The Interplay of Ranks of Submatrices

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***Abstract*— A banded invertible matrix T has a remarkable inverse. All "upper" and "lower" submatrices of T^{-1} have low rank (depending on the bandwidth in T). The exact rank condition is known, and it allows fast multiplication by full matrices that arise in the boundary element method.**

We look for the "right" proof of this property of T^{-1}. Ultimately it reduces to a fact that deserves to be better known: Complementary submatrices of any T and T^{-1} have the same nullity. The last figure in the paper (when T is tridiagonal) shows two submatrices with the same nullity $n-3$. Then C has rank 1. On and above the diagonal of T^{-1}, all rows are proportional.

***Index Terms*— Band matrix, low rank submatrix, fast multiplication.**

I. Introduction

An n by n tridiagonal matrix T is specified by $3n-2$ parameters—its entries on the three central diagonals. In some way T^{-1} must also be specified by $3n-2$ parameters (if T is invertible). It will be especially nice if those parameters can be the "tridiagonal part" of T^{-1}. Fortunately they can. T^{-1} can be built outwards, diagonal by diagonal (and without knowing T), because the inverse of a tridiagonal matrix possesses these two properties:

- ***Upper***: On and above the main diagonal, every 2 by 2 minor of T^{-1} is zero.
- ***Lower***: On and below the main diagonal, every 2 by 2 minor of T^{-1} is zero.

These allow us to fill in the upper triangular part of T^{-1} and then the lower triangular part. Expressed in another way, those properties are statements of *low rank*. Of course the whole matrix T^{-1} has full rank! But it has many submatrices of rank ≤ 1:

- ***Upper***: On and above the main diagonal, every submatrix of T^{-1} has rank ≤ 1:

$$\text{For } j \geq i: \quad (T^{-1})_{ij} = a_i b_j \ \ (2n-1 \text{ parameters})\,.$$

- ***Lower***: On and below the main diagonal, every submatrix of T^{-1} has rank ≤ 1:

$$\text{For } j \leq i: \quad (T^{-1})_{ij} = c_i d_j \ \ (2n-1 \text{ parameters})\,.$$

Equality $a_i b_i = c_i d_i$ along the main diagonal reduces $4n-2$ to $3n-2$, the desired number of parameters.

The friendly but impatient reader will quickly ask for generalizations and proofs. A natural generalization allows T to have a wider band. Suppose $T_{ij} = 0$ for $|i-j| > p$, so that $p = 1$ means tridiagonal. The corresponding conditions on T^{-1} involve zero subdeterminants of order $p+1$. The key is to know which submatrices of T^{-1} are involved when T is banded:

- ***Upper***: Above the pth subdiagonal, every submatrix of T^{-1} has rank $\leq p$.
- ***Lower***: Below the pth superdiagonal, every submatrix of T^{-1} has rank $\leq p$.

Equivalently, all upper and lower minors of order $p+1$ are zero. Again T^{-1} can be completed starting from its "banded part." The count of parameters in T and T^{-1} still agrees.

A next small step would allow the lower triangular part of T to be full. Only the *upper* condition will apply to T^{-1}, coming from the upper condition $T_{ij} = 0$ for $j - i > p$. We will pursue this *one-sided formulation* from now on. Upper conditions on T (above the pth superdiagonal) will be equivalent to upper conditions on T^{-1} (above the pth subdiagonal).

Here is a significant extension. Requiring zero entries is a statement about 1 by 1 minors of T. We could ask instead for all k by k minors of T to vanish (above the pth diagonal). Equivalently, all "upper submatrices" B would have $\operatorname{rank}(B) < k$. *What property does this imply for* T^{-1}? The answer is neat and already known. But this relation between submatrices of T and T^{-1} is certainly not well known. The goal of this paper is to try for a new proof:

Theorem (for invertible T)

All submatrices B above the pth superdiagonal of T have $\operatorname{rank}(B) < k$

if and only if

All submatrices C above the pth subdiagonal of T^{-1} have $\operatorname{rank}(C) < p + k$.

Our tridiagonal case (or Hessenberg case, which is the one-sided version) had $p = 1$ and $k = 1$. Even more special is $p = 0$ and $k = 1$. Then T is lower triangular if and only if T^{-1} is lower triangular (extremely well known!). The case $p = 0$ and $k = 2$ is not so familiar—if T is "rank 1 above the main diagonal" then so is T^{-1}. This also comes from the Woodbury-Morrison formula [20, p. 82], [16, p. 19] for the effect on T^{-1} of a rank one change in T.

We comment in advance about our proof (and a second proof!). For pairs of submatrices B and C of T and T^{-1}, the goal is to show that

$$\operatorname{rank}(B) < k \quad \text{if and only if} \quad \operatorname{rank}(C) < p + k\,. \tag{1}$$

Our proof will use an inequality for ranks of products. Wayne Barrett observed that our lemma is a special case of the Frobenius Rank Inequality

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(B) + \operatorname{rank}(ABC)\,. \tag{2}$$

Then we noticed that (2) follows quickly from our special case. Conceivably this provides a new proof of (2).

Barrett also pointed out that (1) follows immediately from the beautiful observation by Fiedler and Markham [10] that (in this notation)

$$\text{nullity}(B) = \text{nullity}(C)\,. \tag{3}$$

This approach surely gives the best proof of the Theorem. It has the great merit that (3) is an *equality* (of nullities) instead of an inequality (of ranks). After returning to this discussion in Section III, the Theorem is applied to fast multiplication by these "semiseparable" matrices.

Notice that there are an equal number of free parameters in T and T^{-1}. Figure 1 shows how the entry in position $(1, p+2k)$ of both T and T^{-1} is generically the first to be determined from the (equivalent) conditions in the theorem. (This entry is in the upper corner of a square singular submatrix for both matrices. There will be degenerate cases when that entry is not determined, in the same way that specifying three entries of a singular 2 by 2 matrix does not always determine the fourth.) The entries on all earlier diagonals, before position $(1, p+2k)$, can be the free parameters for T and also for T^{-1}.

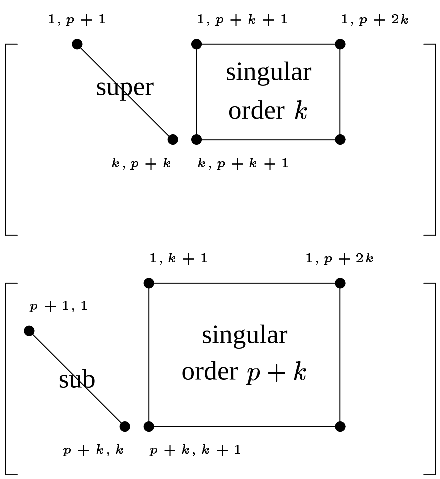


Fig. 1. For both T and T^{-1}, the first entry to be determined from earlier diagonals is in position $(1, p+2k)$. The earlier diagonals are free.

II. Proof of the Theorem

Lemma. Suppose the matrices A and C are M by N and N by L. Then

$$\text{rank}(A) + \text{rank}(C) \le N + \text{rank}(AC)\,. \tag{4}$$

Proof. If AC = zero matrix, the column space of C is contained in the nullspace of A. The dimensions of those spaces give $\text{rank}(C) \le N - \text{rank}(A)$. This is (4) in the case $\text{rank}(AC) = 0$.

To reduce every other case to that one, suppose AC has rank $R > 0$. Then AC can be written as the product $A'C'$ of an M by R matrix and an R by L matrix. Now two block matrices multiply to give zero:

$$\begin{bmatrix} A & A' \end{bmatrix} \begin{bmatrix} C \\ -C' \end{bmatrix} = AC - A'C' = \text{ zero matrix}\,.$$

This returns us to the first case, with $N + R$ columns and rows instead of N:

$$\begin{aligned} \text{rank}(A) + \text{rank}(C) &\le \text{rank}(\begin{bmatrix} A & A' \end{bmatrix}) + \text{rank}(\begin{bmatrix} C \\ -C' \end{bmatrix}) \\ &\le N + R, \text{ which is (4)}\,. \end{aligned}$$

Note. This lemma is a key to our proof of the main Theorem, but (on such a basic subject!) it could not possibly be new. It is the special case $B = I_N$ of the *Frobenius rank inequality* [16, p. 13] for three matrices:

$$\text{rank}(AB) + \text{rank}(BC) \le \text{rank}(B) + \text{rank}(ABC)\,. \tag{5}$$

We noticed that our weaker result (4) leads quickly to the Frobenius inequality (5). Suppose B is any N_1 by N_2 matrix with $\text{rank}(B) = N$. Then B factors into $B_1B_2 = (N_1 \text{ by } N)(N \text{ by } N_2)$. Applying (4) to the matrices AB_1 and B_2C gives

$$\text{rank}(AB_1) + \text{rank}(B_2C) \le N + \text{rank}(ABC)\,. \tag{6}$$

Always $\text{rank}(AB) = \text{rank}(AB_1B_2) \le \text{rank}(AB_1)$. Similarly $\text{rank}(BC) = \text{rank}(B_1B_2C) \le \text{rank}(B_2C)$. So (6) implies (5). □

Now we restate the Theorem and prove it using the rank inequality (4).

> Every submatrix B above the pth superdiagonal of T has $\text{rank}(B) < k$ IFF every submatrix C above the pth subdiagonal of T^{-1} has $\text{rank}(C) < p + k$.

Our proof comes directly from $TT^{-1} = I$. Look at the first M rows of T (with $k \le M < n - p$). They always multiply the last $n - M$ columns of T^{-1} to give a zero submatrix of I:

$$\begin{array}{cc} \begin{bmatrix} \boxed{A} & \boxed{B} \\ * & * \end{bmatrix} & \begin{bmatrix} * & \boxed{C} \\ * & \boxed{D} \end{bmatrix} = \begin{bmatrix} I_M & \boxed{0} \\ * & * \end{bmatrix} \\ M+p \quad n-M-p & M \quad n-M \end{array}$$

Thus $AC + BD = 0$. The lower left entry of B is in row M and column $M + p + 1$, so the submatrix B is above the pth superdiagonal of T. We are given that $\text{rank}(B) < k$ and therefore $\text{rank}(AC) = \text{rank}(BD) < k$. Since A has $M + p$ columns, the Lemma gives

$$\text{rank}(A) + \text{rank}(C) < M + p + k\,. \tag{7}$$

If A has full rank M, it follows that $\text{rank}(C) < p + k$, as we wished to prove.

Note that the lower left entry of C is in row $M + p$ and column $M + 1$. The difference $p - 1$ means that C is immediately above the pth subdiagonal of T^{-1}. As M varies, we capture all the submatrices of T^{-1} above that pth subdiagonal.

In case A fails to have full rank M, perturb it a little. The new T is still invertible and the theorem applies. We have proved that the new submatrix C in T^{-1} has rank less than $p + k$. That rank cannot jump as the perturbation goes to zero, so the actual C also has rank less than $p + k$.

The proof of the converse is similar, but with a little twist. (We need a fifth matrix E.) For the same A, B, C, D with $AC + BD = 0$, we must prove:

$$\text{If } \operatorname{rank}(C) < p + k \text{ then } \operatorname{rank}(B) < k.$$

Since C has $n - M$ columns, and its rank is less than $p + k$, its nullspace has dimension at least $L = n - M - p - k + 1$. Put L linearly independent nullvectors of C into the columns of a matrix E, so that $CE = 0$. Then $ACE = 0$ implies $BDE = 0$ and we can apply the Lemma:

$$\begin{aligned} \operatorname{rank}(B) + \operatorname{rank}(DE) &\leq \text{(number of columns of } B) \\ &= n - M - p \,. \end{aligned} \tag{8}$$

If DE has full column rank L, our conclusion follows:

$$\operatorname{rank}(B) \leq n - M - p - L = k - 1, \text{ as desired}\,.$$

Notice that DE has $n - M - p$ rows, which is at least L. If it happens that $\operatorname{rank}(DE) < L$, perturb D a little to achieve full rank. We don't change C or E, so our proof applies and the new submatrix B (in the new T) has rank less than k. As the perturbation of T^{-1} goes to zero, the new T approaches the actual T. So the actual submatrix B has rank less than k (since the rank can't suddenly increase).

This completes the proof, when M takes all values from 1 to $n - p - k$. That largest M captures the submatrix B in the last k columns of T, above the pth superdiagonal. We have proved that all k by k submatrices of T above that diagonal are singular.

Note on the proof: To see why the matrix E is needed, write out $AC + BD = 0$ in the 3 by 3 case for $p = k = M = 1$:

$$\begin{bmatrix} T_{11} & T_{12} \end{bmatrix} \begin{bmatrix} \\ \text{rank} < 2 \\ \\ \end{bmatrix} + T_{13} \begin{bmatrix} T_{32}^{-1} & T_{33}^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

We must prove $\operatorname{rank}(B) < 1$, in other words $T_{13} = 0$. When these matrices multiply a 2 by 1 nullvector E of that low rank matrix C, the first term ACE disappears to leave

$$T_{13} \begin{bmatrix} \quad D \quad \end{bmatrix} \begin{bmatrix} \\ E \\ \\ \end{bmatrix} = 0\,.$$

This proves $T_{13} = 0$, after possible perturbation of D to make $DE \neq 0$.

III. References and Alternative Proofs

Normally we would comment on the existing literature before adding to it! That is the proper order, especially for a theorem to which many earlier authors have contributed. But the "best" proof was pointed out to us by Wayne Barrett and it becomes particularly easy to describe in terms of the matrices B and C above. The proof will follow directly from a simple and beautiful result of Fiedler and Markham.

The first reference we know is to Edgar Asplund [1] in 1959. He discovered the banded case: bandwidth p in T and rank p submatrices in T^{-1}. Rereading his proof, we see that he anticipated many of the ideas that came later. Combining his theorem with the Woodbury-Morrison formula for a rank k perturbation would complete the proof. Remarkably, it was a paper by his father (in the same journal) that led Asplund to the problem.

The crucial tridiagonal case appears (independently and with earlier references) in the famous 1960 text of Gantmacher and Krein [11, p. 95]. There is a close connection to second-order differential equations, noted in our final section below. Karlin's 1968 book *Total Positivity* [18] refers to T^{-1} as a "Green's matrix". Subsequently Barrett [2] proved from explicit formulas that tridiagonality of T is equivalent to rank 1 submatrices in T^{-1}. The formulas allowed him to clarify all cases of zero entries in T.

The natural extension to block tridiagonal matrices was given in the same year by Ikebe [17]. Then Cao and Stewart [6] allowed $p > 1$ and blocks of varying sizes. The crucial step to $k > 1$ was taken by Barrett and Feinsilver [3]. Their proof of the Theorem is based on the formula for the minors of T^{-1}. Meurant [19] has provided an extremely helpful survey of the literature, and a stable algorithm for computing T^{-1} in the tridiagonal case.

It is interesting to recognize these two approaches: *determinant formulas* or *rank and nullity formulas*. The former come ultimately from Jacobi and Sylvester. They led in [3] to the conditions of unique completion of T^{-1} starting from its central band (for $k = 1$ and any p). We now give the nullity formula (too little known!) which proves the Theorem in a single step.

The Nullity Theorem was given in 1984 by Gustafson [13] in the language of modules over a ring and in 1986 by Fiedler and Markham [10] in matrix language. A neat and simple proof will be in Section 0.7.5 of the forthcoming new edition of [16]. Here is an equivalent statement:

Nullity Theorem

> Complementary submatrices of a matrix and its inverse have the same nullity.

Two submatrices are "*complementary*" when the row numbers not used in one are the column numbers used in the other. If the first submatrix A is M by N, the other submatrix D is $n - N$ by $n - M$. Suppose A is the upper left corner of T, so that D is the lower right corner of the inverse:

$$\begin{array}{ccccc} M \text{ rows} & \begin{bmatrix} A & * \\ * & * \end{bmatrix}^{-1} & = & \begin{bmatrix} * & * \\ * & D \end{bmatrix} \\ & N \text{ columns} & & n - M \text{ columns} \end{array} \tag{9}$$

$$\text{has} \quad \operatorname{nullity}(A) = \operatorname{nullity}(D)\,.$$

Note that all blocks can be rectangular. One partitioning is the "transpose" of the other, to allow block multiplication. A permutation will put both submatrices into the upper right corner (appropriately for our proof):

$$\begin{array}{ccccc} M \text{ rows} & \begin{bmatrix} * & B \\ * & * \end{bmatrix}^{-1} & = & \begin{bmatrix} * & C \\ * & * \end{bmatrix} \\ & N \text{ columns} & & n - M \text{ columns} \end{array} \tag{10}$$

$$\text{has} \quad \operatorname{nullity}(B) = \operatorname{nullity}(C)\,.$$

The submatrices A^{T} and D^{T} (as well as B^{T} and C^{T}) are again complementary, after the entire block matrices are transposed. So the Nullity Theorem applies also to the transposed submatrices (but we don't use it):

$$\text{nullity}(A^{\mathrm{T}}) = \text{nullity}(D^{\mathrm{T}})\,. \qquad (11)$$

Fiedler and Markham begin their proof with direct multiplication of the block matrix and its inverse to produce I. The nullspace matrix that we called E enters in the same way. After two examples, we show how this neater formulation leads instantly to the desired submatrix ranks in T^{-1}.

In the first special case, D (or C) is a 1 by 1 matrix. If its single entry is zero the nullity is 1. By the standard cofactor formula for the entries of the inverse matrix, the determinant of A (or B) of order $n-1$ is also zero. The nullity is therefore at least 1—but why not greater? Because a greater nullity would imply that the whole n by n matrix is not invertible.

A particular form of block matrix arises frequently in applications, with a square zero block on the diagonal. It is interesting to see the Nullity Theorem in action once more:

$$\begin{bmatrix} I_M & S \\ T & 0_N \end{bmatrix}^{-1} = \begin{bmatrix} I_M - S(TS)^{-1}T & S(TS)^{-1} \\ (TS)^{-1}T & -(TS)^{-1} \end{bmatrix}. \quad (12)$$

These block matrices are invertible exactly when the N columns of S and the N rows of T are independent (requiring $M \geq N$). The nullity of 0_N is N. The complementary submatrix $I_M - S(TS)^{-1}T$ must have the same nullity. In this case the N columns of S are a basis for the nullspace of that submatrix:

$$(I_M - S(TS)^{-1}T)S = \text{ zero matrix}\,.$$

Returning to the upper right submatrices B and C in our proof of the main Theorem, we have

$$\text{nullity}(B) = \text{nullity}(C)\,.$$

The matrix B has $n - M - p$ columns, and C has $n - M$ columns. Therefore

$$n - M - p - \text{rank}(B) = n - M - \text{rank}(C)\,. \qquad (13)$$

Then (exactly!) $\text{rank}(C) = p + \text{rank}(B)$. This means that $\text{rank}(B) < k$ if and only if $\text{rank}(C) < p+k$, and the Theorem is proved.

Notice that both proofs extend immediately to *block matrices* T and T^{-1} because they deal one at a time with complementary pairs B and C. Suppose that T has block entries T_{ij} (of any compatible sizes), with T_{ij} = zero block for $j - i > p$. The inverse of this block band matrix T has low rank submatrices C above the pth block subdiagonal. The proof is the same except that the choices of B respect the block form of T. The Mth row of B comes at the end of a block in T.

This block case is of utmost importance in applications.

IV. Fast Matrix Multiplication and Applications

For a tridiagonal matrix T, the upper blocks C in T^{-1} (and also the lower blocks) have $\text{rank}(C) \leq 1$. How quickly can we multiply an n by n matrix T^{-1} of this "semiseparable" form by a vector x? We take this tridiagonal case ($p = 1$ and $k = 0$) as our model, and approach the complexity of $T^{-1}x$ in the most direct way.

If $C = uv^{\mathrm{T}}$ is an M by N matrix of rank one, the product $Cy = u(v^{\mathrm{T}}y)$ requires $M + N$ individual multiplications (rather than MN). We want to partition T^{-1} into blocks C that do not cross the diagonal. The natural choice is to begin with square blocks C_{12} and C_{21} of size $n/2$, in the upper right and lower left corners of T^{-1}. This leaves blocks C_{11} and C_{22} of size $n/2$ on the diagonal of T^{-1}, to be partitioned (recursively) in the same way. The multiplication count $m(n)$ obeys a rule much like the FFT:

$$m(n) = 2\,m\left(\frac{n}{2}\right) + 2\left(\frac{n}{2} + \frac{n}{2}\right)\,. \qquad (14)$$

This recursion is satisfied by $m(n) = 2n \log_2 n$.

The true applications of the Theorem in this paper are not to tridiagonal matrices but to integral equations—often with space dimension greater than one. Now a model problem is the approximate computation of a single or double integral:

$$\int K(s,x) f(x)\, dx \quad \text{or} \quad \iint K(s,t,x,y) f(x,y)\, dx\, dy\,.$$

The kernel K may be the Green's function of an underlying differential equation. We expect to see blocks, rather than scalar entries, when approximating double integrals. The parameters p (for bandwidth) and k (for off-diagonal rank) have "continuous" analogs for an integral operator:

Decay rate: Fast decay away from the diagonal $K(x,x)$ corresponds to low bandwidth.

Smoothness: A slowly varying kernel corresponds to low rank submatrices.

In both cases the word "approximate" should be included. We have matrix analysis rather than matrix algebra.

To summarize the applications to fast solution of integral equations, our best plan is to point to several active groups (with apologies to others). The first group has emphasized the connections to earlier "panel methods" and the delicate partitioning that can sometimes reduce the operation count to $O(n)$—for matrix inversion as well as multiplication. We hope these names and references will help the reader:

1) Hackbusch [14], [15]
2) Chandrasekaran and Gu [7]
3) Tyrtyshnikov [12], [21]
4) Eidelman and Gohberg [8], [9]

V. Tridiagonal Matrices and Differential Equations

The tridiagonal case is the simplest and most important. If we look at the first two columns of T^{-1}, below the first row, then the "lower" statement at the start of our paper means: *Those columns are proportional*. We want to discuss this conclusion directly, and also to recognize the analogous statement for second-order differential equations and their Green's functions.

In the matrix case, the tridiagonal T multiplies the first column of T^{-1} to give zeros below the diagonal. Key point: When the second column of T^{-1} is proportional to the first, multiplication by T gives those zeros again (below the $2,2$

entry). These columns of T^{-1} contain a solution of the "second-order difference equation $Ty = 0$." It is the solution that satisfies the "boundary condition" at the right endpoint.

We see multiples of this homogeneous solution in all columns of T^{-1}, below the diagonal. They meet multiples of the other solution on the main diagonal. That other solution of $Ty = 0$ satisfies the boundary condition at the left endpoint.

Since T can be any invertible tridiagonal matrix, $Ty = 0$ may not look like a second-order difference equation. We could choose three numbers a_k, b_k, c_k to multiply $\Delta^2 u_k, \Delta u_k, u_k$ to match row k of T. It may be useful to compare with the standard approach to second-order differential equations, where the analog of column j in $TT^{-1} = I$ is

$$Ly = a(x)y'' + b(x)y' + c(x)y = \delta(x - a)$$

with boundary conditions at $x = 0$ and $x = 1$. The solution $y(x) = G(x, a)$ is the Green's function and it corresponds to $(T^{-1})_{ij}$.

A good text like [5, pp. 15–18] notes the equivalence between computing G and variation of parameters. The latter begins with two independent solutions $y_1(x)$ and $y_2(x)$ of $Ly = 0$ and finds a particular solution to $Ly = \delta$ of the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)\,. \qquad (15)$$

The underdetermined u_1 and u_2 are constrained by

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0\,. \qquad (16)$$

Substituting (15) into $Ly = \delta(x - a)$ gives

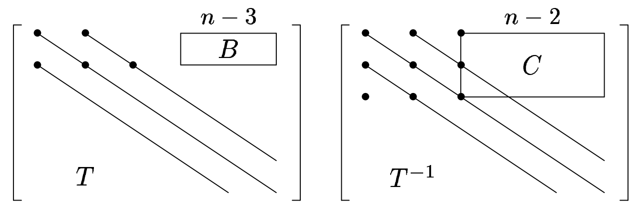
$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = \delta(x - a)\,. \qquad (17)$$

Solving (16) and (17) for u_1' and u_2' involves the nonzero Wronskian of y_1 and y_2. Integration gives $u_1(x)$ and $u_2(x)$, and equation (15) gives $y(x)$.

A few remarks will connect this continuous problem $Ly = \delta$ with the discrete $TT^{-1} = I$. Suppose $y_2(x)$ is chosen to satisfy the boundary condition at the right endpoint $x = 1$, as well as $Ly_2 = 0$. Similarly $y_1(x)$ satisfies $Ly_1 = 0$ and the boundary condition at $x = 0$. With those special choices, the Green's function $G(x, a)$ will be a multiple of $y_2(x)$ for $x \geq a$ and a multiple of $y_1(x)$ for $x \leq a$. Those multiples $u_1(x)$ and $u_2(x)$ correspond to the "proportionality constants" that connect the columns of T^{-1}, above and below its main diagonal. And those constants are given by the first and last *rows* of T^{-1}, which solve the *adjoint* problem based on the transpose of T.

Perhaps we can summarize the discrete case in this way. The matrix T^{-1} is determined by its first and last columns and rows. It has rank 1 above and below the diagonal. Those parameters are reduced to the correct number $3n - 2$ by equality along the diagonal.

To close the circle, we specialize the matrices B and C in this paper to this tridiagonal case. For $M = 2$ rows, B is a 2 by $n-3$ submatrix of zeros and its nullity is $n-3$. Then C is a 3 by $n - 2$ matrix with the same nullity. Therefore its rank is 1! As M varies, all the submatrices C of T^{-1} have rank 1.



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