Symmetry properties of semilinear elliptic equations with isolated singularities

by

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Abstract

In this thesis we use the method of moving planes to establish symmetry properties for positive solutions of semilinear elliptic equations. We give a detailed proof of the result due to Caffarelli, Gidas, and Spruck that a solution in the punctured ball, $B\backslash\{0\}$, behaves asymptotically like its spherical average at the origin. We also show that a solution with an isolated singularity in the upper half space $\mathbb{R}^n_+$ must be cylindrically symmetric about some axis orthogonal to the boundary $\partial\mathbb{R}^n_+$.

Thesis Supervisor: David Jerison
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Chapter 1

Introduction

In this thesis we study positive C^2 solutions of semilinear elliptic equations of the form

$$-\Delta u = g(u) \text{ in } \Omega \subset \mathbb{R}^n, \quad n \geq 3, \tag{1.0.1}$$

where u has an isolated singularity at some point $x_0 \in \Omega$. We are interested in the symmetry properties of solutions in the cases where Ω is either $\mathbb{R}^n \backslash \{0\}$, the punctured ball $B\backslash\{0\}$, or the upper half space $\mathbb{R}^n_+$ minus an interior point $(0, t_0)$.

We prove two results due to Caffarelli, Gidas, and Spruck [1]. In the case where Ω is $\mathbb{R}^n \backslash \{0\}$ and u is a global solution of (1.0.1) with an isolated singularity at the origin, we show that u is radially symmetric about some point (Theorem 2.4.1). Here we do not make any assumptions about the behavior of u at infinity, and this theorem can be thought of as a global result for solutions with two isolated singularities. In the case where Ω is a punctured ball $B\backslash\{0\}$, we use an asymptotic symmetry method to show that if u is a solution of (1.0.1) with an isolated singularity at the origin, then u behaves asymptotically like its spherical average at the origin, that is, $u(x) = (1 + \mathcal{O}(|x|))m(|x|)$ as $x \to 0$, where $m(r) = \frac{1}{|\Sigma_1|}\int_{\Sigma_1} u(ry)d\sigma(y)$ (Theorem 3.5.1).

Finally, in the case where Ω is the upper half space $\mathbb{R}^n_+$ minus a point $(0, t_0)$, we study solutions of (1.0.1) that satisfy a boundary condition of the form $\frac{\partial u}{\partial t} = cu^\beta$ on $\partial\mathbb{R}^n_+$, $c > 0$. We show that if u is a solution of the boundary value problem, then it is cylindrically symmetric about some axis orthogonal to $\partial\mathbb{R}^n_+$ (Theorem 5.2.1). This result is new and is based on several different approaches to the method of moving planes [1], [2], [12].

1.1 Historical survey

This section contains a brief overview of symmetry results proved by the method of moving planes. See [1], [2], [4] [5], and [12] for a more detailed account of the history.

In [4], Gidas, Ni, and Nirenberg established various results including the following two theorems.

Theorem A. (Gidas, Ni, Nirenberg [4]) *In the ball $B_R : |x| < R$ in $\mathbb{R}^n$, let $u > 0$ be a positive solution in $C^2(\bar{B}_R)$ of*

$$\Delta u + f(u) = 0 \text{ with } u = 0 \text{ on } |x| = R.$$

Here f is of class C^1. Then u is radially symmetric about the origin, and

$$\frac{\partial u}{\partial r} < 0 \text{ for } 0 < r < R.$$

Theorem B. (Gidas, Ni, Nirenberg [4]) *Let $v \in C^{2+\alpha}$, $0 < \alpha < 1$, be a positive solution of*

$$\Delta v + f(v) = 0 \text{ in } \mathbb{R}^n,\ n \geq 3,$$

with $v(x) = \mathcal{O}(|x|^{2-n})$ at ∞. Assume that for some $k \geq \frac{n+2}{n-2}$, $g(v) = f(v)v^{-k}$ is Hölder continuous on $0 \leq v \leq v_0$, where v_0 is the maximum of v. Then v is rotationally symmetric about some point, and $v_r < 0$ for $r > 0$, where r is the radial coordinate about that point.

Notice that the nonlinearity they were able to treat in the ball was more general than the nonlinearity they were able to treat in the entire space.

In [5], the same authors extended some of their results from [4] to more general nonlinearities. They also studied solutions in the entire space with isolated singularities. Their results included the following theorem.

Theorem C. (Gidas, Ni, Nirenberg [5]) *Let $a_1, \ldots, a_k \in \mathbb{R}^n$, $n \geq 3$, lie on a line, e.g., the x_n axis. Let $u(x)$ be a positive solution of*

$$-\Delta u = g(u) \text{ in } \mathbb{R}^n \backslash \{a_1, \ldots, a_k\}.$$

Assume

$$u \in C^2(\mathbb{R}^n \backslash \{a_1, \ldots, a_k\}),$$

$$u(x) = \mathcal{O}(1/|x|^m) \text{ at } \infty,\ m > 0,$$

$$u(x) \to \infty \text{ as } x_j \to a_j,\ j = 1, \ldots, k.$$

Assume further that

(i) $g(u)$ *is nondecreasing in* u *for* $u \geq 0$, *and for some* $p > (n+1)/m$, $|g(u)| \leq C|u|^p$ *for* u *small,*

(ii) $\liminf_{t\to\infty} g(t)/t^p > 0$ *for some* $p \geq n/(n-2)$.

Then $u(x)$ *is cylindrically symmetric about the* x_n *axis. Moreover, if* r *denotes the distance from the axis, then* $u_r < 0$ *for* $r > 0$.

An immediate consequence of Theorem C is that a solution with one nonremovable singularity and sufficient decay at infinity is radially symmetric. In [1], Caffarelli, Gidas, and Spruck proved a more general result without any assumptions on the behavior of the solutions at infinity.

Theorem D. (Caffarelli, Gidas, Spruck [1]) *Let* $u \geq 0$ *be a* C^2 *solution of*

$$-\Delta u = g(u) \text{ in } \mathbb{R}^n \backslash \{0\},\ n \geq 3,$$

with an isolated singularity. Assume that $g(t)$ *is a function on* $[0, \infty)$ *satisfying*

(i) $g(t)$ *is nondecreasing,* $g(0) = 0$,

(ii) $t^{-(n+2)/(n-2)} g(t)$ *is nonincreasing,*

(iii) $\liminf_{t\to\infty} g(t)/t^p > 0$ *for some* $p \geq n/(n-2)$,

(iv) $\frac{1}{t_1^{n+2}} g(t_1^{n-2} s_1) \geq \frac{1}{t_2^{n+2}} g(t_2^{n-2} s_2)$ *when* $t_1 \leq t_2$ *and* $s_1 \geq s_2$.

Then either u *is radially symmetric about the origin or* u *is* C^2 *at the origin and radially symmetric about some point.*

In [1], Caffarelli, Gidas, and Spruck also studied the local behavior of solutions of semilinear elliptic equtaions at an isolated singularity. They combined the method of moving planes with a measure theoretic technique to prove an asymptotic symmetry result for solutions at the singularity.

Theorem E. (Caffarelli, Gidas, Spruck [1]) *Let* u *be a positive* C^2 *solution of* $-\Delta u = g(u)$, *in the punctured ball* $B_2\backslash\{0\} \subset \mathbb{R}^n$, $n \geq 3$, *with an isolated singularity at the origin.*

Assume that $g(t)$ *is a function on* $[0, \infty)$ *satisfying:*

(*i*) $g(t)$ *is nondecreasing,* $g(0) = 0$,

(*ii*) $t^{-(n+2)/(n-2)}g(t)$ *is nonincreasing for* $t \geq t_0$.

(*iii*) $\liminf_{t\to\infty} g(t)/t^p > 0$ *for some* $p > n/(n-2)$,

(*iv*) $\frac{1}{t_1^{n+2}}g(t_1^{n-2}s_1) \geq \frac{1}{t_2^{n+2}}g(t_2^{n-2}s_2)$ *when* $t_1 \leq t_2$ *and* $s_1 \geq s_2$.

Then

$$u(x) = (1 + \mathcal{O}(|x|))m(|x|) \text{ as } x \to 0,$$

where $m(r) = \frac{1}{|\Sigma_1|}\int_{\Sigma_1} u(ry)d\sigma(y)$.

In [12], Li and Zhu used the method of moving spheres to prove uniqueness theorems for two semilinear elliptic boundary value problems. Their results were slight improvements of results proved by Escobar in [3].

Theorem F. (Li, Zhu [12]) *Let* $u \in C^2(\mathbb{R}^n_+) \cap C^1(\bar{\mathbb{R}}^n_+)$, $n \geq 3$, *be a nonnegative solution of*

$$-\Delta u = 0 \text{ in } \mathbb{R}^n_+,$$

$$\frac{\partial u}{\partial t} = cu^{n/(n-2)} \text{ on } \partial\mathbb{R}^n_+.$$

When $c \geq 0$, $u = at + b$, *with* $a, b \geq 0$, $a = cb^{n/(n-2)}$. *When* $c < 0$, *either* $u \equiv 0$ *or*

$$u(x,t) = \left(\frac{\varepsilon}{(\varepsilon + t)^2 + |x - x_0|^2}\right)^{(n-2)/2}, \qquad \varepsilon > 0,$$

for some $\varepsilon > 0$, $x_0 \in \mathbb{R}^{n-1}$, *and* $(n-2)t_0 = -\varepsilon c$.

Theorem G. (Li, Zhu [12]) *Let* $u \in C^2(\mathbb{R}^n_+) \cap C^1(\bar{\mathbb{R}}^n_+)$, $n \geq 3$, *be a nonnegative solution of*

$$-\Delta u = n(n-2)u^{(n+2)/(n-2)} \text{ in } \mathbb{R}^n_+,$$

$$\frac{\partial u}{\partial t} = cu^{n/(n-2)} \quad on\ \partial\mathbb{R}^n_+.$$

Then either $u \equiv 0$ *or*

$$u(x,t) = \left(\frac{\varepsilon}{\varepsilon^2 + |(x,t)-(x_0,t_0)|^2}\right)^{(n-2)/2}, \qquad \varepsilon > 0,$$

for some $\varepsilon > 0$, $x_0 \in \mathbb{R}^{n-1}$, *and* $(n-2)t_0 = \varepsilon c$.

1.2 Method of moving planes

The method of moving planes is a technique used to prove symmetry and monotonicity results for solutions of partial differential equations. When we are trying to classify the solutions of a partial differential equation, symmetry and monotonicity results give us additional information about the structure and behavior of the solutions. This kind of information is useful because it may simplify the equations we are interested in solving or tell us that certain types of functions are not solutions.

Suppose you want to show that the solutions of a partial differential equation have reflection symmetry in the direction τ. That is, you want to show that if u is a solution, then $u(x) = u(x - 2(x\cdot\tau - \lambda_0)\tau)$ for some λ_0. The following is a basic description of how to use the method of moving planes to establish this type of result:

Let u be a solution of a partial differential equation. Fix a direction τ, and let $\mathcal{P}(\lambda)$ be the property that: $u(x) < u(x - 2(x\cdot\tau - \lambda)\tau)$ for $x\cdot\tau > \lambda$. The first step in the method of moving planes is to show that the reflection process can be started. That is, we want to show that there exists a $\bar{\lambda}$ so that $\mathcal{P}(\lambda)$ holds for $\lambda > \bar{\lambda}$. This will usually require some kind of apriori information about the solution such as a boundary condition or an assumption on its behavior at infinity. The second step is to continue the reflection process and move the family of hyperplanes up to a critical position λ_0. For concreteness, suppose that $\lambda_0 = \inf\{\bar{\lambda} : \mathcal{P}(\lambda) \text{ holds for } \lambda > \bar{\lambda}\}$, and assume that $\lambda_0 > -\infty$. In this case, the last step is to use the maximum principle to show that $u(x) = u(x - 2(x\cdot\tau - \lambda_0)\tau)$ and $\frac{\partial u}{\partial \tau}(x) < 0$, for $x\cdot\tau > \lambda_0$.

1.3 Plan of the paper

In Chapter 2 we prove some preliminary results and develop the tools that allow us to use the method of moving planes. We close the chapter with two global symmetry results that illustrate the method of moving planes. In Theorem 2.3.1, we show that all positive C^2 solutions of

$$-\Delta u = u^{(n+2)/(n-2)} \tag{1.3.1}$$

in $\mathbb{R}^n$, $n \geq 3$, are radially symmetric about some point. As a corollary, we give a complete characterization of the positive C^2 solutions of (1.3.1) in $\mathbb{R}^n$. In Theorem 2.4.1 (Theorem D in the Historical survey), we show that if u is a positive C^2 solution of

$$-\Delta u = u^p \tag{1.3.2}$$

in $\mathbb{R}^n \backslash \{0\}$, with an isolated singularity at the origin and if $p \in [\frac{n}{n-2}, \frac{n+2}{n-2}]$, then u is radially symmetric about some point. The proof of this theorem involves applying the method of moving planes to the Kelvin transform of u about a regualr point.

In Chapter 3 we study solutions of (1.3.2) in the punctured ball $B_2 \backslash \{0\} \subset \mathbb{R}^n$, $n \geq 3$, where u has an isolated singularity at the origin and $p \in (\frac{n}{n-2}, \frac{n+2}{n-2}]$. We take the Kelvin transform of u about a regular point close to the origin and study the inverted problem in $\mathbb{R}^n \backslash \bar{B}_1$. In Section 3.1 we prove an estimate for the measure of directions along which certain nonnegative superharmonic functions do not have fast decay. In Section 3.2 we prove an extension lemma for C^2 functions in $\bar{B}_2 \backslash B_1$ that allows us to extend a function so that it is superharmonic over a set of sufficiently small measure in B_1. In Section 3.4 we use the method of moving planes to prove a partial symmetry result for the Kelvin transform in certain good directions. In Section 3.5 we prove Theorem 3.5.1 (Theorem E in the Historical survey). The proof involves combining the estimate from Section 3.1 with the reflection lemma from Section 3.4 to show that there are enough good directions to conclude that u behaves asymptotically likes its spherical average, that is, $u(x) = (1 + \mathcal{O}(|x|))m(|x|)$ as $x \to 0$, where $m(r) = \frac{1}{|\Sigma_1|} \int_{\Sigma_1} u(ry)d\sigma(y)$.

In Chapter 4 we use Theorem 3.5.1 to characterize the positive solutions of (1.3.1) in the punctured ball $B \backslash \{0\}$. In Section 4.1, we prove some preliminary estimates on the behavior of the spherical average at the origin. Then, in Section 4.2 we prove Theorem 4.2.4, which tells us that if u is a positive C^2 solution of (1.3.1) in $B_2\backslash\{0\}$, $n \geq 3$, with a nonremovable isolated singularity at the origin, then there is a unique asymptotic constant D_∞ in the interval $-\frac{2}{n}\left(\frac{n-2}{n}\right)^n \leq D_\infty < 0$ so that

$$u(x) = (1 + o(1))\frac{1}{|x|^{(n-2)/2}}\psi_{D_\infty}(-\log|x|) \text{ as } x \to 0,$$

where $\psi_{D_\infty}(t)$ is the singular solution of the differential equation (A.3.3) corresponding to the constant D_∞.

Finally, in Chapter 5 we study the boundary value problem

$$\begin{cases} -\Delta u = u^p \text{ in } \mathbb{R}^n_+\backslash\{(0,t_0)\}, & t_0 > 0, \quad n \geq 3, \\ \frac{\partial u}{\partial t} = cu^\beta \text{ on } \partial\mathbb{R}^n_+, \quad c > 0, & \end{cases} \tag{1.3.3}$$

where $n \leq p(n-2) \leq n+2$, $\beta \geq 1$, and $\beta(n-2) \geq n$. In Section 5.1, we prove some preliminary results that allow us to use the method of moving planes. Then, in Section 5.2 we prove Theorem 5.2.1, which tells us that if u is a positive $C^2(\overline{\mathbb{R}^n_+}\backslash\{(0,t_0)\})$ solution of 1.3.3, then u is cylindrically symmetric about some axis orthogonal to $\partial\mathbb{R}^n_+$. The proof of this result is modeled after the proof of Theorem 2.4.1 in Chapter 2, and it uses techniques from [1], [2], and [12].

Chapter 2

Preliminary results

2.1 Isolated singularities

2.1.1 Local behavior

Let v be a C^2 solution of

$$-\Delta v = f(x,v), \quad v > 0, \tag{2.1.1}$$

in a punctured ball, $B\backslash\{x_0\} \subset \mathbb{R}^n$, $n \geq 3$, with an isolated singularity at the point x_0.

The following lemma gives sufficient conditions for a C^2 solution of (2.1.1) to be a distribution solution in a neighborhood of the singularity. The proof we give is similar to the proof of Lemma 2.1 in [1].

Lemma 2.1.1. *Let v be a positive C^2 solution of $-\Delta v = f(x,v)$ in the punctured ball $B_\delta(x_0)\backslash\{x_0\} \subset \mathbb{R}^n$, $n \geq 3$, where*

(i) $f(x,v) = g(|x|^{n-2}v)/|x|^{n+2}$,

(ii) *$g(t)$ is nondecreasing,* $g(0) = 0$,

(iii) $\liminf_{t\to\infty} g(t)/t^p > 0$ *for some* $p \geq n/(n-2)$,

(iv) $0 \notin B_\delta(x_0)$.

If $R < \delta$, then $f(x,v) \in L_1(B_R(x_0))$, $v \in L_p(B_R(x_0))$, and v is a distribution solution in $B_R(x_0)$.

Proof. Let $\psi(x) \geq 0$ be a smooth function satisfying:

$$\psi(x) = \begin{cases} 0, & |x - x_0| < 1, \\ 1, & |x - x_0| \geq 2. \end{cases}$$

Let $C_0 > 0$ be such that

$$\sup_{\mathbb{R}^n}\{|\psi|, |\nabla\psi|, |\Delta\psi|\} \leq C_0.$$

If we let $\psi_\varepsilon(x) = \psi(x/\varepsilon)$, then $|\psi_\varepsilon| \leq C_0$, $|\nabla\psi_\varepsilon| \leq C_0/\varepsilon$, $|\Delta\psi_\varepsilon| \leq C_0/\varepsilon^2$.

For $k > \max_{\partial B_R(x_0)} u(x)$, let $\phi_k(s)$ be a smooth nonincreasing function satisfying:

$$\phi_k(s) = \begin{cases} 1, & s < k, \\ 0, & s \geq 2k. \end{cases}$$

Set $\Phi_k(t) = \int_0^t \phi_k(s)\,ds$, then $\nabla\Phi_k(v) = \phi_k(v)\nabla v$. Using the fact that ϕ_k is nonincreasing, we have

$$\begin{aligned} \int_{B_R(x_0)} \nabla v \cdot \nabla(\psi_\varepsilon \phi_k(v))\,dx &\leq \int_{B_R(x_0)} \nabla\Phi_k(v) \cdot \nabla\psi_\varepsilon\,dx \\ &= -\int_{B_R(x_0)} \Phi_k(v)\Delta\psi_\varepsilon\,dx \\ &\leq \sup|\Phi_k| \int_{B_{2\varepsilon}(x_0)} |\Delta\psi_\varepsilon|\,dx \\ &= \mathcal{O}(\varepsilon^{n-2}) \end{aligned}$$

so that $\int_{B_R(x_0)} \nabla v \cdot \nabla(\psi_\varepsilon\phi_k(v))\,dx \to 0$ as $\varepsilon \to 0$. We also have,

$$\begin{aligned} \int_{B_R(x_0)} \psi_\varepsilon\phi_k(v) f(x,v)\,dx &= -\int_{B_R(x_0)} \psi_\varepsilon\phi_k(v)\Delta v\,dx \\ &= \int_{B_R(x_0)} \nabla v \cdot \nabla(\psi_\varepsilon\phi_k(v))\,dx - \int_{\partial B_R(x_0)} \nabla v \cdot \nu\,d\sigma. \end{aligned}$$

Taking $\varepsilon \to 0$,

$$\int_{B_R(x_0)\cap\{v<k\}} f(x,v)\,dx \leq -\int_{\partial B_R(x_0)} \nabla v \cdot \nu\,d\sigma.$$

Taking $k \to \infty$, we conclude that $f(x,v) \in L_1(B_R(x_0))$.

By properties (iii) and (iv), there exist $C_1, C_2, C_3, M > 0$ so that

1. If $|x|^{n-2}v > M$ and $x \in B_\delta(x_0)\backslash\{x_0\}$, then $|x|^{p(n-2)}v^p \le C_1 g(|x|^{n-2}v)$.

2. If $x \in B_\delta(x_0)$, then $|x|^{n+2} \le C_2|x|^{p(n-2)}$,

3. $\int_{B_R(x_0)} \frac{M^p}{|x|^{p(n-2)}} dx \le C_3$.

Then

$$\begin{aligned}\int_{B_R(x_0)} v^p \, dx &\le \int_{B_R(x_0)} \frac{M^p}{|x|^{p(n-2)}} \, dx + C_1 \int_{B_R(x_0)} \frac{1}{|x|^{p(n-2)}} g(|x|^{n-2}v) \, dx \\ &\le C_3 + C_1 C_2 \int_{B_R(x_0)} f(x,v) \, dx,\end{aligned}$$

and $v \in L_p(B_R(x_0))$.

To show that v is a distribution solution, we take $\eta \in C_0^\infty(B_R(x_0))$, and we let ψ be as above. Then we have

$$\begin{aligned}\int_{B_R(x_0)} \psi_\varepsilon f(x,v)\eta \, dx &= -\int_{B_R(x_0)} \psi_\varepsilon \eta \Delta v \, dx \\ &= -\int_{B_R(x_0)} v(\psi_\varepsilon \Delta\eta + 2\nabla\psi_\varepsilon \cdot \nabla\eta + \eta\Delta\psi_\varepsilon) \, dx\end{aligned}$$

so that

$$\begin{aligned}\left|\int_{B_R(x_0)} \psi_\varepsilon(f(x,v)\eta + v\Delta\eta) \, dx\right| &\le \int_{B_R(x_0)} v|2\nabla\psi_\varepsilon \cdot \nabla\eta + \eta\Delta\psi_\varepsilon| \, dx \\ &\le \frac{C}{\varepsilon^2} \int_{\{\varepsilon<|x-x_0|<2\varepsilon\}} v \, dx \\ &\le C\varepsilon^{n-2-n/p} \|v\|_{L_p(B_{2\varepsilon}(x_0))}.\end{aligned}$$

Taking $\varepsilon \to 0$, we conclude that $\int_{B_R(x_0)} f(x,v)\eta + v\Delta\eta = 0$ for all $\eta \in C_0^\infty(B_R(x_0))$. □

Lemma 2.1.2. *Let v_1 be a positive C^2 solution of $-\Delta v_1 = f(\lambda(x), v_1)$ in $B_\delta(x_0)\backslash\{x_0\} \subset \mathbb{R}^n$. Assume that v_1 is a distribution solution in $B_\delta(x_0)$. Let v_2 be a positive C^2 solution of $-\Delta v_2 = f(x, v_2)$ in $B_\delta(x_0)$. If $v_1 \ge v_2$, $v_1 \not\equiv v_2$, and $f(\lambda(x), v_1) \ge f(x, v_2)$ in $B_\delta(x_0)\backslash\{x_0\}$, then there exist $r, \varepsilon > 0$ so that $v_1 - v_2 > \varepsilon$ in $B_r(x_0)\backslash\{x_0\}$.*

Proof. Let $y_0 \in B_\delta(x_0)\backslash\{x_0\}$ be such that $v_1(y_0) > v_2(y_0)$. Let

$$\alpha = \delta - |x_0 - y_0|, \quad r = \alpha/4,$$

$$R_1 = |x_0 - y_0| + r, \quad R_2 = R_1 + 2r.$$

Then $R_1 + r < R_2 < \delta$. By assumption, $v_1 - v_2$ is a distribution solution of $\Delta(v_1 - v_2) \le 0$ in $B_{R_2}(x_0)$. By Corollary A.2.2, for $x \in B_r(x_0)\backslash\{x_0\}$, we have

$$\begin{aligned}(v_1 - v_2)(x) &\ge \frac{n}{c_n(R_1 + r)^n} \int_{B_{R_1+r}(x)} (v_1 - v_2)(y)\, dy \\ &\ge \frac{n}{c_n(R_1 + r)^n} \int_{B_{r/2}(y_0)} (v_1 - v_2)(y)\, dy.\end{aligned}$$

Take

$$\varepsilon = \frac{n}{c_n(R_1 + r)^n} \int_{B_{r/2}(y_0)} (v_1 - v_2)(y)\, dy.$$

□

When we study solutions in the entire space, we will need sufficient conditions for a C^2 solution of (2.1.1) with $x_0 = 0$ to be a distribution solution in a neighborhood of the origin.

Lemma 2.1.3. *Let v be a positive C^2 solution of $-\Delta v = f(x, v)$ in the punctured ball $B_\delta\backslash\{0\} \subset \mathbb{R}^n$, $n \ge 3$, where*

(i) $f(x, v) = g(|x|^{n-2}v)/|x|^{n+2}$,

(ii) *$g(t)$ is nondecreasing, $g(0) = 0$,*

(iii) *$\liminf_{t\to\infty} g(t)/t^p > 0$ for some $p \ge n/(n-2)$,*

(iv) *$t^{-(n+2)/(n-2)}g(t)$ is nonincreasing*

If $R < \delta$, then $f(x, v) \in L_1(B_R)$, $v \in L_p(B_R)$, and v is a distribution solution in B_R.

Proof. Using the proof of Lemma 2.1.1, we see that $f(x, v) \in L_1(B_R)$.

By properties (iii) and (iv), there exist $C_1, C_2, M > 0$ so that

1. If $|x|^{n-2}v > M$ and $|x| < \delta$, then $v^p \le C_1 f(x, v)$.

2. If $|x|^{n-2}v \le M$, then $v^{(n+2)/(n-2)} \le C_2 f(x, v)$.

Then

$$
\begin{aligned}
\int_{B_R} v^p\,dx &\le \int_{B_R\cap\{|x|^{n-2}v\le M\}} v^p\,dx + C_1\int_{B_R} f(x,v)\,dx \\
&\le C_3\left(\int_{B_R\cap\{|x|^{n-2}v\le M\}} v^{(n+2)/(n-2)}\,dx\right)^{p(n-2)/(n+2)} + C_1\int_{B_R} f(x,v)\,dx \\
&\le C_2C_3\left(\int_{B_R} f(x,v)\,dx\right)^{p(n-2)/(n+2)} + C_1\int_{B_R} f(x,v)\,dx,
\end{aligned}
\tag{2.1.2}
$$

and $v \in L_p(B_R)$.

Since $v \in L_p(B_R)$, the proof that v is a distribution solution is the same as the proof used in Lemma 2.1.1. □

2.1.2 Global behavior

Let u be a C^2 solution of

$$-\Delta u = g(u), \quad u > 0, \tag{2.1.3}$$

in a punctured ball, $B_2\backslash\{0\} \subset \mathbb{R}^n$, $n \ge 3$, with an isolated singularity at the origin. We will show that the Kelvin transform of u is integrable in $\mathbb{R}^n\backslash\bar{B}_1$. The following lemma gives sufficient conditions for a C^2 solution of (2.1.3) to be integrable in B_1.

Lemma 2.1.4. *Let u be a positive C^2 solution of $-\Delta u = g(u)$ in the punctured ball $B_2(0)\backslash\{0\} \subset \mathbb{R}^n$, $n \ge 3$, where*

(i) *$g(t)$ is nondecreasing, $g(0) = 0$,*

(ii) *$\liminf_{t\to\infty} g(t)/t^p > 0$ for some $p \ge n/(n-2)$.*

Then $g(u) \in L_1(B_1)$ and $u \in L_p(B_1)$.

Proof. Using the proof of Lemma 2.1.1, we see that $g(u) \in L_1(B_1)$. It follows, from property (ii), that $u \in L_p(B_1)$. □

Corollary 2.1.5. *Let $z \in B_{1/2}$, let $v(x) = \frac{1}{|x|^{n-2}}u(z + \frac{x}{|x|^2})$, and let u satisfy the hypotheses of Lemma 2.1.4. Then*

$$\int_{\mathbb{R}^n\backslash\bar{B}_1} \frac{v^p}{|x|^\beta}\,dx \le \int_{B_{3/2}} u^p\,dx, \tag{2.1.4}$$

where $\beta = 2n - p(n-2)$.

2.2 Kelvin transform

Let U be an open set in $\mathbb{R}^n$, $n \geq 3$. Let $z \in U$ and $\widetilde{U} = \{x \neq 0 : z + x/|x|^2 \in U\}$. For $u \in C^2(U)$, we define $v \in C^2(\widetilde{U})$ by

$$v(x) = \frac{1}{|x|^{n-2}} u(z + \frac{x}{|x|^2}), \quad x \in \widetilde{U}. \tag{2.2.1}$$

Then

$$\Delta v(x) = \frac{1}{|x|^{n+2}} \Delta u(z + \frac{x}{|x|^2}), \quad x \in \widetilde{U}. \tag{2.2.2}$$

We call v the Kelvin transform of u about the point z.

Let $y = z + x/|x|^2$. Using (2.2.1), we see that v has the following harmonic asymptotic expansion at ∞:

$$\begin{aligned} v(x) &= \frac{1}{|x|^{n-2}} \left(a_0 + a_i \frac{x_i}{|x|^2} \right) + \mathcal{O}(\frac{1}{|x|^n}), \\ v_{x_i}(x) &= -(n-2) a_0 \frac{x_i}{|x|^n} + \mathcal{O}(\frac{1}{|x|^n}), \\ v_{x_i x_j}(x) &= \mathcal{O}(\frac{1}{|x|^n}), \end{aligned} \tag{2.2.3}$$

where $a_0 = u(z)$, $a_i = u_{y_i}(z)$.

The asymptotic expansion will be used to start the method of moving planes. For $x = (x_1, x')$, we denote the reflection of x about the plane $x_1 = \lambda$ by $x_\lambda = (2\lambda - x_1, x')$.

Lemma 2.2.1. *Let v be a function with the asymptotic expansion* (2.2.3) *in a neighborhood of ∞. If $a_0 > 0$, then there exist positive constants $\bar{\lambda}, R$ so that $v(x_\lambda) > v(x)$ whenever $\lambda \geq \bar{\lambda}$, $|x_\lambda| > R$, and $x_1 > \lambda$.*

After we move the reflection plane in, to a critical point $x_1 = \lambda_*$, we use the asymptotic expansion to bound the set of points x satisfying $v(x_\lambda) < v(x)$, $x_1 > \lambda$, with λ sufficiently close to λ_*.

Lemma 2.2.2. *Let v be a positive C^2 solution of*

$$-\Delta v(x) = f(x) \text{ in } |x| \geq R, \tag{2.2.4}$$

where v has the asymptotic expansion (2.2.3) *in a neighborhood of ∞ with $a_0 > 0$. Suppose that $v(x_{\lambda_*}) > v(x)$ and $f(x_{\lambda_*}) \geq f(x)$ when $|x_{\lambda_*}| \geq R$, $x_1 > \lambda_*$. Then there exist $\varepsilon > 0$,*

$S > R$ so that, if $\lambda_* - \varepsilon < \lambda < \lambda_*$, $|x_\lambda| > S$, and $x_1 > \lambda$, then $v(x_\lambda) > v(x)$.

The proofs we give are similar to the proofs of Lemma 2.3 and Lemma 2.4 in [1].

proof of Lemma 2.2.1. Using the asymptotic expansion, we have

$$\begin{aligned} v(x_\lambda) - v(x) &= a_0\left(\frac{1}{|x_\lambda|^{n-2}} - \frac{1}{|x|^{n-2}}\right) + a_1\frac{2(\lambda - x_1)}{|x_\lambda|^n} \\ &\quad + \sum_{j=1}^{n} a_j x_j \left(\frac{1}{|x_\lambda|^n} - \frac{1}{|x|^n}\right) + \mathcal{O}(\frac{1}{|x_\lambda|^n}). \end{aligned} \tag{2.2.5}$$

If $|x| \geq 2|x_\lambda|$, then

$$v(x_\lambda) - v(x) \geq \frac{1}{2}a_0\frac{1}{|x_\lambda|^{n-2}} + \mathcal{O}(\frac{1}{|x_\lambda|^{n-1}}),$$

and for $|x_\lambda|$ sufficiently large, we have $v(x_\lambda) > v(x)$.

Suppose $|x| < 2|x_\lambda|$ and $x_1 > \lambda$. Then

$$\frac{1}{|x_\lambda|^{n-2}} - \frac{1}{|x|^{n-2}} > \frac{\lambda(x_1 - \lambda)}{|x_\lambda|^n},$$

$$\left| x_j \left(\frac{1}{|x_\lambda|^n} - \frac{1}{|x|^n}\right) \right| < C\frac{\lambda(x_1 - \lambda)}{|x_\lambda|^{n+1}}.$$

Using these estimates and (2.2.5),

$$v(x_\lambda) - v(x) > a_0\frac{\lambda(x_1 - \lambda)}{|x_\lambda|^n} - C_0\frac{\lambda(x_1 - \lambda)}{|x_\lambda|^n}\left[\frac{1}{\lambda} + \frac{1}{|x_\lambda|}\right] - C_2\frac{1}{|x_\lambda|^n}.$$

For λ and $|x_\lambda|$ sufficiently large, we have

$$v(x_\lambda) - v(x) > C_1\frac{\lambda(x_1 - \lambda)}{|x_\lambda|^n} - C_2\frac{1}{|x_\lambda|^n}.$$

If $\lambda(x_1 - \lambda) > C_2/C_1$, then $v(x_\lambda) - v(x) > 0$.

Suppose $\lambda(x_1 - \lambda) \leq C_2/C_1$. Then

$$2\lambda - x_1 \geq \lambda - \frac{C_2}{C_1} \cdot \frac{1}{\lambda},$$

and $2\lambda - x_1 > 0$, for λ sufficiently large. It follows from the asymptotic expansion that

$$\frac{\partial v}{\partial x_1}(y) < 0,$$

when $y_1 \geq 2\lambda - x_1$. In particular, $v(x_\lambda) > v(x)$. □

proof of Lemma 2.2.2. Let $w_\lambda(x) = v(x_\lambda) - v(x)$. Then $w_{\lambda_*}(x) > 0$ and $\Delta w_{\lambda_*}(x) \leq 0$, for $|x_{\lambda_*}| \geq R$, $x_1 > \lambda_*$. By the Hopf boundary lemma,

$$\frac{\partial w_{\lambda_*}}{\partial x_1}(x) > 0, \text{ for } x = (\lambda_*, x'), \quad |x'| = (\lambda_* + R + 1).$$

This allows us to choose $k > 0$ so that, if $|x - (\lambda_*, 0)| = (\lambda_* + R + 1)$ and $x_1 > \lambda_*$, then

$$w_{\lambda_*}(x) > 2k \frac{x_1 - \lambda_*}{|(x_1 - \lambda_*, x')|^n}.$$

It follows from the maximum principle that, if $|x - (\lambda_*, 0)| \geq (\lambda_* + R + 1)$ and $x_1 > \lambda_*$, then

$$w_{\lambda_*}(x) > k \frac{x_1 - \lambda_*}{|(x_1 - \lambda_*, x')|^n}. \tag{2.2.6}$$

By the Hopf boundary lemma, for $|x'| > (\lambda_* + R + 1)$,

$$-2\frac{\partial v}{\partial x_1}(\lambda_*, x') = \frac{\partial w_{\lambda_*}}{\partial x_1}(\lambda_*, x') > \frac{k}{|x'|^n}.$$

Using the asymptotic expansion and the mean-value theorem,

$$\begin{aligned} \frac{\partial v}{\partial x_1}(\lambda, x') &\leq \frac{\partial v}{\partial x_1}(\lambda_*, x') + \frac{C|\lambda - \lambda_*|}{|x'|^n} \\ &\leq -\frac{1}{4}\frac{k}{|x'|^n}, \end{aligned} \tag{2.2.7}$$

when $|\lambda - \lambda_*| < k/(4C)$ and $|x'|$ is sufficiently large.

Therefore, there exist $\varepsilon_1, S_1 > 0$ so that, if

$$\lambda_* - \varepsilon_1 < \lambda < \lambda_*, \quad \lambda < x_1 < \lambda_* + \varepsilon_1, \quad |x_\lambda| > S_1,$$

then $w_\lambda(x) > 0$. This proves the lemma when x_1 is close to λ_*.

To prove the lemma when there is some distance between x_1 and λ_*, suppose

$$\lambda_* - \varepsilon_1 < \lambda < \lambda_*, \quad x_1 > \lambda_* + \varepsilon_1, \quad |x_\lambda| > S_1,$$

and assume $\varepsilon_1 < \lambda_*/2$. Using the asymptotic expansion and the mean-value theorem,

$$v(2\lambda - x_1, x') - v(2\lambda_* - x_1, x') \geq -C_2 \frac{(x_1 - \lambda_*) + C_1}{|(x_1 - \lambda_*, x')|^n} (\lambda_* - \lambda),$$

for $|x_\lambda|$ sufficiently large. By (2.2.6) and the previous estimate,

$$\begin{aligned} w_\lambda(x) &= w_{\lambda_*}(x) + v(2\lambda - x_1, x') - v(2\lambda_* - x_1, x') \\ &> k\frac{x_1 - \lambda_*}{|(x_1 - \lambda_*, x')|^n} - C_2 \frac{(x_1 - \lambda_*) + C_1}{|(x_1 - \lambda_*, x')|^n}(\lambda_* - \lambda) \\ &= \frac{[k - C_2(\lambda_* - \lambda)](x_1 - \lambda_*) - C_2 C_1 (\lambda_* - \lambda)}{|(x_1 - \lambda_*, x')|^n} \\ &> 0, \end{aligned} \tag{2.2.8}$$

for $(\lambda_* - \lambda)$ sufficiently small, $|x_\lambda|$ sufficiently large (where we have assumed $x_1 - \lambda_* > \varepsilon_1$). □

2.3 Classification of positive solutions of $-\Delta u = u^{(n+2)/(n-2)}$ in $\mathbb{R}^n$

In this section, we use the method of moving planes to classify the positive solutions of $-\Delta u = u^{(n+2)/(n-2)}$ in $\mathbb{R}^n$, $n \geq 3$.

Theorem 2.3.1. *Let u be a positive C^2 solution of*

$$-\Delta u = u^{(n+2)/(n-2)} \ \textit{in } \mathbb{R}^n, \quad n \geq 3. \tag{2.3.1}$$

Then u is radially symmetric about some point in $\mathbb{R}^n$.

This result was proved for the first time in [1] as a corollary of a general result for global solutions with one or two isolated singularities. In this section, we focus on the proof of the simpler case, and we give a proof of the general result in the next section.

proof of Theorem 2.3.1. Let v be the Kelvin transform of u about 0. Then

$$-\Delta v = v^{(n+2)/(n-2)} \text{ in } \mathbb{R}^n \backslash \{0\}, \quad n \geq 3. \tag{2.3.2}$$

If $x = (x_1, x')$, let $x_\lambda = (2\lambda - x_1, x')$ and define $v_\lambda(x) = v(x_\lambda)$. We will show that v is symmetric about a plane $x_1 = \lambda_1$ and $v_{x_1}(x) < 0$ when $x_1 > \lambda_1$.

Let $w_\lambda = v_\lambda - v$. Lemma 2.2.1, Lemma 2.1.3, and Lemma 2.1.2 tell us that

$$\lambda_* = \inf\{\bar{\lambda} > 0 : w_\lambda(x) > 0 \text{ for } x_1 > \lambda,\ x \neq 0_\lambda, \text{ and } \lambda \geq \bar{\lambda}\}$$

is well-defined. If $\lambda > \lambda_*$, then $w_\lambda(x) > 0$ for $x_1 > \lambda$, $x \neq 0_\lambda$. Applying the Hopf boundary lemma to w_λ shows that $v_{x_1}(x) < 0$ when $x_1 = \lambda$. Therefore, $v_{x_1}(x) < 0$ when $x_1 > \lambda_*$.

Suppose $\lambda_* > 0$. By definition of λ_*, we have $w_{\lambda_*}(x) \geq 0$ in $x_1 > \lambda_*$, $x_1 \neq 0_{\lambda_*}$. Since $-\Delta w_{\lambda_*} = c(x) w_{\lambda_*}$, where $c(x) > 0$, we have $\Delta w_{\lambda_*}(x) \leq 0$ in $x_1 > \lambda_*$, $x_1 \neq 0_{\lambda_*}$. Suppose $w_{\lambda_*} \not\equiv 0$. It follows from the Strong maximum principle and the Hopf boundary lemma that $w_{\lambda_*} > 0$ in $x_1 > \lambda_*$, $x_1 \neq 0_{\lambda_*}$ and $\frac{\partial w_{\lambda_*}}{\partial x_1}(x) > 0$ on $x_1 = \lambda_*$.

Claim 2.3.2. *There exist $r, \varepsilon > 0$ so that if $\lambda_* - r < \lambda < \lambda_*$, then*

$$w_\lambda(y) > \varepsilon, \quad y \in B_r(0_\lambda) \backslash \{0_\lambda\}.$$

Proof. By Lemma 2.1.3 and Lemma 2.1.2, there exist $r_1, \varepsilon > 0$ so that

$$w_{\lambda_*} > 2\varepsilon \text{ in } B_{r_1}(0_{\lambda_*}) \backslash \{0_{\lambda_*}\}.$$

By continuity, there exists $\delta > 0$ so that, for $y, y' \in \bar{B}_{r_1}(0_{\lambda_*})$ and $|y - y'| < \delta$,

$$|v(y) - v(y')| < \varepsilon.$$

Let $r = \min\{\frac{\delta}{2}, r_1\}$.

Suppose $\lambda_* - r < \lambda < \lambda_*$ and $y \in B_r(0_\lambda) \backslash \{0_\lambda\}$. Let $y' = (y_\lambda)_{\lambda_*}$. Then $|y - y'| < \delta$, and it follows that

$$w_\lambda(y) = w_{\lambda_*}(y') + v(y') - v(y) > \varepsilon.$$

□

By Lemma 2.2.2, there exist $r_0, S > 0$ so that, if $\lambda_* - r_0 < \lambda < \lambda_*$, $|x_\lambda| > S$, and $x_1 > \lambda$, then $w_\lambda(x) > 0$. By definition of λ_*, there exist λ_i and points x^i so that $\lambda_* - r_0 < \lambda_i < \lambda_*$, $\lambda_i \to \lambda$, $(x^i)_1 > \lambda_i$, and $w_{\lambda_i}(x^i) \leq 0$. Taking $r_0 < r/4$, we can find a sequence of points

y^i so that $|y^i| \leq S$, $(y^i)_1 > \lambda_i$, $w_{\lambda_i}(y^i) \leq 0$, $\nabla w_{\lambda_i}(y^i) = 0$, and $|y^i - 0_{\lambda_*}| \geq r/2$. By the boundedness of the y^i, we can find a subsequence, which we denote by y^i, converging to y_0. Notice that $(y_0)_1 \geq \lambda_*$ and $y_0 \neq 0_{\lambda_*}$. It follows that $w_{\lambda_*}(y_0) = \lim w_{\lambda_i}(y^i) \leq 0$, and we conclude that $(y_0)_1 = \lambda_*$. Then

$$0 = \lim_{i \to \infty} \frac{\partial w_{\lambda_i}}{\partial x_1}(y^i) = \frac{\partial w_{\lambda_*}}{\partial x_1}(y_0) > 0,$$

which is a contradiction. Therefore, $w_{\lambda_*} \equiv 0$ and v is symmetric about the plane $x_1 = \lambda_*$.

If $\lambda_* = 0$, then we repeat the above procedure in the $-x_1$-direction. Let

$$\mu_* = \sup\{\bar{\mu} < 0 : v_\mu(x) > v(x) \text{ for } x_1 < \mu,\ x \neq 0_\mu, \text{ and } \mu \geq \bar{\mu}\}.$$

If $\mu_* < 0$, then v is symmetric about the plane $x_1 = \mu_*$ and $v_{x_1}(x) > 0$ when $x_1 > \mu_*$. Otherwise, $\mu_* = 0$ and v is symmetric about the plane $x_1 = 0$.

We apply the method of moving planes in the directions $x_2, \ldots, x_n$, so that v is symmetric about the planes $x_i = \lambda_i$ and $v_{x_i}(x) < 0$ when $x_i > \lambda_i$. This determines a unique point $x_0 = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. If $x_0 \neq 0$, then $\nabla v(x_0) = 0$. Also, $\nabla v \neq 0$ in $\mathbb{R}^n \backslash \{x_0\}$.

Applying the method of moving planes in an arbitrary direction τ, tells us that v is symmetric about a plane $x \cdot \tau = \lambda_\tau$ and $v_\tau(x) < 0$ when $x \cdot \tau > \lambda_\tau$. Since $\nabla v \neq 0$ in $\mathbb{R}^n \backslash \{x_0\}$, the plane $x \cdot \tau = \lambda_\tau$ must pass through x_0. It follows that v is radially symmetric about the point x_0.

To conclude that u is radially symmetric about some point in $\mathbb{R}^n$, there are two cases to consider. If $x_0 = 0$, then $u(y) = |y|^{2-n} v(|y|^{-1})$ in $\mathbb{R}^n \backslash \{0\}$ and u is radially symmetric about the origin. If $x_0 \neq 0$, then v is C^2 in $\mathbb{R}^n$, and it follows that u has a harmonic asymptotic expansioin (2.2.3) in a neighborhood of ∞. Now we can apply the method of moving planes directly to u. Therefore, u is radially symmetric about some point in $\mathbb{R}^n$. □

Corollary 2.3.3. *Let u be a positive C^2 solution of*

$$-\Delta u = u^{(n+2)/(n-2)} \text{ in } \mathbb{R}^n, \quad n \geq 3.$$

Then

$$u(x) = [n(n-2)]^{(n-2)/4} \left(\frac{\alpha}{1 + \alpha^2 |x - x_0|^2} \right)^{(n-2)/2}$$

for some $\alpha > 0$, $x_0 \in \mathbb{R}^n$.

Proof. By the previous theorem, u is radially symmetric about some point x_0. Apply Theorem A.3.2 to $u(x + x_0)$. □

2.4 Symmetry properties of solutions with one or two isolated singularities

The following theorem gives sufficient conditions for a global solution of $-\Delta u = g(u)$ with one or two isolated singularities to be radially symmetric about some point. The proof we give is similar to the proof of Theorem 8.1 in [1].

Theorem 2.4.1. *Let* $u \geq 0$ *be a* C^2 *solution of*

$$-\Delta u = g(u) \text{ in } \mathbb{R}^n \backslash \{0\},\ n \geq 3,$$

with an isolated singularity. Assume that $g(t)$ *is a function on* $[0, \infty)$ *satisfying*

(i) $g(t)$ *is nondecreasing,* $g(0) = 0$,

(ii) $t^{-(n+2)/(n-2)} g(t)$ *is nonincreasing,*

(iii) $\liminf_{t \to \infty} g(t)/t^p > 0$ *for some* $p \geq n/(n-2)$,

(iv) $\frac{1}{t_1^{n+2}} g(t_1^{n-2} s_1) \geq \frac{1}{t_2^{n+2}} g(t_2^{n-2} s_2)$ *when* $t_1 \leq t_2$ *and* $s_1 \geq s_2$.

Then either u *is radially symmetric about the origin or* u *is* C^2 *at the origin and radially symmetric about some point.*

The theorem applies, in particular, to $g(t) = t^p$, $n/(n-2) \leq p \leq (n+2)/(n-2)$.

Proof. Suppose u is not C^2 at the origin. Fix a point $z \neq 0$, and consider the Kelvin transform of u about the point z. Then

$$-\Delta v = f(x, v) \text{ in } \mathbb{R}^n \backslash \{0, -\frac{z}{|z|^2}\},$$

where $f(x, v) = \frac{1}{|x|^{n+2}} g(|x|^{n-2} v)$. Let τ be any direction orthogonal to z. Let $x_\lambda = x - 2(x \cdot \tau - \lambda)\tau$, and define $v_\lambda(x) = v(x_\lambda)$. Let $w_\lambda = v_\lambda - v$. It follows from Lemma 2.1.1, Lemma 2.1.2, Lemma 2.1.3, Lemma 2.2.1, and property (iv) that

$$\lambda_* = \inf\left\{\bar{\lambda} > 0 : w_\lambda(x) > 0 \text{ for } x \cdot \tau > \lambda,\ x \neq 0_\lambda, (-\frac{z}{|z|^2})_\lambda, \text{ and } \lambda \geq \bar{\lambda}\right\}$$

is well-defined. If $\lambda > \lambda_*$, then $w_\lambda(x) > 0$ for $x \cdot \tau > \lambda$, $x \neq 0_\lambda, (-\frac{z}{|z|^2})_\lambda$. Applying the Hopf boundary lemma to w_λ shows that $\frac{\partial v}{\partial \tau}(x) < 0$ when $x \cdot \tau = \lambda$. Therefore, $\frac{\partial v}{\partial \tau}(x) < 0$ when $x \cdot \tau > \lambda_*$.

Suppose $\lambda_* > 0$. Using the definition of λ_*, we have $w_{\lambda_*}(x) \geq 0$ in $x \cdot \tau > \lambda_*$, $x \neq 0_{\lambda_*}, (-\frac{z}{|z|^2})_{\lambda_*}$. By property (iv), $\Delta w_{\lambda_*} \leq 0$ in $x \cdot \tau > \lambda_*$, $x \neq 0_{\lambda_*}, (-\frac{z}{|z|^2})_{\lambda_*}$. If $w_{\lambda_*} \equiv 0$ in $x \cdot \tau > \lambda_*$, $x \neq 0_{\lambda_*}, (-\frac{z}{|z|^2})_{\lambda_*}$, then u is C^2 at the origin. Suppose $w_{\lambda_*} \not\equiv 0$ in $x \cdot \tau > \lambda_*$, $x \neq 0_{\lambda_*}, (-\frac{z}{|z|^2})_{\lambda_*}$. It follows from the Strong maximum principle and the Hopf boundary lemma that

1. $w_{\lambda_*} > 0$ in $x \cdot \tau > \lambda_*$, $x \neq 0_{\lambda_*}, (-\frac{z}{|z|^2})_{\lambda_*}$,

2. $\frac{\partial w_{\lambda_*}}{\partial \tau} > 0$ on $x \cdot \tau = \lambda_*$.

Claim 2.4.2. *There exist $r, b > 0$ so that if $\lambda_* - r < \lambda < \lambda_*$, then*

$$w_\lambda(y) > b,\ 0 < |y - (-\frac{z}{|z|^2})_\lambda| < r$$

and

$$w_\lambda(y) > b,\ 0 < |y - 0_\lambda| < r.$$

Proof. By Lemma 2.1.1 and Lemma 2.1.2, there exists $r_1, b' > 0$ so that

$$w_{\lambda_*} > 2b' \text{ when } 0 < |y - (-\frac{z}{|z|^2})_\lambda| < r_1.$$

By continuity, there exists $\delta > 0$ so that, for $y, y' \in \bar{B}_{r_1}((-\frac{z}{|z|^2})_{\lambda_*})$ and $|y - y'| < \delta$,

$$|v(y) - v(y')| < b'.$$

Let $r' = \min\{\frac{\delta}{2}, r_1\}$.

Suppose $\lambda_* - r' < \lambda < \lambda_*$ and $0 < |y - (-\frac{z}{|z|^2})_\lambda| < r'$. Let $y' = (y_\lambda)_{\lambda_*}$. Then $|y - y'| < \delta$, and it follows that

$$w_\lambda(y) = w_{\lambda_*}(y') + v(y') - v(y) > b'.$$

A similar argument using Lemma 2.1.3 proves the second inequality for some $r'', b'' > 0$. Let $r = \min(r', r'')$, $b = \min(b', b'')$. □

By Lemma 2.2.2, there exist $r_0, S > 0$ so that, if $\lambda_* - r_0 < \lambda < \lambda_*$, $|x_\lambda| > S$, and $x \cdot \tau > \lambda$, then $w_\lambda(x) > 0$. By definition of λ_*, there exist λ_i and points x^i so that $\lambda_* - r_0 < \lambda_i < \lambda_*$, $\lambda_i \to \lambda$, $x^i \cdot \tau > \lambda_i$, and $w_{\lambda_i}(x^i) \leq 0$. Taking $r_0 < r/4$, we can find a sequence of points y^i so that $|y^i| \leq S$, $y^i \cdot \tau > \lambda_i$, $w_{\lambda_i}(y^i) \leq 0$, $|y^i - 0_{\lambda_*}| \geq r/2$, and w_{λ_i} has a local minimum at y^i. By the boundedness of the y^i, we can find a subsequence, which we denote by y^i, converging to y_0.

Notice that $y_0 \cdot \tau \geq \lambda_*$, $y_0 \neq 0_{\lambda_*}, (-\frac{z}{|z|^2})_{\lambda_*}$, and $w_{\lambda_*}(y_0) = 0$. It follows that $y_0 \cdot \tau = \lambda_*$. Since w_{λ_i} has a local minimum at y^i, we have

$$0 = \lim \frac{\partial w_{\lambda_i}}{\partial \tau}(y^i) = \frac{\partial w_{\lambda_*}}{\partial \tau}(y_0) > 0,$$

which is a contradiction. Therefore $\lambda_* = 0$. Since τ was an arbitrary direction orthogonal to z, we conclude that v is cylindrically symmetric about the z-axis.

Let $v_z(x) = \frac{1}{|x|^{n-2}} u(z + \frac{x}{|x|^2})$. For $z \in \mathbb{R}^n \backslash \{0\}$, we have shown that v_z has cylindrical symmetry with respect to the z-axis. It follows that u is cylindrically symmetric about every axis passing through the origin. In particular, u is radially symmetric about the origin.

Suppose u is C^2 at the origin. Then the Kelvin transform of u about the origin has a harmonic asymptotic expansion at ∞. Using the proof of Theorem 2.3.1, we see that u is radially symmetric about some point. □

Chapter 3

Behavior at an isolated singularity

3.1 Superharmonic functions at ∞

In this section, we study the behavior at ∞ of nonnegative superharmonic functions satisfying

$$\int_{\mathbb{R}^n \setminus \bar{B}_1} \frac{v^p}{|x|^\beta}\, dx < \infty \tag{3.1.1}$$

for some $p \geq 1$, $0 \leq \beta < n$. We will use the integral (3.1.1) to estimate the measure of the directions along which v does not have fast decay.

Let Σ_1 denote the unit-sphere in $\mathbb{R}^n$. For $\tau \in \Sigma_1$, let

$$\Gamma(\tau) = \{x \in \mathbb{R}^n : |x - x \cdot \tau| < 3,\ x \cdot \tau > 0\}$$

be the half-infinite cylinder of radius 3 with axis τ, and let $\Gamma_k(\tau) = \Gamma(\tau) \cap (B_{2^{k+1}} \setminus \bar{B}_{2^k})$. Define P_τ^k to be the orthogonal projection along the direction τ onto the plane $x \cdot \tau = 2^k$. For $k \geq 2$ and $\delta, \mu > 0$, we define

$$A(k, \delta, \mu) = \{\tau \in \Sigma_1 : |P_\tau^k(\{v(x) > \delta\} \cap \Gamma_k(\tau))| > \mu\}. \tag{3.1.2}$$

The following theorem gives an upper bound for the measure of $A(k, \delta, \mu)$ in terms of the integral (3.1.1). The proof we give is similar to the proof of Theorem 5.1 in [1].

Theorem 3.1.1. *Let $v \in C^2((\mathbb{R}^n \setminus \bar{B}_1) \setminus \{x_0\})$. Suppose*

$$v(x) \geq 0, \quad \Delta v(x) \leq 0, \qquad x \in (\mathbb{R}^n \setminus \bar{B}_1) \setminus \{x_0\},$$

and

$$\int_{\mathbb{R}^n\setminus\bar{B}_1} \frac{v^p}{|x|^\beta}\,dx < \infty,$$

for some $p \geq 1$, $0 \leq \beta < n$.

If $k \geq 2$ *and* $\delta, \mu > 0$, *then*

$$|A(k,\delta,\mu)| \leq c_n \frac{1}{\delta\mu} 2^{k(\beta-n)/p} \left(\int_{\mathbb{R}^n\setminus\bar{B}_1} \frac{v^p}{|x|^\beta}\,dx \right)^{1/p}, \tag{3.1.3}$$

where c_n *only depends on* n.

Proof. Set $v(x_0) = \liminf_{y\to x_0} v(y)$. Then v is lower semi-continuous, and $\{v(x) > \delta\}$ is an open set. For $\tau \in \Sigma_1$, let $D_{r_k}(\tau) = B_{r_k}(\tau) \cap \Sigma_1$, where $r_k = 2\left(\frac{2^k-(2^{2k}-9)^{1/2}}{2^{k+1}}\right)^{1/2}$. Choose $\tau_i \in A(k,\delta,\mu)$, $i = 1,\ldots,m$, so that

1. $A(k,\delta,\mu) \subset \cup_{i=1}^m D_{r_k}(\tau_i)$,

2. Each $x \in \mathbb{R}^n$ is in at most Θ_n different $D_{r_k}(\tau_i)$, where Θ_n only depends on n.

Then

$$\begin{aligned} |A(k,\delta,\mu)| &\leq \sum_{i=1}^m |D(\tau_i)| \qquad (3.1.4)\\ &\leq |D(\tau_1)| \frac{1}{\mu} \sum_{i=1}^m |P^k_{\tau_i}(\{v(x) > \delta\} \cap \Gamma_k(\tau_i))| \\ &\leq |\Sigma_1| \frac{4^{n-1}}{2^{k(n-1)}} \frac{1}{\mu} \sum_{i=1}^m |P^k_{\tau_i}(\{v(x) > \delta\} \cap \Gamma_k(\tau_i))|. \end{aligned}$$

Let $B(x_0)$ be a ball around x_0 chosen so that, for $i = 1,\ldots,m$,

$$|P^k_{\tau_i}(\bar{B}(x_0) \cap \Gamma_k(\tau_i))| < \frac{1}{2} |P^k_{\tau_i}(\{v(x) > \delta\} \cap \Gamma_k(\tau_i))|.$$

Let $F = \{v(x) > \delta\} \setminus \bar{B}(x_0)$. Then

$$\begin{aligned} |P^k_{\tau_i}(F \cap \Gamma_k(\tau_i))| &\geq |P^k_{\tau_i}(\{v(x) > \delta\} \cap \Gamma_k(\tau_i))| - |P^k_{\tau_i}(\bar{B}(x_0) \cap \Gamma_k(\tau_i))| \qquad (3.1.5)\\ &\geq \frac{1}{2} |P^k_{\tau_i}(\{v(x) > \delta\} \cap \Gamma_k(\tau_i))|. \end{aligned}$$

Let $E = \cup_{i=1}^m F \cap \Gamma_k(\tau_i)$, and let w be the capacitory potential of $\bar{E}$ in $B_{2^{k+2}} \setminus \bar{B}_{2^{k-1}}$ (see [9]):

1. w is harmonic in $(B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}})\backslash\bar{E}$,

2. $w=0$ on $\partial(B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}})$,

3. $w=1$ in E,

4. w is superharmonic (p.59 in [9]) in $B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}$.

Since v is greater than or equal to δ on $\bar{E}$ and superharmonic in $B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}$,

$$\int_{B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}} w\,dx \le \frac{1}{\delta}\int_{B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}} v\,dx. \tag{3.1.6}$$

Lemma 3.1.2.

$$\sum_{i=1}^{m}|P^k_{\tau_i}(F\cap\Gamma_k(\tau_i))| \le \Theta_n(n-2)2^{4-k}\int_{B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}} w\,dx. \tag{3.1.7}$$

Proof. By Theorem 4.21 in [9], there is an increasing sequence $\{w_j\}$ of superharmonic functions on $B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}$ such that (i) each w_j has continuous second partials on $B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}$, (ii) $\lim_{j\to\infty}w_j = w$, and (iii) if U is any neighborhood of ∂E with compact closure in $B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}$, then $w_j=w$ on $(B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}})\backslash\bar{U}$, for j sufficiently large.

Consider the projection $P^k_{\tau_i}(F\cap\Gamma_k(\tau_i))$. This set is open in $\{x\cdot\tau_i=2^k\}$. Let K_i be a compact subset of $P^k_{\tau_i}(F\cap\Gamma_k(\tau_i))$ chosen so that $2|K_i|>|P^k_{\tau_i}(F\cap\Gamma_k(\tau_i))|$. Let V_i be an open set in $\mathbb{R}^n$ chosen so that (i) $\bar{V}_i\subset F\cap\Gamma_k(\tau_i)$ and (ii) $K_i\subset P^k_{\tau_i}(V_i)$. Let $V=\cup_{i=1}^m V_i$.

Fix $\varepsilon>0$. Let U be an open set with smooth boundary chosen so that (i) $\partial\bar{E}\subset U$, (ii) $\bar{U}\subset B_{2^{k+1+\varepsilon}}\backslash\bar{B}_{2^{k-\varepsilon}}$, and (iii) $\bar{V}\subset(E\backslash\bar{U})$. Then

$$|P^k_{\tau_i}((E\backslash\bar{U})\cap\Gamma_k(\tau_i))| > \frac{1}{2}|P^k_{\tau_i}(F\cap\Gamma_k(\tau_i))|. \tag{3.1.8}$$

For j sufficiently large,

$$\begin{aligned}\int_{B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}}|\nabla w_j|^2\,dx &= -\int_{B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}}} w_j\Delta w_j\,dx = -\int_U w_j\Delta w_j\,dx \\ &\le -\int_U \Delta w_j\,dx = -\int_{\partial U}\frac{\partial w_j}{\partial\nu}\,d\sigma \\ &= \int_{\partial((B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}})\backslash\bar{U})}\frac{\partial w_j}{\partial\nu}\,d\sigma - \int_{\partial(B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}})}\frac{\partial w}{\partial\nu}\,d\sigma \\ &= -\int_{\partial(B_{2^{k+2}}\backslash\bar{B}_{2^{k-1}})}\frac{\partial w}{\partial\nu}\,d\sigma,\end{aligned} \tag{3.1.9}$$

where ν is the exterior normal direction.

Using Green's identites,

$$(n-2)\int_{\partial B_R} w\,d\sigma = -\left(R - \frac{R^{n-1}}{(2^{k-1})^{n-2}}\right)\int_{\partial B_{2^{k-1}}} \frac{\partial w}{\partial \nu}\,d\sigma, \qquad 2^{k-1} < R < 2^{k-\varepsilon},$$

$$(n-2)\int_{\partial B_R} w\,d\sigma = \left(\frac{R^{n-1}}{(2^{k+2})^{n-2}} - R\right)\int_{\partial B_{2^{k+2}}} \frac{\partial w}{\partial \nu}\,d\sigma, \qquad 2^{k+1+\varepsilon} < R < 2^{k+2}.$$

Integrating in the variable R,

$$\int_{\partial B_{2^{k-1}}} \frac{\partial w}{\partial \nu}\,d\sigma = \frac{(n-2)}{2^{2k-2}}\frac{1}{\left[\frac{2^{n-n\varepsilon}-1}{n} - \frac{2^{2-2\varepsilon}-1}{2}\right]}\int_{B_{2^k}\backslash \bar{B}_{2^{k-1}}} w\,dx,$$

$$-\int_{\partial B_{2^{k+2}}} \frac{\partial w}{\partial \nu}\,d\sigma = \frac{(n-2)}{2^{2k+4}}\frac{1}{\left[\frac{1-2^{-2+2\varepsilon}}{2} - \frac{1-2^{-n+n\varepsilon}}{n}\right]}\int_{B_{2^{k+2}}\backslash \bar{B}_{2^{k+1}}} w\,dx.$$

Taking $\varepsilon > 0$ sufficiently small,

$$-\int_{\partial(B_{2^{k+2}}\backslash \bar{B}_{2^{k-1}})} \frac{\partial w}{\partial \nu}\,d\sigma \le \frac{(n-2)}{2^{2k-1}}\int_{B_{2^{k+2}}\backslash \bar{B}_{2^{k-1}}} w\,dx. \tag{3.1.10}$$

Given a point $y \in P^k_{\tau_i}((E\backslash\bar{U})\cap\Gamma_k(\tau_i))$, let y^* be the unique point in $(P^k_{\tau_i})^{-1}(y)\cap\partial B_{2^{k+2}}$. Let $\hat{y}$ be any point in $(P^k_{\tau_i})^{-1}(y)\cap(E\backslash\bar{U})$, and let γ_y denote the path from $\hat{y}$ to y^*. Then

$$1 = \left(\int_{\gamma_y} \frac{dw_j}{ds}\,ds\right)^2 \le 2^{k+2}\int_{\gamma_y} |\nabla w_j|^2\,ds.$$

Integrating over $P^k_{\tau_i}((E\backslash\bar{U})\cap\Gamma_k(\tau_i))$,

$$|P^k_{\tau_i}((E\backslash\bar{U})\cap\Gamma_k(\tau_i))| \le 2^{k+2}\int_{P^k_{\tau_i}((E\backslash\bar{U})\cap\Gamma_k(\tau_i))}\int_{\gamma_y} |\nabla w_j|^2\,ds\,dy.$$

Therefore,

$$\sum_{i=1}^m |P^k_{\tau_i}((E\backslash\bar{U})\cap\Gamma_k(\tau_i))| \le \Theta_n 2^{k+2}\int_{B_{2^{k+2}}\backslash \bar{B}_{2^{k-1}}} |\nabla w_j|^2\,dx. \tag{3.1.11}$$

Combining (3.1.8), (3.1.9), (3.1.10), and (3.1.11) proves the lemma. □

Combining (3.1.4), (3.1.5), (3.1.6), and (3.1.7),

$$|A(k,\delta,\mu)| \leq |\Sigma_1| \frac{2^{2n+3}}{2^{kn}} \frac{1}{\mu} \Theta_n(n-2) \frac{1}{\delta} \int_{B_{2^{k+2}} \setminus \bar{B}_{2^{k-1}}} v\, dx.$$

Also,

$$\begin{aligned} \int_{B_{2^{k+2}} \setminus \bar{B}_{2^{k-1}}} v\, dx &\leq \left(\int_{B_{2^{k+2}} \setminus \bar{B}_{2^{k-1}}} |x|^{\beta/(p-1)}\, dx \right)^{(p-1)/p} \left(\int_{B_{2^{k+2}} \setminus \bar{B}_{2^{k-1}}} \frac{v^p}{|x|^\beta}\, dx \right)^{1/p} \\ &\leq \left[\frac{(p-1)|\Sigma_1|}{\beta - n + np} \right]^{(p-1)/p} 2^{(k+2)[\beta - n + np]/p} \left(\int_{\mathbb{R}^n \setminus \bar{B}_1} \frac{v^p}{|x|^\beta}\, dx \right)^{1/p}. \end{aligned}$$

Therefore,

$$|A(k,\delta,\mu)| \leq c_n \frac{1}{\delta\mu} 2^{k(\beta-n)/p} \left(\int_{\mathbb{R}^n \setminus \bar{B}_1} \frac{v^p}{|x|^\beta}\, dx \right)^{1/p}. \tag{3.1.12}$$

where c_n only depends on n. □

3.2 An extension lemma

Let u be a positive C^2 solution of $-\Delta u = g(u)$, in the punctured ball $B_2 \setminus \{0\} \subset \mathbb{R}^n$, $n \geq 3$, with an isolated singularity at the origin, where $g(t)$ is a function on $[0,\infty)$ satisfying:

(i) $g(t)$ is nondecreasing, $g(0) = 0$,

(ii) $t^{-(n+2)/(n-2)} g(t)$ is nonincreasing for $t \geq t_0$,

(iii) $\liminf_{t\to\infty} g(t)/t^p > 0$ for some $p \geq n/(n-2)$

For $z \in B_{1/4}$, consider the Kelvin transform of u about the point z:

$$v_z(x) = \frac{1}{|x|^{n-2}} u(z + \frac{x}{|x|^2}),$$

where $x \in \mathbb{R}^n \setminus (B_1 \cup \{\frac{-z}{|z|^2}\})$. We see that v_z is a positive C^2 solution of

$$-\Delta v_z = f(x, v_z), \qquad 1 \leq |x| \leq 2,$$

where $f(x, v_z) = g(|x|^{n-2} v_z)/|x|^{n+2}$.

Since u is C^2 in $1/4 \le |x| \le 5/4$, there exist δ_0, $M > 0$ so that

$$\delta_0 \le v_z(x) \le \frac{1}{\delta_0}, \qquad 1 \le |x| \le 2, \quad z \in B_{1/4},$$

$$\sup_{1\le|x|\le 2}\left(|v_z| + \sum_{i=1}^{n}|\frac{\partial v_z}{\partial x_i}| + \sum_{i,j=1}^{n}|\frac{\partial^2 v_z}{\partial x_i \partial x_j}|\right) \le M, \qquad z \in B_{1/4}.$$

Lemma 3.2.1. *Let $v > 0$ be a C^2 function in $1 \le |x| \le 2$ satisfying*

$$\delta_0 \le v(x) \le \frac{1}{\delta_0}, \qquad 1 \le |x| \le 2,$$

$$\sup_{1\le|x|\le 2}\left(|v| + \sum_{i=1}^{n}|\frac{\partial v}{\partial x_i}| + \sum_{i,j=1}^{n}|\frac{\partial^2 v}{\partial x_i \partial x_j}|\right) \le M.$$

There exist $\mu_0 > 0$, $m_0 \ge \delta_0 t_0$, so that, if U is an open subset of B_1 with $|U| < \mu_0$, then v has a continuous extension to $|x| \le 2$ satisfying

$$\delta_0/m_0 \le v(x) \le m_0/\delta_0, \qquad |x| \le 2,$$

$$\lim_{\varepsilon\to 0^+}\frac{v(x) - v(x-\varepsilon x)}{\varepsilon} \ge M+1, \qquad x \in \partial B_1,$$

$$-\Delta v(x) \ge \left(\frac{m_0}{\delta_0 t_0}\right)^{(n+2)/(n-2)} g(t_0), \qquad x \in U,$$

where $v \in C^2(U)$.

Proof. By Lemma 6.37 in [7], there exists a function $\bar{v} \in C^2(B_3)$ such that $\bar{v} = v$ in $\bar{B}_2 \backslash B_1$ and

$$\sup_{B_3}\left(|\bar{v}| + \sum_{i=1}^{n}|\frac{\partial \bar{v}}{\partial x_i}| + \sum_{i,j=1}^{n}|\frac{\partial^2 \bar{v}}{\partial x_i \partial x_j}|\right) \le C_1 M.$$

Let $\psi \in C^\infty(\bar{B}_1)$ be a nonnegative smooth function that is continious in $\bar{B}_2$ and chosen so that

$$\min(\delta_0/2, C_1 M) \le (\bar{v} + \psi)(x) \le \max(2/\delta_0, 3C_1 M), \qquad |x| \le 1,$$

$$\psi(x) = 0, \qquad 1 \le |x| \le 2,$$

$$\lim_{\varepsilon\to 0^+}\frac{\psi(x) - \psi(x-\varepsilon x)}{\varepsilon} \ge M + 2 + C_1 M, \qquad x \in \partial B_1,$$

$$\sup_{B_1} |\Delta\psi| \le C_2.$$

Choose $m_0 \ge \delta_0 t_0$ so that $\delta_0/m_0 \le \min(\delta_0/2, CM)$ and $\max(2/\delta_0, 3CM) \le m_0/\delta_0 - 1$. Let

$$h = \left[C_2 + \left(\frac{m_0}{\delta_0 t_0}\right)^{(n+2)/(n-2)} g(t_0) \right] \chi_U.$$

Then (see the discussion in chapter 8 of [7] on Hölder estimates for the first derivatives)

$$\begin{aligned} -\Delta w &= h \qquad \text{in } B_1, \\ w &= 0 \qquad \text{on } \partial B_1, \end{aligned}$$

has a unique solution $w \in C^{1+\alpha}(B_1)$, $0 < \alpha < 1$, with

$$\sup_{B_1} \left(|w| + \sum_{i=1}^{n} |\frac{\partial w}{\partial x_i}| \right) \le C_3 \|h\|_{L_p(B_1)},$$

where $p = n/(1-\alpha)$.

Choose $\mu_0 > 0$ so that $C_3\|h\|_{L_p(B_1)} \le 1$ whenever $|U| < \mu_0$. Extending w to be zero on $\bar{B}_2 \backslash B_1$, it follows that $\tilde{v} + \psi + w$ is the desired extension. □

Lemma 3.2.2. *Let $v_1 > 0$ be a C^2 function satisfying*

$$\delta_0 \le v_1(x) \le \frac{1}{\delta_0}, \qquad 1 \le |x| \le 2,$$

$$\sup_{1\le|x|\le 2} \left(|v_1| + \sum_{i=1}^{n} |\frac{\partial v_1}{\partial x_i}| + \sum_{i,j=1}^{n} |\frac{\partial^2 v_1}{\partial x_i \partial x_j}| \right) \le M,$$

and let v_2 be a C^2 function in $|x| \le 2$.

If v_1 has a continuous extension to $|x| \le 2$ satisfying

$$\lim_{\varepsilon\to 0^+} \frac{v_1(x) - v_1(x - \varepsilon x)}{\varepsilon} \ge M + 1, \qquad x \in \partial B_1,$$

then $v_1 - v_2$ does not have a local minimum on ∂B_1.

Proof. Let $w = v_1 - v_2$. If $x \in \partial B_1$, then

$$\lim_{\varepsilon\to 0^+} \frac{w(x) - w(x - \varepsilon x)}{\varepsilon} > \lim_{\varepsilon\to 0^-} \frac{w(x) - w(x - \varepsilon x)}{\varepsilon}.$$

□

3.3 Defining ε_0

For $\tau \in \Sigma_1$, $0 < \theta < \frac{\pi}{2}$, define $\alpha(\tau, \theta)$ to be the cone with axis τ and half-angle θ:

$$\alpha(\tau, \theta) = \{x \in \mathbb{R}^n : x \cdot \tau > 0, \cos(\theta) < \frac{x \cdot \tau}{|x|}\}.$$

Lemma 3.3.1. *Let $k_0 > 1$. Let v be a function on $\mathbb{R}^n \backslash B_1$, and fix $x \in \mathbb{R}^n \backslash B_1$. Let*

$$\mathcal{D}_1 \subset \left(\alpha(\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_1\right) \text{ and } \mathcal{D}_2 \subset \left(\alpha(-\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_1\right).$$

Suppose for $\tau \in \mathcal{D}_1 \cup \mathcal{D}_2$,

$$v(z) \geq v(z + 2(\lambda - z \cdot \tau)\tau) \text{ whenever } |z| > 1,\ z \cdot \tau < \lambda,\ \lambda \geq k_0.$$

Let $\varepsilon_0 = \frac{1}{2}|\alpha(-\frac{x}{|x|}, \frac{\pi}{8})|$. If $|(\alpha(\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_1) \backslash \mathcal{D}_1| < \varepsilon_0$ and $|(\alpha(-\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_1) \backslash \mathcal{D}_2| < \varepsilon_0$, then

$$v(x) \geq v(y) \text{ whenever } y \in \alpha(\frac{x}{|x|}, \frac{\pi}{8}),\ |y| \geq |x| + 7k_0.$$

Proof. Let $y \in \alpha(\frac{x}{|x|}, \frac{\pi}{8})$ and assume that $|y| \geq |x| + 7k_0$. Let $R = |y| - 2k_0$. Define

$$A_x = \{z \in \Sigma_R : z = x + 2(\lambda - x \cdot \tau)\tau \text{ with } x \cdot \tau < \lambda,\ \lambda \geq k_0,\ \tau \in \mathcal{D}_2\},$$

$$C_x = \{z \in \Sigma_R : z - x \in \alpha(-\frac{x}{|x|}, \frac{\pi}{4})\}.$$

We have

$$R^2 = |z|^2 = |x|^2 - 4\lambda(x \cdot \tau) + 4\lambda^2,$$

so that

$$R^2 \leq (2\lambda + |x|)^2.$$

Since $R \geq |x| + 2k_0$, it follows that

$$2\lambda \geq R - |x| \geq 2k_0$$

and

$$A_x = \left\{ z \in \Sigma_R : \frac{z-x}{|z-x|} \in \mathcal{D}_2 \right\}.$$

By construction,

$$A_x \subset C_x$$

and

$$\left(\alpha(-\frac{x}{|x|}, \frac{\pi}{4}) \cap \Sigma_R \right) \subset C_x \subset \left(\alpha(-\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_R \right).$$

Define

$$A_y = \left\{ z \in \Sigma_R : z = y + 2(\lambda - y \cdot \tau)\tau \text{ with } y \cdot \tau > \lambda,\ \lambda \geq k_0,\ \tau \in \left(\alpha(\frac{y}{|y|}, \frac{\pi}{8}) \cap \mathcal{D}_1 \right) \right\},$$

$$C_y = \{ z \in \Sigma_R : z - y \in \alpha(-\frac{y}{|y|}, \frac{\pi}{8}) \}.$$

We have

$$R^2 = |z|^2 = |y|^2 - 4\lambda(y \cdot \tau) + 4\lambda^2,$$

so that

$$\lambda^2 - (y \cdot \tau)\lambda + \frac{|y|^2 - R^2}{4} = 0.$$

Solving for λ:

$$\lambda = \frac{(y \cdot \tau) + \sqrt{(y \cdot \tau)^2 - (|y|^2 - R^2)}}{2}.$$

Since

$$\sqrt{2}R \geq |y| > R,$$

then

$$y \cdot \tau > \lambda \geq \frac{y \cdot \tau}{2} \geq k_0.$$

By construction,

$$\left(\alpha(-\frac{y}{|y|}, \frac{\pi}{4}) \cap \Sigma_R \right) \subset C_y \subset \left(\alpha(-\frac{y}{|y|}, \frac{3\pi}{8}) \cap \Sigma_R \right).$$

Since

$$R = |y| - 2k_0,$$

then

$$A_y = \{z \in \Sigma_R : \frac{y-z}{|z-x|} \in \alpha(\frac{y}{|y|}, \frac{\pi}{8}) \cap \mathcal{D}_1\},$$

and $A_y \subset C_y$.

Since $y \in \alpha(\frac{x}{|x|}, \frac{\pi}{8})$, we have

$$|C_x \cap C_y| > \left|\left(\alpha(-\frac{x}{|x|}, \frac{\pi}{8}) \cap \Sigma_1\right)\right| R^{n-1}.$$

By assumption,

$$\left|\left(\alpha(-\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_1\right) \backslash \mathcal{D}_2\right| < \varepsilon_0,$$

so that

$$\begin{aligned} |C_x \backslash A_x| &\leq \left|\left(\alpha(-\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_R\right) \backslash \mathcal{D}_2\right| R^{n-1} \\ &< \varepsilon_0 R^{n-1} \\ &< \frac{1}{2}|C_x \cap C_y|. \end{aligned} \tag{3.3.1}$$

Similarly,

$$\begin{aligned} |C_y \backslash A_y| &\leq \left|\left(\alpha(\frac{y}{|y|}, \frac{3\pi}{8}) \cap \Sigma_R\right) \backslash \mathcal{D}_1\right| R^{n-1} \\ &\leq \left|\left(\alpha(\frac{x}{|x|}, \frac{\pi}{2}) \cap \Sigma_R\right) \backslash \mathcal{D}_1\right| R^{n-1} \\ &< \varepsilon_0 R^{n-1} \\ &< \frac{1}{2}|C_x \cap C_y| \end{aligned} \tag{3.3.2}$$

Therefore, $|A_x \cap C_x \cap C_y| > \frac{1}{2}|C_x \cap C_y|$ and $|A_y \cap C_x \cap C_y| > \frac{1}{2}|C_x \cap C_y|$, and it follows that $A_x \cap A_y$ is nonempty. □

3.4 A reflection lemma

Fix $p \in (\frac{n}{n-2}, \frac{n+2}{n-2}]$. Let

$$\beta = 2n - p(n-2),$$

$$\delta = \frac{\delta_0}{2m_0},$$

$$\mu_k = \frac{1}{4}\mu_0 \frac{2^{(k+1)(n-\beta)/2p}}{2^{(n-\beta)/2p} - 1}.$$

Choose k_0 so that

$$\sum_{k=k_0}^{\infty} c_n \frac{1}{\delta \mu_k} 2^{k(n-\beta)/p} \left(\int_{B_{3/2}} u^p \right)^{1/p} < \varepsilon_0,$$

where c_n is the constant from (3.1.3).

If a positive continuous function has a harmonic expansion at infinity, then the reflection process for the method of moving planes can be started in any direction and continued up to some critical point. In particular, the Kelvin transform of a positive C^2 function in $B_2 \backslash \{0\}$ about a regular point in $B_{1/4} \backslash \{0\}$ has this property. In the case where the origin is a nonremovable singularity, the following lemma gives us some control on the size of the critical point (independent of the regular point in $B_{1/4} \backslash \{0\}$).

Lemma 3.4.1. *Let u be a positive C^2 solution of $-\Delta u = g(u)$, in the punctured ball $B_2 \backslash \{0\} \subset \mathbb{R}^n$, $n \geq 3$, with an isolated singularity at the origin, where $g(t)$ is a function on $[0, \infty)$ satisfying:*

(*i*) *$g(t)$ is nondecreasing, $g(0) = 0$,*

(*ii*) *$t^{-(n+2)/(n-2)} g(t)$ is nonincreasing for $t \geq t_0$,*

(*iii*) $\liminf_{t \to \infty} g(t)/t^p > 0$,

(*iv*) *$\frac{1}{t_1^{n+2}} g(t_1^{n-2} s_1) \geq \frac{1}{t_2^{n+2}} g(t_2^{n-2} s_2)$ when $t_1 \leq t_2$ and $s_1 \geq s_2$.*

Let $z \in B_{1/4} \backslash \{0\}$, and consider the Kelvin transform of u about the point z:

$$v(x) = \frac{1}{|x|^{n-2}} u(z + \frac{x}{|x|^2}),$$

where $x \in \mathbb{R}^n \backslash (B_1 \cup \{-\frac{z}{|z|^2}\})$. Let $\mathcal{A} = \cup_{k=k_0}^{\infty} A(k, \delta, \mu_k)$, where $A(k, \delta, \mu)$ is given by (3.1.2). Let $\mathcal{D} = (\alpha(\frac{z}{|z|}, \frac{\pi}{2}) \cap \Sigma_1) \backslash \mathcal{A}$, and fix $\tau \in \mathcal{D}$.

Then either u is C^2 at the origin or

$$v(x) \geq v(x + 2(\lambda - x \cdot \tau)\tau) \text{ whenever } |x| > 1,\ x \cdot \tau < \lambda,\ x \neq -\frac{z}{|z|^2},\ \lambda \geq k_0. \tag{3.4.1}$$

Proof. Define P_τ to be the orthogonal projection along the direction τ onto the plane

$x \cdot \tau = 0$. Let

$$U' = \{P_\tau(x) : v(x) > \delta,\ |x - x \cdot \tau| < 1,\ x \cdot \tau > k_0\},$$

$$U = \{x \in B_1 : P_\tau(x) \in U'\},$$

so that U is an open set with $|U| < \mu_0$ (Theorem 3.1.1).

Using Lemma 3.2.1, we continuously extend v to B_1 so that

$$\delta_0/m_0 \le v(x) \le m_0/\delta_0, \qquad |x| \le 2,$$

$$\lim_{\varepsilon \to 0^+} \frac{v(x) - v(x - \varepsilon x)}{\varepsilon} \ge M + 1, \qquad x \in \partial B_1,$$

$$-\Delta v(x) \ge \left(\frac{m_0}{\delta_0 t_0}\right)^{(n+2)/(n-2)} g(t_0), \qquad x \in U,$$

where $v \in C^2(U)$ and $m_0 \ge \delta_0 t_0$.

Let $x_\lambda = x - 2(x \cdot \tau - \lambda)\tau$, and define $v_\lambda(x) = v(x_\lambda)$. Let $w_\lambda = v_\lambda - v$. Since $v \ge \delta_0/m_0$ in B_2 and $(-\frac{z}{|z|^2}) \cdot \tau < 0$, it follows from Lemma 2.2.1, Lemma 2.1.1, Lemma 2.1.2, and property (iv) that

$$\lambda_* = \inf\left\{\bar\lambda > k_0 : w_\lambda(x) > 0 \text{ for } x \cdot \tau > \lambda,\ x \ne (-\frac{z}{|z|^2})_\lambda, \text{ and } \lambda \ge \bar\lambda\right\}$$

is well-defined. If $\lambda > \lambda_*$, then $w_\lambda(x) > 0$ for $x \cdot \tau > \lambda$, $x \ne (-\frac{z}{|z|^2})_\lambda$.

Suppose $\lambda_* > k_0$. Using the definition of λ_* and the continuity of w_{λ_*}, we have $w_{\lambda_*}(x) \ge 0$ in $x \cdot \tau > \lambda_*$, $x \ne (-\frac{z}{|z|^2})_{\lambda_*}$. By property (iv), $\Delta w_{\lambda_*} \le 0$ in $x \cdot \tau > \lambda_*$, $x \ne (-\frac{z}{|z|^2})_{\lambda_*}$. If $w_{\lambda_*} \equiv 0$ in $x \cdot \tau > \lambda_*$, $x \ne (-\frac{z}{|z|^2})_{\lambda_*}$, $|x_{\lambda_*}| > 1$, then u is C^2 at the origin and we are done. Suppose $w_{\lambda_*} \not\equiv 0$ in $x \cdot \tau > \lambda_*$, $x \ne (-\frac{z}{|z|^2})_{\lambda_*}$, $|x_{\lambda_*}| > 1$. It follows from the Strong maximum principle and the Hopf boundary lemma that

1. $w_{\lambda_*} > 0$ in $x \cdot \tau > \lambda_*$, $x \ne (-\frac{z}{|z|^2})_{\lambda_*}$, $|x_{\lambda_*}| > 1$,

2. $\frac{\partial w_{\lambda_*}}{\partial \tau} > 0$ on $x \cdot \tau = \lambda_*$.

Claim 3.4.2. *There exist $r, b > 0$ so that if $\lambda_* - r < \lambda < \lambda_*$, then*

$$w_\lambda(y) > b,\ 0 < |y - (-\frac{z}{|z|^2})_\lambda| < r.$$

Proof. By Lemma 2.1.1 and Lemma 2.1.2, there exist $b, r_1 > 0$ so that

$$w_{\lambda_*} > 2b \text{ when } 0 < |y - (-\frac{z}{|z|^2})_\lambda| < r_1.$$

By continuity, there exists $\delta > 0$ so that, for $y, y' \in \bar{B}_{r_1}((-\frac{z}{|z|^2})_{\lambda_*})$ and $|y - y'| < \delta$,

$$|v(y) - v(y')| < b.$$

Let $r = \min\{\frac{\delta}{2}, r_1\}$.

Suppose $\lambda_* - r < \lambda < \lambda_*$ and $0 < |y - (-\frac{z}{|z|^2})_\lambda| < r$. Let $y' = (y_\lambda)_{\lambda_*}$. Then $|y - y'| < \delta$, and it follows that

$$w_\lambda(y) = w_{\lambda_*}(y') + v(y') - v(y) > b.$$

□

By Lemma 2.2.2, there exist $r_0, S > 0$ so that, if $\lambda_* - r_0 < \lambda < \lambda_*$, $|x_\lambda| > S$, and $x \cdot \tau > \lambda$, then $w_\lambda(x) > 0$. By definition of λ_*, there exist λ_i and points x^i so that $\lambda_* - r_0 < \lambda_i < \lambda_*$, $\lambda_i \to \lambda$, $x^i \cdot \tau > \lambda_i$, and $w_{\lambda_i}(x^i) \leq 0$. Taking $r_0 < r/4$, we can find a sequence of points y^i so that $|y^i| \leq S$, $y^i \cdot \tau > \lambda_i$, $w_{\lambda_i}(y^i) \leq 0$, $|y^i - 0_{\lambda_*}| \geq r/2$, and w_{λ_i} has a local minimum at y^i. By the boundedness of the y^i, we can find a subsequence, which we denote by y^i, converging to y_0.

Notice that $y_0 \cdot \tau \geq \lambda_*$, $y_0 \neq (-\frac{z}{|z|^2})_{\lambda_*}$, and $w_{\lambda_*}(y_0) = 0$. There are three cases to consider:

1. Suppose $y_0 \cdot \tau = \lambda_*$. Then for (λ, y) sufficiently close to (λ_*, y_0), $\frac{\partial w_\lambda}{\partial \tau}(y)$ is a continuous function. Since w_{λ_i} has a local minimum at y^i, we have

$$0 = \lim \frac{\partial w_{\lambda_i}}{\partial \tau}(y^i) = \frac{\partial w_{\lambda_*}}{\partial \tau}(y_0) > 0,$$

   which is a contradiction.

2. Suppose $y_0 \in \partial B_1(0_{\lambda_*})$. Since $w_{\lambda_*} \geq 0$ in $B_{k_0}(0_{\lambda_*})$, it follows that w_{λ_*} has a local minimum at y_0. By Lemma 3.2.2, w_{λ_*} does not have a local minimum on $\partial B_1(0_{\lambda_*})$, which is a contradiction.

3. Suppose $y_0 \in B_1(0_{\lambda_*})$. By construction, $w_{\lambda_*}(x) \geq \frac{\delta_0}{2m_0}$ for $x \in B_1(0_{\lambda_*}) \backslash U_{\lambda_*}$. It follows

that $y_0 \in U_{\lambda_*}$.

Let $x \in U_{\lambda_*}$. Since g is nondecreasing and $v(x) \le v(x_{\lambda_*}) \le \frac{m_0}{\delta_0}$, we have

$$\left(\frac{\delta_0}{m_0}\right)^{\frac{n+2}{n-2}} \frac{1}{|x|^{n+2}} g(|x|^{n-2}v(x)) \le \frac{g(\frac{m_0}{\delta_0}|x|^{n-2})}{\left(\frac{m_0}{\delta_0}|x|^{n-2}\right)^{\frac{n+2}{n-2}}}.$$

Since $m_0|x|^{n-2}/\delta_0 \ge m_0/\delta_0 \ge t_0$ and $t^{-(n+2)/(n-2)}g(t)$ is nonincreasing for $t \ge t_0$, we have

$$\frac{g(\frac{m_0}{\delta_0}|x|^{n-2})}{\left(\frac{m_0}{\delta_0}|x|^{n-2}\right)^{\frac{n+2}{n-2}}} \le \frac{g(t_0)}{t_0{}^{(n+2)/(n-2)}}.$$

We conclude that

$$-\Delta w_{\lambda_*}(x) = -\Delta v(x_{\lambda_*}) + \Delta v(x) \ge \left(\frac{m_0}{\delta_0 t_0}\right)^{\frac{n+2}{n-2}} g(t_0) - \frac{1}{|x|^{n+2}} g(|x|^{n-2}v(x)) \ge 0.$$

Therefore, w_{λ_*} is superharmonic in U_{λ_*}. Since $w_{\lambda_*}(y_0) = 0$, it follows from the Strong maximum principle that $w_{\lambda_*} \equiv 0$ in U_{λ_*}. By the construction of U and the continuity of w_{λ_*}, we conclude that w_{λ_*} has a local minimum at some point on $\partial B_1(0_{\lambda_*})$, which is a contradiction.

□

3.5 Asymptotic symmetry at the singularity

The following theorem is our main result concerning the asymptotic behavior of solutions of $-\Delta u = g(u)$ at a singularity. The proof we give is similar to the proofs of Theorem 1.1 and Corollary 6.2 in [1].

Theorem 3.5.1. *Let u be a positive C^2 solution of $-\Delta u = g(u)$, in the punctured ball $B_2 \backslash \{0\} \subset \mathbb{R}^n$, $n \ge 3$, with an isolated singularity at the origin.*

Assume that $g(t)$ is a function on $[0, \infty)$ satisfying:

(i) *$g(t)$ is nondecreasing, $g(0) = 0$,*

(ii) *$t^{-(n+2)/(n-2)}g(t)$ is nonincreasing for $t \ge t_0$,*

(iii) *$\liminf_{t\to\infty} g(t)/t^p > 0$ for some $p > n/(n-2)$,*

(iv) $\frac{1}{t_1^{n+2}}g(t_1^{n-2}s_1) \geq \frac{1}{t_2^{n+2}}g(t_2^{n-2}s_2)$ *when* $t_1 \leq t_2$ *and* $s_1 \geq s_2$.

Then

$$u(x) = (1 + \mathcal{O}(|x|))m(|x|) \text{ as } x \to 0,$$

where $m(r) = \frac{1}{|\Sigma_1|}\int_{\Sigma_1} u(ry)d\sigma(y)$.

The theorem applies, in particular, to $g(t) = t^p$, $n/(n-2) < p \leq (n+2)/(n-2)$.

Proof. If u is C^2 at the origin, then

$$u(x) = u(0) + \frac{\partial u}{\partial x_i}(o)x_i + \mathcal{O}(|x|^2) \text{ as } x \to 0.$$

Averaging over Σ,

$$m(|x|) = u(0) + \mathcal{O}(|x|^2) \text{ as } x \to 0.$$

Then

$$\begin{aligned} u(x) &= m(|x|) + \mathcal{O}(|x|) \text{ as } x \to 0 \\ &= (1 + \mathcal{O}(|x|))m(|x|) \text{ as } x \to 0 \end{aligned}$$

and we are done.

Suppose u is not C^2 at the origin. Fix a point $x \in \mathbb{R}\backslash B_1$. Let $\tau_0 = \frac{x}{|x|}$. For $0 < \varepsilon < 1/4$, define

$$v_\varepsilon(x) = \frac{1}{|x|^{n-2}}u(\varepsilon\tau_0 + \frac{x}{|x|^2}),$$

$$A_\varepsilon = \{\tau \in \Sigma_1 : |P_\tau^k(\{v(x) > \delta\} \cap \Gamma_k(\tau))| > \mu\},$$

$$\mathcal{A}_\varepsilon = \cup_{k=k_0}^{\infty} A_\varepsilon(k, \delta, \mu_k),$$

$$\mathcal{D}_\varepsilon = (\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1)\backslash\mathcal{A}.$$

By Theorem 3.1.1 and the way we chose k_0 (see the discussion before Lemma 3.4.1), we have $|\mathcal{A}_\varepsilon| < \varepsilon_0$. In particular,

$$|(\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1)\backslash\mathcal{D}_\varepsilon| < \varepsilon_0.$$

By Lemma 3.4.1, for $\tau \in \mathcal{D}_\varepsilon$, we have

$$v_\varepsilon(z) \geq v_\varepsilon(z + 2(\lambda - z \cdot \tau)) \text{ whenever } |z| > 1,\ z \cdot \tau < \lambda,\ z \neq -\frac{\tau_0}{\varepsilon^2},\ \lambda \geq k_0.$$

Define $\chi_\varepsilon : \Sigma_1 \to \mathbb{R}$ to be the charachteristic function of $\mathcal{D}_\varepsilon$. Let

$$\chi_0 = \limsup_{\varepsilon \to 0} \chi_\varepsilon,$$

$$\mathcal{D}_0 = \{\tau \in \Sigma : \chi_0(\tau) = 1\}.$$

By Fatou's lemma,

$$\begin{aligned}
|(\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1) \backslash \mathcal{D}_0| &= \int_{\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1} 1 - \chi_0 \\
&= \int_{\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1} \liminf_{\varepsilon \to 0} (1 - \chi_\varepsilon) \\
&\leq \liminf_{\varepsilon \to 0} \int_{\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1} 1 - \chi_\varepsilon \\
&= \liminf_{\varepsilon \to 0} |(\alpha(\tau_0, \frac{\pi}{2}) \cap \Sigma_1) \backslash \mathcal{D}_\varepsilon| \\
&< \varepsilon_0.
\end{aligned}$$

Let

$$v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}).$$

Claim 3.5.2. *If $\tau \in \mathcal{D}_0$, then $v(z) \geq v(z + 2(\lambda - z \cdot \tau))$ whenever $|z| > 1$, $z \cdot \tau < \lambda$, $\lambda \geq k_0$.*

Proof. If $\tau \in \mathcal{D}_0$, then there exists a sequence ε_i converging to zero so that $\tau \in \mathcal{D}_{\varepsilon_i}$. Let $\lambda \geq k_0$, and fix $|z| > 1$ with $z \cdot \tau < \lambda$. For ε_i sufficiently small, we have $v_{\varepsilon_i}(z) \geq v_{\varepsilon_i}(z + 2(\lambda - z \cdot \tau))$. It follows from continuity that $v(z) \geq v(z + 2(\lambda - z \cdot \tau))$. □

Applying the same argument in the $-\tau_0$ direction, we find a set $\widetilde{\mathcal{D}}_0 \subset (\alpha(-\tau_0, \frac{\pi}{2}) \cap \Sigma_1)$ with the properties: (1) $|(\alpha(-\tau_0, \frac{\pi}{2}) \cap \Sigma_1) \backslash \widetilde{\mathcal{D}}_0| < \varepsilon_0$ and (2) if $\tau \in \widetilde{\mathcal{D}}_0$, then $v(z) \geq v(z + 2(\lambda - z \cdot \tau))$ whenever $|z| > 1$, $z \cdot \tau < \lambda$, $\lambda \geq k_0$.

It follows from Lemma 3.3.1 that

$$v(x) \geq v(y) \text{ whenever } y \in \alpha(\frac{x}{|x|}, \frac{\pi}{8}),\ |y| \geq |x| + 7k_0.$$

In particular, there exists a constant $C > 0$ (independent of x) so that

$$v(x) \geq v(y) \text{ whenever } |y| \geq |x| + Ck_0.$$

Since $x \in \mathbb{R}^n \backslash B_1$ was arbitrary, we have

$$\sup_{\partial B_{R+Ck_0}} v \leq \inf_{\partial B_R} v \leq \sup_{\partial B_R} v \leq \inf_{\partial B_{R-Ck_0}} v. \tag{3.5.1}$$

By the Strong maximum principle,

$$v(x) \geq \left(\frac{R - Ck_0}{|x|}\right)^{n-2} \inf_{\partial B_{R-Ck_0}} v \; |x| \geq R - Ck_0.$$

It follows that

$$\inf_{\partial B_{R+Ck_0}} v \geq \left(\frac{R - Ck_0}{R + Ck_0}\right)^{n-2} \inf_{\partial B_{R-Ck_0}} v. \tag{3.5.2}$$

Combining 3.5.1 and 3.5.2, we have

$$\sup_{\partial B_{R+Ck_0}} v \leq (1 + \mathcal{O}(1/R)) \inf_{\partial B_{R+Ck_0}} v \text{ as } R \to \infty.$$

It follows that

$$\sup_{\partial B_r} u \leq (1 + \mathcal{O}(r)) \inf_{\partial B_r} u \text{ as } r \to 0.$$

Therefore,

$$u(x) = (1 + \mathcal{O}(|x|)) m(|x|) \text{ as } |x| \to 0.$$

□

Chapter 4

Characterization of the asymptotic behavior at a singularity for positive solutions of $-\Delta u = u^{\frac{n+2}{n-2}}$

Let u be a C^2 solution of $-\Delta u = u^{(n+2)/(n-2)}$ in $B_2\backslash\{0\}$, $n \geq 3$, with a nonremovable isolated singularity at the origin. Let

$$m(r) = \frac{1}{|\Sigma_1|}\int_{\Sigma_1} u(ry)d\sigma(y).$$

Theorem 3.5.1 tells us that

$$u(x) = (1 + \mathcal{O}(|x|))m(|x|) \text{ as } x \to 0. \tag{4.0.1}$$

We will show that $u(x) = (1 + o(1))\phi(|x|)$ as $x \to 0$, where ϕ is a radial singular solution. The proof we give is similar to the proof of Theorem 1.2 given in Section 7 of [1].

4.1 Preliminary estimates

If $r < 2$, then it follows from Lemma 2.1.3 that u is a distribution solution of $-\Delta u = u^{(n+2)/(n-2)}$ in B_r. We have

$$-\int_{B_r} \eta u^{(n+2)/(n-2)}\,dx = \int_{B_r} (\Delta\eta)u\,dx - \int_{\partial B_r} u\frac{\partial\eta}{\partial\nu}\,d\sigma + \int_{\partial B_r} \eta\frac{\partial u}{\partial\nu}\,d\sigma \tag{4.1.1}$$

for all $\eta \in C_0^\infty(B_r)$. Take $\phi \in C^\infty(B_r)$, and let $\psi \in C_0^\infty(B_r)$ with $\psi \equiv 1$ in a neighborhood of the origin. Applying Green's formula to $(1-\psi)\phi$ and (4.1.1) to $\psi\phi$, we see that

$$-\int_{B_r} \phi u^{(n+2)/(n-2)}\,dx = \int_{B_r} (\Delta\phi) u\,dx - \int_{\partial B_r} u\frac{\partial\phi}{\partial\nu}\,d\sigma + \int_{\partial B_r} \phi\frac{\partial u}{\partial\nu}\,d\sigma.$$

Taking $\phi = |x|^2$ and $\phi = 1$:

$$-\int_{B_r} |x|^2 u^{(n+2)/(n-2)}\,dx = 2n\int_{B_r} u\,dx - r\int_{\partial B_r} u\,d\sigma + r^2\int_{\partial B_r} \frac{\partial u}{\partial\nu}\,d\sigma$$

and

$$-\int_{B_r} u^{(n+2)/(n-2)}\,dx = \int_{\partial B_r} \frac{\partial u}{\partial\nu}\,d\sigma.$$

It follows that

$$\int_{B_r} (r^2 - |x|^2) u^{(n+2)/(n-2)}\,dx + 2r\int_{\partial B_r} u\,d\sigma = 2n\int_{B_r} u\,dx \tag{4.1.2}$$

and

$$-m'(r) = \frac{1}{|\Sigma_1| r^{n-1}} \int_{B_r} u^{(n+2)/(n-2)}\,dx > 0. \tag{4.1.3}$$

Since $m(r)$ is decreasing,

$$\begin{aligned} r^{n+2} m^{(n+2)/(n-2)}(r) &= \frac{n(n+2)}{2} m^{(n+2)/(n-2)}(r) \int_0^r (r^2 - t^2) t^{n-1}\,dt \\ &\le \frac{n(n+2)}{2} \int_0^r m^{(n+2)/(n-2)}(t)(r^2 - t^2) t^{n-1}\,dt \\ &= \frac{n(n+2)}{2} \frac{1}{|\Sigma_1|} \int_{B_r} (r^2 - |x|^2) m^{(n+2)/(n-2)}(|x|)\,dx. \end{aligned}$$

We know, from the proof of Theorem 3.5.1, that there exists $R_0 > 0$ so that for $r < R_0$, $m(r) \le 2\,(\inf_{\partial B_r} u)$. Combining this with the previous calculation, we have

$$r^{n+2} m^{(n+2)/(n-2)}(r) \le \frac{n(n+2)}{2} \frac{1}{|\Sigma_1|} \int_{B_r} (r^2 - |x|^2)(2u(x))^{(n+2)/(n-2)}\,dx$$

for $r < R_0$. Combining this with (4.1.2) shows us that

$$r^{n+2} m^{(n+2)/(n-2)}(r) \le C\int_{B_r} u\,dx$$

for $r < R_0$. Using

$$\int_{B_r} u\,dx = |\Sigma_1| \int_0^r m(t)t^{n-1}\,dt$$
$$\leq Cr^{4n/(n+2)} \left(\int_0^r m(t)t^{n-1}\,dt\right)^{(n-2)/(n+2)},$$

we conclude that

$$r^{n+2} m^{(n+2)/(n-2)}(r) \leq Cr^{4n/(n+2)} \left(\int_0^R m^{(n+2)/(n-2)}(t)t^{n-1}\,dt\right)^{(n-2)/(n+2)} \tag{4.1.4}$$

for $r \leq R_0$.

Fix $R \leq R_0$. Let $A = \int_0^R m^{(n+2)/(n-2)}(r)r^{n-1}dr$. Using (4.1.4), we have

$$r^{n-1} m^{(n+2)/(n-2)}(r) \leq Cr^{(n-6)/(n+2)} A^{(n-2)/(n+2)}.$$

Integrating from 0 to R,

$$A \leq CR^{2(n-2)/(n+2)} A^{(n-2)/(n+2)},$$

so that

$$\int_0^R m^{(n+2)/(n-2)}(t)t^{n-1}dt \leq CR^{(n-2)/2}. \tag{4.1.5}$$

Lemma 4.1.1.

$$m(r) = \mathcal{O}(\frac{1}{r^{(n-2)/2}}), \qquad m'(r) = \mathcal{O}(\frac{1}{r^{n/2}}) \qquad \textit{as } r \to 0. \tag{4.1.6}$$

Proof. Since $m(r)$ is decreasing, it follows from (4.1.5) that

$$m(R) \leq C \int_0^R m^{(n+2)/(n-2)}(r)r^{n-1}\,dr \leq C\frac{1}{R^{(n-2)/2}}$$

for $R \leq R_0$.

Moreover, for $R \leq R_0$ sufficiently small, we conclude from (4.0.1), (4.1.3), (4.1.5) that

$$|m'(R)| \leq \frac{C}{R^{n-1}} \int_0^R m^{(n+2)/(n-2)}(r)r^{n-1}dr \leq C\frac{1}{R^{n/2}}.$$

□

4.2 Asymptotic behavior at a nonremovable singularity

Let

$$t = -\log r, \quad r = e^{-t},$$

and consider

$$\psi(t,\theta) = r^{(n-2)/2}u(r,\theta), \qquad \theta \in \Sigma_1.$$

Then

$$\frac{\partial^2 \psi}{\partial t^2} + \Delta_\theta \psi - \left(\frac{n-2}{2}\right)^2 \psi + \psi^{(n+2)/(n-2)} = 0. \tag{4.2.1}$$

Let β denote the spherical average of ψ:

$$\beta(t) = r^{(n-2)/2}m(r) = \frac{1}{|\Sigma_1|}\int_{\Sigma_1} \psi(t,\theta)\, d\theta.$$

Then

$$\psi(t,\theta) = \beta(t)(1+\mathcal{O}(e^{-t})) \text{ as } t \to \infty$$

and

$$\beta'(t) + \frac{n-2}{2}\beta(t) = -r^{n/2}m'(r) \geq 0.$$

It follows from Lemma 4.1.1 that

$$\beta = \mathcal{O}(1), \qquad \beta' = \mathcal{O}(1) \qquad \text{as } t \to \infty.$$

Lemma 4.2.1.

$$\frac{\partial}{\partial t}(\psi - \beta) = \beta\mathcal{O}(e^{-t}), \qquad |\nabla_\theta(\psi-\beta)| = \beta\mathcal{O}(e^{-t}) \qquad \textit{as } t \to \infty.$$

Proof. We have $-\Delta(u-m) = f$, where $f(x) = \frac{1}{\Sigma_1}\int_{\Sigma_1}\left(u^{\frac{n+2}{n-2}}(x) - u^{\frac{n+2}{n-2}}(|x|y)\right) d\sigma(y)$. It follows from Theorem 3.9 in [7], if we take $\Omega : \frac{1}{2}r < |x| < 2r$, that

$$\sup_{|x|=r} |\nabla(u-m)| \leq c\left(\frac{\sup_\Omega |u-m|}{r} + r\sup_\Omega |f|\right).$$

Combining the previous estimate with Theorem 3.5.1, we have

$$\sup_{|x|=r} |\nabla(u-m)| \le c\left(\sup_{\Omega} m + r^2 \sup_{\Omega} m^{(n+2)/(n-2)}\right)$$

.

Claim 4.2.2. $\sup_\Omega u \le c \inf_\Omega u$, *where* c *is independent of* $r < 1$.

Proof. Consider $u > 0$ as a solution of $\Delta u + a(x)u = 0$ in the domain $\frac{1}{4}r < |x| < 4r$. By Lemma 4.1.1, we have $|a(x)| \le c/|x|^2$, in say B_1. Let $K_\rho(x)$ denote the cube in $\mathbb{R}^n$ of side ρ and center x with sides parallel to the coordinate axes. Fix $x_0 \in \Omega$, and let $K = K_{r/12}(x_0)$. By Theorem 1.1 in [13], $\sup_K u \le c \inf_K u$, where c is independent of $r < 1$. We know that $\{1/4 < |x| < 4\}$ can be covered by a finite number of cubes, say N, of side $1/12$ with center at some point in $\{1/4 < |x| < 4\}$. Scaling, we see that Ω can be covered by N cubes of side $r/12$ with center at some point in Ω. It follows that $\sup_\Omega u \le c \inf_\Omega u$, where c is independent of $r < 1$. □

Therefore,

$$\sup_{|x|=r} |\nabla(u-m)| \le c\left(m(r) + r^2 m^{(n+2)/(n-2)}(r)\right).$$

Applying Lemma 4.1.1 to $r^2 m^{4/(n-2)}(r)$, we conclude that

$$|\nabla(u-m)(x)| \le cm(r) \quad \text{for } |x| = r.$$

In particular,

$$|\frac{\partial}{\partial r}(u-m)| \le cm, \qquad |\nabla_\theta(u-m)| \le crm.$$

Since $\psi(t,\theta) - \beta(t) = r^{(n-2)/2}(u(r,\theta) - m(r))$, $r = e^{-t}$, the lemma follows from the previous estimates and Theorem 3.5.1. □

Lemma 4.2.3. *There is a unique asymptotic constant* D_∞ *so that*

$$\beta'^2 = \left(\frac{n-2}{2}\right)^2 \beta^2 - \frac{n-2}{n}\beta^{2n/(n-2)} + D_\infty + (\beta^2 + \beta'^2)\mathcal{O}(e^{-t}) \tag{4.2.2}$$

as $t \to \infty$.

Proof. Multiplying (4.2.1) by $\frac{\partial\psi}{\partial t}$ and integrating over $B_t\backslash\bar{B}_s$:

$$\int_{\Sigma_1}\left(|\frac{\partial\psi}{\partial t}|^2-|\nabla_\theta\psi|^2-\left(\frac{n-2}{2}\right)^2\psi^2+\frac{n-2}{n}\psi^{2n/(n-2)}\right)\Bigg|_s^t d\theta=0.$$

Using Lemma 4.2.1,

$$\left(\beta'^2-\left(\frac{n-2}{2}\right)^2\beta^2+\frac{n-2}{n}\beta^{2n/(n-2)}\right)\Bigg|_s^t=(\beta^2+\beta'^2)\mathcal{O}(e^{-t})$$

as $t\to\infty$. Let

$$D(t)=\beta'^2-\left(\frac{n-2}{2}\right)^2\beta^2+\frac{n-2}{n}\beta^{2n/(n-2)}$$

so that

$$D(t)=D(s)+(\beta^2(s)+\beta'^2(s))\mathcal{O}(e^{-s})+\mathcal{O}(e^{-t})$$

for $t\geq s$ as $s\to\infty$. This determines a unique asymptotic constant $D_\infty=\lim_{t\to\infty}D(t)$. Taking $t\to\infty$ in the previous equation,

$$D_\infty=D(s)+(\beta^2(s)+\beta'^2(s))\mathcal{O}(e^{-s})$$

as $s\to\infty$, which completes the proof of the lemma. □

Notice that $-\frac{2}{n}\left(\frac{n-2}{n}\right)^n\leq D_\infty\leq 0$. If $D_\infty<-\frac{2}{n}\left(\frac{n-2}{n}\right)^n$, then $\beta'^2<0$ for t sufficiently large. If $D_\infty>0$, then $\beta(t)<0$ infinitely often as $t\to\infty$ (see the discussion on radial solutions in the appendix).

If $-\frac{2}{n}\left(\frac{n-2}{n}\right)^n\leq D_\infty<0$, then it follows from the discussion on radial solutions in the appendix that β is asymptotic to a translate of the corresponding singular solution of (A.3.3).

Suppose $D_\infty=0$. We know from (4.2.2) that the behavior of β is determined by the points where $\beta'=0$. In addition, we see that there are $\beta_0,t_0>0$ so that $\beta'\neq 0$ when $\beta<\beta_0$ and $t>t_0$. We also know that there is a point $t_1>t_0$ so that $\beta(t_1)<\beta_0$. Moreover, the arguments given in the appendix tell us that there is a point $t_2>t_0$ so that $\beta(t_2)<\beta_0$ and $\beta'(t_2)<0$. It follows that $\beta(t)<\beta_0$ and $\beta'(t)<0$ for all $t\geq t_2$, and we conclude that

$$\lim_{t\to\infty}\beta(t)=0,$$

$$\lim_{t \to \infty} \beta'(t) = 0,$$

$$\beta'(t) < 0 \text{ for } t > t_2.$$

Let $0 < \lambda < \frac{n-2}{2}$, for t sufficiently large we have $\beta'^2 \geq \lambda^2 \beta^2$ or $\beta'/\beta \leq -\lambda$ so that $\beta = \mathcal{O}(e^{-\lambda t})$ as $t \to \infty$, and hence $\beta' = \mathcal{O}(e^{-\lambda t})$ as $t \to \infty$. Plugging this into (4.2.2), we see that

$$\frac{\beta'^2}{\beta^2} \geq \left(\frac{n-2}{2}\right)^2 + \mathcal{O}(e^{-t}) \text{ as } t \to \infty.$$

For t sufficiently large, say $t \geq t'$, we have

$$\frac{\beta'}{\beta} \leq -\frac{n-2}{2} + ce^{-t},$$

and therefore

$$\beta(t) \leq c_0 e^{-\frac{n-2}{2}t} \text{ for } t \geq t'.$$

It follows that u has a removable singularity at the origin.

Theorem 4.2.4. *Let u be a positive C^2 solution of $-\Delta u = u^{(n+2)/(n-2)}$ in $B_2 \backslash \{0\}$, $n \geq 3$, with a nonremovable isolated singularity at the origin. Then there is a unique asymptotic constant D_∞ in the interval $-\frac{2}{n}\left(\frac{n-2}{n}\right)^n \leq D_\infty < 0$ and a singular solution $\psi(t)$ of (A.3.3) so that*

$$u(x) = (1 + o(1)) \frac{1}{|x|^{(n-2)/2}} \psi_{D_\infty}(-\log|x|)$$

as $x \to 0$.

Proof. By the above discussion,

$$\beta(t) = (1 + o(1)) \psi_{D_\infty}(t)$$

as $t \to \infty$, where D_∞ is the unique asymptotic constant given by $\lim_{t \to \infty} D(t)$ and $\psi_{D_\infty}(t)$ is the singular solution of (A.3.3) corresponding to the constant D_∞. Since

$$u(x) = (1 + \mathcal{O}(|x|)) \frac{1}{|x|^{(n-2)/2}} \beta(-\log|x|)$$

as $x \to 0$, the proof is complete. □

Chapter 5

Upper half space

Let $\mathbb{R}^n_+$ denote the upper half space:

$$\mathbb{R}^n_+ = \{(x,t) : x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\, t > 0\}.$$

We are interested in the following boundary value problem:

$$\begin{cases} -\Delta u = u^p \text{ in } \mathbb{R}^n_+ \backslash \{(0,t_0)\},\, t_0 > 0, \quad n \geq 3, \\ \frac{\partial u}{\partial t} = cu^\beta \text{ on } \partial\mathbb{R}^n_+, \quad c > 0, \end{cases} \tag{5.0.1}$$

where u has an isolated singularity at $(0, t_0)$.

When $u \in C^2(\overline{\mathbb{R}^n_+} \backslash \{(0,t_0)\})$ is a positive solution of (5.0.1) with $n \leq p(n-2) \leq n+2$, $\beta \geq 1$, and $\beta(n-2) \geq n$, we will show that u must be cylindrically symmetric about some axis orthogonal to $\partial\mathbb{R}^n_+$.

5.1 Preliminary results

Let $u \in C^2(\overline{\mathbb{R}^n_+} \backslash \{(0,t_0)\})$, $n \geq 3$, and consider the Kelvin transform of u about the origin:

$$v(x,t) = \frac{1}{|(x,t)|^{n-2}} u\left(\frac{x}{|(x,t)|^2}, \frac{t}{|(x,t)|^2}\right), \quad (x,t) \in \overline{\mathbb{R}^n_+} \backslash \{(0,0), (0, \frac{1}{t_0})\}.$$

Then

$$\Delta v(x,t) = \frac{1}{|(x,t)|^{n+2}} \Delta u\left(\frac{x}{|(x,t)|^2}, \frac{t}{|(x,t)|^2}\right), \quad (x,t) \in \mathbb{R}^n_+ \backslash \{(0, \frac{1}{t_0})\},$$

and

$$\frac{\partial v}{\partial t}(x,0) = \frac{1}{|x|^n}\frac{\partial u}{\partial t}(x,0), \quad x \in \mathbb{R}^{n-1}\backslash\{0\}.$$

If u is a solution of (5.0.1), then v is a solution of

$$\begin{cases} -\Delta v = \frac{1}{|(x,t)|^{n+2-p(n-2)}} v^p \text{ in } \mathbb{R}^n_+\backslash\{(0,t_0)\},\ t_0 > 0, \quad n \geq 3, \\ \frac{\partial v}{\partial t} = c|x|^{\beta(n-2)-n} v^\beta \text{ on } (\partial\mathbb{R}^n_+)\backslash\{(0,0)\}, \quad c > 0. \end{cases} \tag{5.1.1}$$

If u is C^2 at the origin, then there exists an $R_0 > 0$ so that v has a harmonic asymptotic expansion (2.2.3) in $\overline{\mathbb{R}^n_+}\backslash B_{R_0}$. We want to use the asymptotic expansion to start the method of moving planes in directions orthogonal to t. For $x = (x_1, \ldots, x_{n-1})$, we denote the reflection of x about the plane $x_1 = \lambda$ by $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_{n-1})$.

Lemma 5.1.1. *Let v be a function with an asymptotic expansion (2.2.3) in $\overline{\mathbb{R}^n_+}\backslash B_{R_0}$, $n \geq 3$. If $a_0 > 0$, then there exist constants $\bar\lambda > 0$, $R > R_0$ so that $v(x_\lambda, t) > v(x,t)$ whenever $\lambda \geq \bar\lambda$, $(x_\lambda, t) \in \overline{\mathbb{R}^n_+}\backslash \bar B_{R_0}$, and $x_1 > \lambda$.*

Proof. Since x_1 is orthogonal to t, the proof is the same as the proof of Lemma 2.2.1. □

The following lemma gives us some control over the behavior of the Kelvin transform at the origin. The proof we give is similar to the proof of Lemma 2.1 in [12].

Lemma 5.1.2. *Let v be a positive solution of*

$$\begin{cases} -\Delta v = \frac{1}{|(x,t)|^{n+2-p(n-2)}} v^p \textit{ in } \mathbb{R}^n_+\backslash\{(0,\frac{1}{t_0})\}, \\ \frac{\partial v}{\partial t} = c|x|^{\beta(n-2)-n} v^\beta \textit{ on } (\partial\mathbb{R}^n_+)\backslash\{(0,0)\}, \quad c > 0, \end{cases}$$

where $v \in C^2(\overline{\mathbb{R}^n_+}\backslash\{(0,0),(0,\frac{1}{t_0})\})$, $n \geq 3$.

Assume that

1. $n \leq p(n-2) \leq n+2$,

2. $\beta \geq 1$, $\beta(n-2) \geq n$.

Let $2\delta = \min(\frac{1}{t_0}, c^{1/(n-\beta(n-2))})$. Then there exists $m_0 > 0$ so that $v > m_0$ in $B_\delta \cap \mathbb{R}^n_+$.

Proof. By the Hopf boundary lemma, v is positive on $\overline{\mathbb{R}^n_+}\backslash\{(0,0),(0,\frac{1}{t_0})\}$ if and only if it is positive on $\mathbb{R}^n_+\backslash\{(0,\frac{1}{t_0})\}$. It follows that $m = \frac{1}{2}\min\{v(x,t) : |(x,t)| = \delta,\ t \geq 0\}$ is well-defined and positive. Let $m_0 = \min(1, m)$.

For $0 < r < \delta$, consider the comparison function ϕ:

$$\phi(x,t) = 1 - 2\frac{r^{n-2}}{|(x,t)|^{n-2}} + t.$$

Let $\tilde{v} = v - m_0\phi$, and let $\Omega_r = (B_\delta \cap \mathbb{R}^n_+)\backslash \bar{B}_r$. Then

$$\begin{cases} -\Delta\tilde{v} \geq 0 & \text{in } \Omega_r, \\ \frac{\partial \tilde{v}}{\partial t} = c|x|^{\beta(n-2)-n}v^\beta - m_0 & \text{on } \partial\Omega_r \cap \partial\mathbb{R}^n_+. \end{cases}$$

Suppose that $\tilde{v} < 0$ in Ω_r. Then, for some $(x', t') \in \partial\Omega_r$,

$$\tilde{v}(x', t') = \min_{\overline{\Omega}_r} \tilde{v} < 0.$$

On $\partial\Omega_r \cap \partial B_\delta$, we have $\tilde{v} > v - 2m_0 \geq 0$, and on $\partial\Omega_r \cap \partial B_r$, we have $\tilde{v} > v > 0$. Therefore $r < |x'| < \delta$, $t' = 0$, and $\frac{\partial \tilde{v}}{\partial t}(x', 0) \geq 0$. Using the boundary condition for $\tilde{v}$ at $(x', 0)$, we have

$$v(x', 0) \geq \left(|x'|^{n-\beta(n-2)}\frac{m_0}{c}\right)^{1/\beta}.$$

It follows that

$$0 > \tilde{v}(x', 0) = v(x', 0) - \left(m_0 - 2m_0\frac{r^{n-2}}{|x'|^{n-2}}\right) \geq \left(|x'|^{n-\beta(n-2)}\frac{m_0}{c}\right)^{1/\beta} - m_0 \geq 0.$$

Therefore $\tilde{v} \geq 0$ in Ω_r. Since this is true for all $r \in (0, \delta)$, we conclude that $v \geq m_0$ in $B_\delta \cap \mathbb{R}^n_+$. □

The following lemma gives us some control over the behavior of $v_{\lambda_0} - v$ in a neighborhood of the singularity on the boundary at $(0_{\lambda_0}, 0)$. The proof we give is similar to the proof of Lemma 2.3 in [12].

Lemma 5.1.3. *Let v be a positive solution of*

$$\begin{cases} -\Delta v = \frac{1}{|(x,t)|^{n+2-p(n-2)}}v^p \text{ in } \mathbb{R}^n_+\backslash\{(0, \frac{1}{t_0})\}, \\ \frac{\partial v}{\partial t} = c|x|^{\beta(n-2)-n}v^\beta \text{ on } (\partial\mathbb{R}^n_+)\backslash\{(0,0)\}, \quad c > 0, \end{cases}$$

where $v \in C^2(\overline{\mathbb{R}^n_+}\backslash\{(0,0), (0, \frac{1}{t_0})\})$, $n \geq 3$. Assume that

1. $n \leq p(n-2) \leq n+2$,

2. $\beta \geq 1$, $\beta(n-2) \geq n$.

Let $v_\lambda(x,t) = v(x_\lambda, t)$. *Fix* $\lambda_0 > 0$, *and let* $2\delta = \min(\lambda_0, \frac{1}{t_0})$. *Suppose that*

1. $v_{\lambda_0} \geq v$ *in* $B_{2\delta}(0_{\lambda_0}, 0) \cap \mathbb{R}^n_+$

2. $v_{\lambda_0} \not\equiv v(x,t)$ *in* $B_{2\delta}(0_{\lambda_0}, 0) \cap \mathbb{R}^n_+$.

Then there exists $\varepsilon_0 > 0$ *so that* $v_{\lambda_0} - v > \varepsilon_0$ *in* $B_\delta(0_{\lambda_0}, 0) \cap \mathbb{R}^n_+$.

Proof. Let $w = v_{\lambda_0} - v$. Then

$$\begin{cases} -\Delta w \geq \frac{1}{|(x,t)|^{n+2-p(n-2)}} c_0(x,t) w & \text{in } B_{2\delta}(0_{\lambda_0}, 0) \cap \mathbb{R}^n_+, \\ \frac{\partial w}{\partial t} \leq |x|^{\beta(n-2)-n} d_0(x) w & \text{on } (\partial B_{2\delta}(0_{\lambda_0}, 0) \cap \partial \mathbb{R}^n_+) \backslash \{(0_{\lambda_0}, 0)\}, \end{cases}$$

where $c_0(x,t)$ is between $pv_{\lambda_0}^{p-1}(x,t)$ and $pv^{p-1}(x,t)$, and $d_0(x)$ is between $c\beta v_{\lambda_0}^{\beta-1}(x,0)$ and $c\beta v^{\beta-1}(x,0)$.

It follows from the Strong maximum principle that $w > 0$ in $B_{2\delta}(0_{\lambda_0}, 0) \cap \mathbb{R}^n_+$, and it follows from the Hopf boundary lemma that $w > 0$ on $\bar{B}_\delta(0_{\lambda_0}, 0) \cap \overline{\mathbb{R}^n_+}$. In particular, $m = \frac{1}{2}\min\{w(x,t) : |(x - 0_{\lambda_0}, t)| = \delta,\ t \geq 0\}$ is well-defined and positive. Let

$$m_0 = \min(1, m),$$

$$C_0 = (\lambda_0 + 1)^{\beta(n-2)-n} c\beta \left(m_0 + \max_{B_\delta(0_{\lambda_0}, 0)} v \right)^{\beta - 1},$$

$$\mu_0 = \frac{1}{1 + C_0}.$$

For $0 < r < \delta$, consider the comparison function ϕ:

$$\phi(x,t) = \mu_0 - \frac{r^{n-2}}{|(x,t)|^{n-2}} + (1 - \mu_0)t.$$

Let $\tilde{w} = w - m_0\phi$, and let $\Omega_r = (B_\delta \cap \mathbb{R}^n_+) \backslash \bar{B}_r$. Then

$$\begin{cases} -\Delta \tilde{w} \geq 0 & \text{in } \Omega_r, \\ \frac{\partial \tilde{w}}{\partial t} \leq |x|^{\beta(n-2)-n} d_0(x) w - m_0(1 - \mu_0) & \text{on } \partial\Omega_r \cap \partial \mathbb{R}^n_+. \end{cases}$$

Suppose that $\tilde{w} < 0$ in Ω_r. Then, for some $(x', t') \in \partial\Omega_r$,

$$\tilde{w}(x', t') = \min_{\Omega_r} \tilde{w} < 0.$$

On $\partial\Omega_r \cap \partial B_\delta$, we have $\tilde{w} > w - m_0 \geq 0$, and on $\partial\Omega_r \cap \partial B_r$, we have $\tilde{w} > w > 0$. Therefore $r < |x'| < \delta$, $t' = 0$, and $\frac{\partial \tilde{w}}{\partial t}(x', 0) \geq 0$. Since $\tilde{w}(x', 0) < 0$, we have

$$w(x', 0) < m_0\mu_0, \tag{5.1.2}$$

and using the boundary condition,

$$m_0(1 - \mu_0) \leq |x|^{\beta(n-2)-n} d_0(x')w(x', 0). \tag{5.1.3}$$

The inequality (5.1.2) tells us that $v_{\lambda_0}(x', 0) < v(x', 0) + m_0$. Combining this with (5.1.3), we have

$$m_0(1 - \mu_0) \leq C_0 w(x', 0). \tag{5.1.4}$$

Combining (5.1.2) and (5.1.4), tells us that

$$\mu_0 > \frac{1}{1 + C_0},$$

which contradicts our choice of μ. Therefore $\tilde{w} \geq 0$ in Ω_r. Since this is true for all $r \in (0, \delta)$, we conclude that $w \geq m_0\mu_0$ in $B_\delta \cap \mathbb{R}^n_+$. □

5.2 Cylindrical symmetry

In this section, we will show that if $u \in C^2(\overline{\mathbb{R}^n_+}\backslash\{(0, t_0)\})$ is a positive solution of (5.0.1) with an isolated singularity at $(0, t_0)$, then it is cylindrically symmetric about some axis orthogonal to $\partial\mathbb{R}^n_+$. The proof we give is similar to the proof of Theorem 2.4.1. The idea is to use the method of moving planes to show that the Kelvin transform of u about the origin, v, has cylindrical symmetry about an axis orthogonal to $\mathbb{R}^n_+$.

Recall that in the proof of Theorem 2.4.1, we moved a family of hyperplanes $\{x_1 = \lambda\}_{\lambda > 0}$ up to the critical point λ_*, so that either $v_{\lambda_*} \equiv v$ or $v_{\lambda_*} > v$ in $\{x_1 > \lambda\}\backslash\{0_{\lambda_*}\}$. In Lemma 2.2.2 we showed that if $v_{\lambda_*} > v$ in $\{x_1 > \lambda_*\}\backslash\{0_{\lambda_*}\}$, then $v_\lambda > v$ in $\{x_1 >$

$\lambda\} \cap \{|x_\lambda| > S\}$ for $\lambda < \lambda_*$ sufficiently close to λ_*. The proof of this involved comparing $v_{\lambda_*} - v$ with the function $\phi(x) = (x_1 - \lambda_*)/|(x_1 - \lambda_*, x_2, \ldots, x_n)|^n$ in $\{x_1 > \lambda\} \cap \{|x_\lambda| > R\}$, where $v_{\lambda_*} - v$ was superharmonic and v had a harmonic asymptotic expansion. This gave us a uniform bound on the set of points where $v_\lambda - v$ could be nonpositive, which lead to a contradiction in the case where $v_{\lambda_*} \not\equiv v$.

The main difference between the proof of the following theorem and the proof of Theorem 2.4.1 is that in the case where $v_{\lambda_*} \not\equiv v$, instead of comparing $v_{\lambda_*} - v$ with a harmonic function ϕ, we will follow the method of proof used in [2] and analyze the function $\frac{v_{\lambda_*} - v}{g}$, where g is a positive increasing function. The proof also requires a few additional arguments to make sure that nothing goes wrong on the boundary.

Theorem 5.2.1. *Let $u \in C^2(\overline{\mathbb{R}^n_+}\backslash\{(0,t_0)\})$ be a positive solution of*

$$\begin{cases} -\Delta u = u^p \text{ in } \mathbb{R}^n_+\backslash\{(0,t_0)\}, & t_0 > 0, \quad n \geq 3, \\ \frac{\partial u}{\partial t} = cu^\beta \text{ on } \partial\mathbb{R}^n_+, \quad c > 0, \end{cases} \tag{5.2.1}$$

with an isolated singularity at $(0, t_0)$. Assume that

1. *$n \leq p(n-2) \leq n+2$,*

2. *$\beta \geq 1$, $\beta(n-2) \geq n$.*

Then either u is cylindrically symmetric about the t-axis or u is C^2 at $(0, t_0)$ and cylindrically symmetric about some axis orthogonal to $\partial\mathbb{R}^n_+$.

Proof. Let v be the Kelvin transform of u about the origin. Then

$$\begin{cases} -\Delta v = \frac{1}{|(x,t)|^{n+2-p(n-2)}} v^p \text{ in } \mathbb{R}^n_+\backslash\{(0,\frac{1}{t_0})\}, \\ \frac{\partial v}{\partial t} = c|x|^{\beta(n-2)-n} v^\beta \text{ on } (\partial\mathbb{R}^n_+)\backslash\{(0,0)\}. \end{cases}$$

Consider the x_1-direction. For $x = (x_1, \ldots, x_{n-1})$, let $x_\lambda = (2\lambda - x_1, \ldots, x_{n-1})$. Let $v_\lambda(x) = v(x_\lambda)$. We will use the method of moving planes to show that there exists a λ_1 so that $v_{\lambda_1} = v$ in $\mathbb{R}^n_+$.

Let $w_\lambda = v_\lambda - v$, and let

$$\Sigma_\lambda = \left(\{x_1 > \lambda_0\} \cap \mathbb{R}^n_+\right) \backslash\{(0_{\lambda_0}, \frac{1}{t_0})\}.$$

For $\lambda > 0$, we have

$$\begin{cases} -\Delta w_\lambda \geq \frac{1}{|(x,t)|^{n+2-p(n-2)}} c_\lambda(x,t) w_\lambda & \text{in } \Sigma_\lambda, \\ \frac{\partial w_\lambda}{\partial t} \leq |x|^{\beta(n-2)-n} d_\lambda(x) w_\lambda & \text{on } (\partial\Sigma_\lambda \cap \partial\mathbb{R}^n_+)\backslash\{(0_{\lambda_0}, 0)\}, \end{cases}$$

where $c_\lambda(x,t)$ is between $pv_\lambda^{p-1}(x,t)$ and $pv^{p-1}(x,t)$, and $d_\lambda(x)$ is between $c\beta v_\lambda^{\beta-1}(x,0)$ and $c\beta v^{\beta-1}(x,0)$.

It follows from Lemma 2.1.1, Lemma 2.1.2, Lemma 5.1.1, and Lemma 5.1.2 that

$$\lambda_0 = \inf\left\{\bar{\lambda} > 0 : w_\lambda > 0 \text{ in } \Sigma_\lambda \text{ for all } \lambda \geq \bar{\lambda}\right\}$$

is well-defined. If $\lambda > \lambda_0$, then $w_\lambda(x,t) > 0$ for $(x,t) \in \Sigma_\lambda$. Applying the Hopf boundary lemma to w_λ shows that $\frac{\partial v}{\partial x_1}(x,t) < 0$ when $x_1 = \lambda$, $t > 0$. Therefore, $\frac{\partial v}{\partial x_1}(x,t) < 0$ when $x_1 > \lambda_0$, $t > 0$.

Suppose u is not C^2 at $(0,t_0)$. In this case, we want to show that $\lambda_0 = 0$. Suppose $\lambda_0 > 0$. Using the definition of λ_0, we have $w_{\lambda_0}(x,t) \geq 0$ in Σ_0, so that

$$\begin{cases} -\Delta w_{\lambda_0} \geq \frac{1}{|(x,t)|^{n+2-p(n-2)}} c_{\lambda_0}(x,t) w_{\lambda_0} \geq 0 & \text{in } \Sigma_\lambda, \\ \frac{\partial w_{\lambda_0}}{\partial t} \leq |x|^{\beta(n-2)-n} d_{\lambda_0}(x) w_{\lambda_0} & \text{on } (\partial\Sigma_\lambda \cap \partial\mathbb{R}^n_+)\backslash\{(0_{\lambda_0}, 0)\}. \end{cases}$$

Since u is not C^2 at $(0,t_0)$, it follows from the Strong maximum principle and the Hopf boundary lemma that

1. $w_{\lambda_0} > 0$ in Σ_{λ_0},
2. $\frac{\partial w_{\lambda_0}}{\partial x_1} > 0$ on $x_1 = \lambda_0$, $t > 0$,
3. $w_{\lambda_0} > 0$ on $(\{x_1 > \lambda_0\} \cap \{t = 0\}) \backslash \{(0_{\lambda_0}, 0)\}$.

Claim 5.2.2. *There exist $\varepsilon > 0$ and $0 < r < \frac{\lambda_0}{4}$ so that, if $\lambda_0 - r < \lambda < \lambda_0$, then $w_\lambda > \varepsilon$ in $B_r(0_\lambda, \frac{1}{t_0})\backslash\{(0_\lambda, \frac{1}{t_0})\}$ and $B_r(0_\lambda, 0) \cap \mathbb{R}^n_+$.*

Proof. By Lemma 2.1.1 and Lemma 2.1.2, there exists $r_1, \varepsilon' > 0$ so that $w_{\lambda_0} > 2\varepsilon'$ in $B_{r_1}(0_\lambda, \frac{1}{t_0})\backslash\{(0_\lambda, \frac{1}{t_0})\}$. By continuity, there exists $\delta > 0$ so that, if $(x,t), (x',t') \in \bar{B}_{r_1}(0_\lambda, \frac{1}{t_0})$ and $|(x,t) - (x',t')| < \delta$, then $|v(x,t) - v(x',t')| < \varepsilon'$. Let $r' = \min\{\frac{\delta}{2}, r_1\}$. Suppose $\lambda_0 - r' < \lambda < \lambda_0$ and $(x,t) \in B_{r'}(0_\lambda, \frac{1}{t_0})\backslash\{(0_\lambda, \frac{1}{t_0})\}$. Let $(x',t') = ((x_\lambda)_{\lambda_*}, t)$. Then $|(x,t) - (x',t')| < \delta$, and it follows that $w_\lambda(x,t) = w_{\lambda_0}(x',t') + v(x',t') - v(x,t) > \varepsilon'$.

A similar argument using Lemma 5.1.3 shows that $w_\lambda > \varepsilon''$ in $B_{r''}(0_\lambda, 0) \cap \mathbb{R}^n_+$ for some $r'', \varepsilon'' > 0$. Let $\varepsilon = \min(\varepsilon', \varepsilon'')$, $r = \min(r', r'', \frac{\lambda_0}{4})$. □

Let $\widetilde{w}_\lambda = \frac{w_\lambda}{g}$, where $g(x,t) = \log(x_1 + 3)$. For $\lambda_0 - r < \lambda < \lambda_0$, we have

$$\begin{cases} -\Delta \widetilde{w}_\lambda \geq \frac{1}{|(x,t)|^{n+2-p(n-2)}} c_\lambda(x,t) \widetilde{w}_\lambda + \frac{2}{g} \nabla g \cdot \nabla \widetilde{w}_\lambda + \frac{\Delta g}{g} \widetilde{w}_\lambda & \text{in } \Sigma_\lambda, \\ \frac{\partial \widetilde{w}_\lambda}{\partial t} \leq |x|^{\beta(n-2)-n} d_\lambda(x) \widetilde{w}_\lambda & \text{on } (\partial \Sigma_\lambda \cap \partial \mathbb{R}^n_+) \backslash \{(0_{\lambda_0}, 0)\}. \end{cases}$$

Claim 5.2.3. *There exists $S_0 > 0$ so that if $\lambda_0 - r < \lambda < \lambda_0$ and $\inf_{\Sigma_\lambda} \widetilde{w}_\lambda < 0$, then $\widetilde{w}_\lambda$ achieves its infimum over Σ_λ at some point in $\Sigma_\lambda \cap B_{S_0}$.*

Proof. Suppose $\lambda_0 - r < \lambda < \lambda_0$ and $\inf_{\Sigma_\lambda} \widetilde{w}_\lambda < 0$. By the previous claim, $\widetilde{w}_\lambda > 0$ near the two singularities. On $\partial \Sigma_\lambda \cap \{x_1 = \lambda\}$, we have $\widetilde{w}_\lambda = 0$. On $\partial \Sigma_\lambda \cap \{t = 0\}$, if $w(x, 0) < 0$, then $\frac{\partial \widetilde{w}_\lambda}{\partial t}(x, 0) < 0$. By the asymptotic expansion, $\widetilde{w}_\lambda \to 0$ as $|(x,t)| \to \infty$. Therefore, $\widetilde{w}_\lambda$ achieves its infimum over Σ_λ at some point $(x', t') \in \Sigma_\lambda$. At (x', t'), we have

$$-\Delta \widetilde{w}_\lambda(x', t') \geq \left(\frac{1}{|(x',t')|^{n+2-p(n-2)}} c_\lambda(x', t') + \frac{\Delta g(x_1')}{g(x_1')} \right) \widetilde{w}_\lambda(x', t').$$

Since $\widetilde{w}_\lambda$ has a negative minimum at (x', t'), we have $v_\lambda(x', t') < v(x', t')$ and $-\Delta \widetilde{w}_\lambda(x', t') \leq 0$. If $|(x', t')| \geq S$, then

$$\begin{aligned} \frac{1}{|(x',t')|^{n+2-p(n-2)}} c_\lambda(x', t') + \frac{\Delta g(x_1')}{g(x_1')} &\leq \frac{1}{S^{n+2-p(n-2)}} p v^{p-1}(x', t') - \frac{1}{(x_1' + 3)^2 \log(x_1' + 3)} \\ &\leq \frac{C_0}{S^4} - \frac{1}{(x_1' + 3)^2 \log(x_1' + 3)}, \end{aligned}$$

where C_0 is independent of λ. If S is sufficiently large, then $-\Delta \widetilde{w}_\lambda(x', t') > 0$, which is a contradiction. Therefore, there exists an S_0 independent of λ so that $|(x', t')| < S_0$. □

Let $r_0 < r/4$. By definition of λ_0, there exist λ_j with $\lambda_0 - r_0 < \lambda_j < \lambda_0$ converging to λ_0 and points $(x^j, t^j) \in \Sigma_{\lambda_j}$ so that $\widetilde{w}_{\lambda_j}(x^j, t^j) \leq 0$. If $\inf_{\Sigma_{\lambda_j}} \widetilde{w}_{\lambda_j} = 0$, then it follows from the maximum principle (applied to w_{λ_j}) that $\widetilde{w}_{\lambda_j} \equiv 0$. Combining this with the previous claim, we can assume that $\widetilde{w}_{\lambda_j}(x^j, t^j) = \inf_{\Sigma_{\lambda_j}} \widetilde{w}_{\lambda_j}$ and $|(x^j, t^j)| < S_0$. Then we can find a subsequence, which we denote by (x^j, t^j), converging to (x', t'), so that

1. $\widetilde{w}_{\lambda_j}(x^j, t^j) \leq 0$,

2. $\nabla \widetilde{w}_{\lambda_j}(x^j, t^j) = 0$,

3. $|(x^j, t^j) - (0_{\lambda_0}, \frac{1}{t_0})| \geq \frac{r}{2}$.

4. $|(x^j, t^j) - (0_{\lambda_0}, 0)| \geq \frac{r}{2}$.

It follows that $(x', t') \neq (0_{\lambda_0}, \frac{1}{t_0}), (0_{\lambda_0}, 0)$. and hence $\widetilde{w}_{\lambda_0}(x', t') = 0$ and $\nabla \widetilde{w}_{\lambda_0}(x', t') = 0$. We know that $\widetilde{w}_{\lambda_0} > 0$ in Σ_{λ_0} and on $(\{x_1 > \lambda_0\} \cap \{t = 0\}) \backslash \{(0_{\lambda_0}, 0)\}$. Therefore, $x' = (\lambda_0, x'_2, \ldots, x'_{n-1}, 0)$.

Claim 5.2.4.

$$\frac{\partial w_{\lambda_0}}{\partial x_1}(\lambda_0, x_2, \ldots, x_{n-1}, 0) > 0.$$

Proof. Let $h(x,t) = w_{\lambda_0}(x,t) - \varepsilon(x_1 - \lambda_0)(t + \mu)$, where $0 < \varepsilon, \mu < 1$. Let $s = \frac{1}{2}\min(\frac{1}{t_0}, \lambda_0)$, and let $\Omega = B_s(\lambda_0, x_2, \ldots, x_{n-1}, 0) \cap \Sigma_{\lambda_0}$. Using the Hopf boundary lemma for w_{λ_0} in Σ_{λ_0}, choose ε so small that $h(x,t) \geq 0$ on $\partial\Omega \cap \partial B_s(\lambda_0, x_2, \ldots, x_{n-1}, 0)$. Suppose $h(x,t) < 0$ at some point in Ω. Then $h(x^*, t^*) = \min_{\overline{\Omega}} h < 0$. Since $-\Delta h \geq 0$ in Ω, it follows from the maximum principle that $\lambda_0 < x_1^* < \lambda_0 + s$ and $t^* = 0$. Notice that $w_{\lambda_0}(x^*, 0) < \varepsilon(x_1^* - \lambda_0)\mu$. By the Hopf boundary lemma, $\frac{\partial h}{\partial t}(x^*, 0) > 0$. It follows that

$$\begin{aligned} \varepsilon(x_1^* - \lambda_0) &< \frac{\partial w_{\lambda_0}}{\partial t}(x^*, 0) \\ &\leq |x^*|^{\beta(n-2)-n} d_{\lambda_0}(x^*) w_{\lambda_0}(x^*, 0) \\ &\leq C_0 \varepsilon (x_1^* - \lambda_0)\mu. \end{aligned}$$

Choose $\mu = \frac{1}{2}\min(C_0, 1)$. It follows that $h(x,t) \geq 0$ in Ω, and hence in $\overline{\Omega}$. Therefore,

$$\frac{\partial w_\lambda}{\partial x_1}(\lambda_0, x_2, \ldots, x_{n-1}, 0) = \lim_{\delta \to 0^+} \frac{w_{\lambda_0}(\lambda_0 + \delta, x_2, \ldots, x_{n-1}, 0)}{\delta} \geq \varepsilon\mu > 0.$$

□

Combining the previous claim with the Hopf boundary lemma, for $t \geq 0$,

$$\frac{\partial \widetilde{w}_{\lambda_0}}{\partial x_1}(\lambda_0, x_2, \ldots, x_{n-1}, t) = \frac{1}{g(\lambda_0)} \frac{\partial w_{\lambda_0}}{\partial x_1}(\lambda_0, x_2, \ldots, x_{n-1}, t) > 0.$$

Taking $t = t'$ leads to a contradiction. Therefore $\lambda_0 = 0$.

Repeating the above procedure in the $-x_1$-direction, we conclude that

$$v(-x_1, x_2, \ldots, x_{n-1}, t) = v(x_1, x_2, \ldots, x_{n-1}, t) \text{ in } \mathbb{R}^n_+ \backslash \{(0, \frac{1}{t_0})\}.$$

Since the same argument can be applied in any direction orthogonal to t, we conclude that v is cylindrically symmetric about the t-axis. It follows that u is cylindrically symmetric about the t-axis.

Suppose u is C^2 at $(0, t_0)$. If $\lambda_0 > 0$, then proceeding as in the previous case, we conclude that $w_{\lambda_0} \equiv 0$. It follows that u has a harmonic asymptotic expansion (2.2.3) in $\overline{\mathbb{R}^n_+} \backslash B_{R_0}$, for some $R_0 > 0$. Then we can apply the method of moving planes directly to u. We proceed as in the previous argument. If $\lambda_0 > 0$, then $u_{\lambda_0} \equiv u$. If $\lambda_* = 0$, then we apply the method of moving planes in the $-x_1$-direction. It follows, for some λ_1, that u is symmetric about the plane $x_1 = \lambda_1$ and $v_{x_1}(x) > 0$ when $x_1 > \lambda_1$. Applying the method of moving planes in the directions $x_2, \ldots, x_{n-1}$, we conclude that u is symmetric about the planes $x_i = \lambda_i$ and $u_{x_i}(x, t) < 0$ when $x_i > \lambda_i$. This determines a unique axis orthogonal to $\partial\mathbb{R}^n_+$: $\ell = \{(\lambda_1, \ldots, \lambda_{n-1}, t) : t \geq 0\}$. Notice that $(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{n-1}})(x, t) = 0$ if and only if $x = (\lambda_1, \ldots, \lambda_{n-1})$. Let τ be a direction that is orthogonal to $\partial\mathbb{R}^n_+$. Applying the method of moving planes to τ, we see that u is symmetric about a plane $(x, t) \cdot \tau = \lambda_\tau$ and $\frac{\partial u}{\partial \tau}(x, t) < 0$ when $(x, t) \cdot \tau > \lambda_\tau$. It follows that the plane $(x, t) \cdot \tau = \lambda_\tau$ must pass through the axis ℓ. Therefore, u is cylindrically symmetric about the axis ℓ.

If $\lambda_0 = 0$, then we apply the method of moving planes in the $-x_1$-direction, to conclude, for some λ_1, that v is symmetric about the plane $x_1 = \lambda_1$ and $v_{x_1}(x) > 0$ when $x_1 > \lambda_1$. Applying the method of moving planes in the directions $x_2, \ldots, x_{n-1}$, we conclude that v is symmetric about the planes $x_i = \lambda_i$ and $v_{x_i}(x, t) < 0$ when $x_i > \lambda_i$. If any of the λ_i are nonzero, repeating the previous argument shows that u is cylindrically symmetric about some axis orthogonal to $\partial\mathbb{R}^n_+$. Otherwise, a similar argument to the previous argument shows that v is cylindrically symmetric about the t-axis, and it follows that u is cylindrically symmetric about the t-axis. $\square$

Appendix A

Appendix

A.1 Hopf boundary lemma

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. For $u \in C^2(\Omega)$ we define Lu by

$$Lu(x) = a_{ij}(x)u_{x_i x_j}(x) + b_i(x)u_{x_i}(x) + c(x)u(x), \quad a_{ij} = a_{ji}, \tag{A.1.1}$$

where $x = (x_1, \ldots, x_n) \in \Omega$.

We say that L is elliptic at a point $x \in \Omega$ if the coefficient matrix $a_{ij}(x)$ is positive:

$$0 < \lambda(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2, \quad \xi \in \mathbb{R}^n \backslash \{0\},$$

where $\lambda(x)$ and $\Lambda(x)$ are the minimum and maximum eigenvalues of $a_{ij}(x)$. If Λ/λ is bounded in Ω, we say that L is uniformly elliptic in Ω.

Theorem A.1.1 (Hopf boundary lemma [4]). *Let L be a uniformally elliptic operator in Ω. Assume that $|b_i|/\lambda \leq b_0 < \infty$. Let $u \in C^2(\Omega)$ with $u \geq 0$ and $Lu \leq 0$ in Ω. Suppose*

i. *there is a ball B in Ω with a point $x_0 \in \partial\Omega$ on its boundary,*

ii. *u is continuous in $\Omega \cup \{x_0\}$,*

iii. *$u(x_0) = 0$.*

Then if $u > 0$ in B, we have for an outward directional derivative at x_0,

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

in the sense that

$$\liminf_{x\to x_0} \frac{u(x_0)-u(x)}{|x_0-x|} < 0,$$

where x approaches x_0 along a radius in B.

We first prove the theorem in the case where $c \equiv 0$. The proof we give can be found in [7].

Lemma A.1.2. *Let L be an elliptic operator in a bounded domain A. Assume that $c \equiv 0$ and $|b_i|/\lambda \le b_0 < \infty$. Let $u \in C^2(A) \cap C^0(\bar{A})$ with $u \ge 0$ on ∂A. If $Lu \le 0$ in A, then $u \ge 0$ in A.*

Proof. Since $a_{11} \ge \lambda$ and $|b_i|/\lambda \le b_0$, then

$$Le^{\gamma x_1} = (\gamma^2 a_{11} + \gamma b_1(x))e^{\gamma x_1} \ge \lambda(\gamma^2 - \gamma b_0)e^{\gamma x_1}.$$

Choose γ so that $Le^{\gamma x_1} > 0$. For $\varepsilon > 0$, let $v_\varepsilon = u - \varepsilon \cdot e^{\gamma x_1}$. Since $Lv_\varepsilon < 0$ in A, we have

$$\inf_A v_\varepsilon \ge \min_{\partial A} v_\varepsilon.$$

Therefore,

$$\inf_A u \ge -\varepsilon \cdot \min_{\partial A} e^{\gamma x_1}.$$

Taking $\varepsilon \to 0$, we conclude that $u \ge 0$ in A. □

Lemma A.1.3. *Theorem A.1.1 is true in the case where $c \equiv 0$.*

Proof. We can find a ball $B_R(y)$ so that x_0 is on its boundary and $\overline{B_R(y)}\backslash\{x_0\} \subset \Omega$. We condsider the function v defined by

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2},$$

where $r = |x-y|$ and $\alpha > 0$. Then

$$\begin{aligned} Lv(x) &= e^{-\alpha r^2}[4\alpha^2 a_{ij}(x)(x_i-y_i)(x_j-y_j) - 2\alpha(a_{ii}(x) + b_i(x)(x_i-y_i))] \\ &\ge e^{-\alpha r^2}[4\alpha^2\lambda(x)r^2 - 2\alpha(a_{ii}(x) + |b|r)], \end{aligned}$$

where $b = (b_1, \ldots, b_n)$.

Fix $0 < \rho < R$. We choose α so that $Lv \geq 0$ in the annular region $A = B_R(y) \backslash B_\rho(y)$. Since $u > 0$ on $\partial B_\rho(y)$, there is a constant $\varepsilon > 0$ so that $u - \varepsilon v \geq 0$ on $\partial B_\rho(y)$. Then $L(u - \varepsilon v) \leq 0$ in A and $u - \varepsilon v \geq 0$ on ∂A. Applying Lemma A.1.2, we have $u - \varepsilon v \geq 0$. Therefore, if x approaches x_0 in B along a radius, then

$$\liminf_{x \to x_0} \frac{u(x_0) - u(x)}{|x_0 - x|} \leq \varepsilon v'(R) < 0.$$

□

proof of Theorem A.1.1. We consider the function v defined by $v(x) = e^{-\alpha x_1} u(x)$, where $\alpha > 0$. Then

$$Lu = vLe^{\alpha x_1} + e^{\alpha x_1} L' v, \tag{A.1.2}$$

where

$$L' = a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + (b_i + 2\alpha a_{i1}) \frac{\partial}{\partial x_i}.$$

We rewrite equation (A.1.2) as

$$L'v = e^{-\alpha x_1} Lu - u(a_{11}\alpha^2 + b_1 \alpha + c).$$

We choose α so that $L'v \leq 0$. Applying Lemma A.1.3 to L' and v, we conclude that

$$\frac{\partial v}{\partial \nu}(x_0) < 0,$$

and hence $u_\nu(x_0) = e^{-\alpha x_1} v_\nu(x_0) < 0$. □

As a consequence of Theorem A.1.1, we have

Theorem A.1.4 (Strong maximum principle). *Let L be a uniformally elliptic operator in Ω. Assume that $|b_i|/\lambda \leq b_0 < \infty$. Let $u \in C^2(\Omega)$ with $u \geq 0$ and $Lu \leq 0$ in Ω. Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.*

Proof. Consider $A = \{x \in \Omega : u(x) > 0\}$. If $y \in \bar{A} \cap \Omega$, choose $x \in A$ so that $B_{2r}(x) \subset \Omega$, where $|x - y| = r$. Suppose $u = 0$ at some point in $B_{2r}(x)$. Then we can find a ball $B_{r_0}(x)$ and a point x_0 on its boundary so that $u > 0$ in $B_{r_0}(x)$ and $u(x_0) = 0$. Since $u \geq 0$, applying the Hopf boundary lemma leads to a contradiction. Therefore, $u > 0$ in $B_{2r}(x)$, and we conclude that A is open and closed in Ω. □

A.2 Superharmonic functions

Let X be an open set in $\mathbb{R}^n$, $n \geq 3$. The function $u : X \to (-\infty, \infty]$ is lower semi-continuous at $x \in X$ if $u(x) \leq \liminf_{y \to x} u(y)$. The function u is lower semi-continuous in X if u is lower semi-continuous at each point in X. The following theorem (see Theorem 4.1.8 in [10]) gives us a super-mean-value property for distribution solutions of $\Delta u \leq 0$.

Theorem A.2.1. *Let X be an open set in $\mathbb{R}^n$. If $u \in \mathcal{D}'(X)$ is real and $\Delta u \leq 0$, then u is defined by a lower semi-continuous function u_0 in X such that*

$$M(x,r) = \frac{1}{c_n} \int_{|y|=1} u_0(x+ry)\, d\sigma(y), \quad c_n = \int_{|y|=1} d\sigma,$$

is a nonincreasing function of r for $x \in X$ and $0 \leq r < d(x, \partial X)$. Moreover, we can choose u_0 so that if u is continuous at $x \in X$, then $u(x) = u_0(x)$.

Proof. First, we prove the theorem in the case where $u \in C^\infty(X)$ and $\Delta u \leq 0$. Let $e(t) = t^{2-n}/[c_n(n-2)]$, $E(x) = e(|x|)$. For $0 < r < R$, define

$$h(x) = \begin{cases} 0, & |x| > R, \\ e(R) - E(x), & r < |x| < R, \\ e(R) - e(r), & |x| < r, \end{cases}$$

and let $A = B_R(0) \backslash \bar{B}_r(0)$. Then $\nabla h = -\chi_A \nabla E$, and

$$\mathrm{div}(\nabla h) = -(\chi_A \Delta E - \nabla E \cdot \nu d\sigma),$$

where $d\sigma$ is the Euclidean surface measure on ∂A and ν is the exterior unit normal. It follows, for $d(x, \partial X) > R$, that

$$0 \geq (\Delta u) * h(x) = u * (\Delta h)(x) = M(x,R) - M(x,r).$$

Hence, $M(x,r)$ is a nonincreasing for $0 \leq r < d(x, \partial X)$. Also, if $\psi \in C_0^\infty$, $\psi \geq 0$, $\int \psi dx = 1$, $\psi = \psi(|x|)$, and $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$, then

$$u * \psi_\varepsilon(x) = \int_0^\infty \psi(r) \left(\int_{|y|=1} u(x + \varepsilon r y)\, d\sigma(y) \right) r^{n-1}\, dr$$

is a nonincreasing function of ε.

Suppose $u \in \mathcal{D}'(X)$ and $\Delta u \leq 0$. Let $\phi \in C_0^\infty$, $\phi \geq 0$, $\int \phi dx = 1$. Then $u_\phi = u * \phi \in C^\infty$ and Δu_ϕ where u_ϕ is defined. Applying the theorem in the C^∞ case and then letting supp $\phi \to \{0\}$, we conclude that $u * \psi_\varepsilon(x)$ is a nonincreasing function of ε, and

$$\frac{1}{c_n} \int_{|y|=1} (u * \psi_\varepsilon)(x + ry)\, d\sigma(y)$$

is a nonincreasing function of r, for $0 \leq r < d(x, \partial X)$.

We define $u_0(x) = \lim_{\varepsilon \to 0} u * \psi_\varepsilon(x)$. Then u_0 is lower semi-continuous, $M(x,r)$ is a nonincreasing function of r for $x \in X$ and $0 \leq r < d(x, \partial X)$, and if $\eta \geq 0$, $\eta \in C_0^\infty(X)$, then $\langle u, \eta \rangle = \int u_0(x)\eta(x)\, dx$. Moreover, if u is continuous at $x \in X$, then $\lim_{\varepsilon \to 0} u * \psi_\varepsilon(x) = u(x)$. □

Corollary A.2.2. *Let X be an open set in $\mathbb{R}^n$. If $u \in \mathcal{D}'(X)$ is real and $\Delta u \leq 0$, then u is defined by a lower semi-continuous function u_0 in X such that*

$$u_0(x) \geq \frac{n}{c_n R^n} \int_{B_R(x)} u_0(y)\, dy,$$

for $x \in X$ and $0 < R < d(x, \partial X)$. Moreover, we can choose u_0 so that if u is continuous at $x \in X$, then $u(x) = u_0(x)$.

Proof. By Theorem A.2.1, u is defined by a lower semi-continuous function u_0 in X with the following two properties:

1. For $x \in X$ and $0 < r < d(x, \partial X)$,

$$u_0(x) \geq \frac{1}{c_n} \int_{|y|=1} u_0(x + ry)\, d\sigma(y).$$

2. If u is continuous at $x \in X$, then $u(x) = u_0(x)$.

Using property (1.), we have

$$\begin{aligned}
\int_{B_R(x)} u_0(y)\, dy &= \int_0^R \left(\int_{|y|=1} u_0(x + ry)\, d\sigma(y) \right) r^{n-1}\, dr \\
&\leq \int_0^R c_n u_0(x) r^{n-1}\, dr \\
&= \frac{c_n R^n}{n} u_0(x),
\end{aligned}$$

for $x \in X$ and $0 < R < d(x, \partial X)$. □

A.3 Radial solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n \backslash \{0\}$

Suppose u is a radial solution of $\Delta u + u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n \backslash \{0\}$, $n \geq 3$. Then $\phi(r) = u(x)$, where $|x| = r$, is a solution of

$$\phi''(r) + \frac{n-1}{r}\phi'(r) + \phi^{(n+2)/(n-2)}(r) = 0, \qquad r > 0. \tag{A.3.1}$$

If we change coordinates:

$$t = -\log r, \quad r = e^{-t},$$

and consider

$$\psi(t) = e^{-\frac{n-2}{2}t}\phi(e^{-t}),$$

then (A.3.1) becomes

$$\psi''(t) = \left(\frac{n-2}{2}\right)^2 \psi(t) - \psi^{(n+2)/(n-2)}(t), \qquad t \in \mathbb{R}, \quad n \geq 3. \tag{A.3.2}$$

Consider the second-order differential equation:

$$\begin{cases} \psi''(t) = \left(\frac{n-2}{2}\right)^2 \psi(t) - \psi^{(n+2)/(n-2)}(t), & t \in \mathbb{R}, \quad n \geq 3 \\ \psi(0) = \psi_0, \quad \psi'(0) = \psi_1. \end{cases}$$

Let

$$X(t) = (x_1(t), x_2(t)) = (\psi(t), \psi'(t)),$$

$$X_0 = (\psi_0, \psi_1),$$

$$F(X,t) = \left(x_2, \left(\frac{n-2}{2}\right)^2 x_1 - x_1^{(n+2)/(n-2)}\right).$$

Then $\frac{dX}{dt} = F(X,t)$, $X(0) = X_0$. It follows from existence and uniqueness properties of ordinary differential equations ([11]) that (A.3.2) has local existence and uniqueness. Moreover, in any bounded domain containing the point $(X_0, 0)$, the solution of $\frac{dX}{dt} = F(X,t)$ passing through $(X_0, 0)$ may be uniquely extended arbitrarily close to the boundary.

A.3.1 Uniqueness for solutions

Let ψ be a solution of (A.3.2). We have

$$\frac{1}{2}\frac{d}{dt}\left(\psi'^2\right) = \left(\frac{n-2}{2}\right)^2 \frac{1}{2}\frac{d}{dt}\left(\psi^2\right) - \frac{n-2}{2n}\frac{d}{dt}\left(\psi^{2n/(n-2)}\right).$$

Integrating the previous equation:

$$\psi'^2 = \left(\frac{n-2}{2}\right)^2 \psi^2 - \frac{n-2}{n}\psi^{2n/(n-2)} + C_0. \tag{A.3.3}$$

Since $\frac{2n}{n-2} > 2$, there exists M_0 so that $|\psi| < M_0$. It follows from (A.3.3) that there exists M_1 so that $|\psi'| < M_1$.

Let $T > 0$. We know that in $(-M_0, M_0) \times (-M_1, M_1) \times (-T, T)$, the solution of $\frac{dX}{dt} = F(X,t)$ passing through $(X_0, 0)$ may be uniquely extended arbitrarily close to the boundary. Since this is true for all $T > 0$, we conclude that ψ is the unique solution of (A.3.2) passing through $(\psi(0), \psi'(0), 0)$.

A.3.2 Existence for solutions with positive initial value

We know that the second-order differential equation:

$$\begin{cases} \psi''(t) = \left(\frac{n-2}{2}\right)^2 \psi(t) - \psi^{(n+2)/(n-2)}(t), & t \in \mathbb{R}, \quad n \geq 3 \\ \psi(0) = \psi_0, \quad \psi'(0) = \psi_1, \end{cases}$$

has a unique solution in a neighborhood of the origin. We also know that in any bounded domain containing $(\psi_0, \psi_1, 0)$, this solution may be uniquely extended arbitrarily close to the boundary.

Consider

$$y(x) = \left(\frac{n-2}{2}\right)^2 x^2 - \frac{n-2}{n}x^{2n/(n-2)} + C_0, \quad n \geq 3.$$

Then

$$y'(x) = 2x\left[\left(\frac{n-2}{2}\right)^2 - x^{4/(n-2)}\right]$$

and

$$y''(x) = 2\left(\frac{n-2}{2}\right)^2 - 2\left(\frac{n+2}{n-2}\right)x^{4/(n-2)}.$$

We have

$$y'(x) = 0 \text{ when } x = 0, \pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}},$$

$$y(0) = C_0,$$

$$y\left(\pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}\right) = \frac{2}{n}\left(\frac{n-2}{2}\right)^n + C_0,$$

$$y''(0) > 0,$$

$$y''\left(\pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}\right) < 0,$$

$$y''\left(\pm\left(\frac{(n-2)^3}{4(n+2)}\right)^{\frac{n-2}{4}}\right) = 0.$$

The graph of y achieves its maximum at $\left(\pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}, \frac{2}{n}\left(\frac{n-2}{2}\right)^n + C_0\right)$, and it has a local minimum at $(0, C_0)$. The graph is concave up for $x \in \left(-\left(\frac{(n-2)^3}{4(n+2)}\right)^{\frac{n-2}{4}}, \left(\frac{(n-2)^3}{4(n+2)}\right)^{\frac{n-2}{4}}\right)$ and concave down for $x \in \left(-\infty, \left(\frac{(n-2)^3}{4(n+2)}\right)^{\frac{n-2}{4}}\right) \cup \left(\left(\frac{(n-2)^3}{4(n+2)}\right)^{\frac{n-2}{4}}, \infty\right)$.

Returning to the problem of existence for positive solutions, let ψ be the unique local solution of

$$\begin{cases} \psi''(t) = \left(\frac{n-2}{2}\right)^2 \psi(t) - \psi^{(n+2)/(n-2)}(t), & n \geq 3 \\ \psi(0) = \psi_0, \quad \psi'(0) = \psi_1. \end{cases}$$

We know that ψ is a solution of

$$\psi'^2 = \left(\frac{n-2}{2}\right)^2 \psi^2 - \frac{n-2}{n}\psi^{2n/(n-2)} + C_0,$$

where $C_0 = \left((\psi_1)^2 - \left(\frac{n-2}{2}\right)^2 (\psi_0)^2 + \frac{n-2}{2}(\psi_0)^{2n/(n-2)}\right)$. Notice that $C_0 \geq -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$. Suppose $\psi_0 > 0$.

1. Suppose $-\frac{2}{n}\left(\frac{n-2}{2}\right)^n \leq C_0 \leq 0$.

   (a) Suppose $C_0 = -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$. Then $(\psi_0, \psi_1) = \left(\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}, 0\right)$ and $\psi(t) = \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}$.

   (b) Suppose $C_0 = 0$. If $\psi_1 < 0$, then $\psi(t) = [n(n-2)]^{\frac{n-2}{4}}\left(\frac{\alpha e^{-t}}{1+\alpha^2 e^{-2t}}\right)^{\frac{n-2}{2}}$, where $\alpha = \frac{\sqrt{n(n-2)}}{2\psi_0^{2/(n-2)}} - \left(\frac{n(n-2)}{4\psi_0^{4/(n-2)}} - 1\right)^{1/2}$. If $\psi_1 = 0$, then $\psi_0 = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$ and $\psi(t) = [n(n-2)]^{\frac{n-2}{4}}\left(\frac{\alpha e^{-t}}{1+\alpha^2 e^{-2t}}\right)^{\frac{n-2}{2}}$, where $\alpha = 1$. If $\psi_1 > 0$, then $\psi(t) =$

$[n(n-2)]^{\frac{n-2}{4}}\left(\frac{\alpha e^{-t}}{1+\alpha^2 e^{-2t}}\right)^{\frac{n-2}{2}}$, where $\alpha = \frac{\sqrt{n(n-2)}}{2\psi_0^{2/(n-2)}} + \left(\frac{n(n-2)}{4\psi_0^{4/(n-2)}} - 1\right)^{1/2}$.

(c) Suppose $-\frac{2}{n}\left(\frac{n-2}{2}\right)^n < C_0 < 0$. Then there exist constants C_1, C_2:

$$0 < C_1 < \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} < C_2 < \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

so that $C_1 \leq \psi(t) \leq C_2$. Choose $M_0 > C_2$, $M_1 > \frac{n-2}{2}M_0$, $T > 0$, and consider the domain $(0, M_0) \times (-M_1, M_1) \times (-T, T)$. We know that $(\psi(t), \psi'(t), t)$ may be uniquely extended arbitrarily close to the boundary. It follows that $\psi(t)$ may be extended to $(-T, T)$.

Notice that $\psi'(t) = 0$ if and only if $\psi(t) = C_1, C_2$. Also notice that if $\psi(t) = C_1$, then $\psi''(t) > 0$, and if $\psi(t) = C_2$, then $\psi''(t) < 0$.

Claim A.3.1. *$\{t > 0 : \psi(t) = C_1\}$ is an infinte set.*

Proof. Suppose $\{t > 0 : \psi(t) = C_1\}$ is nonempty and finite. Let t' be the largest element. We have $\psi(t') = C_1$, $\psi'(t') = 0$, $\psi''(t') > 0$. For $t > t'$ sufficiently close to t', we have $\psi(t) > C_1$, $\psi'(t') > 0$, $\psi''(t') > 0$. We want to show that $\psi(t'') = C_2$ for some $t'' > t'$: Suppose $\psi(t) < C_2$ for all $t > t'$. Since $\psi'(t) = 0$ if and only if $\psi(t) = C_1, C_2$, then $\psi'(t) > 0$ for all $t > t'$. It follows that $\lim_{t\to\infty}\psi(t) = const_1$ and $\lim_{t\to\infty}\psi'(t) = 0$. Using this and the differential equation, we conclude that $\lim_{t\to\infty}\psi''(t) = const_2 = 0$. Using the differential equation again, we conclude that $const_1 = \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}$. This contradicts the fact that $\lim_{t\to\infty}\psi'(t) = 0$. Therefore, $\psi(t'') = C_2$ for some $t'' > t'$. A similar argument shows that $\psi(t''') = C_1$ for some $t''' > t''$, which is a contradiction.

To see that $\{t > 0 : \psi(t) = C_1\}$ is nonempty, apply the above argument to some $t' > 0$. In particular, this will show that there is some $t'' > t'$ so that $\psi(t'') = C_1, C_2$, and then repeating the argument once more (if necessary) shows that $\psi(t) = C_1$ for some $t > 0$. □

We conclude that, ψ increases from C_1 to C_2 and then decreases from C_2 to C_1 infinitely often as $t \to \infty$. A similar argument shows that ψ exhibits the same behavior as $t \to -\infty$.

2. Suppose $C_0 > 0$. Then there exist constants C_-, C_+:

$$C_- < -\left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}} < \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}} < C_+,$$

so that $C_- \le \psi(t) \le C_+$. Repeating the argument used in the case where $-\frac{2}{n}\left(\frac{n-2}{2}\right)^n < C_0 < 0$ shows that ψ increases from C_- to C_+ and then decreases from C_+ to C_- infinitely often as $t \to \pm\infty$.

A.3.3 Positive radial solutions in $\mathbb{R}^n \backslash \{0\}$

Consider the second-order differential equation:

$$\begin{cases} \phi''(r) + \frac{n-1}{r}\phi'(r) + \phi^{(n+2)/(n-2)}(r) = 0, & r > 0, \quad n \ge 3 \\ \phi(1) = \phi_0, \quad \phi'(1) = \phi_1. \end{cases}$$

Let $C_0 = \left((\phi_1)^2 - (n-2)\phi_0\phi_1 + \frac{n-2}{2}(\phi_0)^{2n/(n-2)}\right)$. For each pair (ϕ_0, ϕ_1) with $\phi_0 > 0$, the differential equation has a unique global solution:

1. If $C_0 = -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, then

$$\phi(r) = \left(\frac{n-2}{2r}\right)^{\frac{n-2}{2}}.$$

2. If $-\frac{2}{n}\left(\frac{n-2}{2}\right)^n < C_0 < 0$, then there exist positive constants $C_1 < C_2$ so that

$$\phi(r) = \frac{1}{r^{\frac{n-2}{2}}}\psi(\log r),$$

where $\psi(t)$ increases from C_1 to C_2 and then decreases from C_2 to C_1 infinitely often as $t \to \pm\infty$.

3. If $C_0 = 0$, then

   (a) If $\phi_1 + \frac{n-2}{2}\phi_0 > 0$, then

$$\phi(r) = [n(n-2)]^{\frac{n-2}{4}}\left(\frac{\alpha}{1+\alpha^2 r^2}\right)^{\frac{n-2}{2}},$$

where $\alpha = \frac{\sqrt{n(n-2)}}{2\phi_0^{2/(n-2)}} - \left(\frac{n(n-2)}{4\phi_0^{4/(n-2)}} - 1\right)^{1/2}$.

(b) If $\phi_1 + \frac{n-2}{2}\phi_0 = 0$, then $\phi_0 = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$ and

$$\phi(r) = [n(n-2)]^{\frac{n-2}{4}} \left(\frac{\alpha}{1+\alpha^2 r^2}\right)^{\frac{n-2}{2}},$$

where $\alpha = 1$.

(c) If $\phi_1 + \frac{n-2}{2}\phi_0 < 0$, then

$$\phi(r) = [n(n-2)]^{\frac{n-2}{4}} \left(\frac{\alpha}{1+\alpha^2 r^2}\right)^{\frac{n-2}{2}},$$

where $\alpha = \frac{\sqrt{n(n-2)}}{2\phi_0^{2/(n-2)}} + \left(\frac{n(n-2)}{4\phi_0^{4/(n-2)}} - 1\right)^{1/2}$.

4. If $C_0 > 0$, then there exist constants $C_- < 0 < C_+$ so that

$$\phi(r) = \frac{1}{r^{\frac{n-2}{2}}}\psi(\log r),$$

where $\psi(t)$ increases from C_- to C_+ and then decreases from C_+ to C_- infinitely often as $t \to \pm\infty$.

Theorem A.3.2. *Let u be a positive C^2 solution of*

$$\Delta u + u^{(n+2)/(n-2)} = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3.$$

If u is radially symmetric about the origin, then

$$u(x) = [n(n-2)]^{(n-2)/4} \left(\frac{\alpha}{1+\alpha^2|x|^2}\right)^{(n-2)/2}$$

for some $\alpha > 0$.

Proof. Let $\phi(r) = u(x)$, where $|x| = r$. Then ϕ, is a positive solution of

$$\phi''(r) + \frac{n-1}{r}\phi'(r) + \phi^{(n+2)/(n-2)}(r) = 0, \qquad r > 0. \tag{A.3.4}$$

Since ϕ does not have a singularity at $r = 0$, it must be of the form $[n(n-2)]^{\frac{n-2}{4}} \left(\frac{\alpha}{1+\alpha^2 r^2}\right)^{\frac{n-2}{2}}$ for some $\alpha > 0$. □

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