

**Expressions for the generating function of the
Donaldson invariants for CP^2 .**

by

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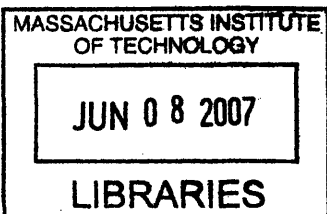
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Abstract

The Donaldson invariants for CP^2 were obtained as the u -plane integral from a $N = 2$ supersymmetric topological $U(1)$ -gauge theory by Moore and Witten. We derive the generating function for the Donaldson invariants of CP^2 as the stationary phase approximation of the low-energy effective $U(1)$ -gauge theory on CP^2 thus obtaining an interpretation of the u -plane integral in terms of determinant line bundles. For the product of the determinant line bundles, the local and global anomalies vanish. Moreover, the product has a canonical trivialization.

We show that the u -plane integral also arises as the stationary phase approximation of a heterotic σ -model on an elliptic curve at the boundary of the Coulomb branch with the target space $CP^1 \times U(1)$. The semi-classical generating function is described in terms of determinant line bundles on the Coulomb branch. We show that in terms of the partition function on the elliptic curve, the blow-up function for the Donaldson invariants derived by Fintushel and Stern arises in a natural way.

Thesis Supervisor: Isadore M. Singer
Title: Institute Professor

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Chapter 1

Introduction and statement of results

1.1 Introduction

The subject of this thesis is the generating function of the Donaldson invariants for CP^2 . Let M be a compact, simply connected, four-dimensional manifold without boundary. Then $H^0(M; \mathbb{Z}) = H^4(M; \mathbb{Z}) = \mathbb{Z}$, $H^1(M; \mathbb{Z}) = H^3(M; \mathbb{Z}) = 0$, and $H^2(M; \mathbb{Z}) = \mathbb{Z}^{b_2}$ where b_2 is the second Betti number. The cup-product defines a symmetric, integer-valued, non-degenerate bilinear form on $H^2(M; \mathbb{Z})$ with determinant ± 1 . In 1983, Donaldson showed that for a simply connected, compact, smooth four-dimensional manifold M without boundary, its intersection form is either indefinite or plus or minus the identity matrix in some basis [2]. Donaldson used the moduli space of instantons (anti-selfdual connections) on $SU(2)$ -principal bundles over M . In 1990, he also defined new invariants for the manifold M by evaluating a characteristic class of a universal $SU(2)$ -bundle on the moduli space of instantons [2].

Given an orientation of M together with an orientation of $H^{2+}(M)$ and b_2^+ odd, the Donaldson invariants give a linear function [5]

$$\Phi : \text{Sym}_* \left(H_0(M; \mathbb{Z}) \oplus H_2(M; \mathbb{Z}) \right) \rightarrow \mathbb{Q} .$$

For $p \in H_0(M; \mathbb{Z})$ and $\Sigma \in H_2(M; \mathbb{Z})$ the invariants can be assembled in a generating function

$$\mathbf{Z}(p, \Sigma) = \sum_{m, n=0}^{\infty} \frac{p^m}{m!} \frac{\Sigma^n}{n!} \Phi(p^m, \Sigma^n) .$$

For $b_2^+ > 1$ the Donaldson invariants are independent of the metric as long as the metric is generic. In the case of $b_2^+ = 1$, the Donaldson invariants depend on the metric of M . It turns out that the positive cone of $H^2(M; \mathbb{R})$ has a chamber structure. The Donaldson invariants depend on the metric only through the chamber in which the period point of the metric on M is located [4]. The period point of the metric is

defined as the one-dimensional space of all selfdual harmonic two-forms with respect to the metric.

For the algebraic surfaces with $b_2^+ = 1$, there is an algorithm called the wall-crossing formula which accounts for the changes in the Donaldson invariants when the period point moves through a system of walls and chambers in the positive cone of $H^2(M; \mathbb{R})$ [6]. Moreover, there is a formula, called the blow-up formula, which accounts for the changes of the Donaldson invariants under blow-ups and blow-downs of the manifold [5]. For CP^2 the positive cone has only one chamber; the Kähler class gives a canonical homology orientation, and since $H^2(M; \mathbb{Z}) = \mathbb{Z}$, Σ is determined by an integer $S = \int_{\Sigma} \omega$ where ω is the Kähler form. Thus, the generating function is a formal power series in the two variables p and S . The computation of the Donaldson invariants for CP^2 can be considered the starting point for the computation on more complicated surfaces.

Using a vanishing result for the Donaldson invariants on rational ruled surfaces [8, Lemma 5.3], Ellingsrud and Göttsche [6] computed the Donaldson invariants for CP^2 of degree smaller or equal to 50. Their results are listed in Sec. 2.3. Their approach was very different from the one that was originally suggested by the physical interpretation of Witten who showed that the Donaldson invariants can be constructed as correlation functions of an $N = 2$ supersymmetric topological $SU(2)$ -Yang-Mills theory on M [16]. One of the main results of the work of Seiberg and Witten [19] was that the moduli space of this topological quantum field theory decomposes into two branches, the Coulomb branch and the Seiberg-Witten branch. Both branches can be interpreted as the moduli spaces of simpler physical theories on M , and the contributions to the Donaldson invariants from each branch can be computed separately [21]. We briefly describe the two branches in Sec. 4.

Since CP^2 admits a metric of positive scalar curvature the contributions from the Seiberg-Witten branch vanish [21]. Thus, for CP^2 the Donaldson invariants can be determined from the Coulomb branch alone. The Coulomb branch is the moduli space of a class of topological $U(1)$ -gauge theories on M which will be described in detail in Sec. 4.1. In [25], Moore and Witten obtained the Donaldson invariants for CP^2 as correlation functions of a universal $U(1)$ -gauge theory, called the low-energy effective $U(1)$ -gauge theory. The universal family depends on M only through $H^2(M; \mathbb{Z})$, and its correlation functions vanish for $b_2^+ > 1$. They found

$$\mathbf{Z}(p, S) = \frac{S}{2\pi} \int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}(\tau)^{\frac{3}{2}}} \frac{\vartheta_4(\tau)^9 \overline{\vartheta_4(\tau)}}{(\vartheta_2(\tau)\vartheta_3(\tau))^4} \exp \left(2pu + \left(\frac{S}{2\pi} \right)^2 \left[-\frac{1}{2} \frac{\widehat{G}_2(\tau)}{(\vartheta_2(\tau)\vartheta_3(\tau))^2} + \frac{4u}{3} \pi^2 \right] \right), \quad (1.1)$$

where $\Gamma_0(4)$ is the congruence subgroup of elements in $SL(2; \mathbb{Z})$ whose mod 4 reduction is lower triangular. We have used the standard notation for the appearing modular functions. A cutoff regularization was applied to regularize the integral. Eq. (1.1) is called the u -plane integral since it is formulated in terms of the complex variable u on the thrice-punctured Riemann sphere $UP = \mathbb{C} - \{-1, 1\}$ and a map

$\tau : \text{UP} \rightarrow H$ from the u -plane to the upper half plane H given by uniformization. We will describe the details in Sec. 3.4. In the appendix, we will present a short MAPLE program that computes the rational coefficients in the formal expansion of $\mathbf{Z}(p, S)$ in Eq. (1.1). The results agree with the results of Ellingsrud and Göttsche [6] up to degree 50. The results of this thesis are:

- We derive the generating function for the Coulomb branch as the stationary phase approximation of the low-energy effective field theory thus obtaining an interpretation of the u -plane integral in terms of determinant line bundles.
- We show that the u -plane integral also arises as the stationary phase approximation of a quantum field theory defined as a heterotic σ -model on an elliptic curve at the boundary of the Coulomb branch with target space $CP^1 \times U(1)$ coupled to an external field. This partition function is described in terms of determinant line bundles on the Coulomb branch.
- We show that the blow-up function derived by Fintushel and Stern in [5, Thm. 4.1] arises in a natural way in terms of the stationary phase approximation of this ‘dual’ quantum field theory on the elliptic curve.

Our construction of the partition function on an elliptic curve at the boundary of the Coulomb branch is similar to Witten’s construction of the elliptic genus [94], the character-valued partition function by Schellekens and Warner [95] and the representation as a one-loop amplitude in a string field theory by Alvarez et al. [97].

1.2 The geometry of the Coulomb branch

First, we will investigate the geometry of the Coulomb branch as it is fundamental for the definition and the understanding of the low-energy effective field theory. The description of the Coulomb branch will be given explicitly in terms of special functions by using the results from [69, 66, 70]. Some of the necessary computations are already present in the physics literature [22]. Our first goal will be to explain these computations in the context of a rational elliptic surface fibered by elliptic curves and the determinant line bundle for the vertical $\bar{\partial}$ -operator on it. We will later use the vertical $\bar{\partial}$ -operator in the definition of a quantum field theory on the elliptic curve at the boundary.

The Coulomb branch is the family of elliptic curves, called the Seiberg-Witten curve, that defines a rational Weierstrass elliptic surface over the u -plane by taking

$$y^2 = x^3 + A(u)x + B(u)$$

as the elliptic fiber E_u with $A(u) = -\frac{u^2}{3} + \frac{1}{4}$, and $B(u) = \frac{u}{12} - \frac{2}{27}u^3$, and the point at infinity as the base point. The two types of singularities that occur are a cubic cusp for $u \rightarrow \infty$ and a rational curve with a single node for $u \rightarrow \pm 1$. The nodes do not give rise to surface singularities. Thus, the Seiberg-Witten curve defines a proper holomorphic submersion of complex manifolds $\pi : Z \rightarrow X$ where X is the set

$X = \{u \in \text{UP}; |u| < R\}$ for some fixed, large R . The family of elliptic curves is the universal curve for $H/\Gamma_0(4)$ and determines by the monodromy action around the singular fibers a representation of $\Gamma_0(4)$. In physics, the unique even spin structure which is invariant under a subgroup $\Gamma_0(4)$, the congruence subgroup consisting of matrices in $SL(2; \mathbb{Z})$ whose mod 4 reduction is upper triangular, is called Periodic-Antiperiodic (PA). It is the unique order two element of the Mordell-Weil group of π and defines a holomorphic line bundle $L \rightarrow Z$ whose square is the canonical line bundle in each fiber. We use the results of Quillen [77], Bismut and Bost [82, 83], and Seeley and Singer [81] to show:

Proposition 1. *Let $\text{DET } \bar{\partial} \rightarrow X$ be the determinant line bundle of the $\bar{\partial}$ -operator along the fiber and $\text{DET } \bar{\partial}_{L,u} \rightarrow X$ the determinant line bundle of the $\bar{\partial}$ -operator along the fiber coupled to L_u . We have the following results:*

1. *The determinant line bundles $\text{DET } \bar{\partial}$ and $\text{DET } \bar{\partial}_{L,u}$ are flat away from the nodal fibers.*
2. *For the line bundle $\text{DET } \bar{\partial}_{L,u}$, the canonical section $\det \bar{\partial}_{L,u}$ trivializes the determinant line bundle. The Quillen metric is*

$$\| \det \bar{\partial}_{L,u} \|_Q = \left| \frac{\vartheta_4(\tau)}{\eta(\tau)} \right| = |\Delta(\tau)|^{\frac{1}{12}} ,$$

where $\Delta(\tau)$ is the modular discriminant of the elliptic curve E_u with $\tau = \tau(E_u)$. The divergence of the Quillen metric is

$$c_1(\text{DET } \bar{\partial}_{L,u}, \|\cdot\|_Q) = \frac{1}{12} \delta_{(u=1)} + \frac{1}{12} \delta_{(u=-1)} .$$

$\delta_{(u=1)}$ is the current supported at $u = 1$ and defined by the equality $\int_Z \delta_{(u=1)} \wedge \alpha = \int_{E_1} \alpha$ for every differential form α with compact support.

3. *For the $\bar{\partial}$ -operator along the fiber, the ζ -function regularized determinant is*

$$\det' \bar{\partial}^* \bar{\partial} = \text{vol}(E_u)^2 \left| \frac{\vartheta_4(\tau)}{\eta(\tau)} \right|^2 .$$

4. *The determinant line bundle $\text{DET } \bar{\partial}$ has a trivializing section s . Its Quillen norm is*

$$\|s\|_Q^2 = \frac{\det' \bar{\partial}^* \bar{\partial}}{\|1\|^2 \|dz\|^2} = |\Delta(\tau)|^{\frac{2}{12}} .$$

Prop. 1 asserts that the curvature of the determinant line bundle of the $\bar{\partial}_\xi$ -operator has current contributions at the nodal fibers and vanishes over the rest of the u -plane. The Coulomb branch controls the parameters of the low-energy effective field theory. In particular, it was determined in [19] that classically the scalar field of the low-energy

effective field theory defines a section \mathbf{a}_u of the elliptic fibration over the open subvariety of points with regular fibers. The section \mathbf{a}_u then defines a flat holomorphic $SU(2)$ -bundle on each regular fiber of Z . There is a way of extending this construction across the nodes [69] which involves an appropriate toroidal compactification at the nodal fibers. We present the details in Sec. 3.9 as we will give a definition of a BPS-state and show:

Proposition 2. *A BPS-state of the low-energy effective field theory for $u \rightarrow 1$ is a $SU(2)/\mathbb{Z}_2$ -bundle $V \rightarrow Z$ with a hermitian Yang-Mills connection on $Z \rightarrow X$ such that $\pi_* c_2(V) = -\delta_{(u=-1)}/4$.*

1.3 The stationary phase approximation and determinant line bundles

Second, we will use the stationary phase approximation to define the semi-classical correlation functions of a supersymmetric quantum field theory in terms of determinant line bundles.

In [26], Baulieu and Singer interpreted the construction of the supersymmetric $SU(2)$ -Yang-Mills gauge theory as the topological gauge fixing of the characteristic class $c_2(E)$ for an $SU(2)$ -vector bundle $E \rightarrow M$. The action

$$\langle c_2(E), [M] \rangle = \frac{1}{8\pi^2} \int_M \text{tr} \left(F_A \wedge F_A \right)$$

is invariant under all orientation preserving diffeomorphisms and not only the group of gauge transformations. By fixing the extended gauge independence by the standard BRST-procedure they obtained Witten's supersymmetric action.

The path integral for a supersymmetric action \mathbf{S} can be defined with mathematical rigor by the stationary phase approximation (or semi-classical approximation): each critical point, i.e., each classical field configuration, will be labeled by some discrete topological data which is called the Pontryagin sector (denoted by k), and some continuous moduli which we will denote by λ . The action at the critical point is denoted by $\mathbf{S}^{(0)}(k, \lambda)$. The quadratic approximation of the action \mathbf{S} around a critical point is the Hessian of \mathbf{S} . It determines a free field theory in the collected variations of the Bose fields $\tilde{\Phi}$ and Fermi fields $\tilde{\Psi}$ which is of the form

$$\mathbf{S}^{(2)} = \int_M \text{vol}_M \left(\langle \tilde{\Phi}, \Delta_{(k,\lambda)} \tilde{\Phi} \rangle + \langle \tilde{\Psi}, \mathcal{D}_{(k,\lambda)} \tilde{\Psi} \rangle \right),$$

where Δ is a family of second-order, elliptic operators and \mathcal{D} a family of real skew-symmetric first-order operator both depending on the moduli (k, λ) . The functional integration over the fluctuations, i.e., over the coordinates of the normal bundle at the critical points, can be considered an infinite-dimensional Gaussian integral. We

define the semi-classical path integral¹

$$\int \left[\mathcal{D}\tilde{\Phi} \mathcal{D}\tilde{\Psi} \right] e^{-\mathbf{S}^{(2)}} \text{ to be } \frac{\text{pfaff}' \mathcal{D}_{k,\lambda}}{\sqrt{\det' \Delta_{k,\lambda}}}.$$

The above expression is a section of the product of the Pfaffians line bundle and the dual of the determinant line bundle over the moduli space or more generally over the configuration space. To proceed further, one has to be able to integrate this section over the moduli space. To do so, the line bundle needs to be flat. In physics, this is called the vanishing of the local anomaly. Since the topology of the moduli space need not to be trivial, the section can still have holonomy around non-trivial loops in the moduli space. To make sure that the bundle is globally trivial as well, the holonomy around all curves needs to be the identity. In physics, this is called the vanishing of the global anomaly. Finally, it is not enough to have a trivial line bundle. We also need a canonical trivialization, i.e., a trivializing section. If all these points are satisfied, we can regard the ratio of determinants as a function on the moduli space. We then carry out the integral over the continuous moduli and the sum over the discrete moduli to obtain the semi-classical approximation of the partition function

$$\begin{aligned} \mathbf{Z} &= \sum_k \int d\lambda e^{-\mathbf{S}^{(0)}(k,\lambda)} \int \left[\mathcal{D}\tilde{\Phi} \mathcal{D}\tilde{\Psi} \right] e^{-\mathbf{S}^{(2)}} \\ &= \sum_k \int d\lambda e^{-\mathbf{S}^{(0)}(k,\lambda)} \frac{\text{pfaff}' \mathcal{D}_{k,\lambda}}{\sqrt{\det' \Delta_{k,\lambda}}}. \end{aligned} \tag{1.2}$$

In the case relevant for the definition of the u -plane integral, physical considerations guarantee that the semi-classical approximation is in fact exact [25].

1.4 The low energy effective theory

We will then apply the description of the correlation functions as sections of determinant line bundles to the low-energy effective field theory which is relevant for the u -plane integral.

The low-energy effective field theory is a family of $N = 2$ supersymmetric $U(1)$ -gauge theories. This family of $U(1)$ -theories is parameterized by a complex coupling constant $\tau \in H$ which is the modular parameter $\tau(E_u)$ for the elliptic fiber E_u of the rational surface Z . The classical bosonic field configuration for each $U(1)$ -theory consists of a line bundle $L \rightarrow M$ with a selfdual connection A and a scalar field with a vacuum expectation value \mathbf{a}_u . We present the full action of the low-energy effective field theory in Sec. 4 and show:

Proposition 3. *The action of the low-energy effective field theory is the topological gauge fixing of the action $\langle \text{Re}(\tau + 1) c_1(L)^2, [M] \rangle$.*

¹The Gaussian integral for the free field theory is actually well-defined and agrees with the ratio of the Pfaffians in the definition we give.

We not only want to determine the partition function, but also the generating function with the insertion of the observables relevant for the Donaldson invariants on CP^2 . The inclusion of the observables leads to an additional factor of $\exp(2pu)$ for the zero-form observable, and $\exp(S^2 \widehat{T}(u))$ for the two-form observable. The insertion of the observable for $S \in H_2(M; \mathbb{R}) \cong \mathbb{R}$ introduces the additional terms in the action and the observable involving an auxiliary field D without any dynamics

$$\int_M \text{vol}_M \left(-\text{Im} \tau \langle D, D \rangle + 2\overline{\Xi} \langle \tilde{\Psi}, *D \wedge \tilde{\Psi} \rangle \right) + 2 \int_{\Sigma} \frac{D}{\omega},$$

where Ξ is the holomorphic cubic form and ω is the period for the A-cycle of the Seiberg-Witten curve. Integrating out the auxiliary field D , we obtain instead of $\mathcal{D}_{k,\lambda}$, an operator $\mathcal{D}_{k,\lambda,S}$ coupled to an external gauge field. The family of perturbed fermionic operators to be used in the quadratic approximation is

$$\mathcal{D}_{k,u,S} = \mathcal{D}_{k,u} + S \mathfrak{A}(u) C_{\omega}, \quad (1.3)$$

where C_{ω} is a real skew-symmetric matrix with eigenvalues ± 1 . $\mathfrak{A} = \overline{\Xi}/(\text{Im} \tau \omega)$ is a function on the u -plane and is the special Kähler structure defined in [71] on the rational elliptic surface Z . For every special coordinate system of $Z \rightarrow X$, we obtain an operator $\mathcal{D}_{k,u,S}$. The operators patch together to give a well-defined family of operators parameterized by the u -plane. We discuss the details in Sec. 3.10 and show:

Proposition 4. *The u -plane integral is the stationary phase approximation for the supersymmetric low-energy effective field theory.*

1. Eq. (1.1) is equal to

$$\sum_{k \in 2H^2(M; \mathbb{Z})} \int_{\text{UP}} d\mathbf{a}_u d\bar{\mathbf{a}}_u e^{-\mathbf{S}_{(k,u)}^{(0)} + 2pu + S^2 \widehat{T}(u)} \frac{\text{pfaff } \mathcal{D}_{k,u,S}}{\sqrt{\det' \Delta_{k,u}}}. \quad (1.4)$$

2. The section

$$\frac{\text{pfaff } \mathcal{D}_{k,u,S}}{\sqrt{\det' \Delta_{k,u}}}$$

is a section of a line bundle without local or global anomaly, and with a canonical section.

A consequence of the vanishing of the global anomaly in Prop. 4 is

Corollary 1. *The integrand in Eq. (1.1) is modular invariant.*

The modular invariance of Eq. (1.1) was already observed by Moore and Witten in [25, Eq. (3.32)].

1.5 A dual theory on the boundary of the moduli space

Finally, we will show that the u -plane integral also arises as the stationary phase approximation of a heterotic σ -model on an elliptic curve at the boundary of the Coulomb branch with a $N = 1/2$ supersymmetry and target space $CP^1 \times U(1)$.

On E_u , the chiral Dirac operator for the trivial Spin-structure is $\bar{\partial} : \Lambda^{(1/2,0)}(E_u) \rightarrow \Lambda^{(1/2,1)}(E_u)$. We denote the chiral Dirac operator for the Spin-structure (P, A) by $\bar{\partial}_{L,u}$. The quantum field theory combines theories previously used in [88, 115]. We prove in Sec. 5.5:

Lemma 1. *The one-loop path integral of the heterotic σ -model on E_u with target space $CP^1 \times U(1)$ coupled to a gravitational and gauge background $V = \mathbb{C}^{12} \rightarrow U(1)$ is*

$$\mathbf{Z}_u = \sqrt{\text{Im } \tau} |\vartheta_4(\tau)|^2 \det^{12} \bar{\partial}_{L,u}. \quad (1.5)$$

Under variation of u , there are no local and global anomalies, and we give a canonical section trivializing of the RHS of Eq. (1.5).

We modify this action by varying the holomorphic structure and changing the regularization of the quantum theory. We obtain a family of two-dimensional supersymmetric theories on the torus E_u parameterized by $H^0(E_u, K^{\otimes 2})$ and $H^{(0,1)}(E_u)$ whose standard complex coordinates are denoted by α and β . We show in Sec. 5.5.1:

Proposition 5. *The one-loop amplitude of the heterotic σ -model on E_u with target space $CP^1 \times U(1)$ in the presence of the background fields α, β is*

$$\mathbf{Z}_u(\alpha, \beta) = \mathbf{Z}_u e^{\alpha u} \frac{\det \left(\bar{\partial} + \frac{\beta d\bar{z}}{\text{vol}(E_u)} \right)}{\det' \bar{\partial}} \frac{\det' \Delta}{\det' \left[-\partial \left(\bar{\partial} + \frac{\beta d\bar{z}}{\text{vol}(E_u)} \right) \right]}. \quad (1.6)$$

Eq. (1.6) is the integrand of the u -plane integral in Eq. (1.1) for $\alpha = p$ and $\beta = S$.

We have used the operator

$$\bar{D}(\check{z}) = \left(\bar{\partial} - \frac{\pi \check{z} d\bar{z}}{\text{vol}(E_u)} \right) : \Lambda^{1,0}(E_u) \rightarrow \Lambda^{1,1}(E_u)$$

and $\square(\check{z}) = *i\bar{D}(\check{z})\partial : \Lambda^{0,0}(E_u) \rightarrow \Lambda^{0,0}(E_u)$. We have considered the determinant line bundle for $\bar{D}(\check{z})$ and $\square(\check{z})$ over the Jacobian of E_u and the pullback on $H^{(0,1)}(E_u)$ with the complex coordinate \check{z} . We use the techniques in [99, 100] to evaluate the ratio $\det \bar{D}(\check{z}) / \det' \square(\check{z})$ in terms of modular functions. The introduction of β stems from the gauging of the $U(1)$ -isometry group of the σ -model on $CP^1 = SU(2)/U(1)$ and the coupling to an external field. The term $\exp(2\alpha u)$ corresponds to a change in the regularization of the one-loop amplitude [93].

We also have an interesting variation of this procedure: we couple the connection of the chiral fermion that gives a contribution $\det \bar{\partial}_{L,u}$ to an external magnetic field γ . We compute the change in the partition function in terms of modular functions in Sec. 5.5.3 and prove:

Proposition 6. *The one-loop amplitude of the heterotic σ -model on E_u with target space $CP^1 \times U(1)$ in the presence of the background fields α , β , and γ is*

$$\widetilde{\mathbf{Z}}_u(\alpha, \beta, \gamma) = \mathbf{Z}_u(\alpha, \beta) \frac{\det \left(\bar{\partial}_{L,u} + \frac{\gamma d\bar{z}}{\text{vol}(E_u)} \right)}{\det \bar{\partial}_{L,u}}. \quad (1.7)$$

Eq. (1.7) is the integrand of the u -plane integral for the blow-up $\widehat{CP^2}$ of CP^2 in one point for $\gamma = E$. For $H^2(\widehat{CP^2}; \mathbb{R}) = H^2(CP^2; \mathbb{R}) \oplus \mathbb{R}\zeta$ with $\zeta^2 = -1$, the homology class $\mathcal{E} \in H_2(M; \mathbb{Z})$ considered in the blow-up formula is related to E by $\int_{\mathcal{E}} \zeta = E$.

As explained in [96], the expressions $\mathbf{Z}_u(\alpha, \beta)$ and $\widetilde{\mathbf{Z}}_u(\alpha, \beta, \gamma)$ require us to integrate this path integral over $\text{Re } \tau$ from 0 to 4, and take the limit $\text{Im } \tau \rightarrow \infty$. The integration serves to impose the left-right mass constraint of the closed string. If one imposes the left-right mass constraint, the generating function becomes

$$\int_0^4 dx \mathbf{Z}_u(\alpha, \beta), \quad (1.8)$$

where $\tau = x + iy$. We expand the partition function in Eq. (1.6) in the form

$$\mathbf{Z}_u(\alpha, \beta) = \sum_{m,n=0}^{\infty} \frac{\alpha^m}{m!} \frac{\beta^{2n+1}}{n!} \text{UP}^{(m,n)}.$$

Each coefficient $\text{UP}^{(m,n)}$ has an asymptotic expansion in $q = \exp(2\pi i\tau)$ in the region $\text{Im } \tau \gg 1$ of the form $\text{UP}^{(m,n)} = \sum_{i,j \geq 0} \text{UP}_{(i,j)}^{(m,n)} q^i \bar{q}^j$. Let the diagonal be the sum of all the terms of the form $q^n \bar{q}^n$. The projection onto the diagonal terms is the integral in Eq. (1.8). After the projection we define for each term $f_{m,n}(y) = \int_0^4 dx \text{UP}_{m,n} = \sum_{l=0}^n 1/y^{l+\frac{3}{2}} \sum_{s \geq 1} a_s e^{-2\pi s^2 y}$ the associated Dirichlet series

$$\mathfrak{D}_{\Lambda_0}(\epsilon, f_{m,n}) = \frac{1}{\Gamma(\epsilon)} \int_{\Lambda_0}^{\infty} f_{m,n}(y) y^{\epsilon-1} dy.$$

The cut-off Λ_0 is introduced to only include the contributions from the cusp $u \rightarrow \infty$ while keeping away from $u \rightarrow \pm 1$. We will show in Sec. 5.6:

Proposition 7. *The regularization used by Moore and Witten in [25] in Eq. (1.1) coincides with the ζ -function regularization. Up to a constant factor we have*

$$\Phi_{CP^2}(p^m S^{2n+1}) = \lim_{\Lambda_0 \rightarrow 0} \mathfrak{D}'_{\Lambda_0}(0, f_{m,n}). \quad (1.9)$$

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Chapter 2

The $N = 1$ Yang-Mills theory

In this chapter, we review the universal bundle construction developed in [58, 40] with the exposition in [59]. We also state the wall-crossing formula derived in [4] and the blow-up formula derived in [5] for the Donaldson invariants. Finally, we give a description of the supersymmetric topological Yang-Mills theory constructed in [16, 26] whose expectation values compute the Donaldson invariants.

2.1 The universal bundle construction

Let M be a simply connected, compact oriented smooth four-manifold without boundary. Let $P \rightarrow M$ be a principal $SU(2)$ -bundle on M with principal right action. $SU(2)$ -bundles on M are classified by their second Chern class $c_2(P)$. Let \mathfrak{A} be the set of connections on P and \mathfrak{G} the group of gauge transformations of P . That is, $\phi \in \mathfrak{G}$ means (i) ϕ is a diffeomorphism of P , (ii) $\phi(pg) = \phi(p)g$ for any $g \in SU(2)$, and (iii) ϕ induces the identity map on M . The action of $\phi \in \mathfrak{G}$ on $A \in \mathfrak{A}$ is denoted by $\phi \cdot A$. Let r denote the fundamental representation of $SU(2)$ on \mathbb{C}^2 . The associated complex vector bundle $E = P \times_r \mathbb{C}^2 \rightarrow M$ has the first Chern class $c_1(E) = 0$. We also have the $SO(3)$ -bundle $\text{Ad } E = P \times_r \mathfrak{su}_2$ for the adjoint representation. We fix a Hermitian metric on E and a compatible unitary connection $d_A = d + A$ induced by the connection A on P . A fixed Riemannian metric g on M gives rise to a Hodge star operator $*$ with $*^2 = (-1)^p$. On p -forms, the adjoint operator is $d_A^* = -*d_A*$. A connection $A \in \mathfrak{A}$ is called anti-selfdual if and only if the curvature $F_A \in \Lambda_M^2(\text{Ad } E)$ on E satisfies $*F_A = -F_A$.

The connection A is called reducible if E reduces to the direct sum $L \oplus L^{-1}$ of a line bundle $L \rightarrow M$ and $A = a \oplus (-a)$ with a connection a on L . We denote by \mathfrak{A}_k the space of all hermitian connections on $E \rightarrow M$ with $c_1(E) = 0$, $k = c_2(E)[M]$. The subspace $\mathfrak{A}_k^* \subset \mathfrak{A}_k$ is the space of irreducible connections. The gauge group \mathfrak{G} acts freely on \mathfrak{A}^* . \mathfrak{G}_0 is the group of gauge transformations leaving a point of P fixed. The action of \mathfrak{G}_0 on \mathfrak{A} is free. We set $\mathfrak{B} := \mathfrak{A}/\mathfrak{G}$, $\mathfrak{B}^* := \mathfrak{A}^*/\mathfrak{G}$, and $\tilde{\mathfrak{B}} := \mathfrak{A}/\mathfrak{G}_0$, $\tilde{\mathfrak{B}}^* := \mathfrak{A}^*/\mathfrak{G}_0$.

The Pauli matrices $(i\sigma^a)_{a=1}^3$ form an anti-Hermitian basis of $\mathfrak{su}(2)$ with $\text{tr}(i\sigma^a)^2 = -2$. We write an Ad E -valued n -form η as $\eta^a(i\sigma^a)$. The adjoint action of $\mathfrak{su}(2)$ is real since

$$\text{ad}(i\sigma^a)(i\sigma^b) = [i\sigma^a; i\sigma^b] = -2\varepsilon^{ab}{}_c i\sigma^c .$$

The positive definite inner product on the Ad E -valued n -forms η_1, η_2 is

$$(\eta_1, \eta_2) = - \int_M \text{tr}(\eta_1 \wedge * \eta_2) .$$

The invariant polynomial $k_2(T) = -1/8\pi^2 \text{tr} T^2$ for $T \in \mathfrak{su}(2)$ determines the first Pontryagin class by

$$p_1(E)[M] = -\frac{1}{8\pi^2} \int_M \text{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2} \int_M \text{vol}_M \left(|F_A^+|^2 - |F_A^-|^2 \right) , \quad (2.1)$$

where $F_A = F_A^a{}_{\mu\nu} dx^\mu \wedge dx^\nu \otimes i\sigma^a$. Similarly, the invariant polynomial $1/2 k_1(T)^2 - k_2(T)$ for $T \in \mathfrak{su}(2)$ with $k_1(T) = i/(2\pi) \text{tr}(T)$ determines the second Chern class. For $SU(2)$, we have $k_1(T) = 0$ and thus $p_1(E) = -c_2(E) = -k$. For a skew-Hermitian matrix $T \in \mathfrak{u}(N)$ the identity

$$\text{tr}_{\text{Mat}(N, \mathbb{C})} \left(\text{ad} T \circ \text{ad} T \right) = 2N \text{tr}_{\mathbb{C}^N} T^2 - 2 \left(\text{tr}_{\mathbb{C}^N} T \right)^2$$

implies $c_2(\text{Ad } E) = 4 c_2(E)$. The action functional of the Yang-Mills theory is

$$\mathbf{S} = \int_M |F_A|^2 = \int_M \text{vol}_M \left(|F_A^+|^2 + |F_A^-|^2 \right) = - \int_M \text{vol}_M \text{tr} \left(F_A \wedge * F_A \right) , \quad (2.2)$$

and the Euler Lagrange equation is $d_A^* F_A = 0$. For $k > 0$, the Yang-Mills action is minimized by the anti-selfdual connections since

$$\mathbf{S} = 2 \|F_A^+\|^2 + 8\pi^2 k .$$

For $k < 0$, the Yang-Mills action is minimized by the selfdual connections since

$$\mathbf{S} = 2 \|F_A^-\|^2 - 8\pi^2 k .$$

The moduli space of anti-selfdual irreducible connections modulo gauge transformations is denoted by \mathfrak{M}_k^* . For a reducible $SU(2)$ -bundle E with reduction to $L \oplus L^{-1}$ we write the connection as $A = -a \otimes i\sigma^3$ and the curvature as $F_A = -f_a \otimes i\sigma^3$ where f_a is a real two form such that $c_1(L) = [\frac{1}{2\pi} f_a]$ and

$$c_2(E)[M] = -\frac{1}{4\pi^2} \int_M f_a \wedge f_a = -c_1(L)^2[M] .$$

There is also a reduction of Ad $E \cong L^{\otimes 2} \oplus \underline{\mathbb{R}}$ such that $c_2(\text{Ad } E)[M] = -c_1(L^{\otimes 2})^2[M]$.

The two bundles $L^{\otimes 2}$ and $(L^{-1})^{\otimes 2}$ are different as $SO(2)$ -bundles, but canonically equivalent as $O(2)$ -bundles by reversing the orientation. Reversing the orientation on $L^{\otimes 2}$ and \mathbb{R} is a $SU(2)$ -bundle automorphism. For $b_2^- = 0$, there is no reducible $SU(2)$ -bundle E with a reduction to $L \oplus L^{-1}$ and with an anti-selfdual connection other than the trivial bundle: assume otherwise that we have such a connection $A = -a \otimes i\sigma^3$ with the curvature $F_A = -f_a \otimes i\sigma^3$ where f_a is a real two-form. The Bianchi identity $d_A F_A = 0$ implies $df_a = 0$. Since F_A is anti-selfdual so is f_a . By the Hodge theorem f_a is a harmonic representative of a non-trivial class in $H^{2-}(M)$ if L is non-trivial, contradicting $b_2^- = 0$.

The Atiyah-Hitchin-Singer complex in dimension four [56] is the three term elliptic complex

$$\Lambda_M^0(\text{Ad } E) \xrightarrow{d_A} \Lambda_M^1(\text{Ad } E) \xrightarrow{\pi_+ d_A} \Lambda_M^{2+}(\text{Ad } E) \quad (2.3)$$

whose cohomology groups are H_A^\bullet . The map π_+ is the orthogonal projection on the self-dual two-forms. It is a complex if F_A is anti-selfdual. The associated elliptic operator of Laplace type on the even forms is Δ_A^{ev} , and on the odd forms Δ_A^{od} with

$$\begin{aligned} \Delta_A^{\text{ev}} &= d_A^* d_A \oplus \pi_+ d_A d_A^* \\ \Delta_A^{\text{od}} &= d_A^* \pi_+ d_A + d_A d_A^* . \end{aligned}$$

The operators Δ_A are Fredholm operators, and the index of the elliptic complex equals

$$\text{Ind } d_A = \dim \ker \Delta_A^{\text{od}} - \dim \ker \Delta_A^{\text{ev}} = 8k - 3(1 + b_2^+) .$$

From here on, we will always assume that b_2^+ is odd so that the index is even and equals $2d_k$. The zeroth cohomology H_A^0 is trivial if and only if A is irreducible. An irreducible connection A is called regular if $\text{coker } \pi_+ d_A = 0$. A local model of a neighborhood of $[A] \in \mathfrak{M}_{\text{regular}}^*$ is given by the intersection of the slice $\ker d_A^* \in T_A \mathfrak{A}^*$, which is locally orthogonal to pure gauge transformations, with $\ker(\pi_+ d_A)$, the linearization of $F_A^+ = 0$. For $[\Psi] \in H_A^1$ and an anti-selfdual, regular connection A this implies

$$d_A^* \Psi = 0 , \quad F_{A+\epsilon\Psi}^+ = O(\epsilon^2) ,$$

and $H_A^2 = 0$. For $b_2^+ > 0$ there are no reducible nor non-regular anti-selfdual connection on a dense subset of metrics [3]. Thus, if g is generic, \mathfrak{M}_k^* is a smooth manifold of dimension $2d_k = 8k - 3(1 + b_2^+)$ [3].

On $M \times \mathfrak{B}^*$ there exists a universal bundle with a universal connection [40, 59]: it is the complex vector bundle $\text{Ad } \mathfrak{E}$ with the structure group $SO(3)$ in the commutative diagram

$$\begin{array}{ccc} \text{Ad } E \times \mathfrak{A}^* & \longrightarrow & \text{Ad } \mathfrak{E} = (\text{Ad } E \times \mathfrak{A}^*)/\mathfrak{G} \\ \downarrow & & \downarrow \\ M \times \mathfrak{A}^* & \longrightarrow & M \times \mathfrak{B}^* \end{array}$$

where \mathfrak{G} acts on $\text{Ad } E \times \mathfrak{A}^*$ by $(v, A) \mapsto (\phi(v), \phi \cdot A)$ and $SU(2)$ acts by $(v, A) \mapsto (\text{ad}(g)v, A)$. The actions of \mathfrak{G} and $SU(2)$ on $P \times \mathfrak{A}^*$ commute. With $m = \pi(v)$ and $X_m \in T_m M$ and $\tau_A \in T_A \mathfrak{A}^*$, a \mathfrak{G} -invariant connection w on $\text{Ad } E \times \mathfrak{A}^* \rightarrow M \times \mathfrak{A}^*$ is given by

$$w_{(g,A)}(X_m, \tau_A) = A(X_m) + \alpha_A(\tau_A)(m),$$

where α is the canonical connection on $\mathfrak{A}^* \rightarrow \mathfrak{B}^*$ given by

$$\alpha_A(\tau_A) = \left(d_A^* d_A\right)^{-1} d_A^*(\tau_A).$$

The connection w and its curvature \mathfrak{F} descend to a connection and curvature on the universal bundle with components

$$\begin{aligned} \mathfrak{F}_{(m,A)}^{2,0}(X_m, Y_m) &= (F_A)_m(X_m, Y_m), \\ \mathfrak{F}_{(m,A)}^{1,1}(\tau_A, X_m) &= \left(\tau_A^{\text{hor}}\right)_m(X_m), \\ \mathfrak{F}_{(m,A)}^{0,2}(\tau_A, \sigma_A) &= -\mathcal{G}_A[\tau_A^{\text{hor}}; * \sigma_A^{\text{hor}}] \end{aligned}$$

where

$$\mathcal{G}_A = \left(d_A^* d_A\right)^{-1}, \quad \tau_A^{\text{hor}} = \left(\tau_A - d_A \alpha_A(\tau_A)\right).$$

We can interpret $A \mapsto \delta A(\tau_A) = \tau_A^{\text{hor}}$ as the exterior covariant derivative acting on the \mathfrak{G} -equivariant form $\tau_A \in \Lambda_{\mathfrak{A}^*}^1$. Since \mathfrak{G} acts freely on \mathfrak{A}^* , it follows that

$$H_{\mathfrak{G}}^*(\mathfrak{A}^*) = H^*(\mathfrak{B}^*).$$

In physics, the component $\mathfrak{F}^{1,1}$ is denoted by Ψ , and $\mathfrak{F}^{0,2} = -\Phi$, such that

$$\delta A = \Psi, \quad \delta \Psi = -d_A \Phi, \quad \delta \Phi = 0. \quad (2.4)$$

The second Chern class of the universal bundle is

$$c_2(\text{Ad } \mathfrak{E}) = \frac{1}{8\pi^2} \text{tr } \mathfrak{F}^2 = \sum_{r=0}^4 \mathfrak{W}^{r,4-r} \quad (2.5)$$

with $\mathfrak{W}^{r,4-r} \in \Lambda_{M \times \mathfrak{B}^*, \mathbb{C}}^{r,4-r}$. Since $c_2(\text{Ad } \mathfrak{E})$ is a closed form, the descent equations

$$d\mathfrak{W}^{r,4-r} = -\delta \mathfrak{W}^{r+1,3-r} \quad (2.6)$$

follow. The μ -map is defined by the slant-product $\mu(\bullet) = c_2(\text{Ad } \mathfrak{E})/\bullet$ [2]. On 2-cycles

and 0-cycles the μ -map gives

$$\begin{aligned}
\mu : H_2(M) &\rightarrow H^2(\mathfrak{B}^*) \\
[\Sigma] &\mapsto W_\Sigma^{2,2} = \int_\Sigma \mathfrak{W}^{2,2} \\
&= \frac{1}{8\pi^2} \int_\Sigma \text{tr} \left(2 \mathfrak{F}^{2,0} \wedge \mathfrak{F}^{0,2} + \mathfrak{F}^{1,1} \wedge \mathfrak{F}^{1,1} \right) \\
&= \frac{1}{8\pi^2} \int_\Sigma \text{tr} \left(-2 \Phi F_A + \Psi \wedge \Psi \right) \\
\mu : H_0(M) &\rightarrow H^4(\mathfrak{B}^*) \\
[p] &\mapsto W_p^{0,4} = \mathfrak{W}^{0,4}(p) \\
&= \frac{1}{8\pi^2} \text{tr} \Phi^2(p).
\end{aligned} \tag{2.7}$$

Classes of the de Rham cohomology of \mathfrak{M}^* can be obtained from $H^\bullet(\mathfrak{B}^*)$ by restriction which we will still denote by $\mu(\Sigma)$ and $\mu(p)$. Moreover, any element of $H^\bullet(\mathfrak{M}^*)$ is induced from $H^\bullet(\mathfrak{B}^*)$ in this way [2]. For the universal $SO(3)$ -bundle $\text{Ad } \mathfrak{E}$ there is a lift, i.e., a well-defined $SU(2)$ -bundle \mathfrak{E} , if and only if $w_2(\text{Ad } \mathfrak{E}) = 0$.

2.2 The Donaldson invariants

To define the integration over the moduli space a compactification of the moduli space of anti-selfdual instantons is needed. The Uhlenbeck compactification is described by the following theorem due to Uhlenbeck [8, Thm. 2.4]:

Theorem 1. *Let $(A_i)_{(i \in \mathbb{N})}$ be a sequence in \mathfrak{M}_k^* . After passing through a subsequence there is a finite collection of points $m_1, \dots, m_l \in M$ with multiplicities $n_1, \dots, n_l > 0$ such that up to gauge transformation $A_i|_{M - \{m_1, \dots, m_l\}}$ converges to an anti-selfdual connection A_∞ . The connection A_∞ can be extended to an anti-selfdual connection on a bundle $E' \rightarrow M$ with $c_1(E') = 0$ and $c_2(E') = k + \sum_{i=1}^l n_i$. Therefore, there exists a topology on*

$$\prod_{n \geq 0} \mathfrak{M}_{k-n}^* \times (M^n / \Sigma_n)$$

such that the closure $\overline{\mathfrak{M}_k^*}$ is compact and called Uhlenbeck compactification.

It can be shown [2, 103] that the μ -map extends over the Uhlenbeck compactification $\overline{\mathfrak{M}_k^*}$, and for a generic metric it has a fundamental class in the stable range $8k > 3(1 + b_2^+)$. Let $2d_k = 4n_1 + 2n_2$, $\Sigma_1, \dots, \Sigma_{n_2} \in H_2(M)$, and $p \in H_0(M)$. The Donaldson invariant is defined by

$$\Phi_{n_1, n_2}^{M, g, k}(p; \Sigma_1, \dots, \Sigma_{n_2}) := \int_{\overline{\mathfrak{M}_k^*}} \mu(p)^{n_1} \wedge \mu(\Sigma_1) \wedge \dots \wedge \mu(\Sigma_{n_2}). \tag{2.8}$$

This map can be extended to a linear function

$$\Phi^{M, g} : \mathbf{A}(M) \rightarrow \mathbb{Q}$$

where

$$\mathbf{A}(M) = \text{Sym}_* \left(H_0(M) \oplus H_2(M) \right)$$

is viewed as a graded algebra where the elements of $H_i(M)$ have degree $(4 - i)/2$. The dependence of the Donaldson invariants on the choice of the metric is captured by the following theorem [8, Thm. 2.6]:

Theorem 2.

1. If $b_2^+ > 1$, then $\Phi_{n_1, n_2}^{M, g}$ is independent of the metric g .
2. If $b_2^+ = 1$, then $\Phi_{n_1, n_2}^{M, g}$ depends only on the chamber of the period point $\omega(g)$, i.e., the point in $H^{2+}(M)/\mathbb{R}^+$ defined by the one-dimensional subspace of g -selfdual harmonic two forms.

We have assumed that we have picked an orientation of $H^{2+}(M)$. Fixing a two-cycle Σ , we can form the generating function

$$\mathbf{Z}(p, \Sigma) = \mathbf{A} \left(e^{p+\Sigma} \right) = \sum_{k>0} \sum_{\substack{n_1, n_2 \geq 0 \\ d_k = 2n_1 + n_2}} \Phi_{n_1, n_2}^{M, g, k}(p; \Sigma) \frac{p^{n_1} \Sigma^{n_2}}{n_1! n_2!}, \quad (2.9)$$

which is independent of the metric if $b_2^+ > 1$, and piecewise constant as a function of g if $b_2^+ = 1$.

2.3 The computation of the Donaldson invariants

In the case $b_2^+ = 1$ the Donaldson invariants depend on the metric g . They depend on the metric through a system of chambers and walls in the positive cone of $H^2(M; \mathbb{R})$. We now restrict ourselves to algebraic surfaces so that for the Kähler metric g there is the Kähler form ω of type $(1, 1)$ and a line bundle with divisor H whose first Chern class is the given Kähler form ω . The change of the metric corresponds to a change of the divisor H .

2.3.1 Walls and chambers

Let $H^2(M)^+$ be the set of all θ with $\theta^2 > 0$. It has two connected components since the intersection form is of type $(1, b_2^-)$. A homology orientation is exactly a choice of one of the components which we call Ω^+ . The period point $\omega(g)$ of the metric g is a point in Ω^+ given by the g -selfdual closed two forms. We fix $k = -p_1(P)$. For $2d_k = 8k - 6 \in \mathbb{Z}_{\geq 0}$, an element $\zeta \in H^2(M; \mathbb{Z})$ is called of type (d_k) if $-k \leq \zeta^2 < 0$. We call $\mathcal{W}_\zeta = \zeta^\perp$ in Ω^+ the wall of type (d_k) . The chambers of type (d_k) are the connected components of Ω^+ after removing all walls of type (d_k) . At a wall, the $SU(2)$ -bundle $E \rightarrow M$ reduces to $L \oplus L^{-1}$ with $c_1(L) = \zeta$ since $k = c_2(P) = -c_1(L)^2 = -\zeta$, and the metric admits a reducible anti-selfdual connection since $\int_M c_1(L) \wedge \omega(g) = 0$.

The dependence of the Donaldson invariants on the period point is described by the following theorem due to Kotschick and Morgan [4]:

Theorem 3.

1. $\Phi_{(d_k)}^{M,g}$ depends only on the chamber of $\omega(g)$.
2. For all ζ of type (d_k) there exists a linear map $\delta_{\zeta, d_k}^M : \mathbf{A}_{d_k}(M) \rightarrow \mathbb{C}$ such that

$$\Phi_{(d_k)}^{M, g_1} - \Phi_{(d_k)}^{M, g_2} = \sum_{\zeta \cdot \omega(g_1) < 0 < \zeta \cdot \omega(g_2)} \delta_{\zeta, d_k}^M .$$

2.3.2 The blow-up formula

The blow-up formulas relate the Donaldson invariants of M to those of $\widehat{M} = M \# \overline{CP^2}$, the blow-up of M in one point. We need to choose a metric on \widehat{M} very close to the metric on the pullback on M to apply the blow-up formulas. Since $H^2(\widehat{M}; \mathbb{R}) = H^2(M; \mathbb{R}) \oplus \mathbb{R} \zeta$ where ζ is the exceptional divisor of the blow-up with $\zeta^2 = -1$, we can identify $H^2(M; \mathbb{R})$ with the elements orthogonal to ζ . If H is a representative of the period point of the metric g on M , then $\widehat{H} = H - \epsilon \text{PD}(\zeta)$ is the period point on \widehat{M} . We write

$$\Phi_{(d_k)}^{M, H} = \Phi_{(d_k)}^{M, g} , \quad \Phi_{(d_k)}^{\widehat{M}, H} = \Phi_{(d_k)}^{\widehat{M}, \widehat{H}} ,$$

where ϵ is sufficiently small so that the result is independent of ϵ . A result due to Fintushel and Stern [5] gives the general relation between the Donaldson invariants on M and \widehat{M} . In the simplest case this leads to the following theorem:

Theorem 4. For $a \in H_2(M)$ and $\mathcal{E} \in \langle \text{PD}(\zeta) \rangle \subset H_2(\widehat{M}; \mathbb{Z})$ it follows

$$\Phi_{(d_k)}^{\widehat{M}, H}(a^{d_k}) = \Phi_{(d_k)}^{M, H}(a^{d_k}) , \quad \Phi_{(d_k)}^{\widehat{M}, H}(\mathcal{E}^2 a^{d_k}) = 0 .$$

Fintushel and Stern [5, Thm. 4.1] then derived the blow-up formula:

Theorem 5. For $z \in \mathbf{A}(M)$ and $\mathcal{E} \in \langle \text{PD}(\zeta) \rangle \subset H_2(\widehat{M}; \mathbb{Z})$ it follows

$$\begin{aligned} \Phi_{(d_k)}^{\widehat{M}, H}(e^{\mathcal{E}} z) &= \sum_{n=0}^{\infty} \frac{\mathcal{E}^n}{n!} \Phi_{(d_k)}^{\widehat{M}, H}(\mathcal{E}^n z) \\ &= \Phi_{(d_k)}^{M, H} \left(e^{-\frac{t^2 p}{6}} \sigma_3(p, t) z \right) \Big|_{t=\mathcal{E}} = \sum_{n=0}^{\infty} \frac{\mathcal{E}^n}{n!} \Phi_{(d_k)}^{M, H} \left(\text{Coeff}_{t^n} \left[e^{-\frac{t^2 p}{6}} \sigma_3(p, t) \right] z \right) . \end{aligned} \quad (2.10)$$

$\sigma_3(u, z)$ is a particular quasi-periodic Weierstrass σ -function associated to the Weierstrass function $\mathcal{P}(z|g_2, g_3)$ and is defined in Sec. 5.1. We will show in Sec. 5 that the blow-up function of Fintushel and Stern arises in a natural way in terms of the stationary phase approximation of a simple quantum field theory defined on an elliptic curve at the boundary of the Coulomb branch.

2.3.3 The results for CP^2

With the help of a vanishing result for rational ruled surface [8, Lemma 5.3] as a starting point, Ellingsrud and Göttsche [6] were able to obtain the Donaldson invariants for CP^2 of degree smaller or equal to 50. Since there are no walls for CP^2 the strategy is to consider the blow up of CP^2 in a point and then to crossover to a chamber where the invariants vanish by the vanishing result. The wall-crossing formula and the blow-up formula account for the changes made during the blow-up and the wall-crossing. Thus, the total changes are the negative of the Donaldson invariants for CP^2 [8, Cor. 5.5.]. In particular, they found the first contributions listed in the following theorem:

Theorem 6.

(k, d_k)	$\mathbf{Z}(p, S) =$
(1, 1)	$-\frac{3}{2}S$
(2, 5)	$S^5 - pS^3 - \frac{13}{8}p^2S$
(3, 9)	$3S^9 + \frac{15}{4}pS^7 - \frac{11}{16}p^2S^5 - \frac{141}{64}p^3S^3 - \frac{879}{256}p^4S$
(4, 13)	$54S^{13} + 24pS^{11} + \frac{159}{8}p^2S^9 + \frac{51}{16}p^3S^7 - \frac{459}{128}p^4S^5 - \frac{1515}{256}p^5S^3 - \frac{36675}{4096}p^6S$
(5, 17)	$2540S^{17} + 694pS^{15} + \frac{487}{2}p^2S^{13} + \frac{2251}{16}p^3S^{11} + \frac{2711}{64}p^4S^9 - \frac{5}{16}p^5S^7 - \frac{3355}{256}p^6S^5 - \frac{143725}{8192}p^7S^3 - \frac{850265}{32768}p^8S$
(6, 21)	$233208S^{21} + 45912pS^{19} + 10625p^2S^{17} + 3036p^3S^{15} + \frac{41103}{32}p^4S^{13} + \frac{1741}{4}p^5S^{11} + \frac{5619}{64}p^6S^9 - \frac{20379}{1024}p^7S^7 - \frac{754141}{16384}p^8S^5 - \frac{904239}{16384}p^9S^3 - \frac{10504593}{131072}p^{10}S$
\vdots	\vdots

Göttsche also computed a closed form for the generating function of the Donaldson invariants in the case of simply connected manifolds with $b_2^+ = 1$ [7].

2.4 The topological field theory

In this section, we show that the generating function in Eq. (2.9) is the partition function with source-terms of a topological quantum field theory. To do so, we will rely on the interpretation of the topological Yang-Mills theory given by Baulieu and Singer [26] which includes the gauge fixing of the Yang-Mills symmetry in the topological field theory. The ghost spectrum is essential for our understanding of the partition function in terms of determinant line bundles, as it will give additional factors of determinants.

The classical topological action for $N = 1$ Yang-Mills theory is the characteristic class $p_1(\text{Ad } E)[M]$ in Eq. (2.1). Being a topological invariant, it possesses a gauge symmetry which is larger than the ordinary Yang-Mills symmetry. It is invariant under all transformations of the bundle $E \rightarrow M$ which lift a diffeomorphism on M . The gauge potential and the curvature then transform in local coordinates according

to

$$A \rightarrow A + d_A \widehat{\epsilon} + \epsilon, \quad F_A = d_A \epsilon - [\widehat{\epsilon}; F_A], \quad (2.11)$$

where $\widehat{\epsilon} \in C^\infty(M, \text{Ad } E)$, $\epsilon \in \Lambda_M^1(\text{Ad } E)$. It was shown in [26] that the full action of the topological field theory can be obtained through BRST-invariant quantization of $p_1(E)[M]$. This means that the full action is

$$\mathbf{S} = p_1(\text{Ad } E)[M] + C_1 \int_M \overline{\mathcal{Q}} \mathbf{V} \text{ vol}_M, \quad (2.12)$$

where $\overline{\mathcal{Q}}$ is the nilpotent fermionic BRST-operator with $(d + \overline{\mathcal{Q}})^2 = 0$, C_1 is a constant, and \mathbf{V} will be determined below. The first term in the action is metric independent whereas the second term introduces a dependence on g on M . A standard result in physics says that the physical theory must be independent of the $\overline{\mathcal{Q}}$ -exact term. Thus, the generating function $\mathbf{Z}(p, \Sigma)$ in Eq. (2.16) is a topological invariant [26, Sec. 4].

2.4.1 The field content

The fact that the parameters in Eq. (2.11) have their own gauge transformation makes the gauge fixing problem more complicated. Therefore, more ghost fields than the usual BRST-ghosts are required. The full set of fields in the topological action can be displayed as follows (cf. [26, Eq. (15)]):

bosonic	A	F_A			
ghost	C	\overline{C}	Ψ	\mathcal{X}	
bosonic	\widehat{b}	Φ	b	$\overline{\Phi}$	
ghost				H	

The bosonic fields are $\widehat{b} \in C^\infty(M, \text{Ad } E)$ and $\Phi \in C^\infty(M, \text{Ad}_{\mathbb{C}} E)$, $b \in \Lambda_M^{2+}(\text{Ad } E)$. The ghost fields are Grassmann-algebra valued forms. Their corresponding commuting forms are denoted by ‘ \rightarrow ’. They are

$$\begin{aligned} C &\rightarrow C^\infty(M, \text{Ad}_{\mathbb{C}} E), & H &\rightarrow C^\infty(M, \text{Ad } E), \\ \Psi &\rightarrow \Lambda_M^1(\text{Ad } E), & \mathcal{X} &\rightarrow \Lambda_M^{2+}(\text{Ad } E). \end{aligned}$$

We have the following BRST-transformation rules:

$$\begin{aligned} \overline{\mathcal{Q}} A &= -d_A C + \Psi, & \overline{\mathcal{Q}} \Psi &= -d_A \Phi - [C, \Psi], & \overline{\mathcal{Q}} \Phi &= -[C, \Phi], \\ \overline{\mathcal{Q}} F_A &= d_A \Psi - [C, F_A], \\ \overline{\mathcal{Q}} C &= -\frac{1}{2}[C, C] + \Phi, \\ \overline{\mathcal{Q}} \overline{C} &= \widehat{b}, & \overline{\mathcal{Q}} \widehat{b} &= 0, \\ \overline{\mathcal{Q}} \mathcal{X} &= b, & \overline{\mathcal{Q}} b &= 0, \\ \overline{\mathcal{Q}} \overline{\Phi} &= H, & \overline{\mathcal{Q}} H &= 0. \end{aligned}$$

If we set the ghost C that fixes the ordinary Yang-Mills symmetry to zero, we obtain the transformation law of the basic multiplet (A, Ψ, Φ) in Eq. (2.4).

2.4.2 The observables

The operator $d + \overline{\mathcal{Q}}$ is exterior differential on $M \times \mathfrak{A}^*/\mathfrak{G}$ with $(d + \overline{\mathcal{Q}})^2 = 0$. $A + C$ is the canonical connection on the universal bundle defined in Sec. 2.1 with the curvature

$$\mathfrak{F} = F_A + \Psi + \Phi.$$

The observables are cohomology classes of $H^*(M \times \mathfrak{A}^*/\mathfrak{G}) = H^*(M) \otimes H^*(\mathfrak{A}^*/\mathfrak{G})$. The classes can be interpreted as local functions of the fields which are $\overline{\mathcal{Q}}$ -invariant modulo d -exact terms. Given any invariant symmetrical exterior polynomial \mathcal{P} of degree l it follows from $(d_{A+C} + \overline{\mathcal{Q}})\mathfrak{F} = 0$ that $(d + \overline{\mathcal{Q}})\mathcal{P}(\mathfrak{F}) = 0$. The expansion in the form-degree r on M and $2l - r$ on $\mathfrak{A}^*/\mathfrak{G}$ shows that for

$$\mathcal{P}(\mathfrak{F}) = \sum_{r=0}^{2l} \mathcal{P}(\mathfrak{F})^{r, 2l-r}$$

the descent equations

$$\overline{\mathcal{Q}}\mathcal{P}(\mathfrak{F})^{r, 2l-r} + d\mathcal{P}(\mathfrak{F})^{r+1, 2l-r-1} = 0$$

are satisfied. Evaluating $\mathcal{P}(\mathfrak{F})^{r, 2l-r}$ on an r -cycle of M gives the component in $H^{2l-r}(\mathfrak{A}^*/\mathfrak{G})$. This is the BRST-interpretation of Eq. (2.6). One can check that in the presence of the ghost-field C the components of $\text{tr } \mathfrak{F}^2$, in particular $W_p^{0,4}$ and $W_\Sigma^{2,2}$ remain unchanged. It follows that

$$W_p^{4,0} = \frac{1}{8\pi^2} \text{tr}_{\text{Mat}(2, \mathbb{C})} \Phi^2(p), \quad (2.13a)$$

$$W_\Sigma^{2,2} = \frac{1}{8\pi^2} \int_\Sigma \text{tr}_{\text{Mat}(2, \mathbb{C})} \left(-2\Phi F_A + \Psi \wedge \Psi \right) \quad (2.13b)$$

remain the same as in Eq. (2.5). The de Rham cohomology classes $\mu(\bullet)$ of \mathfrak{B}^* in Eq. (2.7) are the expectation values of gauge-invariant observables in the topological Yang-Mills theory [16, 17].

2.4.3 The partition function and the generating functional

The partition function for the topological action is formally defined as the path integral

$$\mathbf{Z} = \int \underbrace{[\mathcal{D}A \mathcal{D}b \mathcal{D}\hat{b} \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \mathcal{D}\chi \mathcal{D}\Psi \mathcal{D}H \mathcal{D}C \mathcal{D}\bar{C}]}_{=\mathcal{D}\mathbf{X}} e^{-\mathbf{S}}. \quad (2.14)$$

We have fixed a reference connection A_0 on the bundle $E \rightarrow M$ such that F_{A_0} is anti-selfdual. We define an honest one-form by $B = A - A_0$ and set

$$\mathbf{V} = -\text{tr} \left\{ - * \left[\mathcal{X} \wedge \left(F_A^+ + \frac{1}{2} \rho b \right) \right] + \bar{\Phi} d_A^* \Psi + \bar{C} \left(-d_{A_0}^* B - \frac{1}{2} \widehat{b} \right) \right\} \text{vol}_M, \quad (2.15)$$

where ρ is a real parameter. The full action (2.12) was computed in [26, Eq. (18) and Eq. (22)]. \mathbf{V} is written for the adjoint representation as we are considering the bundle $\text{Ad } E \rightarrow M$. The simplest topological invariant is the partition function itself. It vanishes unless $8k = 3(1 + b_2^+)$ [16].

The correlation functions are the integral representation of the Donaldson invariants in Eq. (2.8). In order to compute the generating function, one introduces source-terms and the generating function

$$\int [\mathcal{D}\mathbf{X}] e^{-\mathbf{S} + p W_p^{0,4} + \Sigma W_\Sigma^{2,2}}.$$

The following theorem is due to Witten [16]:

Theorem 7. *The generating function for the Donaldson invariants on M is equal to the generating function for the correlation functions of the supersymmetric, topological $SU(2)$ -Yang-Mills-theory on M :*

$$\mathbf{Z}(p, \Sigma) = \int [\mathcal{D}\mathbf{X}] e^{-\mathbf{S} + p W_p^{0,4} + \Sigma W_\Sigma^{2,2}}. \quad (2.16)$$

Witten [16] showed that in the evaluation of correlation functions, the path integral is localized to the integral over the BRST-fixed-point set. Thus, the path integral can be computed in the semi-classical limit by carrying out a quadratic expansion about the critical points of the action functional. In [26, Sec. 5] it was shown that the BRST-fixed-point set, i.e., every stationary point of the action \mathbf{S} , constitutes a universal gauge connection plus a tangent vector on the universal bundle described in Sec. 2.1.

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Chapter 3

The geometry of the Seiberg-Witten curve

In this chapter, we investigate the geometry of the Coulomb branch as it is fundamental for the definition and the understanding of the low-energy effective field theory. We will first consider the family of elliptic curves introduced by Seiberg and Witten in [19, 20]. In particular, we will be concerned with the rational elliptic surface constructed by Seiberg and Witten in the context of the $N = 2$ supersymmetric pure $SU(2)$ -Yang-Mills theory.

The description of the Coulomb branch will be given in terms of special functions using the results in [19, 20, 23, 25, 22, 30]. The purpose of Sec. 3.1 through Sec. 3.6 is to develop a consistent notation and put the computations in the context of a rational Weierstrass surface $Z \rightarrow \text{UP}$. In Sec. 3.7 and Sec. 3.8, we will turn to the determinant line bundle for the vertical $\bar{\partial}$ -operator coupled to a holomorphic line bundle on the elliptic surface Z . In Sec. 5, we will use the vertical $\bar{\partial}$ -operator in the definition of the quantum field theory on the elliptic curve at the boundary. In Sec. 3.8, we use the results of Quillen [77], Bismut and Bost [82, 83], Seeley and Singer [81], and Atiyah [80] to show that the determinant line bundle extends across the nodal fibers and to compute its local and global anomaly.

In Sec. 3.9 and Sec. 3.10, we show that a BPS-state in physics defines a $SU(2)/\mathbb{Z}_2$ -bundle $V \rightarrow Z$ together with an Hermitian Yang-Mills connection on the elliptic surface. We use the description of these bundles given in [69, 66, 70]. In Sec. 3.9, we determine the fractional instanton number of the BPS-states by computing the global anomaly of the $\bar{\partial}$ -operator coupled to V . In Sec. 3.10, we show that the construction of the bundle V on the rational surface Z is equivalent to the geometric quantization of the BPS-states.

3.1 The rational Jacobian elliptic surface

In this section, we discuss the family of elliptic curves which is called the Seiberg-Witten curve in physics. Every elliptic curve E can be written in Weierstrass form

as

$$y^2 = x^3 + Ax + B, \quad (3.1)$$

where A and B are numbers such that the discriminant $\Delta(A, B) = 4A^3 + 27B^2$ does not vanish. In the homogeneous coordinates $[X : Y : W]$, Eq. (3.1) becomes

$$WY^2 = X^3 + AXW^2 + BW^3.$$

One can check that the point P with coordinates $[0 : 1 : 0]$ is always a smooth point of the curve. We consider it the base point of the elliptic curve (cf. [69]). P also serves as the origin of the group law on E [65]. The two types of singularities that can occur as Weierstrass cubics are a rational curve with a single node, which appears when the discriminant vanishes, or a cubic cusp when $A = B = 0$.

We study a family of cubic curves, i.e., a Jacobian elliptic surface, parameterized by the base space CP^1 . We fix a line bundle $\mathcal{N} \rightarrow CP^1$. A and B are then promoted to global sections of $\mathcal{N}^{\otimes 4}$ and $\mathcal{N}^{\otimes 6}$ respectively. The discriminant is a section of $\mathcal{N}^{\otimes 12}$. If these sections are generic enough so that they do not always lie in the discriminant locus, we obtain an elliptic fibration $\pi : Z \rightarrow CP^1$ [69]. Each fiber E comes equipped with the base point P , which defines a section S of the elliptic fibration. The bundle \mathcal{N} is the conormal bundle of the section S , i.e., $\pi_*(\nu_S) = \mathcal{N}^{-1}$ [69]. Concretely, we consider the Weierstrass fibration with A and B given by the polynomials

$$A([u : 1]) = -\frac{u^2}{3} + \frac{1}{4}, \quad B([u : 1]) = \frac{u}{12} - \frac{2}{27}u^3,$$

in the coordinate chart $[u : 1] \in CP^1$. Then, Z describes a rational elliptic surface [65, Sec. 4.7]. The discriminant is $\Delta = (u - 1)(u + 1)/4096$. The j -invariant of the elliptic fiber E_u is

$$j(E_u) = \frac{2^8 3^3 A^3}{\Delta} = 64 \frac{(4u^2 - 3)^3}{(u - 1)(u + 1)}. \quad (3.2)$$

Since two elliptic curves are isomorphic if and only if they have the same j -invariant, it follows from Eq. (3.2) that the fibers over u and $-u$ are isomorphic. The isomorphism is given by

$$u \leftrightarrow u' = -u, \quad \begin{aligned} (x, y) &\leftrightarrow (x', y') = (-x, iy) \\ (A, B) &\leftrightarrow (A', B') = (A, -B) \end{aligned}. \quad (3.3)$$

The j -invariant is a function on the base space CP^1 . For the singular fibers at $u = \pm 1$ and $u = \infty$, j has a pole. The local behavior of j is determined by the order of A , B , and Δ . However, the knowledge of the j -function does not determine the elliptic surface. In addition, information about the monodromy is needed which we will discuss in Sec. 3.3.

For $u \rightarrow \pm 1$, the sections A, B do not vanish, whereas Δ has a zero and j a pole of order one. From Kodaira's classification of singular fibers [65], it follows that the

elliptic fibration develops a node at $u = \pm 1$, more specifically a singular fiber of type I_1 . In each smooth fiber, we obtain the following points with $y = 0$, which are the points of order two for the group law of the elliptic curve:

$$x = \frac{u}{6} \pm \frac{\sqrt{u^2 - 1}}{2}, \quad -\frac{u}{3}, \quad y = 0. \quad (3.4)$$

Viewed as elements of the Jacobian torus they correspond to the even Spin-structures, which are labeled (A, A) , (A, P) , (P, A) in physics. We observe that under the diffeomorphism in Eq. (3.3) the point with $x = -\frac{u}{3}$ is invariant, but the other two are interchanged. We will follow the physics language [41, Sec. 4] and call the Spin-structure with $x = -\frac{u}{3}$ the Spin-structure (P, A) . We will explain later that the invariance under the diffeomorphism in Eq. (3.3) is equivalent to the invariance under $\Gamma_0(4)$. The point at infinity corresponds to the identity element of the group law or the odd spin structure (P, P) .

The group law of the elliptic curve determines that adding the order-two points with $x = \frac{u}{6} + \frac{\sqrt{u^2 - 1}}{2}$ and $x = \frac{u}{6} - \frac{\sqrt{u^2 - 1}}{2}$ gives the order-two point with $x = -\frac{u}{3}$

$$(P, A) = (A, P) + (A, A),$$

and

$$(P, P) = (A, P) + (A, A) + (P, A).$$

The second chart of the base space is $[1 : v] \in CP^1$, and the intersection of the two charts is given by $u = 1/v$. For non-zero v we can always replace x and y by xv^2 and yv^3 respectively. The sections A, B then have the following form

$$A([1 : v]) = -\frac{v^2}{3} + \frac{v^4}{4}, \quad B([1 : v]) = \frac{v^5}{12} - \frac{2v^3}{27}.$$

The discriminant is $\Delta = 1/16 v^{10}(v - 1)(v + 1)$. The order-two points are given by

$$x = v \left(\frac{1}{6} \pm \frac{\sqrt{1 - v^2}}{2} \right), \quad -\frac{v}{3} \quad y = 0.$$

For $v \rightarrow 0$, the elliptic curve in the fiber develops a cusp. Writing $A = A'v^2$ and $B = B'v^3$ leaves the j -function invariant, i.e.,

$$j = \frac{2^8 3^3 A^3}{\Delta} = \frac{2^8 3^3 A'^3}{\Delta'}.$$

For $v \rightarrow 0$, the sections A', B' do not vanish, whereas $\Delta' = 1/16 v^4(v - 1)(v + 1)$ has a zero of order four. For

$$y^2 = x^3 + A'x + B',$$

we obtain the following points of order two:

$$x = \frac{1}{6} \pm \frac{\sqrt{1-v^2}}{2}, \quad -\frac{1}{3} \quad y = 0. \quad (3.5)$$

Going from A, B, Δ to A', B', Δ' is called the quadratic twist. After the quadratic twist we have a singularity of type I_4 at $v = 0$. From Kodaira's classification theorem [65, Sec. 4.2], it follows that the elliptic fibration before the quadratic twist has had an I_4^* -singularity at $v = 0$. This proves the following lemma:

Lemma 2. *The family of elliptic curves considered by Seiberg and Witten is a Jacobian rational elliptic surface $\pi : Z \rightarrow CP^1$. In particular, the elliptic fibration has singular fibers of type I_1 (nodal curves) at $u = \pm 1$, and a fiber of type I_4^* at $u = \infty$.*

We will call the punctured base curve the u -plane $UP = CP^1 - \{-1, 1, \infty\}$.

3.2 The Seiberg-Witten differential

In this section, we will describe an explicit parameterization of the family of elliptic curves in Eq. (3.1) in terms of the Weierstrass elliptic function. We will obtain an explicit formula for the periods of the family of elliptic curves by integrating the holomorphic differential over an ordered basis of $H_1(E_u)$.

If we start with the family of elliptic curves given by

$$\eta^2 = \mathfrak{x}^3 - u\mathfrak{x}^2 + \frac{\mathfrak{x}}{4} = \mathfrak{x} \left(\mathfrak{x}^2 - u\mathfrak{x} + \frac{1}{4} \right),$$

the substitution $y = \eta$ and $x + \frac{u}{3} = \mathfrak{x}$ leads to Eq. (3.1)

$$y^2 = x^3 + A(u)x + B(u),$$

with

$$A(u) = -\frac{u^2}{3} + \frac{1}{4}, \quad B(u) = \frac{u}{12} - \frac{2}{27}u^3.$$

The substitution $4\mathbf{y} = y$ and $4\mathbf{x} = x$ converts the equation into the Weierstrass normal form

$$\mathbf{y}^2 = 4\mathbf{x}^3 - g_2(u)\mathbf{x} - g_3(u), \quad (3.6)$$

where

$$g_2(u) = \frac{u^2}{12} - \frac{1}{16}, \quad g_3(u) = -\frac{u}{192} + \frac{1}{216}u^3.$$

We check that the modular discriminant and the j -invariant are

$$\Delta = g_2^3 - 27g_3^2 = \frac{1}{4096}(u-1)(u+1), \quad \mathbf{j} = 1728 \frac{g_2^3}{\Delta} = 64 \frac{(4u^2-3)^3}{(u-1)(u+1)}.$$

The roots of Eq. (3.6) for $y = 0$ are given by

$$e_1 = \frac{u}{24} + \frac{\sqrt{u^2 - 1}}{8}, \quad e_2 = \frac{u}{24} - \frac{\sqrt{u^2 - 1}}{8}, \quad e_3 = -\frac{u}{12}.$$

For u real and $u > 1$ we have $\Delta > 0$ and $e_1 > 0 > e_2 > e_3$. The solutions of Eq. (3.6) are given by the Weierstrass elliptic \mathcal{P} -function and its derivative

$$\mathbf{x} = \mathcal{P}(z | g_2, g_3), \quad \mathbf{y} = \mathcal{P}'(z | g_2, g_3).$$

The Weierstrass function has a pole at $z = 0$, and $\mathcal{P}(z) - z^{-2}$ is analytic in a neighborhood of $z = 0$ and vanishes for $z = 0$. Throughout this chapter we adapt the notation of [1, Sec. 17, 18] for the Weierstrass elliptic function and the complete elliptic integrals. The map $z \rightarrow (\mathbf{x}, \mathbf{y})$ is a parameterization of the elliptic curve by the parallelogram for the modular parameter τ in the complex plane. We introduce $e_i = \mathcal{P}(\omega_i | g_2, g_3)$ such that $0 = \mathcal{P}'(\omega_i | g_2, g_3)$, and

$$\omega = \omega_1, \quad \omega + \omega' = \omega_2, \quad \omega' = \omega_3.$$

The Weierstrass elliptic function \mathcal{P} is an elliptic function of order two with periods 2ω and $2\omega'$. We will also use the notation $\mathcal{P}(z | \omega, \omega')$. The definition of g_2, g_3 is equivalent to

$$g_2 = 60 \sum'_{n_1, n_2} \frac{1}{(2n_1\omega + 2n_2\omega')^4}, \quad g_3 = 140 \sum'_{n_1, n_2} \frac{1}{(2n_1\omega + 2n_2\omega')^6}.$$

Under the rescaling

$$\omega_1 \rightarrow c\omega_1, \quad \omega_2 \rightarrow c\omega_2,$$

τ and \mathbf{j} remain unchanged, but g_2 and g_3 are rescaled by

$$g_2 \rightarrow c^4 g_2, \quad g_3 \rightarrow c^6 g_3.$$

Thus, the Weierstrass elliptic function identifies E_u in Eq. (3.6) with points in the class of parallelograms $\langle 2c\omega, 2c\omega' \rangle$ for the modular parameter τ in the complex plane, i.e., $z \in \mathbb{C} / \langle 2c\omega, 2c\omega' \rangle$ with $c \in \mathbb{C}^*$.

Remark 1. *The family of elliptic curves considered by Fintushel and Stern in [5] is related to the family in Eq. (3.6) as follows: the coordinate y in [5, Eq. (4.2)] is $y = 8\mathbf{x}$ and $y' = 8\mathbf{y}$; the parameter of the family of the elliptic curve is $x = u/2$.*

The connection to the complete elliptic integrals is given by introducing the two parameters

$$m = \frac{e_2 - e_3}{e_1 - e_3} = (u - \sqrt{u^2 - 1})^2 = \frac{1}{(u + \sqrt{u^2 - 1})^2}, \quad m' = 1 - m.$$

To invert the relation one has to choose a square root of m . We pick

$$u = \frac{1}{2} \left(\sqrt{m} + \frac{1}{\sqrt{m}} \right),$$

and obtain using [1, Sec. 18.8]

$$\omega = \frac{K(m)}{\sqrt{e_1 - e_3}} = 2\sqrt{2} m^{\frac{1}{4}} K(m), \quad \omega' = 2\sqrt{2} i m^{\frac{1}{4}} K'(m),$$

where the complete elliptic integrals of the first and second kind are given by

$$K(m) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \mid m\right), \quad E(m) = \frac{\pi}{2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1 \mid m\right).$$

The complementary integrals are $K'(m) = K(1 - m)$ and $E'(m) = E(1 - m)$. Thus, the modular parameter is

$$\tau = \frac{\omega'}{\omega} = i \frac{K'(m)}{K(m)}.$$

The derivatives of the complete elliptic integrals are

$$\frac{d}{dm} K(m) = \frac{1}{2mm'} \left(E(m) - m' K(m) \right), \quad (3.7a)$$

$$\frac{d}{dm} E(m) = \frac{1}{2m} \left(E(m) - K(m) \right), \quad (3.7b)$$

and the Legendre relation is

$$E(m) K'(m) + E'(m) K(m) - K(m) K'(m) = \frac{\pi}{2}. \quad (3.8)$$

The Weierstrass ζ -function $\zeta(z|g_2, g_3)$ is defined as the solution of

$$\zeta'(z|g_2, g_3) = -\mathcal{P}(z|g_2, g_3).$$

with a pole at $z = 0$, $\zeta(z) - z^{-1}$ analytic in a neighborhood of $z = 0$, and vanishing for $z = 0$. The function $\zeta(z)$ is not elliptic as it is not periodic, but quasi-periodic (cf. [40, Sec. 19]). The quasi-periods are

$$\eta_i = \zeta(\omega_i|g_2, g_3),$$

and $\eta = \eta_1$, $\eta + \eta' = \eta_2$, and $\eta' = \eta_3$. In terms of complete elliptic integrals [1,

Sec. 18.8], they are

$$\begin{aligned}\eta &= \frac{K(m)}{3\omega} \left[3E(m) + (m-2)K(m) \right], \\ \eta' &= i \frac{K'(m)}{K(m)} \eta - \frac{\pi i}{2\omega}.\end{aligned}$$

It follows

$$\begin{aligned}\eta &= \frac{1}{6\sqrt{2}m^{\frac{1}{4}}} \left[3E(m) + (m-2)K(m) \right], \\ \eta' &= \frac{iK'(m)}{6\sqrt{2}m^{\frac{1}{4}}K(m)} \left[3E(m) + (m-2)K(m) \right] - \frac{\pi i}{4\sqrt{2}m^{\frac{1}{4}}K(m)}.\end{aligned}$$

The Legendre relation in Eq. (3.8) becomes

$$\eta\omega' - \eta'\omega = \frac{\pi i}{2}.$$

The fiber E_u over each u can be described as follows: We have two sheets of the \mathbf{x} -space for to the two roots $\pm \mathbf{y}$ which satisfy Eq. (3.6). These two sheets are joined pairwise together by two cuts connecting the points with $\mathbf{y} = 0$. The first cut runs from e_1 to e_2 on one sheet and back on the other sheet and is called the B-cycle. The second cut runs from e_3 to ∞ on one sheet and back on the other sheet. The line running from e_3 to e_2 on one sheet and back on the other sheet is called the A-cycle. The holomorphic differential or differential of the first kind is given by $d\mathbf{x}/\mathbf{y}$. It follows by substitution that

$$\begin{aligned}\int_{\text{B-cycle}} \frac{d\mathbf{x}}{\mathbf{y}} &= 2 \int_{e_1}^{e_2} \frac{d\mathbf{x}}{\mathbf{y}} = 2 \int_{\omega_1}^{\omega_2} dz = 2\omega', \\ \int_{\text{A-cycle}} \frac{d\mathbf{x}}{\mathbf{y}} &= 2 \int_{e_3}^{e_2} \frac{d\mathbf{x}}{\mathbf{y}} = 2 \int_{\omega_3}^{\omega_2} dz = 2\omega.\end{aligned}$$

The meromorphic differential or differential of the second kind is given by $\mathbf{x} d\mathbf{x}/\mathbf{y}$. We check that

$$\begin{aligned}\int_{\text{B-cycle}} \frac{\mathbf{x} d\mathbf{x}}{\mathbf{y}} &= 2 \int_{e_1}^{e_2} \frac{\mathbf{x} d\mathbf{x}}{\mathbf{y}} = -2 \int_{\omega_1}^{\omega_2} \zeta'(z|g_2, g_3) dz = -2\eta', \\ \int_{\text{A-cycle}} \frac{\mathbf{x} d\mathbf{x}}{\mathbf{y}} &= 2 \int_{e_3}^{e_2} \frac{\mathbf{x} d\mathbf{x}}{\mathbf{y}} = -2 \int_{\omega_3}^{\omega_2} \zeta'(z|g_2, g_3) dz = -2\eta.\end{aligned}$$

Let us define the meromorphic one-form

$$\lambda_{SW} = \frac{2}{3} u \frac{d\mathbf{x}}{\mathbf{y}} + 8g \mathbf{x} \frac{d\mathbf{x}}{\mathbf{y}},$$

where $g \in \mathbb{R}$ will be determined, and

$$\begin{aligned}\mathbf{a}_D &= \int_{\text{B-cycle}} \lambda_{SW} = \frac{4}{3} u \omega' - 16 g \eta' = \frac{2}{3} \left(\frac{1}{\sqrt{m}} + \sqrt{m} \right) \omega' - 16 g \eta', \\ \mathbf{a} &= \int_{\text{A-cycle}} \lambda_{SW} = \frac{4}{3} u \omega - 16 g \eta = \frac{2}{3} \left(\frac{1}{\sqrt{m}} + \sqrt{m} \right) \omega - 16 g \eta.\end{aligned}$$

A short calculation using Eqs. (3.7) shows that

$$\frac{d\mathbf{a}}{dm} = \frac{\sqrt{2}}{3 m^{\frac{5}{4}}} \left\{ (g+2)(1+m) \frac{E(m)}{1-m} - (gm-m+3) K(m) \right\}.$$

Similarly, a calculation using Eqs. (3.7), (3.8) shows

$$\begin{aligned}\frac{d\mathbf{a}_D}{dm} &= i \frac{K'(m)}{K(m)} \frac{\sqrt{2}}{3 m^{\frac{5}{4}}} \left\{ (g+2)(1+m) \frac{E(m)}{1-m} - (gm-m+3) K(m) \right. \\ &\quad \left. - \frac{\pi}{2(1-m)} (g+2)(1+m) \frac{1}{K'(m)} \right\}.\end{aligned}$$

$g = -2$ is the unique value for which

$$\frac{\frac{d\mathbf{a}_D}{dm}}{\frac{d\mathbf{a}}{dm}} = i \frac{K'(m)}{K(m)} = \tau.$$

In the following, we set $g = -2$. It follows

$$\frac{d\mathbf{a}}{du} = \frac{d\mathbf{a}}{dm} \frac{dm}{du} = -\frac{\sqrt{2}(1-m)}{m^{\frac{5}{4}}} K(m) \left(-\frac{4m^{\frac{3}{2}}}{1-m} \right) = 2\omega.$$

Similarly, we have

$$\frac{d\mathbf{a}_D}{du} = -i \frac{\sqrt{2}(1-m)}{m^{\frac{5}{4}}} K'(m) \left(-\frac{4m^{\frac{3}{2}}}{1-m} \right) = 2\omega'.$$

Differentiating the Legendre relation

$$\omega' \mathbf{a} - \omega \mathbf{a}_D = 16\pi i,$$

we obtain

$$0 = \mathbf{a} d(\tau \omega) - \mathbf{a}_D d\omega = (\mathbf{a} \tau - \mathbf{a}_D) d\omega + \mathbf{a} \omega d\tau.$$

Using the Legendre relation again, we obtain

$$\mathbf{a} d\tau = -\frac{16\pi i}{\omega^2} d\omega. \tag{3.9}$$

We summarize the computations of this section:

Lemma 3. *For the family of elliptic curves*

$$\mathbf{y}^2 = \mathbf{x}^3 - g_2(u) \mathbf{x} - g_3(u) ,$$

with

$$g_2(u) = \frac{u^2}{12} - \frac{1}{16} , \quad g_3(u) = -\frac{u}{192} + \frac{1}{216}u^3 ,$$

an identification of the fiber E_u with the fundamental parallelogram for the modular parameter τ in the complex plane, i.e., $z \in \mathbb{C} / \langle 2\omega, 2\omega' \rangle$ with $\tau = \omega'/\omega$, is given by

$$\mathbf{x} = \mathcal{P}(z|g_2, g_3) , \quad \mathbf{y} = \mathcal{P}'(z|g_2, g_3) .$$

The canonical holomorphic differential on each fiber E_u is the one-form $d\mathbf{x}/\mathbf{y}$. The A-cycle is given by the cut $e_3 \rightleftharpoons e_2$, i.e., connecting the two points with $\mathbf{x} = e_3$ and $\mathbf{x} = e_2$, and the B-cycle by $e_1 \rightleftharpoons e_2$. The periods are

$$\int_{\text{B-cycle}} \frac{d\mathbf{x}}{\mathbf{y}} = 2\omega' = 2\sqrt{2} m^{\frac{1}{4}} K'(m) , \quad (3.10a)$$

$$\int_{\text{A-cycle}} \frac{d\mathbf{x}}{\mathbf{y}} = 2\omega = 2\sqrt{2} m^{\frac{1}{4}} K(m) . \quad (3.10b)$$

The relation between τ , u , and m is

$$\begin{aligned} m &= (u - \sqrt{u^2 - 1})^2 , \\ u &= \frac{1}{2} \left(\sqrt{m} + \frac{1}{\sqrt{m}} \right) , \\ \tau &= i \frac{K'(m)}{K(m)} . \end{aligned}$$

Moreover, we have the meromorphic one-form

$$\lambda_{SW} = \frac{2}{3} u \frac{d\mathbf{x}}{\mathbf{y}} - 16 \mathbf{x} \frac{d\mathbf{x}}{\mathbf{y}} ,$$

such that

$$\int_{\text{B-cycle}} \lambda_{SW} = \mathbf{a}_D , \quad \int_{\text{A-cycle}} \lambda_{SW} = \mathbf{a} , \quad (3.11)$$

and

$$\frac{d}{du} \int_{\text{B-cycle}} \lambda_{SW} = 2\omega' , \quad \frac{d}{du} \int_{\text{A-cycle}} \lambda_{SW} = 2\omega . \quad (3.12)$$

On the total space $Z \rightarrow \text{UP}$, the two form $du \wedge dx/y$ is exact with

$$du \wedge \frac{dx}{y} = \frac{3}{2} d\lambda_{SW} = d \left(u \frac{dx}{y} - \frac{128}{3} x \frac{dx}{y} \right). \quad (3.13)$$

3.3 The monodromy representation

In this section, we will compute the monodromy representation around the singular fibers for the family of elliptic curves defined in Eq. (3.1).

For the regular fibers, we have $\dim H^1(E_u) = 2$ and an ordered basis $\mathcal{A} = \{\alpha_u, \beta_u\}$ of $H_1(E_u)$ with the intersection numbers

$$\alpha_u \cdot \alpha_u = 0, \quad \beta_u \cdot \beta_u = 0, \quad \alpha_u \cdot \beta_u = -\beta_u \cdot \alpha_u = 1.$$

The ordered basis $\{\alpha_u, \beta_u\}$ is obtained as follows: we have two sheets of the x -space according to the two square roots $\pm y$ which satisfy the equation $y^2 = x^3 + Ax + B$. These two sheets are joined together by two cuts connecting the points with $y = 0$. We choose for the two cuts the line

$$l_1 : (x, y) = \left(4e_3 = -\frac{u}{3}, 0 \right) \Leftrightarrow (x, y) = (\infty, \infty),$$

and the line

$$l_2 : (x, y) = \left(4e_1 = \frac{u}{6} + \frac{\sqrt{u^2 - 1}}{2}, 0 \right) \Leftrightarrow (x, y) = \left(4e_2 = \frac{u}{6} - \frac{\sqrt{u^2 - 1}}{2}, 0 \right).$$

The A-cycle α_u is $4e_3 \Leftrightarrow 4e_2$, the line connecting the point with $x = 4e_3$ (or $\mathbf{x} = e_3$) and the point with $x = 4e_1$, and the B-cycle β_u is $4e_1 \Leftrightarrow 4e_2$, the line l_2 . We observe that for $|u| < \infty$ the A-cycle can never shrink to zero, whereas for $u \rightarrow \pm 1$, the B-cycle shrinks to zero.

To determine the action of the monodromy on the symplectic basis \mathcal{A} of the homology one observes the change on a closed loop around the singular values $u = \pm 1, \infty$. This amounts to constructing a family of diffeomorphisms $F_t : E_u \rightarrow E_{u(t)}$ with $F_0 = \text{id}$ and continuous in t . The monodromy operator is defined as F_1 . Since it leaves the intersection form invariant its differential can be represented by a matrix $M = F_{1,*} \in SL(2; \mathbb{Z})$. We can determine this matrix by Picard-Lefschetz theory.

While the order two point with x -coordinate

$$x = 4e_3 = -\frac{u(t)}{3} = -\frac{1}{3} + O(\epsilon),$$

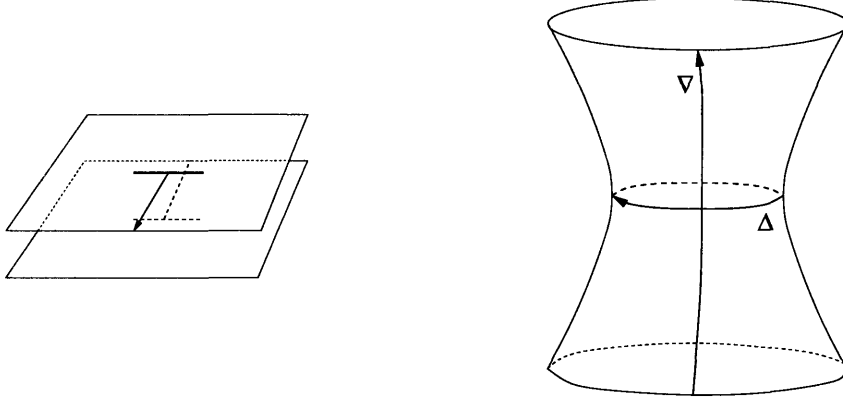


Figure 3-1: The two x -coordinate sheets joined along a cut and the cut-cylinder

for $u(t) = 1 + \epsilon e^{2\pi it}$ stays at the same point, the order two points with

$$\begin{aligned}
 x &= 4e_1 = \frac{u(t)}{6} + \frac{\sqrt{u(t)^2 - 1}}{2} = \frac{1}{6} + \frac{\sqrt{2}}{2} \sqrt{\epsilon} e^{\pi it} + O(\epsilon), \\
 x &= 4e_2 = \frac{u(t)}{6} - \frac{\sqrt{u(t)^2 - 1}}{2} = \frac{1}{6} - \frac{\sqrt{2}}{2} \sqrt{\epsilon} e^{\pi it} + O(\epsilon),
 \end{aligned}$$

turn in a half-circle into each other. Thus, the B-cycle, $4e_1 \rightleftharpoons 4e_2$, is transformed into itself. The A-cycle, $4e_3 \rightleftharpoons 4e_2$, picks up a contribution $4e_1 \rightleftharpoons 4e_2$.

Since for $u \rightarrow 1$ the B-cycle vanishes, it is easier to describe the situation in terms of the complementary torus which is obtained as the boundary of the complement of the solid torus embedded into \mathbb{R}^3 . For this complementary torus, the A-cycle vanishes for $u \rightarrow 1$ and the complementary torus develops a node. The situation can be described by a cut cylinder where two copies of the plane are joined along one cut. Each copy of the plane is homeomorphic to the half-cylinder. The cut represents the part of the torus developing the node, i.e., the circumference of the cylinder. The circumference is called the vanishing cycle Δ of Picard-Lefschetz. The component of the B-cycle intersecting the cylinder is a line ∇ and is called the vanishing cocycle. We then have $\nabla \cdot \Delta = \alpha_u \cdot \beta_u = 1$. The situation is shown in Fig. 3-1. For $u(t)$ as t progresses, the cut will start turning counterclockwise where F_t remains the identity outside of a large compact set containing the cut as shown in Fig. 3-2. It follows that for $u = 1 + \epsilon e^{2\pi it}$ for $t \rightarrow 1$, we obtain for the transformation of the A-cycle and B-cycle of the original torus

$$\begin{aligned}
 \Delta &\mapsto \Delta \\
 \nabla &\mapsto -\Delta + \nabla.
 \end{aligned}$$

Thus, the monodromy action on $\mathcal{A} \subset H_1(E_u)$ when encircling the point $u = 1$ is the

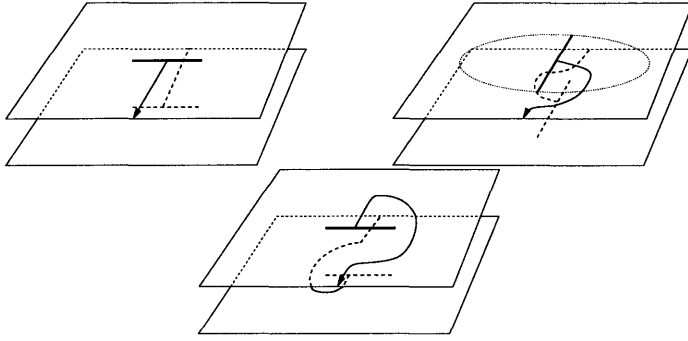


Figure 3-2: The monodromy for $u(t) = 1 + \epsilon e^{2\pi it}$.

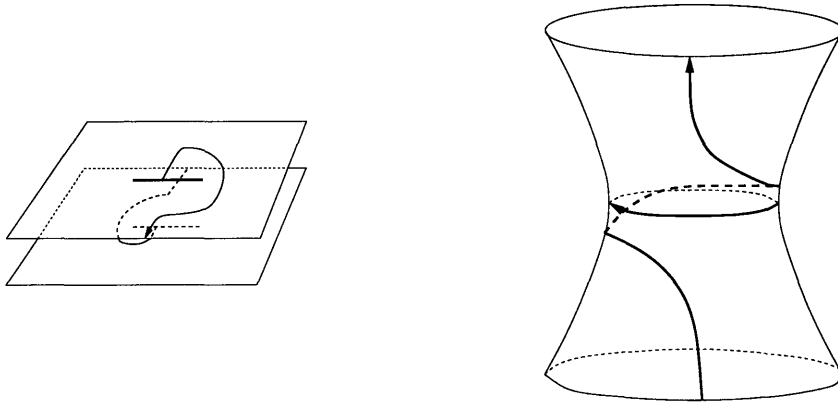


Figure 3-3: The two x -coordinate sheets joined along a cut for $u = 1 + \lim_{t \rightarrow 1} \epsilon e^{2\pi it}$.

matrix

$$M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

For $u \rightarrow \infty$ or ($v \rightarrow 0$) the elliptic curve develops a cusp since both the A-cycle, $4e_3 \rightleftharpoons 4e_2$, and the B-cycle, $4e_1 \rightleftharpoons 4e_2$, shrink to zero. However, after carrying out the quadratic twist with $v = \epsilon e^{2\pi it}$ the A-cycle becomes

$$4e'_3 = -\frac{1}{3} \rightleftharpoons 4e'_2 = \frac{1}{6} - \frac{\sqrt{1-v^2}}{2} = -\frac{1}{3} + \frac{\epsilon^2}{4} e^{4\pi it} + O(\epsilon^4),$$

and the B-cycle becomes

$$\begin{aligned} 4e'_1 &= \frac{1}{6} + \frac{\sqrt{1-v^2}}{2} = \frac{1}{2} - \frac{\epsilon^2}{4} e^{4\pi it} + O(\epsilon^4) \\ &\rightleftharpoons 4e'_2 = \frac{1}{6} - \frac{\sqrt{1-v^2}}{2} = -\frac{1}{3} + \frac{\epsilon^2}{4} e^{4\pi it} + O(\epsilon^4). \end{aligned}$$

For $v \rightarrow 0$ the A-cycle shrinks to zero. In particular, for $t \rightarrow 1$ the point with $x = 4e'_2$

turns in a circle around the point with $x = 4e'_3$ twice whence transforming the A-cycle into itself. However, the B-cycle connecting the point with $x = 4e'_1$ with the point with $x = 4e'_2$ picks up a contribution from the A-cycle. The monodromy action for $v = \lim_{t \rightarrow -1} \epsilon e^{2\pi it}$ after the quadratic twist is

$$M_\infty'^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

We denote this action as the inverse matrix since we go around the point $v = 0$ counterclockwise which is equivalent to encircling $u = \infty$ clockwise. The quadratic twist itself can be formally obtained by a change of variables according to $x = x'v$ and $y = y'v^{\frac{3}{2}}$. Then, the monodromy picks up an extra minus sign under a full rotation $v \rightarrow \lim_{t \rightarrow -1} v e^{2\pi it}$ since the two sheets covering the plane are swapped. In conclusion, we have

$$M_\infty^{-1} = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}.$$

The matrix M_∞^{-1} describes the monodromy action on \mathcal{A} when going around $v = 0$ counterclockwise – which is equivalent to going around $u = \infty$ clockwise. Finally, since $M_\infty = M_1 M_{-1}$ we obtain

$$M_{-1} = M_1^{-1} M_\infty = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}.$$

The general theorem for punctured discs implies that the elliptic fibration is determined by the j -function and the monodromy action [65]. Since $\pi_1(\text{UP})$ is free on two generators, we can pick as generators the loops encircling (counterclockwise) the points $u = 1$ and $u = \infty$. We have proved:

Lemma 4. *The family of elliptic curves constructed by Seiberg and Witten determines by the monodromy action around the singular fibers a representation*

$$M : \pi_1(\text{UP}) \rightarrow SL(2; \mathbb{Z}).$$

The monodromy matrices up to conjugation by elements in $SL(2; \mathbb{Z})$ are determined by the Kodaira type of the singular fiber. For $u = \pm 1$ the degenerate fiber is of type I_1 , and the monodromy matrices M_1, M_{-1} are conjugate to T with

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In particular,

$$M_1 = STS^{-1}, \quad M_{-1} = (T^2S)T(T^2S)^{-1},$$

with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

For $u = \infty$, the degenerate fiber is of type I_4^* , the monodromy matrix M_∞^{-1} is $-T^4$. The three monodromy matrices M_γ around the three boundary circles $\gamma \in \pi_1(\text{UP})$ satisfy

$$M_{-1}, M_1, M_\infty^{-1} \in \Gamma_0(4) ,$$

where $\Gamma_0(4)$ is the subgroup of lower triangular elements in $SL(2; \mathbb{Z})$ modulo 4.

3.4 The u -plane

In this section, we describe the modular parameter and the discriminant for the family of elliptic curves defined in Eq. (3.1) in terms of the Jacobi ϑ -functions.

The inverse relation between the modular parameter τ and m in Lemma 3 is

$$m = \lambda(\tau) ,$$

where λ is the elliptic modular λ -function invariant under $\Gamma(2)$. The λ -function is defined for all τ in the upper half-plane H by

$$\lambda(\tau) = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)} .$$

The ϑ -function is defined as

$$\vartheta(z, R) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 R + 2\pi i z n} ,$$

and the Jacobi ϑ -functions $\vartheta_i(\tau)$ are

$$\vartheta_2(R) = \vartheta_{10}(R) = e^{\frac{\pi i R}{4}} \vartheta\left(\frac{R}{2}, R\right) = \sum_{n \in \mathbb{Z}} Q^{(n+\frac{1}{2})^2} , \quad (3.14a)$$

$$\vartheta_3(R) = \vartheta_{00}(R) = \vartheta(0, R) = \sum_{n \in \mathbb{Z}} Q^{n^2} , \quad (3.14b)$$

$$\vartheta_4(R) = \vartheta_{01}(R) = \vartheta\left(\frac{1}{2}, R\right) = \sum_{n \in \mathbb{Z}} Q^{n^2} (-1)^n , \quad (3.14c)$$

with $Q = e^{\pi i R}$. In general, we write

$$\vartheta_{ab}(R) = e^{\pi i R \frac{a^2}{4} + 2\pi i \frac{ab}{4}} \vartheta\left(\frac{a}{2} R + \frac{b}{2}, R\right) .$$

The $\Gamma(2)$ -modular invariance of λ means that

$$\lambda(\tau + 2) = \lambda(\tau) , \quad \lambda\left(\frac{\tau}{2\tau + 1}\right) = \lambda(\tau) . \quad (3.15)$$

Thus, choosing the square root \sqrt{m} it follows that

$$u = \frac{1}{2} \left(\sqrt{m} + \frac{1}{\sqrt{m}} \right) = \frac{1}{2} \frac{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)}{[\vartheta_2(\tau) \vartheta_3(\tau)]^2} , \quad (3.16)$$

and from Eq. (3.10) that

$$\omega = \sqrt{2} \pi \vartheta_2(\tau) \vartheta_3(\tau) .$$

We also compute the discriminant

$$\begin{aligned} \Delta &= \frac{1}{4096} (u^2 - 1) = 2^{-14} \left(\sqrt{m} - \frac{1}{\sqrt{m}} \right)^2 = 2^{-14} \left(\frac{\vartheta_3^4(\tau) - \vartheta_2^4(\tau)}{\vartheta_2^2(\tau) \vartheta_3^2(\tau)} \right)^2 \\ &= 2^{-14} \left(\frac{\vartheta_4^4(\tau)}{\vartheta_2^2(\tau) \vartheta_3^2(\tau)} \right)^2 = (2\pi)^{12} \frac{[\frac{1}{2} \vartheta_2(\tau) \vartheta_3(\tau) \vartheta_4(\tau)]^8}{[2\sqrt{2}\pi \vartheta_2(\tau) \vartheta_3(\tau)]^{12}} \\ &= (2\pi)^{12} \frac{\eta^{24}(\tau)}{(2\omega)^{12}} = \frac{1}{2^{18}} \left(\frac{\vartheta_4(\tau)}{\eta(\tau)} \right)^{12} , \end{aligned} \quad (3.17)$$

where we have used the identity

$$\vartheta_4^4(R) = \vartheta_3^4(R) - \vartheta_2^4(R) ,$$

and the Dedekind η -function

$$\eta(R) = e^{\frac{\pi i R}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n R}) ,$$

that satisfies

$$\eta^3(R) = \frac{1}{2} \vartheta_2(R) \vartheta_3(R) \vartheta_4(R) .$$

We have the useful identity

$$\Delta = \frac{\pi^4}{\omega^4} \vartheta_4(\tau)^8 = \left(\frac{du}{da} \right)^2 \left(\frac{i}{\pi} \frac{du}{d\tau} \right) . \quad (3.18)$$

We have the following transformation rules for the ϑ -functions and the η -function

$$\vartheta_{ab}(R+1) = e^{\frac{\pi i a}{4}} \vartheta_{a(b+a+1)}(R), \quad (3.19a)$$

$$\vartheta_{ab}\left(-\frac{1}{R}\right) = e^{\pi i a b} \sqrt{-i R} \vartheta_{ba}(R), \quad (3.19b)$$

$$\eta(R+1) = e^{\frac{\pi i}{12}} \eta(R), \quad (3.19c)$$

$$\eta\left(-\frac{1}{R}\right) = \sqrt{-i R} \eta(R), \quad (3.19d)$$

and

$$\vartheta_{(a+2n)(b+2m)}(R) = e^{2\pi i a m} \vartheta_{ab}(R). \quad (3.20)$$

In conclusion, we have the following transformation rules:

τ	$\mapsto \tau_T = T \cdot \tau$	$= \tau + 1$	
$\vartheta_2(\tau)$	$\mapsto \vartheta_2(\tau_T)$	$= \sqrt{i} \vartheta_2(\tau)$	
$\vartheta_3(\tau)$	$\mapsto \vartheta_3(\tau_T)$	$= \vartheta_4(\tau)$	
$\vartheta_4(\tau)$	$\mapsto \vartheta_4(\tau_T)$	$= \vartheta_3(\tau)$	
$u = J(\tau)$	$\mapsto u_T = J(\tau_T)$	$= -\frac{1 - \vartheta_2^4(\tau) + \vartheta_4^4(\tau)}{2 [\vartheta_2(\tau) \vartheta_4(\tau)]^2}$	
τ	$\mapsto \tau_{T^2} = T^2 \cdot \tau$	$= \tau + 2$	
$\vartheta_2(\tau)$	$\mapsto \vartheta_2(\tau_{T^2})$	$= i \vartheta_2(\tau)$	
$\vartheta_3(\tau)$	$\mapsto \vartheta_3(\tau_{T^2})$	$= \vartheta_3(\tau)$	
$\vartheta_4(\tau)$	$\mapsto \vartheta_4(\tau_{T^2})$	$= \vartheta_4(\tau)$	
$u = J(\tau)$	$\mapsto u_{T^2} = J(\tau_{T^2})$	$= -u$	
τ	$\mapsto \tau_{T^3} = T^3 \cdot \tau$	$= \tau + 3$	
$u = J(\tau)$	$\mapsto u_{T^3} = J(\tau_{T^3})$	$= -u_T$	
τ	$\mapsto \tau_{T^4} = T^4 \cdot \tau$	$= \tau + 4$	
$u = J(\tau)$	$\mapsto u_{T^4} = J(\tau_{T^4})$	$= u$	
τ	$\mapsto \tau_S = S \cdot \tau$	$= -1/\tau$	
$\vartheta_2(\tau)$	$\mapsto \vartheta_2(\tau_S)$	$= \sqrt{-i\tau} \vartheta_4(\tau)$	
$\vartheta_3(\tau)$	$\mapsto \vartheta_3(\tau_S)$	$= \sqrt{-i\tau} \vartheta_3(\tau)$	
$\vartheta_4(\tau)$	$\mapsto \vartheta_4(\tau_S)$	$= \sqrt{-i\tau} \vartheta_2(\tau)$	
$u = J(\tau)$	$\mapsto u_S = J(\tau_S)$	$= \frac{1}{2} \frac{\vartheta_3^4(\tau) + \vartheta_4^4(\tau)}{[\vartheta_3(\tau) \vartheta_4(\tau)]^2}$	
τ	$\mapsto \tau_{T^2 S} = T^2 S \cdot \tau$	$= -1/\tau$	
$u = J(\tau)$	$\mapsto u_{T^2 S} = J(\tau_{T^2 S})$	$= -u_S$	

(3.21)

The function u is defined on $H/\Gamma_0(4)$. Since we have picked a square root of $\lambda(\tau)$ in the definition of u , the function is defined on $H/\Gamma_0(4)$ as opposed to $H/\Gamma(2)$.

Remark 2. *The fundamental domain for $\Gamma_0(4)$ is*

$$H/\Gamma_0(4) \cong \mathcal{F} \cup T \cdot \mathcal{F} \cup T^2 \cdot \mathcal{F} \cup T^3 \cdot \mathcal{F} \cup S \cdot \mathcal{F} \cup T^2 S \cdot \mathcal{F}.$$

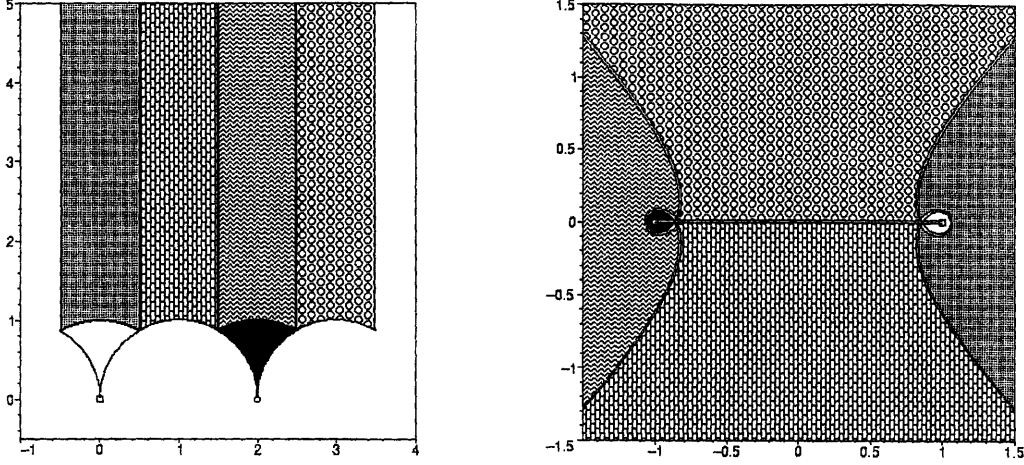


Figure 3-4: The mapping from $H/\Gamma_0(4)$ (with the six copies of the fundamental domain) to the u -plane (with the points $u = 1$ (\square) and $u = -1$ (\circ) removed)

The $\Gamma_0(4)$ -invariant bijective map

$$\begin{aligned} J: H &\rightarrow \text{UP} = \mathbb{C}P^1 - \{-1, 1, \infty\} \\ \tau &\mapsto u. \end{aligned}$$

is depicted in Fig. 3-4, where the u -plane is covered by six charts.

The isomorphism in Eq. (3.3) is the action $\tau \rightarrow \tau+2$ or $u \rightarrow -u$ which corresponds to the projective action of the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

For $\text{Im } \tau \rightarrow \infty$ and $\text{Re } \tau$ fixed, we have $q = e^{\pi i \tau} \rightarrow 0$. The first terms in an asymptotic expansion of u and ω are

$$u \sim \frac{1}{8q^{\frac{1}{4}}} + \frac{5q^{\frac{1}{4}}}{2} + O(q^{\frac{3}{4}}), \quad \omega \sim 2\sqrt{2}\pi q^{\frac{1}{4}} + 4\sqrt{2}\pi q^{\frac{5}{4}} + O(q^{\frac{9}{4}}).$$

Thus, we have $u \rightarrow \infty$, for $\text{Im } \tau \rightarrow \infty$ and $\text{Re } \tau$ fixed. On the other hand, for $\tau = \alpha + i\beta$ we have

$$\tau_S = \frac{-\alpha + i\beta}{\alpha^2 + \beta^2},$$

whence $\tau_S \rightarrow 0$ for $\text{Im } \tau = \beta \rightarrow \infty$. The first terms in an asymptotic expansion of u_S are

$$u_S \sim 1 + 32q + O(q^2),$$

Thus, we have $u_S \rightarrow 1$ for $\tau_S \rightarrow 0$.

If we define the vectors of the periods and the quasi-periods (cf. Lemma 3)

$$\mathbf{\Omega} = (\omega'(\tau), \omega(\tau))^t, \quad (3.22a)$$

$$\mathbf{H} = (\mathbf{a}_D, \mathbf{a}) = \left(\frac{2u}{3} \omega'(\tau) + 32 \eta'(\tau), \frac{2u}{3} \omega(\tau) + 32 \eta(\tau) \right)^t, \quad (3.22b)$$

one can check that they transform in the representation $\pi_1(\text{UP}) \rightarrow SL(2; \mathbb{Z})$ determined in Lemma 4:

τ	$\rightarrow \tau + 4$	(3.23)
u	$\rightarrow \lim_{t \rightarrow -1} u e^{2\pi i t}$	
$\mathbf{\Omega}, \mathbf{H}$	$\rightarrow M_\infty^{-1} \cdot \mathbf{\Omega}, \mathbf{H} = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix} \cdot \mathbf{\Omega}, \mathbf{H}$	
τ	$\rightarrow STS^{-1} \tau = \frac{\tau}{-\tau+1}$	(3.23)
u	$\rightarrow 1 + \lim_{t \rightarrow 1} (u-1) e^{2\pi i t}$	
$\mathbf{\Omega}, \mathbf{H}$	$\rightarrow M_1 \cdot \mathbf{\Omega}, \mathbf{H} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \mathbf{\Omega}, \mathbf{H}$	
τ	$\rightarrow T^2 STS^{-1} T^{-2} = \frac{-\tau+4}{-\tau+3}$	(3.23)
u	$\rightarrow -1 + \lim_{t \rightarrow 1} (u+1) e^{2\pi i t}$	
$\mathbf{\Omega}, \mathbf{H}$	$\rightarrow M_{-1} \cdot \mathbf{\Omega}, \mathbf{H} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \cdot \mathbf{\Omega}, \mathbf{H}$	

3.5 The integrable system

In this section, we describe the Seiberg-Witten curve as an integrable system for the universal curve $H/\Gamma_0(4)$.

Let (z, τ) be the complex coordinates on $\mathbb{C} \times H$, and $\omega : H \rightarrow \mathbb{C}$ a holomorphic function of τ with some additional properties specified below. We also set $\omega'(\tau) = \tau \omega(\tau)$. Let $\mathbb{Z} \times \mathbb{Z}$ act on $\mathbb{C} \times H$ trivially on H and by translation by $2n_1 \omega + 2n_2 \omega'$ on $\mathbb{C} \times \{\tau\}$. Then,

$$\mathfrak{Z}' = (\mathbb{C} \times H) / (\mathbb{Z} \times \mathbb{Z}) \rightarrow H$$

is a holomorphic fibration with fiber $\mathbb{C} / \langle 2\omega, 2\omega' \rangle$. Locally, we can use the complex coordinate $z \in \mathbb{C}$ to define the holomorphic differential dz . The cohomology $H^0(E_\tau, K)$ is one-dimensional for all τ where $K = T^{*(1,0)} E_\tau$. They fit together into a line bundle, and dz is a local holomorphic section with norm square $4 \operatorname{Im} \tau |\omega(\tau)|^2$. Equivalently, we have a family of positive polarizations given by $dz \wedge d\bar{z}$ on each fiber such that

$$\int_{E_\tau} dz \wedge d\bar{z} = -8i \operatorname{Im} \tau |\omega(\tau)|^2.$$

For all τ we can define a symplectic basis $\mathcal{A} = \{\alpha_\tau, \beta_\tau\} \in H_1(E_\tau)$ as the images of

$[0, 2\omega(\tau)] \times \{\tau\}$ and $[0, 2\omega'(\tau)] \times \{\tau\}$ such that

$$\int_{\alpha_\tau} dz = 2\omega(\tau), \quad \int_{\beta_\tau} dz = 2\omega'(\tau).$$

The action of the modular group $PSL(2; \mathbb{Z})$ on $\tau \in H$ lifts to an $SL(2; \mathbb{Z})$ -action on \mathfrak{Z}' . Conversely, a change of the symplectic basis $\{\alpha_\tau, \beta_\tau\}$ transforms τ by an element of a discrete subgroup in $SL(2; \mathbb{Z})$. In the case of the family of elliptic curves considered by Seiberg and Witten, the discrete subgroup is $\Gamma_0(4) \subset SL(2; \mathbb{Z})$. We obtain as the well-defined quotient the modular elliptic surface of index six

$$\mathfrak{Z} = \mathfrak{Z}'/\Gamma_0(4) \rightarrow H/\Gamma_0(4).$$

We identify this modular elliptic surface with the holomorphic fibration $Z \rightarrow \text{UP}$ as follows: we identify the base space with $H/\Gamma_0(4)$ by the map $J : \text{UP} \rightarrow H/\Gamma_0(4)$. To identify the holomorphic fibration $Z \rightarrow \text{UP}$ with $\mathfrak{Z} \rightarrow H/\Gamma_0(4)$, we need a holomorphic symplectic two form $\eta \in \Omega^{(2,0)}(Z)$ which identifies $T^*\text{UP}$ with the invariant vector fields along the fibers. As shown by Witten [30], for the Jacobian family of elliptic curves in Eq. (3.1) there is such a global holomorphic two form η , given by

$$\eta = du \wedge \frac{dx}{y}, \quad (3.24)$$

where (x, y) are the Weierstrass coordinates in Eq. (3.6) on the fiber. η is a holomorphic two-form η and $\eta \wedge \bar{\eta}$ the volume form. We have shown in Lemma 3 that $[\eta] \in H^2(Z; \mathbb{C})$ is trivial.

Now, we change our point of view and fix the square torus on which $SL(2; \mathbb{Z})$ acts by diffeomorphism. Then $dz = 2\omega dr + 2\omega' ds$ is the holomorphic differential, and $dr \wedge ds$ is the relative Kähler form. The matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z})$$

acts on the coordinates $(\tau; s, r) \in H \times \mathbb{R}^2$ and the periods in Eq. (3.22) by

$$\begin{aligned} \tau &\mapsto \tau' = M \cdot \tau \\ r &\mapsto r' = ar - bs \\ s &\mapsto s' = ds - cr \end{aligned} \quad \Omega = \begin{pmatrix} \omega'(\tau) \\ \omega(\tau) \end{pmatrix} \mapsto \Omega' = \begin{pmatrix} \omega'(\tau') \\ \omega(\tau') \end{pmatrix} = M \cdot \begin{pmatrix} \omega'(\tau) \\ \omega(\tau) \end{pmatrix}.$$

The induced action takes dz to dz . Thus, we can canonically identify the holomorphic differential dz with du of the cotangent bundle $T^*\text{UP}$. The metric on $Z \rightarrow \text{UP}$ is

$$g = du.d\bar{u} \oplus \chi(u, \bar{u}) \left((dr)^2 \oplus (ds)^2 \right).$$

with $\chi(u, \bar{u}) = \text{Im } \tau |\omega|^2$. χ vanishes if and only if $u = \pm 1$ or $u \rightarrow \infty$. At the nodal points $u = \pm 1$, the metric becomes singular. We also determine the differentials invariant under the monodromy action of M_1 and M_∞ , and the whole one-parameter groups in $PSL(2; \mathbb{R})$ generated by them. For M_∞^{-1} the action of the one-parameter group is the flow of $\dot{u} = -2\pi i u$ on the upper half plane. The invariant differential ω_{M_∞} is du/u . Similarly, $\omega_{M_1} = du/(u-1)$. This construction can be summarized in the following Lemma:

Lemma 5. *The family of elliptic curves constructed by Seiberg and Witten is a modular elliptic surface denoted by*

$$\pi : Z \rightarrow \text{UP},$$

where the generic fiber is a two-torus \mathbb{T}^2 and π is holomorphic. We have the following description as the polarized universal family of elliptic curves:

$$\begin{array}{ccc} Z = \coprod (u, E_u) & \xleftarrow[\substack{x=\mathcal{P}(z|g_2, g_3) \\ y=\mathcal{P}'(z|g_2, g_3)}]{} & \mathfrak{Z} = \coprod \left(\tau, \mathbb{C} / \langle 2\omega(\tau), 2\omega'(\tau) \rangle \right) / \Gamma_0(4) \\ \downarrow \pi & & \downarrow \\ u \in \text{UP} & \xleftarrow{J} & \tau \in H/\Gamma_0(4) \\ \downarrow j(E_u) & & \downarrow j(\tau) \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

The family of positive polarizations is given by $dz \wedge d\bar{z}$, and the holomorphic symplectic two-form is $\eta = du \wedge dx/y$.

3.6 Topological invariants of the Seiberg-Witten curve

In this section, we compute several topological invariants of the elliptic surface defined in Eq. (3.1) and of its minimal resolution.

We need to understand the divisors and sections of the rational elliptic surface $Z \rightarrow \text{UP}$. The Mordell-Weil group $MW(\pi)$ is the Abelian group of sections with the zero-section S as the identity element. The Neron-Severi group is the image of divisors modulo linear equivalence in $H^2(Z; \mathbb{Z})$. We can explicitly describe the Abelian group of sections and the divisors in the positive Kähler cone. We determine the surface singularities of $Z \rightarrow CP^1$. To do so, we start with the polynomial in $(u, [X : Y : W])$ given by

$$F_u(X, Y, W) = -Y^2 W + X^3 + \left(-\frac{u^2}{3} + \frac{1}{4} \right) X W^2 + \left(\frac{u}{12} - \frac{2u^3}{27} \right) W^3.$$

The elliptic surface is the hyper-surface $F = 0$. The singular points are those at

which all the partial derivatives simultaneously vanish. We observe that the system of equations

$$\frac{\partial F}{\partial X} = 0, \quad \frac{\partial F}{\partial Y} = 0, \quad \frac{\partial F}{\partial W} = 0, \quad \frac{\partial F}{\partial u} = 0,$$

has no solutions. Thus, Z has no surface singularities for $|u| < \infty$. Similarly, we can look at the polynomial

$$F_v(X, Y, W) = -Y^2 W + X^3 + \left(-\frac{v^2}{3} + \frac{v^4}{4}\right) X W^2 + \left(\frac{v^5}{12} - \frac{2v^3}{27}\right) W^3.$$

We observe that the system of equations

$$\frac{\partial F}{\partial X} = 0, \quad \frac{\partial F}{\partial Y} = 0, \quad \frac{\partial F}{\partial W} = 0, \quad \frac{\partial F}{\partial v} = 0,$$

has the solution

$$(v, [X : Y : W]) = (0, [0 : 0 : 1]).$$

Thus, the point $(v = 0, [0 : 0 : 1])$ is the only singular point of the surface $Z \rightarrow CP^1$. It is always the case given a Weierstrass fibration which is rational that the minimal resolution of its surface singularities is a smooth rational surface that dominates the plane, i.e., a blow-up of CP^2 at nine points. However, these points may be infinitely near to each other¹. This is because on such a surface, the anti-canonical class $-K$ is the fiber class, and is therefore effective and moves in a pencil. Therefore any rational curve C on the surface must have $C^2 \geq -2$, since $C^2 \geq C^2 + C \cdot K = -2$ as $C \cdot K \leq 0$. Hence, since self-intersection numbers can only rise when one blows down, it must be the case that any rational curve on the minimal model has $C^2 \geq -2$. This means that the minimal model is either one of the Hirzebruch surfaces F_0, F_2 , or CP^2 . But one blow-up of F_0 or F_2 gives rise to a surface which also can be blown down to CP^2 . The result is that all such rational elliptic surfaces with section are blow-ups of the plane with exactly nine blow-ups.

The rational elliptic surface obtained by the minimal resolution of the Jacobian elliptic surface described above has the following Hodge diamond

$$\begin{array}{ccccc}
 & & 1 & & 1 \\
 & q & & q & & 0 & & 0 \\
 p_g & & h^{1,1} & & p_g = 0 & & 10 & & 0 \\
 & q & & q & & 0 & & 0 \\
 & & 1 & & & & & & 1
 \end{array} .$$

The signature of the elliptic surface is given in terms of its Chern classes $c_1 = c_1(K)$

¹We thank Rick Miranda for explaining this point to us in great detail.

and $c_2 = c_2(K)$ as

$$\text{sign}(Z) = \frac{c_1^2 - 2c_2}{3}.$$

The anti-canonical class $-K$ is the fiber class, thus $c_1^2 = 0$. The Euler number is given by the Euler numbers of the singular fibers F_i by $c_2 = \sum e(F_i)$. Thus,

$$\text{sign}(Z) = -\frac{2}{3} \left(e(F_{-1}) + e(F_1) + e(F_\infty) \right).$$

Let us compare the situation with the standard elliptic surface $Z_{\text{std}} \rightarrow CP^1$ obtained by blowing up 9 general points in the plane. Each exceptional fiber is a nodal cubic, and the singularity type is I_1 . Each of these singular fibers F contributes equally, $\text{sign}(F) = 0$ and $e(F) = 1$. For the singular fibers in an elliptic fibration, one can define the Meyer's signature invariant (cf. [80]). It is the quantity

$$[M_\gamma] \rightarrow \phi(M_\gamma) = \frac{2}{3} e(F_\gamma) + \text{sign}(F_\gamma),$$

where $\gamma \in \pi_1(\text{UP})$, $[\cdot]$ denotes the congruence class of M_γ , and F_γ is the exceptional fiber of the elliptic fibration encircled by γ . The geometric meaning of the quantity as monodromy of the determinant line bundle of an elliptic operator along the fiber will be discussed in Sec. 3.8. We obtain $\phi(M_{-1}) = \phi(M_1) = 2/3$. Atiyah [80] also showed that $\phi(M_\infty^{-1}) = \phi(-T^4) = -4/3$.

Finally, it is always the case that for a rational elliptic surface, the zero-section S and the smooth fiber-class F generate a two-dimensional lattices that splits off the Neron-Severi group which is $I \oplus -I$ where I is the one-dimensional lattice generated by α with $\alpha^2 = 1$. The two orthogonal generators are S and $G = S + F$ such that

$$S^2 = -1 \quad (\text{generates } -I), \quad G^2 = 1 \quad (\text{generates } I), \quad S \cdot G = 0.$$

In particular, this is not a hyperbolic plane over Z . The surface is extremal in the sense that the zero-section and the components of the fibers generate the Neron-Severi group over the rational numbers \mathbb{Q} . It has an order two Mordell-Weil group $MW(\pi)$; there is exactly one other section - other than the section at infinity which we have taken to be the zero-section of the Mordell-Weil group. In the u -chart, we can compute this section. We know that $MW(\pi)$ is generated by sections of the form (cf. [68])

$$x = b_2 u^2 + b_1 u + b_0, \quad y = c_3 u^3 + c_2 u^2 + c_1 u + c_0.$$

Substituting into the Weierstrass equation yields the rational solution

$$x = -\frac{u}{3}, \quad y = 0.$$

We take as the generator of $MW(\pi)$ over the rational numbers \mathbb{Q} the section with

$x = -\frac{u}{3}$. This proves:

Lemma 6.

1. The minimal resolution $\hat{\pi} : \hat{Z} \rightarrow CP^1$ is $CP^2 \# 9\overline{CP^1}$, i.e., the complex projective space blown up at nine points.
2. The section S and the smooth fiber-class F generate a two-dimensional lattices that splits off the Neron-Severi group which is $I \oplus -I$ (where I is the one-dimensional lattice generated by x with $x^2 = 1$). The two orthogonal generators are S and $G = S + F$ such that

$$S^2 = -1 \quad (\text{generates } -I), \quad G^2 = 1 \quad (\text{generates } I), \quad S \cdot G = 0.$$

3. The Mordell-Weil group $MW(\pi)$ has order two and is generated by the Spin-structures (P, A) and (P, P) .

3.7 The determinant line on an elliptic curve

In this section, we discuss the determinant line bundle of the $\bar{\partial}$ -operator coupled to a flat holomorphic line bundle on the elliptic curve E_u where u is fixed.

We consider the elliptic curve E_u with periods 2ω and $2\omega'$ and $\tau = \frac{\omega'}{\omega}$. Let $\xi = \xi^1 + i\xi^2$ be the complex coordinate on the normalized elliptic curve with periods 1 and τ , and z with $\xi = \frac{z}{2\omega}$ the complex coordinate on the elliptic curve with periods 2ω and $2\omega'$. The functions φ with the periodicity

$$\begin{aligned} \varphi^{(\nu_1, \nu_2)}(\xi^1 + 1, \xi^2) &= -e^{\pi i \nu_1} \varphi^{(\nu_1, \nu_2)}(\xi^1, \xi^2), \\ \varphi^{(\nu_1, \nu_2)}(\xi^1 + \text{Re } \tau, \xi^2 + \text{Im } \tau) &= -e^{\pi i \nu_2} \varphi^{(\nu_1, \nu_2)}(\xi^1, \xi^2) \end{aligned}$$

are

$$\begin{aligned} \varphi_{n_1, n_2}^{(\nu_1, \nu_2)}(\xi^1, \xi^2) = \\ \exp 2\pi i \left\{ \left(n_1 + \frac{1 - \nu_1}{2} \right) \xi^1 + \frac{1}{\text{Im } \tau} \left(n_2 + \frac{1 - \nu_2}{2} - \text{Re } \tau \left(n_1 + \frac{1 - \nu_1}{2} \right) \right) \xi^2 \right\}. \end{aligned}$$

They constitute a complete system of eigenfunctions for $4\partial_\xi \bar{\partial}_\xi$ with $2\bar{\partial}_\xi = \partial_{\xi^1} + i\partial_{\xi^2}$. The eigenvalues of $2\bar{\partial}_\xi$ are

$$\lambda_{n_1, n_2}^{(\nu_1, \nu_2)} = \frac{2\pi}{\text{Im } \tau} \left\{ \left(n_1 + \frac{1 - \nu_1}{2} \right) \tau - \left(n_2 + \frac{1 - \nu_2}{2} \right) \right\}. \quad (3.25)$$

Since $2\bar{\partial} = 2\bar{\partial}_z = \frac{1}{\omega} \bar{\partial}_\xi$, the functions $\varphi_{n_1, n_2}^{(\nu_1, \nu_2)}$ are eigenfunctions of $2\bar{\partial}$ for the eigenvalues

$$\frac{\pi}{\text{Im } \tau \omega} \left\{ \left(n_1 + \frac{1 - \nu_1}{2} \right) \tau - \left(n_2 + \frac{1 - \nu_2}{2} \right) \right\}.$$

The holomorphic line bundle of positive spinors on E_u can be interpreted as a holomorphic square root $K^{1/2}$ of the bundle of $(1,0)$ -forms $K = T^{*(1,0)}E_u$. The chiral Dirac operators are

$$\begin{aligned}\not{D}^+ &= \bar{\partial} : C^\infty(K^{1/2}) \rightarrow C^\infty(K^{1/2} \otimes \bar{K}), \\ \not{D}^- &= -\partial : C^\infty(\overline{K^{1/2}}) \rightarrow C^\infty(\overline{K^{1/2}} \otimes K).\end{aligned}$$

Equivalently, we view the situation as follows: we take the preferred odd Spin-structure (P, P) as a reference $K_0^{1/2}$ and then twist the Dirac operator by a flat holomorphic line bundle $W_{(\nu_1, \nu_2)}$ of order two. The twisted chiral Dirac operator is

$$\not{D}_{(\nu_1, \nu_2)}^+ = \bar{\partial}_{(\nu_1, \nu_2)} : C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)}) \rightarrow C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)} \otimes \bar{K}) \quad (3.26)$$

If we set

$$\begin{aligned}\varphi_{n_1, n_2}^{(\nu_1, \nu_2)+}(z) &= \sqrt{dz} \varphi_{n_1, n_2}^{(\nu_1, \nu_2)}(z) \in C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)}), \\ \varphi_{n_1, n_2}^{(\nu_1, \nu_2)-}(z) &= \sqrt{d\bar{z}} \varphi_{n_1, n_2}^{(\nu_1, \nu_2)}(z) \in C^\infty(\overline{K_0^{1/2}} \otimes W_{(\nu_1, \nu_2)}),\end{aligned}$$

it follows

$$\begin{aligned}\not{D}_{(\nu_1, \nu_2)}^+ \varphi_{n_1, n_2}^{(\nu_1, \nu_2)+}(z) &= \lambda_{n_1, n_2}^{(\nu_1, \nu_2)} \varphi_{n_1, n_2}^{(\nu_1, \nu_2)}(z) \sqrt{dz} \otimes d\bar{z}, \\ \not{D}_{(\nu_1, \nu_2)}^- \varphi_{n_1, n_2}^{(\nu_1, \nu_2)-}(z) &= \overline{\lambda_{n_1, n_2}^{(\nu_1, \nu_2)}} \varphi_{n_1, n_2}^{(\nu_1, \nu_2)}(z) \sqrt{d\bar{z}} \otimes dz.\end{aligned}$$

Using the Kähler form, we can identify $\sqrt{dz} \otimes d\bar{z}$ with $\sqrt{d\bar{z}}$. Thus, the operator $(-4\partial\bar{\partial})_{(\nu_1, \nu_2)}$ has the eigenvalues

$$\left(\frac{\pi}{\text{Im } \tau |\omega|} \right)^2 \left| \left(n_1 + \frac{1-\nu_1}{2} \right) \tau - \left(n_2 + \frac{1-\nu_2}{2} \right) \right|^2.$$

The ζ -function regularization gives

$$\ln \det(-4\partial\bar{\partial})_{(\nu_1, \nu_2)} = -\zeta'(0) + \ln \left(\frac{\pi}{\text{Im } \tau |\omega|} \right)^2 \zeta(0),$$

where

$$\zeta_{(\nu_1, \nu_2)}(s) = \sum_{n_1, n_2} \frac{1}{\left[\left(n_1 + \frac{1-\nu_1}{2} \right)^2 \text{Im}^2 \tau + \left(\left(n_1 + \frac{1-\nu_1}{2} \right) \text{Re } \tau - \left(n_2 + \frac{1-\nu_2}{2} \right) \right)^2 \right]^s}, \quad (3.27)$$

which is absolutely convergent for $\text{Re } s > 1$. When $(\nu_1, \nu_2) = (1, 1)$ it is understood that the summation does not include $n_1 = n_2 = 0$. It was shown in [84, Sec. 5A and

App. E] that $\zeta(0) = 0$ for $(\nu_1, \nu_2) \neq (1, 1)$, and $\zeta(0) = -1$ for $(\nu_1, \nu_2) = (1, 1)$, and

$$\det(-4\partial\bar{\partial})_{(1,1)} = \text{Im}^2 \tau |\omega|^2 |\eta(\tau)|^4, \quad (3.28a)$$

$$\det(-4\partial\bar{\partial})_{(\nu_1, \nu_2)} = \left| \frac{\vartheta_{\nu_1 \nu_2}(\tau)}{\eta(\tau)} \right|^2. \quad (3.28b)$$

3.7.1 The Quillen metric

The chiral Dirac operator $\bar{\partial}_{(\nu_1, \nu_2)}$ has numerical index zero. Since it is also a skew-symmetric operator it has a mod-2 index, which is $\dim \ker \bar{\partial}_{(\nu_1, \nu_2)}$. For the odd Spin-structure (P, P) , we have

$$\dim \ker \bar{\partial}_{(1,1)} = 1,$$

and for the even Spin-structures (P, A) , (A, A) , and (A, P) , we have

$$\dim \ker \bar{\partial}_{(\nu_1, \nu_2)} = 0.$$

The operator is invertible for $(\nu_1, \nu_2) \neq (1, 1)$. The canonical section $\det(2\bar{\partial})_{(\nu_1, \nu_2)}$ trivializes the determinant line bundle such that

$$\|\det(2\bar{\partial})_{(\nu_1, \nu_2)}\|_Q^2 = \det(-4\partial\bar{\partial})_{(\nu_1, \nu_2)}.$$

For $(\nu_1, \nu_2) \neq (1, 1)$,

$$\det(2\bar{\partial})_{(\nu_1, \nu_2)} = \frac{\vartheta_{\nu_1, \nu_2}(0|\tau)}{\eta(\tau)}.$$

There is an extension to the Quillen construction by Belavin and Knizhnik (cf. [41, 85]) when the operator has a kernel and cokernel. We pick a holomorphic varying basis (ϕ) for $\ker \bar{\partial}_{1,1}$. The kernel consists of the constant functions: we take the constant zero mode $\phi = 1$ such that $\|\phi\|^2 = \text{vol}(E_u)$. By Serre duality we can identify the cokernel $\ker \bar{\partial}_{1,1}^*$ with the dual of the space of holomorphic one-forms. We pick $\chi = dz/\omega$ as a basis of the holomorphic one-forms with norm $\|dz\|^2 = \text{Im} \tau$. We obtain for the Quillen norm of the section \mathfrak{s}

$$\|\mathfrak{s}\|_Q^2 = \frac{\det_{\zeta}(-4\partial\bar{\partial})_{(1,1)}}{\|\phi\|^2 \|\chi\|^2} \|\mathfrak{s}\|^2 = |\eta(\tau)|^4 \|\mathfrak{s}\|^2.$$

$\|\chi\|^2$ appears in the denominator because of the Serre duality [41]. It is again possible to factorize the RHS holomorphically in τ given the explicit holomorphic trivialization. By choosing a different trivialization given by $\chi = dz$ with $\|dz\|^2 = \text{vol}(E_u)$ we obtain

$$\|\mathfrak{s}\|_Q^2 = \left| \frac{\eta(\tau)^2}{2\omega} \right|^2 \|\mathfrak{s}\|^2 = 2\pi \left| \frac{\vartheta_4(\tau)}{\eta} \right|^2 \|\mathfrak{s}\|^2.$$

We will consider the Seiberg-Witten family of elliptic curves. As shown above $dz/(2\omega)$ does not transform trivially under the action of $SL(2; \mathbb{Z})$ whereas dz does. Only the second trivialization gives a global trivialization on the elliptic surface.

3.7.2 The skew-adjoint operator

If $W_{(\nu_1, \nu_2)}$ is an element of order two, then the operator $\not\partial_{(\nu_1, \nu_2)}^+$ is the chiral Dirac operator with respect to a different Spin-structure. It follows that

$$\lambda_{n_1, n_2}^{(\nu_1, \nu_2)} = -\lambda_{-n_1+1-2\nu_1, -n_2+1-2\nu_2}^{(\nu_1, \nu_2)}$$

in Eq. (3.25), and thus the spectrum of $\not\partial^- \not\partial^+$ consists of eigenvalues $|\lambda_{(n_1, n_2)}|^2$ with multiplicity two. The determinant of the Dirac operator has a natural holomorphic square root, and $\det \not\partial^+$ is the square of a function $\text{pfaff } \not\partial^+$. The square root in Eq. (3.28b) is given by the relations

$$\begin{aligned} \vartheta_{00}(\tau)/\eta(\tau) &= \left(\frac{\eta(\tau)^2}{\eta(\tau/2)\eta(2\tau)} \right)^2, \\ \vartheta_{01}(\tau)/\eta(\tau) &= \left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^2, \\ \vartheta_{10}(\tau)/\eta(\tau) &= \left(\exp(i\pi\tau/8) \frac{\eta(2\tau)}{\eta(\tau)} \right)^2. \end{aligned}$$

The metric on E_u induces a metric on K , $K_0^{1/2}$, and $W_{(\nu_1, \nu_2)}$, and it gives rise to an anti-linear isomorphism

$$\begin{aligned} j : C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)} \otimes \bar{K}) &\rightarrow C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)}) \\ \text{by } \langle \eta_1, j(\eta_2) \rangle &= \int_{E_u} \eta_1 \eta_2 \end{aligned}$$

where $\eta_1 \in C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)})$, $\eta_2 \in C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)} \otimes \bar{K})$, and $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on $C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)})$. The first order elliptic differential operator

$$P_{(\nu_1, \nu_2)} = j \circ \not\partial_{(\nu_1, \nu_2)}^+ : C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)}) \rightarrow C^\infty(K_0^{1/2} \otimes W_{(\nu_1, \nu_2)})$$

is a complex anti-linear, skew-adjoint, bounded Fredholm operator with respect to $\text{Re} \langle \cdot, \cdot \rangle$. This is often denoted by $P_{(\nu_1, \nu_2)} \in \mathcal{F}_2 R$ (cf. [47]).

We describe the flat holomorphic connection $\mathfrak{a}(L)$ on the bundle $L_u = \mathcal{O}(Q_u - P_u)$ given a point $Q_u \in E_u$. The coordinates of the points $Q_u - P_u$ are

$$x_u = 4\mathcal{P}(z_{Q_u} | g_2(u), g_3(u)), \quad y_u = 4\mathcal{P}'(z_{Q_u} | g_2(u), g_3(u)). \quad (3.29)$$

z_{Q_u} is only defined modulo multiples of 2ω and $2\omega'$. The holonomy of the flat bundle

$L_u = \mathcal{O}(Q_u - P_u)$ is characterized by their transition functions around the A-cycle α_u and the B-cycle β_u , which can be taken to be constant phases. We identify a section along α_u with $e^{2\pi i\theta}$ times the section, and the section along β_u with $e^{-2\pi i\phi}$ times the section. A dual basis \mathcal{A}^* is comprised by the one-forms ρ and σ such that

$$\begin{aligned} \int_{\alpha_u} \rho &= 1, & \int_{\beta_u} \rho &= 0, \\ \int_{\alpha_u} \sigma &= 0, & \int_{\beta_u} \sigma &= 1, \end{aligned}$$

with

$$\begin{aligned} \rho &= -\frac{\bar{\tau}}{\tau - \bar{\tau}} \frac{dz}{2\omega} + \frac{\tau}{\tau - \bar{\tau}} \frac{d\bar{z}}{2\bar{\omega}} = dr, \\ \sigma &= \frac{1}{\tau - \bar{\tau}} \frac{dz}{2\omega} - \frac{1}{\tau - \bar{\tau}} \frac{d\bar{z}}{2\bar{\omega}} = ds. \end{aligned}$$

The holomorphic connection on $L_u = \mathcal{O}(Q_u - P_u)$ is described in terms of the harmonic one-forms on E_u

$$\begin{aligned} \mathfrak{a}(L)_u &= 2\pi i\theta \rho - 2\pi i\phi \sigma \\ &= \pi \left(-\frac{2\phi\bar{\omega} + 2\theta\bar{\omega}'}{\text{vol}(E_u)} dz + \frac{2\phi\omega + 2\theta\omega'}{\text{vol}(E_u)} d\bar{z} \right) \\ &= \frac{\pi}{\text{vol}(E_u)} (-\bar{z}_{Q_u} dz + z_{Q_u} d\bar{z}), \end{aligned} \tag{3.30}$$

with $\text{vol}(E_u) = 4 \text{Im} \tau |\omega|^2$ and $z_{Q_u} = 2\phi\omega + 2\theta\omega'$. We obtain $L_u = K_0^{1/2} \otimes W_{(\nu_1, \nu_2)}$ for

$$\theta = \frac{\pm 1 + \nu_1}{2}, \quad \phi = \frac{\pm 1 - \nu_2}{2}, \quad z_{Q_u} = -\nu_2\omega + \nu_1\omega' + (\pm\omega \pm \omega').$$

We have proved:

Lemma 7. *For $z_{Q_u} = 0$ which corresponds to the Spin-structure (P, P) in each smooth fiber it follows*

$$\det(-4\partial\bar{\partial})_{(1,1)} = \text{Im}^2 \tau |\omega|^2 |\eta(\tau)|^4 = \text{vol}(E_u)^2 \left| \frac{\eta^2(\tau)}{2\omega} \right|^2 = \text{vol}(E_u)^2 \left| \Delta(\tau)^{\frac{1}{12}} \right|^2.$$

For $z_{Q_u} = \omega'$ which corresponds to the Spin-structure (P, A) in each smooth fiber it follows

$$\begin{aligned} \det(-\partial\bar{\partial})_{(0,1)} &= \left| \frac{\vartheta_4(\tau)}{\eta(\tau)} \right|^2 = 4 \left| \frac{\eta^2(\tau)}{\vartheta_2(\tau)\vartheta_3(\tau)} \right|^2 = 8 \left| 2\pi \frac{\eta^2(\tau)}{2\omega} \right|^2 \\ &= 8 \left| \Delta(\tau)^{\frac{1}{12}} \right|^2. \end{aligned}$$

We also write down the content of the bozonization theorem [115, Thm. 1.3] as it applies to the Seiberg-Witten family of elliptic curves:

Lemma 8. *For each spin-structure (ν_1, ν_2) on the elliptic curve E_u with u fixed, it follows*

$$\left(\frac{\text{vol}(E_u) \text{Im } \tau}{\det'(-4\partial\bar{\partial})} \right)^{\frac{1}{2}} |\vartheta_{(\nu_1\nu_2)}(\tau)|^2 = \left| \frac{\vartheta_{(\nu_1\nu_2)}(\tau)}{\eta(\tau)} \right|^2 = \det(\not{\partial}^- \not{\partial}^+)_{(\nu_1\nu_2)} .$$

3.8 The vertical Dolbeault operator

In this section, we discuss the determinant line bundle of the vertical $\bar{\partial}$ -operator coupled to a flat holomorphic line bundle on the elliptic surface $Z \rightarrow \text{UP}$. We also compute the global anomaly of the determinant line bundle.

3.8.1 The line bundles on $Z \rightarrow \text{UP}$

Let us examine when the bundle of the form $\mathcal{O}(Q_u - P_u)$ with $z_{Q_u} = 2\phi\omega + 2\theta\omega'$ where ϕ, θ are constants lifts to a globally defined line bundle. As we encircle the singularities at $u = \pm 1$ in a cycle γ the periods transform as

$$\Omega = \begin{pmatrix} \omega' \\ \omega \end{pmatrix} \rightarrow \Omega' = M_\gamma \cdot \begin{pmatrix} \omega' \\ \omega \end{pmatrix} . \quad (3.31)$$

With

$$z_{Q_u} = (2\theta \ 2\phi) \cdot \Omega , \quad (3.32)$$

ϕ and θ transform as

$$(2\theta \ 2\phi) \rightarrow (2\theta \ 2\phi) \cdot M_\gamma^{-1} \quad (3.33)$$

In order to give a well-defined section z_{Q_u} on UP, z_{Q_u} can only be shifted by multiples of 2ω and $2\omega'$ which still gives the same coordinates x_u and y_u in Eqs. (3.29). Let us define the vectors

$$\mathbf{v}_1 = (m \ 0), \quad \mathbf{v}_{-1} = (-d \ 2d), \quad \mathbf{v}_\infty = (0 \ c), \quad (3.34)$$

with $m, d, c \in \mathbb{R}^+$. \mathbf{v}_1 is the eigenvector of the monodromy matrix M_1^{-1} with eigenvalue 1 with respect to multiplication from the right. Similarly, \mathbf{v}_{-1} is the eigenvector of the monodromy matrix M_{-1}^{-1} with eigenvalue 1. Finally, \mathbf{v}_∞ is the eigenvector of the monodromy matrix M_∞^{-1} with eigenvalue -1 . We observe that

$$\begin{aligned} \mathbf{v}_1 \cdot M_1^{-1} - \mathbf{v}_1 &= 0 , \\ \mathbf{v}_1 \cdot M_{-1}^{-1} - \mathbf{v}_1 &= (2m \ -4m) , \\ \mathbf{v}_1 \cdot M_\infty^{-1} - \mathbf{v}_1 &= (-2m \ 4m) , \end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_{-1} \cdot M_1^{-1} - \mathbf{v}_{-1} &= (2d \ 0), \\ \mathbf{v}_{-1} \cdot M_{-1}^{-1} - \mathbf{v}_{-1} &= 0, \\ \mathbf{v}_{-1} \cdot M_\infty^{-1} - \mathbf{v}_{-1} &= (2d \ -8d),\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_\infty \cdot M_1^{-1} - \mathbf{v}_\infty &= (c \ 0), \\ \mathbf{v}_\infty \cdot M_{-1}^{-1} - \mathbf{v}_\infty &= (c \ -2c), \\ \mathbf{v}_\infty \cdot M_\infty^{-1} - \mathbf{v}_\infty &= (0 \ -2c).\end{aligned}$$

Since we need invariance under the monodromy action up to multiples of two we have to set $m, d \in \mathbb{Z}$ and $c \in 2\mathbb{Z}$. We obtain $\mathbf{v}_1 = (1 \ 0)$, $\mathbf{v}_{-1} = (1 \ -2)$. Thus, the only non-trivial element that gives rise to a globally defined line bundle is $z_{Q_u} = \omega'$. The line bundle $\mathcal{O}(Q_u - P_u)$ corresponds to the even spin structure (P, A) in each smooth fiber.

3.8.2 The vertical $\bar{\partial}$ -operator coupled to a line bundle

We determine the Quillen metric of the determinant line bundle of the Dolbeault operators along the fiber coupled to the line bundle $\mathcal{O}(Q_u - P_u)$. For the vertical canonical line bundle of $\pi : Z \rightarrow \text{UP}$ for all $|u| < R < \infty$, we apply the results of Bismut and Bost [82] to show that we can include the nodal fibers at $u = \pm 1$. We denote by $\overline{\text{UP}} = \text{UP} \cup \{\pm 1\}$. We have a proper, surjective, holomorphic map of complex manifolds $\pi : Z \rightarrow \overline{\text{UP}}$ such that for all $u \in \mathbb{C}$ the fiber $E_u = \pi^{-1}(u)$ is a reduced curve with the singularities at $u = \pm 1$ being ordinary double points. We set $\omega_{Z/\overline{\text{UP}}} = \omega_Z \otimes (\pi^* \omega_{\overline{\text{UP}}})^{-1}$ where $\omega_Z, \omega_{\overline{\text{UP}}}$ denote the canonical line bundles of Z and $\overline{\text{UP}}$ respectively. $\omega_{Z/\overline{\text{UP}}}$ restricts to ω_{E_u} on each smooth fiber. Eq. (3.13) shows that ω_Z is trivial whence $\omega_{Z/\overline{\text{UP}}} = (\pi^* \omega_{\overline{\text{UP}}})^{-1}$. A choice of a Hermitian metric $\|\cdot\|$ on $\omega_{Z/\overline{\text{UP}}}$ restricts to the Kähler metric $i dz \wedge d\bar{z} / \text{vol}(E_u)$ on each smooth fiber.

A vertical nowhere-vanishing differential holomorphic one-form is given in the coordinates of Eq. (3.6) by $dz = dx/y$. The metric on $\omega_{Z/\overline{\text{UP}}}$ is $\|dz\| = 1$. In particular, it is a nowhere vanishing section of $\omega_{Z/\overline{\text{UP}}}$ and defines a trivialization. It follows that

$$c_1\left(\omega_{Z/\overline{\text{UP}}}, \|\cdot\|\right) = 0.$$

Let $(L, \|\cdot\|)$ be a holomorphic line bundle with Hermitian C^∞ -metric over Z . Then there exists a unique connection \mathfrak{a} on L which is compatible with the holomorphic structure and the metric. The Hermitian metric $\|\cdot\|_\xi$ on $L|_{E_u}$ determines L^2 -scalar products on the fiber E_u for $C^\infty(E_u; L)$ by

$$(\sigma_1, \sigma_2)_{L^2} = \int_{E_u} \langle \sigma_1, \sigma_2 \rangle \frac{i dz \wedge d\bar{z}}{\text{vol}(E_u)},$$

The $\bar{\partial}$ -operator on E_u with coefficients in L is

$$\bar{\partial}_{L,u} : C^\infty(E_u; L) \rightarrow C^\infty(E_u; L \otimes \bar{\omega}_{E_u}) ,$$

its adjoint is $\bar{\partial}_{L,u}^*$. We can form the following holomorphic line bundle

$$\lambda(L) \rightarrow \overline{\text{UP}} \quad \text{with fibers} \quad \lambda(L)_u = \Lambda^{\max} H^1(E_u, L) \otimes \left(\Lambda^{\max} H^0(E_u, L) \right)^{-1} .$$

Using the results of [82], the bundle $\lambda(L)$ extends across the points with nodal fibers. For $u \neq \pm 1$ we can interpret $\lambda(L)$ as the determinant line bundle $\text{DET } \bar{\partial}_{L,u}$ as follows: the line bundle $\lambda(L)$ is the determinant line bundle $\text{DET } \bar{\partial}_{L,u}$ over $\mathbb{C} - \{\pm 1\}$ with

$$H^0(E_u, L) \cong \ker \bar{\partial}_{L,u} , \quad H^1(E_u, L) \cong \ker \bar{\partial}_{L,u}^* .$$

The Quillen metric $\|\cdot\|_Q$ of the determinant line bundle is given by the regularized determinant

$$\|\cdot\|_Q = \sqrt{\det'(\bar{\partial}_{L,u}^* \bar{\partial}_{L,u})} \|\cdot\|_{L^2} .$$

Since L and $\omega_{Z/\overline{\text{UP}}}$ are equipped with a Hermitian C^∞ -metric, the Quillen metric is a Hermitian C^∞ -metric on the holomorphic fibers $\lambda(L)_u$ for $u \in \text{UP}$. The currents $\delta_{(\pm 1)}$ on Z are defined by saying that for every differential form α with compact support the equality

$$\int_Z \delta_{(\pm 1)} \wedge \alpha = \int_{E_{u=\pm 1}} \alpha$$

holds. One defines the first Chern-class $c_1(\text{DET } \bar{\partial}_{L,u}, \|\cdot\|_Q)$ as a current of type $(1, 1)$ by

$$c_1(\text{DET } \bar{\partial}_{L,u}, \|\cdot\|_Q) = \frac{1}{2\pi i} \partial \bar{\partial} \log \|\sigma\|_Q^2 , \quad (3.35)$$

where σ is a local non-vanishing holomorphic section. The results of [82, Thm. 2.1] and [83] can be applied to the situation with $L \rightarrow Z$ and $Z \rightarrow CP^1$. Thus,

$$\begin{aligned} & c_1(\text{DET } \bar{\partial}_{L,u}, \|\cdot\|_Q) \\ &= - \int_\pi \left[\text{ch}(L) \wedge \text{Todd}(\omega_{Z/\overline{\text{UP}}}; \|\cdot\|) \right]_{(4)} - \frac{\text{rank}(L)}{12} \delta_{(1)} - \frac{\text{rank}(L)}{12} \delta_{(-1)} \\ &= \int_\pi c_2(L, \|\cdot\|_\xi) - \frac{\text{rank}(L)}{12} \delta_{(1)} - \frac{\text{rank}(L)}{12} \delta_{(-1)} . \end{aligned} \quad (3.36)$$

We have proved:

Proposition 8. *Let $L_u = \mathcal{O}(Q_u - P_u)$ be the flat holomorphic line bundle $L \rightarrow Z$ with $x_u = -\frac{u}{3}$, $y_u = 0$. Let U be in $\overline{\text{UP}}$, and σ be a non-vanishing C^∞ -section of*

$\lambda(L)$ normalized such that $\|\sigma\| = 1$. It follows

$$\|\sigma\|_Q = |\Delta(\tau)|^{\frac{1}{12}}.$$

There exists a $c \in \mathbb{R}^+$ such that

$$\|\sigma\|_Q \sim c |u \pm 1|^{\frac{1}{6}} \quad (u \rightarrow \pm 1).$$

The holonomy of the section σ around $u = \pm 1$ is $e^{2\pi i \frac{1}{12}}$, and

$$c_1(\text{DET } \bar{\partial}_{L,u}, \|\cdot\|_Q) = -\frac{1}{12} \delta_{(1)} - \frac{1}{12} \delta_{(-1)}.$$

So far, we have considered the family of operators $\bar{\partial}_{L,u}$. Since L is not trivial, $\bar{\partial}_{L,u}$ is invertible, and we have computed the determinant of the Laplacian

$$\det \bar{\partial}_{L,u}^* \bar{\partial}_{L,u} = |\Delta^{\frac{1}{12}}(\tau)|^2 = \left| \frac{\vartheta_4(\tau)}{\eta(\tau)} \right|^2.$$

The RHS is a holomorphic function of τ . We have the canonical section $\det \bar{\partial}_{L,u}$. The holomorphic section

$$\frac{\eta(\tau)}{\vartheta_4(\tau)} \det \bar{\partial}_{L,u}$$

is the flat trivialization of the determinant line bundle on H with norm 1. The line bundle is acted on naturally by $SL(2; \mathbb{Z})$, and the canonical section is invariant up to roots of unity. It follows that we can consider the determinant line bundle on the quotient $H/\Gamma_0(4)$. For $u \in \text{UP}$ the local anomaly, i.e., the curvature of the determinant line bundle vanishes. Thus, the fundamental loops in $H/\Gamma_0(4)$ give rise to well-defined monodromies around $u = \pm 1, \infty$. The global anomaly is the monodromy of the canonical section $\det \bar{\partial}_{L,u}$ after quotienting out $\Gamma_0(4)$. The bundle $\text{DET } \bar{\partial}_{L,u}$ is isomorphic to a trivial line bundle on the universal covering space $\widetilde{\text{UP}}$, the section becomes a function on the universal covering space $\widetilde{\text{UP}}$.

If we consider the operator $\bar{\partial}_u$, we have to separate out the zero eigenvalue [80, Sec. 5]. There is a factorization of the determinant line bundle $\text{DET } \bar{\partial}_u$ as the tensor product of \mathcal{L}' and \mathcal{H} , corresponding to the non-zero and zero eigenvalues respectively. The bundle \mathcal{L}' has a holomorphic section $\det'(2\bar{\partial})_{(1,1)}$ whose norm was computed in Eq. (3.28) to be

$$\sqrt{\text{vol}(E_u)} \left| \Delta(\tau)^{\frac{1}{12}} \right| = \|dz\| \left| \frac{\eta^2(\tau)}{2\omega(\tau)} \right|.$$

On the other hand, we can identify \mathcal{H}^* as the bundle given by the holomorphic differential dz along the fibers. Thus, \mathcal{H} has the holomorphic section $(dz)^{-1}$ whose norm squared is $1/(4 \text{Im } \tau |\omega|^2)$. The product $(dz)^{-1} \det'(2\bar{\partial})_{(1,1)}$ is therefore a holomorphic section of \mathcal{L} with norm $|\frac{\eta^2(\tau)}{2\omega(\tau)}|$. Viewed as a elliptic surface over $H/\Gamma_0(4)$ we can

identify $dz/2\omega$ with $d\tau$ as they have the same behavior under the action of $SL(2; \mathbb{Z})$. This point of view was taken in [80]. However, as $dz = dX/Y$ is globally defined, only the first choice gives us a global trivialization of $\ker \bar{\partial}^*$. Identifying \mathcal{H}^* with $T^*\text{UP}$, by identifying dz with du , it follows that $(\text{DET } \bar{\partial}_u)^*$ can be identified with $T^*\text{UP}$ with a norm for which the section

$$\frac{\eta^2(\tau)}{2\omega(\tau)} du \quad (3.37)$$

has norm 1. The line bundle \mathcal{L}^* is acted on naturally by $SL(2; \mathbb{Z})$. The section (3.37) is invariant up to roots of unity. It follows that we can consider the determinant line bundle on the quotient H/M_A of the upper half-plane by the group generated by a monodromy matrix M_A . Now, if ω_A is the natural M_A -invariant differential it defines a standard trivialization of the determinant line bundle on the quotient. In this trivialization the section is given by

$$\frac{\eta^2(\tau)}{2\omega(\tau)} \frac{du}{\omega_A}. \quad (3.38)$$

For $A = M_\infty^{-1}$, we find using the results of Sec. 3.5

$$\frac{\eta^2(\tau)}{2\omega(\tau)} \frac{du}{\omega_A} = \frac{\eta^2(\tau)}{2\omega(\tau)} \frac{du}{\frac{du}{u}} = u \frac{\eta^2(\tau)}{2\omega(\tau)}.$$

For $\tau \rightarrow \tau + 4$ we have $\omega(\tau) \rightarrow -\omega(\tau)$, $u \rightarrow u$, and $\eta(\tau) \rightarrow \exp(4\pi i/12) \eta(\tau)$. We obtain

$$\begin{array}{c} \tau \rightarrow \tau + 4 \\ u \frac{\eta^2(\tau)}{2\omega(\tau)} \rightarrow e^{\pi i (\frac{2}{3}-1)} u \frac{\eta^2(\tau)}{2\omega(\tau)} = e^{2\pi i (-\frac{1}{6})} u \frac{\eta^2(\tau)}{2\omega(\tau)}. \end{array} \quad (3.39)$$

For $A = M_1$, we have

$$\frac{\eta^2(\tau)}{2\omega(\tau)} \frac{du}{\omega_A} = \frac{\eta^2(\tau)}{2\omega(\tau)} \frac{du}{\frac{du}{u-1}} = (u-1) \frac{\eta^2(\tau)}{2\omega(\tau)}.$$

For $\tau \rightarrow STS^{-1}\tau = \tau/(1-\tau)$ we have $\omega(\tau) \rightarrow (1-\tau)\omega(\tau)$, $u-1 \rightarrow u-1$, and

$$\eta \left(-\frac{1}{-\frac{1}{\tau} + 1} \right) = e^{\frac{\pi i}{12}} \sqrt{(1-\tau)} \eta(\tau).$$

We obtain

$$\begin{array}{c} \tau \rightarrow \tau/(1-\tau) \\ (u-1) \frac{\eta^2(\tau)}{2\omega(\tau)} \rightarrow e^{2\pi i (\frac{1}{12})} (u-1) \frac{\eta^2(\tau)}{2\omega(\tau)}. \end{array} \quad (3.40)$$

Eq. (3.40) determines the holonomy of the holomorphic section around $u = 1$ (and similarly around $u = -1$). The computed monodromy is equal to the monodromy of

$$\frac{\eta^2(\tau)}{2\omega(\tau)} = \Delta(\tau)^{\frac{1}{12}}.$$

For $u \rightarrow \infty$, we have $\tau \rightarrow \infty$, $q \rightarrow 0$, and find the asymptotics

$$\begin{aligned} u &= \frac{1}{8\sqrt{q}} \left(1 + O(q)\right), \\ \text{Im } \tau &= -\frac{1}{2\pi} \ln |q|^2, \\ \vartheta_3(\tau) \vartheta_4(\tau) &= 1 + O(q^2). \end{aligned}$$

In terms of the S-dual variables, we find $u_S \rightarrow 1$, and $\tau_S \rightarrow 0$ and find the asymptotics

$$\begin{aligned} u_S - 1 &= 32q \left(1 + O(q)\right), \\ q &= \frac{u_S - 1}{32} \left(1 + O(u_S - 1)\right), \\ \text{Im } \tau &= -\frac{1}{\pi} \ln |u_S - 1| \left(1 + o(u_S - 1)\right), \\ \text{Im } \tau_S |\omega(\tau_S)|^2 &= -2\pi \ln |u_S - 1| \left(1 + o(u_S - 1)\right). \end{aligned}$$

since $\text{Im } \tau_S |\omega(\tau_S)|^2 = 2\pi^2 \text{Im } \tau |\vartheta_3(\tau) \vartheta_4(\tau)|^2$. We have proved:

Proposition 9. *For the Dolbeault operator $\bar{\partial}_u$ along the fiber of $Z \rightarrow \text{UP}$, the determinant line bundle of the scalar Laplacian Δ_u has a standard trivialization around the loops encircling $u = \pm 1$, defined by the one-parameter groups for $M_{\pm 1}$. Since the determinant line bundle is flat it gives rise to a well-defined logarithmic monodromy. The ζ -function regularized determinant is*

$$\det' \Delta_u = \text{vol}(E_u)^2 |\Delta(\tau)|^{\frac{1}{6}}.$$

In particular, there exists a $c \in \mathbb{R}^+$ such that

$$\det' \Delta_u \sim c \left(\ln |u \pm 1|\right)^2 |u \pm 1|^{\frac{1}{6}} \quad (u \rightarrow \pm 1).$$

Remark 3. *The signature operator is $D = \bar{\partial} + \bar{\partial}_1$ where $\bar{\partial}_1$ is the $\bar{\partial}$ -operator on $(1,0)$ -forms. But on each fiber E_u multiplication by $d\bar{z}$ converts $\bar{\partial}$ into $\bar{\partial}_1$. Their determinant line bundles are isomorphic and so the the determinant line bundle of D is flat. The operator D is essentially two copies of $\bar{\partial}$, and the holonomy is twice the holonomy computed in Eqs. (3.39) and (3.40), which is half the Meyer's signature invariant.*

Remark 4. *The results of Prop. 8 and Prop. 9 are in agreement with [82, Thm. 2], [83, Sec. 13(c)].*

3.9 The Yang-Mills connection on the family of elliptic curves

In this section, we discuss the rank-two holomorphic vector bundles V with trivial determinant on the Jacobian elliptic fibration $Z \rightarrow \text{UP}$. We will consider bundles V for the more general gauge group $SU(2)/\mathbb{Z}_2$. We will compute the rational instanton number using the global anomaly of the $\bar{\partial}$ -operator along the fiber coupled to V .

Let V be a vector bundle with a Hermitian Yang-Mills connection such that V is semi-stable. This case was described in [70]: in the generic case, we have a reduction $V|_{Z_u} \cong L_u \oplus L_u^{-1}$ where L_u is a holomorphic line bundle on E_u . The analogue of a Yang-Mills connection in two dimensions is a flat $SU(2)$ -connection on V_u . Because of the semi-stability, the line bundle L_u has degree zero. We equip V_u with a connection and first consider the case that the holomorphic structure is decomposable. We use the base point P_u to identify the bundle L_u with $\mathcal{O}(Q_u - P_u)$ for a point $Q_u \in Z_u$. Similarly, we have $L_u^{-1} \cong \mathcal{O}(-Q_u - P_u)$.

Let us denote the coordinates of $Q_u - P_u$ as (x_u, y_u) . Then, the point $-Q_u + P_u$ has coordinates $(x_u, -y_u)$. The coordinate x_u can be described as solution of the equation $w_u = a_{0,u} + a_{2,u}x = 0$ with $w_u \in H^0(E_u, \mathcal{O}(2P_u))$. The function w is meromorphic with a pole only at the point at infinity P_u . To generalize this construction to all fibers, we have to promote a_i to a section of $\mathcal{N}^{-\otimes i}$, and Q_u becomes a section of the dual elliptic fibration. Moreover, the fiber over $u = \pm 1$ degenerate to a node. Therefore, we consider Q_u a section of the open sub-variety of points regular in their fibers, and then give an appropriate toroidal compactification at the nodal fibers. The spectral cover C is the two sheeted (possibly branched) cover of UP given by the hyper-surface $w = 0$ in Z .

The bundles produced in this way have the property that the restriction to most, but not all, fibers carry a flat $SU(2)$ -connection, i.e., the unique holomorphic connection determined by $Q_u \in Z_u$. At the branch points with $2Q_u = 0$, we obtain for the restriction of V to Z_u

$$V|_{Z_u} \cong \mathcal{O}(Q_u - P_u) \otimes \mathbf{F}_2 ,$$

where \mathbf{F}_2 is the unique non-trivial holomorphic extension bundle of $\underline{\mathbb{C}}$ by $\underline{\mathbb{C}}$, i.e.,

$$0 \rightarrow \underline{\mathbb{C}} \rightarrow \mathbf{F}_2 \rightarrow \underline{\mathbb{C}} \rightarrow 0 .$$

This bundle does not admit a flat $SU(2)$ -connection [70]. This means that one has to replace these fibers by non-isomorphic, S-equivalent bundles. Then fitting these together we can produce a stable bundle which carries a Hermitian Yang-Mills connection, the Yang-Mills connection on $V \rightarrow Z$ [70].

3.9.1 The Poincaré line bundle

The connection A_u on $V|_{E_u} = \mathcal{O}(Q_u - P_u) \oplus \mathcal{O}(-Q_u - P_u)$ in the generic case with $2Q_u \neq 0$ is

$$A_u = \frac{\pi}{\text{vol}(E_u)} \left(-\bar{z}_{Q_u} dz \otimes \sigma^3 + z_{Q_u} d\bar{z} \otimes \sigma^3 \right) , \quad (3.41)$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The application of the results in [70, Sect. 2.3 and Sect. 7.4] shows that conversely any generic rank-two vector bundle V with vanishing first Chern class on the elliptic fibration $Z \rightarrow \text{UP}$ can be reconstructed from its spectral cover C defined by $\pm Q_u - P_u$: we start with $C \times_{CP^1} Z \subset Z \times_{CP^1} Z$. The two-fold covering map π_2 from $C \times_{CP^1} Z$ to Z is obtained by forgetting the first coordinate. We define the Poincaré line bundle $\mathbf{P} \rightarrow \xi^* Z$ by the commutative diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\text{Jac}(E_u)} & \xi^* Z & \longleftarrow & \mathbf{P} \\ E_u \downarrow \pi & & E_u \downarrow & & \\ \text{UP} & \xleftarrow[\xi]{\text{Jac}(E_u)} & Y & & \end{array} \quad (3.42)$$

It is the unique holomorphic line bundle whose fiber over $(u, \check{\xi})$ is the flat bundle $W_{\check{\xi}} \rightarrow E_u$ for the character $\xi = \alpha + \tau\beta \mapsto e^{2\pi i(\alpha\theta - \beta\phi)}$ for $\check{\xi} = \phi + \tau\theta \in \text{Jac}(E_u)$. Over (u, ξ) , it is the flat line bundle $W_{-\xi} \rightarrow \text{Jac}(E_u)$ for the character $\check{\xi} = \phi + \tau\theta \mapsto e^{2\pi i(\alpha\theta - \beta\phi)}$ for $\xi = \alpha + \tau\beta \in E_u$. The flat metric on E_u is the metric along $\xi^* Z/Y$ which lifts to the constant flat metric along the fibers of $Y \times \mathbb{C} \rightarrow Y$, so the curvature $\Omega^{(\xi^* Z/Y)}$ of the vertical line bundle vanishes. The local coordinate

$$\check{z} = 2\omega \check{\xi} = 2\phi\omega + 2\theta\omega' .$$

does not lift to the total space Y , however, $d\check{z}$ is well-defined. The unitary curvature of the Poincaré line bundle \mathbf{P} is

$$\Omega^{\mathbf{P}} = \frac{\pi}{\text{vol}(E_u)} (dz \wedge d\check{z} - d\bar{z} \wedge d\check{z}) .$$

Locally, the unitary connection is

$$\omega^{\mathbf{P}} = \frac{\pi}{\text{vol}(E_u)} (-\check{z} dz + \check{z} d\bar{z} + z d\check{z} - \bar{z} d\check{z}) .$$

The bundle $V \rightarrow Z$ is the push-forward of the the Poincaré line-bundle restricted to $C \times_{CP^1} Z$ to Z which we denote by $\pi_{2*}(\mathbf{P})$. Thus, in the semi-stable case the bundle V is defined by the spectral cover as $V = \pi_{2*}(\mathbf{P})$. For a generic connection over a

smooth fiber, i.e., $u \neq \pm 1, \infty$, the kernel of $\ker \bar{\partial}_{a(u)}$ vanishes. The dimension jumps only when either Q_u is a point of order two (and then $V|_{E_u} = \mathbf{F}_2$ and it jumps by one) or $Q_u = P_u$ (and then $V|_{E_u} = \mathbb{C} \oplus \mathbb{C}$ and it jumps by two). From index theory, we know that the number of jumping points counted with multiplicity is computed by the first Chern number of the index bundle. From [70, Eq. 7.67] we also have

$$c_2(V) = -\frac{1}{2}\pi_{2*}(c_1(\mathbf{P})^2) .$$

It remains to explain how we treat the singular fibers over $u = \pm 1, \infty$. The Yang-Mills density is

$$-2 \operatorname{tr}(F_A \wedge *F_A) = |F_A|^2 * 1 .$$

We will require the gauge connection to approach an asymptotic state close to the singular fibers. An asymptotic state is a trivial flat connection $A_{0,u}$ on a topologically trivial split bundle

$$A_{0,u} = \frac{\pi}{\operatorname{vol}(E_u)} \left(-\bar{z}_{Q_{0,u}} dz \otimes \sigma^3 + z_{Q_{0,u}} d\bar{z} \otimes \sigma^3 \right) ,$$

for $z_{Q_{0,u}} = 2\phi_0 \omega + 2\theta_0 \omega'$ with ϕ_0, θ_0 constant. The connection is of pure gauge since

$$A_{0,u} = 2\pi i \theta_0 dr \otimes \sigma^3 - 2\pi i \phi_0 ds \otimes \sigma^3 .$$

In the remainder of the section, we will determine the possible asymptotic states for the gauge connection.

3.9.2 The behavior for $u \rightarrow \infty$

We look at the generic connections A subject to the following condition: The connection becomes pure gauge for $|u| \rightarrow \infty$ and A approaches a trivial flat connection A_0 on a topologically trivial split bundle over $Z_{|u|>1}$ such that for $|u| \rightarrow \infty$ we have

$$|A - A_0| = O(|u|^{-1} \ln |u|) . \tag{3.43}$$

The constant flat connection on each fiber E_u determines uniquely a holomorphic structure on the restriction of V to the fiber. The bundle must split holomorphically as the sum of two line bundles uniquely up to ± 1 . The assumption (3.43) implies that

$$|F_A| = O(|u|^{-2}) .$$

Then, we have a finite-energy contribution for $|u| \rightarrow \infty$, i.e., there exists a $c \in \mathbb{R}^+$ such that for $R \gg 1$

$$\begin{aligned} \int_{Z_{|u|>R}} L_{\text{YM}} &= \int_{|u|>R} du \wedge d\bar{u} \int_{E_u} dz \wedge d\bar{z} |F_A|^2 \leq c \int_{|u|>R} du \wedge d\bar{u} \frac{\ln |u|}{|u|} |F_A|^2 \\ &\leq c \int_{|u|>R} du \wedge d\bar{u} \frac{\ln |u|}{|u|^5} \leq 2\pi c \int_{r>R} dr \frac{\ln r}{r^4} < \infty. \end{aligned}$$

We have used the asymptotics for $u \rightarrow \infty$

$$\begin{aligned} u &= \frac{1}{8\sqrt{q}} \left(1 + O(q)\right), \\ q &= \frac{1}{64u^2} \left(1 + O(u^{-1})\right), \\ \omega &= 2\sqrt{2}\pi q^{\frac{1}{4}} \left(1 + O(q)\right) = \frac{\pi}{\sqrt{u}} \left(1 + O(u^{-1})\right), \\ \text{Im } \tau &= -\frac{1}{2\pi} \ln |q|^2 = \frac{2}{\pi} \ln |u| \left(1 + O(u^{-1})\right), \\ \text{Im } \tau |\omega|^2 &= 2\pi \frac{\ln |u|}{|u|} \left(1 + O(u^{-1})\right), \end{aligned}$$

Because of

$$|A - A_0| = \frac{\pi |\check{z}_{Q_u} - \check{z}_{Q_{0,u}}|}{\text{vol}(E_u)}.$$

we have to have

$$|\check{z}_{Q_u} - \check{z}_{Q_{0,u}}| = O\left(\frac{\ln^2(u)}{u^2}\right)$$

to satisfy the bound in Eq. (3.43). The interpretation of such a choice is that over the region in which $|u| > R$ is large the bundle becomes the trivial holomorphic split bundle. The restriction of the extended bundle to the added divisor is the split bundle of the flat trivial line bundle. The condition of having a trivial bundle over the fiber at infinity can be viewed as follows: having a rank two vector bundles on the singular I_4^* -fiber, each component is a smooth CP^1 . Thus, on each component one will have a split bundle with the unique holomorphic structure. Now the question is, how these bundles are glued together. If each bundle is trivial, then there is no issue, as the result is a trivial bundle which splits. This is because the gluing data would have no homology, as there are no cycles in the dual graph. However, if there are non-trivial bundles on the components, then in order for the global bundle to split there should be more sophisticated matching conditions at the intersection points.

Let us determine which asymptotic states $z_{Q_{0,u}} = 2\phi_0 \omega + 2\theta_0 \omega'$ with ϕ_0, θ_0 constant are possible. As we move around the point $u = \infty$, the periods transform

according to

$$\Omega \rightarrow \Omega' = M_\infty \cdot \Omega .$$

ϕ_0 and θ_0 transform as

$$(2\theta_0 \ 2\phi_0) \rightarrow (2\theta_0 \ 2\phi_0) \cdot M_\infty^{-1} = (-2\theta_0 \ , \ -8\theta_0 - 2\phi_0) \ ,$$

whence

$$(2\theta_0 \ 2\phi_0) - (2\theta_0 \ 2\phi_0) \cdot M_\infty^{-1} = (4\theta_0, \ 8\theta_0 + 4\phi_0) \ .$$

The allowed asymptotic states are obtained for $\phi_0, \theta_0 \in \{0, \frac{1}{2}\}$ since multiples of two will shift $z_{Q_0,u}$ by multiples of 2ω and $2\omega'$. If we exclude the trivial connection $\phi_0 = \theta_0 = 0$, the allowed asymptotic states correspond to the non-trivial spin-structures (P, A) , (A, P) , (A, A) . This can also be seen from the fact that the coordinates $(x_u, y_u = 0)$ in Eq. (3.4) remain unchanged for $u = \lim_{t \rightarrow 1} R e^{2\pi i t}$ with $R > 1$.

In physics, the spin structures are called BPS-states for $|u| \rightarrow \infty$. The states with $z_{Q_0,u} = \omega' + 2n\omega$ and $z_{Q_0,u} = \omega' + (2n + 1)\omega$ are two families of BPS-states (monopoles) they correspond to the spin structures (P, A) and (A, A) . The spin structure (A, P) corresponds to an electrically charged particle with $z_{Q_0,u} = \omega$. This explains a remark in [30, Lec. II-18, Sec. I.7] which describes the BPS-states at infinity. The spin structures (A, P) and (A, A) only define a $SU(2)$ -bundle for $|u| > 1$ and transform into each other when going around $u = \pm 1$.

The bundle $V \rightarrow Z$ with $V_u = \mathcal{O}(Q_u - P_u) \oplus \mathcal{O}(-Q_u - P_u)$ is well-defined if $Q_u - P_u$ for $|u| \rightarrow \infty$ approaches asymptotically an order-two point or the trivial bundle in the fiber. The asymptotics of the Quillen metric for the coupled Dolbeault operator $\bar{\partial}_{V,u}$ for $|u| \rightarrow \infty$ can be computed from the Quillen metric for the the coupled Dolbeault operator $\bar{\partial}_{V_0,u}$ where V_0 is the well-defined bundle on $Z_{|u|>1}$ given by an asymptotic state. We have proved:

Proposition 10. *The Quillen metric for the Dolbeault operator $\bar{\partial}_{V_0,u}$ coupled to an asymptotic state is given by*

$$\det(\bar{\partial}_{V_0,u}^* \bar{\partial}_{V_0,u}) = \left| \frac{\vartheta_b(\tau)}{\vartheta_4(\tau)} \right|^4 \left| \Delta(\tau)^{\frac{2}{12}} \right|^2$$

with $b = 4$ for $z_{Q_0,u} = \omega'$, and $b = 2, 3$ for $z_{Q_0,u} = \omega, \omega + \omega'$. The flat holomorphic trivialization of DET $\bar{\partial}_{V_0,u}$ for $u > 1$ gives

$$\det \bar{\partial}_{V_0,u} = \frac{\vartheta_b^2(\tau)}{\vartheta_4^2(\tau)} \Delta(\tau)^{\frac{2}{12}} = \frac{\vartheta_b^2(\tau)}{\eta^2(\tau)} \ .$$

For $\tau \rightarrow \tau + 4$, the monodromy of the canonical section around $u = \infty$ is trivial since

$$\frac{\vartheta_b^2(\tau + 4)}{\vartheta_4^2(\tau + 4)} = \frac{\vartheta_b^2(\tau)}{\vartheta_4^2(\tau)} \ .$$

The regularized determinant

$$\det(\bar{\partial}_{V_{0,u}}^* \bar{\partial}_{V_{0,u}}) = |\sigma|^4 \left| \Delta(\tau)^{\frac{2}{12}} \right|^2 ,$$

for the possible asymptotic states for $|u| \rightarrow \infty$ are summarized in the table (3.44).

(ν_1, ν_2)	(θ, ϕ)	Spin –struct.	$z_{Q_{0,u}}$	$ \sigma ^4$
(1, 1)	(0, 0)	(P, P)	0	$\text{vol}^4(E_u)$
(0, 1)	$(\frac{1}{2}, 0)$	(P, A)	ω'	1
(1, 1)	$(\frac{1}{2}, \frac{1}{2})$	(A, A)	$\omega + \omega'$	$ \vartheta_3(\tau)/\vartheta_4(\tau) ^4$
(1, 0)	$(0, \frac{1}{2})$	(A, P)	ω	$ \vartheta_2(\tau)/\vartheta_4(\tau) ^4$

(3.44)

3.9.3 The behavior for $u \rightarrow \pm 1$.

We look at the generic connections A subject to the following condition: The connection becomes pure gauge for $|u| \rightarrow \pm 1$ and A approaches a trivial flat connection A_0 on a topologically trivial split bundle over $Z_{|u \pm 1| > \epsilon}$ such that

$$|A - A_0| = O(|u \pm 1|) . \quad (3.45)$$

The constant flat connection on each fiber E_u determines uniquely a holomorphic structure on the restriction of V to the fiber. The bundle must split holomorphically as the sum of two line bundles uniquely up to ± 1 . The assumption (3.45) implies that

$$|F_A| = O(1) .$$

Then, we have a finite-energy contribution from a small punctured ball $B_\epsilon(\pm 1)$ of radius ϵ around $u = \pm 1$. There exists a $c \in \mathbb{R}^+$ such that

$$\begin{aligned} \int_{Z_{B_\epsilon(\pm 1)}} L_{\text{YM}} &= \int_{|u \pm 1| < \epsilon} du \wedge d\bar{u} \int_{E_u} dz \wedge d\bar{z} |F_A|^2 \\ &\leq -c \int_{|u \pm 1| < \epsilon} du \wedge d\bar{u} \ln |u \pm 1| |F_A|^2 \\ &\leq -c \int_{|u \pm 1| < \epsilon} du \wedge d\bar{u} \ln |u \pm 1| \leq -c \int_{0 \leq r < \epsilon} dr \ln r < \infty . \end{aligned}$$

We have used that for $u \rightarrow \pm 1$ the volume of the fiber E_u has the asymptotics

$$\text{Im } \tau |\omega(\tau)|^2 = -2\pi \ln |u \pm 1| \left(1 + o(u \pm 1) \right) .$$

Let us determine which asymptotic states $z_{Q_{0,u}} = 2\phi_0 \omega + 2\theta_0 \omega'$ with ϕ_0, θ_0 constant are possible. As we move around the singularities at $u = \pm 1$ the periods transform according to Eq. (3.31) and Eq. (3.33). Let us examine when the bundle of the form $V_{0,u} = \mathcal{O}(Q_{0,u} - P_u) \oplus \mathcal{O}(-Q_{0,u} - P_u)$ over $Z \rightarrow B_\epsilon(\pm 1) \subset \text{UP}$ is well-defined. Let us look at the more general question of which covectors $(2\theta_0 \ 2\phi_0)$ are invariant

under the monodromy action up to action of the center \mathbb{Z}_2 . The \mathbb{Z}_2 -action is the action of the constant gauge transformation $i\sigma^2$ on $B_\epsilon(\pm 1)$ that changes the sign of the gauge potential

$$A_{0,u} \rightarrow -A_{0,u} = (i\sigma^2)^\dagger A_{0,u} (i\sigma^2).$$

For the vectors defined in Eq. (3.34) we check that

$$\mathbf{v}_1 \cdot M_1^{-1} \pm \mathbf{v}_1 = 0, \quad \mathbf{v}_1 \cdot M_{-1}^{-1} \pm \mathbf{v}_1 = (4m - 4m),$$

and

$$\mathbf{v}_{-1} \cdot M_1^{-1} \pm \mathbf{v}_{-1} = (0 \ 4d), \quad \mathbf{v}_{-1} \cdot M_{-1}^{-1} \pm \mathbf{v}_{-1} = 0.$$

Since we require invariance under the monodromy action only up to the action of \mathbb{Z}_2 it follows $m, d \in \frac{1}{2}\mathbb{Z}$. The non-trivial central charges $z^{(m/2)} = \frac{1}{2}\omega'$ and $z^{(d/2)} = -\omega + \frac{1}{2}\omega'$ give rise to a $SU(2)$ -bundle on a small disc around $u = \pm 1$.

Remark 5. *The central charge $z^{(m/2)} = \frac{1}{2}\omega'$ defines a bundle with trivial holonomy around $u = 1$, and $z^{(d/2)} = -\omega + \frac{1}{2}\omega'$ defines a bundle with trivial holonomy around $u = -1$. The bundle for $z^{(m/2)}$ (and $z^{(d/2)}$ respectively) have a non-trivial holonomy -1 around $u = \infty$ and $u = -1$ (and $u = \infty$ and $u = 1$ respectively). This explains a remark in [30, Lec. II-18, Sec. I.7] that the BPS-states around $u = \pm 1$ do not exist for $|u| \rightarrow \infty$.*

Remark 6. *The bundle $V \rightarrow Z$ with $V_u = \mathcal{O}(Q_u - P_u) \oplus \mathcal{O}(-Q_u - P_u)$ is well-defined if $Q_u - P_u$ for $u \rightarrow 1$ (and $u \rightarrow -1$ respectively) approaches $Q_u^{(m/2)} - P_u$ (and $Q_u^{(d/2)} - P_u$ respectively).*

3.9.4 The $SU(2)/\mathbb{Z}_2$ -bundles on the Seiberg-Witten curve

In this section, we will compute the asymptotics of the Quillen metric for the coupled Dolbeault operator $\bar{\partial}_{V,u}$ for $u \rightarrow \pm 1$. It can be computed from the Quillen metric for the the coupled Dolbeault operator $\bar{\partial}_{V_0,u}$ where V_0 is the well-defined bundle on $B_\epsilon(\pm 1) \subset \text{UP}$ for the asymptotic states $Q_{0,u}$ determined in Sec. 3.9.3.

We describe the case involving the quarter periods. $Q_{0,u}$ coincides with an order two point if and only if $u = \pm 1$ or $u \rightarrow \infty$. Thus, we have

$$\forall u \in B_\epsilon(\pm 1) \subset \text{UP} : 2Q_{0,u} \neq 0.$$

Let us compute the coordinates of the points on the Jacobian for

$$z^{(m/2)} = \frac{1}{2}\omega' \quad \text{and} \quad z^{(d/2)} = -\omega + \frac{1}{2}\omega'.$$

An identity [1] for the Weierstrass function implies

$$\mathcal{P}\left(\frac{1}{2}\omega'(u) \mid g_2(u), g_3(u)\right) = e_3 - H_3,$$

where $H_3^2 = 2e_3^2 + e_1 e_2 = 1/64$. Thus, we obtain

$$\mathcal{P}\left(\frac{1}{2}\omega'(u) \mid g_2(u), g_3(u)\right) = -\frac{u}{12} - \frac{1}{8}.$$

The points $\pm Q_u^{(m/2)} - P_u$ for $\pm z^{(m/2)} = \pm \frac{1}{2}\omega'$ have the coordinates

$$\mathbf{x}_u^{(m/2)} = -\frac{u}{12} - \frac{1}{8}, \quad \mathbf{y}_u^{(m/2)} = \pm \frac{i}{8}\sqrt{u+1}.$$

Similarly, for $z^{(d/2)} = -\omega + \frac{1}{2}\omega'$ we use the identity

$$\mathcal{P}\left(-\omega(u) + \frac{1}{2}\omega'(u) \mid g_2(u), g_3(u)\right) = e_3 + H_3.$$

Thus, we obtain

$$\mathcal{P}\left(-\omega(u) + \frac{1}{2}\omega'(u) \mid g_2(u), g_3(u)\right) = -\frac{u}{12} + \frac{1}{8}.$$

The points $\pm(Q_u^{(d/2)} - P_u)$ for $\pm z^{(d/2)} = \pm(-\omega + \frac{1}{2}\omega')$ have the coordinates

$$\mathbf{x}_u^{(d/2)} = -\frac{u}{12} - \frac{1}{8}, \quad \mathbf{y}_u^{(d/2)} = \pm \frac{i}{8}\sqrt{u-1}.$$

The bundles

$$\begin{aligned} V_u^{(m)} &= \mathcal{O}(Q_u^{(m/2)} - P_u) \oplus \mathcal{O}(-Q_u^{(m/2)} - P_u), \\ V_u^{(d)} &= \mathcal{O}(Q_u^{(d/2)} - P_u) \oplus \mathcal{O}(-Q_u^{(d/2)} - P_u) \end{aligned}$$

do not exist as $SU(2)$ -bundles on UP because of the non-trivial monodromy. However, the associated adjoint bundles $\text{Ad} V$ exist as $SO(3)$ -bundle. For the gauge group $SU(2)/\mathbb{Z}_2$, the first homotopy group is $\pi_1(SU(2)/\mathbb{Z}_2) = \mathbb{Z}_2$.

Going around the point $u = 1$ leaves the bundle $V^{(m)}$ invariant whereas the monodromy action around $u = -1$ and $u = \infty$ is the non-trivial element $[-1]$ in $\pi_1(SU(2)/\mathbb{Z}_2)$. Similarly, going around the point $u = -1$ leaves the bundle $V^{(d)}$ invariant whereas the monodromy action around $u = 1$ and $u = \infty$ is $[-1]$. Thus, we have proved:

Lemma 9. *The BPS-state for $u = \pm 1$ define non-trivial $SU(2)/\mathbb{Z}_2$ -bundles on $Z \rightarrow \text{UP}$.*

For the asymptotic states $\pm z^{(m/2)} = \pm \frac{1}{2}\omega'$, we find $\nu_1 = \mp \frac{1}{2}$ and $\nu_2 = 1$ in Eq. (3.28b),

and thus for the ζ -regularized determinant

$$\begin{aligned} \det(\bar{\partial}_{V,u}^* \bar{\partial}_{V,u}) &\sim \left| \frac{\vartheta_{-\frac{1}{2}1}(\tau)}{\eta(\tau)} \frac{\vartheta_{\frac{1}{2}1}(\tau)}{\eta(\tau)} \right|^2 = \left| \frac{\vartheta_{-\frac{1}{2}1}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}1}(\tau)}{\vartheta_4(\tau)} \right|^2 \left| \frac{\vartheta_4(\tau)}{\eta(\tau)} \right|^4 \\ &= \left| \frac{\vartheta_{-\frac{1}{2}1}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}1}(\tau)}{\vartheta_4(\tau)} \right|^2 \left| \Delta(\tau)^{\frac{2}{12}} \right|^2. \end{aligned}$$

For $\pm z^{(d/2)} = \pm(-\omega + \frac{1}{2}\omega')$, we find $\nu_1 = \mp \frac{1}{2}$ and $\nu_2 = 0$, and

$$\det(\bar{\partial}_{V,u}^* \bar{\partial}_{V,u}) \sim \left| \frac{\vartheta_{-\frac{1}{2}0}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}0}(\tau)}{\vartheta_4(\tau)} \right|^2 \left| \Delta(\tau)^{\frac{2}{12}} \right|^2.$$

Therefore, we investigate the logarithmic monodromy of the section

$$\sigma_b(\tau) = \frac{\vartheta_{-\frac{1}{2}b}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}b}(\tau)}{\vartheta_4(\tau)}$$

with $b = 0, 1$. Eq. (3.19) implies that for $\tau \rightarrow \tau + 4$

$$\frac{\vartheta_{-\frac{1}{2}b}(\tau + 4)}{\vartheta_4(\tau + 4)} \frac{\vartheta_{\frac{1}{2}b}(\tau + 4)}{\vartheta_4(\tau + 4)} = \frac{\vartheta_{-\frac{1}{2}b+6}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}b+2}(\tau)}{\vartheta_4(\tau)} = \frac{\vartheta_{-\frac{1}{2}b}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}b}(\tau)}{\vartheta_4(\tau)}.$$

Thus, the monodromy around $u = \infty$ is trivial since

$$\sigma_b(\tau + 4) = \sigma_b(\tau).$$

Eq. (3.19) implies for $\tau \rightarrow STS^{-1}\tau$ that

$$\begin{aligned} \frac{\vartheta_{-\frac{1}{2}b}(STS^{-1}\tau)}{\vartheta_4(STS^{-1}\tau)} \frac{\vartheta_{\frac{1}{2}b}(STS^{-1}\tau)}{\vartheta_4(STS^{-1}\tau)} &= \frac{\vartheta_{b-\frac{1}{2}}(TS^{-1}\tau)}{\vartheta_2(TS^{-1}\tau)} \frac{\vartheta_{b\frac{1}{2}}(TS^{-1}\tau)}{\vartheta_2(TS^{-1}\tau)} \\ &= e^{\frac{\pi i}{2}(b-1)} \frac{\vartheta_{b(\frac{1}{2}+b)}(S^{-1}\tau)}{\vartheta_2(S^{-1}\tau)} \frac{\vartheta_{b(\frac{3}{2}+b)}(S^{-1}\tau)}{\vartheta_2(S^{-1}\tau)}. \end{aligned}$$

Using Eq. (3.20) and $b = 0, 1$, we obtain

$$\begin{aligned} \frac{\vartheta_{-\frac{1}{2}b}(STS^{-1}\tau)}{\vartheta_4(STS^{-1}\tau)} \frac{\vartheta_{\frac{1}{2}b}(STS^{-1}\tau)}{\vartheta_4(STS^{-1}\tau)} &= e^{\frac{\pi i}{2}(b-1)} \frac{\vartheta_{b(-\frac{1}{2}+b)}(S^{-1}\tau)}{\vartheta_2(S^{-1}\tau)} \frac{\vartheta_{b(\frac{1}{2}-b)}(S^{-1}\tau)}{\vartheta_2(S^{-1}\tau)} \\ &= e^{\frac{\pi i}{2}(b-1)} \frac{\vartheta_{(-\frac{1}{2}+b)b}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{(\frac{1}{2}-b)b}(\tau)}{\vartheta_4(\tau)}. \end{aligned}$$

Thus, the monodromy around $u = 1$ is

$$\sigma_b(STS^{-1}\tau) = e^{-\frac{\pi i}{2}\delta_{b,0}} \sigma_b(\tau).$$

Finally, we compute the monodromy for $\tau \rightarrow T^2STS^{-1}T^{-2}\tau$. Eqs. (3.19), (3.20) imply that

$$\sigma_b(T^2\tau') = -\sigma_{b'}(\tau') ,$$

where $b' = 1 - b$. Thus, the monodromy around $u = -1$ is

$$\sigma_b(T^2STS^{-1}T^{-2}\tau) = e^{-\frac{\pi i}{2}\delta_{b,1}} \sigma_b(\tau) .$$

We have proved:

Proposition 11. *For the Dolbeault operator $\bar{\partial}_{V,u}$ coupled to $V_u = \mathcal{O}(Q_u - P_u) \oplus \mathcal{O}(-Q_u - P_u)$ on $Z \rightarrow B_\epsilon(\pm 1) \subset \text{UP}$ restricted to a small ball in vicinity of the nodal points $u = \pm 1$, the Quillen metric is given by the ζ -function regularized determinant*

$$\det(\bar{\partial}_{V,u}^* \bar{\partial}_{V,u}) = \left| \frac{\vartheta_{-\frac{1}{2}b}(\tau)}{\vartheta_4(\tau)} \frac{\vartheta_{\frac{1}{2}b}(\tau)}{\vartheta_4(\tau)} \right|^2 \left| \Delta(\tau)^{\frac{2}{12}} \right|^2 . \quad (3.46)$$

$Q_u - P_u$ must approach either $z_u^{(m/2)} = \frac{1}{2}\omega'$ ($b = 1$ in Eq. (3.46)) or $z_u^{(d/2)} = -\omega + \frac{1}{2}\omega'$ ($b = 0$ in Eq. (3.46)) to give a $SU(2)/\mathbb{Z}_2$ -bundle over Z . The first Chern class of the index bundle in Eq. (3.35) is

$$c_1(\det \bar{\partial}_{V,u}, \|\cdot\|_Q) = \pi_* c_2(V) - \frac{1}{6}\delta_{(1)} - \frac{1}{6}\delta_{(-1)} . \quad (3.47)$$

The logarithmic monodromy of the holomorphic section of the determinant line bundle determines the current contributions to the first Chern class

$$\pi_* c_2 \left(V \Big|_{B_\epsilon(u_0)} \right) = \begin{cases} \frac{1}{4}\delta_{(-1)} & \text{if } b = 1, \text{ and } u_0 = -1 \\ \frac{1}{4}\delta_{(1)} & \text{if } b = 0, \text{ and } u_0 = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Remark 7. Eq. (3.47) generalizes [70, Eq. (7.72)] to include the case of nodal fibers. The current contribution to Chern class is 1/4 and agrees with a result in [31, Eq. (3.37)].

3.10 The geometric quantization

In this section, we apply the results of [71] to the Seiberg-Witten family of elliptic curves. This will give a quantum mechanical description of the BPS-states found in Sec. 3.8.

There is a globally defined, real, closed non-vanishing two-form form Ω on UP. In terms of the holomorphic two form η , Ω is given by

$$\Omega = \int_{E_u} \eta \wedge \bar{\eta} = 8i \operatorname{Im} \tau |\omega(u)|^2 du \wedge d\bar{u} .$$

We integrate η over the A-cycle and B-cycle to obtain

$$\int_{\text{A-cycle}} \eta = 2\omega du, \quad \int_{\text{B-cycle}} \eta = 2\omega' du.$$

They supply a set of holomorphic sections in every patch of the u -plane UP. Such a basis is unique up to $SL(2; \mathbb{Z})$ -transformations. Under a change of the coordinate chart on $U \cap \tilde{U} \subset \text{UP}$ the basis must transform according to the fundamental representation of $SL(2; \mathbb{Z})$ by

$$\tilde{\Omega} = \begin{pmatrix} \tilde{\omega}' \\ \tilde{\omega} \end{pmatrix} = M \cdot \begin{pmatrix} \omega' \\ \omega \end{pmatrix}. \quad (3.48)$$

Therefore, the sections

$$\mathfrak{s}_1^* = \omega'(u) du = (1 \ 0) \cdot \Omega \otimes du, \quad \mathfrak{s}_2^* = \omega(u) du = (0 \ 1) \cdot \Omega \otimes du$$

define a basis of non-vanishing holomorphic sections of a rank-two $SL(2; \mathbb{Z})$ -bundle $\mathfrak{H}^* \rightarrow \text{UP}$ where Ω was defined in Eq. (3.22). We set $d\mathbf{a} = \omega du$ and $d\mathbf{a}_D = \omega' du$ such that

$$\tau = \frac{\omega'}{\omega} = \frac{d\mathbf{a}_D}{d\mathbf{a}}.$$

This notation should not suggest that $d\mathbf{a}$ is integrable, i.e., that we can find a globally well-defined function \mathbf{a} . On an open set $U \subset \text{UP}$, we can always integrate and find holomorphic functions $(\mathbf{a}, \mathbf{a}_D)$ such that on U we have $\omega = \frac{d\mathbf{a}}{du}$ and $\omega' = \frac{d\mathbf{a}_D}{du}$. However, the transition functions are (a priori) of the form

$$\tilde{\mathbf{H}} = \begin{pmatrix} \tilde{\mathbf{a}}_D \\ \tilde{\mathbf{a}} \end{pmatrix} = M \cdot \begin{pmatrix} \mathbf{a}_D \\ \mathbf{a} \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}, \quad (3.49)$$

where $M \in SL(2; \mathbb{Z})$ and $c, d \in \mathbb{C}$, i.e., they transform in the fundamental representation of $ISL(2; \mathbb{Z})$. We have locally a Kähler potential \mathfrak{K} such that $\Omega = i\partial\bar{\partial}\mathfrak{K}$ for $\mathfrak{K} = \frac{1}{2} \text{Im}(\mathbf{a}_D \bar{\mathbf{a}})$. We can integrate again to obtain a local holomorphic function \mathcal{F} , called the holomorphic prepotential, such that

$$\mathbf{a}_D = \frac{\partial \mathcal{F}}{\partial \mathbf{a}}.$$

3.10.1 The special Kähler connection

A basis of holomorphic sections of the dual bundle $\mathfrak{H} \rightarrow \text{UP}$ is given by $\partial/\partial\mathbf{a}$ and $\partial/\partial\mathbf{a}_D$. We have the identity

$$\int_{E_u} dz \wedge d\bar{z} = - \int_{\text{A-cycle}} dz \int_{\text{B-cycle}} d\bar{z} + \int_{\text{A-cycle}} d\bar{z} \int_{\text{B-cycle}} dz.$$

By choosing $dp = \operatorname{Re} \omega du$ and $dq = -\operatorname{Re} \omega' du$, we find

$$\Omega = 2 dp \wedge dq = \frac{1}{2} \left(d\mathbf{a}_D \wedge d\bar{\mathbf{a}} - d\mathbf{a} \wedge d\bar{\mathbf{a}}_D \right) = i \operatorname{Im} \tau d\mathbf{a} \wedge d\bar{\mathbf{a}} .$$

We check that

$$\frac{\partial}{\partial \mathbf{a}} = \frac{1}{2} \left(\frac{\partial}{\partial p} - \tau \frac{\partial}{\partial q} \right) , \quad \frac{\partial}{\partial \mathbf{a}_D} = \frac{1}{\tau} \frac{\partial}{\partial \mathbf{a}} ,$$

and

$$\frac{\partial}{\partial q} = \frac{i}{\operatorname{Im} \tau} \left(\frac{\partial}{\partial \mathbf{a}} - \frac{\partial}{\partial \bar{\mathbf{a}}} \right) , \quad \frac{\partial}{\partial p} = \frac{i}{\operatorname{Im}(-1/\tau)} \left(\frac{\partial}{\partial \mathbf{a}_D} - \frac{\partial}{\partial \bar{\mathbf{a}}_D} \right) .$$

A flat torsion-free connection ∇ of $\mathfrak{H}_{\mathbb{R}}$ is defined by

$$\nabla \frac{\partial}{\partial p} = \nabla \frac{\partial}{\partial q} = 0 , \tag{3.50}$$

and $d^\nabla(\operatorname{id}) = 0$ in each chart. We view $(\mathfrak{H}, \nabla, \Omega)$ as a special Kähler structure on UP (cf. [71]): in the context of the special Kähler geometry, the collection of coordinates $\{\mathbf{a}^U\}$ over the patches $U \subset \text{UP}$ of an open cover is called the adapted special coordinates. $\{\mathbf{a}^U\}$ is called conjugate special coordinates, and $\{p^U, q^U\}$ flat Darboux coordinates. The form Ω is a Kähler form on UP and ∇ is a torsion-free symplectic connection on the Kähler manifold (UP, Ω) . The connection is symplectic since the Kähler is locally of the form $\Omega = 2 dp \wedge dq$. One checks that

$$\nabla \frac{\partial}{\partial \mathbf{a}} = -\frac{1}{2} d\tau \otimes \frac{\partial}{\partial q} , \quad \nabla \frac{\partial}{\partial \mathbf{a}_D} = -\frac{1}{2} d \left(-\frac{1}{\tau} \right) \otimes \frac{\partial}{\partial p} .$$

The expression $\nabla \frac{\partial}{\partial \mathbf{a}}$ is a \mathfrak{H} -valued one-form, and it follows

$$d^\nabla \left(\nabla \frac{\partial}{\partial \mathbf{a}} \right) = d^\nabla \left(\nabla \frac{\partial}{\partial \mathbf{a}_D} \right) = 0$$

which proves flatness. Let $\pi^{(1,0)} \in \Omega^{(1,0)}(T_{\mathbb{C}}\text{UP})$ be the projection onto the $(1, 0)$ part of the complexified tangent bundle. $\pi^{(1,0)}$ is a one-form with values in the tangent bundle $T(\text{UP})$. The flatness of ∇ implies $d^\nabla \pi^{(1,0)} = 0$. There is a holomorphic cubic form Ξ on UP which encodes the extent to which the flat connection ∇ fails to preserve the complex structure

$$\Xi = -\Omega \left(\pi^{(1,0)}, \nabla \pi^{(1,0)} \right) \in \Gamma \left(\text{UP}, \operatorname{Sym}^3 T^*(\text{UP}) \right) . \tag{3.51}$$

In local coordinates, the section Ξ is

$$\begin{aligned}\Xi &= -\Omega\left(d\mathbf{a} \otimes \frac{\partial}{\partial \mathbf{a}}, \nabla\left(d\mathbf{a} \otimes \frac{\partial}{\partial \mathbf{a}}\right)\right) \\ &= (d\mathbf{a})^{\otimes 3} \frac{1}{2} \frac{\partial \tau}{\partial \mathbf{a}} \Omega\left(\frac{\partial}{\partial \mathbf{a}}, \frac{\partial}{\partial q}\right) = \frac{1}{4} \frac{\partial \tau}{\partial \mathbf{a}} (d\mathbf{a})^{\otimes 3}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\Xi &= \Xi_{aaa} (d\mathbf{a})^{\otimes 3} = \frac{1}{4} \frac{\partial \tau}{\partial \mathbf{a}} (d\mathbf{a})^{\otimes 3} \\ &= \Xi_{uuu} (du)^{\otimes 3} = \frac{1}{4} \frac{\partial \tau}{\partial u} \omega^2 (du)^{\otimes 3}.\end{aligned}$$

The canonical section Ξ_{uuu} is invariant under the monodromy action as $\omega^2 d\tau$ is invariant under $SL(2; \mathbb{Z})$.

3.10.2 The Levi-Civita connection

We compare the connection ∇ with the Levi-Civita connection D on the tangent bundle of UP. The Levi-Civita connection is defined by

$$d\left[\Omega\left(\frac{\partial}{\partial \mathbf{a}}, \frac{\partial}{\partial \bar{\mathbf{a}}}\right)\right] = \Omega\left(D\frac{\partial}{\partial \mathbf{a}}, \frac{\partial}{\partial \bar{\mathbf{a}}}\right) - \Omega\left(\frac{\partial}{\partial \mathbf{a}}, D\frac{\partial}{\partial \bar{\mathbf{a}}}\right).$$

In contrast to Eq. (3.51), it follows that the Levi-Civita connection satisfies

$$-\Omega(\pi^{(1,0)}, D\pi^{(1,0)}) = 0$$

since it is the unique connection which is compatible with the metric and the complex structure. We compute

$$D\frac{\partial}{\partial \mathbf{a}} = -\frac{i}{2 \operatorname{Im} \tau} d\tau \otimes \frac{\partial}{\partial \mathbf{a}}, \quad D\frac{\partial}{\partial \bar{\mathbf{a}}} = \frac{i}{2 \operatorname{Im} \tau} d\bar{\tau} \otimes \frac{\partial}{\partial \bar{\mathbf{a}}}.$$

From the relations

$$(\nabla - D)\frac{\partial}{\partial \mathbf{a}} = \frac{i}{2 \operatorname{Im} \tau} d\tau \otimes \frac{\partial}{\partial \bar{\mathbf{a}}}, \quad (\nabla - D)\frac{\partial}{\partial \bar{\mathbf{a}}} = -\frac{i}{2 \operatorname{Im} \tau} d\bar{\tau} \otimes \frac{\partial}{\partial \mathbf{a}},$$

it follows that D cannot be flat. We find

$$\begin{aligned}d^D\left(D\frac{\partial}{\partial \mathbf{a}}\right) &= -\frac{i}{2} d\left(\frac{1}{\operatorname{Im} \tau}\right) \wedge d\tau \otimes \frac{\partial}{\partial \mathbf{a}} - \frac{i}{2 \operatorname{Im} \tau} d\tau \wedge D\frac{\partial}{\partial \mathbf{a}} \\ &= -\frac{1}{4 \operatorname{Im}^2 \tau} d\tau \wedge d\bar{\tau} \otimes \frac{\partial}{\partial \mathbf{a}}.\end{aligned}$$

The Riemannian curvature R of the Kähler metric is

$$R = \begin{pmatrix} R_a^a{}_{a\bar{a}} & 0 \\ 0 & R_{\bar{a}}^{\bar{a}}{}_{a\bar{a}} \end{pmatrix} d\mathbf{a} \wedge d\bar{\mathbf{a}}.$$

We check that

$$R_a^a{}_{a\bar{a}} = d\mathbf{a} \left\{ d^D \left(D \frac{\partial}{\partial \mathbf{a}} \right) (\partial_{\mathbf{a}}, \partial_{\mathbf{a}}) \right\} = -\frac{1}{4 \operatorname{Im}^2 \tau} \left| \frac{\partial \tau}{\partial \mathbf{a}} \right|^2.$$

Pulling down the summation-index using the Kähler metric we find

$$R_{a\bar{a}a\bar{a}} = \operatorname{Im} \tau R_a^a{}_{a\bar{a}} = -\frac{1}{4 \operatorname{Im} \tau} \left| \frac{\partial \tau}{\partial \mathbf{a}} \right|^2.$$

Similarly, we obtain

$$\begin{aligned} d^D \left(D \frac{\partial}{\partial u} \right) &= -\frac{1}{4 \operatorname{Im}^2 \tau} d\tau \wedge d\bar{\tau} \otimes \frac{\partial}{\partial u}, \\ R_u^u{}_{u\bar{u}} &= -\frac{1}{4 \operatorname{Im}^2 \tau} \left| \frac{\partial \tau}{\partial u} \right|^2, \\ R_{u\bar{u}u\bar{u}} &= -\frac{1}{4 \operatorname{Im} \tau} \left| \omega \frac{\partial \tau}{\partial u} \right|^2 \end{aligned}$$

The scalar curvature is obtained by contracting the summation-indices using the inverse Kähler metric

$$\rho = -4 \operatorname{Im}^{-2} \tau R_{a\bar{a}a\bar{a}} = \frac{1}{\operatorname{Im}^3 \tau} \left| \frac{\partial \tau}{\partial \mathbf{a}} \right|^2.$$

3.10.3 Prequantization

The expression

$$\zeta = \frac{1}{2} \left(\mathbf{a} \frac{\partial}{\partial p} - \mathbf{a}_D \frac{\partial}{\partial q} \right) \tag{3.52}$$

defines locally a complex vector field that satisfies

$$\nabla \zeta = \frac{1}{2} \left(d\mathbf{a} \otimes \frac{\partial}{\partial p} - d\mathbf{a}_D \otimes \frac{\partial}{\partial q} \right) = d\mathbf{a} \otimes \frac{\partial}{\partial \mathbf{a}} = du \otimes \frac{\partial}{\partial u} = \pi^{(1,0)}.$$

Eqs. (3.11), (3.12) show that \mathbf{a} and \mathbf{a}_D can be computed as integrals over the basic cycles of a globally defined holomorphic one-form λ_{SW} . Thus, \mathbf{a} and \mathbf{a}_D are globally defined on the whole UP. The vector field ζ in Eq. (3.52) is globally defined on the

whole u -plane and equals

$$\zeta = \mathbf{a} \frac{\partial}{\partial \mathbf{a}} - \frac{8\pi i}{\omega} \frac{\partial}{\partial q}.$$

The fact that $\nabla \zeta = \pi^{(1,0)}$ is the Legendre relation for the periods in Eq. (3.9). We consider ζ the trivialization of a holomorphic line bundle $\mathcal{L} \rightarrow \text{UP}$ whose holomorphic sections are of the form $\mathfrak{s} = f(u) \zeta$. We check that

$$\nabla^{(0,1)} \mathfrak{s} = \nabla_{\partial_{\bar{\mathbf{a}}}} (f(u) \zeta) = f(u) \nabla_{\partial_{\bar{\mathbf{a}}}} \zeta = 0.$$

We define a holomorphic inclusion $\mathcal{L} \rightarrow \mathfrak{H}$ by mapping $\zeta \mapsto \partial_{\mathbf{a}}$. Since we have

$$\nabla_{\partial_{\mathbf{a}}} \zeta = \partial_{\mathbf{a}},$$

the holomorphic inclusion is an immersion. An Hermitian metric on $\mathcal{L} \rightarrow \text{UP}$ is defined by

$$\|\zeta\|^2 = \Omega(\zeta, \bar{\zeta}) = -\frac{1}{2}(\mathbf{a} \bar{\mathbf{a}}_D - \bar{\mathbf{a}}_D \mathbf{a}) = -i \operatorname{Im}(\mathbf{a} \bar{\mathbf{a}}_D) = -2i \mathfrak{K},$$

with $\mathfrak{K} = \operatorname{Im}(\mathbf{a} \bar{\mathbf{a}}_D)/2$. \mathfrak{K} is the Kähler potential on the Hermitian line bundle since

$$\Omega = \frac{i}{2} \partial \bar{\partial} \|\zeta\|^2 = \partial \bar{\partial} \mathfrak{K}.$$

$\mathcal{L} \rightarrow \text{UP}$ is a holomorphic Hermitian line bundle with connection ∇ and curvature $-2\pi i \Omega$. $(\text{UP}, \Omega, \mathcal{L})$ is called a Hodge manifold. In physics, it is called the geometric prequantization. Because of Eq. (3.50) the flat sections in \mathfrak{H}^* are locally given by

$$\mathfrak{s}_{n_m, n_e}^* = -n_e dp + n_m dq,$$

where n_e, n_m are constants. In physics, the flat sections (3.53) are called prequantum states. Using the holomorphic symplectic two-form the corresponding flat section \mathfrak{s} of \mathfrak{H} is

$$\begin{aligned} \mathfrak{s}_{n_m, n_e} &= n_m \frac{\partial}{\partial p} + n_e \frac{\partial}{\partial q} \\ &= \frac{i}{\operatorname{Im} \tau} \left(\bar{z}_{Q_u} \frac{\partial}{\partial \mathbf{a}} - z_{Q_u} \frac{\partial}{\partial \bar{\mathbf{a}}} \right), \end{aligned}$$

where $z_{Q_u} = n_e + \tau n_m$ is called the central charge. The Kähler form defines a skew-symmetric, non-degenerate real form Q on $\mathfrak{H}_{\mathbb{R}} \rightarrow \text{UP}$ by

$$Q(n_m \partial_p \oplus n_e \partial_q, n'_m \partial_p \oplus n'_e \partial_q) = (n_m \ n_e) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} n'_m \\ n'_e \end{pmatrix}.$$

$\mathfrak{H} \rightarrow \text{UP}$ is a flat holomorphic rank-two bundle with a non-degenerate skew-symmetric form Q of type (1,1) with respect to the complex structure that satisfies $\nabla Q = 0$. We define a flat sub-manifold $\Lambda \subset \mathfrak{H}_{\mathbb{R}}$ which intersects each fiber in a full lattice such that

$$Q|_{\Lambda \times \Lambda} \subset \mathbb{Z}.$$

In physics, Λ is called the charge lattice. The space of solution for the scalar field of the low-energy effective field theory are sections of Λ . Under the identification of dz with du by the holomorphic two-form η the state $\mathfrak{s}_{n_m, n_e}^*$ is an invariant vector field along the fibers of π . For z_{Q_u} defined in Eqs. (3.32), (3.34) the connection on the bundle $V \rightarrow Z$ can viewed as local flat sections of the flat sub-manifold Λ which intersect each fiber in the integer charge lattice or prequantized states of the classical field theory.

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Chapter 4

The $N = 2$ Yang-Mills theory

One of the main results of the work of Seiberg and Witten is that the moduli space of supersymmetric states of the twisted, pure, $N = 2$ supersymmetric $SU(2)$ -Yang-Mills theory on a compact four dimensional smooth manifold decomposes into two branches, the Coulomb branch and the Seiberg-Witten branch. Both branches can be interpreted as the moduli spaces of simpler physical theories, and the contributions to the Donaldson invariants from each branch can be computed separately [21].

The Seiberg-Witten branch is the moduli space of the solutions of the Seiberg-Witten monopole equations modulo gauge transformations. It has been studied in great detail (cf. [9] for an introduction). A generating function for the Seiberg-Witten invariants can be defined. For CP^2 or CP^2 with a small number of points blown up, the manifold admits a metric of positive scalar curvature, and by theorem due to Witten the contributions from the Seiberg-Witten branch vanish in some chamber [10, Lemma 21]. If $b_2^- < 9$ there is no wall-crossing at all, and the contributions from the Seiberg-Witten branch vanish [8]. We will not discuss the contribution from the Seiberg-Witten branch any further.

Thus, for CP^2 the Donaldson invariants are determined by the Coulomb branch alone. In this chapter, we will derive the partition function for the Coulomb branch as the stationary phase approximation of the low-energy effective field theory thus we obtain an interpretation of the u -plane integral in terms of determinant line bundles. For the product of the determinant line bundles, the local and global anomalies vanish.

4.1 The topological gauge-fixing and reduction

In this section, we start with the Lagrangian for the twisted, pure, $N = 1$ supersymmetric $SU(2)$ -gauge theory considered in [26, Eq. (21)] that we have described in Sec. 2.4. We look at the gauge group reduction to $U(1)$ and relate it to the Lagrangian for the $N = 2$ supersymmetric, high-energy field theory.

4.1.1 The longitudinal part of the action

Let us assume that there is a reduction of the $SU(2)$ -bundle $E \rightarrow M$ to $N \oplus N^{-1} \rightarrow M$. The $\overline{\mathcal{Q}}$ -exact term in the Lagrangian in [26, Eq. (21)] which was introduced in Sec. 2.4 becomes

$$\begin{aligned} \mathbf{S} = & -\frac{1}{2} \int_M \text{tr} \left\{ \frac{\text{Im } \tau}{8\pi} \left(\frac{1}{2} * (F_A^+ \wedge F_A^+) - \frac{1}{2} * (D \wedge D) + \frac{1}{2} (d_{A_0}^* B)^2 \right. \right. \\ & + \overline{\Phi} \Delta_A^{(0)} \Phi + \frac{1}{2} \overline{C} d_{A_0}^* d_A C - \frac{1}{4} [\Phi, \overline{\Phi}]^2 - \frac{1}{8} [\Phi, \overline{C}] [C, \overline{\Phi}] \\ & + * (\mathcal{X} \wedge d_A \Psi) + H d_A^* \Psi - \frac{i}{2} * ([\Phi, \mathcal{X}] \wedge \mathcal{X}) \\ & \left. \left. - \frac{i}{2} * ([\overline{\Phi}, \Psi] \wedge * \Psi) - \frac{i}{2} [\Phi, H] H \right\} \text{vol}_M, \end{aligned}$$

where the fields take values in the line bundle $\text{Ad } E$. We use the capital letters (H, Ψ, \mathcal{X}) for the fermionic fields. We fix a reference connection A_0 on the bundle $E \rightarrow M$, such that F_{A_0} is harmonic. B is the one-form $B = A - A_0$. The Lagrangian is identical with the one presented by Witten in [16, Eq. (2.13)] extended by the contributions from the ghost fields C, \overline{C} . From the reduction of the $SU(2)$ -bundle E , it follows that

$$\text{Ad } E \cong N^2 \oplus \underline{\mathbb{R}}.$$

where $\underline{\mathbb{R}}$ is a trivial real oriented line bundle with the trivial, flat connection. The bundles $N^{\otimes 2}$ and $(N^{-1})^{\otimes 2}$ are different as $SO(2)$ -bundles, but canonically isomorphic as $O(2)$ -bundles by reversing the orientation. Reversing the orientation on $N^{\otimes 2}$ and $\underline{\mathbb{R}}$ is a $SO(3)$ bundle automorphisms. Let L denote the line bundle $L = N^{\otimes 2}$. A section $\mathfrak{z} \in C^\infty(M, \text{Ad } E)$ can be decomposed as

$$\mathfrak{z} = \mathfrak{z}^a (i\sigma^a) = \mathfrak{z}^3 (i\sigma^3) + \underbrace{(\mathfrak{z}^1 - i\mathfrak{z}^2)}_{\mathfrak{z}^+} (i\sigma^+) + \underbrace{(\mathfrak{z}^1 + i\mathfrak{z}^2)}_{\mathfrak{z}^-} (i\sigma^-),$$

where $i\sigma^\pm = 1/2(i\sigma^1) \mp 1/2\sigma^2$ are local sections of L (plus sign) and L^{-1} (minus sign). The following relations hold:

$$[i\sigma^3; i\sigma^\pm] = \pm 2i (i\sigma^\pm), \quad \text{tr}(i\sigma^\pm i\sigma^\mp) = -1, \quad \text{tr}(i\sigma^\pm i\sigma^\pm) = 0. \quad (4.1)$$

Remark 8. The decomposition for the $U(1)$ -reducible connection $A \otimes i\sigma^3$ is given by

$$d_A \Big|_{i\sigma^3} = d, \quad d_A \Big|_{i\sigma^+} = \underbrace{\partial + 2iA}_{d_L \text{ on } L}, \quad d_A \Big|_{i\sigma^-} = \underbrace{\partial - 2iA}_{d_{-L} \text{ on } L^{-1}}.$$

The connection reduces to a connection on the line bundle $L = N^{\otimes 2} \rightarrow M$.

4.1.2 The high-energy field theory

Let us assume that the fields take values in the trivial bundle $\underline{\mathbb{R}}$ of $\text{Ad } E$. Thus, the non-vanishing components of the fields are proportional to $i\sigma^3$ with $\text{tr}(i\sigma^3)^2 = -2$. Thus, the fields are differential forms on M and their corresponding ghost fields. We denote the fermionic fields that take values in $\underline{\mathbb{R}}$ by (η, ψ, χ) . We call the contribution to the action from these fields the longitudinal part. The Lagrangian of the gauge-fixed action (2.12) simplifies to

$$\begin{aligned} \overline{\mathcal{Q}}\mathbf{V} &= -b \wedge \left(F_A^+ + \frac{1}{2}\rho b \right) - \chi \wedge \pi_+ d\psi \\ &+ \left(-\eta d^*\psi + \bar{\phi} \Delta^{(0)}\phi + \bar{c} \Delta^{(0)}c - \widehat{b} d^*A - \frac{1}{2}\widehat{b}^2 - \bar{c} d^*\psi \right) \text{vol}_M . \end{aligned} \quad (4.2)$$

The Euler-Lagrange equations for ψ are $d\psi = 0$ and $d^*\psi = 0$. Since the manifold M is simply connected, the field ψ vanishes classically. In the semi-classical approximation, ψ is a form in the orthogonal complement to the kernel of $d + d^*$. By definition, the ghost field c does not have a zero mode and c is in the orthogonal complement of the constant functions on M . In the path integral in Eq. (2.16), we shift the integration variable $c \rightarrow c - \Delta^{-1}d^*\psi$. The field \widehat{b} appears in the action without derivatives. The Euler-Lagrange equation of \widehat{b} is algebraic, and the field \widehat{b} can be integrated out. The action (4.2) becomes

$$-b \wedge \left(F_A^+ + \frac{1}{2}\rho b \right) - \chi \wedge \pi_+ d\psi + \left(-\eta d^*\psi + \bar{\phi} \Delta^{(0)}\phi + \bar{c} \Delta^{(0)}c + \frac{1}{2}(d^*A)^2 \right) \text{vol}_M .$$

The first term can be diagonalized by introducing a selfdual two-form D with $b = -F_A^+ + D$ and $\rho = 1$. It follows

$$\mathbf{V} = \frac{1}{2} \left\{ -\langle \chi, D + F_A^+ \rangle + 2\bar{\phi} d^*\psi - \bar{c} d^*A \right\} \text{vol}_M ,$$

and the action in Eq. (4.2) becomes

$$\begin{aligned} \overline{\mathcal{Q}}\mathbf{V} &= \frac{1}{2} F_A^+ \wedge F_A^+ - \frac{1}{2} D \wedge D + \chi \wedge \pi_+ d\psi \\ &+ \left(\eta d^*\psi + \bar{\phi} \Delta^{(0)}\phi + \frac{1}{2} \bar{c} \Delta^{(0)}c + \frac{1}{2}(d^*A)^2 \right) \text{vol}_M . \end{aligned} \quad (4.3)$$

We have shown:

Lemma 10. *For $b_2^- = 0$, the gauge-fixed topological action after reduction of the*

gauge group to $U(1)$ becomes

$$\begin{aligned}
\mathbf{S} &= i\pi \langle \text{Re } \tau c_1(L)^2, [M] \rangle - \bar{\mathcal{Q}} \int_M \frac{\text{Im } \tau}{16\pi} \{ \langle \chi, D + F_A^+ \rangle - 2\bar{\phi} d^* \psi + \bar{c} d^* A \} \text{vol}_M \\
&= \frac{i \text{Re } \tau}{16\pi} \int_M F_A^+ \wedge F_A^+ + \frac{\text{Im } \tau}{8\pi} \int_M \left(\frac{1}{2} \langle F_A^+, F_A^+ \rangle + \bar{\phi} \Delta^{(0)} \phi + \frac{1}{2} (d^* A)^2 \right. \\
&\quad \left. - \frac{1}{2} \langle D, D \rangle + \langle \chi, \pi_+ d\psi \rangle + \eta d^* \psi + \frac{1}{2} \bar{c} \Delta^{(0)} c \right) \text{vol}_M .
\end{aligned}$$

The BRST-transformations in Sec. 2.4 with $b = -F_A^+ + D$ and $\rho = 1$ become

$$\begin{aligned}
\bar{\mathcal{Q}}A &= -dc + \psi, & \bar{\mathcal{Q}}\psi &= -d\phi, & \bar{\mathcal{Q}}\phi &= 0, \\
\bar{\mathcal{Q}}c &= \phi \\
\bar{\mathcal{Q}}\bar{c} &= -d^*A, \\
\bar{\mathcal{Q}}\chi &= -F_A^+ + D, & \bar{\mathcal{Q}}D &= \bar{\mathcal{Q}}F_A^+ = \pi_+ d\psi, \\
\bar{\mathcal{Q}}\bar{\phi} &= \eta, & \bar{\mathcal{Q}}\eta &= 0.
\end{aligned} \tag{4.4}$$

4.2 The Coulomb branch and the low-energy effective field theory

The Coulomb branch is the moduli space of a family of a $N = 2$ supersymmetric $U(1)$ -theories called the low-energy effective field theory. The family of the $U(1)$ -theories is parameterized by a complex coupling constant in the upper half-plane $\tau \in H$. In this section, we describe the low-energy effective field theory on the Kähler manifold $M + CP^2$. The low-energy effective field theory can be derived from two conditions: the positivity of the free field theory in the variations around a critical point and the absence of a local anomaly.

4.2.1 The field content

The family of $N = 2$ supersymmetric $U(1)$ -theories has the following field content: For each coupling constant τ , let A be the connection on a line bundle $L \rightarrow M$ with curvature F_A . We assume that $c_1(L) \in H^2(M; \mathbb{Z}) \cong \mathbb{Z}$ and even. \mathcal{A} is the space of unitary $U(1)$ -connections on $L \rightarrow M$, \mathcal{G} the group of smooth maps $g : M \rightarrow S^1$. If p is the choice of a base point in M , \mathcal{G}_0 is the group of pointed gauge transformations with $g(p) = \mathbb{I}$. The group \mathcal{G}_0 acts freely on \mathcal{A} . The space of gauge equivalence classes of connections is $\mathcal{B}^* = \mathcal{A}/\mathcal{G}_0$.

The other fields of the theory will be differential forms on M . The gauge group acts on the connection and trivially on the other fields. In addition to the connection, the classical bosonic fields of the theory are a complex valued scalar and its conjugate, and a selfdual two-form

$$\phi, \bar{\phi} \in C^\infty(M; \mathbb{C}), \quad D \in \Lambda^{2+}(M).$$

The selfdual two-form D is an auxiliary field without any dynamics, and will be integrated out. The fermionic fields (η, ψ, χ) are Grassmann-algebra valued forms. The corresponding commuting form – denoted by \rightarrow – are

$$\eta \rightarrow C^\infty(M), \quad \psi \rightarrow \Lambda^1(M), \quad \chi \rightarrow \Lambda^{2+}(M).$$

A super-multiplets in the twisted, $N = 2$ supersymmetric theory consists of a $N = 1$ vector multiplet and a chiral multiplet consisting of the collection of fields

$$\left\{ \psi_\mu, A_\mu \right\}, \quad \left\{ (\eta, \chi), (\phi, D) \right\}.$$

The remaining $U(1)$ -gauge invariance of the action requires the additional gauge fixing and the introduction of the Faddeev-Popov ghost fields c, \bar{c} . Classically, every field satisfies an Euler-Lagrange equation. In addition, every field has a semi-classical contribution which is the linear fluctuation orthogonal to the classical field configuration.

4.2.2 The descent procedure

From the BRST-transformations (4.4) one can build local functions in the fields which are $\overline{\mathcal{Q}}$ -invariant modulo d -exact terms. This is called the descent procedure [16]. A short calculation confirms:

Lemma 11. *The general form of the local functions in the fields are given by a set of holomorphic functions $\mathcal{P} = \{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ and the following family of invariant polynomials*

$$\begin{aligned} \mathfrak{H}_{\mathcal{P}}^{4,0} &= \mathcal{P}_0(\phi), \\ \mathfrak{H}_{\mathcal{P}}^{3,1} &= -\mathcal{P}'_0(\phi) \psi, \\ \mathfrak{H}_{\mathcal{P}}^{2,2} &= \frac{1}{2!} \mathcal{P}''_0(\phi) \psi^2 - \mathcal{P}'_0(\phi) F_A - \mathcal{P}_1(\phi) b, \\ \mathfrak{H}_{\mathcal{P}}^{1,3} &= -\frac{1}{3!} \mathcal{P}'''_0(\phi) \psi^3 + \mathcal{P}''_0(\phi) \psi \wedge F_A \\ &\quad + \mathcal{P}'_1(\phi) \psi \wedge b + \mathcal{P}_1(\phi) d\chi + \mathcal{P}_2(\phi) *d\eta, \\ \mathfrak{H}_{\mathcal{P}}^{0,4} &= \frac{1}{4!} \mathcal{P}_0^{(IV)}(\phi) \psi^4 - \frac{1}{2!} \mathcal{P}'''_0(\phi) \psi^2 \wedge F_A + \frac{1}{2!} \mathcal{P}''_0(\phi) F_A^2 \\ &\quad - \frac{1}{2!} \mathcal{P}''_1(\phi) \psi^2 \wedge b - \mathcal{P}'_1(\phi) F_A \wedge b - \mathcal{P}'_1(\phi) \psi \wedge d\chi \\ &\quad - \mathcal{P}'_2(\phi) \psi \wedge *d\eta + \mathcal{P}_2(\phi) d *d\bar{\phi} + \mathcal{P}_3(\phi) b^2 \end{aligned}$$

such that

$$\overline{\mathcal{Q}} \mathfrak{H}_{\mathcal{P}}^{4,0} = 0, \quad \overline{\mathcal{Q}} \mathfrak{H}_{\mathcal{P}}^{4-j,j} = d \mathfrak{H}_{\mathcal{P}}^{5-j,j-1}, \quad 0 = d \mathfrak{H}_{\mathcal{P}}^{0,4},$$

with $1 \leq j \leq 4$.

Proof. The proof follows from

$$(d + \overline{\mathcal{Q}}) \left(\mathfrak{H}^{4,0} + \mathfrak{H}^{3,1} + \mathfrak{H}^{2,2} + \mathfrak{H}^{1,3} + \mathfrak{H}^{0,4} \right) = 0 .$$

Since $(d + \overline{\mathcal{Q}})^2 = 0$, it follows that d and $\overline{\mathcal{Q}}$ anti-commute. \square

Similarly, the general form of a topological gauge-fixing term can be determined:

Remark 9. *The general form of a topological gauge fixing term is given by a set of anti-holomorphic functions $\overline{\mathcal{A}} = \{\overline{\mathcal{A}}_0, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2, \overline{\mathcal{A}}_3\}$ and*

$$\mathbf{V}_{\overline{\mathcal{A}}} = \overline{\mathcal{A}}_0(\overline{\phi}) \chi \wedge F_A + \overline{\mathcal{A}}_1(\overline{\phi}) \chi \wedge b + \overline{\mathcal{A}}_2(\overline{\phi}) d * \psi + \overline{\mathcal{A}}_3 \bar{c} d * A + \overline{\mathcal{A}}_4(\overline{\phi}) \chi^3 \text{vol}_M$$

such that

$$\begin{aligned} \overline{\mathcal{Q}} \mathbf{V}_{\overline{\mathcal{A}}} &= \overline{\mathcal{A}}_0(\overline{\phi}) b \wedge F_A - \overline{\mathcal{A}}_0(\overline{\phi}) \chi \wedge d\psi + \overline{\mathcal{A}}'_0(\overline{\phi}) \eta \chi \wedge F_A \\ &+ \overline{\mathcal{A}}_1(\overline{\phi}) b \wedge b + \overline{\mathcal{A}}'_1(\overline{\phi}) \eta \chi \wedge b - \overline{\mathcal{A}}_2(\overline{\phi}) d * d\phi + \overline{\mathcal{A}}'_2(\overline{\phi}) \eta d * \psi \\ &+ \overline{\mathcal{A}}_3(\overline{\phi}) (d^* A) d * A + \overline{\mathcal{A}}_3 \bar{c} d * dc - \overline{\mathcal{A}}_3 \bar{c} d * \psi - \overline{\mathcal{A}}'_3 \bar{c} \eta d * A \\ &+ \frac{3}{2} \overline{\mathcal{A}}_4(\overline{\phi}) \langle \chi, \mathbb{L}_b \chi \rangle \text{vol}_M + \overline{\mathcal{A}}'_4(\overline{\phi}) \eta \chi^3 \text{vol}_M . \end{aligned}$$

For $\chi = 1/2 \chi_{\mu\nu} dx^\mu \wedge dx^\nu$ and $b = 1/2 b_{\mu\nu} dx^\mu \wedge dx^\nu$, we have defined

$$\chi^3 = \chi_{\mu\nu} \chi^{\nu\rho} \chi_\rho{}^\mu , \quad (\mathbb{L}_b \chi)_{\mu\nu} = b_{\rho[\mu} \chi_{\nu]}{}^\rho .$$

The general action is given by the sum of the four-observable, constructed from \mathcal{P} and the $\overline{\mathcal{Q}}$ -exact term from the general gauge-fixing. After shifting the integration variable \bar{c} , we obtain the following result:

Lemma 12. *The general form of a supersymmetric topological gauge-fixed action is*

$$8\pi \mathbf{S} = \int_M (i \mathfrak{H}_P^{0,4} + i \overline{\mathcal{Q}} \mathbf{V}_{\overline{\mathcal{A}}}) = \mathbf{S}_{\text{bose}} + \mathbf{S}_{\text{fermi}} + \mathbf{S}_{i.a.}$$

with

$$\begin{aligned} 8\pi \mathbf{S}_{\text{bose}} &= \int_M \left(\frac{1}{2!} \mathcal{P}''_0(\phi) F_A^2 + \mathcal{P}_3(\phi) b^2 - \mathcal{P}'_1(\phi) F_A \wedge b \right. \\ &+ \overline{\mathcal{A}}_0(\overline{\phi}) b \wedge F_A + \overline{\mathcal{A}}_1(\overline{\phi}) b \wedge b - \overline{\mathcal{A}}_3 (d^* A)^2 \text{vol}_M \\ &\left. - \mathcal{P}_2(\phi) * \Delta \overline{\phi} + \overline{\mathcal{A}}_2(\overline{\phi}) * \Delta \phi - \overline{\mathcal{A}}_3 \bar{c} * \Delta c \right) , \end{aligned}$$

and

$$\begin{aligned} 8\pi \mathbf{S}_{\text{fermi}} &= \int_M \left(-\mathcal{P}'_1(\phi) \psi \wedge * d^* \chi - \mathcal{P}'_2(\phi) \psi \wedge * d\eta \right. \\ &\left. - \overline{\mathcal{A}}_0(\overline{\phi}) \chi \wedge * d\psi - \overline{\mathcal{A}}'_2(\overline{\phi}) \eta * d^* \psi \right) , \end{aligned}$$

and

$$8\pi \mathbf{S}_{i.a.} = \int_M \left(-\frac{1}{2} \mathcal{P}_0'''(\phi) \psi^2 \wedge F_A - \frac{1}{2} \mathcal{P}_1''(\phi) \psi^2 \wedge b \right. \\ \left. + \bar{\mathcal{A}}_0'(\bar{\phi}) \eta \chi \wedge F_A + \bar{\mathcal{A}}_1'(\bar{\phi}) \eta \chi \wedge b + \frac{3}{2} \bar{\mathcal{A}}_4(\bar{\phi}) \langle \chi, \mathbf{C}_b \chi \rangle \text{vol}_M \right).$$

We obtain the following result:

Proposition 12. *The family of topological actions*

$$\int_M \frac{i \text{Re } \tau}{16\pi} F_A \wedge F_A$$

is invariant under all orientation-preserving diffeomorphisms. Thus, it requires additional topological gauge fixing. We require:

1. The free field theory in the variations of the bosonic fields is described by a self-adjoint second order elliptic operator with positive-definite symbol.
2. The free field theory in the variations of the fermionic fields is described by a skew-adjoint first order elliptic operator \mathbf{D} with skew-symmetric real symbol.
3. The determinant line bundle of \mathbf{D} has no local anomaly.

Then, the gauge-fixed action is

$$8\pi \mathbf{S} = \int_M \left(\frac{i \text{Re } \tau}{2} F_A \wedge F_A + \left\{ \frac{\text{Im } \tau}{2} \left(\|F_A^+\|^2 + \|d^* A\|^2 \right) \right. \right. \\ \left. \left. + \text{Im } \tau \langle d\phi, d\bar{\phi} \rangle + \frac{\text{Im } \tau}{2} \langle dc, d\bar{c} \rangle - \text{Im } \tau D^2 \right. \right. \\ \left. \left. + \text{Im } \tau \langle \psi, d^* \chi \rangle + \text{Im } \tau \langle \psi, d\eta \rangle - \frac{i}{2} \frac{d\bar{\tau}}{d\bar{\phi}} \eta \chi \wedge (D + F_A^+) \right. \right. \\ \left. \left. - \frac{i}{2} \frac{d\tau}{d\phi} \psi^2 \wedge (D + F_A^-) - \frac{i}{4\sqrt{2}} \frac{d\bar{\tau}}{d\bar{\phi}} \chi \wedge \mathbf{C}_{D-F_A^+} \chi \right\} \text{vol}_M \right)$$

Proof. If we fix the coefficient of the scalar field to

$$i \left(-\mathcal{P}_2(\phi) * \Delta \bar{\phi} + \bar{\mathcal{A}}_2(\bar{\phi}) * \Delta \phi \right) = \text{Im } \tau \langle d\phi, d\bar{\phi} \rangle,$$

it follows that

$$\mathcal{P}_2'(\phi) = \frac{\tau}{2}, \quad \bar{\mathcal{A}}_2'(\bar{\phi}) = \frac{\bar{\tau}}{2}.$$

Let τ be given in terms of the prepotential \mathcal{F} by

$$\tau(\phi) = \frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi^2}.$$

If one diagonalizes the terms in \mathbf{S}_{bose} involving F_A and D by choosing $b = -F_A^+ + D$, it follows that the unique solution is

$$\mathcal{P}_0 = \mathcal{F}, \quad \mathcal{P}'_1 = \tau, \quad \mathcal{P}_3 = -\frac{\tau}{2},$$

and

$$\bar{\mathcal{A}}_0 = -\bar{\tau}, \quad \bar{\mathcal{A}}_1 = -\frac{\bar{\tau}}{2},$$

such that

$$\begin{aligned} 8\pi \mathbf{S}_{\text{bose}} &= \int_M \left(\frac{i\bar{\tau}}{2} F_A^{+2} + \frac{i\tau}{2} F_A^{-2} - \text{Im } \tau D^2 + \text{Im } \tau \langle d\phi, d\bar{\phi} \rangle \text{vol}_M \right. \\ &\quad \left. + (\text{terms involving } \bar{\mathcal{A}}_3) \right) \\ &= \int_M \left(\frac{i \text{Re } \tau}{2} F_A \wedge F_A + \frac{\text{Im } \tau}{2} F_A^{+2} + \text{Im } \tau \langle d\phi, d\bar{\phi} \rangle - \text{Im } \tau D^2 \right. \\ &\quad \left. + (\text{terms involving } \bar{\mathcal{A}}_3) \right). \end{aligned}$$

Relaxing the assumption $b_2^- = 0$, the same argument gives

$$\begin{aligned} 8\pi \mathbf{S}_{\text{bose}} &= \int_M \left(\frac{i \text{Re } \tau}{2} F_A \wedge F_A + \frac{\text{Im } \tau}{2} \|F_A^+\|^2 \text{vol}_M + \text{Im } \tau \langle d\phi, d\bar{\phi} \rangle \text{vol}_M \right. \\ &\quad \left. - \text{Im } \tau D^2 + (\text{terms involving } \bar{\mathcal{A}}_3) \right). \end{aligned}$$

The part in the action $\mathbf{S}_{\text{fermi}}$ involving η and ψ becomes

$$i \left(-\mathcal{P}'_2(\phi) \psi \wedge *d\eta - \bar{\mathcal{A}}'_2(\bar{\phi}) \eta *d^*\psi \right) = \text{Im } \tau \langle \psi, d\eta \rangle \text{vol}_M.$$

The part in the action $\mathbf{S}_{\text{fermi}}$ involving the anti-commuting fields ψ and χ becomes

$$i \left(-\mathcal{P}'_1(\phi) \psi \wedge *d^*\chi - \bar{\mathcal{A}}'_0(\bar{\phi}) \chi \wedge *d\psi \right) = \text{Im } \tau \langle \psi, d^*\chi \rangle \text{vol}_M.$$

The integrand in the interaction terms becomes

$$\begin{aligned} i \left(\bar{\mathcal{A}}'_0(\bar{\phi}) \eta \chi \wedge F_A + \bar{\mathcal{A}}'_1(\bar{\phi}) \eta \chi \wedge b \right) &= i \bar{\mathcal{A}}'_1(\bar{\phi}) \eta \chi \wedge (F_A^+ + D) \\ &= -\frac{i}{2} \frac{d\bar{\tau}}{d\bar{\phi}} \eta \chi \wedge (F_A^+ + D). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & i \left(-\frac{1}{2} \mathcal{P}_0'''(\phi) \psi^2 \wedge F_A - \frac{1}{2} \mathcal{P}_1''(\phi) \psi^2 \wedge b \right) = -\frac{i}{2} \mathcal{P}_1''(\phi) \psi^2 \wedge (F_A^- + D) \\ & = -\frac{i}{2} \frac{d\tau}{d\phi} \psi^2 \wedge (F_A^- + D) = \frac{i}{2} \frac{d\tau}{d\phi} \langle \psi, *D \wedge \psi \rangle . \end{aligned}$$

We set $\bar{\mathcal{A}}_3 = -\frac{1}{2} \text{Im } \tau$ so that the linearization of

$$\frac{\text{Im } \tau}{2} \|F_A\|^2 + \frac{\text{Im } \tau}{2} \|d^*A\|^2$$

is a positive definite elliptic operator. The Pfaffian of the ghost contribution c, \bar{c} will cancel the determinant of the scalar field. We observe that all kinetic terms and interaction terms in ψ^2 and $\eta\chi$ are fixed. The only remaining term is the interaction term χ^2 . This term can be chosen such that the local anomaly vanishes. We will prove this statement in Sec. 4.8. \square

Remark 10. *The additional topological term present in the action of Moore and Witten is related to the family of topological actions*

$$i \pi \langle \text{Re } \tau c_1(L)^2, [M] \rangle .$$

by a shift $\tau \rightarrow \tau + 1$. Since

$$\langle c_1(L)^2, [M] \rangle \equiv c_1(L) \cdot w_2(M) \quad (2) ,$$

the family of topological actions becomes

$$i \pi \langle \text{Re}(\tau + 1) c_1(L)^2, [M] \rangle .$$

4.2.3 The different normalizations

Let us assume that the bosonic part of the action in Prop. 12 is fixed. Rescaling the supersymmetry operator by $\tilde{\mathcal{Q}} = \mu_{\bar{\mathcal{Q}}} \bar{\mathcal{Q}}$ and the fermionic fields (η, ψ, χ) according to

$$\begin{pmatrix} \eta \\ \chi \\ \psi \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\eta} \\ \tilde{\chi} \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} \mu_\eta & & \\ & \mu_\chi & \\ & & \frac{1}{\mu_{\bar{\mathcal{Q}}}} \end{pmatrix} \begin{pmatrix} \eta \\ \chi \\ \psi \end{pmatrix} ,$$

and $\tilde{c} = c/\mu_{\bar{\mathcal{Q}}}$ changes the BRST-transformations in Eq. (4.4) to

$$\begin{aligned} \tilde{\mathcal{Q}}A &= -d\tilde{c} + \tilde{\psi} , & \tilde{\mathcal{Q}}\tilde{\psi} &= -\mu_{\bar{\mathcal{Q}}}^2 d\phi , & \tilde{\mathcal{Q}}\phi &= 0 , \\ \tilde{\mathcal{Q}}\tilde{\chi} &= \frac{\mu_{\bar{\mathcal{Q}}}}{\mu_\chi} (-F_A^+ + D) , & \tilde{\mathcal{Q}}D &= \tilde{\mathcal{Q}}F_A^+ = \pi_+ d\tilde{\psi} , & & \\ \tilde{\mathcal{Q}}\tilde{\phi} &= \frac{\mu_{\bar{\mathcal{Q}}}}{\mu_\eta} \tilde{\eta} , & \tilde{\mathcal{Q}}\tilde{\eta} &= 0 . & & \end{aligned} \quad (4.5)$$

The supersymmetric action then changes to

$$\begin{aligned}
8\pi \mathbf{S} &= \int_M \left(\frac{i \operatorname{Re} \tau}{2} F_A \wedge F_A + \left\{ \frac{\operatorname{Im} \tau}{2} \left(\|F_A^+\|^2 + \|d^* A\|^2 \right) \right. \right. \\
&+ \operatorname{Im} \tau \langle d\phi, d\bar{\phi} \rangle + \frac{\operatorname{Im} \tau}{2} |\mu_{\overline{\mathcal{Q}}}|^2 \langle dc, d\bar{c} \rangle - \operatorname{Im} \tau D^2 \\
&+ \operatorname{Im} \tau \frac{\mu_\chi}{\mu_{\overline{\mathcal{Q}}}} \langle \psi, d^* \chi \rangle + \operatorname{Im} \tau \frac{\mu_\eta}{\mu_{\overline{\mathcal{Q}}}} \langle \psi, d\eta \rangle - \frac{i}{2} \mu_\chi \mu_\eta \frac{d\bar{\tau}}{d\phi} \eta \chi \wedge (D + F_A^+) \\
&\left. \left. - \frac{i}{2} \frac{i}{2} \frac{1}{\mu_{\overline{\mathcal{Q}}}^2} \frac{d\tau}{d\phi} \psi^2 \wedge (D + F_A^-) - \frac{i}{2\sqrt{2}} \mu_\chi^2 \frac{d\bar{\tau}}{d\phi} \chi \wedge \mathfrak{C}_{D-F_A^+} \chi \right\} \operatorname{vol}_M \right).
\end{aligned}$$

For $\mu_{\overline{\mathcal{Q}}} = -2\sqrt[4]{2}i$, $\mu_\chi = 2\sqrt[4]{2}$, and $\mu_\eta = 2/\sqrt[4]{2}$, the BRST-relations (4.5) are the relations used in [25]. Choosing $\mu_{\overline{\mathcal{Q}}} = -\sqrt[4]{2}i$, $\mu_\chi = \sqrt[4]{2}$, and $\mu_\eta = 1/\sqrt[4]{2}$, we obtain the following result:

Corollary 2. *Under the hypothesis of Prop. 12 and with the BRST-transformations*

$$\begin{aligned}
\overline{\mathcal{Q}}A &= -dc + \psi, & \overline{\mathcal{Q}}\psi &= \sqrt{2} d\phi, & \overline{\mathcal{Q}}\phi &= 0, \\
\overline{\mathcal{Q}}\chi &= i(F_A^+ - D), & \overline{\mathcal{Q}}D &= \overline{\mathcal{Q}}F_A^+ = \pi_+ d\psi, \\
\overline{\mathcal{Q}}\bar{\phi} &= -i\eta, & \overline{\mathcal{Q}}\eta &= 0,
\end{aligned} \tag{4.6}$$

the twisted, $N = 2$ supersymmetric action becomes

$$\begin{aligned}
8\pi \mathbf{S} &= \int_M \left(\frac{i \operatorname{Re} \tau}{2} F_A \wedge F_A + \left\{ \frac{\operatorname{Im} \tau}{2} \left(\|F_A^+\|^2 + \|d^* A\|^2 \right) \right. \right. \\
&+ \operatorname{Im} \tau \langle d\phi, d\bar{\phi} \rangle + \frac{\operatorname{Im} \tau}{\sqrt{2}} \langle dc, d\bar{c} \rangle - \operatorname{Im} \tau D^2 \\
&+ \frac{i}{\sqrt{2}} \left(\sqrt{2} \operatorname{Im} \tau \langle \psi, d^* \chi \rangle + \operatorname{Im} \tau \langle \psi, d\eta \rangle - \frac{1}{2} \frac{d\bar{\tau}}{d\phi} \eta \chi \wedge (D + F_A^+) \right. \\
&\left. \left. + \frac{1}{2} \frac{d\tau}{d\phi} \sqrt{2} \psi^2 \wedge (D + F_A^-) - \frac{1}{4} \frac{d\bar{\tau}}{d\phi} \chi \wedge \mathfrak{C}_{D-F_A^+} \chi \right\} \operatorname{vol}_M \right).
\end{aligned}$$

4.3 The source terms

In this section, we derive the gauge invariant observables for the action in Cor. 2 by integrating the local function \mathfrak{H} in Lemma 11.

We denote the vacuum expectation value of ϕ by \mathbf{a} . On the Coulomb branch the expectation value of the Higgs field breaks $SU(2)$ to $U(1)$ by the Higgs-effect. One maps the observables of the Donaldson theory to the observables of the low energy theory. Let us start with the general zero-observable

$$H_p^{4,0} = i p \mathfrak{H}^{4,0} = i p \frac{u(\mathbf{a})}{16\pi}. \tag{4.7}$$

For $b_1 = 0$, the next non-trivial observable is obtained from integrating

$$H_{\Sigma}^{2,2} = i \int_{\Sigma} \mathfrak{H}^{2,2} = \frac{i}{16\pi} \int_{\Sigma} \left[\sqrt{2} u''(\mathbf{a}) \psi^2 - 2 u'(\mathbf{a}) (b + F_A) \right]. \quad (4.8)$$

There is an additional source term $H_{\Sigma \cdot \Sigma}$, called the contact term, connected to the intersection $\Sigma \cdot \Sigma$. For a Kähler manifold, the Kähler class ω gives a canonical cohomology orientation, and $\Sigma \in H^{2+}(M; \mathbb{Z})$ is determined by the integer $S = \int_{\Sigma} \omega$. The contact term is

$$H_{\Sigma \cdot \Sigma} = S^2 T(\mathbf{a}), \quad (4.9)$$

with

$$T(\phi) = \frac{1}{4} \left(2u - \phi \frac{\partial u}{\partial \phi} \right). \quad (4.10)$$

The following Lemma was proved in [29]:

Lemma 13. *The insertion of $\exp(i \int_M \mathfrak{H}^{0,4})$ followed by integrating out the field D is equivalent to the insertion of the contact term $\exp(H_{\Sigma \cdot \Sigma})$.*

In the remainder of the section, we will consider the two cases of the high-energy action and the low-energy effective action that correspond to two different choices of the prepotential \mathcal{F}

4.3.1 The high-energy field theory

The high-energy field theory is given by

$$\mathcal{F}(\phi) = \frac{1}{2} \tau_0 \phi^2, \quad u(\phi) = \frac{1}{2} \phi^2.$$

It follows $T = 0$ in Eq. (4.10). The action in Cor. 2 becomes

$$\begin{aligned} \mathbf{S} &= i \pi \operatorname{Re} \tau_0 \langle c_1(L)^2, [M] \rangle \\ &+ \frac{\operatorname{Im} \tau_0}{8\pi} \int_M \left\{ \frac{1}{2} \langle F_A^+, F_A^+ \rangle + \frac{1}{2} (d^* A)^2 - \frac{1}{2} \langle D, D \rangle \right. \\ &\left. + \bar{\phi} \Delta^{(0)} \phi + \frac{1}{2} \bar{c} \Delta^{(0)} c + i \langle \chi, \pi_+ d\psi \rangle + \frac{i}{\sqrt{2}} \eta d^* \psi \right\} \operatorname{vol}_M. \end{aligned}$$

The observables are

$$\begin{aligned} H_p^{4,0} &= \frac{p}{32\pi} \mathbf{a}^2, \\ H_{\Sigma}^{2,2} &= -\frac{\mathbf{a}}{8\pi} \int_{\Sigma} D, \\ H_{\Sigma \cdot \Sigma} &= 0. \end{aligned}$$

4.3.2 The low-energy effective field theory

The low-energy effective action is obtained as follows: $\tau(u)$ is the modular parameter of the elliptic curve E_u in the elliptic fibration $Z \rightarrow \text{UP}$. The vacuum expectation value of ϕ is \mathbf{a} from Lemma 3. with $\omega = d\mathbf{a}/du$ in Eq. (3.9) and

$$-16\pi i u(\phi) = 2\mathcal{F}(\phi) - \phi \frac{\partial \mathcal{F}(\phi)}{\partial \phi}. \quad (4.11)$$

We obtain the source terms

$$\begin{aligned} H_p^{4,0} &= \frac{i p u}{16\pi}, \\ H_\Sigma^{2,2} &= -\frac{i}{8\pi} \int_\Sigma u' D, \\ H_{\Sigma,\Sigma} &= S^2 T(u). \end{aligned}$$

One still has to integrate out the auxiliary field D . Because D is a selfdual two-form the algebraic Euler-Lagrange equation for the field D yields

$$\text{Im } \tau D = 2i S u' \omega. \quad (4.12)$$

If we replace the auxiliary field D in the action by its critical value, we obtain

$$\int_M -\frac{\text{Im } \tau}{16\pi} \langle D, D \rangle + H_\Sigma^{2,2} + H_{\Sigma,\Sigma} = S^2 \widehat{T}(u),$$

with

$$\widehat{T}(u) = \frac{1}{4} \left(\frac{[u']^2}{2\pi \text{Im } \tau} + 2u - \mathbf{a} u' \right).$$

By integrating out the auxiliary field D we obtain through the interaction terms of D and $\eta\chi$ a perturbed skew-adjoint operator \mathcal{D}_S in the fermionic path integral (cf. Sec. 1.4). In Sec. 4.7, we will describe the operator \mathcal{D}_S as an operator coupled to an external gauge potential proportional to S .

Remark 11. *When integrating out the auxiliary field D there is also an additional contribution in Eq. (4.12) proportional to the zero-modes $\eta_0\chi_0$. We will show in Sec. 4.11 that for $M = CP^2$ this term does not contribute to the path integral in the semi-classical approximation.*

4.4 The Kähler geometry and heat kernel coefficients of the Laplacian

In this section, we describe some geometric and analytic consequences of the Kähler property for a four-dimensional manifold.

Let g denote the Kähler metric on a complex two-dimensional Kähler manifold M . Let the real, harmonic Kähler form be $K = \frac{i}{2} g_{a\bar{b}} dz^a \wedge dz^{\bar{b}}$. The volume form is $\text{vol}_M = \frac{1}{2} K \wedge K$. The Ricci curvature ρ is

$$\rho_{\mu\nu} = R_{\mu\alpha\nu\beta} g^{\alpha\beta},$$

where R is the Riemann curvature tensor. The natural line bundle $L_\omega \rightarrow M$ is related to the Kähler form by

$$\omega = c_1(L_\omega^*) = \frac{i}{2\pi} \Omega(L_\omega^*) = \frac{1}{\pi} K.$$

The operator d can be decomposed as $d = \partial + \bar{\partial}$. The adjoints are

$$\partial^* = - * \bar{\partial} * = - * \# \partial * \#, \quad \bar{\partial}^* = - * \partial * = - * \# \bar{\partial} * \#,$$

where the Hodge star-operator maps $\Lambda^{p,q}$ into $\Lambda^{2-q,2-p}$, and $\#$ is the complex conjugation. An operator of type $(1,1)$ is given by $L_K(x) = K \wedge x$ and its adjoint is denoted by Λ_K . The Kähler identities [62] are

$$-i * (K \wedge \alpha^{(0,1)}) = \alpha^{(0,1)}, \quad i * (K \wedge \alpha^{(1,0)}) = \alpha^{(1,0)}. \quad (4.13)$$

It follows that

$$[\Lambda_K, \partial] = i\bar{\partial}^*, \quad [\Lambda_K, \bar{\partial}] = -i\partial^*. \quad (4.14)$$

Since K is closed it follows $[L_\omega, \partial] = [L_\omega, \bar{\partial}] = 0$, it follows that

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = 2\Delta_{\bar{\partial}},$$

and

$$[L_K, \Delta_{\bar{\partial}}] = 0.$$

For $t > 0$, $e^{-t\Delta_{\bar{\partial}}^{(0,p)}}$ are operators with C^∞ -kernel. The heat traces

$$h[t, \Delta_{\bar{\partial}}^{(0,p)}] = \text{tr}_{L^2} e^{-t\Delta_{\bar{\partial}}^{(0,p)}} = \sum_{n=1}^{\infty} e^{-t|\lambda_n^{(0,p)}|^2}$$

exist and are analytic for $t > 0$. There exists an asymptotic series for $t \rightarrow 0+$ of the form

$$h[t, \Delta_{\bar{\partial}}^{(0,p)}] = \frac{1}{(2\pi t)^2} \sum_{n \geq 0} a_n^{(0,p)} t^n. \quad (4.15)$$

We define the ζ -function

$$\zeta_{(0,p)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{tr}_{L^2} \left[e^{-t\Delta^{(0,p)}} - P_0^{(0,p)} \right],$$

where $P_0^{(0,p)}$ is the projector onto the set of zero modes. From Eq. (4.15) and $\Gamma(s) = s + O(s^2)$ for $s \rightarrow 0$, it follows that

$$\zeta_{(0,0)}(0) = \frac{1}{4\pi^2} a_2^{(0,0)} - 1, \quad \zeta_{(0,2)}(0) = \frac{1}{4\pi^2} a_2^{(0,2)} - b^{(0,2)},$$

and

$$(s \zeta_{(0,0)}(s+1)) \Big|_{s=0} = \frac{1}{4\pi^2} a_1^{(0,0)}, \quad (s \zeta_{(0,0)}(s+2)) \Big|_{s=0} = \frac{1}{4\pi^2} a_0^{(0,0)}.$$

In [53], the following proposition was proved:

Proposition 13.

1. The scalar curvature $\tau = \rho_{\mu\nu} g^{\mu\nu}$ is a spectral invariant of order one.
2. The spectral invariants of order two are

$$\tau^2, \quad \|\rho\|^2 = \rho_{\mu\nu} \rho^{\mu\nu}, \quad \|R\|^2, \quad \Delta\tau = \sum R_{ijij,kk}.$$

Moreover, it was shown by Donnelly in [53] that

$$a_0^{(0,0)} = 1, \quad a_0^{(0,2)} = 1,$$

and

$$a_1^{(0,0)} = \frac{1}{12} \tau, \quad a_1^{(0,2)} = -\frac{1}{6} \tau, \tag{4.16a}$$

$$a_2^{(0,0)} = \frac{1}{288} \tau^2 - \frac{1}{720} \|\rho\|^2 + \frac{1}{720} \|R\|^2 + \frac{1}{720} \Delta\tau, \tag{4.16b}$$

$$a_2^{(0,2)} = \frac{1}{72} \tau^2 - \frac{1}{45} \|\rho\|^2 + \frac{1}{720} \|R\|^2 - \frac{1}{80} \Delta\tau. \tag{4.16c}$$

4.4.1 The complex projective space CP^2

The Kähler metric on CP^2 is the Fubini-Study metric

$$g = g_{a\bar{b}} dz^a dz^{\bar{b}} = \frac{\sum dz^a dz^{\bar{a}}}{1 + \sum z^c z^{\bar{c}}} - \frac{z^{\bar{a}} dz^a z^{\bar{b}} dz^b}{(1 + \sum z^c z^{\bar{c}})^2}.$$

The Ricci curvature is $\rho_{\mu\nu} = 6 g_{\mu\nu}$. The scalar curvature is $\tau = 24$. The spectral invariants of order two are

$$\tau^2 = 576, \quad \|\rho\|^2 = 144, \quad \|R\|^2 = 192, \quad \Delta\tau = \sum R_{ijij,kk} = 0.$$

The Kähler form satisfies $\int_{CP^2} \omega \wedge \omega = 1$. The Chern classes of the tangent bundle are $c_1(CP^2) = 2\omega$ and $c_2(CP^2) = 3\omega^2$ such that

$$C_1^2 = \int_{CP^2} c_1^2 = 4, \quad C_2 = \int_{CP^2} c_2 = 3.$$

It follows

$$\int_{CP^2} \text{vol}_{CP^2} = \int_{CP^2} \frac{\pi^2}{2} \omega \wedge \omega = \frac{\pi^2}{2}.$$

Our conventions for the Fubini-Study metric and the curvature agree with the ones used in [54]. We also define a form $\widehat{\omega} = \frac{1}{\sqrt{2}}K = \frac{\pi}{\sqrt{2}}\omega$ such that $\text{vol}_{CP^2} = \widehat{\omega}^2$ and

$$\langle \widehat{\omega}, \widehat{\omega} \rangle = *(\widehat{\omega} \wedge *\widehat{\omega}) = 1.$$

The coefficients in Eqs. (4.16) are

$$a_1^{(0,0)} = 2, \quad a_1^{(0,2)} = -3, \quad a_2^{(0,0)} = \frac{31}{15}, \quad a_2^{(0,2)} = \frac{76}{15}.$$

4.5 Scaling and $N = 2$ supersymmetry

In this section, we describe the action of the global supersymmetry on the field content of the twisted, $N = 2$ supersymmetric, pure, low-energy effective field theory on $M = CP^2$. The word ‘pure’ indicates that we will describe a theory with the minimal field content without additional matter fields. The word ‘twisted’ indicates that the field content consists of various differential forms of different degrees instead of spinors. $N = 2$ supersymmetry refers to the global supersymmetry.

4.5.1 The critical points

The family of bosonic, gauge-invariant actions on M for the $U(1)$ -connection A and the complex valued function ϕ on M is given by

$$\begin{aligned} & \mathbf{S}_{\text{bos}}(\tau)[A, \phi, w_2(M)] \\ &= \frac{1}{2g^2} \int_M \frac{1}{2} F_A \wedge *F_A + d\phi \wedge *d\bar{\phi} + i \frac{\theta}{32\pi^2} \int_M F_A \wedge F_A + \frac{i}{4} \int_M F_A \wedge w_2(M) \end{aligned}$$

which equals

$$\begin{aligned} & \frac{i\bar{\tau}}{16\pi} \int_M F_A^+ \wedge F_A^+ + \frac{i\tau}{16\pi} \int_M F_A^- \wedge F_A^- \\ & + \frac{i}{4} \int_M F_A \wedge w_2(M) + \frac{\text{Im } \tau}{8\pi} \int_M d\phi \wedge *d\bar{\phi} \quad (4.17) \end{aligned}$$

with $\tau = \theta/(2\pi) + i4\pi/g^2 \in H$, and $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ is the second Stiefel-Whitney class. The action is defined on $\mathcal{B}^* \times C^\infty(M; \mathbb{C})$. Since $c_1(L) = [\frac{1}{2\pi}F_A] \in H^2(M; \mathbb{Z})$, it follows that

$$\text{Im } \mathbf{S}_{\text{bos}}(\tau)[A, \phi] = \frac{1}{8}\theta \langle c_1^2(L), [M] \rangle + \frac{\pi}{2} c_1(L) \cdot w_2(M) .$$

The minima for $\text{Re } \mathbf{S}_{\text{bos}}(\tau)$ under the constraint of fixed $\text{Im } \mathbf{S}_{\text{bos}}(\tau)[A, \phi]$ were determined by Moore and Witten [25]: 1) For the fermionic fields (η, ψ, χ) the zero-modes (η_0, ψ_0, χ_0) are given by the equations

$$d\eta_0 = 0, \quad d\psi_0 = 0, \quad d^*\psi_0 = 0, \quad d\chi_0 = 0 .$$

Since χ is a selfdual two-form it must be harmonic and a constant multiple of the Kähler form $\chi_0 = \theta_0 \omega$ for a Grassmann-algebra valued constant θ_0 with $\theta_0^2 = 0$. Similarly, η_0 is a Grassmann-algebra valued constant, and $\psi_0 = 0$ vanishes since $b_1 = 0$. The ghost-fields c, \bar{c} have vanishing zero-modes by definition.

2) For the bosonic fields $(A, \phi, \bar{\phi})$ the zero-modes $(A_0, \mathbf{a}, \bar{\mathbf{a}})$ are given by the equations

$$d\phi = 0, \quad F_{A_0}^- = 0 .$$

Since $b_2^- = 0$ vanishes, the unique non-trivial minima in $\mathcal{B}^* \times C^\infty(M; \mathbb{C})$ for each τ are given by a complex constant $\phi = \mathbf{a}$ and a gauge equivalence class of a connection $[A]$ such that the curvature F_A is self-dual. The Kähler class ω is a selfdual harmonic two-form normalized such that $\int_M \omega \wedge \omega = 1$ and $w_2(M) = \frac{1}{2}\omega$ since CP^2 is not Spin. We assume that the class $[A]$ satisfies $\frac{1}{2\pi}F_{A(2k)} = 2k\omega$ and $C_1(L) = 2k$. The zero-mode D_0 is fixed by integrating out the auxiliary field D . Using Eq. (3.14), the instanton sum becomes

$$\sum_{k=-\infty}^{\infty} e^{-\mathbf{S}_{\text{bos}}(\tau)[A^{(2k)}, \mathbf{a}, w_2(M)]} = \sum_{k=-\infty}^{\infty} e^{-i\pi \bar{\tau} k^2 - \frac{i\pi k}{2}} = \overline{\vartheta_4(\tau)} . \quad (4.18)$$

The linear fluctuations of the fermionic fields are

$$\begin{aligned} \eta &= \eta_0 + \epsilon \eta' , \\ \psi &= \epsilon \psi' , \\ \chi &= \theta_0 \omega + \epsilon \chi' , \end{aligned}$$

with

$$\int_M \text{vol}_M \eta' = \int_M \text{vol}_M \langle \omega, \chi' \rangle = 0 .$$

The variations of the bosonic field are

$$\begin{aligned} A &= A_0 + \epsilon A' , \\ F_A &= F_{A_0}^+ + \epsilon dA' , \\ \phi &= \mathbf{a} + \epsilon \phi' , \\ \bar{\phi} &= \bar{\mathbf{a}} + \epsilon \bar{\phi}' , \end{aligned}$$

with

$$\int_M \text{vol}_M \phi' = 0 .$$

We expand the action \mathbf{S} in powers of ϵ up to order two:

$$\mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}^{(1)} + \epsilon^2 \mathbf{S}^{(2)} + O(\epsilon^3) .$$

where $\mathbf{S}^{(1)} = 0$ since we have expanded around a critical point.

4.5.2 The behavior under rescaling

The full quantum theory is expected to be invariant under rescaling of the metric. Thus, it will be important to determine how the action and the field content change in a one-parameter family of metrics $g \rightarrow t^2 g$ for $t \rightarrow \infty$. This means that the inverse and the determinant of the metric tensor, and the volume form scale as

$$g^{-1} \rightarrow t^{-2} g^{-1} , \quad \det g \rightarrow t^8 \det g , \quad \text{vol}_M \rightarrow t^4 \text{vol}_M .$$

We write a p -form a in local coordinates as $a = \frac{a_{\nu_1 \dots \nu_p}}{p!} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p}$, and the wedge product of a p -form a and q -form b as

$$a \wedge b = \frac{a_{\nu_1 \dots \nu_p} b_{\mu_1 \dots \mu_q}}{p! q!} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} . \quad (4.19)$$

The Hodge-star operator acts on a p -form by

$$*a = \frac{1}{(4-p)!} \frac{\sqrt{|g|}}{p!} \epsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_4} a^{\nu_1 \dots \nu_p} dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_4} . \quad (4.20)$$

For two k -forms a, b we define

$$\langle a, b \rangle = *(a \wedge *b) = \langle b, a \rangle = \frac{1}{k!} a_{\nu_1 \dots \nu_k} b^{\nu_1 \dots \nu_k} , \quad (4.21a)$$

$$(a, b) = \int_M a \wedge *b = \int_M \text{vol}_M \langle a, b \rangle , \quad (4.21b)$$

where $\text{vol}_M = *1$ and $*^2 = (-1)^k$ on k -forms. The zero-modes $(\mathbf{a}, A_0, \eta_0, \chi_0)$ of the fields (ϕ, A, η, χ) can be normalized in a way invariant under rescaling of the metric since the equations of motion are conformally invariant. Similarly, ω has a topological

interpretation and remains invariant under rescaling. The Laplace-Beltrami operator on a real function φ is

$$\Delta_d \varphi = d^* d \varphi = -\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right) .$$

It rescales according to $\Delta_d \rightarrow t^{-2} \Delta_d$, and so do the eigenvalues $2|\lambda_n^{(0)}|^2$. On a compact manifold, Δ_d is a positive definite selfadjoint operator with discrete spectrum. We denote the pairwise L^2 -orthogonal eigenfunctions to the eigenvalues $\lambda_n^{(0)}$ by φ'_n . We fix the transformation of the eigenfunctions to be $\varphi'_n \rightarrow t^{-1} \varphi'_n$ under rescaling. Because of the rescaling of the volume form we have a transformation under rescaling of $\|\varphi'_n\|^2 \rightarrow t^2 \|\varphi'_n\|^2$. It follows that the kinetic term for a scalar field $\|d\varphi'_n\|^2 = 2|\lambda_n^{(0)}|^2 \|\varphi'_n\|^2$ is invariant under rescaling.

We are interested in the spectrum of the Laplace operator Δ^+ relevant for the Yang-Mills complex on one-forms and two-forms:

$$\begin{aligned} \Delta^{(0)+} \Big|_{\Lambda^0} &= d^* d \Big|_{\Lambda^0} = \Delta_d \Big|_{\Lambda^0} , \\ \Delta^{(1)+} \Big|_{\Lambda^1} &= 2 d^* \pi_+ d + d d^* \Big|_{\Lambda^1} , \\ \Delta^{(2)+} \Big|_{\Lambda^{2+}} &= 2 \pi_+ d d^* \Big|_{\Lambda^{2+}} . \end{aligned}$$

The Laplace operator on a Kähler manifold satisfies $\Delta^{(0)+} = 2\Delta_{\bar{\partial}}^{(0,0)}$ and

$$\Delta^{(2)+} \Big|_{\Lambda^{2+}} = (\pi_+ d d^* + \pi_+ d^* d) \Big|_{\Lambda^{2+}} = \pi_+ \Delta_d \Big|_{\Lambda^{2+}} = 2 \pi_+ \Delta_{\bar{\partial}} \Big|_{\Lambda^{2+}} .$$

On a Kähler manifold, there is the following orthogonal sum decomposition of the self-dual two-forms

$$\Lambda^{2+} = \Lambda^{(2,0)} \oplus \Lambda^{(1,1)+} \oplus \Lambda^{(0,2)} ,$$

where $\Lambda^{(1,1)+}$ are the forms proportional to the Kähler form ω . Because of the identity

$$[L_K, \Delta_{\bar{\partial}}] = 0 ,$$

there is an isomorphism between $C^\infty(M)$ and $\Lambda^{(1,1)+}(M)$ given by the multiplication with the Kähler form: for each eigenvalue $2|\lambda_n^{(0)}|^2$ and the eigenfunction φ'_n of $\Delta^{(0)+} = 2\Delta_{\bar{\partial}}$, there is the two-form $f_n'^{(1,1)} = \varphi'_n \widehat{\omega}$. We check that

$$\begin{aligned} \langle \widehat{\omega}, \Delta^{(2)+} f_n'^{(1,1)} \rangle &= 2 \langle \widehat{\omega}, \pi_+ \Delta_{\bar{\partial}} f_n'^{(1,1)} \rangle = 2 \langle \widehat{\omega}, \Delta_{\bar{\partial}} L_{\widehat{\omega}} \varphi'_n \rangle = 2 \langle \widehat{\omega}, \Delta_{\bar{\partial}} (\varphi'_n \widehat{\omega}) \rangle \\ &= 2 \Delta_{\bar{\partial}} \varphi'_n = 2 |\lambda_n^{(0)}|^2 \varphi'_n . \end{aligned}$$

We have proved:

Lemma 14. *On CP^2 there is a one-to-one correspondence between the eigenfunction φ'_n of $\Delta^{(0)+}$ for the eigenvalue $2|\lambda_n^{(0)}|^2$ and the eigenform $f'_n \in \Lambda^{(1,1)+}$ of $\Delta^{(2)+}$ for the eigenvalue $2|\lambda_n^{(0)}|^2$ given by $\varphi'_n \mapsto f'_n{}^{(1,1)} = \varphi'_n \widehat{\omega}$ and $f'_n{}^{(1,1)} \mapsto \varphi'_n = \langle f'_n{}^{(1,1)}, \widehat{\omega} \rangle$ such that*

$$\|\varphi'_n\|^2 = \|f'_n{}^{(1,1)}\|^2, \quad \langle \Delta^{(2)+} f'_n{}^{(1,1)}, \widehat{\omega} \rangle = \Delta^{(0)+} \varphi'_n.$$

Under the rescaling of the metric $g \rightarrow t^2 g$, the bosonic fields transform as

$$\widehat{\omega} \rightarrow t^2 \widehat{\omega}, \quad f'_n{}^{(1,1)} \rightarrow t f'_n{}^{(1,1)}, \quad \varphi'_n \rightarrow t^{-1} \varphi'_n.$$

Remark 12. *If the eigenvector f'_K of $\Delta^{(2)+}$ is a section of $\Lambda^{(2,0)} \oplus \Lambda^{(0,2)}$, it follows that $\langle \omega, f'_k \rangle = 0$ for dimensional reasons. Using the real and imaginary part, there are two orthogonal eigenvectors $f'_k{}^{[(0,2),I]}$, $f'_k{}^{[(0,2),II]}$ of $\Delta^{(2)+}$ for each eigenvalue $2|\lambda_k^{(0,2)}|^2$.*

Remark 13. *There is an isomorphism between the eigenspaces of $\Delta^{(1)+}$ spanned by the orthogonal eigenforms $A_n^{(+)}$ and $A_K^{(-)}$ and the eigenfunctions φ'_n of $\Delta^{(0)+}$ for the eigenvalue $2|\lambda_n^{(0)}|^2$ and f'_K of $\Delta^{(2)+}$ for the eigenvalue $2|\lambda_k^{(0,2)}|^2$ given by*

$$A_n^{(+)} = \frac{d \varphi'_n}{\sqrt{2} \lambda_n^{(0)}}, \quad A_K^{(-)} = \frac{d^* f'_K}{\lambda_K^{(2)}}.$$

and

$$\frac{d^* A_n^{(+)}}{\sqrt{2} \lambda_n^{(0)}} = \varphi'_n, \quad \frac{d A_K^{(-)}}{\lambda_K^{(2)}} = f'_K. \quad (4.22)$$

It follows

$$\|A_n^{(+)}\|^2 = \|\varphi'_n\|^2, \quad \|f'_K\|^2 = \|A_K^{(-)}\|^2, \quad (A_n^{(+)}, A_K^{(-)}) = 0.$$

Applying the Hodge star to a k -form increases the scaling dimension of a field by the metric dependence in Eq. (4.20) by $t^{2(2-k)}$. Thus, we have the following transformation under rescaling for the terms involving the gauge

$$A_{\bullet}^{(\pm)} \rightarrow t^{-1} A_{\bullet}^{(\pm)}, \quad (A_{\bullet}^{(\pm)}, \Delta^{(1)} A_{\bullet}^{(\pm)}) = |\lambda^{(\bullet)}|^2 \|A_{\bullet}^{(\pm)}\|^2 \rightarrow (A_{\bullet}^{(\pm)}, \Delta^{(1)} A_{\bullet}^{(\pm)}).$$

4.5.3 The fermionic fluctuations

For the fermionic fields, the transformation under rescaling is different because the kinetic terms in the action are different. We define $\eta'_n = \eta_n \lambda_n^{(0)} \varphi'_n / \sqrt{2}$ where η_n are Grassmann-algebra valued constants with $\eta_n^2 = 0$ invariant under rescaling. Similarly, we define $\chi'_K = \theta_K \lambda_K^{(2)} f'_K / \sqrt{2}$ where θ_K are Grassmann-algebra valued constants with $\theta_K^2 = 0$ invariant under rescaling. We have proved:

Corollary 3. *Under the rescaling of the metric by $g \rightarrow t^2 g$, the fermionic fluctuations transform as*

$$\begin{aligned}\eta'_n &\rightarrow t^{-2} \eta'_n, \\ \chi'_K &\rightarrow \chi'_K,\end{aligned}$$

The fermionic fluctuations of ψ are given by

$$\psi_n'^{(+)} = \psi_n^{(+)} \frac{d\varphi'_n}{\sqrt{2}\lambda_n^{(0)}}, \quad \psi_K'^{-} = \psi_K^{(-)} \frac{d^* f'_K}{\lambda_K^{(2)}}, \quad (4.23)$$

where $\psi_\bullet^{(\pm)}$ are Grassmann-algebra valued constants with $\psi_\bullet^{(\pm)2} = 0$ and invariant under rescaling. It follows

$$\left(\psi_\bullet^{(\pm)}, \psi_\bullet^{(\mp)}\right) = 0, \quad \left(\psi_K'^{-}, d\eta'_m\right) = 0, \quad \left(\psi_n'^{+}, d^* \chi'_K\right) = 0.$$

We have the following transformation under rescaling:

$$\begin{aligned}\psi_\bullet^{(\pm)} &\rightarrow t^{-1} \psi_\bullet^{(\pm)}, \\ \int \text{vol}_M \langle \psi_n'^{+}, d\eta'_m \rangle &\rightarrow \int \text{vol}_M \langle \psi_n'^{+}, d\eta'_m \rangle, \\ \int \text{vol}_M \langle \chi'_K, \sqrt{2} d\psi_L'^{-} \rangle &\rightarrow \int \text{vol}_M \langle \chi'_K, \sqrt{2} d\psi_L'^{-} \rangle, \\ \int \text{vol}_M \langle \eta'_n \omega, \chi_m'^{(1,1)} \rangle &\rightarrow t^{-2} \int \text{vol}_M \langle \eta'_n \omega, \chi_m'^{(1,1)} \rangle, \\ \int \text{vol}_M \langle \psi_n'^{(\pm)}, *\omega \wedge \psi_m'^{(\mp)} \rangle &\rightarrow t^{-2} \int \text{vol}_M \langle \psi_n'^{(\pm)}, *\omega \wedge \psi_m'^{(\mp)} \rangle, \\ \int \text{vol}_M \langle \chi'_K, \mathbb{C}_\omega \chi'_L \rangle &\rightarrow t^{-2} \int \text{vol}_M \langle \chi'_K, \mathbb{C}_\omega \chi'_L \rangle.\end{aligned}$$

We notice that that the kinetic terms are invariant under rescaling:

Lemma 15. *Under the rescaling of the metric by $g \rightarrow t^2 g$, the kinetic terms*

$$\begin{aligned}&\left(\psi_n'^{+}, d\eta'_m\right), \quad \left(\chi'_K, \sqrt{2} d\psi_K'^{-}\right) \\ &\left(A_n'^{(\pm)}, \Delta^{(1)+} A_n'^{(\pm)}\right), \quad \left(\varphi'_n, \Delta^{(0)+} \varphi'_n\right)\end{aligned}$$

are invariant.

We summarize the results of this section:

Proposition 14. *The $N = 1$ global supersymmetry relates the bosonic and fermionic semi-classical fields in each $N = 1$ multiplet: there is a one-to-one correspondence between the fluctuations of A and ψ ,*

$$A_\bullet'^{(\pm)} \leftrightarrow \psi_\bullet'^{(\pm)}.$$

There is a one-to-one correspondence between the fluctuations of ϕ and η , and the two-forms D and χ

$$\phi'_n \leftrightarrow \eta'_n, \quad D'_K \leftrightarrow \chi'_K.$$

The $N = 2$ global supersymmetry relates the bosonic and fermionic fields from different $N = 1$ multiplets: there is a one-to-one correspondence between the bosonic and fermionic fluctuations

$$\begin{aligned} A_n'^{(+)} &\leftrightarrow \eta'_n, \\ A_K'^{-} &\leftrightarrow \chi'_K, \\ \psi_n'^{+}, \psi_n'^{-}(1,1) &\leftrightarrow \phi'_n, \\ \psi_k'^{-}[(0,2),I], \psi_k'^{-}[(0,2),II] &\leftrightarrow D_k'^{[(0,2),I]}, D_k'^{[(0,2),II]} \end{aligned}$$

by Eqs. (4.22), (4.23).

4.6 The Yang-Mills complex

In this section, we present a short review of the torsion of the uncoupled Yang-Mills complex on $M = CP^2$. This section is based on the results in [39, 56, 38].

Let the operator

$$D : \Lambda^1(M) \rightarrow C^\infty(M) \oplus \Lambda^{2+}(M)$$

be defined by

$$D A = \begin{pmatrix} d^* A \\ \sqrt{2} \pi_+ d A \end{pmatrix}, \quad D^* \begin{pmatrix} \phi \\ f \end{pmatrix} = d\phi + \sqrt{2} d^* f,$$

where the map π_+ denotes the orthogonal projection on the self-dual two-forms. The operator D fits in the elliptic \mathbb{Z}_2 -graded complex

$$\Lambda^1(M) \xrightarrow{D} C^\infty(M) \oplus \Lambda^{2+}(M). \quad (4.24)$$

The associated Laplace operators are

$$\Delta^{(\text{ev})} = D D^* = \Delta^{(0)+} + \Delta^{(2)+}, \quad \Delta^{(\text{od})} = D^* D = \Delta^{(1)+}. \quad (4.25)$$

The index of D is

$$\text{Ind } D = \dim \ker \Delta^{(\text{od})} - \dim \ker \Delta^{(\text{ev})} = -(1 + b_2^+) = -\frac{\chi + \sigma}{2}.$$

4.6.1 The heat kernel

The Laplace operators in Eq. (4.25) are self-adjoint, second-order elliptic operators. Their leading symbols are positive definite. Then, for $t > 0$, $e^{-t\Delta^{(\text{od})}}$ and $e^{-t\Delta^{(\text{ev})}}$ are operators with C^∞ -kernel. The heat traces

$$h[t, \Delta^{(\text{od})}] = \text{tr}_{L^2} e^{-t\Delta^{(\text{od})}} = \sum_{\lambda^{(\text{od})}} e^{-t\lambda^{(\text{od})}}$$

exists and is analytic for $t > 0$. There exists an asymptotic series as $t \rightarrow 0+$ of the form

$$h[t, \Delta^{(\text{od})}] = \frac{1}{4\pi^2 t^2} \sum_{n \geq 0} a_n^{(\text{od})} t^n, \quad (4.26)$$

As the Laplace operators have the same spectrum except for the zero modes, it follows that

$$\begin{aligned} \text{Ind } D &= \dim \ker \Delta^{(\text{od})} - \dim \ker \Delta^{(\text{ev})} = -(1 + b_2^+) = -\frac{\chi + \sigma}{2} \\ &= h[t, \Delta^{(\text{od})}] - h[t, \Delta^{(\text{ev})}] = \frac{a_2^{(\text{od})}}{4\pi^2} - \frac{a_2^{(\text{ev})}}{4\pi^2}. \end{aligned}$$

We define the ζ -function

$$\zeta_{\Delta^{(\text{od})}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}_{L^2} [e^{-t\Delta^{(\text{od})}} - P_0^{(\text{od})}]$$

where $P_0^{(\text{od})}$ is the projector onto the set of zero modes. We also set

$$\zeta(s) = \sum_{n=1}^{\infty} (2|\lambda_n^{(0)}|^2)^{-s} + \sum_{K=1}^{\infty} (2|\lambda_K^{(2)}|^2)^{-s}, \quad (4.27)$$

where $\{2|\lambda_n^{(0)}|^2\}$ are the eigenvalues of $\Delta^{(0)+}$ on functions and $\{2|\lambda_K^{(2)}|^2\}$ the eigenvalues of $\Delta^{(2)+}$ on self-dual two-forms on M . The non-zero spectra of $\Delta^{(\text{od})}$ and $\Delta^{(\text{ev})}$ are identical, and it follows that $\zeta(s) = \zeta_{\Delta^{(\text{od})}}(s)$ and $\zeta_{\Delta^{(\text{ev})}}(s) = \zeta(s) - 2$.

4.6.2 The rescaled Laplace operator

The ζ -function regularized determinant is defined by

$$\det \Delta^{(\text{od})} = -\frac{d}{ds} \Big|_{s=0} \zeta'_{\Delta^{(\text{od})}}(s).$$

For $\alpha \in \mathbb{R}^+$ the positive definite, self-adjoint Laplace operator on one-forms

$$\alpha D^* D = \alpha \Delta^{(\text{od})}$$

has real and positive eigenvalues $\alpha \lambda_n^2$. The eigenvalues are rescaled by a positive constant. In terms of ζ -functions the functional determinant changes as

$$\ln \det \left[\alpha \Delta^{(\text{od})} \right] = -\zeta'(0) + \ln(\alpha) \zeta(0),$$

which implies

$$\det \left[\alpha \Delta^{(\text{od})} \right] = \alpha^{\zeta(0)} \det \Delta^{(\text{od})} = \alpha^{\zeta(0)} \det' \Delta^{(0)+} \det' \Delta^{(2)+}.$$

Remark 14. *The difference between $\det[\alpha \Delta^{(\text{od})}]$ and the functional determinant*

$$\det \left[\alpha (dd^* + d^*\pi_+d) \right]$$

is a real number independent of α . $1/\sqrt{\det[\alpha \Delta^{(\text{od})}]}$ is up to a normalization the contribution of the path integral over variations of the gauge potential A .

A family of operators

$$\mathbf{D}_\alpha : C^\infty(M) \oplus \Lambda^{2+}(M) \oplus \Lambda^1(M) \rightarrow C^\infty(M) \oplus \Lambda^{2+}(M) \oplus \Lambda^1(M)$$

for $\alpha \in \mathbb{R}^+$ is defined by

$$\mathbf{D}_\alpha \begin{pmatrix} \eta \\ \chi \\ \psi \end{pmatrix} = \alpha \left(\begin{array}{c|c} 0 & D \\ -D^* & 0 \end{array} \right) \begin{pmatrix} \eta \\ \chi \\ \psi \end{pmatrix}. \quad (4.28)$$

The positive definite, self-adjoint Laplacian

$$\Delta_\alpha = \mathbf{D}_1^* \mathbf{D}_\alpha = \begin{pmatrix} \alpha d^*d & 0 & 0 \\ 0 & 2\alpha \pi_+ dd^* & 0 \\ 0 & 0 & \alpha [dd^* + 2d^*\pi_+d] \end{pmatrix}.$$

has the ζ -regularized determinant

$$\det' \Delta_\alpha = \frac{1}{\alpha^2} (\alpha^{2\zeta(0)} \det' \Delta^{(0)+} \det' \Delta^{(2)+})^2.$$

4.6.3 The determinant line bundle

For the family of operators \mathbf{D}_α , the determinant line bundle

$$\text{DET } \mathbf{D}_\alpha \rightarrow \mathbb{R}^+, \quad (4.29)$$

has a canonical section $\det \mathbf{D}_\alpha$ since the numerical index of \mathbf{D}_α is zero. The Quillen metric is defined by

$$\|\det \mathbf{D}_\alpha\|_Q^2 = \det' \Delta_\alpha.$$

As α is a real variable, the determinant line bundle is flat. It follows:

Lemma 16. *An orientation of $H^0(M) \oplus H^{2+}(M)$ determines a canonical trivialization of the determinant line bundle of \mathbf{D}_α . It follows that*

$$\frac{\det \mathbf{D}_\alpha}{\det' \left(\alpha^2 \Delta^{(\text{od})} \right)} = \frac{1}{\alpha}. \quad (4.30)$$

On a Kähler manifold, $C^\infty(M)$ is isomorphic to $\Lambda^{(1,1)+}(M)$ by Lemma 14. Thus, the eigenvalue $\lambda_n^{(0)}$ occurs with multiplicity two. Using the complex conjugation on M , it follows that the eigenvalue $\lambda_k^{(0,2)}$ occurs with multiplicity two as well. Thus,

$$\zeta(s) = 2 \sum_{n=1}^{\infty} (2|\lambda_n^{(0)}|^2)^{-s} + 2 \sum_{k=1}^{\infty} (2|\lambda_k^{(0,2)}|^2)^{-s},$$

and since $b^{(0,2)} = 0$ we have

$$\zeta_{(0,0)}(s) = \sum_{n=1}^{\infty} |\lambda_n^{(0)}|^{-2s} - 1, \quad \zeta_{(0,2)}(s) = \sum_{n=1}^{\infty} |\lambda_n^{(0,2)}|^{-2s},$$

where $\zeta_{(0,0)}(s) = \zeta_{\Delta_{\bar{\delta}}^{(0,0)}}(s)$ and $\zeta_{(0,2)}(s) = \zeta_{\Delta_{\bar{\delta}}^{(0,2)}}(s)$. It follows

$$\det \left[\alpha \Delta^{(\text{od})} \right] = (\sqrt{2}\alpha)^{\zeta(0)} \left(\det' \Delta_{\bar{\delta}}^{(0,0)} \det' \Delta_{\bar{\delta}}^{(0,2)} \right)^2.$$

4.6.4 The Pfaffian line bundle.

The operator \mathbf{D}_α is a real skew-adjoint Fredholm operator. The space of real skew-adjoint Fredholm operators is often denoted by $\mathcal{F}_1 R$ (cf. [47]). The determinant line bundle (4.29) has a natural square root, the Pfaffian line bundle

$$\text{PFAFF } \mathbf{D}_\alpha \rightarrow \mathbb{R}^+.$$

It is a flat \mathbb{R} -bundle defined by the homomorphism $\pi_1(\mathcal{F}_1 R) = \mathbb{Z}_2 \rightarrow \text{End } \mathbb{R}$ which takes [1] to multiplication with -1 . Geometrically, as the parameters of the operator vary a number of pairs eigenvalues that switch their sign. This number (mod 2) labels the class of $\pi_1(\mathcal{F}_1 R) = \mathbb{Z}_2$ [47, Sec. III]. The holonomy is the number of eigenspace rearrangements (mod 2) [47, Sec. VI]. For the operator \mathbf{D}_α the class is [0]. Thus, the Pfaffian line bundle has no local nor global holonomy. We denote the canonical section by $\text{pfaff } \mathbf{D}_\alpha$. We have proved:

Lemma 17. *The Pfaffian line bundle of \mathbf{D}_α has no local nor global holonomy and such that*

$$\frac{\text{pfaff } \mathbf{D}_\alpha}{\sqrt{\det' \left[\alpha^2 \Delta^{(\text{od})} \right]}} = \frac{1}{\sqrt{\alpha}}. \quad (4.31)$$

4.7 The determinant line bundle for the u -plane integral

In this section, we explain how the uncoupled Yang-Mills complex on CP^2 is modified in the low-energy effective field theory.

A family of complex operators

$$\mathbf{D}_{\alpha,\xi} : C^\infty(M; \mathbb{C}) \oplus \Lambda_{\mathbb{C}}^{2+}(M) \oplus \Lambda_{\mathbb{C}}^1(M) \rightarrow C^\infty(M; \mathbb{C}) \oplus \Lambda_{\mathbb{C}}^{2+}(M) \oplus \Lambda_{\mathbb{C}}^1(M)$$

is defined by

$$\mathbf{D}_{\alpha,\xi} \begin{pmatrix} \eta \\ \chi \\ \psi \end{pmatrix} = [\mathbf{D}_\alpha + \xi C_\omega^{(1)} - \bar{\xi} C_\omega^{(2)} + \xi C_\omega^{(3)}] \begin{pmatrix} \eta \\ \chi \\ \psi \end{pmatrix},$$

for $\alpha \in \mathbb{R}^+$, $\xi \in \mathbb{C}^*$. The operators $C_\omega^{(i)}$ are

$$C_\omega^{(1)} = \begin{pmatrix} 0 & * (\widehat{\omega} \wedge \bullet) & 0 \\ -\widehat{\omega} \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_\omega^{(2)} = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * (\widehat{\omega} \wedge \bullet) \end{pmatrix},$$

$$C_\omega^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathfrak{C}_{\widehat{\omega}} \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with

$$\mathfrak{C}_{\widehat{\omega}} \chi' = \frac{1}{2!} \left(\mathfrak{C}_{\widehat{\omega}} \chi' \right)_{\mu\nu} dx^\mu \wedge dx^\nu, \quad \left(\mathfrak{C}_{\widehat{\omega}} \chi' \right)_{\mu\nu} = \widehat{\omega}^\alpha_{[\nu} \chi'_{\mu] \alpha},$$

and $\widehat{\omega}$ was defined in Sec. 4.4.1. One easily checks the following Lemma:

Lemma 18.

1. $\sqrt{2} * (\widehat{\omega} \wedge *) = -i \Big|_{\Lambda^{1,0}} \oplus i \Big|_{\Lambda^{0,1}}$
2. $\mathfrak{C}_{\widehat{\omega}} = -i \Big|_{\Lambda^{2,0}} \oplus i \Big|_{\Lambda^{0,2}}$
3. *The operators $C_\omega^{(i)}$ are skew-adjoint.*

The adjoint operator of $\mathbf{D}_{\alpha,\xi}$ is

$$\mathbf{D}_{\alpha,\xi}^* = -\mathbf{D}_\alpha - \bar{\xi} C_\omega^{(1)} + \xi C_\omega^{(2)} - \bar{\xi} C_\omega^{(3)}.$$

Lemma 19. *The following relations hold for the skew-adjoint operators $C_\omega^{(i)}$:*

1. $C_\omega^{(1)} \cdot C_\omega^{(2)} = C_\omega^{(2)} \cdot C_\omega^{(1)} = C_\omega^{(1)} \cdot C_\omega^{(3)} = C_\omega^{(3)} \cdot C_\omega^{(1)} = C_\omega^{(2)} \cdot C_\omega^{(3)} = C_\omega^{(3)} \cdot C_\omega^{(2)} = 0$

2. If $P_\omega^{(2)+}$ is the projection on the self-dual two-forms proportional to ω then

$$\begin{aligned} C_\omega^{(1)2} &= \begin{pmatrix} -1 & & \\ & -P_\omega^{(2)+} & \\ & & 0 \end{pmatrix}, & C_\omega^{(2)2} &= \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \\ C_\omega^{(3)2} &= \begin{pmatrix} 0 & & \\ & -(1 - P_\omega^{(2)+}) & \\ & & 0 \end{pmatrix}. \end{aligned}$$

Proof. The Lemma follows from $\mathfrak{C}_\omega \widehat{\omega} = 0$, $\langle \widehat{\omega}, \mathfrak{C}_\omega \chi' \rangle = 0$, and Eq. (4.13). \square

Lemma 20. *The following relation holds*

$$\mathbf{D}_1 (C_\omega^{(1)} + C_\omega^{(3)}) = C_\omega^{(2)} \mathbf{D}_1.$$

Proof. We compute

$$\mathbf{D}_1 (C_\omega^{(1)} + C_\omega^{(3)}) - C_\omega^{(2)} \mathbf{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{31} & X_{32} & 0 \end{pmatrix},$$

with

$$\begin{aligned} X_{31} &= \sqrt{2} d^* (\widehat{\omega} \bullet) + \sqrt{2} * (\widehat{\omega} \wedge d \bullet), \\ X_{32} &= -d * (\widehat{\omega} \wedge \bullet) + 2 * (\widehat{\omega} \wedge d^* \bullet) - \sqrt{2} d^* \mathfrak{C}_\omega \bullet. \end{aligned}$$

Since ω is closed, it follows $X_{31} = 0$. We apply X_{32} to an eigenfunction $\phi'_n \widehat{\omega}$ to obtain

$$X_{32} (\phi'_n \widehat{\omega}) = -d \phi'_n - 2 * [\widehat{\omega} \wedge * (\widehat{\omega} \wedge d \phi'_n)],$$

which vanishes by Eq. (4.13). We apply X_{32} to an eigenfunction $\chi'^{(2,0)}$ of type (2, 0) and obtain

$$\begin{aligned} X_{32} (\chi'^{(2,0)}) &= -2 * [\widehat{\omega} \wedge * (\bar{\partial} \chi'^{(2,0)})] - \sqrt{2} d^* \mathfrak{C}_\omega \chi'^{(2,0)} \\ &= \sqrt{2} i * (\bar{\partial} \chi'^{(2,0)}) + \sqrt{2} i d^* \chi'^{(2,0)} = 0. \end{aligned}$$

Using the complex conjugation the same follows for (0, 2)-forms whence $X_{32} = 0$. \square

The following Lemma is a consequence of Lemma 19 and Lemma 20:

Lemma 21. *The positive definite, selfadjoint Laplacian $\Delta_{\alpha,\xi} = \mathbf{D}_{1,\xi/\alpha}^* \mathbf{D}_{\alpha,\xi}$ is*

$$\Delta_{\alpha,\xi} = \alpha \begin{pmatrix} \Delta^{(0)+} & 0 & 0 \\ 0 & \Delta^{(2)+} & 0 \\ 0 & 0 & \Delta^{(1)+} \end{pmatrix} + \frac{|\xi|^2}{\alpha}. \quad (4.32)$$

For the family of operators $\mathbf{D}_{\alpha,\xi}$ for the parameters $(\alpha, \xi) \in \mathbb{R}^+ \times \mathbb{C}$, the determinant line bundle

$$\text{DET } \mathbf{D}_{\alpha,\xi} \rightarrow \mathbb{R}^+ \times \mathbb{C}^*$$

has a canonical section $\det \mathbf{D}_{\alpha,\xi}$ since the operator is invertible. The Quillen metric is defined by

$$\|\det \mathbf{D}_{\alpha,\xi}\|_Q^2 = \det \mathbf{\Delta}_{\alpha,\xi}. \quad (4.33)$$

The functional determinant will be computed in Eq. (4.39).

Lemma 22. *For the operator $\mathbf{D}_{\alpha,\xi}$, there is the skew-adjoint, complex anti-linear operator*

$$\mathcal{D}_{\alpha,\xi} = \mathbf{D}_{\alpha,\xi} \circ \# = \begin{pmatrix} 0 & \mathbf{Q}_{\alpha,\xi} \\ \mathbf{P}_{\alpha,\xi} & 0 \end{pmatrix} \circ \#, \quad (4.34)$$

where $\#$ denotes complex conjugation. The operator

$$\mathbf{Q}_{\alpha,\xi} = \sqrt{2} \begin{pmatrix} -i \frac{\xi}{\sqrt{2}} & 0 & \alpha \partial^* \\ 0 & -i \frac{\xi}{\sqrt{2}} & \alpha \partial \\ -\alpha \partial & -\alpha \partial^* & i \frac{\bar{\xi}}{\sqrt{2}} \end{pmatrix}$$

acts on the sections

$$\frac{1}{\sqrt{2}} \left(C^\infty(M, \mathbb{C}) + i\Lambda_{\bar{\omega}}\Lambda^{(1,1)+} \right) \oplus \Lambda^{(2,0)} \oplus \Lambda^{(1,0)},$$

and

$$\mathbf{P}_{\alpha,\xi} = \sqrt{2} \begin{pmatrix} i \frac{\xi}{\sqrt{2}} & 0 & \alpha \bar{\partial}^* \\ 0 & i \frac{\xi}{\sqrt{2}} & \alpha \bar{\partial} \\ -\alpha \bar{\partial} & -\alpha \bar{\partial}^* & -i \frac{\bar{\xi}}{\sqrt{2}} \end{pmatrix} \quad (4.35)$$

acts on the sections

$$\frac{1}{\sqrt{2}} \left(C^\infty(M, \mathbb{C}) - i\Lambda_{\bar{\omega}}\Lambda^{(1,1)+} \right) \oplus \Lambda^{(0,2)} \oplus \Lambda^{(0,1)}.$$

Proof. First, it follows that

$$\left(\mathbf{D}_{\alpha,\xi} \circ \# \right)^* = \# \circ \mathbf{D}_{\alpha,\xi}^* = \overline{\mathbf{D}_{\alpha,\xi}^*} \circ \# = -\mathbf{D}_{\alpha,\xi} \circ \#,$$

as $-\overline{\mathbf{P}_{\alpha,\xi}^*} = \mathbf{Q}_{\alpha,\xi}$. On a Kähler manifold, the operator $\mathbf{D}_{\alpha,\xi}$ acts on sections that can be decomposed in the components

$$C^\infty(M; \mathbb{C}) \oplus \Lambda^{(1,1)+} \oplus \Lambda^{(2,0)} \oplus \Lambda^{(0,2)} \oplus \Lambda^{(1,0)} \oplus \Lambda^{(0,1)}$$

as the operator

$$\left(\begin{array}{cc|cc|cc} 0 & \xi * (\widehat{\omega} \wedge \bullet) & 0 & 0 & \alpha \partial^* & \alpha \bar{\partial}^* \\ -\xi \widehat{\omega} & 0 & 0 & 0 & \sqrt{2} \alpha \widehat{\omega} * (\widehat{\omega} \wedge \bar{\partial} \bullet) & \sqrt{2} \alpha \widehat{\omega} * (\widehat{\omega} \wedge \partial \bullet) \\ \hline 0 & 0 & \xi \mathcal{L}_{\widehat{\omega}} & 0 & \sqrt{2} \alpha \partial & 0 \\ 0 & 0 & 0 & \xi \mathcal{L}_{\widehat{\omega}} & 0 & \sqrt{2} \alpha \bar{\partial} \\ \hline -\alpha \partial & -\sqrt{2} \alpha \bar{\partial}^* & -\sqrt{2} \alpha \bar{\partial}^* & 0 & -\bar{\xi} * (\widehat{\omega} \wedge \bullet) & 0 \\ -\alpha \bar{\partial} & -\sqrt{2} \alpha \partial^* & 0 & -\sqrt{2} \alpha \partial^* & 0 & -\bar{\xi} * (\widehat{\omega} \wedge \bullet) \end{array} \right)$$

Using the Kähler identities

$$\sqrt{2} * (\widehat{\omega} \wedge \bar{\partial} \rho^{(1,0)}) = -i \partial^* \rho^{(1,0)}, \quad \sqrt{2} \bar{\partial}^* (\widehat{\omega} f) = i \partial f,$$

and identifying $C^\infty(M; \mathbb{C})$ with $\Lambda^{(1,1)+} \cong \widehat{\omega} C^\infty(M; \mathbb{C})$, the operator can be written in the form

$$\left(\begin{array}{cc|cc|cc} 0 & \xi & 0 & 0 & \alpha \partial^* & \alpha \bar{\partial}^* \\ -\xi & 0 & 0 & 0 & -i \alpha \partial^* & i \alpha \bar{\partial}^* \\ \hline 0 & 0 & -i \xi & 0 & \sqrt{2} \alpha \partial & 0 \\ 0 & 0 & 0 & i \xi & 0 & \sqrt{2} \alpha \bar{\partial} \\ \hline -\alpha \partial & -i \alpha \bar{\partial} & -\sqrt{2} \alpha \bar{\partial}^* & 0 & i \bar{\xi} & 0 \\ -\alpha \bar{\partial} & i \alpha \partial & 0 & -\sqrt{2} \alpha \partial^* & 0 & -i \bar{\xi} \end{array} \right).$$

The decomposition of the operator given in Eq. (4.34) follows. \square

Lemma 23. *The operator $\mathcal{D}_{\alpha, \xi}$ is of the form $z \rightarrow (A + iB)\bar{z}$ where A and B are real skew-adjoint operators. The space of complex skew-adjoint, anti-linear Fredholm operator on a complex Hilbert space is denoted by $\mathcal{F}_2 R$ (cf. [47]). The kernel of $\mathcal{D}_{\alpha, \xi}$ is a Clifford module with the \mathbb{Z}_2 -index*

$$\text{ind}_2(\mathcal{D}_{\alpha, \xi}) = \dim_{\mathbb{C}} \ker \mathbf{D}_{\alpha, \xi} \pmod{2} \in \pi_0(\mathcal{F}_2 R) \cong \mathbb{Z}_2.$$

For $b_1 = 0$, the kernel is $H^0(M; \mathbb{C}) \oplus H^{2+}(M; \mathbb{C})$ at $\xi = 0$, and no kernel otherwise. For $b_2^+ = 1$, it follows $\text{ind}_2(\mathcal{D}_{\alpha, \xi}) = [1 + b_2^+] = [0]$.

The natural function to consider is the the complex Pfaffian (cf. [47]). We construct the complex Pfaffian line bundle

$$\text{PFAFF } \mathcal{D}_{\alpha, \xi} \rightarrow \mathbb{R}^+ \times \mathbb{C}^*.$$

with induced metric and connection. We denote the canonical section by $\text{pfaff } \mathcal{D}_{\alpha, \xi}$. The Pfaffian is constructed by setting $\text{pfaff } \mathcal{D}_{\alpha, \xi} = \det \mathbf{P}_{\alpha, \xi}$. The complex determinant on the RHS is obtained from

$$\|\det \mathbf{P}_{\alpha, \xi}\|_Q^2 = \det \Delta_{\bar{\partial}, \alpha, \xi},$$

for the positive definite, selfadjoint Laplacian $\Delta_{\bar{\partial}, \alpha, \xi} = \mathbf{P}_{1, \xi/\alpha}^* \mathbf{P}_{\alpha, \xi}$.

4.8 The vanishing of the anomaly

In this section, we prove that the local and global anomalies for the u -plane integral vanish.

4.8.1 The Bismut-Freed connection

Let us recall the construction of the Bismut-Freed connection [43]: For each point (α, ξ) we have the spectrum of the associated Laplacian $\Delta_{\alpha, \xi}$. On an open set \mathcal{U} around (α, ξ) , the sums of eigenspaces of $\Delta_{\alpha, \xi}$ below a cutoff Λ , form smooth vector bundles $\mathcal{H}_{\mathcal{U}}^+$ and $\mathcal{H}_{\mathcal{U}}^-$. The operator $\mathbf{D}_{\alpha, \xi}$ maps $\mathcal{H}_{\mathcal{U}}^+$ to $\mathcal{H}_{\mathcal{U}}^-$. There is a canonical isomorphism over \mathcal{U}

$$(\Lambda^{\max} \mathcal{H}_{\mathcal{U}}^+)^{-1} \otimes \Lambda^{\max} \mathcal{H}_{\mathcal{U}}^- \cong \underline{\mathbb{C}}.$$

The Hilbert space geometry defines standard connections and metrics on $\mathcal{H}_{\mathcal{U}}^-$ and $\mathcal{H}_{\mathcal{U}}^+$. However, these do not patch together on the overlaps. Instead the L^2 -metric $\|\cdot\|$ has to be modified to obtain the Quillen metric which agrees on all overlaps. The L^2 -connection ∇ of the determinant line bundle is modified. It is changed to the connection $\nabla_Q = \nabla + \Gamma$ with

$$\nabla_Q \left(\det \mathbf{D}_{\alpha, \xi} \right) = \Gamma_Q \det \mathbf{D}_{\alpha, \xi}, \quad (4.36)$$

where Γ is the Bismut-Freed connection. The connection ∇_Q is compatible with the Quillen metric

$$d \left\| \det \mathbf{D}_{\alpha, \xi} \right\|_Q^2 = 2 \left\langle \det \mathbf{D}_{\alpha, \xi}, \nabla_Q \left(\det \mathbf{D}_{\alpha, \xi} \right) \right\rangle.$$

To make our notation more compact we introduce the operator

$$\mathcal{D}_{\alpha, \xi} = \begin{pmatrix} 0 & \mathbf{D}_{\alpha, \xi}^* \\ \mathbf{D}_{\alpha, \xi} & 0 \end{pmatrix}.$$

We define a purely imaginary one-form γ with the components

$$\begin{aligned} \gamma_\xi &= \text{LIM}_{\epsilon \rightarrow 0} \int_\epsilon^\infty dt \text{Str} \left[\mathcal{D}_{1, \xi/\alpha} \frac{\partial}{\partial \xi} (\mathcal{D}_{\alpha, \xi}) e^{-t/\alpha \mathcal{D}_{\alpha, \xi}^2} \right] \\ &= - \text{LIM}_{\epsilon \rightarrow 0} \int_{\epsilon/\alpha}^\infty dt \frac{d}{dt} \text{Str} \left[\mathcal{D}_{\alpha, \xi} \frac{\partial}{\partial \xi} (\mathcal{D}_{\alpha, \xi}) (\mathcal{D}_{\alpha, \xi}^2)^{-1} e^{-t \mathcal{D}_{\alpha, \xi}^2} \right] \\ &= \text{LIM}_{\epsilon \rightarrow 0} \text{Str} \left[\mathcal{D}_{\alpha, \xi} \frac{\partial}{\partial \xi} (\mathcal{D}_{\alpha, \xi}) (\mathcal{D}_{\alpha, \xi}^2)^{-1} e^{-\epsilon/\alpha \mathcal{D}_{\alpha, \xi}^2} \right], \end{aligned}$$

and

$$\gamma_\alpha = \text{LIM}_{\epsilon \rightarrow 0} \text{Str} \left[\mathcal{D}_{\alpha, \xi} \frac{\partial}{\partial \alpha} (\mathcal{D}_{\alpha, \xi}) (\mathcal{D}_{\alpha, \xi}^2)^{-1} e^{-\epsilon/\alpha \mathcal{D}_{\alpha, \xi}^2} \right],$$

where LIM denotes the limit for $t \rightarrow 0$ after subtracting the poles. Str is the supertrace

$$\text{Str} [\bullet] = \text{Tr} \left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \bullet \right],$$

where Tr is the combination of the L^2 -trace and the matrix trace. It follows from [43, Prop. 2.4] that the one-form of the Quillen connection in Eq. (4.36) is

$$\Gamma_Q = -\frac{1}{2} \left(\gamma + d \zeta'_{\Delta_{\alpha,\xi}}(0) \right).$$

The following proposition holds:

Lemma 24. *The purely imaginary one-form γ satisfies $\gamma_\alpha = 0$ and*

$$\gamma_\xi = \bar{\xi} \lim_{\epsilon \rightarrow 0} \text{Tr} \left[\Gamma^5 \Delta_{\alpha,\xi}^{-1} e^{-\epsilon \Delta_{\alpha,\xi}} \right] = \frac{2}{\bar{\xi}},$$

and $\gamma_{\bar{\xi}} = -\bar{\gamma}_\xi$ where

$$\Gamma^5 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Proof. Since $\Delta_{\alpha,\xi}$ is diagonal the eigenvalue $2\alpha|\lambda_n^{(0)}|^2 + |\xi|^2/\alpha$ has multiplicity four. The eigenfunctions are $(\phi'_n, 0, 0)^t$, $(0, 0, A_n^+)^t$ and $(0, f_n^{(1,1)}, 0)^t$, $(0, 0, A_n^{(1,1)-})^t$. Similarly, the eigenvalue $2\alpha|\lambda_k^{(0,2)}|^2 + |\xi|^2/\alpha$ has multiplicity four. The eigenfunctions are $(0, f_n^{[(0,2),I]}, 0)^t$, $(0, 0, A_n^{[(0,2),I]-})^t$ and $(0, f_n^{[(0,2),I]}, 0)^t$, $(0, 0, A_n^{[(0,2),I]-})^t$. The eigenfunctions containing ϕ' and f' are eigenfunctions of Γ^5 for the eigenvalue 1, the eigenfunctions containing A' are eigenfunctions of Γ^5 with the eigenvalue -1 .

There is also the eigenvalue $|\xi|^2/\alpha$ of $\Delta_{\alpha,\xi}$ with multiplicity two, and eigenfunctions $(1, 0, 0)^t$, $(0, \omega, 0)^t$ which are also eigenfunction of Γ^5 with eigenvalue 1.

Because of $\mathbf{D}_{\alpha,\xi}^* = -\overline{\mathbf{D}_{\alpha,\xi}}$ it follows that $\overline{\Delta_{\alpha,\xi}} = \mathbf{D}_{1,\xi/\alpha} \mathbf{D}_{\alpha,\xi}^*$. A short computation using Lemma 19 yields

$$\begin{aligned} \gamma_\alpha = \lim_{\epsilon \rightarrow 0} \text{Str} \left[\left\{ -\bar{\xi} \left(C_\omega^{(1)} + C_\omega^{(3)} \right) \mathbf{D}_1 + \bar{\xi} \mathbf{D}_1 C_\omega^{(2)} \right. \right. \\ \left. \left. + \xi \mathbf{D}_1 \left(C_\omega^{(1)} + C_\omega^{(3)} \right) - \xi C_\omega^{(2)} \mathbf{D}_1 \right\} (\alpha \Delta_{\alpha,\xi})^{-1} e^{-\epsilon \Delta_{\alpha,\xi}} \right] = 0. \end{aligned} \quad (4.37)$$

Similarly, one computes

$$\begin{aligned} \gamma_\xi &= -\bar{\xi} \lim_{\epsilon \rightarrow 0} \text{Str} \left[\left(C_\omega^{(1)2} + C_\omega^{(3)2} - C_\omega^{(2)2} \right) (\alpha \Delta_{\alpha,\xi})^{-1} e^{-\epsilon \Delta_{\alpha,\xi}} \right] \\ &= \frac{\bar{\xi}}{\alpha} \lim_{\epsilon \rightarrow 0} \text{Tr} \left[\Gamma^5 \Delta_{\alpha,\xi}^{-1} e^{-\epsilon \Delta_{\alpha,\xi}} \right]. \end{aligned}$$

For all eigenvalues of $\Delta_{\alpha,\xi}$ except $|\xi|^2/\alpha$ the eigenfunctions come in pairs which have

the eigenvalue ± 1 of Γ^5 so that these contributions cancel when taking the trace. Thus, the only contribution comes from the eigenvalue $|\xi|^2/\alpha$. It follows

$$\gamma_\xi = 2\bar{\xi} \operatorname{LIM}_{\epsilon \rightarrow 0} \frac{1}{|\xi|^2} e^{-\epsilon|\xi|^2/\alpha} = \frac{2}{\xi}.$$

□

4.8.2 The curvature of the determinant line bundle

It follows from [43, Prop. 2.5] that the curvature of the Quillen connection in Eq. (4.36) is

$$\Omega_{(\alpha_0, \xi_0)} = d\Gamma_Q = -\frac{1}{2} d\gamma. \quad (4.38)$$

An immediate consequence of Lemma 24 is the following Lemma:

Lemma 25. *The curvature of the Bismut-Freed connection on $\operatorname{DET} \mathbf{D}_{\alpha, \xi}$ vanishes. In physics, this is called the vanishing of the local anomaly.*

Remark 15. *The vanishing of the local anomaly is a consequence of the fact that ω is a harmonic, self-dual two-form. The existence of the form ω is the condition for the $N = 2$ global supersymmetry in the semi-classical limit explained in Sec. 4.5.*

4.8.3 The holonomy of the determinant line bundle

Lemma 26. *The operator $\mathbf{D}_{\alpha, \xi}$ is invertible for $\xi \neq 0$ and there is a trivialization of the determinant line bundle such that*

$$\frac{\det \mathbf{D}_{\alpha, \xi}}{\det' \left(\alpha \Delta^{(\text{od})} \right)} = \frac{\xi^2}{\alpha} \alpha^{-\frac{|\xi|^2}{\alpha} n_1(|\xi|, \alpha)} e^{-\frac{|\xi|^4}{\alpha^2} n_2} \quad (4.39)$$

with $n_1(|\xi|, \alpha) = (s \zeta(s+1)) \Big|_{s=0} - \frac{|\xi|^2}{2\alpha} (s \zeta(s+2)) \Big|_{s=0}$ and $n_2 = (s \zeta(s+2)) \Big|_{s=0}$.

Proof. The ζ -function for $\Delta_{\alpha, \xi}$ is

$$\zeta_{\Delta_{\alpha, \xi}}(s) = \frac{2\alpha^s}{|\xi|^{2s}} + 4 \sum_{n=1}^{\infty} \frac{1}{\left(2\alpha \left| \lambda_n^{(0)} \right|^2 + |\xi|^2/\alpha \right)^s} + 4 \sum_{k=1}^{\infty} \frac{1}{\left(2\alpha \left| \lambda_k^{(0,2)} \right|^2 + |\xi|^2/\alpha \right)^s}.$$

The expansion of the ζ -function yields

$$\begin{aligned}
\zeta_{\Delta_{\alpha,\xi}}(s) &= \frac{2\alpha^s}{|\xi|^{2s}} + \frac{4}{(2\alpha)^s} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(0)}|^{2s}} \frac{1}{\left[1 + \left(|\xi|/(\sqrt{2\alpha} |\lambda_n^{(0)}|)\right)^2\right]^s} \\
&+ \frac{4}{(2\alpha)^s} \sum_{k=1}^{\infty} \frac{1}{|\lambda_k^{(0,2)}|^{2s}} \frac{1}{\left[1 + \left(|\xi|/(\sqrt{2\alpha} |\lambda_k^{(0,2)}|)\right)^2\right]^s} \\
&= \frac{2\alpha^s}{|\xi|^{2s}} + \frac{2}{\alpha^s} \zeta(s) - \frac{2|\xi|^2}{\alpha^{s+1}} s \zeta(s+1) + \frac{|\xi|^4}{\alpha^{s+2}} (s^2 + s) \zeta(s+2) \\
&+ \sum_{m \geq 3}^{\infty} \frac{2|\xi|^{2m}}{\alpha^{s+m}} q_m(s) s \zeta(s+m)
\end{aligned}$$

where q_m are polynomials of degree $m - 1$ in s . Since the ζ -function $\zeta(s)$ has simple poles only at $s = 1, 2$ [49], it follows

$$\zeta_{\Delta_{\alpha,\xi}}(0) = 2 + 2\zeta(0) - \frac{2|\xi|^2}{\alpha} (s \zeta(s+1)) \Big|_{s=0} + \frac{|\xi|^4}{\alpha^2} (s \zeta(s+2)) \Big|_{s=0}, \quad (4.40)$$

and

$$\begin{aligned}
\ln \det \Delta_{\alpha,\xi} &= -\zeta'_{\Delta_{\alpha,\xi}}(0) = -2\zeta'(0) + 2 \ln(|\xi|^2) \\
&+ \ln(\alpha) \left(-2 + 2\zeta(0) - \frac{2|\xi|^2}{\alpha} (s \zeta(s+1)) \Big|_{s=0} + \frac{|\xi|^4}{\alpha^2} (s \zeta(s+2)) \Big|_{s=0} \right) \\
&- \frac{2|\xi|^4}{\alpha^2} (s \zeta(s+2)) \Big|_{s=0}.
\end{aligned} \quad (4.41)$$

□

Lemma 27. *There is a trivialization of the Pfaffian line bundle such that*

$$\frac{\text{pfaff } \mathbf{D}_{\alpha,\xi}}{\sqrt{\det'(\alpha \Delta^{(\text{od})})}} = \frac{\xi}{\sqrt{\alpha}} \alpha^{-\frac{|\xi|^2}{2\alpha} n_1(|\xi|,\alpha)} e^{-\frac{|\xi|^4}{2\alpha^2} n_2} \quad (4.42)$$

with $n_1(|\xi|, \alpha) = (s \zeta(s+1)) \Big|_{s=0} - \frac{|\xi|^2}{2\alpha} (s \zeta(s+2)) \Big|_{s=0}$ and $n_2 = (s \zeta(s+2)) \Big|_{s=0}$.

Proof. The positive definite, selfadjoint elliptic operator of Laplace type is

$$\Delta_{\bar{\partial},\alpha,\xi} = - \left[\mathbf{P}_{1,\bar{\xi}/\alpha} \mathbf{P}_{\alpha,\xi} \right] = 2 \begin{pmatrix} \alpha \Delta_{\bar{\partial}}^{(0,0)} + \frac{|\xi|^2}{2\alpha} & 0 & 0 \\ 0 & \alpha \Delta_{\bar{\partial}}^{(0,2)} + \frac{|\xi|^2}{2\alpha} & 0 \\ 0 & 0 & \alpha \Delta_{\bar{\partial}}^{(0,1)} + \frac{|\xi|^2}{2\alpha} \end{pmatrix}.$$

Thus, it follows

$$\zeta'_{\Delta_{\bar{\delta},\alpha,\xi}}(0) = \frac{1}{2}\zeta'_{\Delta_{\alpha,\xi}}(0).$$

□

Remark 16. Since ξ has an integer exponent there is no holonomy when encircling $\xi = 0$ by $\xi \mapsto \lim_{t \rightarrow 1} \xi e^{2\pi it}$. In physics, this is called the vanishing of the global anomaly.

Corollary 4. For $b_2^+ = 1$, the limit of the ratio of determinants of under the rescaling $\xi \rightarrow t^{-2}\xi$ and $t \rightarrow \infty$ is

$$\lim_{t \rightarrow \infty} \frac{t^{2b_2^+} \text{pfaff } \mathbf{D}_{\alpha, t^{-2}\xi}}{\sqrt{\det'(\alpha \Delta^{(1)+})}} = \frac{\xi}{\sqrt{\alpha}}. \quad (4.43)$$

Proof. It follows from Lemma 27 that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{t^{2b_2^+} \text{pfaff } \mathbf{D}_{\alpha, t^{-2}\xi}}{\sqrt{\det'(\alpha \Delta^{(1)+})}} &= \frac{\xi}{\sqrt{\alpha}} \lim_{t \rightarrow \infty} t^{b_2^+ - 1} \alpha^{-\frac{|\xi|^2}{2\alpha t^4} n_1(t^{-2}|\xi|, \alpha)} e^{-\frac{|\xi|^4}{2\alpha^2 t^4} n_2} \\ &= \frac{\xi}{\sqrt{\alpha}}. \end{aligned}$$

□

4.9 The coupling to gravity

In this section, we determine the gravitational corrections in the definition of the u -plane integral for $M = CP^2$. We discuss the relevant terms in the action, show that they can be obtained from determinant line bundles, and compute the trivializing sections.

4.9.1 The transversal part of the Yang-Mills Lagrangian

First, we consider the transversal part of the gauge-fixed action. The discussion will hold for the high-energy theory as well as for the low-energy effective field theory.

The quadratic approximation contains only terms with purely bosonic or fermionic variations. Since we want to determine the gravitational part of the action only, the coupling to the line bundle is irrelevant [16]. This means we will replace all sections of the line bundle $L^{\otimes 2} \rightarrow M$ in Sec. 4.1 by sections of the trivial \mathbb{C} -bundle over M . For a section $\xi = \xi^k (i\sigma^k)$ with real coefficients ξ^k , it follows that $\xi^+ = \bar{\xi}^-$ is a section of a trivial complex line bundle. The hermitian metric on the line bundle is

$$\langle \xi_1, \xi_2 \rangle = -\text{tr}(\xi_1^- \xi_2^+ + \xi_1^+ \xi_2^-) = \text{Re}(\bar{\xi}_1^+ \xi_2^+).$$

The transversal contributions from the quadratic approximation in the ghost fields C^\pm, \bar{C}^\pm will cancel the transversal contributions from the scalar fields $\Phi^\pm, \bar{\Phi}^\pm$. The remaining terms stem from the transversal components in the gauge-fixed action:

1) For the high-energy theory, an expansion $\Phi = \mathbf{a}/2i\sigma^3 + O(\epsilon)$ and $A = \epsilon B$ using Eqs. (4.1) gives the mass term

$$-\text{tr} \left(\Phi \Delta_A^{(0)} \Phi \right) = -\frac{1}{2} |\mathbf{a}|^2 \text{tr}(B^+ B^- + B^- B^+) + O(\epsilon).$$

We obtain for the gravitational contribution of the transversal parts of the action in Eq. (2.12) the terms

$$\begin{aligned} \mathbf{S}^t = \int_M & \left(\frac{1}{2} \langle B^-, (\Delta^{(1)} + |\mathbf{a}|^2) B^+ \rangle \right. \\ & + \sqrt{2} \langle \mathcal{X}^-, \pi_+ d\Psi^+ \rangle + \langle H^-, d^* \Psi^+ \rangle \\ & \left. + \langle \mathcal{X}^-, \mathbf{a} \mathcal{X}^+ \rangle + \langle \Psi^-, \bar{\mathbf{a}} \Psi^+ \rangle + \langle H^-, \mathbf{a} H^+ \rangle \right) \text{vol}_M. \end{aligned}$$

2) The generalization to the low-energy effective action is obtained by replacing the interaction terms in ϕ by the same interaction terms but in $u'(\phi)$ where u is a holomorphic function. The action becomes

$$\begin{aligned} \mathbf{S}'^t = \int_M \text{vol}_M & \left(\langle B^-, (\Delta^{(1)} + |u'|^2) B^+ \rangle + \sqrt{2} \langle \mathcal{X}^-, d\Psi^+ \rangle + \langle H^-, d^* \Psi^+ \rangle \right. \\ & \left. + \langle \mathcal{X}^-, u' \mathcal{X}^+ \rangle + \langle \Psi^-, \bar{u}' \Psi^+ \rangle + \langle H^-, u' H^+ \rangle \right). \end{aligned}$$

Two families of operators

$$\mathbf{D}_\mathbf{a}^\pm : C^\infty(M; \mathbb{C}) \oplus \Lambda_{\mathbb{C}}^{2+}(M) \oplus \Lambda_{\mathbb{C}}^1(M) \rightarrow C^\infty(M; \mathbb{C}) \oplus \Lambda_{\mathbb{C}}^{2+}(M) \oplus \Lambda_{\mathbb{C}}^1(M)$$

for $\mathbf{a} \in \mathbb{C}^*$ are defined by are

$$\mathbf{D}_\mathbf{a}^\pm \begin{pmatrix} H^+ \\ \mathcal{X}^+ \\ \Psi^+ \end{pmatrix} = [\mathbf{D}_1 \pm B(\mathbf{a})] \begin{pmatrix} H^+ \\ \mathcal{X}^+ \\ \Psi^+ \end{pmatrix},$$

with

$$B(\mathbf{a}) = \begin{pmatrix} \mathbf{a} & 0 & 0 \\ 0 & \mathbf{a} & 0 \\ 0 & 0 & \bar{\mathbf{a}} \end{pmatrix}.$$

To treat the low-energy effective field theory we only have to replace \mathbf{a} by $u'(\mathbf{a})$. The operator

$$\mathcal{D}_\mathbf{a} = \begin{pmatrix} 0 & -\mathbf{D}_\mathbf{a}^- \\ \mathbf{D}_\mathbf{a}^+ & 0 \end{pmatrix} \circ \#$$

is self-adjoint and complex anti-linear since

$$(\mathbf{D}_a^+ \circ \#)^* = \# \circ \mathbf{D}_a^{+*} = \# \circ (-\mathbf{D}_1 + B(\bar{\mathbf{a}})) = -\# \circ (\mathbf{D}_1 - B(\bar{\mathbf{a}})) = -\mathbf{D}_a^- \circ \# .$$

For $\mathbf{a} \neq 0$ the operator has neither kernel nor cokernel. At $\mathbf{a} = 0$ the dimension of the kernel and cokernel jumps by $1 + b_2^+ = (\chi + \sigma)/2$. The canonical orientation of these complex vector spaces gives rise to a canonical trivialization of the determinant line bundle.

Lemma 28. *The following relation holds*

$$B(\mathbf{a})\mathbf{D}_1 = \mathbf{D}_1 B(\bar{\mathbf{a}}) .$$

Lemma 29. *The positive definite, selfadjoint elliptic operator of Laplace type satisfies*

$$\mathcal{D}_a^2 = \begin{pmatrix} -\mathbf{D}_a^- \overline{\mathbf{D}_a^+} & 0 \\ 0 & -\mathbf{D}_a^+ \overline{\mathbf{D}_a^-} \end{pmatrix} = \begin{pmatrix} \Delta_a & 0 \\ 0 & \Delta_a \end{pmatrix}$$

with

$$\Delta_a = \begin{pmatrix} \Delta^{(0)+} + |\mathbf{a}|^2 & 0 & 0 \\ 0 & \Delta^{(2)+} + |\mathbf{a}|^2 & 0 \\ 0 & 0 & \Delta^{(1)+} + |\mathbf{a}|^2 \end{pmatrix} .$$

For the family of operators \mathcal{D}_a , the determinant line bundle

$$\text{DET } \mathbf{D}_a^+ \rightarrow \mathbb{C}^* , \tag{4.44}$$

has the canonical section $\det \mathbf{D}_a^+ = \text{pfaff } \mathcal{D}_a$ which is the Pfaffian of \mathcal{D}_a . The Quillen metric is

$$\|\det \mathbf{D}_a^+\|_Q^2 = \det \Delta_a .$$

The L^2 -connection ∇ of the determinant line bundle is modified. The connection $\nabla_Q = \nabla + \Gamma$ satisfies

$$\nabla_Q \left(\det \mathbf{D}_a^+ \right) = \Gamma_Q \det \mathbf{D}_a^+ ,$$

where Γ is the Bismut-Freed connection compatible with the Quillen metric. From [43, Prop. 2.4] it follows that the one-form of the Quillen connection on (4.44) is

$$\Gamma_Q = -\frac{1}{2} (\gamma + d\zeta'_{\Delta_a}(0)) .$$

From Lemma 28 it follows

Lemma 30. *The Bismut-Freed connection on the determinant line bundle (4.44) vanishes $\gamma = 0$. The curvature of the determinant line bundle vanishes.*

Lemma 31. For $\mathbf{a} \in \mathbb{C}^*$, we have the trivial flat determinant line bundle

$$\text{DET} \left(\Delta^{(1)+} + |\mathbf{a}|^2 \right) \rightarrow \mathbb{C},$$

and the flat determinant line bundle of the family of Dirac operators $\mathcal{D}_{\mathbf{a}}^+$

$$\text{DET } \mathbf{D}_{\mathbf{a}}^+ \rightarrow \mathbb{C}.$$

For $\chi + \sigma$ even, the product bundle

$$\text{DET } \mathbf{D}_{\mathbf{a}}^+ \otimes \left[\text{DET} \left(\Delta^{(1)} + |\mathbf{a}|^2 \mathbb{I} \right) \right]^{-2} \rightarrow \mathbb{C}$$

has a holomorphic trivialization such that

$$\frac{\det \mathbf{D}_{\mathbf{a}}^+}{\det \left(\Delta^{(1)+} + |\mathbf{a}|^2 \right)} = \mathbf{a}^{\frac{\chi + \sigma}{2}}. \quad (4.45)$$

Proof. There are also the eigenvalue $|\mathbf{a}|^2$ of multiplicity $1 + b_2^+$, and $|\mathbf{a}|^2$ of multiplicity b_1 . All eigenvalues are strictly greater than zero. We define the following ζ -function for $R \in \mathbb{R}$

$$\zeta(s, R^2) = \sum_{n=0}^{\infty} \frac{1}{(2\lambda_n^2 + R^2)^s}.$$

It follows from

$$\zeta(s, R^2) = \zeta(s) - R^2 s \zeta(s+1) + \frac{R^4}{2} (s^2 + s) \zeta(s+2) + O(R^3)$$

that

$$\begin{aligned} \zeta(0, R) &= \zeta(0) - R^2 \left(s \zeta(s+1) \right) \Big|_{s=0} + \frac{R^4}{2} \left(s \zeta(s+2) \right) \Big|_{s=0}, \\ \zeta'(0, R) &= \zeta'(0) + R^4 \left(s \zeta(s+2) \right) \Big|_{s=0}, \end{aligned}$$

as the ζ -function only has poles at $s = 1, 2$. The determinant of the generalized Laplace operator $\Delta_{\mathbf{a}}$ is the regularized product of its eigenvalues

$$\ln \det \Delta_{\mathbf{a}} = -\zeta'_{\Delta_{\mathbf{a}}}(0),$$

with

$$\zeta_{\Delta_{\mathbf{a}}}(s) = 2 \zeta(s, |\mathbf{a}|^2) + (1 + b_2^+) \frac{1}{|\mathbf{a}|^{2s}} + b_1 \frac{1}{|\mathbf{a}|^{2s}}.$$

We obtain

$$\exp -\zeta'_{\Delta_{\mathbf{a}}}(0) = |\mathbf{a}|^{2(1+b_2^+)} |\mathbf{a}|^{2b_1} \exp \{-4\zeta'(0, |\mathbf{a}|^2)\} .$$

Similarly, we obtain for the determinant from the bosonic fields

$$\det \left(\Delta^{(1)+} + |\mathbf{a}|^2 \mathbb{I} \right) = |\mathbf{a}|^{2b_1} \exp \{-2\zeta'(0, |\mathbf{a}|^2)\} .$$

It follows that

$$\frac{\det |\mathbf{D}_{\mathbf{a}}^+|}{\det \left(\Delta^{(1)+} + |\mathbf{a}|^2 \right)} = |\mathbf{a}|^{\frac{x+\sigma}{2}} .$$

By Lemma 30 the curvature of the determinant line bundle vanishes. \square

Remark 17. *The curvature of the determinant line bundle vanishes. In physics, this is called the vanishing of the local anomaly. Since $\frac{x+\sigma}{2}$ is an integer there is no holonomy $\mathbf{a} \mapsto \lim_{t \rightarrow 1} \mathbf{a} e^{2\pi i t}$. In physics, this is called the vanishing of the global anomaly. The contribution to path integral from the transversal part of the high-energy theory is*

$$\frac{\det \mathbf{D}_{\mathbf{a}}^+}{\det \left(\Delta^{(1)+} + |\mathbf{a}|^2 \right)} = \int [\mathcal{D}B^+ \mathcal{D}H^+ \mathcal{D}\Psi^+ \mathcal{D}\mathcal{X}^+] e^{-S^t} = \mathbf{a}^{\frac{x+\sigma}{2}} .$$

The contribution to the path integral from the transversal part of the low-energy effective field theory is

$$\frac{\det \mathbf{D}_{u'(\mathbf{a})}^+}{\det \left(\Delta^{(1)+} + |u'(\mathbf{a})|^2 \right)} = \int [\mathcal{D}B^+ \mathcal{D}H^+ \mathcal{D}\Psi^+ \mathcal{D}\mathcal{X}^+] e^{-S'^t} = \left(\frac{d\mathbf{a}}{du} \right)^{-\frac{x+\sigma}{2}} . \quad (4.46)$$

4.9.2 The Seiberg-Witten contribution

There is another gravitational renormalization term coming from the Seiberg-Witten monopoles. In this section, we determine the Seiberg-Witten contribution for CP^2 .

On a Kähler manifold, there is a canonical Spin^c -connection whose square is the anti-canonical bundle. In [10, Sec. 3.7], the Dirac operator was shown to be

$$\not{D} = \sqrt{2} \begin{pmatrix} 0 & 0 & \bar{\partial}^* \\ 0 & 0 & \bar{\partial} \\ -\bar{\partial} & -\bar{\partial}^* & 0 \end{pmatrix} , \quad (4.47)$$

and act on the sections

$$S^+(M) \oplus S^-(M) = (C^\infty(M, \mathbb{C}) \oplus \Lambda^{(0,2)}) \oplus \Lambda^{(0,1)} .$$

If we include a complex mass-term, we obtain the operator

$$\not{\phi}_{\mathbf{m}} = \sqrt{2} \begin{pmatrix} i\mathbf{m} & 0 & \bar{\partial}^* \\ 0 & i\mathbf{m} & \bar{\partial} \\ -\bar{\partial} & -\bar{\partial}^* & -i\bar{\mathbf{m}} \end{pmatrix}. \quad (4.48)$$

We will determine in Sec. 4.11 that

$$\mathbf{m} = (\omega^4 \Delta(\tau))^{\frac{1}{8}} = \sqrt{\pi} \vartheta_4(\tau).$$

From Eq. (3.18) in Sec. 3.9.2, it follows that \mathbf{m} has an asymptotic behavior

$$\begin{aligned} \mathbf{m} &\sim \left(\frac{\pi^4}{u^2} \frac{u^2}{4096} \right)^{\frac{1}{8}} = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad (\text{for } u \rightarrow \infty), \\ \mathbf{m} &\sim \left((\sqrt{2}\pi)^4 \frac{2(1 \mp u)}{4096} \right)^{\frac{1}{8}} \quad (\text{for } u \rightarrow \pm 1). \end{aligned} \quad (4.49)$$

The Seiberg-Witten monopoles become massless at $u = \pm 1$. The proofs of Lemma 22 and Lemma 26 also prove the following Lemma:

Lemma 32. *The curvature of the determinant line bundle $\text{DET } \not{\phi}_{\mathbf{m}}$ vanishes. There is no global holonomy for $\mathbf{m} \mapsto \lim_{t \rightarrow 1} \mathbf{m} e^{2\pi i t}$. The contribution to path integral is*

$$\frac{\det \not{\phi}_{\mathbf{m}}}{\det (\Delta^{(1)+} + |\mathbf{m}|^2)} = \frac{\mathbf{m}^{1+b(0,2)} \bar{\mathbf{m}}^{b(0,1)}}{|\mathbf{m}|^{2b(0,1)}} = \mathbf{m}^{1-b(0,1)+b(0,2)} = \mathbf{m}^\sigma. \quad (4.50)$$

4.10 The partition function

In this section, we determine the topological part of the path integral for the twisted $N = 2$ supersymmetric, pure $U(1)$ -gauge theory on CP^2 in the semi-classical approximation.

Remark 18. *The path integral including the source terms for the twisted $N = 2$ supersymmetric, pure $U(1)$ -gauge theory on $M = CP^2$ in the semi-classical approximation is*

$$\begin{aligned} \tilde{\mathbf{Z}}_u(p, S) &= \sum_{L \in 2H^2(M; \mathbb{Z})} \int_{\mathbb{C}} da d\bar{a} \int [\mathcal{D}\phi' \mathcal{D}\bar{\phi}' \mathcal{D}A' \mathcal{D}D \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}H \mathcal{D}\psi \mathcal{D}\chi] \\ &\times \exp \left(-\mathbf{S} + 2pu + S^2 T(u) \right), \end{aligned} \quad (4.51)$$

where \mathbf{S} is the action determined in Sec. 4.2, Cor. 2, and $2pu + S^2 T(u)$ are the observables obtained from the descent procedure in Sec. 4.3.

The path integral (4.51) has the form which was described in Eq. (1.2) in Sec. 1.3: the evaluation of the path integral in the semi-classical approximation consists of the

integral over the continuous moduli \mathbf{a} , $\bar{\mathbf{a}}$, and the sum over the discrete moduli, the lattice points of $2H^2(M; \mathbb{Z})$. For the further evaluation, one first has to integrate out the field D : the selfdual two-form D appears in the action quadratically without any derivatives. Following the formalism of the path integrals, one can integrate D out by a Gaussian integration. The effect is to replace $D = 2i S u'(\mathbf{a}) \omega / \text{Im } \tau$. Secondly, the integration $[\mathcal{D}\phi' \mathcal{D}\bar{\phi}']$ over the variations of the complex scalar field orthogonal to the zero mode gives the functional determinant

$$\left[\det' \left(\frac{\text{Im } \tau}{8\pi} \Delta^{(0)} \right) \right]^{-1}.$$

It cancels the same term in the numerator from the integration over the ghost-fields $[\mathcal{D}c \mathcal{D}\bar{c}]$ which by definition do not contain any zero modes. It follows that

$$\begin{aligned} \tilde{\mathbf{Z}}_u(p, S) &= \sum_{L \in 2H^2(M; \mathbb{Z})} \int_{\mathbb{C}} d\mathbf{a} d\bar{\mathbf{a}} \int [\mathcal{D}A' \mathcal{D}H \mathcal{D}\psi \mathcal{D}\chi] \\ &\times \exp \left(-\mathbf{S}^{(0)} - \mathbf{S}^{(2)} + 2p u + S^2 \hat{T}(u) \right), \end{aligned} \quad (4.52)$$

where the action at a critical point is given by

$$\mathbf{S}^{(0)} = \frac{i \bar{\tau}}{16\pi} \|F_A^+\|^2 + \frac{i}{4} (F_A^+, w_2(TM)). \quad (4.53)$$

The quadratic approximation $\mathbf{S}^{(2)}$ of the action is given by a skew-adjoint first-order operator of Dirac type in the fermionic fields, and a second-order, positive definite, self-adjoint operator in the bosonic fields. The free field theory defined by $\mathbf{S}^{(2)}$ consists of three contributions: the longitudinal part and the transversal part of the gauge-fixed action under the dimensional reduction of the gauge group. These two contribution were discussed in Sec. 4.8 and Sec. 4.9.1. The third contribution comes from the Seiberg-Witten monopoles which were discussed in Sec. 4.9.2. Each factor in this product has been shown to have a vanishing local and global anomaly (cf. Lemma 27, Lemma 31, and Lemma 32). Thus, we have proved:

Lemma 33. *The path integral in Eq. (4.51) is*

$$\begin{aligned} \tilde{\mathbf{Z}}_u(p, S) &= \underbrace{\sum_{L \in 2H^2(M; \mathbb{Z})} \int_{\mathbb{C}} d\mathbf{a} d\bar{\mathbf{a}}}_{\substack{\text{sum over Pontryagin sectors,} \\ \text{integration over zero-modes}}} \underbrace{e^{p u(\mathbf{a}) + S^2 \hat{T}(u)}}_{\text{source terms}} \underbrace{e^{-\mathbf{S}^{(0)}}}_{\text{action at stationary point}} \\ &\times \frac{\det \not{D}_{\mathbf{m}}}{\det [\Delta^{(1)+} + |\mathbf{m}(\tau)|^2]} \frac{\det D_{u'(\mathbf{a})}^+}{\det [\Delta^{(1)+} + |u'(\mathbf{a})|^2]} \frac{\text{pfaff } \not{D}_{\text{Im } \tau, \xi}}{\sqrt{\det [\text{Im } \tau \Delta^{(1)+}]}} \end{aligned} \quad (4.54)$$

one-loop approximation,
Seiberg-Witten contribution
one-loop approximation,
transversal contribution
one-loop approximation,
longitudinal contribution

where

$$\xi = -\frac{1}{2} \frac{d\bar{\tau}}{d\bar{\mathbf{a}}} \langle D + F_A^+, \widehat{\omega} \rangle = -\frac{1}{2} \frac{d\bar{\tau}}{d\bar{\mathbf{a}}} \left(\langle F_A^+, \widehat{\omega} \rangle + \frac{2\sqrt{2}iS}{\pi \operatorname{Im} \tau} u'(\mathbf{a}) \right). \quad (4.55)$$

The linear theory of the quadratic approximation is not topological [25]. However, the metric independent part of the partition function can be extracted by looking at the behavior in a one-parameter family of metrics $g_t = t^2 g_0$, for fixed g_0 with $t \rightarrow \infty$. We observed in Sec. 4.5 that the different interaction terms in the linear theory defined by $\mathbf{D}_{\operatorname{Im} \tau, \xi}$ are not invariant under rescaling. The interaction term $\xi \widehat{\omega}$ is invariant under rescaling from its definition as a topological quantity. On the other hand, $\widehat{\omega}$ scales in the same way as the metric does, $\widehat{\omega} \rightarrow t^2 \widehat{\omega}$ from its definition $*(\widehat{\omega} \wedge \widehat{\omega}) = 1$ which involves the metric. Accordingly, ξ scales as $\xi \rightarrow t^{-2} \xi$. Moreover, the zero-modes χ_0 are invariant under rescaling, but the fluctuations χ' are scaled by $\chi' \rightarrow t^{-2} \chi'$. This introduces an additional factor of $t^{2b_2^+}$. The action $\mathbf{S}^{(0)}$ in Eq. (4.53) at the stationary point is invariant under rescaling. We have proved the following proposition:

Proposition 15. *The topological invariant part of the path integral in Eq. (4.51) is*

$$\begin{aligned} \mathbf{Z}_u(p, S) &= \lim_{t \rightarrow \infty} \widetilde{\mathbf{Z}}_u(p, S) \\ &= \lim_{t \rightarrow \infty} \sum_{L \in 2H^2(M; \mathbb{Z})} \int_{\mathbb{C}} d\mathbf{a} d\bar{\mathbf{a}} e^{p u(\mathbf{a}) + S^2 \widehat{T}(u)} e^{-\mathbf{S}^{(0)}} \\ &\times \frac{\det \not{D}_{\mathbf{m}}}{\det [\Delta^{(1)+} + |\mathbf{m}(\tau)|^2]} \frac{\det \mathbf{D}_{u'(\mathbf{a})}^+}{\det [\Delta^{(1)+} + |u'(\mathbf{a})|^2]} \frac{t^{2b_2^+} \operatorname{pfaff} \not{D}_{\operatorname{Im} \tau, t^{-2} \xi}}{\sqrt{\det [\operatorname{Im} \tau \Delta^{(1)+}]} }. \end{aligned} \quad (4.56)$$

4.11 The evaluation of the path integral

In this section, we evaluate the topological part of the path integral for the twisted supersymmetric, pure $U(1)$ -gauge theory on $M = CP^2$ in the semi-classical approximation in terms of the Jacobi ϑ -functions and related functions.

Using Eqs.(4.42), (4.46), and (4.50) and Cor. 4, one evaluates Eq. (4.56) and obtains

$$\mathbf{Z}_u(p, S) = \sum_{L \in 2H^2(M; \mathbb{Z})} \int_{\mathbb{C}} d\mathbf{a} d\bar{\mathbf{a}} e^{2p u + S^2 \widehat{T}(u)} e^{-\mathbf{S}^{(0)}} \mathbf{m}^\sigma \left(\frac{du}{d\mathbf{a}} \right)^{\frac{\chi+\sigma}{2}} \frac{\xi}{\sqrt{\operatorname{Im} \tau}}. \quad (4.57)$$

Since $H^{2+}(M; \mathbb{Z}) = \mathbb{Z}$ it follows $F_A^+ = 4\pi k \omega$ for $k \in \mathbb{Z}$. It follows

$$\mathbf{S}^{(0)} = i\pi \bar{\tau} k^2 + i \frac{\pi k}{2}.$$

ξ in Eq. (4.55) depends on k since

$$\xi = -\frac{1}{2} \frac{d\bar{\tau}}{d\bar{\mathbf{a}}} \langle D + F_A^+, \hat{\omega} \rangle = -\frac{1}{2} \frac{d\bar{\tau}}{d\bar{\mathbf{a}}} \left(4\sqrt{2}k + \frac{2\sqrt{2}iS}{\pi \operatorname{Im} \tau} u'(\mathbf{a}) \right).$$

Because of

$$\sum_{k \in \mathbb{Z}} k e^{-\mathbf{S}^{(0)}} = \sum_{k \in \mathbb{Z}} k e^{-i\pi\bar{\tau}k^2 - i\pi k w_2} = 0, \quad (4.58)$$

it follows

$$\sum_{k \in \mathbb{Z}} \frac{\xi}{\sqrt{\operatorname{Im} \tau}} e^{-\mathbf{S}^{(0)}} = -\frac{\sqrt{2}i}{\pi} S \frac{d\bar{\tau}}{d\bar{\mathbf{a}}} \frac{u'(\mathbf{a})}{\operatorname{Im} \tau} \frac{1}{\sqrt{\operatorname{Im} \tau}} \overline{\vartheta(w_2|\tau)} = \xi_0 \frac{1}{\sqrt{\operatorname{Im} \tau}} \overline{\vartheta(w_2|\tau)}.$$

Eq. (4.56) simplifies to

$$\mathbf{Z}_u(p, S) = \int_{\mathcal{C}} [d\mathbf{a} d\bar{\mathbf{a}}] e^{2p u + S^2 \hat{T}(u)} \frac{1}{\sqrt{\operatorname{Im} \tau}} \mathbf{m} \overline{\vartheta_4(\tau)} \left(\frac{du}{d\mathbf{a}} \right)^2 \xi_0. \quad (4.59)$$

The defining property of the family is that the group $SL(2; \mathbb{Z})$ acts on it by electro-magnetic duality. The electro-magnetic duality is realized as follows: the coupling constant $\operatorname{Im} \tau$ of each $U(1)$ -theory in the family is viewed as the modular parameter $\tau = \tau(E_u)$ of the elliptic fiber of the rational surface Z . The action of the modular group on τ is the electro-magnetic duality transformation that relates different $U(1)$ -theories in this family. The transformation on the coupling constant is accompanied by a duality transformation on the fields of the $U(1)$ -theory to specify the isomorphism between the full physical theories. The expectation value of the scalar field transforms in the monodromy representation of $\Gamma_0(4)$ and thus is a section of the Jacobian elliptic fibration over the u -plane. Eq. (4.59) simplifies to

$$\begin{aligned} \mathbf{Z}_u(p, S) &= \int_{\Gamma_0(4) \backslash H} \frac{|d\tau|^2}{\operatorname{Im} \tau^2} e^{2p u + S^2 \hat{T}(u)} \sqrt{\operatorname{Im} \tau} \mathbf{m} \overline{\vartheta_4(\tau)} \operatorname{Im} \tau \left| \frac{d\mathbf{a}}{d\tau} \right|^2 \left(\frac{du}{d\mathbf{a}} \right)^2 \xi_0 \\ &= -\sqrt{2}S \int_{\Gamma_0(4) \backslash H} \frac{|d\tau|^2}{\operatorname{Im} \tau^2} e^{2p u + S^2 \hat{T}(u)} \sqrt{\operatorname{Im} \tau} \mathbf{m} \overline{\vartheta_4(\tau)} \left(\frac{i}{\pi} \frac{du}{d\tau} \right) \left(\frac{du}{d\mathbf{a}} \right)^2. \end{aligned} \quad (4.60)$$

Remark 19.

1. To ensure the invariance of the integrand under the $\Gamma_0(4)$ -transformation in Eq. (3.23) the source terms have to be invariant. In particular, it follows that $u = u(\tau)$ is invariant under $\Gamma_0(4)$.
2. The contribution from the $U(1)$ -photon partition function to the path integral in

Eq. (4.59) is

$$\frac{1}{\sqrt{\text{Im } \tau}} \overline{\vartheta_4(\tau)}.$$

It is not invariant under the $\Gamma_0(4)$ -transformation in Eq. (3.23). In physics, this is called the Konishi anomaly. Together with the contribution from the Seiberg Witten monopoles, the $U(1)$ -photon partition function combines to

$$\sqrt{\text{Im } \tau} \mathbf{m} \overline{\vartheta_4(\tau)},$$

which is invariant under $\Gamma_0(4)$. We set $\mathbf{m} = \vartheta(w_2|\tau)$. It is the section of the dual of the bundle $\mathcal{O}(\Theta) \rightarrow H$ with Θ -divisor $\Theta = w_2$ and with the metric invariant under $\Gamma_0(4)$ is given by

$$\|1_{\mathcal{O}(\Theta)}\|^2 = \sqrt{\text{Im } \tau} |\vartheta(w_2|\tau)|^2.$$

3. For the term

$$\left(\frac{i}{\pi} \frac{du}{d\tau} \right) \left(\frac{du}{d\mathbf{a}} \right)^2,$$

Eq. (3.18) proves the invariance under the action of $\Gamma_0(4)$.

We have proved:

Proposition 16. Eq. (4.51) simplifies to

$$\mathbf{Z}_u(p, S) = C S \int_{\Gamma_0(4) \backslash H} \frac{|d\tau|^2}{\text{Im } \tau^2} e^{2p u + S^2 \hat{T}(u)} \sqrt{\text{Im } \tau} |\vartheta_4(\tau)|^2 \Delta,$$

for a constant C . The integrand in Eq. (4.61) is invariant under $\Gamma_0(4)$.

Remark 20. The u -plane integral Eq. (4.61) has to be regularized. We will explain its regularization in Sec. 5.6.

Remark 21. The physical theory as defined in Sec. 4.2 depends on the special Kähler structure of the rational elliptic surface Z . The special Kähler geometry on UP does not define a classical field theory for the $N = 2$ vector multiplet; instead it describes a family of $U(1)$ -theories. For every special coordinate patch, one obtains a classical Lagrangian and a classical field theory. The change of special coordinates must be accompanied by a duality transformation on the gauge field in the vector multiplet. This duality transformation only make sense as the holonomy of the special Kähler connection in Sec. 3.10 is contained in the integral symplectic group.

4.11.1 The strong coupling limit

The strong coupling limit is the limit of the family of supersymmetric $U(1)$ -theories for $u \rightarrow \pm 1$. Using a duality transformation $S \in SL(2; \mathbb{Z})$ one can relate this limit to the weak coupling limit $u \rightarrow \infty$. Let the line bundle $L_\omega \rightarrow M$ be the complex line bundle with the first Chern class $c_1(L_\omega) = \omega$. Let A_ω be a connection on L with $F_{A_\omega}/2\pi = \omega$. The instanton sum for $w_2 = 1$ in Eq. (4.18) is

$$\sum_{k=-\infty}^{\infty} e^{-S_{\text{bos}}(\tau)[A^{(2k)}, \mathbf{a}, w_2(M)]} = \overline{\vartheta_4(\tau)}. \quad (4.61)$$

We have assumed that the family of the twisted $N = 2$ supersymmetric, pure $U(1)$ -Yang-Mills theory on which the electro-magnetic duality is realized as $\Gamma_0(4)$ acting as follows: on the coupling constant $\text{Im } \tau$ of each $U(1)$ -theory, the action of $\Gamma_0(4)$ on τ is the projective action given by Eq. (3.23). The action of $S \in SL(2; \mathbb{Z})$ on τ is $\tau \rightarrow -1/\tau$. The transformation on the coupling constant is accompanied by a duality transformation on the fields of the $U(1)$ -theory to specify the isomorphism between the full physical theories:

$$\begin{aligned} L &\rightarrow \tilde{L} = L \otimes L_\omega, \\ \frac{1}{2\pi} F_{A^{(2k)}}^+ = 2k \omega &\rightarrow \frac{1}{2\pi} F_{\tilde{A}^{(2k+1)}}^+ = (2k+1) \omega, \\ \mathbf{a} &\rightarrow \tilde{\mathbf{a}} = -\mathbf{a}_D, \\ w_2 &\rightarrow w_2 - 1, \\ S_{\text{bos}}(\tau)[A, \mathbf{a}, w_2(M)] &\rightarrow S_{\text{bos}}(\tau)[A + A_\omega, -\mathbf{a}_D, w_2(M) - \omega]. \end{aligned}$$

The Poisson re-summation formula for the ϑ -function implies

$$\vartheta_4\left(-\frac{1}{\tau}\right) = \vartheta_4(S \cdot \tau) = \sqrt{-i\tau} \vartheta_2(\tau).$$

After carrying out the electro-magnetic duality transformations on the vacuum fields the instanton sum becomes

$$\begin{aligned} \sum_{k=-\infty}^{\infty} e^{-S_{\text{bos}}(\tau)[A^{(2k)} + A_\omega, -\mathbf{a}_D, w_2(M) - \omega]} \\ = \sum_{k=-\infty}^{\infty} e^{-i\pi \bar{\tau} (k + \frac{1}{2})^2 - i\pi (k + \frac{1}{2}) (w_2 - 1)} = \overline{\vartheta_2(\tau)}. \quad (4.62) \end{aligned}$$

This shows that the partition function for the electro-magnetic dual fields corresponds to the partition function for the S-dual coupling constant $S \cdot \tau = -1/\tau$.

Remark 22. *The instanton sum in Eq. (4.18) as the sum over line bundles whose mod two reduction is zero. Equivalently, this is the sum over all squares of line bundles. The dual instanton sum corresponds to a sum over all line bundle whose mod two*

reduction is $w_2(M)$. Equivalently this is the sum over all squares of Spin^C -structures. The physical picture is that the $U(1)$ -gauge connection from the Coulomb branch for $u \rightarrow \infty$ is the magnetic dual of the $U(1)$ -gauge connection used in the definition of the Seiberg-Witten monopoles for $u \rightarrow 1$.

Chapter 5

The dual theory on the boundary

In this chapter, we will construct a conformal field theory on an elliptic curve E_u for large $|u| \gg 1$ as a heterotic σ -model with target space $CP^1 \times U(1)$. We will modify the action by varying the holomorphic structure and comparing different regularizations of the theory. This defines a family of action densities parameterized by $H^0(E_u, K^{\otimes 2})$ and $H^{(0,1)}(E_u)$ whose standard complex coordinates are denoted by α and β . When u is varied, there is no local or global anomaly present, and it make sense to ask what the limiting partition function is for $u \rightarrow \infty$. We will show using ζ -function regularization that the limit of the partition function is well-defined and gives in fact the u -plane integral.

5.1 The blow-up function as a current on an elliptic curve

In this section, we compute certain ratios of regularized determinants on the elliptic curve E_u which will later play a central role in the evaluation of the partition function in Sec. 5.5. The computations are carried out in this section and will be continued in the next section. We learnt about this method of regularizing functional determinants on Riemann surfaces from [99, 98].

On the fixed elliptic curve E_u with $u \neq \pm 1$, we compute the ratio of functional determinants

$$R(\check{z}, \nu_1, \nu_2) = \ln \left[\frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\bar{z}}{\text{vol}(E_u)} \right)_{(\nu_1, \nu_2)}}{\det \bar{\partial}_{(\nu_1, \nu_2)}} \right]. \quad (5.1)$$

We have set the $(0, 1)$ -part of the connection equal to $-\pi \check{z} / \text{vol}(E_u) d\bar{z}$ (cf. Eq. (3.30)). We also assume that for $\check{z} = 2\phi\omega + 2\theta\omega'$ we always choose $-1 < 2\phi, 2\theta < 1$. The expression in Eq. (5.1) makes sense as long as ν_1 and ν_2 are integers since then $\text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}})$ is canonically isomorphic to K^{-1} , so 1 denotes a canonical section of $\text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \otimes K$ (cf. [92, Ex. (1.5)]).

In the case $(\nu_1, \nu_2) \neq (1, 1)$, the operators in Eq. (5.1) do not have any kernel or cokernel. We have the holomorphic standard trivialization of the two determinant

line bundles which will allow us to write R as function on $H^{(0,1)}(E_u)$. Formally, we would like to write

$$\begin{aligned} \frac{\det\left(\bar{\partial} - \frac{\pi\check{z}d\check{z}}{\text{vol}(E_u)}\right)_{(\nu_1, \nu_2)}}{\det' \bar{\partial}_{(1,1)}} &= \frac{\prod_{n_1, n_2} \left(\frac{1}{2\bar{\omega}} \lambda_{n_1, n_2}^{(\nu_1, \nu_2)} - \frac{\pi\check{z}}{\text{vol}(E_u)}\right)}{\prod'_{n_1, n_2} \frac{1}{2\bar{\omega}} \lambda_{n_1, n_2}^{(1,1)}} \\ &= -\frac{\pi}{\text{vol}(E_u)} (\check{z} + \check{z}_{(\nu_1, \nu_2)}) \prod'_{n_1, n_2} \left(1 - \frac{\check{z} + \check{z}_{(\nu_1, \nu_2)}}{\Omega_{n_1, n_2}}\right) \end{aligned} \quad (5.2)$$

where $\check{z}_{(\nu_1, \nu_2)} = (1 - \nu_2)\omega - (1 - \nu_1)\omega'$ and $\Omega_{n_1, n_2} = 2n_1\omega + 2n_2\omega'$. The prime denotes the omission of the zero mode eigenvalue. The expression (5.2) is divergent. If we employ a cutoff N the ratio of determinants is

$$\begin{aligned} \frac{\det\left(\bar{\partial} - \frac{\pi\check{z}}{\text{vol}(E_u)}\right)_{(\nu_1, \nu_2)}}{\det' \bar{\partial}_{(1,1)}} &= -\frac{\pi}{\text{vol}(E_u)} \sigma\left(\check{z} + \check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega'\right) \\ &\times \exp\left[-(\check{z} + \check{z}_{(\nu_1, \nu_2)}) \sum_{n_1, n_2}^N \frac{1}{\Omega_{n_1, n_2}} - \frac{(\check{z} + \check{z}_{(\nu_1, \nu_2)})^2}{2} \sum_{n_1, n_2}^N \frac{1}{\Omega_{n_1, n_2}^2}\right], \end{aligned}$$

where $\sigma(z \mid \omega, \omega')$ is the Weierstrass σ -function given by

$$\sigma(z \mid \omega, \omega') = \prod_{n_1, n_2 = -\infty}^{\infty} \left[\left(1 - \frac{z}{\Omega_{n_1, n_2}}\right) \exp\left(\frac{z}{\Omega_{n_1, n_2}} + \frac{z^2}{2\Omega_{n_1, n_2}^2}\right) \right].$$

The Weierstrass σ -function satisfies

$$\frac{\sigma'(z \mid \omega, \omega')}{\sigma(z \mid \omega, \omega')} = \zeta(z \mid \omega, \omega'),$$

where the Weierstrass ζ -function ζ was defined in Sec. 3.2. σ is an entire function which vanishes at the origin. The Weierstrass σ -function is not an elliptic function since it is not periodic, but quasi periodic. We also define the functions (for $i = 1, 2, 3$)

$$\sigma_i(z) = e^{-\eta_i z} \frac{\sigma(z + \omega_i \mid \omega, \omega')}{\sigma(\omega_i \mid \omega, \omega')}.$$

In Sec. 2.3.2, we used the notation $\sigma_3(u, z)$ to exemplify that the periods are sections of a flat line bundle over the u -plane, and $\sigma_3(u, z)$ is a function in the two variables u and z . The entire issue revolves on how to handle the diverging sums when $N \rightarrow \infty$. The above expressions tells us that the ambiguity in different regularization is a quadratic polynomial in $\check{z} + \check{z}_{(\nu_1, \nu_2)}$. We define the regularized determinant by the

differential equation

$$\begin{aligned} \frac{\partial^3}{\partial \check{z}^3} \ln \left(\frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(\nu_1, \nu_2)}}{\det \bar{\partial}_{(\nu_1, \nu_2)}} \right) &= 2 \sum_{n_1, n_2} \frac{1}{(\check{z} + \check{z}_{(\nu_1, \nu_2)} - \Omega_{n_1, n_2})^3} \\ &= -\mathcal{P}'(\check{z} + \check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega'). \end{aligned}$$

This allows us to express the ratio of determinant as

$$\frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(\nu_1, \nu_2)}}{\det \bar{\partial}_{(\nu_1, \nu_2)}} = \frac{\sigma(\check{z} + \check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega')}{\sigma(\check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega')} e^{P(\check{z})},$$

where P is the quadratic polynomial $p_2 \check{z}^2 + p_1 \check{z}$ whose coefficients depend on ν_1, ν_2 . There is no constant coefficient in P if we impose $R(0, \nu_1, \nu_2) = 0$ for $\check{z} = 0$. It is easy to resolve the ambiguity in the definition of the determinant. If we go around in a cycle $\check{z}_{(\nu_1, \nu_2)} \rightarrow \check{z}_{(\nu_1, \nu_2)} + \Omega_{nm}$ on the torus the ratio of determinants has to remain unchanged. This shows that $p_1 = \zeta(\check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega')$. Thus, R equals

$$\frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(\nu_1, \nu_2)}}{\det \bar{\partial}_{(\nu_1, \nu_2)}} = \frac{\sigma(\check{z} + \check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega')}{\sigma(\check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega')} e^{p_2 \check{z}^2 + \zeta(\check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega') \check{z}}.$$

R satisfies the identity

$$R(\check{z} \mid \nu_1 + 2m, \nu_2 + 2n) = R(\check{z} \mid \nu_1, \nu_2),$$

and the initial conditions

$$\begin{aligned} R(0 \mid \nu_1, \nu_2) &= 0, \quad R'(0 \mid \nu_1, \nu_2) = 0, \\ R''(0 \mid \nu_1, \nu_2) &= 2p_2 - \mathcal{P}'(\check{z}_{(\nu_1, \nu_2)} \mid \omega, \omega'). \end{aligned} \tag{5.3}$$

Since \mathcal{P} is an elliptic function the coefficient p_2 does not depend on ν_1, ν_2 . We have

$$\frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(\nu_1, \nu_2)}}{\det \bar{\partial}_{(\nu_1, \nu_2)}} = \exp \left[(2p_2 - [(1 - \nu_2)e_1 + (1 - \nu_1)e_3]) \frac{\check{z}^2}{2} + O(\check{z}^3) \right],$$

where we have used that $e_1 = \mathcal{P}(\omega \mid \omega, \omega')$ and $e_3 = \mathcal{P}(\omega' \mid \omega, \omega')$.

In the case $(\nu_1, \nu_2) = (1, 1)$, let $\mathbf{1}$ denote the subspace in $\Lambda^0(E_u)$ spanned by the constant functions and $\mathbf{1}^\perp$ its orthogonal complement. For $(\nu_1, \nu_2) = (1, 1)$ the ratio of determinants is formally

$$\frac{\det_{\mathbf{1}^\perp} \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(1,1)}}{\det' \bar{\partial}_{(1,1)}} = \prod'_{n_1, n_2} \left(1 - \frac{\check{z}}{\Omega_{n_1, n_2}} \right)$$

which is divergent. We define the regularized determinant by the differential equation

$$\begin{aligned} \frac{\partial^3}{\partial \check{z}^3} \ln \left(\frac{\det_{\mathbf{1}^\perp} \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(1,1)}}{\det' \bar{\partial}_{(1,1)}} \right) &= 2 \sum'_{n_1, n_2} \frac{1}{(\check{z} - \Omega_{n_1, n_2})^3} \\ &= -\mathcal{P}'(\check{z} \mid \omega, \omega') - \frac{2}{\check{z}^3}. \end{aligned}$$

Thus, we have

$$\frac{\det_{\mathbf{1}^\perp} \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(1,1)}}{\det' \bar{\partial}_{(1,1)}} = \frac{\sigma(\check{z} \mid \omega, \omega')}{\check{z}} e^{p_2 \check{z}^2}.$$

The linear coefficient in the exponential vanishes because we require that the determinant is odd under $\check{z} \rightarrow -\check{z}$. Then R satisfies the initial conditions

$$R(0 \mid 1, 1) = 0, \quad R'(0 \mid 1, 1) = 0, \quad R''(0 \mid 1, 1) = 2p_2. \quad (5.4)$$

Since $\mathcal{P}(z) - z^{-2}$ is analytic in a neighborhood of the origin and vanishes at $z = 0$ the initial conditions (5.4) are the regularization of the initial conditions (5.3) for $\check{z}_{(\nu_1, \nu_2)} \rightarrow 0$. If we drop the restriction to $\mathbf{1}^\perp$ we obtain

$$\frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right)_{(1,1)}}{\det' \bar{\partial}_{(1,1)}} = -\frac{\pi}{\text{vol}(E_u)} \sigma(\check{z} \mid \omega, \omega') e^{p_2 \check{z}^2}. \quad (5.5)$$

We can fix p_2 once and for all, by requiring $R''(0 \mid i, j) = 0$ for some fixed $\nu_1 = i$ and $\nu_2 = j$. If $R''(0 \mid 0, 1) = 0$ we obtain $2p_2 = e_3$ which is invariant under $\Gamma_0(4)$. The value $(\nu_1, \nu_2) = (0, 1)$ corresponds to the Spin-structure (P, A) in each smooth fiber. this point was explained in Sec. 3.9. Moreover, $\mathcal{O}(Q_u - P_u)$ is a non-trivial holomorphic line bundle which is well-defined on UP. We have proved:

Lemma 34. *For the bundle $L_u = \mathcal{O}(Q_u - P_u)$ where $Q_u - P_u$ is given by*

$$z_{Q_u} = \check{z}_{(\nu_1, \nu_2)} - \check{z},$$

and the coupled Dolbeault operator $\bar{\partial}_{L,u}$, the ratios of regularized determinants

$$e^{R(\check{z}, \nu_1, \nu_2)} = \frac{\det \bar{\partial}_{L,u}}{\det \bar{\partial}_{(\nu_1, \nu_2)}}$$

and

$$\text{vol}(E_u) \frac{\det \left(\bar{\partial} - \frac{\pi \check{z} d\bar{z}}{\text{vol}(E_u)} \right)_{(1,1)}}{\det' \bar{\partial}_{(1,1)}},$$

are given in the table (5.6):

(ν_1, ν_2)	(θ, ϕ)	Spin-struct.	$\check{z}_{(\nu_1, \nu_2)}$	$\exp R(\check{z}, \nu_1, \nu_2)$		
(1, 1)	(0, 0)	(P, P)	0	$-\pi$	$\exp \left(\frac{\check{z}^2 e_3}{2} \right)$	$\sigma \left(\check{z} \mid \omega, \omega' \right)$
(0, 1)	$(\frac{1}{2}, 0)$	(P, A)	ω'		$\exp \left(\frac{\check{z}^2 e_3}{2} \right)$	$\sigma_3 \left(\check{z} \mid \omega, \omega' \right)$
(1, 1)	$(\frac{1}{2}, \frac{1}{2})$	(A, A)	$\omega + \omega'$		$\exp \left(\frac{\check{z}^2 e_2}{2} \right)$	$\sigma_2 \left(\check{z} \mid \omega, \omega' \right)$
(1, 0)	$(0, \frac{1}{2})$	(A, P)	ω		$\exp \left(\frac{\check{z}^2 e_1}{2} \right)$	$\sigma_1 \left(\check{z} \mid \omega, \omega' \right)$

(5.6)

We have imposed the initial conditions

$$R(0 \mid \nu_1, \nu_2) = 0, \quad R'(0 \mid \nu_1, \nu_2) = 0, \quad R''(0 \mid \nu_1, \nu_2) = 0.$$

for $(\nu_1, \nu_2) \neq (1, 1)$, and

$$R(0 \mid 1, 1) = 0, \quad R'(0 \mid 1, 1) = 0, \quad R''(0 \mid 1, 1) = e_3.$$

Remark 23. Using $e_3 = u/6$ we obtain

$$\exp \left(\frac{\check{z}^2 e_3}{2} \right) \sigma_3 \left(\check{z} \mid \omega, \omega' \right) = \exp \left(-\frac{\check{z}^2 u}{12} \right) \sigma_3 \left(\check{z} \mid \omega, \omega' \right). \quad (5.7)$$

As explained in Sec. 3.2, the Seiberg-Witten family of elliptic curves was also used in [5]. Fintushel and Stern used the family of elliptic curves with the parameters x and t with $x = u/2$ and $t = \check{z}$. We obtain in Eq. (5.7) the blow-up function for the $SU(2)$ -Donaldson invariants of [5, Thm. 4.1]. $(\nu_1, \nu_2) = (1, 1)$ corresponds to the trivial Spin-structure (P, P) in each smooth fiber. In this case,

$$\exp \left(\frac{\check{z}^2 e_3}{2} \right) \sigma \left(\check{z} \mid \omega, \omega' \right) = \exp \left(-\frac{\check{z}^2 u}{12} \right) \sigma \left(\check{z} \mid \omega, \omega' \right). \quad (5.8)$$

is the blow-up function for the $SO(3)$ -Donaldson invariants of [5, Thm. 4.1]. Thus, the blow-up functions of the Donaldson theory arise naturally in evaluating the ratio of functional determinants on the Seiberg-Witten family of elliptic curves.

5.2 The variation of the determinant of the Laplacian

In this section, we evaluate the ratio of determinants

$$\frac{\det \square(\check{z})}{\det' \Delta} \quad (5.9)$$

where we have used the operator

$$\bar{D}(\check{z}) = \left(\bar{\partial} - \frac{\pi \check{z} d\check{z}}{\text{vol}(E_u)} \right) : \Lambda^{(1,0)}(E_u) \rightarrow \Lambda^{(1,1)}(E_u)$$

to define

$$\square(\check{z}) = *i\bar{D}(\check{z})\partial : \Lambda^{(0,0)}(E_u) \rightarrow \Lambda^{(0,0)}(E_u),$$

and $\Delta = \square(0)$.

The determinant line bundle $\text{DET } \bar{D}(\check{z}) \rightarrow \text{Jac}(E_u)$ was described in [100, Sec. 6.2]. Taking the standard covering map $\pi : H^{(0,1)}(E_u) \rightarrow \text{Jac}(E_u)$ we have a pull-back line bundle which can be trivialized by the standard trivialization. If we write $\check{z} = 2\phi\omega + 2\theta\omega'$ then $(2\phi, 2\theta)$ are real coordinates of the Jacobian torus looked at as the square torus of period 1. It follows that the Quillen curvature two-form is $2\pi d\phi \wedge d\theta$ (cf. [100]). The operator D varies holomorphically on $H^{0,1}(E_u)$.

The ζ -function regularization gives

$$\begin{aligned} & \ln \left(\frac{\det' \square(\check{z})}{\det' \Delta} \right) \\ &= - [\zeta_{\square(\check{z})}(s) - \zeta_{(1,1)}(s)]'_{s=0} + 2 \ln \left(\frac{\pi}{\text{Im } \tau |\omega|} \right) [\zeta_{\square(\check{z})}(s) - \zeta_{(1,1)}(s)]_{s=0}, \end{aligned}$$

where the difference of the ζ -functions (cf. Eq.(3.27)) equals

$$\zeta_{\square(\check{z})}(s) - \zeta_{(1,1)}(s) = \sum'_{n_1, n_2} \frac{1}{\left[\left(n_1 + n_2 \tau - \frac{\check{z}}{2\omega} \right) \overline{(n_1 + n_2 \tau)} \right]^s} - \sum'_{n_1, n_2} \frac{1}{|n_1 + n_2 \tau|^{2s}}.$$

Using the identity for the hyper-geometric function

$$\frac{1}{\left(1 - \frac{\check{z}}{\Omega_{n_1, n_2}} \right)^s} = \frac{1}{2\pi i \Gamma(s)} \int_{-i\infty}^{i\infty} dr \Gamma(s+r) \Gamma(-r) \left(-\frac{\check{z}}{\Omega_{n_1, n_2}} \right)^r,$$

we compute

$$\begin{aligned}\zeta_{\square(\bar{z})}(s) - \zeta_{(1,1)}(s) &= \sum'_{n_1, n_2} \frac{1}{|n_1 + n_2\tau|^{2s}} \sum_{k=1}^{\infty} \frac{(s)_{2k} \bar{z}^{2k}}{(2k)! (2\omega)^{2k} (n_1 + n_2\tau)^{2k}} \\ &= \sum_{k=1}^{\infty} \frac{(s)_{2k} \bar{z}^{2k}}{(2k)! (2\omega)^{2k}} \sum'_{n_1, n_2} \frac{1}{|n_1 + n_2\tau|^{2s}} \frac{1}{(n_1 + n_2\tau)^{2k}}.\end{aligned}$$

In particular, we have

$$[\zeta_{\square(\bar{z})}(s) - \zeta_{(1,1)}(s)]_{s=0} = 0.$$

Since $\partial_s|_{s=0}(s)_{2k} = (2k-1)!$ we obtain

$$\begin{aligned}[\zeta_{\square(\bar{z})}(s) - \zeta_{(1,1)}(s)]'_{s=0} &= \sum_{k=1}^{\infty} \frac{\bar{z}^{2k}}{2k (2\omega)^{2k}} \lim_{s \rightarrow 0} \Phi_{2k}(\tau, 2s) \\ &= \frac{\bar{z}^2}{2(2\omega)^2} \widehat{G}_2(q^2) + \sum_{k=2}^{\infty} \frac{\bar{z}^{2k}}{2k (2\omega)^{2k}} G_{2k}(q^2),\end{aligned}$$

where we have used the almost modular functions

$$\begin{aligned}\lim_{s \rightarrow 0} \Phi_{2k}(\tau, 2s) &= \lim_{s \rightarrow 0} \sum'_{n_1, n_2} \frac{1}{|n_1 + n_2\tau|^{2s}} \frac{1}{(n_1 + n_2\tau)^{2k}} \\ &= \begin{cases} \widehat{G}_2(q^2) = G_2(q^2) - \frac{\pi}{\text{Im}\tau} & \text{if } k = 1 \\ G_{2k}(q^2) & \text{if } k \geq 2 \end{cases},\end{aligned}$$

for the Eisenstein functions $G_k(q^2)$ with $q = e^{\pi i\tau}$. Our notation follows the notation used in [95]. We have the following identity (cf. [95, Eq. (7.8)])

$$\begin{aligned}\frac{\vartheta_1(v|\tau)}{v \vartheta_1'(0|\tau)} e^{\frac{\pi v^2}{2\text{Im}\tau}} &= \exp\left(-\sum_{k=1}^{\infty} \frac{v^{2k}}{2k} G_{2k}(q^2) + \frac{\pi v^2}{2\text{Im}\tau}\right) \\ &= \exp\left(-\frac{v^2}{2} \widehat{G}_2(q^2)\right) \exp\left(-\sum_{k=2}^{\infty} \frac{v^{2k}}{2k} G_{2k}(q^2)\right) \\ &= \exp\left(-\frac{v^2}{2} \widehat{G}_2(q^2)\right) \frac{1}{w} \sigma(w|\omega, \omega')\end{aligned}$$

where $v = \frac{w}{2\omega}$. Thus, the ratio of determinants in Eq. (5.9) is

$$\frac{\det' \square(\bar{z})}{\det' \Delta} = \frac{\vartheta_1(v|\tau)}{v \vartheta_1'(0|\tau)} e^{\frac{\pi v^2}{2\text{Im}\tau}}$$

with $v = \check{z}/2\omega$. Using the normalized Eisenstein series $\widehat{G}_2(q^2) = \frac{\pi^2}{3}\widehat{E}_2(q^2)$, we obtain

$$\begin{aligned} \frac{\det' \square(\check{z})}{\det' \Delta} &= \frac{\vartheta_1(v|\tau)}{v\vartheta_1'(0|\tau)} e^{\frac{\pi v^2}{2\text{Im}\tau}} = e^{-\check{z}^2 \left[\frac{1}{24} \left(\frac{2\pi}{2\omega} \right)^2 \widehat{E}_2(q^2) \right]} \frac{1}{\check{z}} \sigma\left(\check{z} \mid \omega, \omega'\right) \\ &= e^{-\frac{\check{z}^2}{8} \frac{\widehat{E}_2(q^2)}{24h^2}} \frac{1}{\check{z}} \sigma\left(\check{z} \mid \omega, \omega'\right). \end{aligned} \quad (5.10)$$

Combining Eqs. (5.5) and (5.10) we have proved:

Lemma 35.

$$\begin{aligned} &\det'(\partial) \det \overline{D}(\check{z}) \frac{\text{vol}(E_u)}{\det' \square(\check{z})} \\ &= -\pi \check{z} \exp\left(\frac{\check{z}^2}{8} \frac{\widehat{E}_2(q^2)}{24h^2} + \frac{\check{z}^2}{2} e_3\right) = -\pi \check{z} e^{-\frac{\check{z}^2}{8} \widehat{T}}, \end{aligned}$$

with

$$\widehat{T} = -\frac{1}{24} \left(\frac{\widehat{E}_2(q^2)}{h^2} + 96e_3 \right) = -\frac{1}{24} \left(\frac{\widehat{E}_2(q^2)}{h^2} - 8u \right).$$

Corollary 5.

$$\begin{aligned} &\sqrt{\det'(\partial) \det \overline{D}(\check{z}_+) \frac{\text{vol}(E_u)}{\det' \square(\check{z}_+)}} \sqrt{\det'(\partial) \det \overline{D}(\check{z}_-) \frac{\text{vol}(E_u)}{\det' \square(\check{z}_-)}} \\ &= 2\sqrt{2} \pi S e^{S^2 \widehat{T}}, \end{aligned}$$

with $z_{\pm} = \pm 2\sqrt{2}iS$.

5.3 The σ -model with target space $U(1)$

In this section, we derive the partition function for the heterotic σ -model on E_u with the target space $U(1)$ coupled to a gauge background. This computation is an application of a more general computation carried out in [96, Sec. 3] if we specialize the target space to $U(1)$.

5.3.1 The action of the σ -model

Let z be the complex coordinate z on E_u , x a map from E_u to $U(1)$. We also introduce a fermionic field λ . It is a section of $\overline{K}_0^{-1/2}$ with the square root $K_0^{1/2}$ of the canonical bundle K determined by the trivial Spin-structure. The chiral Dirac operator on the Riemann surface E_u is

$$\partial : \Lambda^{(0, \frac{1}{2})}(E_u) \rightarrow \Lambda^{(1, \frac{1}{2})}(E_u).$$

The action for the $N = \frac{1}{2}$ supersymmetric, σ -model in the trivial gravitational background is

$$\int_{E_u} \frac{dz d\bar{z}}{4\pi i} (\partial x \bar{\partial} x + \lambda \partial \lambda) . \quad (5.11)$$

The supersymmetry transformation is

$$\delta x = \epsilon \lambda , \quad \delta \lambda = \epsilon \partial x , \quad (5.12)$$

where ϵ is a local holomorphic section of $\bar{K}_0^{1/2}$. Note that because E_u is a torus the supersymmetry transformation laws in Eq. (5.12) and x is periodic around any cycle in E_u , it follows that λ must belong to the odd Spin-structure on E_u .

We couple the right moving system with a $N = \frac{1}{2}$ supersymmetry to a left moving sector. The boundary conditions for the left sector can be chosen to be any of the odd or even Spin-structures on E_u . None of the choices will spoil the topological character of the result since the topological invariance is entirely governed by the supersymmetry of the right sector [98, Sec. 4]. The left moving fermion is a section of $K_0^{1/2} \otimes x^*(V)$ with the square root $K^{1/2}$ of the canonical bundle K determined by the trivial Spin-structure on E_u . $V = TU(1)^{\otimes n} = \mathbb{R}^{\otimes n}$ is the n th power of the tangent bundle of $U(1)$ with the trivial connection. The action of the heterotic σ -model on E_u with the target space $U(1)$ coupled to a gauge background is

$$\mathbf{S}^{U(1)} = \int_{E_u} \frac{dz d\bar{z}}{4\pi i} (\partial x \bar{\partial} x + \lambda \partial \lambda + \langle \psi, \bar{\partial} \psi \rangle) . \quad (5.13)$$

We also include the effect of diffeomorphisms on E_u . The operator $\hat{\Lambda}$ generating diffeomorphisms is simply $\Lambda = \Lambda \frac{\partial}{\partial x}$. Since V is connected to the tangent bundle of $U(1)$, $\hat{\Lambda}$ also acts by a gauge transformation (Λ_{AB}) on the fermions Ψ^A . The generalized action was computed in [96, Eq. (4.1)]

$$\begin{aligned} \mathbf{S}_{\Lambda}^{U(1)} = \int_{E_u} \frac{dz d\bar{z}}{4\pi i} & \left(\partial x \bar{\partial} x + \lambda \partial \lambda + \langle \psi, \bar{\partial} \psi \rangle \right. \\ & \left. + \frac{1}{\text{vol}(E_u)} [i \Lambda \partial x + \Lambda_{AB} \psi^A \psi^B] \right) . \end{aligned} \quad (5.14)$$

We look at the quadratic fluctuations of the action around the critical points. The quadratic term $\frac{\partial \Lambda}{\partial x} x' \partial x'$ arising from $\Lambda \partial x$ in the action is a total derivative and vanishes because of the periodic boundary conditions for x (cf. [96, Sec. 4]). The quadratic part of the action is

$$\mathbf{S}^{U(1), (2)} = \int_{E_u} \frac{dz d\bar{z}}{4\pi i} \left(\partial x \bar{\partial} x + \lambda \partial \lambda + \left\langle \psi, \left(\bar{\partial} \otimes \mathbb{I}_n + \frac{\Lambda_{AB} d\bar{z}}{\text{vol}(E_u)} \right) \psi \right\rangle \right) . \quad (5.15)$$

Performing the Gaussian integration over the fermions, the contribution to the path

integral is

$$\mathbf{Z}_{\text{fermi}}^{U(1)} = \int [\mathcal{D}\psi \mathcal{D}\lambda] e^{-\mathbf{S}^{U(1), (2)}} = \left[\det'(\partial) \det \left(\bar{\partial} \otimes \mathbb{I}_n + \frac{\Lambda_{AB} d\bar{z}}{\text{vol}(E_u)} \right) \right]^{\frac{1}{2}}, \quad (5.16)$$

where we have removed the fermionic zero modes of λ . The explicit computation of the determinants was carried out in Sec. 3.7.

5.3.2 The instanton contribution

For the bosonic part, there is an instanton contribution which comes from the evaluation of the action at the critical points, and summing over all of the topologically different critical points. The computation of the instanton contribution is well-known in the literature (cf. [88, 115]). We show in this section that the bosonic contribution to the σ -model defined in Sec. 5.3.1 is the bosonic theory considered in [88] coupled to the line bundle $L_u \rightarrow E_u$.

Due to the marking of the elliptic curve E_u we have a homology basis \mathcal{A} of $H_1(E_u)$ with the periods $(2\omega, 2\omega')$ and the modular parameter $\tau = \frac{\omega'}{\omega}$. The one-forms ρ and σ comprise a dual basis \mathcal{A}^* such that

$$\begin{aligned} \int_{\alpha_u} \rho &= 1, & \int_{\beta_u} \rho &= 0, \\ \int_{\alpha_u} \sigma &= 0, & \int_{\beta_u} \sigma &= 1. \end{aligned}$$

The flat holomorphic connection \mathfrak{a}_u on the line bundle $L_u = \mathcal{O}(Q_u - P_u)$ is given by the point $Q_u \in E_u$ with

$$\mathfrak{a}_u = 2\pi i \theta \rho - 2\pi i \phi \sigma.$$

The image of \mathfrak{a}_u in the Jacobian of E_u is

$$\begin{aligned} H^1(E_u; \mathbb{R})/H^1(E_u; \mathbb{Z}) &\rightarrow \text{Jac}(E_u) = \mathbb{C}/\langle 1, \tau \rangle \\ \mathfrak{a}_u = 2\pi i \theta \rho - 2\pi i \phi \sigma &\mapsto \check{\xi} = \frac{z_{Q_u}}{2\omega} = \phi + \tau \theta. \end{aligned}$$

On the Jacobian, there is a natural hermitian norm \mathcal{H} given by

$$\mathcal{H}(\check{\xi}_1, \check{\xi}_2) = \frac{\overline{\check{\xi}_1} \check{\xi}_2}{\text{Im } \tau} = \langle \check{\xi}_1, \check{\xi}_2 \rangle + i (\check{\xi}_1, \check{\xi}_2),$$

for $\check{\xi}_1, \check{\xi}_2 \in \text{Jac}(E_u)$ where

$$\langle \check{\xi}_1, \check{\xi}_2 \rangle = \frac{\text{Re}(\overline{\check{\xi}_1} \check{\xi}_2)}{\text{Im } \tau}, \quad (\check{\xi}_1, \check{\xi}_2) = \frac{\text{Im}(\overline{\check{\xi}_1} \check{\xi}_2)}{\text{Im } \tau}.$$

We evaluate the partition function

$$\int_{\Phi} \mathcal{D}\phi e^{-S^{U(1)}(\phi)}$$

over the space of fields $\Phi = \{\phi : E_u \rightarrow U(1)\}$. As proved in [115, Sec. 1.3], we can decompose the space of fields

$$\Phi = \bigcup_{h \in H^1(E_u; \mathbb{Z})} \Phi_{\omega_h}, \quad \text{with } \Phi_{\omega_h} = \left\{ \phi \mid \frac{\phi^{-1} d\phi}{2\pi i} - \omega_h \text{ is exact} \right\}.$$

The measure of the path integral decomposes into

$$\int_{\Phi} = \sum_{H^1(E_u; \mathbb{Z})} \int_{\Phi_{\omega_h}}.$$

If we choose a $\phi_{\omega_h} \in \Phi$ with $\phi_{\omega_h}^{-1} d\phi_{\omega_h} / (2\pi i) = \omega_h$ then

$$\Phi_{\omega_h} = \Phi_0 \cdot \phi_{\omega_h}, \quad \text{with } \Phi_0 = \left\{ \phi = e^{2\pi i x} \mid x : E_u \rightarrow \mathbb{R} \right\},$$

where x is the map used in Sec. 5.3.1. We set

$$\mathbf{S}_{\text{bos}}^{U(1)}(\phi_{\omega_h} e^{2\pi i x}) = \frac{\langle \omega_h, \omega_h \rangle}{2} + \frac{\langle dx, dx \rangle}{2} + i \frac{(\omega_h + dx, \check{\xi})}{2}, \quad (5.17)$$

with $\check{\xi} \in \text{Jac}(E_u)$. The action in Eq. (5.17) includes the instanton contribution and quadratic approximation

$$\mathbf{S}_{\text{bos}}^{U(1)}(\phi_{\omega_h} e^{2\pi i x}) = \mathbf{S}_{\text{bos}}^{U(1), (0)}(\omega_h, \check{\xi}) + \mathbf{S}_{\text{bos}}^{U(1), (2)}(x, \check{\xi}),$$

with

$$\mathbf{S}_{\text{bos}}^{U(1), (0)}(\omega_h, \check{\xi}) = \frac{\langle \omega_h, \omega_h \rangle}{2} + i \frac{(\omega_h, \check{\xi})}{2}, \quad (5.18a)$$

$$\mathbf{S}_{\text{bos}}^{U(1), (2)}(x, \check{\xi}) = \frac{\langle dx, dx \rangle}{2} + i \frac{(dx, \check{\xi})}{2}. \quad (5.18b)$$

The action $\mathbf{S}_{\text{bos}}^{U(1), (2)}(\omega_h, \check{\xi})$ is the bosonic part of the action in Eq. (5.16) for $\Lambda = 2\omega \check{\xi}$. It was also considered in [88, Sec. 4.C] where its partition function was computed. For $\check{\xi} = \frac{1}{2}$ the path integral becomes

$$\mathbf{Z}_{\text{bos}}^{U(1)} = \int_{\Phi} \mathcal{D}\phi e^{-\mathbf{S}_{\text{bos}}^{U(1)}(\phi)} = \sum_{H^1(E_u; \mathbb{Z})} e^{-\frac{\langle \omega_h, \omega_h \rangle}{2} + i \frac{(\omega_h, \check{\xi})}{2}} \int \mathcal{D}x e^{-\frac{\langle dx, dx \rangle}{2}}. \quad (5.19)$$

The instanton part of the partition function was evaluated in [88, Eq. (4.13)]. For

$\tilde{\xi} = \frac{1}{2}$ we obtain

$$\sum_{H^1(E_u; \mathbb{Z})} e^{-\frac{\langle \omega_h, \omega_h \rangle}{2} + i \frac{\langle \omega_h, \tilde{\xi} \rangle}{2}} = \sqrt{\text{Im } \tau} \left| \vartheta_4(\tau) \right|^2. \quad (5.20)$$

Carrying out the Gaussian integration in Eq. (5.19) yields

$$\mathbf{Z}_{\text{bos}}^{U(1)} = \frac{1}{\sqrt{2\pi}} \left(\frac{\text{vol}(E_u)}{\det' \Delta} \right)^{\frac{1}{2}} \sqrt{\text{Im } \tau} \left| \vartheta_4(\tau) \right|^2. \quad (5.21)$$

5.3.3 The ghost contribution

The action in Eq. (5.13) is an action in which the general coordinate and $N = \frac{1}{2}$ supersymmetry gauges have been fixed. Thus, we have to introduce the following ghost fields [96, Sec. 3]: the general coordinate ghosts and anti-ghosts b, c and \bar{b}, \bar{c} and the chiral supersymmetry ghosts and anti-ghosts $\bar{\beta}, \gamma$. The presence of the gauge fields does not affect the ghost sector. Thus, the action is

$$\mathbf{S}_{\text{gh}}^{U(1)} = \int_{E_u} \frac{dz d\bar{z}}{4\pi i \text{vol}(E_u)} (\bar{b} \partial \bar{c} + b \bar{\partial} c + \bar{\beta} \partial \gamma). \quad (5.22)$$

Here, b, c are sections of $K^{1/2}$ with the square root $K^{1/2}$ of the canonical bundle K determined by the trivial Spin-structure on E_u . Similarly, γ is a complex function on E_u , and $\bar{\beta}$ is a section of $\Lambda^{(0,1)}(E_u)$. The path integral over the ghost fields is (cf. [96, Eq. (4.14)])

$$\begin{aligned} \mathbf{Z}_{\text{gh}}^{U(1)} &= \int [Db D\bar{b} Dc D\bar{c} D\bar{\beta} D\gamma] e^{-\mathbf{S}_{\text{gh}}^{U(1)}} = \left[\frac{\det'(\partial) \det'(\bar{\partial})}{\text{vol}(E_u) \det'(\partial)} \right]^{\frac{1}{2}} \\ &= \left[\frac{\det'(\bar{\partial})}{\text{vol}(E_u)} \right]^{\frac{1}{2}}. \end{aligned} \quad (5.23)$$

5.3.4 The partition function

The full action of the heterotic σ -model on E_u with target space $U(1)$ is

$$\mathbf{S}^{U(1)} = \mathbf{S}_{\Lambda}^{U(1)} + \mathbf{S}_{\text{gh}}^{U(1)},$$

where $\mathbf{S}_{\Lambda}^{U(1)}$ and $\mathbf{S}_{\text{gh}}^{U(1)}$ are given in Eq. (5.14) and Eq. (5.22). To avoid the problems in defining the path-integrals we will treat the partition function as the one-loop Gaussian approximation of a string field theory. The Spin-structure (P, A) on E_u gives a well-defined holomorphic line bundle $L_u \rightarrow Z$. The corresponding ϑ -characteristics is $\vartheta_4(\tau)$ and corresponds to $\Lambda = \omega$ in Eq. (5.14) and $\tilde{\xi} = 1/2$ in Eq. (5.17). We also set $n = 24$. It follows:

Lemma 36. *The path integral of the heterotic σ -model on E_u to $U(1)$ with the target*

space $U(1)$ coupled to a gauge background $V = \mathbb{C}^{12} \rightarrow U(1)$ described in Sec. 5.3.1 is

$$\mathbf{Z}^{U(1)} = \frac{1}{\sqrt{2\pi}} \sqrt{\text{Im } \tau} \left| \vartheta_4(\tau) \right|^2 \Delta(\tau). \quad (5.24)$$

Proof. We have used that

$$\det \left(\bar{\partial} \otimes \mathbb{I}_n + \frac{\Lambda_{AB} d\bar{z}}{\text{vol}(E_u)} \right) = \det^n(\bar{\partial}_{L,u}),$$

where L_u is the even Spin-structure (P, A) on E_u . Combining Eqs. (5.16), (5.21), (5.23) it follows

$$\begin{aligned} \mathbf{Z}_u^{U(1)} &= \left[\frac{\text{vol}(E_u) \det(\bar{\partial} \otimes \mathbb{I}_n + \Lambda_{AB})}{\det'(\bar{\partial})} \right]^{\frac{1}{2}} \\ &\times \left[\frac{\det'(\bar{\partial})}{\text{vol}(E_u)} \right]^{\frac{1}{2}} \sqrt{\text{Im } \tau} \left| \vartheta_4(\tau) \right|^2 \\ &= \sqrt{\text{Im } \tau} \left| \vartheta_4(\tau) \right|^2 \det^{\frac{n}{2}}(\bar{\partial}_\xi). \end{aligned}$$

The equality of the determinant of $\bar{\partial}$ coupled to the line bundle L_u and the discriminant of E_u was proved in Sec. 3.7. \square

Remark 24. *The cancellation of the $\det'(\partial)$ -terms in Eq. (5.16) is called the world sheet supersymmetry in physics.*

5.4 The σ -model with target space CP^1

In this section, we evaluate the partition function for the σ -model on E_u with target space CP^1 . The evaluation is an application of the computations in [100] where we have chosen the target space $Y = SU(2)/U(1) = CP^1$. The σ -model with target space $SU(2)/U(1)$ has first been considered in [107].

We will use the following normalization: deleting the point at infinity, we may stereographically project S^2 onto the complex plane and use the complex projective coordinate $w = x^1 + ix^2$. The Kähler form can then be written as $K = \frac{i}{2} \partial \bar{\partial} \ln(1 + w\bar{w})$. On CP^1 , we can choose the vierbeins e^1 and e^2 as in [54, Examples 3.4 (2)] such that the Kähler form becomes $K = e^1 \wedge e^2$. On CP^1 , we have $K = \text{vol}_{CP^1}$ (cf. [54, Examples 3.4 (2)]). We also have $c_1(CP^1) = \frac{2}{\pi} K$ and thus $\int_{CP^1} K = \pi$ [54, Examples 6.3 (2)].

5.4.1 The action

Let z be the complex coordinate z on E_u , dx the differential of a map from TE_u to TY . We also introduce a fermionic field λ . It is a section of $\bar{K}_0^{-1/2} \otimes x^*(TM)$ with the

square root $K_0^{1/2}$ of the canonical bundle K determined by the trivial Spin-structure. The chiral Dirac-operator $\nabla_X^{(1,0)}$ on the Riemann surface E_u is

$$\left(\nabla_X^{(1,0)}\lambda\right)^\mu = \partial\lambda^\mu + \partial x^\lambda \Gamma_{\lambda\nu}^\mu \lambda^\nu ,$$

where Γ is the Christoffel symbol of the Levi-Civita connection on TY .

Remark 25. *Since the target space is complex one-dimensional every three-forms on Y vanishes. In the framework of [100], it follows that $C = dB = 0$ where B is a real two-form on Y .*

The action for the $N = \frac{1}{2}$ supersymmetric, non-linear σ -model is

$$\int_{E_u} \frac{dz d\bar{z}}{4\pi i} g_{\mu\nu} \left(\partial x^\mu \bar{\partial} x^\nu + (\nabla_X^{(1,0)}\lambda)^\mu \lambda^\nu \right) . \quad (5.25)$$

The supersymmetry transformation is

$$\delta x = \epsilon \lambda , \quad \delta \lambda = \epsilon \bar{\partial} x , \quad (5.26)$$

where ϵ is a local holomorphic section of $\bar{K}_0^{-1/2}$. Note that because E_u is a torus the supersymmetry transformation laws in Eq. (5.12) and x is periodic around any cycle in E_u , it follows that ψ must belong to the unique odd Spin-structure on E_u .

We couple the $N = \frac{1}{2}$ right moving system with a left moving sector. The boundary conditions for the left sector can be chosen to be any of the odd or even Spin-structures on E_u . The left moving fermion ψ is a section of $K_0^{1/2} \otimes x^*(TY)$ with the square root $K_0^{1/2}$ of the canonical bundle K determined by the trivial Spin-structure. The action of the σ -model on E_u with the target space $Y = CP^1$ coupled to a gravitational background is

$$\begin{aligned} \mathbf{S}^{CP^1} = \int_{E_u} \frac{dz d\bar{z}}{4\pi i} & \left[g_{\mu\nu} \left(\partial x^\mu \bar{\partial} x^\nu + (\nabla_X^{(1,0)}\lambda)^\mu \lambda^\nu + (\nabla_X^{(0,1)}\psi)^\mu \psi^\nu \right) \right. \\ & \left. + R_{\mu\nu\alpha\beta} \lambda^\mu \lambda^\nu \psi^\alpha \psi^\beta \right] , \end{aligned} \quad (5.27)$$

where R is the Riemann curvature tensor of the Levi-Civita connection Γ on Y .

We look at the quadratic fluctuations of the action around the critical points. We must exercise care since x and λ have zero modes. We write

$$x = x_0 + x' , \quad \lambda = \lambda_0 + \lambda' .$$

The quadratic part of the action is

$$\begin{aligned} \mathbf{S}^{CP^1, (2)} = \int_{E_u} \frac{dz d\bar{z}}{4\pi i} g_{\mu\nu} & \left(\partial x'^\mu [(\bar{\partial} + \mathcal{R}) x']^\nu \right. \\ & \left. + \lambda'^\mu \partial \lambda'^\nu + \psi^\mu [(\bar{\partial} + \mathcal{R}) \psi]^\nu \right) , \end{aligned} \quad (5.28)$$

with

$$(\mathcal{R}\psi)^\mu = \frac{1}{2} R^\mu{}_{\nu\alpha\beta} \lambda_0^\alpha \lambda_0^\beta \psi^\nu .$$

Performing the Gaussian integration over the fluctuations in the fields, the path integral becomes

$$\int [\mathcal{D}x' \mathcal{D}\psi \mathcal{D}\lambda'] e^{-\mathbf{S}^{CP^1, (2)}} = \left[\frac{\det'(\partial \otimes \mathbb{I}_2) \det(\bar{\partial} + \mathcal{R})}{\det'[-\partial(\bar{\partial} + \mathcal{R})]} \right]^{\frac{1}{2}}, \quad (5.29)$$

which agrees with [98, Eq. (4.7)]. The bosonic part of the measure for the full path integral at x_0 re-expressed in terms of the normal bundle data is (cf. [100, Eq. (4.5)])

$$[\mathcal{D}x] = \frac{\text{vol}(E_u)}{2\pi} d^2x_0 [\mathcal{D}x'] ,$$

where $[\mathcal{D}x']$ is the measure on the space of maps orthogonal to the constant map. Similarly, $\frac{\sqrt{d\bar{z}}}{\sqrt{\text{vol}(E_u)}} \in \Lambda^{(1/2,0)}(E_u)$ is a normalized holomorphic spinor since

$$\langle \sqrt{d\bar{z}}, \sqrt{d\bar{z}} \rangle = \text{vol}(E_u) .$$

Thus, we choose $\lambda_0^\mu = \widehat{\lambda}_0^\mu \otimes \frac{\sqrt{d\bar{z}}}{\sqrt{\text{vol}(E_u)}}$ and conclude that

$$[\mathcal{D}\lambda] = d^2\widehat{\lambda}_0 [\mathcal{D}\lambda'] .$$

The path integral is

$$\begin{aligned} \mathbf{Z}^{CP^1} &= \int [\mathcal{D}x \mathcal{D}\lambda \mathcal{D}\psi] e^{-\mathbf{S}^{CP^1}} \\ &= \frac{\text{vol}(E_u)}{2\pi} \int_{CP^1} d^2x_0 d^2\widehat{\lambda}_0 \left[\frac{\det'(\partial \otimes \mathbb{I}_2) \det(\bar{\partial} + \mathcal{R})}{\det'[-\partial(\bar{\partial} + \mathcal{R})]} \right]^{\frac{1}{2}} \\ &= \frac{\text{vol}(E_u)}{2\pi} \det'(\partial) \int_{CP^1} d^2x_0 d^2\widehat{\lambda}_0 \left[\frac{\det(\bar{\partial} + \mathcal{R})}{\det'[-\partial(\bar{\partial} + \mathcal{R})]} \right]^{\frac{1}{2}}, \end{aligned} \quad (5.30)$$

where

$$\left(\mathcal{R}_{\mu\nu} \right)_{\mu,\nu=1}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \widehat{\lambda}_0^1 \widehat{\lambda}_0^2 \otimes \frac{d\bar{z}}{\text{vol}(E_u)} .$$

The eigenvalues of \mathcal{R} are $\pm \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}$ with $r = 2\pi \widehat{\lambda}_0^1 \widehat{\lambda}_0^2$. It follows that

$$\begin{aligned} \mathbf{Z}^{CP^1} &= \frac{\text{vol}(E_u)}{2\pi} \det'(\partial) \int_{CP^1} d^2 x_0 d^2 \widehat{\lambda}_0 \\ &\times \left[\frac{\det\left(\bar{\partial} + \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}\right)}{\det'[-\partial\left(\bar{\partial} + \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}\right)]} \right]^{\frac{1}{2}} \left[\frac{\det\left(\bar{\partial} - \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}\right)}{\det'[-\partial\left(\bar{\partial} - \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}\right)]} \right]^{\frac{1}{2}}. \end{aligned} \quad (5.31)$$

Applying Lemma 35 and Lemma 5, we obtain

$$\begin{aligned} \mathbf{Z}^{CP^1} &= \frac{1}{2\pi} \int_{CP^1} d^2 x_0 d^2 \widehat{\lambda}_0 \frac{\text{vol}(E_u) \det'(\partial) \det\left(\bar{\partial} + \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}\right)}{\det'[-\partial\left(\bar{\partial} + \frac{2\pi i r d\bar{z}}{\text{vol}(E_u)}\right)]} \\ &= \int_{CP^1} d^2 x_0 d^2 \widehat{\lambda}_0 r e^{\frac{r^2}{2} \widehat{T}}. \end{aligned} \quad (5.32)$$

Using the correspondence $\widehat{\lambda}_0^1 \widehat{\lambda}_0^2 \leftrightarrow e^1 \wedge e^2 = K$, the partition function in Eq. (5.31) can be evaluated

$$\mathbf{Z}^{CP^1} = \int_{CP^1} r e^{\frac{r^2}{2} \widehat{T}} = 2\pi^2. \quad (5.33)$$

We have proved:

Lemma 37. *The path integral of the non-linear σ -model on E_u with target space $SU(2)/U(1)$ coupled to a gravitational background is*

$$\mathbf{Z}^{CP^1} = 2\pi^2. \quad (5.34)$$

Remark 26. *The partition function in Eq. (5.32) is known as the string theoretic derivation of the Euler characteristic of the loop space of Y [98].*

5.5 The one-loop amplitude

The combination of the the two σ -models defined in Sec. 5.3 and Sec. 5.4 determines a heterotic σ -model on the elliptic curve E_u with target space $N = SU(2)/U(1) \times U(1)$. The classical bosonic fields are maps $X : \Sigma \rightarrow N$ and the classical action is

$$\mathbf{S}_{\text{bos}} = \int \frac{dz d\bar{z}}{4\pi i} g_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu.$$

The Lagrangian for the $N = 1/2$ supersymmetric theory consisting of only right-moving spinors is

$$\mathbf{S} = \int \frac{dz d\bar{z}}{4\pi i} g_{\mu\nu} \left[\partial X^\mu \bar{\partial} X^\nu - \lambda^\mu \nabla^{(1,0)} \lambda^\nu \right].$$

with $\lambda^\mu \in \bar{K}_0^{1/2} \otimes X^*(TN)$ where the square root of the canonical bundle K is given by the trivial spin structure. The connection is given by

$$(\nabla^{(1,0)}\lambda)^\mu = \partial\lambda^\mu + \partial X^\alpha \Gamma_{\alpha\beta}^\mu \lambda^\beta ,$$

where Γ is the Christoffel symbol of the Levi-Civita connection on N . The $N = \frac{1}{2}$ supersymmetry is often referred to as a $(1, 0)$ -supersymmetry in the right-moving sector [109]. It has been proved in [108, 110], that parallelised group manifolds carry an extended supersymmetry due to their complex structures. In the general setting of the heterotic σ -model [109], we treat the $U(1)$ -factor in N as an internal manifold and CP^1 as the space-time of a string field theory. The $N = \frac{1}{2}$ supersymmetry is enough to preserve the supersymmetry of the space-time. The topological invariance of the path integral is entirely governed by this supersymmetry [98].

The σ -model on E_u with the target space $U(1)$ is coupled to a gauge background. The left-moving fermions are section $\psi^A \in \bar{K}^{1/2} \otimes X_{U(1)}^*(V)$ with $V = \underline{\mathbb{C}}^{12} \rightarrow S^1$. The σ -model on E_u with the target space CP^1 is the non-linear σ -model used for computing the Euler characteristic of the loop space of the space-time. It follows from the results in Sec. 5.3 and Sec. 5.4:

Lemma 38. *The path integral of the heterotic σ -model on E_u with target space $CP^1 \times U(1)$ coupled to a gravitational and gauge background described above is*

$$\mathbf{Z}_u = \mathbf{Z}^{CP^1} \mathbf{Z}^{U(1)} = \sqrt{\text{Im } \tau} \left| \vartheta_4(\tau) \right|^2 \Delta(\tau) , \quad (5.35)$$

where C is a constant.

Under the change of u , there are anomalies which would give rise to inconsistencies:

1. There is no local anomaly. The Quillen determinant line bundle for the Dirac-operator for a non-trivial Spin-structure is flat [77]. Thus, we do not have any local anomaly.
2. On each elliptic curve E_u with $u \in U$ we have the standard holomorphic trivialization of the various determinant line bundles. These canonical trivializations patch together to give a canonical trivialization. The issue is whether there is such a canonical trivializing section for the kernel and the cokernel of the uncoupled $\bar{\partial}$ -operator. Such a trivialization is given by the holomorphic two form η in Lemma 3.
3. There is no global anomaly. We can evaluate Eq. (5.35) and obtain that the action is invariant under the action of $\Gamma_0(4)$. Encircling the non-trivial loops in $\pi_1(\text{UP})$, the monodromy acts as described in Eqs. 3.23. However, one can check that \mathbf{Z}_u does not acquire a phase.

In the remainder of this section, we modify the action of Sec. 5.3 and Sec. 5.4 by changing the holomorphic structure and the regularization and compute the changes to the partition function in Eq. (5.35).

5.5.1 The change of the holomorphic structure

In [107], the σ -model with target space $SU(2)/U(1) \cong CP^1$ was considered. The isometry group $U(1)$ is a global symmetry of the nonlinear σ -model and can be gauged [107]. We consider the coupling to an external field β . The presence of the external field leads to the modification

$$\bar{\partial} \rightarrow \bar{D}(\beta) = \bar{\partial} \pm \frac{2\pi i \sqrt{2}\beta d\bar{z}}{\text{vol}(E_u)}, \quad (5.36)$$

$$r \rightarrow r + \beta, \quad (5.37)$$

in Eq. (5.32) where the factor of $\sqrt{2}$ has been inserted for normalization. We interpret the operator $\bar{D}(\beta)$ as a flat complexified $U(1)$ -connection of type $(0, 1)$ and $\beta d\bar{z}$ as a holomorphic tangent vector to the holomorphic connection on $K^{\frac{1}{2}}$ where the square root of the canonical bundle is determined by an even or odd Spin-structure. Then, $\text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}})$ is canonically isomorphic to K^{-1} , so 1 denotes a canonical section of $\text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \otimes K$ (cf. [92, Ex. (1.5)]). On the elliptic curve E_u , there is only one choice for a holomorphic tangent vector up to a constant which is $d\bar{z}$ [73]. The ∂ -operator is left unchanged.

Equivalently, we view the situation as follows: let $V \rightarrow Z$ be a semi-stable vector bundle with a Hermitian Yang-Mills connection described in Sec. 3.9. There is a reduction $V|_{Z_u} \cong L_u \oplus L_u^{-1}$ where L_u is a holomorphic line bundle on E_u . We use the base point P_u to identify the bundle L_u with $\mathcal{O}(Q_u - P_u)$ for a point $Q_u \in Z_u$. The flat connection A_u on $V|_{E_u} = \mathcal{O}(Q_u - P_u) \oplus \mathcal{O}(-Q_u - P_u)$ is

$$A_u = \frac{\pi}{\text{vol}(E_u)} (-\bar{z}_{Q_u} dz \otimes \sigma^3 + z_{Q_u} d\bar{z} \otimes \sigma^3), \quad (5.38)$$

We define an endomorphism valued $(0, 1)$ -form in the adjoint representation $\mathbf{h} = \beta d\bar{z} \otimes i\sigma^3$. The flat connection A_u and the holomorphic tangent vector \mathbf{h} form a bosonic Higgs vacuum on E_u

$$F_{A_u} = [\mathbf{h}; \mathbf{h}^*] = 0, \quad \bar{\partial}_{A_u}^\dagger \mathbf{h} = 0.$$

The coupling to the external field is obtained by changing the $(0, 1)$ -part of the connection according to

$$\bar{\partial} \otimes \mathbb{I} + A_u^{(0,1)} \rightarrow \bar{\partial} \otimes \mathbb{I} + A_u^{(0,1)} + \mathbf{h}.$$

5.5.2 The comparison of different regularizations

For the quantum theory in the external gravitational field, there are quadratic and logarithmic divergences that can be renormalized by adding further terms to the action. Different regularizations will introduce certain counter-terms in the one-loop amplitude and we obtain a one-parameter family of actions. As explained in [93, Eq. (2.6)], these terms are proportional to the volume and the Euler characteristic of

E_u . The latter vanishes as E_u is a torus.

Let $\delta\kappa d\bar{z}^2 + \overline{\delta\kappa} dz^2$ be a quadratic differential on E_u . A quadratic differential corresponds to a deformation of the metric by a change of the volume form

$$|dz|^2 \rightarrow |dz + \delta\kappa d\bar{z}|^2 + O(\delta\kappa^2) = |dw|^2 .$$

The new complex variable w on the torus is the solution to the differential equation $\partial_{\bar{z}}w = \delta\kappa$. The usual metric on the space of traceless two-tensors is

$$\|\delta\kappa d\bar{z}^2 + \overline{\delta\kappa} dz^2\|_{\text{Sym}^2(E_u)}^2 = \int_{E_u} d\bar{z} \wedge dz \delta\kappa \overline{\delta\kappa} = 8 \text{ vol}(E_u) |\delta\kappa|^2 .$$

The identification with a change in the coordinate on the u -plane is given by $\delta u = \text{vol}(E_u) \delta\kappa$. $H^1(E_u; K^{-1})$ is the space of Beltrami differentials on E_u . Its dual is the space of quadratic differentials $H^0(E_u; K^{\otimes 2})$. For the quadratic differential $\delta\kappa d\bar{z}^2 + \overline{\delta\kappa} dz^2$ the corresponding Beltrami differential is $\mu = \delta\kappa \partial_{\bar{z}} \otimes dz$. The change in the action corresponds to the evaluation of the quadratic differential

$$\int_{E_u} d\bar{z}^2 \mu = \delta\kappa \text{ vol}(E_u) = \delta u .$$

Thus, under a change of the regularization of the quantum theory on E_u , the action will change by a term proportional to δu . For $|u| > 1$, the flow generated by the monodromy matrix M_∞ defines a canonical trivialization of $H^0(E_u; K^{\otimes 2})$ as u varies. We showed in Sec. 3.5 that the invariant differential for M_∞ is du/u . Using the trivialization the additional source term in the action is proportional to

$$\int_{E_u} d\bar{z}^{\otimes 2} \mu = \frac{du}{u} = u .$$

Let α be the coordinate on the space of Beltrami differentials or its dual $H^0(E_u; K^{\otimes 2})$ given by writing the elements as $\alpha d\bar{z}^{\otimes 2}$. Multiplication of the action $\exp(-\mathbf{S})$ with $e^{2\alpha u}$ does not present any problems in the context of local anomalies. Since u is invariant under the action of $\Gamma_0(4)$, it does not give rise to a global anomaly.

It follows:

Lemma 39. *Changing the holomorphic structure and the regularization define a family of actions*

$$\mathbf{S}[\bar{\partial}] \mapsto \mathbf{S} \left[\bar{\partial} + \frac{2i \sqrt{2} \beta d\bar{z}}{\text{vol}(E_u)} \right] - 2\alpha u ,$$

parameterized by $H^0(E_u, K^{\otimes 2})$ and $H^{(0,1)}(E_u)$ with complex coordinates α and β . The change in the partition function (5.35) is

$$\mathbf{Z}_u \mapsto \mathbf{Z}_u(\alpha, \beta) = \mathbf{Z}_u e^{2\alpha u + \beta^2 \hat{T}} , \quad (5.39)$$

Proof. The modification $r \rightarrow \tilde{r} = r + \sqrt{2} \beta$ leads to

$$\begin{aligned} \mathbf{Z}^{CP^1}(\beta) &= \frac{1}{2\pi} \int_{CP^1} d^2 x_0 d^2 \hat{\lambda}_0 \frac{\text{vol}(E_u) \det'(\partial) \det \left(\bar{\partial} + \frac{2\pi i \tilde{r} d\bar{z}}{\text{vol}(E_u)} \right)}{\det' \left[-\partial \left(\bar{\partial} + \frac{2\pi i \tilde{r} d\bar{z}}{\text{vol}(E_u)} \right) \right]} \\ &= \int_{CP^1} d^2 x_0 d^2 \hat{\lambda}_0 \tilde{r} e^{\frac{\tilde{r}^2}{2} \hat{T}} = 2\pi^2 e^{\beta^2 \hat{T}} . \end{aligned}$$

□

Using Lemma 34, Lemma 35 and Eq. (3.17) it follows:

Proposition 17. *The one-loop amplitude of the heterotic σ -model on E_u with target space $CP^1 \times U(1)$ in the presence of the background fields α, β is*

$$\mathbf{Z}_u(\alpha, \beta) = C_0 \sqrt{\text{Im } \tau} \frac{\vartheta_4(\tau)^9 \overline{\vartheta_4(\tau)}}{h^4} \exp \left(2\alpha u + \beta^2 \hat{T}(u) \right) , \quad (5.40)$$

where C_0 is a constant and $h(\tau) = \frac{1}{2} \vartheta_2(\tau) \vartheta_3(\tau) = \omega(\tau)/(2\sqrt{2}\pi)$ and

$$\begin{aligned} T(u) &= -\frac{1}{24} \left(\frac{E_2}{h^2(\tau)} - 8u \right) , \\ \hat{T}(u) &= -\frac{1}{24} \left(\frac{\hat{E}_2}{h^2(\tau)} - 8u \right) , \quad \hat{E}_2 = E_2 - \frac{3}{\pi \text{Im } \tau} . \end{aligned}$$

Corollary 6. *The amplitude in Eq. (5.40) is the integrand of the u -plane integral in Eq. (1.1) if we set $\alpha = p$ and $\beta = S$ up to an overall factor of S .*

Remark 27. *The missing factor of S in Eq. (5.40) means that in the limit $S = 0$ and $p = 0$ the generating function $\mathbf{Z}_u(\alpha, \beta)$ gives the Euler characteristic of the loop space of CP^1 whereas the integrand of the u -plane integral in Eq. (1.1) vanishes.*

Remark 28. *The partition function in Eq. (5.40) has the same form as the anomaly generating functionals constructed in [95, 96]. Through the insertion of the terms proportional to β we have assembled in the partition function not only the massless, but also the massive levels. In the limit of vanishing β , the partition function in Eq. (5.40) reduces to the semi-classical path-integral derived in Sec. 5.4.*

Remark 29. *As explained in [96, Sec. 2], we have to integrate $\mathbf{Z}_u(\alpha, \beta)$ over $\text{Re}(\tau)$ from 0 to 4 to impose the left-right mass constraint. Ultimately, it means that only the massless level contributes since β acts only on the left-movers. We will carry out this step carefully in Eq. (5.44) of Sec. 5.6. If one imposes the left-right mass constraint, the partition function is*

$$\int_0^4 dx \mathbf{Z}_u(\alpha, \beta) , \quad (5.41)$$

with $\tau = x + iy$. It makes sense to ask what the limiting partition function is in the high temperature limit $\text{Im } \tau \rightarrow \infty$. We will show in Sec. 5.6 that using the ζ -function regularization the limit of Eq. (5.41) is well-defined and gives the u -plane integral.

5.5.3 The blow-up function of Fintushel and Stern

We explain an interesting modification of this procedure. We can modify the connection of one of the complex chiral fermion that gives a contribution $\det(\not{\partial}_{L,u}^+)$ by the holomorphic tangent vector γ . We interpret the modification as the gauging of the $U(1)$ in Sec. 5.3 and the coupling to an external magnetic field γ . It follows:

Proposition 18. *The one-loop amplitude of the heterotic σ -model on E_u with target space $CP^1 \times U(1)$ in the presence of the background fields α , β , and γ is*

$$\widetilde{\mathbf{Z}}_u(\alpha, \beta, \gamma) = \mathbf{Z}_u(\alpha, \beta) \frac{\det \not{\partial}_\xi^+(2i\sqrt{2}\gamma)}{\det \not{\partial}_\xi^+}.$$

yields

$$\widetilde{\mathbf{Z}}_u(\alpha, \beta, \gamma) = \mathbf{Z}_u(\alpha, \beta) e^{-\frac{u\gamma}{12}} \sigma_3(\gamma | \omega, \omega'). \quad (5.42)$$

If we set $\gamma = E$ for $E = \int_{\mathcal{E}} \zeta$, then $\widetilde{\mathbf{Z}}_u(\alpha, \beta, \gamma)$ is the integrand for the u -plane integral that gives the Donaldson invariants for the blow-up of CP^2 in one point in Eq. (2.10).

Proof. Lemma 34 and the comparison with [5, Thm. 4.1] and [25, Sec. 6] shows that Eq. (5.42) is the integrand for the generating function of the Donaldson invariant for CP^2 blown up in one point. \square

5.6 The regularization of the u -plane integral

The u -plane integral is

$$C_0 \int_{H/\Gamma_0(4)} \frac{|d\tau|^2}{\text{Im}(\tau)^{\frac{3}{2}}} \frac{\vartheta_4(\tau)^9 \overline{\vartheta_4(\tau)}}{h^4} S \exp\left(2pu + S^2 \widehat{T}(u)\right), \quad (5.43)$$

where $h(\tau)$ and $\widehat{T}(u)$ were defined in Prop. 17. In this section, we explain how the integral in Eq. (5.43) is regularized.

Eq. (5.43) is a generating function in the formal variables p and S . If we set

$$\text{UP} = C_0 \sum_{m,n=0}^{\infty} \frac{p^m S^{2n+1}}{m! (2n+1)!} \int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}^2 \tau} \text{UP}^{(m,n)},$$

the expansion of Eq. (5.43) yields

$$\begin{aligned} \mathbb{UP} = \sum_{m,n=0}^{\infty} C_0 \frac{p^m S^{2n+1}}{m! (2n+1)!} \int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}^2 \tau} \sqrt{\text{Im} \tau} \frac{(2n+1)!}{n!} \\ \times 2^m u^m \widehat{T}^n(u) \left(\frac{\vartheta_4^9(\tau)}{h^4} \right) \overline{\vartheta_4(\tau)}. \end{aligned}$$

Thus, we will be concerned with the integration of the functions

$$\mathbb{UP}^{(m,n)} = \frac{(2n+1)!}{n!} 2^m \sqrt{\text{Im} \tau} u^m \widehat{T}^n(u) \left(\frac{\vartheta_4^9(\tau)}{h(\tau)^4} \right) \overline{\vartheta_4(\tau)}.$$

The integrals are in general divergent because of the behavior of $\mathbb{UP}_{m,n}$ at $u = \pm 1, \infty$. Since we will be integrating over a part of the fundamental domain of $\Gamma_0(4)$, it is useful to introduce the variables $\tau = x + iy$. We are only interested in the contribution coming from the upper part of the fundamental domain for $y > y_0 > 1$. We observe that the x -integration then runs from 0 to 4 and thus

$$\int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}^2 \tau} = \int^{\text{reg}} \frac{dy}{y^2} \int_0^4 dx.$$

Lemma 40. $\int_0^4 dx \mathbb{UP}^{(m,n)} = 0$ for $m + n \equiv 1(2)$.

Proof. The shift $x \rightarrow x + 2$ is the modular transformation $\tau \rightarrow \tau + 2$. In Sec. 3.4, we have shown that

$$\tau \rightarrow \tau + 2, \quad u \rightarrow -u, \quad \begin{array}{l} T(u) \rightarrow -T(u) \\ \widehat{T}(u) \rightarrow -\widehat{T}(u) \end{array}, \quad \begin{array}{l} \vartheta_4(\tau) \rightarrow \vartheta_4(\tau) \\ h^4(\tau) \rightarrow h^4(\tau) \end{array}.$$

Changing $u \rightarrow -u$ is equivalent to changing $p \rightarrow -p$, and $S \rightarrow \pm iS$ and multiplying the integrand with $\mp i$. \square

The next step is to provide an expansion of the integrand in Eq. (5.43) in terms of q, \bar{q}, y .

Lemma 41. *The following relation holds:*

$$u^m \widehat{T}^n \frac{\vartheta_4^9(\tau)}{h^4} = \sum_{l=0}^n \frac{1}{y^l} \binom{n}{l} \frac{1}{(8\pi)^l} q^{\frac{n-m}{4} - \frac{l+1}{2}} \mathbf{R}^{(m,n,l)}(q),$$

where the function \mathbf{R} is holomorphic in $q^{\frac{1}{2}}$ and equals

$$\begin{aligned} \mathbf{R}^{(m,n,l)}(q) &= q^{-\frac{n-m}{4} + \frac{l+1}{2}} \frac{u^m T^{n-l} \vartheta_4^9(\tau)}{h^{2l+4}} = \sum_{k=0}^{\infty} \mathbf{R}_k^{(m,n,l)} q^{\frac{k}{2}}, \\ \text{with } \mathbf{R}_k^{(m,n,l)} &= \text{Coeff}_{q^{\frac{k}{2}}} \left(q^{-\frac{n-m}{4} + \frac{l+1}{2}} \frac{u^m T^{n-l} \vartheta_4^9(\tau)}{h^{2l+4}} \right), \end{aligned}$$

Proof. We compute

$$\begin{aligned} u^m \widehat{T}^n \frac{\vartheta_4^9(\tau)}{h^4} &= \frac{u^m \vartheta_4^9(\tau)}{h^4} \left(T + \frac{1}{8\pi h^2} \frac{1}{y} \right)^n \\ &= \sum_{l=0}^n \frac{1}{y^l} \binom{n}{l} \frac{1}{(8\pi)^l} \frac{u^m T^{n-l} \vartheta_4^9(\tau)}{h^{2l+4}}. \end{aligned}$$

For $q = \exp(2\pi i\tau)$ all functions have absolute convergent power series expansion holomorphic in $q^{\frac{1}{2}}$. In particular, we have

$$\begin{aligned} T &= q^{\frac{1}{4}} \sum_{k=0}^{\infty} t_k q^{\frac{k}{2}}, & h &= q^{\frac{1}{8}} \sum_{k=0}^{\infty} h_k q^{\frac{k}{2}}, \\ u &= q^{-\frac{1}{4}} \sum_{k=0}^{\infty} u_k q^{\frac{k}{2}}, & \vartheta_4(\tau) &= \sum_{k=0}^{\infty} (-1)^k q^{\frac{k^2}{2}}. \end{aligned}$$

It follows that

$$u^m \widehat{T}^n \frac{\vartheta_4^9(\tau)}{h^4} = \sum_{l=0}^n \frac{1}{y^l} \binom{n}{l} \frac{1}{(8\pi)^l} q^{\frac{n-m}{4} - \frac{l+1}{2}} \mathbf{R}(q).$$

□

Thus, the integrand $\mathbf{UP}^{(m,n)}$ can be written as an absolute convergent power series

$$\begin{aligned} &u^m \widehat{T}^n(u) \left(\frac{\vartheta_4^9(\tau)}{h^4} \right) \overline{\vartheta_4(\tau)} \\ &= \sum_{l=0}^n \frac{1}{y^l} \binom{n}{l} \frac{1}{(8\pi)^l} \sum_{r=0}^{\infty} \mathbf{R}_r^{(m,n,l)} \sum_{s=0}^{\infty} (-1)^s q^{\frac{n-m}{4} - \frac{l+1}{2} + \frac{r}{2}} \bar{q}^{\frac{s^2}{2}}. \end{aligned}$$

The integration with respect to x yields

$$\int_0^4 dx q^{\frac{a}{2}} \bar{q}^{\frac{b}{2}} = \begin{cases} 4 e^{-2\pi a y} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}.$$

The integration with respect to x is a projection onto the diagonal terms in $\mathbf{UP}^{(m,n)}$, i.e., the terms with the same power in q and \bar{q} . As the section only depends on q, \bar{q}, y the integration will remove the dependence on x . If we define

$$\begin{aligned} 4 \mathbf{UP}_{\text{diag}}^{(m,n)}(iy) &= \int_0^4 dx \mathbf{UP}^{(m,n)} \\ &= \int_0^4 dx \frac{(2n+1)!}{n!} 2^m \sqrt{\text{Im } \tau} u^m \widehat{T}^n(u) \left(\frac{\vartheta_4^9(\tau)}{h^4} \right) \overline{\vartheta_4(\tau)}, \end{aligned}$$

we write

$$4 \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n)}(iy) = \frac{2^{m+1} (2n+1)!}{n!} \sum_{l=0}^n \binom{n}{l} \frac{1}{(8\pi)^l} \sum_{s=0}^{\infty} \mathbf{R}_{s^2+l+1-\frac{n-m}{2}}^{(m,n,l)} (-1)^s \Theta \left(s^2 + l + 1 - \frac{n-m}{2} \right) \frac{e^{-2\pi s^2 y}}{y^{l-\frac{1}{2}}}, \quad (5.44)$$

where for $k \in \mathbb{Z}$ we have

$$\Theta(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The following lemma follows:

Lemma 42. *The diagonal term $\mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n)}(iy)$ can be decomposed*

$$\begin{aligned} \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n)}(iy) &= \left(\mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n)}(iy) \right)_{\text{sing}} + \left(\mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n)}(iy) \right)_{\text{reg}} \\ &= \sum_{l=0}^n \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l,0)} \frac{1}{y^{l-\frac{1}{2}}} + \sum_{l=0}^n \frac{1}{y^{l-\frac{1}{2}}} \sum_{s=1}^{\infty} \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l,s)} e^{-2\pi s^2 y}, \end{aligned}$$

with

$$\mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l,s)} = \frac{2^{m-1} (2n+1)! (-1)^s}{(8\pi)^l l! (n-l)!} \mathbf{R}_{(s^2+l+1-\frac{n-m}{2})}^{(m,n,l)} \Theta \left(s^2 + l + 1 - \frac{n-m}{2} \right). \quad (5.45)$$

Remark 30. *The singular part contains only a finite number of terms which are not exponentially suppressed. For l fixed, we have the cusp form*

$$\mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l)}(iy) = \sum_{s=0}^{\infty} \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l,s)} e^{-2\pi s^2 y},$$

such that

$$\mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n)}(iy) = \sum_{l=0}^n \frac{1}{y^{l+\frac{1}{2}}} 4 \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l)}(iy).$$

In the remainder of this section we explain the regularization of the u -plane integral.

5.6.1 The ζ -function regularization.

By projecting onto the diagonal terms we obtain the functions

$$f_{m,n}(y) = \int_0^4 dx \frac{\mathbb{U}\mathbb{P}^{(m,n)}}{y} = \sum_{l=0}^n \frac{1}{y^{l+\frac{3}{2}}} \sum_{s \geq 1} \mathbb{U}\mathbb{P}_{\text{diag}}^{(m,n),(l,s)} e^{-2\pi s^2 y}. \quad (5.46)$$

The associated Dirichlet series are

$$\mathfrak{D}_{\Lambda_0}(\epsilon, f_{m,n}) = \frac{1}{\Gamma(\epsilon)} \int_{\Lambda_0}^{\infty} f_{m,n}(y) y^{\epsilon-1} dy .$$

The cut-off Λ_0 is introduced to only include the contributions from the cusp $u \rightarrow \infty$ while keeping away from $u \rightarrow \pm 1$. We define the ζ -function regularization of the u -plane integral as

$$\lim_{\Lambda_0 \rightarrow 0} \mathfrak{D}'_{\Lambda_0}(0, f_{m,n}) . \quad (5.47)$$

In [25, Sec. 9.1] a certain regularization procedure was applied: the u -plane integral was evaluated by writing the integrand as total divergence and then picking up the term at infinity. One uses integration by parts for

$$\frac{\partial}{\partial \bar{\tau}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) .$$

We will only consider the boundary component coming from the line segment with x running from 0 to 4 at $y = y_0$. The other boundary components are the singular contributions from $y \rightarrow 0$. Thus, the regularization procedure used in [25, Sec. 9.1] is defined by

$$\int_{H/\Gamma_0(4)}^{\text{reg}} |d\tau|^2 \frac{\partial}{\partial \bar{\tau}} F(x, y) = \frac{i}{2} \lim_{y_0 \rightarrow \infty} \int_0^4 dx F(x, y_0) .$$

Applied to the u -plane integral, one uses the identity

$$\frac{\partial}{\partial \bar{\tau}} \mathbb{E}^{(l)} = \frac{\overline{\vartheta_4(\tau)}}{y^{\frac{3}{2}}} \widehat{E}_2^l ,$$

with

$$\mathbb{E}^{(l)} = \sum_{j=0}^l \binom{l}{j} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} + l)} \left(\frac{6i}{\pi} \right)^j E_2^{l-j} \frac{\partial^j}{\partial \tau^j} \mathcal{H}(\tau) + \sum_{\substack{k \geq 0 \\ \text{finite}}} \frac{C_k}{y^{k+\frac{1}{2}}}$$

for the non-holomorphic Hurwitz function $\mathcal{H}(\tau)$ of weight $(\frac{3}{2}, 0)$. The terms in the u -plane integral are of the form

$$\begin{aligned} \int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}^{\frac{3}{2}} \tau} \frac{\vartheta_4^9(\tau)}{h(\tau)^4} \frac{\widehat{E}_2^j}{h^{2j}(\tau)} u^{m-n-j} \overline{\vartheta_4(\tau)} &= \int_{H/\Gamma_0(4)}^{\text{reg}} |d\tau|^2 \frac{\vartheta_4^9(\tau) u^{m-n-j}}{h(\tau)^{4+2j}} \frac{\partial}{\partial \bar{\tau}} \mathbb{E}^{(l)}[\mathcal{H}] \\ &= \int_{H/\Gamma_0(4)}^{\text{reg}} |d\tau|^2 \frac{\partial}{\partial \bar{\tau}} \left(\frac{\vartheta_4^9(\tau) u^{m-n-j}}{h(\tau)^{4+2j}} \mathbb{E}^{(l)}[\mathcal{H}] \right) . \end{aligned} \quad (5.48)$$

Thus, we obtain

$$\begin{aligned} \int_{H/\Gamma_0(4)}^{\text{reg}} |d\tau|^2 \frac{\partial}{\partial \bar{\tau}} \left(\frac{v_4^9(\tau) u^{m-n-j}}{h(\tau)^{4+2j}} \mathbb{E}^{(l)}[\mathcal{H}] \right) &= \frac{i}{2} \lim_{y_0 \rightarrow \infty} \int_0^4 dx \frac{v_4^9(\tau) u^{m-n-j}}{h(\tau)^{4+2j}} \mathbb{E}^{(l)}[\mathcal{H}] \\ &= 2i \lim_{y_0 \rightarrow \infty} \text{Coeff}_{q^0} \left\{ \frac{v_4^9(\tau) u^{m-n-j}}{h(\tau)^{4+2j}} \mathbb{E}^{(l)}[\mathcal{H}] \right\}. \end{aligned} \quad (5.49)$$

Lemma 43. *The regularization procedure applied by Moore and Witten in [25] is up to a constant factor the ζ -function regularization in Eq. (5.47).*

Proof. Using the incomplete Γ -function [40, Eqs. (6.5.3) and (6.5.12)], we compute for $l \in \mathbb{N}$

$$\begin{aligned} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \frac{1}{\Gamma(\epsilon)} \int_{\Lambda_0}^{\infty} dy y^{\epsilon-1} \frac{e^{-2\pi s^2 y}}{y^{l+\frac{1}{2}}} &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \frac{(2\pi s^2)^{l-\epsilon+\frac{1}{2}} \Gamma(\epsilon - l - \frac{1}{2}, 2\pi s^2 \Lambda_0)}{\Gamma(\epsilon)} \\ &= (2\pi s^2)^{l+\frac{1}{2}} \Gamma(-l - \frac{1}{2}, 2\pi s^2 \Lambda_0), \end{aligned}$$

and thus

$$\lim_{\Lambda_0 \rightarrow 0} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \frac{1}{\Gamma(\epsilon)} \int_{\Lambda_0}^{\infty} dy y^{\epsilon-1} \frac{e^{-2\pi s^2 y}}{y^{l+\frac{1}{2}}} = (2\pi s^2)^{l+\frac{1}{2}} \Gamma(-l - \frac{1}{2}).$$

We also obtain

$$\lim_{\Lambda_0 \rightarrow 0} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \frac{1}{\Gamma(\epsilon)} \int_{\Lambda_0}^{\infty} dy y^{\epsilon-1} \frac{1}{y^{l+\frac{1}{2}}} = 0.$$

An integral function F for $\frac{e^{-2\pi s^2 y}}{y^{l+\frac{3}{2}}}$ is

$$\begin{aligned} F(y) &= \int \frac{e^{-2\pi s^2 y}}{y^{l+\frac{3}{2}}} dy = (2\pi s^2)^l \int \frac{e^{-\tilde{y}}}{\tilde{y}^{l+\frac{3}{2}}} d\tilde{y} \\ &= (2\pi s^2)^l \left[\Gamma\left(-l - \frac{1}{2}\right) - \Gamma\left(-l - \frac{1}{2}, 2\pi s^2 y\right) \right]. \end{aligned}$$

Thus, $f(y) = \frac{\partial}{\partial y} F(y) = \frac{e^{-2\pi s^2 y}}{y^{l+\frac{3}{2}}}$ and we obtain

$$\begin{aligned} \int_{H/\Gamma_0(4)}^{\text{reg}} |d\tau|^2 f(y) &= \int_{H/\Gamma_0(4)}^{\text{reg}} |d\tau|^2 \frac{\partial}{\partial \bar{\tau}} F(y) = \frac{i}{2} \lim_{y_0 \rightarrow \infty} \int_0^4 dx F(y_0) \\ &= 2i \lim_{y_0 \rightarrow \infty} F(y_0) = 2i (2\pi s^2)^l \Gamma\left(-l - \frac{1}{2}\right). \end{aligned}$$

□

We summarize the results of this section:

Proposition 19. *The u -plane integral is the generating function*

$$\mathbb{UP} = C_0 \sum_{m,n=0}^{\infty} \frac{\alpha^m \beta^{2n+1}}{m! (2n+1)!} \int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}^2 \tau} \mathbb{UP}^{(m,n)}.$$

with

$$\mathbb{UP}^{(m,n)} = \frac{(2n+1)!}{n!} 2^m \sqrt{\text{Im} \tau} u^m \widehat{T}^n(u) \left(\frac{\vartheta_4(\tau)}{\omega^4} \right) \overline{\vartheta_4(\tau)}.$$

The regularization used in [25] coincides with the ζ -function regularization

$$\int_{H/\Gamma_0(4)}^{\text{reg}} \frac{|d\tau|^2}{\text{Im}^2 \tau} \mathbb{UP}^{(m,n)} = \frac{i}{2} \lim_{\Lambda_0 \rightarrow 0} \mathfrak{D}'_{\Lambda_0}(0, f_{m,n}) \quad (5.50)$$

where $f_{m,n}$ was defined in Eq. (5.46). The only contributions to this integral come from the exponentially suppressed terms

$$\frac{\left(\mathbb{UP}_{\text{diag}}^{(m,n)}(iy) \right)_{\text{reg}}}{y} = \sum_{l=0}^n \frac{1}{y^{l+\frac{1}{2}}} \sum_{s=1}^{\infty} \mathbb{UP}_{\text{diag}}^{(m,n),(l,s)} e^{-2\pi s^2 y}$$

with the coefficients $\mathbb{UP}_{\text{diag}}^{(m,n),(l,s)}$ defined in Eq. (5.45).

In the appendix, we present a short MAPLE-program that computes the Donaldson invariants of CP^2 using the methods of [25] to evaluate Eq. (5.43).

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Chapter 6

The conclusion and an open question

In this thesis, we have found an equality between the u -plane integral that computes the Donaldson invariants on CP^2 and its blow-ups and the one-loop amplitude in a heterotic string field theory on the elliptic curve at the boundary of the Coulomb branch. We conjecture that this equality stems from a superconformal quantum theory on the six-dimensional space $X = \mathbb{T}^2 \times M$ in the two limits where \mathbb{T}^2 is either much smaller or much bigger than any characteristic radius of M . The six-dimensional theory would serve as a F-theory and different compactifications lead to different low-energy field theories. We briefly describe the bosonic field content of the quantum theory in the two limits.

Interpretation as combined gauge and gravitational anomaly. First, we describe the partition function for the \mathbb{T}^2 -invariant supersymmetric $U(1)$ -gauge theory on X . The universal $U(1)$ -gauge theory on M that defines the u -plane integral is naturally defined as a $N = 1$ supersymmetric topological $U(1)$ -gauge theory on the six-dimensional space $X = \mathbb{T}^2 \times M$ that combines the gravitational and gauge anomaly.

Let $\mathcal{A}^{\mathbb{T}^2}$ be the space of \mathbb{T}^2 -invariant connections $H = A + \phi d\xi + \bar{\phi} d\bar{\xi}$ on X where ξ is the complex coordinate on the two-torus \mathbb{T}^2 and A is a connection on $L \rightarrow M$, and ϕ is a complex function on M . Then $L \times \mathcal{A}^{\mathbb{T}^2}$ is the universal line bundle over $M \times \mathbb{T}^2 \times \mathcal{A}^{\mathbb{T}^2}$. The bosonic action is the L^2 -norm of the curvature G for the \mathbb{T}^2 -invariant connection H is

$$\begin{aligned} G &= F_A + d\phi \wedge d\xi + d\bar{\phi} \wedge d\bar{\xi}, \\ \int_{M \times \mathbb{T}^2} \text{vol}_6 |G|^2 &= \int_M \text{vol}_M \text{Im } \tau \left(|F_A|^2 + |d\phi|^2 \right), \end{aligned} \quad (6.1)$$

where the torus is oriented and has an area of $\text{Im } \tau$. The action in Eq. (6.1) is the bosonic Lagrange density used in the low-energy effective field theory in Sec. 4.2.

Let \mathbb{T}^2 be equipped with a spin structure. On \mathbb{T}^2 , we consider the space of flat metrics $\text{Met}(\mathbb{T}^2)$ and the principal $\text{Spin}(2)$ -bundle $\mathfrak{F} \rightarrow \text{Met}(\mathbb{T}^2) \times \mathbb{T}^2$ of Spin-frames

for the metrics on \mathbb{T}^2 . Restricting to the group of diffeomorphisms $\text{Diff}(\mathbb{T}^2)$ which preserves the Spin-structure, the action lifts to an action on \mathfrak{F} . This is the context of the gravitational anomaly in [111]. The combination of the two constructions defines a universal $\text{Spin}(2) \times U(1)$ -principal bundle \mathcal{L} in the commuting diagram

$$\begin{array}{ccc} \mathfrak{F} \times L \times \mathcal{A}^{\mathbb{T}^2} & \longrightarrow & \mathcal{L} = (\mathfrak{F} \times L \times \mathcal{A}^{\mathbb{T}^2})/\mathcal{H} \\ \downarrow & & \downarrow \\ \text{Met}(\mathbb{T}^2) \times \mathbb{T}^2 \times M \times \mathcal{A}^{\mathbb{T}^2} & \longrightarrow & \mathfrak{M} = (\text{Met}(\mathbb{T}^2) \times \mathbb{T}^2 \times M \times \mathcal{A}^{\mathbb{T}^2})/\mathcal{H} \end{array}$$

together with a canonical connection. \mathcal{H} is the group of all \mathbb{T}^2 -invariant automorphisms of the bundle L leaving a base point fixed. The action of the automorphisms is diagonal since ϕ is an ordinary function on M . We obtain $\mathfrak{M} = (\text{Met}(\mathbb{T}^2) \times \mathbb{T}^2) / \text{Diff}_0(\mathbb{T}^2) \times M \times \mathcal{A} / \mathcal{G}_0$. The last factor is contractible since M is simply connected. If we choose as the Spin-structure on \mathbb{T}^2 the unique $\Gamma_0(4)$ -invariant Spin-structure (P, A) we obtain in fact the Seiberg-Witten curve times the manifold M . This shows that the presence of the electro-magnetic duality in the low-energy effective field theory can be understood in the context of the combined gauge and gravitational anomaly on X .

Turning to the fermionic part of the action, it is known that the dimensional reduction of the full, untwisted, high-energy, $N = 1$ supersymmetric Yang-Mills action on $\mathbb{T}^2 \times \mathbb{R}^4$ gives the high energy, $N = 2$ supersymmetric Lagrangian on \mathbb{R}^4 [13]. Thus, on a Spin-manifold M the Lagrangian of the untwisted $N = 2$ supersymmetric Lagrangian has a natural interpretation as dimensional reduction of the $N = 1$ supersymmetric Yang-Mills action on $\mathbb{T}^2 \times \mathbb{R}^4$. However, we cannot assume that M is a Spin-manifold and have to use a description which is independent of the Spin-structure. In physics, this is a topological twisted. The Spin^c -structure on M combines with the Spin-structure on \mathbb{T}^2 and determines the complex chiral spinor bundles $S_{6,\mathbb{C}}^\pm$ on X . We ask whether the $N = 2$ supersymmetric Lagrangian on M in Sec. 3 is the dimensional reduction of a twisted, $N = 1$ supersymmetric theory on X with both gauge and gravitational coupling.

Interpretation as a theory of 3-forms. We also describe the partition function for a theory of selfdual three-forms on X which was defined in [114, 115]. This theory is the three-form analogue of the $U(1)$ -gauge theory on the six-dimensional manifold $X = M \times \mathbb{T}^2$ [112, 113]. There is no Lagrangian whose critical points are the self-dual three-forms H with $*H = iH$ [115]. However, there is the action

$$\mathbf{S} = \frac{1}{2\pi} \int_{M \times \mathbb{T}^2} H \wedge *H \quad (6.2)$$

which is conformally invariant since the Hodge star-operator is conformally invariant in the middle dimension.

A differential character β is a two-form modulo gauge invariance $\beta \rightarrow \beta + d\epsilon^{(1)}$ [114]. The curvature $H = d\beta$. β is the de Rham representative of a characteristic class

that takes values in $H^3(X; \mathbb{Z})$. Using the product metric on X we identify $H^3(X; \mathbb{R})$ with the harmonic forms, and define a Hodge star-operator with eigenvalues $\pm i$. There is a Lagrangian decomposition $\Gamma = \Gamma_1 \oplus \Gamma_2$ of the lattice of $H^3(X; \mathbb{Z})$ in $H^3(X; \mathbb{R})$ with $\Gamma_1 = H^2(M; \mathbb{Z}) \otimes \rho$ and $\Gamma_2 = H^2(M; \mathbb{Z}) \otimes \sigma$ where ρ, σ comprise the dual basis of the choice of the A and B -cycle on \mathbb{T}^2 in Sec. 3. The lattice Γ has a skew-form $(x, y) = \int_X x \wedge y$. The skew form plus the metric determine a complex structure on $H^3(X, U(1))$. For $M = CP^2$ with the Kähler form ω introduced in Sec. 4.4, we can decompose every element $\gamma \in \Gamma$ as $\gamma = n_1(\omega \otimes \rho) + n_2(\omega \otimes \sigma)$ and define a quadratic function $q : \Gamma \rightarrow \mathbb{R}/\mathbb{Z}$ by

$$q(\gamma) = \int_{M \times \mathbb{T}^2} \gamma_1 \wedge \gamma_2 ,$$

whence $q(x + y) = q(x) + q(y) + (x, y)$. Since $\int_M \omega \wedge \omega = 1$, it follows that

$$\langle \omega \otimes \alpha, \omega \otimes \alpha \rangle = 2 \operatorname{Im} \tau .$$

Moreover, any other quarf differs from q by any element $\delta \in \Gamma \otimes \mathbb{Z}/2$. Let $L_q \rightarrow H^3(X, U(1))$ be the line bundle whose curvature is the skew-form (x, y) . The partition function is the path integral

$$\mathbf{Z} = \int \mathcal{D}\beta e^{-\mathbf{S} + \frac{i}{2}(\delta, d\beta)} \quad (6.3)$$

over the group of differential characters β . It was shown in [115, Eq. (1.7)] that the path integral in Eq. (6.3) is

$$\mathbf{Z} = \left(\sum_{\gamma \in \Gamma} e^{-\frac{1}{2}\langle \gamma, \gamma \rangle + \frac{i}{2}(\delta, \gamma)} \right) \frac{\operatorname{vol} H^2(X; \mathbb{R}/\mathbb{Z})}{\sqrt{\det' d^* d|_{\Omega^2/\Omega_{\text{cl}}^2}}},$$

where

$$\operatorname{vol} H^2(X; \mathbb{R}/\mathbb{Z}) = \operatorname{vol} (H^0(\mathbb{T}^2) \otimes H^2(M) \otimes \mathbb{R}/\mathbb{Z}) = (\operatorname{vol} \mathbb{T}^2)^{\frac{1}{2}} \operatorname{vol} H^2(M) .$$

We can use the Poisson resummation to evaluate the instanton part of the partition function. If we choose $\delta = \frac{1}{2}\omega \otimes \rho$, we obtain

$$\mathbf{Z} = \sqrt{\operatorname{Im} \tau} |\vartheta_4(\tau)|^2 \frac{\sqrt{\operatorname{vol} \mathbb{T}^2} \operatorname{vol} H^2(M)}{\sqrt{\det' d^* d|_{\Omega^2/\Omega_{\text{cl}}^2}}} .$$

If we restrict $\Omega^2(X)/\Omega_{\text{cl}}^2(X)$ to the subspace of the two-forms $f\omega$ where $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ is a non-constant function on \mathbb{T}^2 , the partition function is proportional to

$$\sqrt{\operatorname{Im} \tau} |\vartheta_4(\tau)|^2 \frac{\sqrt{\operatorname{vol} \mathbb{T}^2}}{\sqrt{\det' \Delta|_{\mathbb{T}^2}}} .$$

which is equal to the bosonic contribution we obtained in Sec. 5.3. In the limit with \mathbb{T}^2 much bigger than any characteristic radius of M , the partition function in Eq. (6.3) reduces to the bosonic part of the partition function on the elliptic curve at the boundary of the Coulomb branch for $\text{Im } \tau \rightarrow \infty$. On the other hand, Witten [116] has argued that in the limit where \mathbb{T}^2 is much smaller than any characteristic radius of M the important modes of H are invariant under translations of \mathbb{T}^2 , and the theory is reduced to a $U(1)$ -gauge theory on M with coupling $\text{Im } \tau$.

However, to obtain the full six-dimensional twisted topological field theory, certain fields must be added for the topological gauge fixing. Following [116], in dimension six there is for each choice of a simply-laced gauge group G a superconformal field theory that is a nonlinear supersymmetric version of the self-dual theory of three-forms in Eq. (6.2). We ask whether the partition function of the superconformal six-dimensional theory can be computed and the isomorphisms relating the different quantum theories represented by the arrows in the diagram (6.4) can be determined.

$$\begin{array}{ccc}
 \text{superconformal field theory} & \longrightarrow & N = 2 \text{ low-energy effective} \\
 \text{on } CP^2 \times E \text{ for } SU(2) & & U(1)\text{-gauge theory on } CP^2 \\
 & & \text{with } \tau = \tau(E) \\
 \downarrow & & \\
 \text{heterotic } \sigma\text{-model on } E & & \\
 \text{to } SU(2)/U(1) \times U(1) & &
 \end{array} \tag{6.4}$$

Appendix A

The Donaldson invariants for CP^2

A.1 The MAPLE-program

```
> with(LinearAlgebra):
> # Definition of the Eisenstein series in  $q^{1/2}$ 
> EE:=[1, -24, -72, -96, -168, -144, -288, -192, -360, -312, -432,
-288, -672, -336, -576, -576, -744, -432, -936, -480, -1008,
-768, -864, -576, -1440, -744, -1008, -960, -1344, -720, -1728,
-768, -1512, -1152, -1296, -1152, -2184, -912, -1440, -1344,
-2160, -1008, -2304, -1056, -2016, -1872, -1728]:
> EE1:=convert(EE, Matrix):
> NE:=ColumnDimension(EE1)-1:
> E2:=sum(EE[i+1]*q^i,i=0..NE);

E2 := 1 - 24 q - 72 q2 - 96 q3 - 168 q4 - 144 q5 - 288 q6 - 192 q7
- 360 q8 - 312 q9 - 432 q10 - 288 q11 - 672 q12 - 336 q13 - 576 q14
- 576 q15 - 744 q16 - 432 q17 - 936 q18 - 480 q19 - 1008 q20 - 768 q21
- 864 q22 - 576 q23 - 1440 q24 - 744 q25 - 1008 q26 - 960 q27 - 1344 q28
- 720 q29 - 1728 q30 - 768 q31 - 1512 q32 - 1152 q33 - 1296 q34 - 1152 q35
- 2184 q36 - 912 q37 - 1440 q38 - 1344 q39 - 2160 q40 - 1008 q41
- 2304 q42 - 1056 q43 - 2016 q44 - 1872 q45 - 1728 q46

> # Hurwitz polynomial coefficients
> H:=[-1/12,0,0,1/3,1/2,0,0,1,1,0,0,1,4/3,0,0,2,3/2,0,0,1,2,0,0,
3,2,0,0,4/3,2,0,0,3,3,0,0,2,5/2,0,0,4,2]:
> H1:=convert(H, Matrix):
> N:=ColumnDimension(H1)-1:
> Hurwitz:=sum(H[i+1]*q^i,i=0..N);
```

$$\begin{aligned}
\text{Hurwitz} &:= -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + q^7 + q^8 + q^{11} + \frac{4}{3}q^{12} + 2q^{15} \\
&+ \frac{3}{2}q^{16} + q^{19} + 2q^{20} + 3q^{23} + 2q^{24} + \frac{4}{3}q^{27} + 2q^{28} + 3q^{31} \\
&+ 3q^{32} + 2q^{35} + \frac{5}{2}q^{36} + 4q^{39} + 2q^{40}
\end{aligned}$$

```

> # Moore/Witten's function H_(infinity,0);
> HH:=sum(H[4*i+1]*q^(2*i),i=0..floor(N/4))
+1/2*sum(H[4*i+1]*q^(i/2)*(-1)^i,i=0..floor(N/4))
-sum(H[4*i]*q^(2*i-1/2),i=1..floor(N/4));

```

$$\begin{aligned}
HH &:= \frac{q}{2} + \frac{5q^2}{4} + q^3 + \frac{5q^4}{2} + q^5 + \frac{4q^6}{3} + \frac{3q^8}{2} + 2q^{10} + 2q^{12} \\
&+ 2q^{14} + 3q^{16} + \frac{5q^{18}}{2} + 2q^{20} - \frac{\sqrt{q}}{4} - q^{(3/2)} - q^{(5/2)} - 2q^{(7/2)} \\
&- \frac{5q^{(9/2)}}{4} - q^{(11/2)} - 2q^{(15/2)} - q^{(19/2)} - 3q^{(23/2)} - \frac{4q^{(27/2)}}{3} \\
&- 3q^{(31/2)} - 2q^{(35/2)} - 4q^{(39/2)} - \frac{1}{8}
\end{aligned}$$

```

> # Moore/Witten's function H_(infinity,0)(q,j)
> HHH:=proc(j)
local Z,k; global HH;
Z:=collect(HH,q);
for k from 1 to j do Z:=q*diff(Z,q) end do;
Z:=map(simplify,collect(expand(Z),q));
end proc:

```

```

> #
> assume(q>0):
> Koeff:=proc(Z)
local A; global q;
A:=convert(series(Z,q),polynom);
subs(q=0,map(simplify,
expand(A-convert(series(A,q=0,0),polynom))));
end proc:

```

```

> # Defining the functions h,u,T
> h:=1/2*JacobiTheta2(0,(q)^(1/2))*JacobiTheta3(0,(q)^(1/2));

```

$$h := \frac{1}{2} \text{JacobiTheta2}(0, \sqrt{q}) \text{JacobiTheta3}(0, \sqrt{q})$$

```

> u:=1/2*(JacobiTheta2(0,(q)^(1/2))^4+JacobiTheta3(0,(q)^(1/2))^4)
/(JacobiTheta2(0,(q)^(1/2))*JacobiTheta3(0,(q)^(1/2)))^2;

```

$$u := \frac{1}{2} \frac{\text{JacobiTheta2}(0, \sqrt{q})^4 + \text{JacobiTheta3}(0, \sqrt{q})^4}{\text{JacobiTheta2}(0, \sqrt{q})^2 \text{JacobiTheta3}(0, \sqrt{q})^2}$$

> T:=-1/24*(E2/h^2-8*u):

> series(T,q);

$$\begin{aligned} & q^{-(1/4)} - 2q^{-(3/4)} + 6q^{-(5/4)} - 16q^{-(7/4)} + 37q^{-(9/4)} \\ & - 78q^{-(11/4)} + 158q^{-(13/4)} - 312q^{-(15/4)} + 594q^{-(17/4)} \\ & - 1090q^{-(19/4)} + 1952q^{-(21/4)} + O(q^{-(23/4)}) \end{aligned}$$

> Z:=(x,t,j)->JacobiTheta4(0,(q)^(1/2))^9
/h^(4+2*j)*T^(x-j)*u^t*HHH(j);

$$Z := (x, t, j) \rightarrow \frac{\text{JacobiTheta4}(0, \sqrt{q})^9 T^{(x-j)} u^t \text{HHH}(j)}{h^{(4+2j)}}$$

> # Computation of Seiberg-Witten invariants for CP^2

> MooreWitten:=proc(k)

local d,SW,t,x, summe,j,sw; global Z;

d:=4*k-3;SW:=0: for t from 0 while d-1>=0 do

x:=floor((d-1)/2): summe:=0:

for j from 0 to x do

summe:=summe+1/(x-j)!*GAMMA(3/2)/

(j!*GAMMA(3/2+j))*1/2^j*Koeff(Z(x,t,j)):

end do:

sw:=-2^(t-1)*(2*x+1)!*summe;

SW:=SW+sw*S^d*p^t:

d:=d-2:

end do:

SW;

end proc:

A.2 The MAPLE-output

> MooreWitten(1);

$$-\frac{3S}{2}$$

> MooreWitten(2);

$$S^5 - S^3 p - \frac{13}{8} S p^2$$

> MooreWitten(3);

$$3S^9 + \frac{15}{4} S^7 p - \frac{11}{16} S^5 p^2 - \frac{141}{64} S^3 p^3 - \frac{879}{256} S p^4$$

> MooreWitten(4);

$$54S^{13} + 24S^{11}p + \frac{159}{8}S^9p^2 + \frac{51}{16}S^7p^3 - \frac{459}{128}S^5p^4 \\ - \frac{1515}{256}S^3p^5 - \frac{36675}{4096}Sp^6$$

> MooreWitten(5);

$$2540S^{17} + 694S^{15}p + \frac{487}{2}S^{13}p^2 + \frac{2251}{16}S^{11}p^3 + \frac{2711}{64}S^9p^4 \\ - \frac{5}{16}S^7p^5 - \frac{3355}{256}S^5p^6 - \frac{143725}{8192}S^3p^7 - \frac{850265}{32768}Sp^8$$

> MooreWitten(6);

$$233208S^{21} + 45912S^{19}p + 10625S^{17}p^2 + 3036S^{15}p^3 \\ + \frac{41103}{32}S^{13}p^4 + \frac{1741}{4}S^{11}p^5 + \frac{5619}{64}S^9p^6 - \frac{20379}{1024}S^7p^7 \\ - \frac{754141}{16384}S^5p^8 - \frac{904239}{16384}S^3p^9 - \frac{10504593}{131072}Sp^{10}$$

> MooreWitten(7);

$$35825553S^{25} + \frac{21975543}{4}S^{23}p + \frac{15224337}{16}S^{21}p^2 + \frac{12159687}{64}S^{19}p^3 \\ + \frac{11618625}{256}S^{17}p^4 + \frac{15077511}{1024}S^{15}p^5 + \frac{19602561}{4096}S^{13}p^6 \\ + \frac{20676279}{16384}S^{11}p^7 + \frac{11107665}{65536}S^9p^8 - \frac{28437201}{262144}S^7p^9 \\ - \frac{169509159}{1048576}S^5p^{10} - \frac{757633329}{4194304}S^3p^{11} - \frac{4334081031}{16777216}Sp^{12}$$

> MooreWitten(8);

$$\begin{aligned}
& 8365418914 S^{29} + 1047342410 S^{27} p + \frac{1157569571}{8} S^{25} p^2 \\
& + \frac{357034013}{16} S^{23} p^3 + \frac{499796309}{128} S^{21} p^4 + \frac{25506259}{32} S^{19} p^5 \\
& + \frac{423516455}{2048} S^{17} p^6 + \frac{245576651}{4096} S^{15} p^7 + \frac{537423737}{32768} S^{13} p^8 \\
& + \frac{118590907}{32768} S^{11} p^9 + \frac{131266019}{524288} S^9 p^{10} - \frac{498648655}{1048576} S^7 p^{11} \\
& - \frac{4800905323}{8388608} S^5 p^{12} - \frac{2551074181}{4194304} S^3 p^{13} - \frac{115237180987}{134217728} S p^{14}
\end{aligned}$$

> MooreWitten(9);

$$\begin{aligned}
& 2780195996868 S^{33} + 293334321858 S^{31} p + \frac{67261095005}{2} S^{29} p^2 \\
& + \frac{67539891519}{16} S^{27} p^3 + \frac{37480404303}{64} S^{25} p^4 + \frac{1455758501}{16} S^{23} p^5 \\
& + \frac{258640401}{16} S^{21} p^6 + \frac{14239101477}{4096} S^{19} p^7 + \frac{14274421501}{16384} S^{17} p^8 \\
& + \frac{7420640919}{32768} S^{15} p^9 + \frac{7179481275}{131072} S^{13} p^{10} + \frac{10830752675}{1048576} S^{11} p^{11} \\
& - \frac{218792349}{4194304} S^9 p^{12} - \frac{8125524657}{4194304} S^7 p^{13} - \frac{34310453897}{16777216} S^5 p^{14} \\
& - \frac{561608698989}{268435456} S^3 p^{15} - \frac{3135392459541}{1073741824} S p^{16}
\end{aligned}$$

> MooreWitten(10);

$$\begin{aligned}
& 1253558847090600 S^{37} + 114049802084088 S^{35} p + 11151310348527 S^{33} p^2 \\
& + \frac{2356053433779}{2} S^{31} p^3 + \frac{4330247481231}{32} S^{29} p^4 + \frac{136302640305}{8} S^{27} p^5 \\
& + \frac{19010805303}{8} S^{25} p^6 + \frac{5969251539}{16} S^{23} p^7 + \frac{560820990153}{8192} S^{21} p^8 \\
& + \frac{120608801397}{8192} S^{19} p^9 + \frac{227818311585}{65536} S^{17} p^{10} + \frac{108448319625}{131072} S^{15} p^{11} \\
& + \frac{380576939595}{2097152} S^{13} p^{12} + \frac{30600598425}{1048576} S^{11} p^{13} - \frac{24269489295}{8388608} S^9 p^{14} \\
& - \frac{128245327215}{16777216} S^7 p^{15} - \frac{3958008534873}{536870912} S^5 p^{16} - \frac{3928400321367}{536870912} S^3 p^{17} \\
& - \frac{43427017514031}{4294967296} S p^{18}
\end{aligned}$$

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