \vdash n, cl(T) = 1\}$. Let $\mathcal{S}^2_{l,\gamma,n} := \{\text{SRHT } T : sh(T) = \gamma, l(ty(T)) = l, ty(T) \vdash n, cl(T) = 2\}$.

We will define a bijection $f : \mathcal{S}^1_{l,\gamma,n} \leftrightarrow \mathcal{S}^2_{l,\gamma,n}$. Take some $T_1 \in \mathcal{S}^1_{l,\gamma,n}$ of type α:

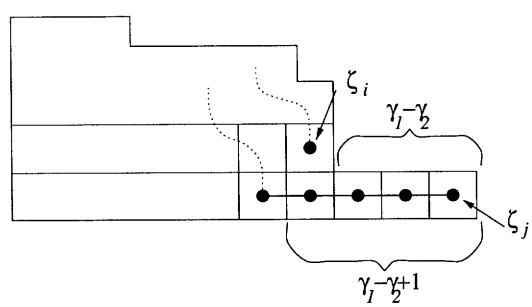


i.e., there exists a special rim hook ξ_i, say of size α_i, that occupies both the γ_1 position of the bottom row and the γ_2 position of the second-from-bottom row. Then there must be another special rim hook ξ_j of size α_j ($j \neq i$) that ends in the $\gamma_2 - 1$ position of the bottom row.

Define $\sigma \vdash n$ such that

$$\sigma_k = \begin{cases} \alpha_k - (\gamma_1 - \gamma_2 + 1) & \text{if } k = i \\ \alpha_k + \gamma_1 - \gamma_2 + 1 & \text{if } k = j \\ \alpha_k & \text{otherwise.} \end{cases}$$

Note that $l(\sigma) = l$ and $\sigma \neq \alpha$. We can then construct a special rim hook tabloid $T_2 \in \mathcal{S}^2_{l,\gamma,n}$ of shape γ and type σ:



where a special rim hook ζ_j of size σ_j occupies the γ_1 and $\gamma_2 - 1$ positions of the bottom row, such that ζ_j is ξ_j plus the last $\gamma_1 - \gamma_2 + 1$ boxes of the

bottom row. A special rim hook ζ_i of size σ_i ends in the γ_2 position of the second-from-bottom row such that ζ_i is ξ_i minus the last $\gamma_1 - \gamma_2 + 1$ boxes of the bottom row.

This is clearly a bijection, and it is easy to see that $wt(T_1) = -wt(T_2)$. Also, $\mathcal{S}^1_{l,\gamma,n} \uplus \mathcal{S}^2_{l,\gamma,n}$ (disjoint union) consists of all rim hook tabloids of shape γ and type of l parts. So

$$\begin{aligned}
\sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} K^{-1}_{\alpha,\gamma} &= \sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} \sum_{\substack{SRHT\ T\\ sh(T)=\gamma\\ ty(T)=\alpha}} wt(T)\\
&= \sum_{\substack{SRHT\ T\\ sh(T)=\gamma\\ l(ty(T))=l\\ ty(T)\vdash n}} wt(T)\\
&= \sum_{T_1\in\mathcal{S}^1_{l,\gamma,n}} wt(T_1) + \sum_{T_2\in\mathcal{S}^2_{l,\gamma,n}} wt(T_2).
\end{aligned}$$

This sum is 0 from the above construction. ■

Theorem 1.6 *Take $\gamma \vdash n$. Then*

$$[s_\gamma]Z[B_{(n)}] = \begin{cases} (-1)^{n-l(\gamma)+1} + nD_{n-l(\gamma)} & \textit{if } \gamma \textit{ is a hook}\\ 0 & \textit{otherwise.}\end{cases}$$

Proof By Theorem 1.2,

$$[s_\gamma]Z[B_{(n)}] = \frac{\sum_{\pi\in\mathcal{D}_n}\chi^\gamma(\pi)}{f^\gamma}\chi^\gamma_{(n)}.$$

By the Murnaghan-Nakayama rule,

$$\chi^\gamma_{(n)} = \begin{cases} (-1)^{l(\gamma)-1} & \text{if } \gamma \text{ is a hook}\\ 0 & \text{otherwise.}\end{cases}$$

So

$$[s_\gamma]Z[B_{(n)}] = \begin{cases} (-1)^{l(\gamma)-1}\frac{\sum_{\pi\in\mathcal{D}_n}\chi^\gamma(\pi)}{f^\gamma} & \text{if } \gamma \text{ is a hook}\\ 0 & \text{otherwise.}\end{cases}$$

Let us take $\gamma \vdash n$ to be a hook. The graph of $\overline{B}_{(n)}$ is a directed n-cycle, so when considering subgraphs of $\overline{B}_{(n)}$ of type $(\alpha;\beta)$, the only possibilities

are $\beta = \emptyset$ (and $\alpha \vdash n$) or $\beta = (n)$ (and $\alpha = \emptyset$). Now let us compute the cycle indicator the other way. By Theorem 1.3, we have

$$Z[B_{(n)}] = \sum_{\alpha \vdash n} \prod_j m_j(\alpha)! r_{\alpha,\emptyset}(\overline{B}_{(n)}) f_\alpha + (-1)^n r_{\emptyset,(n)}(\overline{B}_{(n)}) p_n.$$

Before we proceed with the proof, we must prove the following claim:

Claim 1.6 *Take $\alpha \vdash n$.*

$$r_{\alpha,\emptyset}(\overline{B}_{(n)}) = \frac{n(l(\alpha)-1)!}{\prod_j m_j(\alpha)!}$$

Proof [Claim 1.6] We know that $r_{\alpha,\emptyset}(\overline{B}_{(n)})$ is the number of ways of taking an n-cycle and cutting edges to form paths of lengths $\alpha_1, \alpha_2, \ldots$. Say $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{k+1} > 0)$. Fix some $1 \leq i \leq k+1$. There are n ways to choose the first point of the α_i-path. The number of ways to split the rest of the n-cycle into paths of type $\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{k+1})$ is $\frac{k!}{\prod_j m_j(\alpha')!}$. Since there are $m_{\alpha_i}(\alpha)$ paths of length α_i,

$$\begin{aligned} r_{\alpha,\emptyset}(\overline{B}_{(n)}) &= \frac{n \cdot k!}{m_{\alpha_i}(\alpha) \prod_j m_j(\alpha')!} \\ &= \frac{n(l(\alpha)-1)!}{\prod_j m_j(\alpha)!}. \end{aligned}$$

■

Given Claim 1.6 and $f_\alpha = \sum_{\gamma \vdash n} s_\gamma K^{-1}_{\alpha,\gamma'}$,

$$Z[B_{(n)}] = \sum_{\alpha \vdash n} n(l(\alpha)-1)! \sum_{\gamma \vdash n} s_\gamma K^{-1}_{\alpha,\gamma'} + (-1)^n \sum_{\gamma \vdash n} s_\gamma \chi^\gamma_{(n)}.$$

$$\begin{aligned} \text{So } [s_\gamma] Z[B_{(n)}] &= (-1)^n \chi^\gamma_{(n)} + \sum_{\alpha \vdash n} n(l(\alpha)-1)! K^{-1}_{\alpha,\gamma'} \\ &= (-1)^n (-1)^{l(\gamma)-1} + \sum_{l=1}^{n} n(l-1)! \sum_{\substack{\alpha \vdash n \\ l(\alpha)=l}} K^{-1}_{\alpha,\gamma'}. \end{aligned}$$

By Theorem 1.4, we have

$$\begin{aligned} [s_\gamma] Z[B_{(n)}] &= (-1)^{n-l(\gamma)+1} + \sum_{l=1}^{n-l(\gamma)+1} n(l-1)!(-1)^{n-l(\gamma)-l+1} \binom{n-l(\gamma)}{l-1} \\ &= (-1)^{n-l(\gamma)+1} + n \sum_{l=1}^{n-l(\gamma)+1} \frac{(n-l(\gamma))!}{(n-l(\gamma)-l+1)!} (-1)^{n-l(\gamma)-l+1}. \end{aligned}$$

Here we reindex, letting $i = n - l(\gamma) - l + 1$. Then we have

$$\begin{aligned}[s_\gamma]Z[B_{(n)}] &= (-1)^{n-l(\gamma)+1} + n \sum_{i=0}^{n-l(\gamma)} (n - l(\gamma))! \frac{(-1)^i}{i!} \\ &= (-1)^{n-l(\gamma)+1} + nD_{n-l(\gamma)}.\end{aligned}$$

■

Note that this lemma shows that $Z[B_{(n)}]$ is Schur-positive. In fact, the corresponding representation has degree $n!$. If it *is* a permutation representation, the number of orbits is D_n. We *do* know that $Z[B_{(n)}]$ is not h-positive in general. For example, $Z[B_{(4)}] = 3h_{1^4} - 8h_{21^2} + 8h_{31} + 2h_{2^2} + 4h_4$.

We know that in general, $Z[B_\lambda]$ is not Schur-positive if $\lambda \neq (n)$. For example, $Z[B_{31}] = 9s_4 - 3s_{22} - 3s_{1111}$. See Tables .1 and .2 in the Appendix for a table of cycle indicators for $n \leq 6$.

Corollary 1.3 *Take $\gamma \vdash n$ where γ is a hook. Then*

$$\sum_{\pi \in \mathcal{D}_n} \chi^\gamma(\pi) = f^\gamma \left[(-1)^n + (-1)^{l(\gamma)-1} n D_{n-l(\gamma)}\right].$$

1.3.2 Ménage Board

The problème des ménages can be described as follows. Suppose there are n married couples. The men are seated around a circular table, leaving one seat open in between each consecutive pair of men. Each woman, bored of her husband's conversational skills, wishes to sit next to any man but her own husband. In how many ways can this be done?

Definition 1.1 *We define the $n \times n$* ménage board *to be $B^n_{\text{men}} := \{(i,j) \in [n]^2 \mid j \not\equiv i \text{ or } i+1 \bmod n\}$.*

The problème des ménages can then be restated as counting the number of B^n_{men}-compatible permutations in S_n. It is well-known (see, *e.g.*, [Sta86, p.73]) that this number is $\sum_{i=0}^n (-1)^i (n-i)! \frac{2n}{2n-i}\binom{2n-i}{i}$. But what if we wish to classify these permutations by cycle type?

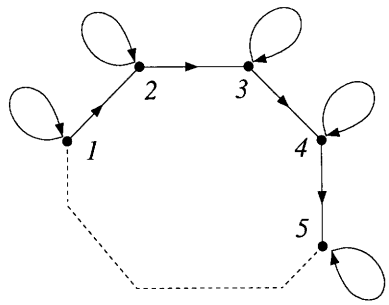
Theorem 1.7

$$Z[B^n_{\text{men}}] = \sum_{\gamma \vdash n} s_\gamma \left[\sum_{\substack{\lambda \vdash n \\ \text{hook}}} \frac{D_{n-l(\lambda)}}{(n-1)!} \sum_{\pi \in \mathcal{D}_n} \chi^\lambda(\pi)\chi^\gamma(\pi) + (-1)^n \chi^\gamma_{(n)}\right]$$

Proof By Theorem 1.3, we have

$$Z[B^n_{\text{men}}] = \sum_{\alpha,\beta}(-1)^{|\beta|}\prod_j m_j(\alpha)! r_{\alpha,\beta}(\overline{B}^n_{\text{men}}) f_\alpha p_\beta.$$

Take some graph of type $(\alpha;\beta)$ in the sum. The only cycles in $\overline{B}^n_{\text{men}}$ are the 1-cycles and the n-cycle. This is because the graph of $\overline{B}^n_{\text{men}}$ looks like:



So the only possibilities are:

$$\beta = \begin{cases} \emptyset & \text{iff } \alpha \vdash n \\ 1^i & \text{iff } \alpha \vdash n-i \text{ where } 1 \le i \le n \\ (n) & \text{iff } \alpha = \emptyset. \end{cases}$$

Therefore,

$$\begin{aligned} Z[B^n_{\text{men}}] &= \sum_{\alpha\vdash n}\prod_j m_j(\alpha)! r_{\alpha,\emptyset}(\overline{B}^n_{\text{men}}) f_\alpha + \\ &\quad \sum_{i=1}^{n}\sum_{\alpha\vdash n-i}(-1)^i\prod_j m_j(\alpha)! r_{\alpha,1^i}(\overline{B}^n_{\text{men}}) f_\alpha p_1^i + (-1)^n r_{\emptyset,(n)}(\overline{B}^n_{\text{men}}) p_n. \end{aligned}$$

Let

$$b_\mu := \sum_{\alpha\vdash n}\prod_j m_j(\alpha)! r_{\alpha,\emptyset}(\overline{B}^n_{\text{men}})\sum_{\lambda\vdash n} K^{-1}_{\alpha\lambda'}\frac{1}{z_\mu}\chi^\lambda_\mu,$$

and let

$$a_\mu := \sum_{i=1}^{m_1(\mu)}\sum_{\alpha\vdash n-i}(-1)^i\prod_j m_j(\alpha)! r_{\alpha,1^i}(\overline{B}^n_{\text{men}})\sum_{\lambda\vdash n-i} K^{-1}_{\alpha\lambda'}\frac{1}{z_{\mu\backslash 1^i}}\chi^\lambda_{\mu\backslash 1^i}.$$

Since $f_\alpha = \sum_\mu p_\mu \sum_\lambda K^{-1}_{\alpha\lambda'}\frac{1}{z_\mu}\chi^\lambda_\mu$, we know that

$$Z[B^n_{\text{men}}] = \sum_{\mu\vdash n} p_\mu b_\mu + \sum_{\substack{\mu\vdash n\\ m_1(\mu)\ge 1}} p_\mu a_\mu + (-1)^n p_n.$$

We can rewrite this so that

$$Z[B^n_{\text{men}}] = \sum_{\substack{\mu\vdash n\\ m_1(\mu)\geq 1}} p_\mu b_\mu + \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} p_\mu b_\mu + \sum_{\substack{\mu\vdash n\\ m_1(\mu)\geq 1}} p_\mu a_\mu + (-1)^n p_n.$$

By definition of B^n_{men} and $Z[B^n_{\text{men}}]$, we know that $[p_\mu]Z[B^n_{\text{men}}] = 0$ where $m_1(\mu) \geq 1$. So $\sum_{\substack{\mu\vdash n\\ m_1(\mu)\geq 1}} p_\mu b_\mu = 0$ and $\sum_{\substack{\mu\vdash n\\ m_1(\mu)\geq 1}} p_\mu a_\mu = 0$. Thus, we have

$$Z[B^n_{\text{men}}] = \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} p_\mu \sum_{\alpha\vdash n}\prod_j m_j(\alpha)! r_{\alpha,\emptyset}(\overline{B}^n_{\text{men}}) \sum_{\lambda\vdash n} K^{-1}_{\alpha\lambda'}\frac{1}{z_\mu}\chi^\lambda_\mu + (-1)^n p_n.$$

Since $r_{\alpha,\emptyset}(\overline{B}^n_{\text{men}}) = r_{\alpha,\emptyset}(\overline{B}_{(n)}) = \frac{n(l(\alpha)-1)!}{\prod_j m_j(\alpha)!}$, we know that

$$Z[B^n_{\text{men}}] = \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} p_\mu \sum_{\alpha\vdash n} n(l(\alpha)-1)! \sum_{\lambda\vdash n} K^{-1}_{\alpha\lambda'}\frac{1}{z_\mu}\chi^\lambda_\mu + (-1)^n p_n.$$

Changing to the Schur function basis, we get

$$Z[B^n_{\text{men}}] = \sum_{\gamma\vdash n} s_\gamma \sum_{\alpha\vdash n} n(l(\alpha)-1)! \sum_{\lambda\vdash n} K^{-1}_{\alpha,\lambda'} \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} \frac{1}{z_\mu}\chi^\gamma_\mu\chi^\lambda_\mu + (-1)^n\chi^\gamma_{(n)}.$$

It follows that

$$\begin{aligned}
[s_\gamma]Z[B^n_{\text{men}}] &= \sum_{\alpha\vdash n} n(l(\alpha)-1)! \sum_{\lambda\vdash n} K^{-1}_{\alpha,\lambda'} \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} \frac{1}{z_\mu}\chi^\gamma_\mu\chi^\lambda_\mu + (-1)^n\chi^\gamma_{(n)}\\
&= \sum_{l=1}^{n} n(l-1)! \sum_{\lambda\vdash n}\sum_{\substack{\alpha\vdash n\\ l(\alpha)=l}} K^{-1}_{\alpha,\lambda'} \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} \frac{1}{z_\mu}\chi^\gamma_\mu\chi^\lambda_\mu + (-1)^n\chi^\gamma_{(n)}.
\end{aligned}$$

Then by Theorem 1.4,

$$\begin{aligned}
[s_\gamma]Z[B^n_{\text{men}}] &= \sum_{l=1}^{n} n(l-1)! \sum_{\substack{\lambda\vdash n\\ \text{hook}}} (-1)^{n-l-l(\lambda)+1}\binom{n-l(\lambda)}{l-1} \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} \frac{1}{z_\mu}\chi^\gamma_\mu\chi^\lambda_\mu +\\
&\quad +(-1)^n\chi^\gamma_{(n)}\\
&= \sum_{l=1}^{n} \sum_{\substack{\lambda\vdash n\text{hook}\\ l(\lambda)\leq n-l+1}} \frac{(-1)^{n-l-l(\lambda)+1}(n-l(\lambda))!}{(n-1)!(n-l(\lambda)-l+1)!} \sum_{\substack{\mu\vdash n\\ m_1(\mu)=0}} \frac{n!}{z_\mu}\chi^\gamma_\mu\chi^\lambda_\mu +
\end{aligned}$$

$$
\begin{aligned}
& +(-1)^n \chi^\gamma_{(n)} \\
= & \sum_{\substack{\lambda \vdash n \\ \text{hook}}} \frac{(-1)^{n-l(\lambda)+1}}{(n-1)!} \sum_{\substack{\mu \vdash n \\ m_1(\mu)=0}} \frac{n!}{z_\mu} \chi^\gamma_\mu \chi^\lambda_\mu \sum_{l=1}^{n-l(\lambda)+1} \frac{(-1)^l (n-l(\lambda))!}{(n-l(\lambda)+1-l)!} + \\
& +(-1)^n \chi^\gamma_{(n)}.
\end{aligned}
$$

Reindexing,

$$
\begin{aligned}
\sum_{l=1}^{n-l(\lambda)+1} \frac{(-1)^l (n-l(\lambda))!}{(n-l(\lambda)+1-l)!} &= (-1)^{n-l(\lambda)+1} \sum_{l=0}^{n-l(\lambda)} (n-l(\lambda))! \frac{(-1)^l}{l!} \\
&= (-1)^{n-l(\lambda)+1} D_{n-l(\lambda)}.
\end{aligned}
$$

So

$$
\begin{aligned}
[s_\gamma] Z[B^n_{\text{men}}] &= \sum_{\substack{\lambda \vdash n \\ \text{hook}}} \frac{D_{n-l(\lambda)}}{(n-1)!} \sum_{\substack{\mu \vdash n \\ m_1(\mu)=0}} \frac{n!}{z_\mu} \chi^\gamma_\mu \chi^\lambda_\mu + (-1)^n \chi^\gamma_{(n)} \\
&= \sum_{\substack{\lambda \vdash n \\ \text{hook}}} \frac{D_{n-l(\lambda)}}{(n-1)!} \sum_{\pi \in \mathcal{D}_n} \chi^\lambda(\pi) \chi^\gamma(\pi) + (-1)^n \chi^\gamma_{(n)}.
\end{aligned}
$$

■

Corollary 1.4 *Let* $g_{\lambda,m} = |\{\sigma \in S_n \,|\, \rho(\sigma) = \lambda, \sigma \text{ is } B^n_{\text{men}} - \text{compatible}\}|$. *Then*

$$
g_{\lambda,m} = \frac{1}{z_\lambda} \sum_{\gamma \vdash n} \chi^\gamma_\lambda \sum_{\substack{\mu \vdash n \\ \text{hook}}} \frac{D_{n-l(\mu)}}{(n-1)!} \sum_{\pi \in \mathcal{D}_n} \chi^\mu(\pi) \chi^\gamma(\pi) + (-1)^n \chi^\gamma_{(n)}.
$$

Proof Follows the proof of Corollary 1.1. ■

1.4 Wreath Products

Take a finite abelian group G with $|G| = m$. Say $G = \{g_1, g_2, \ldots, g_m\}$. Assume g_1 = identity. We will define the *wreath product* of G with S_n, denoted $G \wr S_n$, as follows: $G \wr S_n := \{(h_1, h_2, \ldots, h_n; \pi) \,|\, h_i \in G \text{ and } \pi \in S_n\}$. Note that $|G \wr S_n| = m^n \cdot n!$. Group operations are defined as follows:

- For $(h_1, \ldots, h_n; \pi), (h'_1, \ldots, h'_n; \sigma) \in G \wr S_n$,
  $(h_1, \ldots, h_n; \pi) \cdot (h'_1, \ldots, h'_n; \sigma) = (h_{\pi^{-1}(1)} h'_{\pi^{-1}(1)}, \ldots, h_{\pi^{-1}(n)} h'_{\pi^{-1}(n)}; \pi\sigma)$.
- $(h_1, \ldots, h_n; \pi)^{-1} = (h^{-1}_{\pi(1)}, \ldots, h^{-1}_{\pi(n)}; \pi^{-1})$.

We will define for $G \wr S_n$ the analogue of the cycle type of an element of S_n. Consider $w = (h_1, h_2, \ldots, h_n; \pi) \in G \wr S_n$. Given any l-cycle $(i_1 i_2 \cdots i_l)$ of π, the corresponding element $h_{i_1} h_{i_2} \cdots h_{i_l} \in G$ is in some conjugacy class of G.

Fix some conjugacy class g_i of G. (Since G is abelian, each element of G is its own conjugacy class.) Let α^i be the partition whose parts consist of the cycle-lengths of the cycles of π whose corresponding element in G is g_i. Then type$(w) := (\alpha^1, \alpha^2, \ldots, \alpha^m)$. (Note that $\rho(\pi) = \cup_{i=1}^m \alpha^i$.) Any two elements of $G \wr S_n$ are conjugate if and only if they have the same type. Let $Z_{\alpha^1, \alpha^2, \ldots, \alpha^m}$ denote the order of the centralizer of the conjugacy class $(\alpha^1, \alpha^2, \ldots, \alpha^m)$. We know (see, *e.g.*, [Mac95, p. 171]) that $Z_{\alpha^1, \alpha^2, \ldots, \alpha^m} = |G|^{l(\rho(\pi))} \prod_{i=1}^m z_{\alpha^i}$.

We define a generalization of the power-sum symmetric function that is mentioned in [Mac95, p. 172]. Take some conjugacy class, say g_j, of G. Then we define $p_i(g_j) := \sum_\chi \chi(g_j) p_i(x^\chi)$, where the sum is taken over all irreducible characters of G and $p_i(x^\chi)$ denotes the regular i-th power-sum symmetric function in the variables $x_1^\chi, x_2^\chi, \ldots$. For $\alpha = (\alpha_1, \alpha_2, \ldots)$ a partition, $p_\alpha(g_j) = p_{\alpha_1}(g_j) p_{\alpha_2}(g_j) \cdots$. For partitions $\alpha^1, \alpha^2, \ldots, \alpha^m$ such that $\sum_i |\alpha^i| = n$, we define a generalized power-sum symmetric function

$$P_{\alpha^1, \alpha^2, \ldots, \alpha^m} := p_{\alpha^1}(g_1) p_{\alpha^2}(g_2) \cdots p_{\alpha^m}(g_m). \tag{1.2}$$

Just as we can think of a permutation in S_n as a placement of n non-attacking rooks on $[n]^2$, we can think of an element $w = (h_1, \ldots, h_n; \pi)$ as a placement of n labelled non-attacking rooks on $[n]^2$ viz., if $\pi(i) = j$, then we place a rook labelled h_i in position (i, j).

Consider the following generalization of B-compatibility to the group $G \wr S_n$: instead of removing positions from $[n]^2$, we label some positions with elements of G, and we allow no rooks with those labels on those positions. Formally, given some subset $A \subseteq [n]^2$, we define a function $L : A \to G$ such that $L(i, j)$ is the label in position (i, j) of the board B. We will say that $(h_1, \ldots, h_n; \pi) \in G \wr S_n$ is B-compatible if $h_i \neq L(i, \pi(i))$, if the latter is defined. If $L(i, \pi(i))$ is not defined, h_i can be any element of G. We define the generalized cycle indicator to be $Z^G[B] := \sum_w P_{\text{type}(w)}$, where the sum is taken over B-compatible elements $w \in G \wr S_n$.

Example 1.7 Let $G = \mathbf{Z}_2$ and let $A = \{(1,1),(1,3),(2,1),(3,2)\}$. Define $L : A \to \mathbf{Z}_2$ such that $L(1,1) = +1$, $L(1,3) = +1$, $L(2,1) = -1$, and $L(3,2) = -1$. Then B looks like:

+		+
−		
	−	

We say that $(h_1, h_2, h_3; \pi) \in \mathbf{Z}_2 \wr S_3$ is B-compatible if

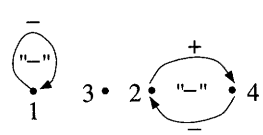
- $h_1 \neq L(1, \pi(1))$
- $h_2 \neq L(2, \pi(2))$
- $h_3 \neq L(3, \pi(3))$

For example, $(-1, +1, -1; (13)(2))$ is B-compatible:

		−
	+	
−		

We can think of any labelled subset of $[n]^2$ with label function L as a graph, viz., $i \to j$ if $\pi(i) = j$, and edge $i \to j$ is labelled $L(i,j)$. When the subset is a placement of at most n labelled non-attacking rooks, then the graph consists of labelled disjoint paths and cycles:

−			
			+
	−		



Then we give each cycle (or path) a label, which is the conjugacy class of the product of the labels of the edges in the cycle (or path). In our example above, the path/cycle labels are in quotes. We say that the graph has *type* $(\alpha^0, \alpha^1, \ldots, \alpha^m; \beta^1, \beta^2, \ldots, \beta^m)$, where α^0 is the partition of singletons (1-paths), α^i $(i > 0)$ is the partition whose parts are the lengths of paths labelled g_i, and β^j is the partition whose parts are the lengths of the cycles labelled g_j. (Note that $m_1(\alpha^i) = 0$ for $i > 0$.) Again, referring to the example above, the graph is of type $(1, \emptyset, \emptyset; \emptyset, 21)$.

Claim 1.7

$$p_\alpha(x) = \frac{1}{|G|^{l(\alpha)}} \sum_{\substack{\beta^1,\ldots,\beta^m \\ \cup_{i=1}^m \beta^i = \alpha}} \prod_{j\geq 1} \frac{m_j(\alpha)!}{m_j(\beta^1)!\cdots m_j(\beta^m)!} P_{\beta^1,\ldots,\beta^m}.$$

Proof Fix an irreducible character χ of G. Then we know (see [Mac95, p. 174]) that $p_i(x^\chi) = \frac{1}{|G|}\sum_{j=1}^m \chi(g_j)p_i(g_j)$. Let $x = x^{triv}$ where $triv$ denotes the trivial character of G. Then $p_i(x) = \frac{1}{|G|}(p_i(g_1)+p_i(g_2)+\cdots+p_i(g_m))$. So

$$p_\alpha(x) = \frac{1}{|G|^{l(\alpha)}}(p_{\alpha_1}(g_1)+\cdots+p_{\alpha_1}(g_m))(p_{\alpha_2}(g_1)+\cdots+p_{\alpha_2}(g_m))\cdots.$$

It follows from (1.2) that

$$p_\alpha(x) = \frac{1}{|G|^{l(\alpha)}} \sum_{\substack{\beta^1,\ldots,\beta^m \\ \cup_{i=1}^m \beta^i = \alpha}} \prod_{j\geq 1} \frac{m_j(\alpha)!}{m_j(\beta^1)!\cdots m_j(\beta^m)!} P_{\beta^1,\ldots,\beta^m}.$$

■

With the above definitions, we can prove the wreath product analogue of Theorem 1.3. Much of the notation used here is from [SS93].

Theorem 1.8 *Given S a placement of at most n labelled non-attacking rooks with label function L, we define $Z^G_{\supseteq S} := \sum_w P_{\mathrm{type}(w)}$, where the sum is taken over all $w = (h_1,\ldots,h_n;\pi) \in G \wr S_n$ such that if $L(i,\pi(i))$ is defined, $h_i = L(i,\pi(i))$; otherwise, h_i can be any element of G. Say that S is a graph of type $(\alpha^0,\ldots,\alpha^m;\beta^1,\ldots,\beta^m)$. Then*

$$Z^G_{\supseteq S} = (-1)^{|\alpha|-l(\alpha)}|G|^{l(\alpha)}\prod_i m_i(\alpha)! f_\alpha(x) P_{\beta^1,\ldots,\beta^m},$$

where α denotes $\cup_{i=1}^m \alpha^i$.

Proof Each i-cycle labelled "g_k" in S contributes a factor of $p_i(g_k)$ to $Z^G_{\supseteq S}$, so we need only consider the case in which S is of type $(\alpha^0,\ldots,\alpha^m;\emptyset,\ldots,\emptyset)$. Let α denote $\cup_{i=0}^m \alpha^i$ in this proof.

Let $I_\alpha(\lambda)$ denote the number of unlabelled subgraphs of type $(\alpha;\emptyset)$ in the unlabelled graph of λ. Then

$$\frac{|G|^n n!}{Z_{\lambda^1,\ldots,\lambda^m}} I_\alpha(\lambda) = \text{ number of inclusions } S_1 \subset S_2,$$

where S_2 is a labelled digraph of type $(\emptyset, \ldots, \emptyset; \lambda^1, \ldots, \lambda^m)$ and S_1 is a digraph of type $(\alpha; \emptyset)$ inheriting labels from S_2 and λ denotes $\cup_{i=1}^m \lambda^i$.

Since $\frac{n!}{\prod_i m_i(\alpha)!}$ is the total number of unlabelled digraphs of type $(\alpha; \emptyset)$, and $|G|^{n-l(\alpha)}$ is the number of ways of labelling an unlabelled digraph of type $(\alpha; \emptyset)$,

$$\begin{aligned}
[P_{\lambda^1,\ldots,\lambda^m}] Z^G_{\supseteq S} &= \frac{\frac{|G|^n n!}{Z_{\lambda^1,\ldots,\lambda^m}} I_\alpha(\lambda)}{\frac{n!}{\prod_i m_i(\alpha)!} |G|^{n-l(\alpha)}} \\
&= \frac{\prod_i m_i(\alpha)!}{Z_{\lambda^1,\ldots,\lambda^m}} |G|^{l(\alpha)} I_\alpha(\lambda).
\end{aligned}$$

Since $Z_{\lambda^1,\ldots,\lambda^m} = |G|^{l(\lambda)} z_{\lambda^1} \cdots z_{\lambda^m}$,

$$[P_{\lambda^1,\ldots,\lambda^m}] Z^G_{\supseteq S} = \frac{\prod_i m_i(\alpha)!}{z_{\lambda^1} \cdots z_{\lambda^m} |G|^{l(\lambda)}} |G|^{l(\alpha)} I_\alpha(\lambda).$$

From [Ste92], we know that $f_\alpha(x) = \sum_\gamma \frac{(-1)^{|\alpha|-l(\alpha)}}{z_\gamma} I_\alpha(\gamma) p_\gamma(x)$, so using Claim 1.7, we get

$$\begin{aligned}
f_\alpha(x) &= \sum_{\gamma \vdash n} \frac{(-1)^{|\alpha|-l(\alpha)}}{z_\gamma} I_\alpha(\gamma) \frac{1}{|G|^{l(\gamma)}} \sum_{\substack{\lambda^1,\ldots,\lambda^n \\ \cup_{i=1}^m \lambda^i = \gamma}} \prod_{j \geq 1} \frac{m_j(\gamma)!}{m_j(\lambda^1)! \cdots m_j(\lambda^m)!} P_{\lambda^1,\ldots,\lambda^m} \\
&= \sum_{\substack{\lambda^1,\ldots,\lambda^m \\ |\cup_{i=1}^m \lambda^i| = n}} \frac{(-1)^{|\alpha|-l(\alpha)}}{z_\lambda} I_\alpha(\lambda) \frac{1}{|G|^{l(\lambda)}} \prod_j \frac{m_j(\lambda)!}{m_j(\lambda^1)! \cdots m_j(\lambda^m)!} P_{\lambda^1,\ldots,\lambda^m}.
\end{aligned}$$

Since $z_\lambda = z_{\lambda^1} \cdots z_{\lambda^m} \prod_j \frac{m_j(\lambda)!}{m_j(\lambda^1)! \cdots m_j(\lambda^m)!}$,

$$[P_{\lambda^1,\ldots,\lambda^m}] f_\alpha(x) = \frac{(-1)^{|\alpha|-l(\alpha)}}{z_{\lambda^1} \cdots z_{\lambda^m}} I_\alpha(\lambda) \frac{1}{|G|^{l(\lambda)}}.$$

So

$$[P_{\lambda^1,\ldots,\lambda^m}] Z^G_{\supseteq S} = (-1)^{|\alpha|-l(\alpha)} |G|^{l(\alpha)} \prod_i m_i(\alpha)! [P_{\lambda^1,\ldots,\lambda^m}] f_\alpha(x).$$

Therefore,

$$Z^G_{\supseteq S} = (-1)^{|\alpha|-l(\alpha)} |G|^{l(\alpha)} \prod_i m_i(\alpha)! f_\alpha(x).$$

■

Theorem 1.9

$$Z^G[B] = \sum_{(\alpha^0,\ldots,\alpha^m;\gamma^1,\ldots,\gamma^m)} (-1)^{|\gamma|}|G|^{l(\alpha)} \prod_i m_i(\alpha)! r_{\alpha^0,\ldots,\alpha^m;\gamma^1,\ldots,\gamma^m}(\overline{B}) f_\alpha(x) P_{\gamma^1\ldots,\gamma^m},$$

where $r_{\alpha^0,\ldots,\alpha^m;\gamma^1,\ldots,\gamma^m}(\overline{B})$ *denotes the number of subgraphs of* $\overline{B}$ *of type* $(\alpha^0,\ldots,\alpha^m;\gamma^1,\ldots,\gamma^m)$ *and* α *denotes* $\cup_{i=0}^m \alpha^i$ *and* γ *denotes* $\cup_{i=1}^m \gamma^i$.

Proof Using the standard inclusion-exclusion argument,

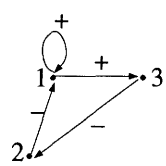
$$Z^G[B] = \sum_{S\subseteq\overline{B}} (-1)^{|S|} Z^G_{\supseteq S},$$

where the sum is taken over all labelled non-attacking rook placements S on $\overline{B}$. Use Theorem 1.2, noting that $|S| = |\cup_{i=0}^m \alpha^i| + |\cup_{j=1}^m \beta^j| - l(\cup_{i=0}^m \alpha^i)$ for S of type $(\alpha^0,\ldots,\alpha^m;\beta^1,\ldots,\beta^m)$. ■

Example 1.8 Let our board B be given by:

+		+
−		
	−	

Then the graph of $\overline{B}$ will look like:



We use Figure 1.1 to calculate $Z^{\mathbf{Z}_2}[B] = (p_1(x)-p_1(y))^3+(p_1(x)+p_1(y))^2(p_1(x)-p_1(y)) + 2(p_1(x)+p_1(y))(p_1(x)-p_1(y))^2 + 2(p_2(x)+p_2(y))(p_1(x)+p_1(y)) + 3(p_2(x)+p_2(y))(p_1(x)-p_1(y)) + 2(p_1(x)+p_1(y))(p_2(x)-p_2(y)) + 3(p_1(x)-p_1(y))(p_2(x)-p_2(y)) + 4(p_3(x)+p_3(y)) + 5(p_3(x)-p_3(y))$, which simplifies to

$$Z^{\mathbf{Z}_2}[B] = 9p_3(x) + 10p_{21}(x) + 4p_{1^3}(x) - 4p_{1^2}(x)p_1(y) - 2p_2(x)p_1(y) - p_3(y).$$

Now let us verify Theorem 1.9. The following is a table of terms in $Z^{\mathbf{Z}_2}[B]$ of Theorem 1.9 with non-zero $r_{\alpha^0,\alpha^1,\alpha^2;\gamma^1,\gamma^2}(\overline{B})$ values.

$w \in \mathbf{Z}_2 \wr S_3$	$P_{\mathrm{type}(w)}$
$(-++;(1)(2)(3))$	$P_{1^2;1}$
$(-+-;(1)(2)(3))$	$P_{1;1^2}$
$(--+;(1)(2)(3))$	$P_{1;1^2}$
$(---;(1)(2)(3))$	$P_{\emptyset;1^3}$
$(+++;(12)(3))$	$P_{21;\emptyset}$
$(++-;(12)(3))$	$P_{2;1}$
$(-++;(12)(3))$	$P_{1;2}$
$(-+-;(12)(3))$	$P_{\emptyset;21}$
$(-++;(13)(2))$	$P_{1;2}$
$(-+-;(13)(2))$	$P_{21;\emptyset}$
$(--+;(13)(2))$	$P_{\emptyset;21}$
$(---;(13)(2))$	$P_{2;1}$
$(-++;(1)(23))$	$P_{2;1}$
$(--+;(1)(23))$	$P_{\emptyset;21}$
$(+++;(123))$	$P_{3;\emptyset}$
$(++-;(123))$	$P_{\emptyset;3}$
$(+-+;(123))$	$P_{\emptyset;3}$
$(+--;(123))$	$P_{3;\emptyset}$
$(-++;(123))$	$P_{\emptyset;3}$
$(-+-;(123))$	$P_{3;\emptyset}$
$(--+;(123))$	$P_{3;\emptyset}$
$(---;(123))$	$P_{\emptyset;3}$
$(-++;(132))$	$P_{\emptyset;3}$

Figure 1.1: Table of B-compatible elements of $\mathbf{Z}_2 \wr S_3$

α^0	α^1	α^2	γ^1	γ^2	$r_{\alpha^0,\alpha^1,\alpha^2;\gamma^1,\gamma^2}(\overline{B})$
1	2	$\emptyset$	$\emptyset$	$\emptyset$	1
1	$\emptyset$	2	$\emptyset$	$\emptyset$	2
1^2	$\emptyset$	$\emptyset$	1	$\emptyset$	1
1^3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1
$\emptyset$	3	$\emptyset$	$\emptyset$	$\emptyset$	1
$\emptyset$	$\emptyset$	2	1	$\emptyset$	1
$\emptyset$	$\emptyset$	3	$\emptyset$	$\emptyset$	2
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	3	1

By Theorem 1.9 and with the help of J. Stembridge's symmetric function package "SF" for *Maple*,

$$\begin{aligned}
Z^{\mathbf{Z}_2}[B] &= 2^2 3 f_{21}(x) - 2!2^2 f_{1^2}(x) P_{1;\emptyset} + 3!2^3 f_{1^3}(x) + 2f_3(x) - 2f_2(x)P_{1;\emptyset} + \\
&\quad 2 \cdot 2f_3(x) - P_{3;\emptyset} \\
&= 12(-p_{21}(x) - p_3(x)) - 8\left(\frac{1}{2}p_{1^2}(x) + \frac{1}{2}p_2(x)\right)(p_1(x) + p_1(y)) \\
&\quad +48\left(\frac{1}{2}p_{21}(x) + \frac{1}{3}p_3(x) + \frac{1}{6}p_{1^3}(x)\right) - 2p_2(x)(p_1(x) + p_1(y)) + 4p_3(x) \\
&\quad -(p_3(x) + p_3(y)) \\
&= 9p_3(x) + 10p_{21}(x) + 4p_{1^3}(x) - 4p_{1^2}(x)p_1(y) - 2p_2(x)p_1(y) - p_3(y),
\end{aligned}$$

which is what we want.

1.5 Some Related Generating Functions

Given partitions α, β, and λ such that $|\alpha| + |\beta| = |\lambda|$, we can think of λ as a graph consisting of disjoint cycles of lengths $\lambda_1, \lambda_2, \ldots$. Let $r_{\alpha,\beta}(\lambda)$ denote the number of subgraphs of λ of type $(\alpha; \beta)$.

Lemma 1.2 *Let* $f(u, x_1, x_2, \ldots) := \sum_{j \geq 1} x_j u^j$. *Then*

$$\sum_{\alpha \vdash n} r_{\alpha,\emptyset}((n))\, x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots = [u^{n-1}] \frac{\frac{\partial f}{\partial u}}{1 - f}.$$

Proof Define $F(t,u,x_1,x_2,\ldots) := \frac{1}{t}\sum_{l\geq 1}(tf)^l$. Then

$$\begin{aligned}
F &= \sum_{l\geq 1}(x_1ut + x_2u^2t + \cdots)^l\frac{1}{t} \\
&= \sum_{l\geq 1} t^{l-1}(x_1u + x_2u^2 + \cdots)^l \\
&= \sum_{l\geq 1} t^{l-1} \sum_{m_1+m_2+\cdots=l} \binom{l}{m_1, m_2, \ldots} x_1^{m_1}x_2^{m_2}\cdots u^{m_1+2m_2+3m_3+\cdots} \\
&= \sum_{l\geq 1} t^{l-1} \sum_{m_1+m_2+\cdots=l} l!\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots u^{m_1+2m_2+3m_3+\cdots}.
\end{aligned}$$

Hence,

$$\frac{\partial F}{\partial u} = \sum_{l\geq 1} t^{l-1} \sum_{m_1+m_2+\cdots=l} l!(m_1+2m_2+\cdots)\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots u^{(m_1+2m_2+3m_3+\cdots)-1}.$$

Integrating with respect to t, we get

$$\int_{t=0}^1 \frac{\partial F}{\partial u}dt = \sum_{l\geq 1}\sum_{m_1+m_2+\cdots=l} \frac{(m_1+2m_2+\cdots)(l-1)!}{m_1!m_2!\cdots} x_1^{m_1}x_2^{m_2}\cdots u^{(m_1+2m_2+\cdots)-1}.$$

So

$$\begin{aligned}
[u^{n-1}]\int_{t=0}^1 \frac{\partial F}{\partial u}dt &= \sum_{l\geq 1}\sum_{\substack{m_1+m_2+\cdots=l \\ m_1+2m_2+\cdots=n}} \frac{n(\sum_j m_j - 1)!}{m_1!m_2!\cdots} x_1^{m_1}x_2^{m_2}\cdots \\
&= \sum_{\alpha\vdash n} r_{\alpha,\emptyset}((n))\, x_1^{m_1(\alpha)}x_2^{m_2(\alpha)}\cdots.
\end{aligned}$$

It is easy to see that $F(t,u,x_1,x_2,\ldots) = \frac{f}{1-tf}$. Taking the partial derivative of this with respect to u and integrating with respect to t, we get

$$\begin{aligned}
\int_{t=0}^1 \frac{\partial F}{\partial u}dt &= \int_{t=0}^1 \frac{\partial}{\partial u}\left(\frac{f}{1-tf}\right)dt \\
&= \int_{t=0}^1 \frac{\frac{\partial f}{\partial u}}{(1-tf)^2}dt \\
&= \left.\frac{\frac{\partial f}{\partial u}}{f(1-tf)}\right|_{t=0}^1 \\
&= \frac{\frac{\partial f}{\partial u}}{1-f}.
\end{aligned}$$

■

Theorem 1.10 *Let* $f_i(u_i, x_1, x_2, \ldots) := \sum_{j\geq 1} x_j u_i^j$. *Then*

$$\sum_{\lambda}\left(\sum_{\alpha\vdash|\lambda|} r_{\alpha,\emptyset}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots\right) m_\lambda(u) = \prod_i \left(1 + \frac{u_i \frac{\partial f_i}{\partial u_i}}{1-f_i}\right),$$

where the outer sum is taken over all partitions λ.

Proof Let $G_i := u_i \frac{\frac{\partial f_i}{\partial u_i}}{1-f_i}$, and let $\mathcal{F}_i(n) := [u_i^n]G_i$, the generating function of Lemma 1.2. Then

$$\begin{aligned} r_{\alpha,\emptyset}(\lambda) &= [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots]\mathcal{F}_1(\lambda_1)\mathcal{F}_2(\lambda_2)\cdots \\ &= [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots][u_1^{\lambda_1} u_2^{\lambda_2}\cdots u_{l(\lambda)}^{\lambda_{l(\lambda)}}]G_1 G_2\cdots G_{l(\lambda)} \\ &= [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots][m_\lambda(u_1, u_2, \ldots, u_{l(\lambda)})]G_1 G_2\cdots G_{l(\lambda)}. \end{aligned}$$

So

$$G_1 G_2\cdots G_k = \sum_{\substack{\lambda \\ l(\lambda)=k}}\left(\sum_{\alpha\vdash|\lambda|} r_{\alpha,\emptyset}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots\right) m_\lambda(u_1, u_2, \ldots, u_k).$$

Therefore,

$$\sum_{i_1<i_2<\cdots<i_k} G_{i_1} G_{i_2}\cdots G_{i_k} = \sum_{\substack{\lambda \\ l(\lambda)=k}}\left(\sum_{\alpha\vdash|\lambda|} r_{\alpha,\emptyset}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots\right) m_\lambda(u),$$

where $m_\lambda(u)$ denotes the monomial symmetric function in infinitely many variables $u_1, u_2, \ldots$. Summing over all k, we get

$$\prod_i (1+G_i) = \sum_{\lambda}\left(\sum_{\alpha\vdash|\lambda|} r_{\alpha,\emptyset}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)}\cdots\right) m_\lambda(u).$$

■

We know that

$$\begin{aligned} \sum_{\beta\vdash n} r_{\emptyset,\beta}((n))\, y_1^{m_1(\beta)} y_2^{m_2(\beta)}\cdots &= \sum_{\beta\vdash n} \delta_{\beta,(n)} y_1^{m_1(\beta)} y_2^{m_2(\beta)}\cdots \\ &= y_n. \end{aligned}$$

So in order to calculate $r_{\alpha,\beta}(\lambda)$, we can consider any n-cycle of λ and impose either a path structure or a cycle structure on it. Therefore, we get

$$r_{\alpha,\beta}(\lambda) = [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots] \prod_n \left([u^{n-1}] \frac{\frac{\partial f}{\partial u}}{1-f} + y_n \right)^{m_n(\lambda)}.$$

Theorem 1.11 *Let f_i be defined as in Theorem 1.10. Then*

$$\sum_\lambda \left(\sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta|=|\lambda|}} r_{\alpha,\beta}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots \right) m_\lambda(u) =$$

$$\prod_i \left(1 + \frac{u_i \frac{\partial f_i}{\partial u_i}}{1-f_i} + \sum_{m \geq 1} y_m u_i^m \right),$$

where the outer sum is taken over all partitions λ.

Proof Let

$$\mathcal{H}_i(n) := [u_i^{n-1}] \left(\frac{\frac{\partial f_i}{\partial u_i}}{1-f_i} + \sum_{m \geq 1} y_m u_i^{m-1} \right),$$

and define

$$\mathcal{G}_i := \frac{u_i \frac{\partial f_i}{\partial u_i}}{1-f_i} + \sum_{m \geq 1} y_m u_i^m.$$

Then we have $[u_i^n]\mathcal{G}_i = \mathcal{H}_i(n)$, so

$$\begin{aligned} r_{\alpha,\beta}(\lambda) &= [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots] \mathcal{H}_1(\lambda_1) \mathcal{H}_2(\lambda_2) \cdots \\ &= [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots][u_1^{\lambda_1} u_2^{\lambda_2} \cdots u_{l(\lambda)}^{\lambda_{l(\lambda)}}] \mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_{l(\lambda)} \\ &= [x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots][m_\lambda(u_1, u_2, \ldots, u_{l(\lambda)})] \mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_{l(\lambda)}. \end{aligned}$$

Therefore,

$$\mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_k = \sum_{\substack{\lambda \\ l(\lambda)=k}} \left(\sum_{\alpha \vdash |\lambda|} r_{\alpha,\beta}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots \right) m_\lambda(u_1, u_2, \ldots, u_k),$$

which gives us

$$\sum_{i_1 < i_2 < \cdots < i_k} \mathcal{G}_{i_1} \mathcal{G}_{i_2} \cdots \mathcal{G}_{i_k} = \sum_{\substack{\lambda \\ l(\lambda)=k}} \left(\sum_{\alpha \vdash |\lambda|} r_{\alpha,\emptyset}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots \right) m_\lambda(u).$$

Summing over all k, we get

$$\prod_i (1+\mathcal{G}_i) = \sum_\lambda \left(\sum_{\alpha \vdash |\lambda|} r_{\alpha,\emptyset}(\lambda) x_1^{m_1(\alpha)} x_2^{m_2(\alpha)} \cdots y_1^{m_1(\beta)} y_2^{m_2(\beta)} \cdots \right) m_\lambda(u).$$

■

Chapter 2

Unlabelled Interval Orders

2.1 Introduction

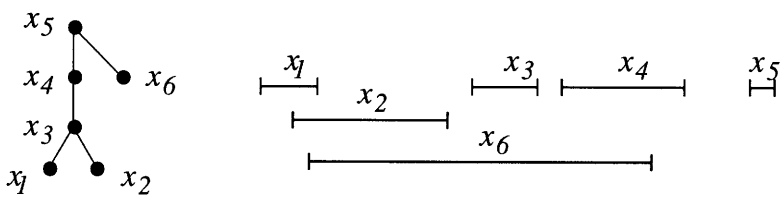
Interval orders have been studied purely for the mathematics and also for their various applications in the social sciences. It has long been known which posets have interval representations in which all of the intervals are of equal length. These posets are called *semiorders*. Here, we determine which posets are $(1, 1, \ldots, 1, 1 + \alpha)$-interval orders.

2.2 Background

A poset $P = (X, <_P)$ is called an *interval order* if there exists a function F from X to the set of non-degenerate closed intervals on the real line such that $F(x) = [a_x, b_x]$, and x is less than y in P if and only if $[a_x, b_x]$ lies completely to the left of $[a_y, b_y]$. Any interval representation of P may also be expressed as an ordered pair of real functions (f, l) such that $f(y)$ is the left endpoint of the interval corresponding to $y \in X$ and $l(y)$ is the length of that interval. $(F(y) = [f(y), f(y) + l(y)].)$

Suppose $|X| = k$ with $X = \{x_1, x_2, \ldots, x_k\}$. P is said to be an $(l_1, l_2, \ldots, l_k)$-interval order if there exist closed real intervals $I_1, \ldots, I_k$ with the length of I_i being l_i such that there is a bijection $F : X \leftrightarrow \{I_1, \ldots, I_k\}$ with $F(y_1)$ lying completely to the left of $F(y_2)$ for $y_1, y_2 \in X$ if and only if y_1 is less than y_2 in P.

Example 2.1



Interval orders that have an interval representation in which all intervals have equal length (which we can take to be 1) are called *semiorders* and have been studied quite a bit. Let $\mathbf{i}+\mathbf{j}$ denote an i-chain and a j-chain which are disjoint from one another such that no relations hold between elements of the i-chain and elements of the j-chain. We say that $S = (X_S, <_S)$ is an induced subposet of $P = (X, <_P)$ if $X_S \subseteq X$ and given $x, y \in X_S$, $x <_P y$ if and only if $x <_S y$. The following are two well-known results about interval orders (see, *e.g.*,[Tro92, p. 86 and p. 193]):

Theorem 2.1 *A poset P is an interval order if and only if P does not contain $\mathbf{2}+\mathbf{2}$ as an induced subposet.*

Theorem 2.2 *An interval order $P = (X, <_P)$ is a semiorder if and only if P does not contain $\mathbf{3}+\mathbf{1}$ as an induced subposet.*

The problem we investigate is that of characterizing $(1, 1, \ldots, 1+\alpha)$-interval orders, where $0 < \alpha < 1$. Before we state the main result, we state some relevant theorems.

Theorem 2.3 *[Fis85, p. 126] Suppose $P = (X, <_P)$ is a finite interval order, where $|X| = k$. Let ρ denote the set of inequalities of the form $\sum_{y \in X_1} l(y) < \sum_{y \in X_2} l(y)$, where $X_1, X_2 \subseteq X$, which* must *be satisfied by any length function l of an interval representation (f, l) of P. Then P is an $(l_1, l_2, \ldots, l_k)$-interval order if and only if there exists a function $l : X \to \{l_1, \ldots, l_k\}$ and a permutation $\pi \in S_k$ with $l(x_i) = l_{\pi(i)}$ $(1 \leq i \leq k)$ such that l satisfies the inequalities in ρ.*

Corollary 2.1 *Let P be an $(l_1, l_2, \ldots, l_k)$-interval order such that $i < l_m < i+1$ for some $i \in \mathbf{Z}$ and $l_n \in \mathbf{Z}\, \forall n \neq m$. Then P is an $(l_1, \ldots, l_{m-1}, l_m{}', l_{m+1}, \ldots, l_k)$-interval order, for any $i < l_m{}' < i+1$.*

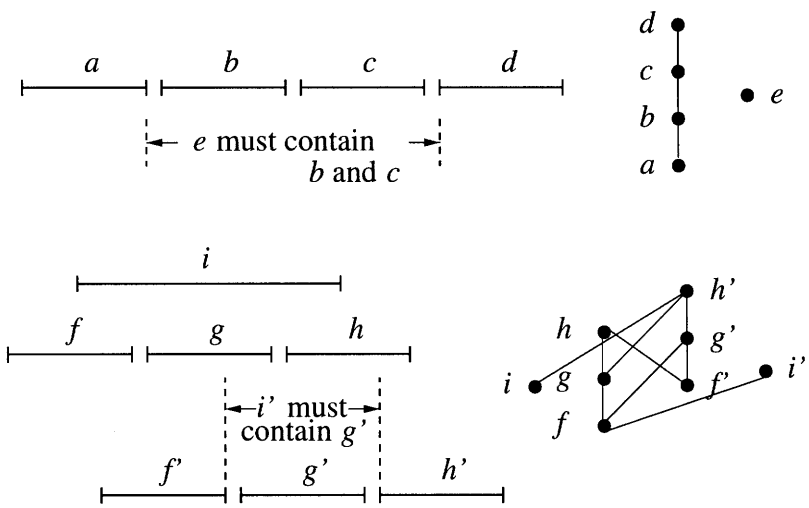
We are now ready to state our main result.

2.3 $(1, 1, \ldots, 1, 1+\alpha)$-Interval Orders

Theorem 2.4 *Let $P = (X, <_P)$ be an interval order. Fix $0 < \alpha < 1$. Then P is a $(1, \ldots, 1, 1+\alpha)$-interval order if and only if*

1. *P does not contain a $\mathbf{4+1}$ and*
2. *If P contains more than one $\mathbf{3+1}$, then they share the same $\mathbf{1}$. (This excludes a finite number of induced subposets.)*

If P *does* contain a $\mathbf{4+1}$ or any $\mathbf{3+1}$s that do not share the same $\mathbf{1}$, then it is clear from the following that P is not a $(1, \ldots, 1, 1+\alpha)$-interval order:



In the case in which P contains no $\mathbf{3+1}$, Theorem 2.4 is easy to see. This is because P is a semiorder, so there exists an interval representation of P with intervals of length 1. Take the right-most interval in this representation and extend it by α.

Also, because of Corollary 2.1, we need only show that if P satisfies 1. and 2. of Theorem 2.4, then P is a $(1, 1, \ldots, 1, 1+\alpha)$-interval order, for *some* $0 < \alpha < 1$.

Given $y \in P$, define $D(y) = \{z \in X \mid z <_P y\}$, the set of elements below y in P. Similarly, define $U(y)$ in P to be the set of elements above y in P. Following W. Trotter [Tro92], we say that $y_1, y_2 \in X$ have *duplicated holdings in P* if $D(y_1) = D(y_2)$ in P and $U(y_1) = U(y_2)$ in P. (When considering interval representations of an interval order P, we can assume that no two

elements in P have duplicated holdings in P, since any such elements may be represented by the same interval.) Define the binary relation $<_{H_P}$ such that $y_1 <_{H_P} y_2$ if $D(y_1) \subset D(y_2)$ (read "$D(y_1)$ is strictly contained in $D(y_2)$") in P or if $U(y_2) \subset U(y_1)$ in P. (Note that H_P is a partial order if P is a semiorder.) This relation has some very nice properties which we will see throughout this section.

Let $P = (X, <_P)$ be an interval order with at least one $\mathbf{3+1}$ and no $\mathbf{4+1}$. Suppose further that all $\mathbf{3+1}$'s have the same $\mathbf{1}$, which we denote by x. Consider all of the $\mathbf{3+1}$'s. Let X^1 be the set of all least elements in the $\mathbf{3}$-chains in a $\mathbf{3+1}$ of P. Similarly, let X^2 be the set of middle elements, and let X^3 be the set of all maximal elements. Let us write $y_1 \|_P y_2$ if y_1 and y_2 are incomparable in P. More formally,

Definition 2.1 *Let $I_x = \{y \in X : y \|_P x\}$. Then*

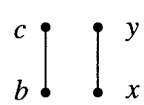
$$X^1 := \{y \in I_x : \ \exists b, c \in I_x \text{ such that } y <_P b <_P c\}.$$

$$X^2 := \{y \in I_x : \ \exists a, c \in I_x \text{ such that } a <_P y <_P c\}.$$

$$X^3 := \{y \in I_x : \ \exists a, b \in I_x \text{ such that } a <_P b <_P y\}.$$

Claim 2.1 *$D(x) \subset D(b)$ and $U(x) \subset U(b)$ for all $b \in X^2$.*

Proof Fix some $b \in X^2$. We know there exists no $y \in X$ such that $x <_P y <_P b$. So suppose that $y \|_P b$ for some $y \in U(x)$. There exists a $c \in X^3$ such that $b <_P c$. Since $b \not<_P y$, we know that $c \not<_P y$; and since $x \|_P c$, we know that $y \not<_P c$. Thus, there exists a $\mathbf{2+2}$:



which contradicts the fact that P is an interval order, so $U(x) \subset U(b)$. Similarly, $D(x) \subset D(b)$. ■

Given L a linear order on X, we say that L is a linear extension of P if $y_1 <_L y_2$ for any $y_1, y_2 \in X$ such that $y_1 <_P y_2$.

Definition 2.2 *Since $P - x$ is a semiorder, H_{P-x} is a partial order. Let L' be a linear extension of H_{P-x}, and define $b_{\mathcal{L}}$ to be the least element of the elements of X^2 in the ordering L'. By Claim 2.1, we can define a linear extension L of P extending L' such that:*

- *for* $y_1, y_2 \in X - x$, $y_1 <_L y_2$ *if* $y_1 <_{L'} y_2$ *and*
- *for* $y \in X - x$, $x <_L y$ *if* $b_{\mathcal{L}} \leq_{L'} y$ *and*
- *for* $y \in X - x$, $y <_L x$ *if* $y <_{L'} b_{\mathcal{L}}$.

Unless otherwise noted, let us assume that $P = (X, <_P)$ is an interval order with $|X| = k$ and $X = \{x_1, x_2, \ldots, x_k\}$. Suppose the x_i's are labelled so that $x_1 <_L x_2 <_L \cdots <_L x_k$, where L is the linear extension of P in Definition 2.2. Let $A := \{y \in X - x_k : y \parallel_P x_k\}$. For $X_1, X_2 \subseteq X$, let us say $X_1 \not\subset X_2$ if $X_1 = X_2$ or if X_1 is not strictly contained in X_2.

Lemma 2.1 *There exists* $i > 1$ *such that*

$$A = \begin{cases} \{x, x_i, x_{i+1}, \ldots, x_{k-1}\} & \text{if } x_k \in X^3 \\ \{x_i, x_{i+1}, \ldots, x_{k-1}\} & \text{otherwise} \end{cases}$$

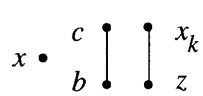
Furthermore, $b <_L x_i$ *for all* $b \in X^2$, *and* $x <_L x_{i-1}$.

Proof The proof follows from Claims 2.2-2.5. ■

Claim 2.2 *If* $b \in X^2$, *then* $b <_P x_k$.

Proof Suppose there exists a $b \in X^2$ such that $b \parallel_P x_k$. Then we know that there exists some $c \in X^3$ such that $b <_P c$ and $c \parallel_P x_k$.

Since $c <_L x_k$, we have $D(x_k) \not\subset D(c)$ in $P - x$. We know that $D(x_k) \neq D(c)$ in $P - x$ since $b \notin D(x_k)$, so there exists $z \in X - x$ such that $z <_P x_k$ and $z \not<_P c$ and hence, $z \not<_P b$. Since $x_k \parallel_P b, c$, we know that $b, c \not<_P z$. Hence, there exists a $\mathbf{2+2}$:



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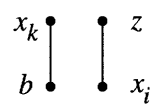
Claim 2.3 *Suppose* $x_i \in A$ *such that* $x_i \neq x$. *Then* $x_{i+1} \neq x$.

Proof Suppose that $x_{i+1} = x$. Then $x_i <_L b$ for all $b \in X^2$. So $U(x_i) \not\subset U(b)$ in $P - x$ for all $b \in X^2$. By Claim 2.2, we know that $x_k \in U(b)$ for all $b \in X^2$. Fix $b \in X^2$. Since $x_k \notin U(x_i)$, there exists $z \in X - x$ such that $x_i <_P z$ and $b \not<_P z$. Since $x_i \parallel_P x_k$, we know that $z \not<_P b$. Similarly, $z \not<_P x_k$ and $x_i \not<_P b$. Finally, since $b \not<_P z$, we know that $b \not<_P x_i$; hence, there exists a $\mathbf{2} + \mathbf{2}$:



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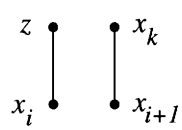
Claim 2.4 *Suppose $x_i \in A$ such that $x_i \neq x$. Then $x_{i+1} \in A$.*

Proof Suppose $x_{i+1} <_P x_k$. We know that $x_i \parallel_P x_{i+1}$ and $x_i <_L x_{i+1}$. In particular, $U(x_i) \not\subset U(x_{i+1})$ in $P - x$. We know that $U(x_i) \neq U(x_{i+1})$ in $P - x$, so there exists $z \in X - x$ such that $x_i <_P z$ and $x_{i+1} \not<_P z$. Since $x_i \parallel_P x_k$, we know that $z \not<_P x_{i+1}$ and $z \not<_P x_k$. So $z \parallel_P x_k$ and $z \parallel_P x_{i+1}$, and there exists a $\mathbf{2} + \mathbf{2}$:



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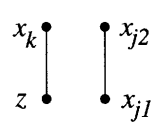
Claim 2.5 *All elements of A (except for x, if $x \in A$) are pairwise incomparable.*

Proof Suppose not. Suppose there exists $x_{j_1}, x_{j_2} \in A$ (where $x_{j_1}, x_{j_2} \neq x$) such that $x_{j_1} <_P x_{j_2}$. Since $x_{j_2} <_L x_k$, $D(x_k) \not\subset D(x_{j_2})$ in $P - x$. Because $D(x_k) \neq D(x_{j_2})$ in $P - x$, there exists a $z \in X - x$ such that $z <_P x_k$ and $z \not<_P x_{j_2}$ and hence, $z \not<_P x_{j_i}$. Also, since $x_{j_1}, x_{j_2} \parallel_P x_k$, we know $x_{j_1}, x_{j_2} \not<_P z$. Then $x_{j_1} <_P x_{j_2}, z <_P x_k$ form a $\mathbf{2} + \mathbf{2}$:



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A proof of Theorem 2.2 that appears in [Tro92, p. 193] proves the following stronger theorem:

Theorem 2.5 *[Tro92] Let $P = (X, <_P)$ be a finite interval order with no $\mathbf{3+1}$. Let L_s be a linear extension of H_P. Then there exists an injection: $h : X \to \mathbf{R}$ such that*

- $h(y_1) < h(y_2)$ *iff* $y_1 <_{L_s} y_2$
- $h(y_1) + 1 < h(y_2)$ *iff* $y_1 <_P y_2$
- $h(y_1) + 1 \neq h(y_2)$ *if* $y_1 \|_P y_2$

To prove Theorem 2.4, we will prove the following stronger theorem, which is similar to Theorem 2.5:

Theorem 2.6 *Let $P = (X, <_P)$ be a finite interval order with $X = \{x_1, x_2, \ldots, x_k\}$. Let L be the linear extension of P of Definition 2.2, and without loss of generality, assume $x_1 <_L x_2 <_L \cdots <_L x_k$. Fix $0 < \alpha < 1$ and assume:*

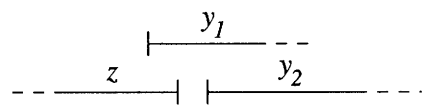
- *P contains at least one $\mathbf{3+1}$,*
- *P does not contain a $\mathbf{4+1}$, and*
- *If P contains more than one $\mathbf{3+1}$, then they share the same $\mathbf{1}$, which we denote by x.*

Then there exists an injection $f : X \to \mathbf{R}$ such that

1. $f(y_1) < f(y_2)$ *if*
   - $y_1 <_L y_2$ *and* $y_1, y_2 \in \{x_{i-1}, x_i, \ldots, x_k\}$ *or*
   - $y_1 <_L x_{i-1}$ *and* $y_2 = x_{i-1}$
2. $f(y_1) + 1 < f(y_2)$ *if and only if* $y_1 <_P y_2$ *and* $y_1 \neq x$
3. $f(y_1) + 1 \neq f(y_2)$ *if* $y_1 \|_P y_2$ *and* $y_1 \neq x$
4. $f(x) + 1 + \alpha < f(y_2)$ *if and only if* $x <_P y_2$

5. $f(x) + 1 + \alpha \neq f(y_2)$ *if* $x \parallel_P y_2$.

Before we prove this theorem, let us attempt to give an idea why this linear extension L should work. First of all, for $y_1, y_2 \in X$, if $D(y_1) \subset D(y_2)$, we must have $f(y_1) < f(y_2)$ for any interval representation (f, l). Furthermore, there exists $z \in X$ such that $f(y_1) \leq f(z) + l(z) < f(y_2)$:



Similarly, if $U(y_2) \subset U(y_1)$, we must have $f(y_1) + l(y_1) < f(y_2) + l(y_2)$; furthermore, there exists $w \in X$ such that $f(y_1) + l(y_1) < f(w) \leq f(y_2) + l(y_2)$.

So it seems like a good idea to have L be some extension of H_P. But H_P is not a partial order since P is not a semiorder, so we must be careful. We know that $P - x$ is a semiorder, so a linear extension of $P - x$ which extends H_{P-x} is desired. The problem remains of where to place x. We know that the left end of the interval corresponding to x must be less than the left ends of the intervals corresponding to the elements of X^2, (*i.e.*, $f(x) < f(y)\ \forall y \in X^2$). These are the reasons behind the choice of L. We will now proceed with the proof.

Proof [Theorem 2.6] We will prove this theorem by induction on the size of X. Assume the theorem holds for all $P = (X, <_P)$ with $|X| \leq k-1$. Take an interval order $P = (X, <_P)$ with $|X| = k$ and $X = \{x_1, x_2, \ldots, x_k\}$. Let L and $b_{\mathcal{L}}$ be as defined in Definition 2.2. Say $x_1 <_L x_2 <_L \cdots <_L x_k$. Let $A = \{y \in X : y \parallel_P x_k\}$. By Lemma 2.1, we know there exists $i > 1$ such that

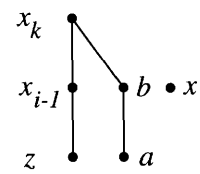
$$A = \begin{cases} \{x, x_i, x_{i+1}, \ldots, x_{k-1}\} & \text{if } x_k \in X^3 \\ \{x_i, x_{i+1}, \ldots, x_{k-1}\} & \text{otherwise} \end{cases}$$

Case 0: Suppose $x \in A$ and $P - x_k$ is a semiorder. We will need the following claims.

Claim 2.6 *We have* $x_k \in X^3$ *and* $x_{i-1} \in X^2$.

Proof We know there exist $a \in X^1, b \in X^2, c \in X^3$ such that $a <_P b <_P c$. We know from Claim 2.2 that $b <_P x_k$, which implies that $x_k \in X^3$. Suppose $x_{i-1} \notin X^2$. We know there exists some $a \in X^1$

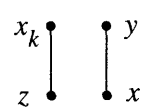
and $b \in X^2$ such that $a <_P b <_P x_k$. Since $x_{i-1} \notin X^2$, we know that $a \not<_P x_{i-1}$. We have $x_{i-1} \not<_P x$, and since $x \parallel_P x_k$, we have $x \not<_P x_{i-1}$, so $x \parallel_P x_{i-1}$. Because $b <_L x_{i-1}$, we know $D(x_{i-1}) \not\subset D(b)$ in $P - x$. Now $D(x_{i-1}) \neq D(b)$ in $P - x$, so there exists $z \in X - x$ such that $z <_P x_{i-1}$ and $z \not<_P b$ and hence, $z \not<_P a$ and $x_{i-1} \not<_P b$. Since $a \not<_P x_{i-1}$, we know $a, b \not<_P z$. There does not exist a $\mathbf{4} + \mathbf{1}$, so $b \not<_P x_{i-1}$ and $x_{i-1} \not<_P a$. Hence, there exists a $\mathbf{2} + \mathbf{2}$:



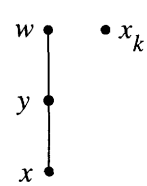
■

Claim 2.7 *If there exists $y \in X$ such that $x <_P y$, then y and x_k have duplicated holdings in $P - x$.*

Proof We know that $y <_L x_k$, so $D(x_k) \not\subset D(y)$ in $P - x$ and $U(y) \not\subset U(x_k)$ in $P - x$. Assume that $D(x_k) \neq D(y)$ in $P - x$. There exists $z \in X - x$ such that $z <_P x_k$ and $z \not<_P y$ and thus $z \not<_P x$. Because $x \parallel_P x_k$, we have $x, y \not<_P z$ and $y \not<_P x_k$. Thus, we have a $\mathbf{2} + \mathbf{2}$:



Now assume that $U(y) \neq U(x_k)$ in $P - x$. Then there exists $w \in X - x$ such that $y <_P w$ and $x_k \not<_P w$. This gives a $\mathbf{3} + \mathbf{1}$ in which x_k is the $\mathbf{1}$:

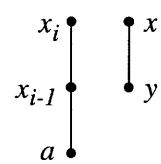


■

In particular, this claim shows that if $x \in A$, then we can assume that $x \not<_P y$ for all $y \in X$. Furthermore, for any $x_j \in A$ where $x_j \neq x$, we know that $x_j \not<_P x$. So we can assume that if $x \in A$, then $x_j \parallel_P x$ for any $x_j \in A$ with $x_j \neq x$.

Claim 2.8 *There exists a linear extension L'' of H_{P-x_k} such that $x_{i-1} <_{L''} x_i <_{L''} \cdots <_{L''} x_k$, and if $y <_L x_{i-1}$, then $y <_{L''} x_{i-1}$.*

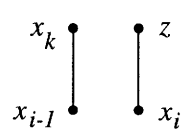
Proof Take any $y <_L x_{i-1}$. From Claim 2.6, we know that $x_{i-1} \in X^2$, so there is some $a \in X^1$ with $a <_P x_{i-1}$. Assume that $x_{i-1} <_{H_{P-x_k}} y$. We can also assume that $y \parallel_P x_{i-1}$. If $y <_P x$, then we can show that there exists a $\mathbf{2}+\mathbf{2}$:



Thus, $y \parallel_P x$. Now $D(x_{i-1}) \subset D(y)$ in $P - x_k$ implies that $D(x_{i-1}) \subset D(y)$ in $P - x$, which is a contradiction. Similarly, $U(y) \subset U(x_{i-1})$ in $P - x_k$ implies that $U(y) \subset U(x_{i-1})$ in $P - x$, which is a contradiction. So $x_{i-1} \not<_{H_{P-x_k}} y$ for any $y <_L x_{i-1}$.

We know that $D(x) \subset D(x_{i-1})$ in P, so $D(x) \subset D(x_{i-1})$ in $P - x_k$. Thus, $x_{i-1} \not<_{H_{P-x_k}} x$.

Suppose $x_i <_{H_{P-x_k}} x_{i-1}$. If $D(x_i) \subset D(x_{i-1})$ in $P - x_k$, then the same holds in $P - x$. This is a contradiction. Now $U(x_{i-1}) \subset U(x_i)$ in $P - x_k$, so there is a $z \in X - x_k$ such that $x_i <_P z$ and $x_{i-1} \not<_P z$ and hence $x_k \not<_P z$. Since $x_{i-1}, x_k \parallel_P x_i$, we know $z \not<_P x_{i-1}, x_k$. There exists a $\mathbf{2}+\mathbf{2}$:



Take $j_1, j_2 \in \{i, i+1, \ldots, k\}$ such that $x_{j_1} <_L x_{j_2}$. Suppose $x_{j_2} <_{H_{P-x_k}} x_{j_1}$. If $D(x_{j_2}) \subset D(x_{j_1})$ in $P - x_k$, then the same holds in $P - x$, which is a contradiction. Similarly, since $x_{j_1}, x_{j_2} \parallel_P x_k, x$, we get a contradiction if $U(x_{j_1}) \subset U(x_{j_2})$ in $P - x_k$. So $x_{j_2} \not<_{H_{P-x_k}} x_{j_1}$. ■

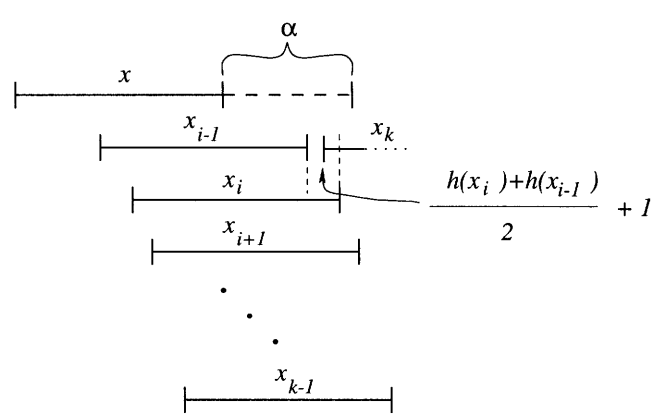
By Claim 2.7, we can assume that $x \parallel_P x_j$ for all $x_j \in A$ where $x_j \neq x$. By Theorem 2.5 and Claim 2.8, there exists an injection $h : X - x_k \to \mathbf{R}$ such that:

1. $h(y_1) < h(y_2)$ if $y_1 <_L y_2$ and $y_1, y_2 \in \{x_{i-1}, x_i, \ldots, x_{k-1}\}$,
2. $h(y) < h(x_{i-1})$ for all y such that $y <_L x_{i-1}$,
3. $h(y_1) + 1 < h(y_2)$ if and only if $y_1 <_P y_2$,
4. $h(y_1) + 1 \neq h(y_2)$ if $y_1 \parallel_P y_2$.

Let $\alpha = \frac{h(x_i)+h(x_{i-1})}{2} - h(x) + \delta$, where $0 < \delta < h(x_{k-1}) - h(x_i)$. Then $0 < \alpha < 1$, and $h(x) + 1 + \alpha > \frac{h(x_i)+h(x_{i-1})}{2} + 1$. We extend the length of the interval corresponding to x by α and place the left end of the interval corresponding to x_k so that it is less than the right end of the new interval corresponding to x and the right end of x_i, but greater than the right end of x_{i-1} and the left end of x_{k-1}. Let

$$f(x_k) = \frac{h(x_i) + h(x_{i-1})}{2} + 1,$$

and $f(x_j) = h(x_j)$ for $0 < j < k$. The interval representation looks like:



By Claim 2.7, we can assume that there does not exist a $y \in X$ such that $x <_P y$. So extending the length of the interval corresponding to x does not affect anything.

Now we must consider the cases in which $P - x_k$ is not a semiorder. Let $L(P - x_k)$ denote the linear extension L restricted to elements of $P - x_k$. Note that it is a linear extension of $P - x_k$.

Claim 2.9 *There exists a linear extension $L*$ of H_{P-x-x_k} such that*

- *for $y_1, y_2 \in X - x - x_k$, $y_1 <_{L(P-x_k)} y_2$ if $y_1 <_{L*} y_2$ and*
- *for $y \in X - x - x_k$, $x <_{L(P-x_k)} y$ if $b_{\mathcal{L}} \leq_{L*} y$ and*
- *for $y \in X - x - x_k$, $y <_{L(P-x_k)} x$ if $y <_{L*} b_{\mathcal{L}}$.*

Proof Straightforward. ■

Let A' denote the set of elements incomparable to x_{k-1}. Then there exists $i' > 1$ such that

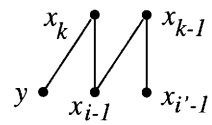
$$A' = \begin{cases} \{x, x_{i'}, x_{i'+1}, \ldots, x_{k-2}\} & \text{if } x_{k-1} \in X^3 \\ \{x_{i'}, x_{i'+1}, \ldots, x_{k-2}\} & \text{otherwise} \end{cases} .$$

By induction, there exists an injection $h : X - x_k \to \mathbf{R}$ for α with $0 < \alpha < 1$ such that

1. $h(y_1) < h(y_2)$ if
   - $y_1 <_L y_2$ and $y_1, y_2 \in \{x_{i'-1}, x_{i'}, \ldots, x_{k-1}\}$ or
   - $y_1 <_L x_{i'-1}$ and $y_2 = x_{i'-1}$
2. $h(y_1) + 1 < h(y_2)$ if and only if $y_1 <_P y_2$ and $y_1 \neq x$
3. $h(y_1) + 1 \neq h(y_2)$ if $y_1 \|_P y_2$ and $y_1 \neq x$
4. $h(x) + 1 + \alpha < h(y_2)$ if and only if $x <_P y_2$
5. $h(x) + 1 + \alpha \neq h(y_2)$ if $x \|_P y_2$.

Claim 2.10 $x_{i'-1} \leq_L x_{i-1}$.

Proof Suppose that $x_{i-1} <_L x_{i'-1}$. This implies that $x_{i'-1} \|_P x_k$. We know $x_{i'-1} <_P x_{k-1}$. Since $x_{k-1} <_L x_k$, we know $D(x_k) \not\subset D(x_{k-1})$ in $P - x$. Now $D(x_k) \neq D(x_{k-1})$ in $P - x$, so there exists $y \in X - x$ such that $y <_P x_k$ and $y \not<_P x_{k-1}$ and hence $y \not<_P x_{i'-1}$. We have $x_k \|_P x_{i'-1}, x_{k-1}$, so $x_{k-1}, x_{i'-1} \not<_P y$. There is a $\mathbf{2+2}$:



■

Note that

- $\{x_{i-1}, x_i, \ldots, x_{k-1}\} \subseteq \{x_{i'-1}, x_{i'}, \ldots, x_{k-1}\}$, so $h(y_1) < h(y_2)$ if $y_1 <_L y_2$ and $y_1, y_2 \in \{x_{i-1}, x_i, \ldots, x_{k-1}\}$ and
- if $y_1 <_L x_{i'-1}$, then $h(y_1) < h(x_{i'-1}) < h(x_{i-1})$, and if $x_{i'-1} <_L y_1 <_L x_{i-1}$, then $y_1, x_{i-1} \in \{x_{i'-1}, x_{i'}, \ldots, x_{k-1}\}$, so $h(y_1) < h(x_{i-1})$.

So $h(y_1) < h(y_2)$ if

- $y_1 <_L y_2$ and $y_1, y_2 \in \{x_{i-1}, x_i, \ldots, x_{k-1}\}$ or
- $y_1 <_L x_{i-1}$ and $y_2 = x_{i-1}$.

From Claim 2.5, we know that $h(x_i) < h(x_{i+1}) < \cdots < h(x_{k-1}) < h(x_i) + 1$.

For the construction in Cases 1 and 2, see Figure 2.1.

Case 1: If $h(x) + 1 + \alpha < h(x_i) + 1$ and if $x <_P x_k$, then let $f(x_j) = h(x_j)$ for $0 < j < k$. We place the left end of the interval corresponding to x_k so that it is greater than the left end of x_{k-1}, the right end of x_{i-1}, and the right end of x, but less than the right end of x_i. To do this, we let

$$f(x_k) = \max\left\{\frac{h(x_i) + h(x_{i-1})}{2} + 1, \frac{h(x_i) + h(x_{k-1}) + 1}{2}, \frac{h(x_i) + h(x) + \alpha}{2} + 1\right\}.$$

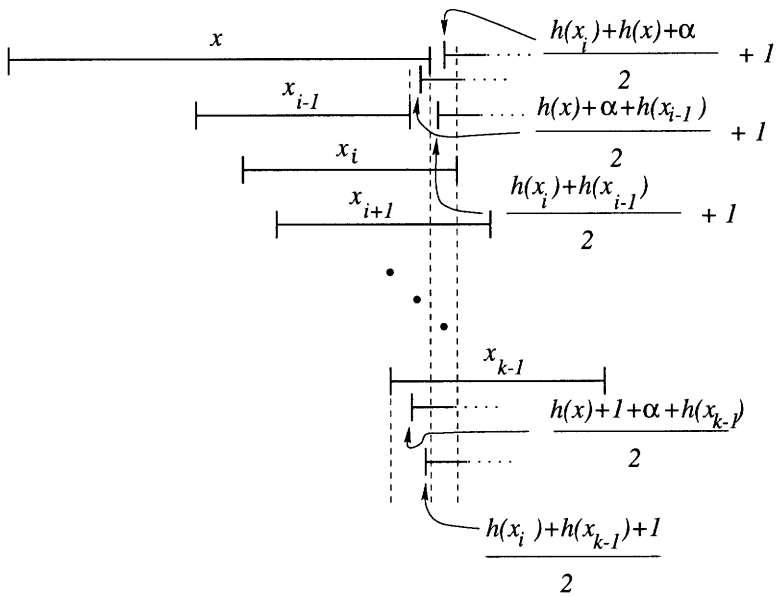


Figure 2.1: Case 1 and Case 2

Case 2: If $h(x) + 1 + \alpha < h(x_i) + 1$ and if $x \parallel_P x_k$ and if $h(x_{i-1}) + 1 < h(x) + 1 + \alpha$, then let $f(x_j) = h(x_j)$ for $0 < j < k$. We wish to place the left end of the interval corresponding to x_k so that it is greater than the left end of x_{k-1} and the right end of x_{i-1}, but less than the right ends of x_i and x. To do this, we let

$$f(x_k) = \max\left\{\frac{h(x) + \alpha + h(x_{i-1})}{2} + 1, \frac{h(x) + 1 + \alpha + h(x_{k-1})}{2}\right\}.$$

For the construction in Cases 3 and 4, see Figure 2.2.

Case 3: If $h(x) + 1 + \alpha > h(x_i) + 1$ and if $x \parallel_P x_k$, then let $f(x_j) = h(x_j)$ for $0 < j < k$. We wish to place the left end of the interval corresponding to x_k so that it is greater than the left end of x_{k-1} and the right end of x_{i-1}, but less than the right ends of x_i and x. To do this, we let

$$f(x_k) = \max\left\{\frac{h(x_i) + h(x_{k-1}) + 1}{2}, \frac{h(x_i) + h(x_{i-1})}{2} + 1\right\}.$$

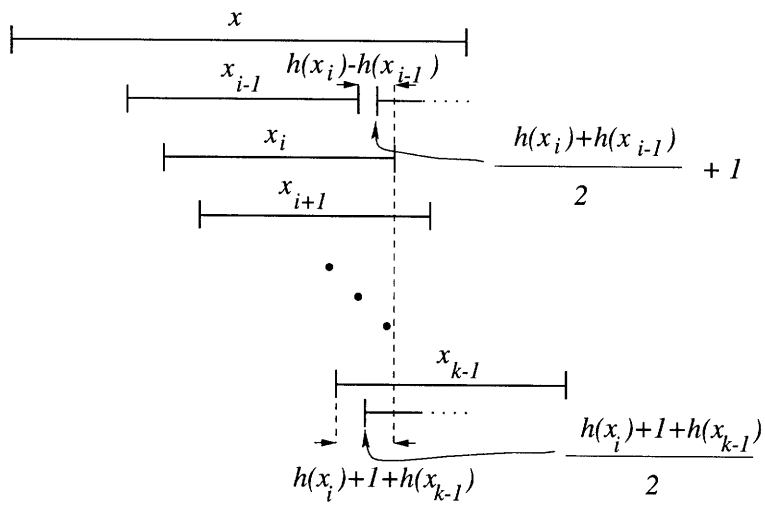


Figure 2.2: Case 3 and Case 4

Case 4: If $h(x) + 1 + \alpha > h(x_i) + 1$ and if $x <_P x_k$, then we shrink the interval corresponding to x by a small amount so that the new length $\alpha' + 1$ satisfies $1 < \alpha' + 1 < 2$, and the right end of the new interval is less than the right end of x_i. More formally, let $\alpha' = \alpha - (h(x) + 1 + \alpha - h(x_i) - 1) - \delta$, where $0 < \delta < \min\{h(x_i) + 1 - h(x_{k-1}), h(x_i) - h(x_{i-1})\}$. Then $0 < \alpha' < 1$, and $h(x) + \alpha' + 1 < h(x_i) + 1$. We can now go to Case 1.

■ The techniques used in the proof of this theorem certainly generalize to other interval orders, and we are currently working on these generalizations.

$n = 2$
$Z[B_2] = s_2 + s_{11}$
$Z[B_{11}] = s_2 - s_{11}$
$n = 3$
$Z[B_3] = 2s_3 + s_{21} + 2s_{111}$
$Z[B_{21}] = 2s_3 - 2s_{111}$
$Z[B_{111}] = 2s_3 - 2s_{21} + 2s_{111}$
$n = 4$
$Z[B_4] = 9s_4 + 3s_{31} + s_{211} + 3s_{1111}$
$Z[B_{31}] = 9s_4 - 3s_{22} - 3s_{1111}$
$Z[B_{22}] = 9s_4 + 3s_{31} + 6s_{22} - s_{211} - 3s_{1111}$
$Z[B_{211}] = 9s_4 - 3s_{31} - s_{211} + 3s_{1111}$
$Z[B_{1111}] = 9s_4 - 9s_{31} + 6s_{22} + 3s_{211} - 3s_{1111}$
$n = 5$
$Z[B_5] = 44s_5 + 11s_{41} + 4s_{311} + s_{2111} + 4s_{11111}$
$Z[B_{41}] = 44s_5 - 4s_{32} - 4s_{221} - 4s_{11111}$
$Z[B_{32}] = 44s_5 + 11s_{41} + 4s_{32} + 4s_{221} - s_{2111} - 4s_{11111}$
$Z[B_{311}] = 44s_5 - 11s_{41} - 4s_{32} + 4s_{221} - s_{2111} + 4s_{11111}$
$Z[B_{221}] = 44s_5 + 4s_{32} - 8s_{311} - 4s_{221} + 4s_{11111}$
$Z[B_{2111}] = 44s_5 - 22s_{41} + 4s_{32} + 4s_{221} + 2s_{2111} - 4s_{11111}$
$Z[B_{11111}] = 44s_5 - 44s_{41} + 20s_{32} + 24s_{311} - 20s_{221} - 4s_{2111} + 4s_{11111}$

Figure .1: Cycle indicators of boards avoiding a fixed permutation in S_n $(2 \leq n \leq 5)$

$n = 6$
$Z[B_6] = 265s_6 + 53s_{51} + 13s_{411} + 5s_{3111} + s_{21111} + 5s_{111111}$
$Z[B_{51}] = 265s_6 - 15s_{42} - 5s_{321} - 5s_{2211} - 5s_{111111}$
$Z[B_{42}] = 265s_6 + 53s_{51} + 15s_{42} + 11s_{33} - 7s_{222} + 5s_{2211} - s_{21111} - 5s_{111111}$
$Z[B_{411}] = 265s_6 - 53s_{51} - 15s_{42} + 11s_{33} + 7s_{222} + 5s_{2211} - s_{21111} + 5s_{111111}$
$Z[B_{33}] = 265s_6 + 53s_{51} + 13s_{411} - 22s_{33} + 10s_{321} - 5s_{3111} + 14s_{222} - s_{21111} - 5s_{111111}$
$Z[B_{321}] = 265s_6 - 13s_{411} - 11s_{33} - 5s_{3111} - 7s_{222} + 5s_{111111}$
$Z[B_{3111}] = 265s_6 - 106s_{51} + 13s_{411} + 11s_{33} + 10s_{321} - 5s_{3111} - 7s_{222} + 2s_{21111} - 5s_{111111}$
$Z[B_{222}] = 265s_6 + 53s_{51} + 45s_{42} - 26s_{411} + 33s_{33} - 10s_{3111} + 21s_{222} - 15s_{2211} + s_{21111} + 5s_{111111}$
$Z[B_{2211}] = 265s_6 - 53s_{51} + 15s_{42} - 26s_{411} - 11s_{33} + 10s_{3111} + 7s_{222} + 5s_{2211} + s_{21111} - 5s_{111111}$
$Z[B_{21111}] = 265s_6 - 159s_{51} + 45s_{42} + 26s_{411} - 11s_{33} + 10s_{3111} - 7s_{222} - 15s_{2211} - 3s_{21111} + 5s_{111111}$
$Z[B_{111111}] = 265s_6 - 265s_{51} + 135s_{42} + 130s_{411} - 55s_{33} - 80s_{321} - 50s_{3111} + 35s_{222} + 45s_{2211} + 5s_{21111} - 5s_{111111}$

Figure .2: Cycle indicators of boards avoiding a fixed permutation in S_6

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