

**AN EFFICIENT SEMI-ANALYTIC METHOD FOR
THE CALCULATION OF INTERLAMINAR
STRESSES AROUND HOLES**

by

KEVIN JAY SAEGER

B.S., Tri-State University (1984)

S.M., Massachusetts Institute of Technology (1986)

**SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND
ASTRONAUTICS IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY**

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August, 1989

© MASSACHUSETTS INSTITUTE OF TECHNOLOGY 1989

Signature of Author _____

Aeronautics, August, 1989

Certified by _____

Thesis Supervisor, _____

Prof. Paul A. Lagace
Aeronautics and Astronautics

Certified by _____

Prof. John Dugundji
Professor of Aeronautics and Astronautics

Certified by _____

Jerome

Prof. James W. Mar
Education

Accepted by _____

Wachman

Departmental Graduate Committee
OF TECHNOLOGY

SEP 29 1989

LIBRARIES

Aero

WITHDRAWN

M.I.T.

LIBRARIES

AN EFFICIENT SEMI-ANALYTIC METHOD FOR THE CALCULATION OF INTERLAMINAR STRESSES AROUND HOLES

by

KEVIN JAY SAEGER

Submitted to the Department of Aeronautics and Astronautics on
August 22, 1989 in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

ABSTRACT

A generic procedure is developed which provides a methodology for the approximation of interlaminar stresses in a laminate of arbitrary geometry. The existence of interlaminar stresses due to generic in-plane gradients is demonstrated as part of the methodology. As an example of the capability of the methodology, the interlaminar stress state caused by the presence of a circular hole in a laminated material subjected to a far-field applied load is investigated analytically. This is accomplished by explicitly assuming the through-the-thickness variations of the in-plane stresses in a given ply, but still allowing these stresses to be arbitrary functions of the in-plane coordinates. Using the equations of equilibrium, the remaining three stresses, which are the interlaminar stresses, are then found. The principle of minimum complementary energy is used to obtain a set of coupled differential equations for the in-plane variation of the in-plane stresses. These resulting differential equations are solved via perturbation techniques in order to obtain a closed form solution. The solution is found to compare favorably with an exact solution for an idealized case as well as to an existing finite element solution. This solution is implemented in a computer program, SLASH, which is used for parametric studies which include laminate delamination initiation analyses. The delamination initiation studies explore the effects of stacking sequence, load angle, and lamination angle. The developed solution technique is extremely fast, requiring 18 seconds of execution time on an IBM-PC[®] for the case of a twenty-ply laminate, with ten different angular locations examined.

Thesis Supervisor: Prof. Paul A. Lagace

Associate Professor of Aeronautics and Astronautics

ACKNOWLEDGEMENTS

This is the section of the thesis where I attempt to acknowledge and thank all those who helped to bring this document to fruition. This is an impossible task, yet, I will try to thank those whose contributions are indelible in my memory. First, I would like to thank my two outstanding UROP students, Adam Sawicki and Brian Quinn. Both were able to suffer the slings and arrows of my abuses and still produce a tremendous quantity of quality work. Next, I would like to thank Professors Dugundji and Mar who are always supportive and accessible. I would like to thank all my office mates in 41-219C, past and present, who have made life at TELAC not only bearable, but also enjoyable. In particular, I would like to thank Pierre Minguet who has been a constant compatriot, both intellectually and, more importantly, recreatively. Another graduate student, Narendra Bhat provided the constant questioning and skepticism required in order to polish my work. I would also like to express my sincere gratitude to my advisor, Prof. Paul Lagace, who demonstrated a vitalizing interest in my work, allowed me the freedom to pursue my interests, and worked diligently to provide me with the necessary funding. I cannot think of an area in which he has failed me as an advisor.

Finally, I would like to thank Philipp Podsiadlowski, a true and trusted companion over the last five years, and my loving wife Kelly. Kelly has constantly provided the love and support required for me to have achieved what I have.

The pronouncement of mortal man:

"I am not God;

I am not God, that I should prevail.

Why, I am the most stupid of men,
and have not even human intelligence;

Neither have I learned wisdom,
nor have I the knowledge of the Holy One.

Who has gone up to heaven and come down again-
who has cupped the wind in his hands?

Who has bound up the waters in a cloak-
who has marked out all the ends of the Earth?

What is his name, what is his son's name,
if you know it?"

FOREWORD

This work was performed in the Technology Laboratory for Advanced Composites (TELAC) of the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology. This work was sponsored by Boeing Military Aircraft under Boeing Purchase Order 644740.

TABLE OF CONTENTS

<u>CHAPTER</u>		<u>PAGE</u>
1	INTRODUCTION	18
2	PREVIOUS WORK	22
	2.1 Historical Overview - Straight Free Edge Solution	22
	2.2 Simplified Models for the Straight Free Edge Problem	28
	2.3 Solutions for Interlaminar Stresses Near Cutouts	30
	2.4 Prediction of Delamination Initiation	33
3	GENERAL FORMULATION	35
	3.1 Interlaminar Stresses Arising in Gradient Stress Fields	37
	3.1.1 Necessity of Interlaminar Stresses in Gradient Stress Fields	37
	3.1.2 Approximate Solution for Interlaminar Stresses in Gradient Stress Fields	42
	3.2 Reduced Eigenfunction Solution Technique (REST)	48
	3.2.1 Solution of the Companion Problem	52
	3.2.2 Choice of the Tensor $\mathbf{F}$	55
4	SOLUTION FOR CIRCULAR HOLES	60
	4.1 Analytic Expression for In-Plane Stresses in the Presence of a Circular Hole	61
	4.2 Interlaminar Stresses Due to Gradient Stress Field	65
	4.3 Interlaminar Stresses Due to Residual Stresses - - - Companion Solution	67
	4.3.1 Formulation of Equations	70
	4.3.2 Solution of System of Equations	77
	4.4 Computer Implementation	88
5	SOLUTION VERIFICATION	91
	5.1 Validity of Perturbation Solution	92
	5.2 Comparison with Idealized Problem	109

TABLE OF CONTENTS (Continued)

<u>CHAPTER</u>		<u>PAGE</u>
	5.3 Comparison with Finite Element Model	114
	5.4 Summary	127
6	PARAMETRIC STUDIES	128
	6.1 Importance of In-Plane Gradient Stress Field Contribution	128
	6.2 Stress State Characterization	137
	6.2.1 General Characteristics	137
	6.2.2 Effect of Geometry	154
	6.3 Quadratic Delamination Initiation Criterion	164
	6.4 $[0/90]_s$ Laminates	168
	6.5 Quasi-Isotropic Laminates	177
	6.6 $[\pm\theta/0]_s$ Laminates	183
	6.7 Summary	188
7	CONCLUSIONS AND RECOMMENDATIONS	191
	7.1 Assessment of Methodology	191
	7.2 Conclusions	193
	7.3 Recommendations	195
	REFERENCES	197
APPENDIX A	Material Properties used in Analyses	203
APPENDIX B	Source Code Listing for S.L.A.S.H.	206
APPENDIX C	Source Code Listing for PLOTTER	247

LIST OF FIGURES

<u>FIGURE</u>		<u>PAGE</u>
3.1	Geometry for problem of laminated plate under loading	36
3.2	Illustration for Bernoulli-Euler beam theory showing (a) the displacement assumption, (b) the resulting extensional stress, and (c) the resulting shear stress.	43
3.3	Schematic representation of a generic free-edge problem.	49
3.4	Illustration of solution technique methodology using the principle of superposition.	51
4.1	Solution methodology for interlaminar stresses due to a stress-free edge.	62
4.2	Geometry of a typical hole problem.	63
4.3	S.L.A.S.H. program overview	89
5.1	Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.125.	93
5.2	Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.250.	96
5.3	Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.500.	97
5.4	Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 1.000.	98

LIST OF FIGURES

<u>FIGURE</u>		<u>PAGE</u>
5.5	Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.125.	100
5.6	Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.250.	101
5.7	Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.500.	102
5.8	Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 1.000.	103
5.9	Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.125.	105
5.10	Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.250.	106
5.11	Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.500.	107
5.12	Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 1.000.	108
5.13	Illustration of (a) the geometry and (b) the applied loading of the idealized isotropic problem.	110

LIST OF FIGURES

<u>FIGURE</u>		<u>PAGE</u>
5.14	Comparison of exact solution and solution obtained via SLASH for an idealized problem with ϵ equal to 1.000.	115
5.15	Comparison of exact solution and solution obtained via SLASH for an idealized problem with ϵ equal to 0.125.	116
5.16	Geometry of problem solved via Finite Element analysis by Lucking, Hoa, and Sankar [47].	119
5.17	Comparison of finite element solution and SLASH solution for $\sigma_{\theta z}$ at 63° from the load axis.	120
5.18	Comparison of finite element solution and SLASH solution for σ_{rz} at 63° from the load axis.	121
5.19	Comparison of finite element solution and SLASH solution for σ_{zz} at 63° from the load axis.	123
5.20	Comparison of finite element solution and SLASH solution for $\sigma_{\theta z}$ at 81° from the load axis.	124
5.21	Comparison of finite element solution and SLASH solution for σ_{rz} at 81° from the load axis.	125
5.22	Comparison of finite element solution and SLASH solution for σ_{zz} at 81° from the load axis.	126
6.1	Components of $\sigma_{\theta z}$ at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 63° from load axis.	130
6.2	Components of $\sigma_{\theta z}$ at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 81° from load axis.	132
6.3	Components of σ_{rz} at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 63° from load axis.	133

LIST OF FIGURES

<u>FIGURE</u>		<u>PAGE</u>
6.4	Components of σ_{rz} at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 81° from load axis.	134
6.5	Components of σ_{zz} at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 63° from load axis.	135
6.6	Components of σ_{zz} at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 81° from load axis.	136
6.7	Constant resultant shear contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $+45^\circ/-45^\circ$ interface.	138
6.8	Constant resultant shear contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $-45^\circ/0^\circ$ interface.	139
6.9	Constant resultant shear contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface.	140
6.10	Variation of $\sigma_{\theta z}$ with angular location for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.	143
6.11	Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $+45^\circ/-45^\circ$ interface.	144
6.12	Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $-45^\circ/0^\circ$ interface.	145
6.13	Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface.	146

LIST OF FIGURES

FIGURE		PAGE
6.14	Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the midplane.	147
6.15	Variation of σ_{zz} with angular location for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.	149
6.16	Plot of $\sigma_{\theta z}$ at various radial positions at $-45^\circ/0^\circ$ interface for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.	151
6.17	Plot of σ_{zz} at various radial positions at the midplane for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.	152
6.18	Plot of $\sigma_{\theta\theta}$, at a position on the 90° radial corresponding to the hole edge, as a function of the through-the-thickness location for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.	153
6.19	Variation of $\sigma_{\theta z}$ at hole edge at $+45^\circ/-45^\circ$ interface for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.	155
6.20	Variation of $\sigma_{\theta z}$ at hole edge at $-45^\circ/0^\circ$ interface for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.	156
6.21	Variation of $\sigma_{\theta z}$ at hole edge at $0^\circ/90^\circ$ interface for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.	157
6.22	Plot of $\sigma_{\theta z}$ along 50° radial at the $-45^\circ/0^\circ$ interface of a uniaxially-loaded $[\pm 45/0/90]_s$ laminate for various hole sizes.	159
6.23	Plot of $\sigma_{\theta z}$ along 70° radial at the $-45^\circ/0^\circ$ interface for a $[\pm 45/0/90]_s$ laminate for various hole sizes.	160

LIST OF FIGURES

<u>FIGURE</u>		<u>PAGE</u>
6.24	Variation of σ_{zz} at the hole edge at the midplane for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.	161
6.25	Plot of σ_{zz} along 70° radial at the midplane for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate for various hole sizes.	162
6.26	Plot of σ_{zz} at the midplane along 90° radial for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate for various hole sizes.	163
6.27	Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied along the laminate axis.	169
6.28	Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied at 15° from the laminate axis.	171
6.29	Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied at 30° from the laminate axis.	172
6.30	Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied at 45° from the laminate axis.	173
6.31	Illustration of (a) the geometry of the laminate loading and (b) the laminate stresses as a function of the load angle.	174
6.32	Delamination initiation prediction of (a) load and (b) angular location for $[0/90]_s$ laminates with 26.8 mm holes.	176
6.33	Delamination initiation prediction of (a) load and (b) angular location for $[0/90/\pm 45]_s$ laminates with 26.8 mm holes.	179

LIST OF FIGURES

<u>FIGURE</u>		<u>PAGE</u>
6.34	Delamination initiation prediction of (a) load and (b) angular location for $[\pm 45/0/90]_s$ laminates with 26.8 mm holes.	181
6.35	Delamination initiation prediction of (a) load and (b) angular location for $[0/\pm 45/90]_s$ laminates with 26.8 mm holes.	182
6.36	Resultant shear contours for a uniaxially-loaded $[\pm 15/0]_s$ laminate with a 26.8 mm hole at the $+15^\circ/-15^\circ$ interface.	184
6.37	Resultant shear contours for a uniaxially-loaded $[\pm 15/0]_s$ laminate with a 26.8 mm hole at the $-15^\circ/0^\circ$ interface.	185
6.38	Resultant shear contours for a uniaxially-loaded $[\pm 45/0]_s$ laminate with a 26.8 mm hole at the $+45^\circ/-45^\circ$ interface.	186
6.39	Resultant shear contours for a uniaxially-loaded $[\pm 45/0]_s$ laminate with a 26.8 mm hole at the $-45^\circ/0^\circ$ interface.	187
6.40	Prediction of delamination initiation stress for $[\pm \theta/0]_s$ laminates with 26.8 mm holes.	189

LIST OF TABLES

<u>TABLE</u>		<u>PAGE</u>
5.1	AS4/3501-6 Material Data	95
5.2	T300/5208 Material Data	118
6.1	Interlaminar Strength Parameters for AS4/3501-6	166
6.2	Equivalent Laminates for Various Load Angles	178

NOMENCLATURE

a^i	Magnitude of residual stress in the i^{th} ply
A	Coefficient matrices for differential equation for the contribution due to a stress free edge
b^i	Integration constants used in expression for interlaminar shear stress for the i^{th} ply
c^i	Integration constants used in expression for interlaminar normal stress for the i^{th} ply
d^i	Thickness weighted combinations of the a^i 's, b^i 's, and c^i 's
e^i	Compliance weighted combinations of the d^i 's
$E_{\alpha\beta\sigma\gamma}$	Two-dimensional laminate stiffness tensor
f_i	Magnitude of i^{th} eigenvector for zeroth order solution
F_0	Zeroth order perturbation solution for free edge component solution
F_1	First order correction solution for free edge component solution
$F_{\alpha\beta}$	Generic functions for the decay shapes of the in-plane stresses
g_i	Magnitude of i^{th} eigenvector for first order correction term
$I_{\alpha\beta\sigma\gamma}$	Tensor used to measure the difference between laminate and ply stiffnesses and compliances
$J_{\alpha\beta\sigma\gamma}$	Thickness weighted summation of the I tensor
$K_{\alpha\beta\sigma\gamma}$	Thickness weighted summation of the J tensor
n_β	Free edge surface normal

$N_{\alpha\beta}$	In-plane laminate plate loading
p	Far-field stress magnitude used in expression by Savin for in-plane stresses around holes
R	Hole radius
R^i	Residual stress vector
S_{ijkl}	Three-dimensional laminate compliance tensor
t^i	Thickness of the i^{th} ply
t^*	Effective ply thickness
u	Displacement in the x_1 direction
v	Displacement in the x_2 direction
W	Vector measuring first order residuals left from zeroth order perturbation solution
x	Distance from a stress free edge
x_α	In-plane rectangular coordinate system
Y_α	Orthonormal eigenvectors in the plane normal to the eigenvector ϕ
z	Coordinate measuring through-the-thickness distances in a laminate
Z_α	Vectors obtained via the transform of the Y vectors by the matrices of the zeroth order solution
z_α	Component of Z_α in the direction of W
$\delta_{\alpha\beta}$	Kronecker delta function
ϵ	Perturbation parameter given by the ratio of the effective ply thickness to the hole radius

$\varepsilon_{\alpha\beta}$	Laminate strains
λ_i	The i^{th} eigenvalue for the free edge component solution
Π_c	Complementary energy
ρ	Radial distance from hole edge
σ_{ij}	Stress tensor
ϕ_i	The i^{th} eigenvector corresponding to the i^{th} eigenvalue
Φ_0	Complex function used in solution for in-plane stresses around holes
Ψ_0	Complex function used in solution for in-plane stresses around holes

Chapter 1

Introduction

The use of advanced composite materials, especially graphite/epoxy, in the primary structures of general and military aerospace vehicles is increasing. With this increase in use has come an increased emphasis in the study of the various failure mechanisms of these materials. Work in the past has centered on what might be termed obvious failure mechanisms. Such mechanisms are basic strength properties, notch sensitivity, and fatigue, to name a few. These mechanisms could be called obvious because all of these fields have been dealt with before in metal structures, and the designer has a basic concept of what methodologies for analysis should be used, even though the precise phenomenon might not be known.

A failure mechanism, unique to laminated materials, with which the average designer of metal structures has little or no knowledge is delamination. Delamination is a separation of the individual plies of a laminate, resulting in the loss of the combined action of the plies in the delaminated region. The resulting loss in the load-carrying capability of a delaminated part can vary from being unnoticeable, as is the case for a part loaded in tension which contains an imbedded delamination [56]., to being catastrophic, as is the case for various laminates loaded in tension due solely to the presence of a straight stress free edge [55]. Another serious consequence of delamination is that stiffness properties, such as the bending stiffness of the laminate, can be greatly degraded.

Delamination can arise in many different situations, such as an impact event. However, just as important a cause are interlaminar stresses which

result wherever a stress free edge occurs. Another source of these interlaminar stresses which has previously received practically no attention, are those arising whenever gradients occur in the in-plane stress field. A potentially serious problem occurs whenever interlaminar stresses are present, because the ability of a laminate to resist delamination is matrix dominated, and since the matrix tends to be much weaker than the fibers, this could very well be the weakest link in the composite strength chain.

Interlaminar stresses along straight stress free edges have been examined by numerous investigators [1-37]. Much less common is work dealing with the interlaminar stresses at curved boundaries [38-52]. Practically all of the literature available presents a specific numerical example for a particular laminate using finite element methods. The task of this work is to provide a more generic approach to the problem of the calculation of interlaminar stresses for the purpose of predicting delamination initiation. Some accuracy will be sacrificed so that an efficient and insightful solution can be obtained which will allow a broader view of the interlaminar stress problem.

Of special interest to this investigation are the interlaminar stresses arising in composite laminates containing circular holes. The need for efficiency for this problem is especially acute because of the enormous computational effort which is associated with these problems whenever Finite Element methods are used.

Initially in this report, different approaches to the interlaminar stress problem in general, with emphasis on work dealing with the interlaminar stresses around holes, will be detailed. The important trends and findings of these previous studies will be used in order to focus this current research on

problems which have significant impact on delamination initiation. Also, these findings can be used to provide a better approximation to the interlaminar stress fields. This review of the pertinent literature appears in Chapter Two.

In Chapter Three, a generic derivation is made which provides an approximate solution methodology for the interlaminar stresses for an arbitrary problem. New methodologies are presented for approximating the interlaminar stresses in gradient stress fields as well as those occurring near a stress free edge. The techniques which will be used throughout the following text, as well as justification for these techniques, are given.

The specific case of the solution for the interlaminar stresses in the vicinity of circular holes is solved in Chapter Four. Using the methodologies of Chapter Three, a fast and efficient solution is derived. Also described here is the computer implementation of these equations. A specific example of how the approximations of the previous chapter are used is provided in this chapter.

The material in Chapter Five seeks to verify and find the limitations of the solution obtained in Chapter Four. The solution is compared to an exact elasticity solution for a special isotropic case and the limitations on the applicability of the solution are assessed. The effects on the accuracy of the solution caused by material orthotropy and laminate inhomogeneity is investigated by comparing the current solution with an existing finite element solution.

In Chapter Six, various case studies for different laminate families and load and geometry configurations are performed. The effects of layup and ply orientation on the interlaminar stresses and the delamination initiation stress

are investigated. The high efficiency of the solution technique allows this current investigation to consider a large number of such configurations.

Finally, in Chapter Seven, the results of the previous chapters are enumerated and explained. Both the advantages and the shortcomings of the current analysis will be listed, and suggestions for improving the deficiencies are given. Three appendices are also given, in two, listings of the FORTRAN programs "SLASH," which was formulated in the course of this research, and "PLOTTER," utilized to manipulate the results, are given. In the first appendix, the material properties utilized in various analyses within the manuscript are summarized.

Chapter Two

Previous Work

2.1 Historical Overview - Straight Free Edge Solution

The problem of interlaminar stresses has been studied now for over twenty years. The fact that interlaminar stresses may be present in composite laminates was first of concern for problems involving the bending of plates [1,2]. These papers showed that classical approaches for calculating the interlaminar shear stresses due to beam bending were adequate except in pathological cases where the beam aspect ratio was very small.

Work on what is now commonly envisioned as being the interlaminar stress problem, the stresses arising at a straight free edge, was first performed in the early 1970's by Pipes and Pagano [3,4]. These were finite difference solutions, and they sparked a great deal of interest in the topic of interlaminar stresses.

Studies which immediately followed were mostly performed using finite element approaches [e.g., 5-7]. Because of a general lack of understanding of the problem and its complexities, and because of different modelling methods for these finite element solutions, controversy soon arose. Such questions as what the sign of the interlaminar normal stress should be for various laminates and which, if any, of the interlaminar stresses were singular at the free edge dominated the discussion. These finite element solutions generally used rather course meshes, contributing to the inaccuracies.

Next followed some analytical approaches to try to account for the discrepancies previously mentioned [8-10]. These approaches tended to use

Airy stress function to model highly idealized problems. Some new light was shed on the problems, but the practical relevance of these solutions was lacking. Also, these studies did little to clear up the singularity questions.

In the late 1970's, the analyses started gaining in complexity, and the problem of interlaminar stresses at straight free edges was being brought into better focus. Hsu and Herakovitch [11] were able to derive an approximate analytical solution for angle-ply laminates. These investigators found that weak singularities might be present for both of the interlaminar shear stresses.

Wang and Crossman [12], investigated five different laminates $[0/90]_s$, $[90/0]_s$, $[\pm 45]_s$, $[\pm 45/0/90]_s$, and $[90/0/\pm 45]_s$ laminates using a finite element analysis. What sets this work apart is that great care was taken to accurately model the boundary layer near the free edge. These investigators used a total of 392 constant strain triangular finite elements with 226 nodes and 452 degrees of freedom to model the problem. From this analysis they were led to conclude that possibly all of the interlaminar stresses, including the interlaminar normal stress, might be singular at the free edge.

A more sophisticated analytical study was performed by Wang [13] for a $[0/90]_s$ laminate. Wang assumed displacements to be in the form of the sum of Legendre polynomials. Using Galerkin's method, the coefficients of these terms were calculated. For a complete set of these polynomials, one would expect a very accurate solution. However, in order to simplify the calculations, only two modes were taken for the displacements in the thickness direction. The resulting solutions for various numbers of modes in the x direction, the direction which measures the distance from the free edge, are compared.

These solutions revealed some convergence difficulties near the free edge at the interface, thereby limiting the accuracy of the solution.

Pagano [14,15], using a similar approach, was able to obtain a system of $13N$ ordinary differential equations for the displacements, where N is the number of sublayers into which the laminate is divided. The form of the equations, which resulted from singular perturbations, proved to be unstable for large values of N . Useful solutions were not obtained with this method until Soni and Chu [16] slightly reformulated the problem. Even in this case, almost nine minutes of execution time were required on a personal computer for the solution of a twenty-ply laminate.

Because of the convergence problems associated with the global problem as was being solved in [13-16], some investigators began using perturbation techniques to examine the local region at the intersection of the ply interface and straight free edge. Wang and Choi [17] used an eigenfunction expansion technique to determine the stress at the interface. They analytically found the existence of a logarithmic singularity. Similarly, Dempsey and Sinclair [18], using an Airy stress function approach, found that terms like $(\log x)^2$ and $(\log x)^3$, etc. may also be present. Such higher order logarithmic terms were then included in the analysis of Ting and Chou [19].

Zwiers, Ting, and Spilker [20] looked locally at the interface between two adjacent plies in a laminate. They found weak singularities between plies of dissimilar angles. These singularities were of the form of logarithmic singularities, and singularities of the form of x^{-b} . However, the largest value they found for b , using a graphite/epoxy type material, was 0.0541. This is a very weak singularity. Also, these investigators found that although singularities other than logarithmic may arise, the coefficients of these terms

tended to be rather small, and often were zero. A complete analysis of the entire laminate was required in order to determine these constants. The logarithmic singularities were found to be always present, and the coefficients of these terms depended only on the two local plies. Zwiers and Ting [21] later expanded this analysis to the case of loaded holes and found similar results.

Kanwal [22] performed a generic derivation for the boundary value problems in composite media, the straight free edge problem being a specific example. He was able to derive integral equations, the solution of which would yield the interlaminar stresses. This offered technique was complete, but probably unsolvable.

With these new analytical foundations, the straight free edge problem was now beginning to become better understood. A great step forward in this understanding came with the summary paper of Whitcomb, Raju, and Goree [23], commenting on the reliability of the finite element method for calculating the free edge stresses. This paper showed that logarithmic singularities existed in very simple problems which could be used to model the free edge problem. In addition, this paper compared finite element solutions with exact analytical solutions for these simplified solutions and found that a properly modelled finite element solution was accurate except in a region within two elements of the free edge, at the interface. This two element distance was independent of element size, as long as a suitably fine discretization had been used. Simultaneously, this paper showed that the finite element solutions could be employed, but that for an accurate solution near the free edge, a very fine mesh was required.

A natural extension of this work was later published by Whitcomb and Raju [24]. These investigators noticed that although it was nearly impossible

to have a physical "feel" for what the interlaminar stresses should be, given a generic straight free edge problem, a more expressive problem could be formulated which would yield identical results. What Whitcomb and Raju did was to break up the straight free edge problem into two parts. The first part was the solution from classical laminated plate theory which is easily obtainable. This solution satisfied equilibrium and boundary conditions everywhere except at the straight free edge. At the free edge, the transverse stresses which are required for displacement compatibility between the plies in a laminate are still present. The integral averages of these stresses through the thickness are zero, each stress was not identically zero as was required. To compensate for this, a companion problem was formulated which took the negative of these stresses which were present at the free edge and applied them to the side of a laminate whose other sides are unloaded. Using the method of superposition, the classical laminated plate theory solution and the solution to the companion problem could then be added together to obtain a complete solution satisfying all boundary conditions.

The advantage of this type of formulation was that the loadings which were driving the interlaminar stresses could now be easily seen. From simply looking at this loading, the correct signs for the interlaminar stresses which had previously been considered a horribly complex and unresolved issue, could easily be seen. Also, much simpler finite element schemes could now be used.

Most of the remaining work on the straight free edge problem has been various finite difference [25] and finite element solutions [26-29]. Notable among these for its complexity is the work of Lin, Hwang, and Ilcewicz [28] which investigated the effects of cyclic loading and viscoelastic material properties on the interlaminar stress state.

As can be seen, almost all of the preceding work has been based on highly discretized numerical procedures such as finite elements and finite differences, or on purely analytical techniques. Little work has been done to try to find some compromise between the two approaches.

Finite element solutions can, if formulated correctly, provide a fairly detailed description of the stress state. However, a great deal of post-processing is usually required in order to gain any insight into the problem, and also, because of the fine meshes required near the free edge at ply interfaces, run times can be prohibitive. Even if formulated correctly, many of the available finite element solutions fail to exactly match the boundary conditions.

On the other hand, most of the analytical solutions presented have a comparatively high degree of accuracy, but can be used only for highly idealized problems such as the case for angle-ply or cross-ply laminates. Also, some solutions, such as those in the form of infinite series, can still require a large degree of post processing in order to gain an understanding of the interlaminar stress state.

The work, both through analytical means and with the use of finite elements, on the nature of the possible singularity at the free edge indicates that if such a singularity exists, it is very weak. Because of this, it is only of academic and not practical concern. A large deal of computational effort is required to capture this behavior, yet the singularity is so weak and it acts over such a small portion of the laminate that its effect on laminate behavior is minimal. It should be remembered that the ultimate goal for these analyses is the prediction of the onset of delamination. Many methods for the prediction of the onset of delamination deal with some type of average of either the stresses

or the strain energy over a finite distance away from the free edge. The existence of a weak singularity will have little effect on these calculations.

Work by Bar-Yoseph, et al, [30-32] used techniques to join local asymptotic analyses and finite element analyses in an attempt to obtain a more efficient and more accurate solution. Hybrid finite elements were used which incorporated assumed stress states in order to get a better estimate for the interlaminar stress state, retaining such features as traction continuity between plies, while keeping the total number of required elements small. This technique was later expanded by Bar-Yoseph and Sitton [31] to include nonlinear material properties in the analysis.

This approach taken by Bar-Yoseph, et al, represents an attempt to sacrifice some of the accuracy of an exact analytical solution for some of the flexibility of problem types which was available with finite elements. What is lacking, however, is an even more generic approach which affords appropriate accuracy and insight while keeping the required computation time manageable

2.2 Simplified Models for the Straight Free Edge Problem

Simplified approximate techniques for determining the interlaminar stresses at a straight free edge have only appeared relatively recently. These methods are willing to sacrifice the high degree of accuracy sought by previous investigators in exchange for greatly improved execution time and more easily interpreted solutions.

Ueng and Zhang [33], performed what is probably the crudest approximation one could use and still expect sensible answers. These

investigators studied the interlaminar stresses in finite width coupons by assuming that the in-plane stresses which were required to go to zero at the free edge decayed linearly in the region within a specified number of laminate thicknesses from the free edge. The remaining stresses were then found using equilibrium equations. Although solutions are shown to match a finite element analysis fairly well, there is no reason to assume a technique such as this would have any general applicability.

A much more rigorous approach to this problem, using a similar technique, was performed by Kassapoglou and Lagace [34]. In this method, the in-plane stresses were again assumed, but were assumed this time in the form of two decaying exponentials. What set this analysis apart was that the decay rates of these functions were not specified *a priori*. The functions that were assumed were chosen so that both the equilibrium equations and the boundary conditions were satisfied. The principle of minimum complementary energy was then used to determine what the decay rates of the functions should be. This approach resulted in comparatively simple algebraic equations which needed to be solved in order to determine the aforementioned decay rates. Solutions obtained using this methodology showed commendable agreement with finite element solutions, and in a typical case required less than one second of run time on a personal computer as compared to 12 CPU seconds on a UNIVAC-1108 for the finite element analysis of Wang and Crossman [12].

Kassapoglou has extended this approach for combined loading cases [35] and bending [36]. In both cases exponentials are again assumed and the principle of minimum complementary energy is used to obtain expressions for the decay rates.

Lagace and Saeger [37] were able to improve this methodology while simultaneously simplifying it. Instead of assuming an explicit functional form for the in-plane stresses, as was done in [33-36], they allowed these stresses to be arbitrary functions. Again, the principle of minimum complementary energy was used, resulting this time in a set of coupled differential equations instead of the algebraic equations obtained. Because of the form of these differential equations, the method of Lagace and Saeger proved to be more easily solved than that of Kassapoglou and Lagace. Because a more generic approach was taken, the solution obtained in [37] is also more accurate.

These studies on the state of stress at a straight free edge have enjoyed varied success. Recent work has dispelled many of the earlier misunderstandings which were prevalent in the early work, and the general nature of the stress state is now known. However, the transition from highly detailed, and computationally expensive, analyses to analyses more suited for actual design purposes, especially preliminary design purposes, have been slow to come. The approximate methods described in Section 2.2 have attempted to fill this need. The sacrifice which is made to reduce the computational load of this problem is an accompanying decrease in accuracy. This tradeoff may or may not be a concern, depending on the specific goal to be satisfied.

2.3 Solutions for Interlaminar Stresses Near Cutouts

Solutions for the interlaminar stresses around cutouts are much rarer than the solutions for the straight free edge problem. The earliest investigation seems to have been a finite element analysis by Dana and Barker

[38]. This analysis used three dimensional twenty node brick elements. Similar work was later performed by Rybicki and Schmueser [39,40], sampling a wider range of laminates, but lumping $\pm\theta$ plies together as a singular equivalent ply. These investigators were concerned only with the interlaminar normal stress, thought to be the sole contributor to delamination, and did not include interlaminar shear data.

Following this, Tang [41] used a perturbation analysis to investigate the problem. He broke the problem up into two cases, one involving only the stresses $\sigma_{r\theta}$ and $\sigma_{\theta z}$, and the other involving the remaining stresses. These problems were coupled later through material constants. The stresses were assumed to be exponential in the r direction and sinusoidal in the z -direction. The work was found to compare well with the finite element results of Rybicki and Hopper [42]. This work had some serious shortcomings, however. Because of the assumed form of the stresses, Tang was not able to satisfy the condition that σ_{zz} was equal to zero at the top and bottom of the laminate. Also, the decay rates of the solutions were not a function of the material properties.

Altus and Bar-Yoseph [43] solved the complete equations of elasticity using finite differences for the case of an elliptical cutout in an orthotropic plate. A slightly more efficient solution was obtained by Barsoum and Freese [44] using special interlaminar shear elements along with the logarithmic singularity proposed by Oplinger [45]. This analysis required a stiffness matrix that was, at most, one-third the size of an equivalent standard three-dimensional model, thus reducing the storage requirements and the associated run time. Carlson [46], using 90 three-dimensional brick elements per ply in a twenty-eight ply laminate, was able to find the steep gradients in

the interlaminar stresses which had been found previously for a straight free edge.

Lucking, Hoa, and Sankar [47], also using twenty node isoparametric brick elements, investigated the effect of the ratio of the hole radius to the plate thickness for $[0/90]_s$ laminates. They found that for a fixed laminate thickness, increasing the radius of the hole had the effect of increasing the magnitude of the interlaminar shear stress and normal stress, as well as influencing the zone of influence over which they act.

Bar-Yoseph and Avrashi [48] used a variational-perturbation approach for the three-dimensional stress analysis around an elliptical cutout. Using only the zeroth order perturbation solution, the stresses were assumed to retain the same angular distribution present at the free edge. The stress distribution in z was assumed to be the sum of N Legendre polynomials. The decay rates of the stresses were then solved for using the principle of minimum complementary energy. This resulted in a $3 \times (\text{number of plies}) \times N$ eigenvalue problem. Only the first two Legendre polynomials were used for the interpolation in the z -direction. This left governing equations of the singular perturbation type which are ill-conditioned.

Both Klang and Hyer [49] and Ericson, et al [50] have used singular finite elements to investigate the interlaminar stress state around holes. The latter found that, as in the straight free edge case, the singularities were very weak, with values for b , in the form r^{-b} , of around 0.03. This analysis gives approximately the same distribution over the hole boundary of the interlaminar normal stress as do analyses based on nonsingular finite elements.

Only two non-intensively numeric solutions could be found in the literature. One solution by Zhang and Ueng [51], although concise, is probably greatly limited in applicability. These investigators again just assumed an explicit form for the in-plane stresses and then calculated the interlaminar stresses using the equilibrium equations. No attempt was made to find an optimal form for these in-plane stresses. Also, it was assumed that the interlaminar shear stress had a root r singularity, which is far worse than that encountered anywhere else in the literature.

The other simplified solution was performed by Kassapoglou [52]. Here, he used an explicit assumed distribution for stresses in the form of the isotropic hole solution plus an arbitrary power of r . This approach proved to be only marginally successful. He was not able to obtain the orthotropic solution for the in-plane stresses for the laminate, and the interlaminar stresses did not have a very high degree of accuracy.

The need for a simple and efficient solution for the case of holes has not yet been fulfilled.

2.4 Prediction of Delamination Initiation

The development of criteria for delamination initiation has seen two main approaches: strain energy release rate [53] and strength of materials [54,55]. These two approaches require vastly different information of the interlaminar stress state.

In the case of strain energy release rate approaches, the strain energy must be calculated for both a fully laminated specimen and then for a specimen in which a delamination has begun to propagate. The difference

between the strain energy in these two situations is then used to calculate the strain energy release rate. In the case of a straight free edge, where the stress does not vary along the load direction, this method can still be practically applied. For example, a fairly simple finite element solution can be performed for the laminated specimen, and then analyses could be performed for the criticality of a given interface by placing initiated delaminations at the given interface and repeating the calculation. Although this method might be practical, its efficiency is somewhat low. When applied to the case of a hole, however, this problem becomes intractable. Not only must delaminations be initiated at each interface of interest, but also at each angular location of interest. The number of problems to be solved to adequately analyze such a situation is tremendous.

For predicting the onset of delamination, Lagace [55] has demonstrated that the strength of materials approach when used with an average stress concept is superior to strain energy release rate approaches. Whereas the previous technique required the solution of multiple problems, the exact number of which depended on the number of interfaces of interest, this latter approach requires only one problem to be solved. Another advantage of this approach is that if an average stress concept is used, then the exact point values of the stress are no longer an overriding concern, and the question of the nature of the singularity loses importance. For the case of a straight free edge, the methods of solution for the interlaminar stresses given by Lagace, et al, are easily incorporated into this type of initiation prediction, and as stated previously, this prediction of the initiation stress is the ultimate goal of the analysis of the interlaminar stress state.

Chapter 3

General Formulation

Interlaminar stresses may be thought of as arising from two distinct mechanisms, in-plane gradients in the stress field, and the presence of a stress-free edge. Classical laminated plate theory cannot account for these stresses, yet a careful examination of this theory reveals relatively straightforward methods for approximating them. The reason that it is natural to break the interlaminar stress problem up into these two components is that in the first case, the decay rate of the interlaminar stresses will be related to in-plane parameters, such as the size of a hole or slit, or how quickly applied loadings vary. In the second case, the decay rate will be closely associated with the stacking sequence and location of various plies in the laminate and the decay rate will be more related to the ply/laminate thickness. In both cases, the magnitude of the interlaminar stresses are greatly affected by stacking sequence, however, only in the second case does the stacking sequence affect the decay rates of these stresses. The following sections detail the formulation for the solution for the interlaminar stresses for a generic problem. A sketch of a typical composite laminate showing the in-plane and out-of-plane stresses as well as lamination planes is given in Figure 3.1.

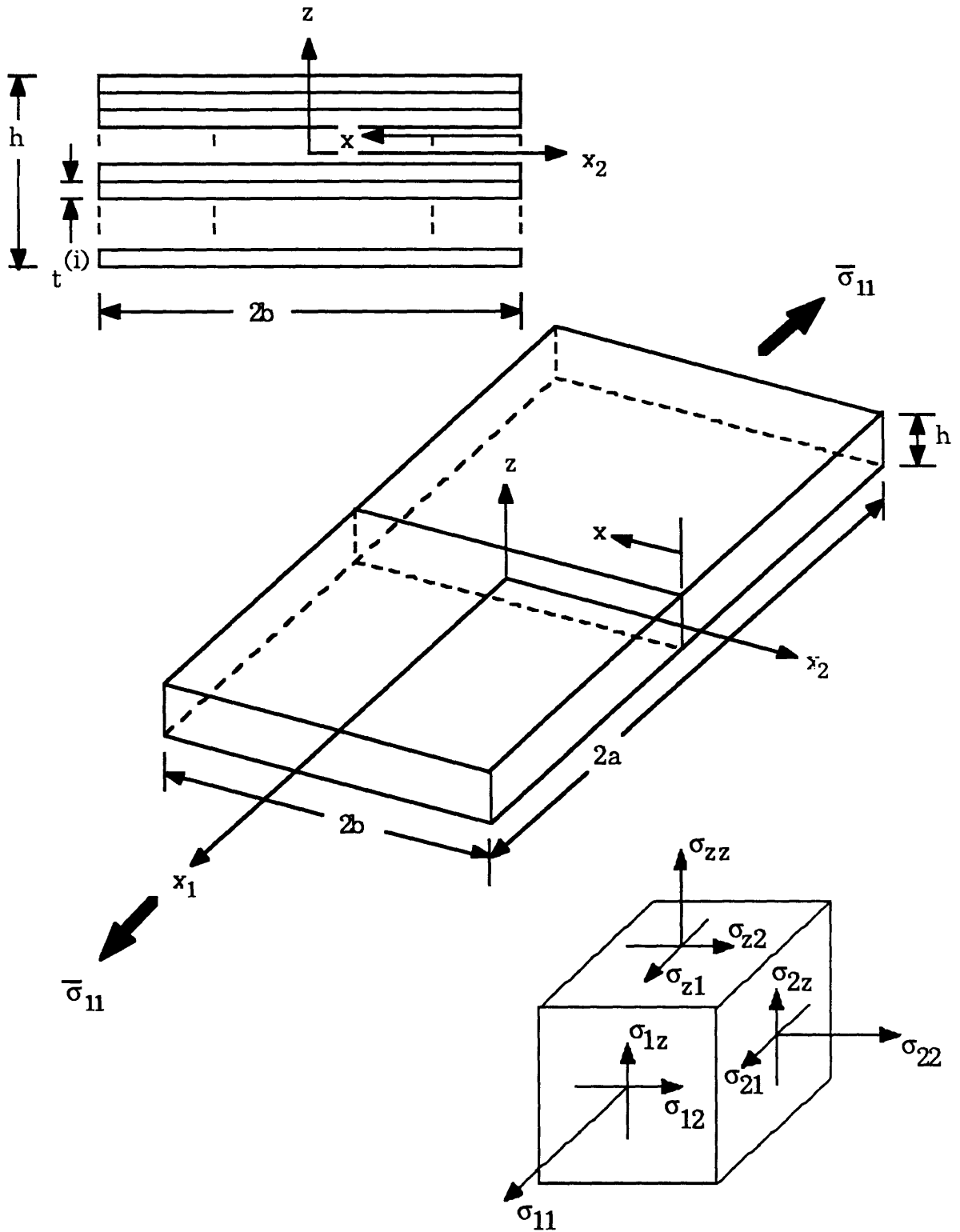


Figure 3.1 Geometry for problem of a laminated plate under loading.

3.1 Interlaminar Stresses Arising in Gradient Stress Fields

3.1.1 Necessity of Interlaminar Stresses in Gradient Stress Fields

A commonly overlooked source of interlaminar stress is an in-plane gradient stress field. In order to investigate how in-plane gradient stress fields cause interlaminar stresses, the assumptions of classical laminated plate theory need to be reviewed. To begin, assume that a two-dimensional problem has been solved using smeared properties from classical laminated plate theory. Then, the in-plane loading will satisfy the condition that:

$$N_{\alpha\beta,\beta} = 0 \quad (3.1)$$

Where $N_{\alpha\beta}$ are the in-plane loadings. As is the case with standard tensor notation, the use here of Greek subscripts denotes planar problems where the subscripts take on the values 1 or 2. Latin subscripts will take on the values from 1 to 3. A comma indicates a derivative. The laminate stresses are then defined as:

$$\sigma_{\alpha\beta}^L = \frac{N_{\alpha\beta}}{t^L} \quad (3.2)$$

where the superscript L denotes laminate quantities. Thus, t^L is the laminate thickness. From classical laminated plate theory, the laminate strains, which are assumed to be constant through the thickness of the laminate when there is no bending, are given by:

$$\epsilon_{\alpha\beta}^L = S_{\alpha\beta\sigma\gamma}^L \sigma_{\sigma\gamma}^L \quad (3.3)$$

where $S_{\alpha\beta\sigma\gamma}$ represents the two dimensional compliance tensor. The stresses in the i^{th} ply are then given by:

$$\sigma_{\alpha\beta}^i = E_{\alpha\beta\sigma\gamma}^i \epsilon_{\sigma\gamma}^L \quad (3.4)$$

Here, $E_{\alpha\beta\sigma\gamma}^i$ denotes the components of the two dimensional stiffness tensor for the i^{th} ply. Combining equations 3.3 and 3.4, an expression for the ply stresses is obtained in terms of the laminate stresses as:

$$\sigma_{\alpha\beta}^i = E_{\alpha\beta\lambda\kappa}^i S_{\lambda\kappa\sigma\gamma}^L \sigma_{\sigma\gamma}^L \quad (3.5)$$

This quantity of the stiffness tensor multiplied by the compliance tensor is encountered frequently in the following discussions. For this reason, it will be useful to define a new tensor, $I_{\alpha\beta\sigma\gamma}$, by the relationship:

$$I_{\alpha\beta\sigma\gamma}^i = E_{\alpha\beta\kappa\lambda}^i S_{\kappa\lambda\sigma\gamma}^L \quad (3.6)$$

It can be seen from Equation 3.5 that the I tensor, when multiplied by the laminate stress, gives the amount of stress which is carried in a given ply. For example, the term I_{1122}^i gives the magnitude of the stress σ_{11} in the i^{th} ply due to a unit loading σ_{22} applied to the laminate. The tensor I , has the property that:

$$\frac{1}{t} \sum_i t^i I_{\alpha\beta\sigma\gamma}^i = \frac{1}{2} (\delta_{\alpha\sigma} \delta_{\beta\gamma} + \delta_{\alpha\gamma} \delta_{\beta\sigma}) \quad (3.7)$$

where δ_{mn} is the Kronecker delta function defined by:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (3.8)$$

The expression given in Equation 3.7 has only six nonzero terms. If the symbol $I_{\alpha\beta\sigma\gamma}^L$ is used to denote the sum in this equation, then the terms I_{1111}^L and I_{2222}^L are equal to 1.0, and the terms I_{1212}^L , I_{1221}^L , I_{2112}^L , and I_{2121}^L are equal to 0.5. This property gives an identity relationship between the in-plane extensional stresses, and if the symmetry in the in-plane shear stresses is used, a similar identity relationship is obtained for these stresses. This arises from the fact that the laminate stress may be written as the weighted sum of the ply stresses via:

$$\sigma_{\alpha\beta}^L = \frac{1}{t} \sum_i t^i \sigma_{\alpha\beta}^i \quad (3.9)$$

If Equation 3.5 is substituted into the right-hand side of Equation 3.9, the following expression results:

$$\sigma_{\alpha\beta}^L = \frac{1}{t} \sum_i t^i \left(E_{\alpha\beta\lambda\kappa}^i S_{\lambda\kappa\sigma\gamma}^L \right) \sigma_{\sigma\gamma}^L \quad (3.10)$$

The quantity in parentheses has already been defined as the $\mathbf{I}$ tensor given in Equation 3.6. Making this substitution, an identity relation between the in-plane stresses is found as:

$$\sigma_{\alpha\beta}^L = \left(\frac{1}{t} \sum_i t_i I_{\alpha\beta\sigma\gamma}^i \right) \sigma_{\sigma\gamma}^L \quad (3.11)$$

Equilibrium of the i^{th} ply requires :

$$\sigma_{mn,n}^i = 0 \quad (3.12)$$

This set of equilibrium equations can be rewritten in terms of in-plane and out-of-plane terms as:

$$\sigma_{\alpha\beta,\beta}^i + \sigma_{\alpha 3,3}^i = I_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\beta}^L + \sigma_{\alpha 3,3}^i = 0 \quad (3.13 \text{ a})$$

and

$$\sigma_{3\beta,\beta}^i + \sigma_{33,3}^i = 0 \quad (3.13 \text{ b})$$

In Equation 3.13a, the tensor representing the in-plane ply stresses, σ^i , has been replaced in the right hand side of the equation by the tensor $\mathbf{I}\sigma^L$, by using the definitions of Equations 3.5 and 3.6. Because the material properties are assumed not to vary in the in-plane directions, the subsequent differentiation with respect to x_β acts only on the σ^L tensor.

These equations are much stricter than Equation 3.1, in the sense that solutions for the in-plane equilibrium equations of Equation 3.1 will satisfy the conditions of Equations 3.13 only in very special circumstances. The two most notable of these special cases is, one, the case where σ^L has no gradients in

the in-plane directions and thus the term $I_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\beta}^L$ is zero thus allowing the $\sigma_{\alpha 3}^i$ to be zero; and, two, the case where I^i is the identity tensor described in Equation 3.7 thus giving ply stresses which are identical to the laminate stresses which are known to satisfy equilibrium without the need for interlaminar shear stresses. If these criteria are not met, then out-of-plane stresses will be required for ply equilibrium.

As an example of this phenomenon, assume that a quasi-isotropic laminated plate occupies the entire x_1 - x_2 plane, and also assume that this plate is loaded at infinity so as to achieve the following state of stress:

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\}^L = \left\{ \begin{array}{c} x_1 \\ -x_1 \\ -x_2 \end{array} \right\} \quad (3.14)$$

This stress state satisfies equilibrium, and for an isotropic material, it also satisfies compatibility. For this problem, it is initially taken that all stresses acting in the x_3 direction are identically zero. Substituting this stress into Equation 3.13a, the following ply equilibrium equations are obtained:

$$\sigma_{\alpha 3,3}^i = -I_{\alpha 111}^i + I_{\alpha 212}^i + I_{\alpha 221}^i + I_{\alpha 122}^i \quad (3.15)$$

Taking the first of these equations by using α equal to one, and using the fact that the laminate is quasi-isotropic to simplify the expression by using the equality between E_{1111} and E_{2222} as well as S_{1111} and S_{2222} , this equation becomes:

$$\sigma_{13,3}^i = -\left(E_{1111}^i - E_{1122}^i\right)\left(S_{1111}^L - S_{1122}^L\right) + 4E_{1212}^i S_{1212}^L \quad (3.16)$$

If each ply of this laminate is made of the same material, then the right-hand side of this equation is identically zero due to the inverse relationship between **S** and **E**. Assuming that the laminate has the quasi-isotropic layup $[\pm 60/0]_s$ and is made of AS4/3501-6 graphite/epoxy, whose properties are given in Appendix A, then equilibrium for the top 60° ply requires:

$$\sigma_{13,3}^{+60^\circ} = 1.4976 \quad [\text{Pa} \cdot \text{m}^{-1}] \quad (3.17)$$

In other words, for the $+60^\circ$ ply to be in equilibrium for this ply, a linearly varying through the thickness σ_{13} is required whose slope is equal to approximately 1.5 times the in-plane σ_{11} variation. Depending on the thickness of the ply, this stress could be significant. This demonstrates that the equations of equilibrium on a ply by ply basis are not, in general, satisfied by a stress state that satisfies smeared equilibrium, compatibility, displacement continuity between plies, as well as traction continuity between plies. A procedure for correcting for this situation is developed in the next section.

3.1.2 Approximate Solution for Interlaminar Stresses in Gradient Stress Fields

A problem arises in Bernoulli-Euler beam theory which is entirely analogous to the gradient stress field problem for interlaminar stresses. A sketch of such a problem is given in Figure 3.2. In Bernoulli-Euler beam

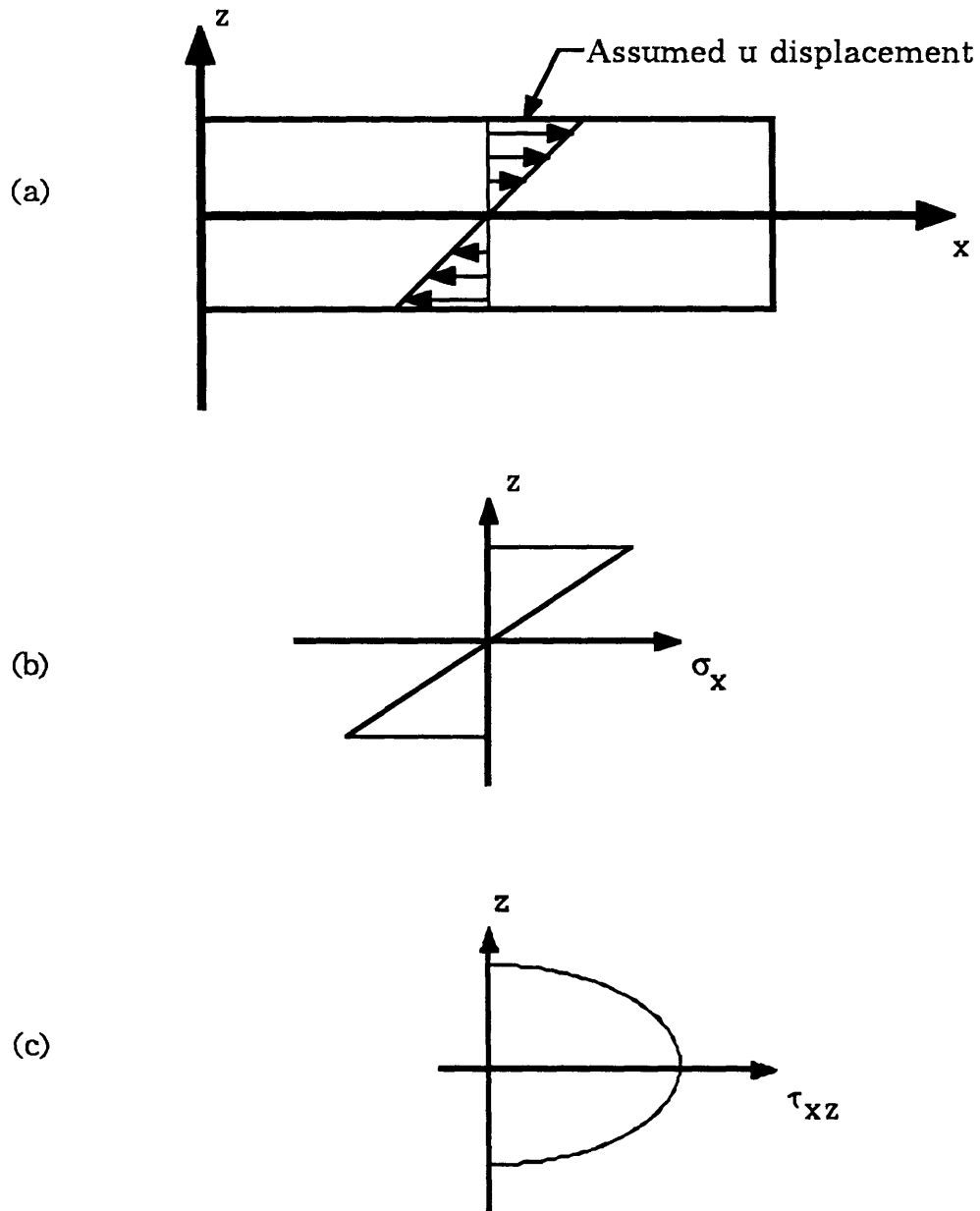


Figure 3.2 Illustration for Bernoulli-Euler beam theory showing (a) the displacement assumption, (b) the resulting extensional stress, and (c) the resulting shear stress.

theory, the displacement field is assumed. In particular, the u displacement of the beam is assumed to be linear with z . This displacement field gives a stress σ_x which is also linear in z . Assuming that the out-of-plane stresses (in this case, stresses acting on the y -face or in the y -direction) are zero, then equilibrium requires that:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (3.18)$$

The usual procedure for approximating the shear stress τ_{xz} is simply to use the assumed distribution for σ_x in Equation 3.18. This differential equation is then solved for the shear stress subject to the boundary conditions that at the top and bottom surfaces of the beam the shear stress is zero. It should be noted that the resulting stress field, which now satisfies equilibrium, violates the original displacement field assumption (unless the beam is infinitely stiff in shear). However, for many beam problems, this approximation yields a sufficiently accurate answer for practical purposes. It may be seen that the need for this shear stress stemmed from the fact that σ_x varied in the x -direction, i.e. an in-plane gradient stress field.

In classical laminated plate theory, the displacement field has again been assumed. Namely, the strains in the laminate are assumed to be constant in the z direction for the case in which no bending is present. If this assumption is retained, then Equations 3.13a and 3.13b can be thought of as equations for the interlaminar stresses. Taking terms dealing with the laminate stresses, σ^L , to the right hand side of the equations represented in Equation 3.13a and integrating with respect to z , one obtains:

$$\sigma_{\alpha z}^i = - \int_z I_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\beta}^L dz \quad (3.19)$$

Performing the required integration, and making use of the fact that the top and bottom of the laminate are traction free, the interlaminar shear stresses may be written as:

$$\sigma_{\alpha z}^i = - \left(I_{\alpha\beta\sigma\gamma}^i z^i + \sum_{j=1}^{i-1} I_{\alpha\beta\sigma\gamma}^j t^j \right) \sigma_{\sigma\gamma,\beta}^L \quad (3.20)$$

Where z^i is the distance in the z direction from the bottom of the i^{th} ply. These stresses can be easily evaluated at the i^{th} interface, and they are found to be:

$$\sigma_{\alpha z}^i(z^i = t^i) = - \left(\sum_{j=1}^i I_{\alpha\beta\sigma\gamma}^j t^j \right) \sigma_{\sigma\gamma,\beta}^L \quad (3.21)$$

With these expressions for the interlaminar shear stresses, the interlaminar normal stress can be calculated via Equation 3.13b, resulting in:

$$\sigma_{zz,z}^i = - \sigma_{\alpha z,\alpha}^i = - \left(I_{\alpha\beta\sigma\gamma}^i z^i + \sum_{j=1}^{i-1} I_{\alpha\beta\sigma\gamma}^j t^j \right) \sigma_{\sigma\gamma,\alpha\beta}^L \quad (3.22)$$

Equation 3.22 can be integrated with respect to z and, after applying the boundary conditions that the top and bottom of the laminate are traction free, the interlaminar normal stress can be written as:

$$\sigma_{zz}^i = \left\{ \frac{1}{2} \Gamma_{\alpha\beta\sigma\gamma}^i (z^i - t^i)^2 + \left(\sum_{j=1}^i \Gamma_{\alpha\beta\sigma\gamma}^j t^j \right) z^i - \frac{1}{2} \sum_{j=1}^i \Gamma_{\alpha\beta\sigma\gamma}^j (t^j)^2 + \sum_{j=1}^{i-1} \left(\sum_{k=1}^j \Gamma_{\alpha\beta\sigma\gamma}^k t^k \right) t^j \right\} \sigma_{\sigma\gamma,\alpha\beta}^L \quad (3.23)$$

Again evaluating this stress at the i^{th} interface yields:

$$\sigma_{zz}^i(z^i = t^i) = \left\{ -\frac{1}{2} \sum_{j=1}^i \Gamma_{\alpha\beta\sigma\gamma}^j (t^j)^2 + \sum_{j=1}^i \left(\sum_{k=1}^j \Gamma_{\alpha\beta\sigma\gamma}^k t^k \right) t^j \right\} \sigma_{\sigma\gamma,\alpha\beta}^L \quad (3.24)$$

The results of this section can be summarized in compact form if two new tensors are first defined:

$$J_{\alpha\beta\sigma\gamma}^i = - \sum_{j=1}^i \Gamma_{\alpha\beta\sigma\gamma}^j t^j \quad (3.25)$$

This tensor represents the thickness weighted sum of the $\mathbf{I}$ tensor from the bottom of the laminate to the i^{th} ply. Another useful tensor which is similar to the $\mathbf{J}$ tensor but which has a different weighting function is given by:

$$K_{\alpha\beta\sigma\gamma}^i = - \sum_{j=1}^i J_{\alpha\beta\sigma\gamma}^j t^j - \frac{1}{2} \sum_{j=1}^i \Gamma_{\alpha\beta\sigma\gamma}^j (t^j)^2 \quad (3.26)$$

Using these two definitions, the interlaminar stresses may be written as:

$$\sigma_{\alpha z}^i(z^i = t^i) = J_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\beta}^L \quad (3.27 \text{ a})$$

and

$$\sigma_{zz}^i(z^i = t^i) = K_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\alpha\beta}^L \quad (3.27 \text{ b})$$

It should be noted that the tensors **J** and **K** are both just thickness-weighted summations of the **I** tensor. The **I** tensor has real physical significance in that it provides a measure of the difference in the elastic constants between an individual ply and the entire laminate. When these constants are identical, no interlaminar stresses will result. This was shown in the example problem worked in the last section. When the plies were all made of the same isotropic material, it was shown that all of the equations of ply equilibrium were satisfied. Conversely, the greater the difference in the elastic constants, the larger will be the magnitude of the elements of **I** and hence, larger interlaminar stresses.

This procedure provides only a first order correction for the interlaminar stresses. It was assumed that the classical laminated plate theory displacement field, i.e. displacements are constant through the thickness of the laminate, was correct. Also, it was assumed that the out-of-plane stresses were equal to zero. These displacements were then used to calculate the in-plane stresses in a given ply. It was these ply stresses which were used to obtain an approximate solution for the out-of-plane stresses. The fact that these out-of-plane stresses exist implies that the original displacement field assumption must be modified. This would result in slightly different in-plane stresses, which would now vary in the z-direction in a given ply, and would, in turn, lead to new out-of-plane stresses. This suggests that an iterative procedure could be used to obtain more and more accurate estimates for the out-of-plane stresses. However, for the case in which the through the thickness direction is small compared to in-plane dimensions,

such corrections should be negligible, just as the quadratic approximation for the shear stress is fairly accurate for a slender beam.

3.2 Reduced Eigenfunction Solution Technique (REST)

The more commonly considered source of interlaminar stresses is a stress-free edge. After a typical planar problem has been solved using classical laminated plate theory, the equations of elasticity will have been satisfied only in a through-the-thickness average sense. This is done by assuming that the plate which has been subjected to a given in-plane loading is made of a homogeneous smeared material. The problem with the laminate equilibrium equations not exactly satisfying individual ply equilibrium was discussed in the preceding section. This section will deal with the fact that stress boundary conditions have also only been satisfied in an average sense.

Of special interest here is the situation in which the stress boundary condition requires a stress-free edge as shown in Figure 3.3. If the smeared material property problem has been solved correctly, then the stress-free condition will have been solved in an average sense, namely:

$$N_{\alpha\beta} n_{\beta} = 0 \quad (3.28)$$

where $\mathbf{n}$ is the unit normal vector to the surface. However, on a ply-by-ply basis, the surface normal and surface tangent stresses will not necessarily be equal to zero, as is required. The problem of interest thus becomes that of forcing each individual ply to have the correct stress-free boundary conditions.

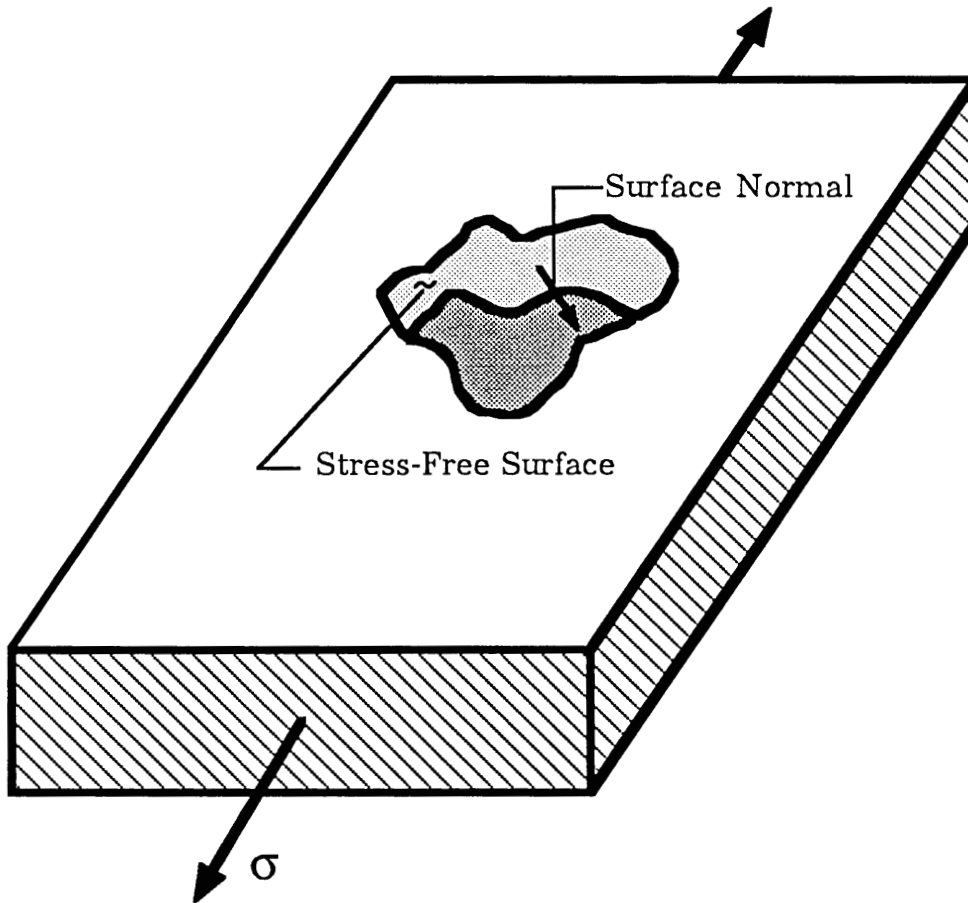


Figure 3.3 Schematic representation of a generic free-edge problem.

To perform this task, a companion problem is formulated. First, the problem is solved using smeared ply properties. Then, the stresses on a ply-by-ply basis can be found by using the $\mathbf{I}$ tensor of Equation 3.6. For convenience, x_2 will be used to denote the coordinate which is normal to the stress-free edge. The stresses acting on this face are the in-plane stresses σ_{22} , and σ_{12} , and possibly, after the corrections of the previous section, σ_{2z} . These stresses are termed residual stresses. These residual stresses have the property that when they are integrated through the thickness of a laminate at a stress-free edge, they average to zero. However, these calculated stresses are not identically equal to zero on a point by point basis as they should be.

For simplicity, dealing with only one of these stresses at a time, assume that a residual σ_{22} is present. Then, the companion problem will be defined as a problem in which a laminate is subjected to a loading which is the negative of this σ_{22} , applied at what will become the stress-free edge. Using the principle of superposition, this solution is added with the solution found using classical laminated plate theory. The resulting sum of these two solutions will yield a solution which satisfies the required boundary conditions as well as equilibrium on a ply-by-ply basis. This is possible because the addition of any two equilibrium stress states will result in a stress state which is also in equilibrium, and because the companion problem has stresses acting normal to the free edge which are the negative of the residual stresses present in the classical laminated plate theory solution with the sum of these two solutions resulting in a zero stress state at the free edge. This technique of adding the two solutions is shown pictorially in Figure 3.4.

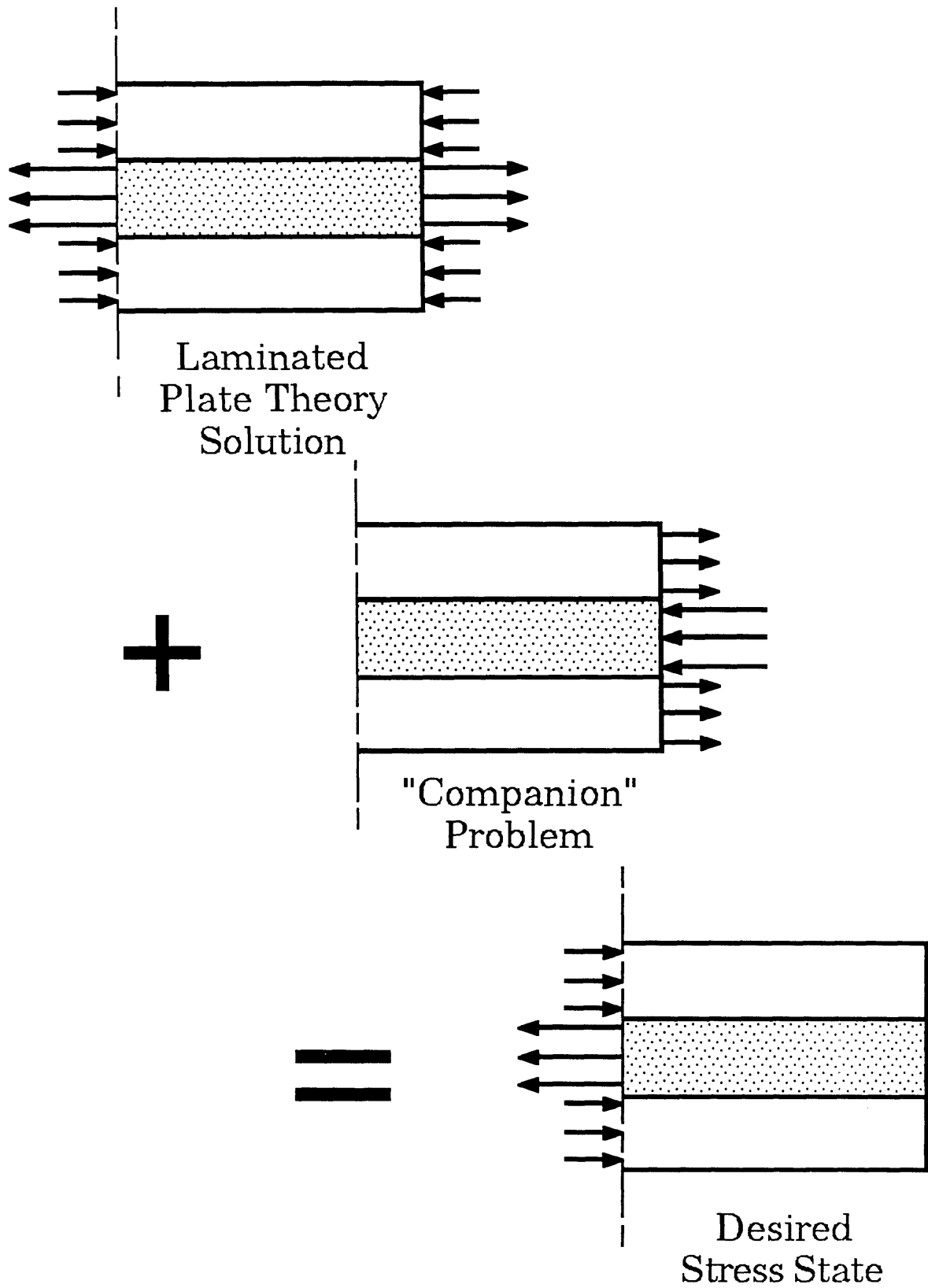


Figure 3.4 Illustration of solution technique methodology using the principle of superposition.

3.2.1 Solution of the Companion Problem

The companion problem, which is to be solved, represents, in its most general form, not only all of the usual complexities of three-dimensional elasticity but also the need to solve this large set of equations for each ply with the requirement that appropriate portions of the solutions must match at ply interfaces. The quest for such a solution analytically would lead to a system of equations whose exact solution, if it could be found, would be tremendously complex. For this reason, much of the work that has been done heretofore has been numerical attempts to solve these equations. In the current work, a different approach is taken. First the equations to be solved are simplified in the hope that the resulting equations can be solved analytically.

The first assumption that is made is that, within a given ply, the in-plane stresses do not vary in the z-direction. With this one assumption, along with the usual assumptions of linear elasticity, great progress can be made in obtaining a more orderly set of equations to be solved. Mathematically, this assumption may be written as:

$$\sigma_{\alpha\beta}^i = F_{\alpha\beta}^i(x_1, x_2) \quad (3.29)$$

to show explicitly that the in-plane stresses are not a function of z within a given ply. After substituting these expressions for the in-plane stresses into the equations of equilibrium from Equation 3.13a, integrating with respect to z, and applying stress-free boundary conditions to the top and bottom surfaces of the laminate, the interlaminar shear stresses may be written as:

$$\sigma_{\alpha z}^i = -F_{\alpha\beta,\beta}^i (z^i - t^i) - \sum_{j=1}^i F_{\alpha\beta,\beta}^j t^{\beta j} \quad (3.30)$$

This solution for the interlaminar shear stresses satisfies equilibrium, stress continuity between plies, and also stress-free top and bottom surfaces of the laminate. In a similar manner, the interlaminar normal stress may be found by substituting the stresses from Equation 3.30 into the equation for equilibrium in the z-direction which is given by:

$$\sigma_{zz,z}^i = -\sigma_{\alpha z,\alpha}^i \quad (3.31)$$

After this substitution has been performed, and the resulting equation has been integrated with respect to z, again making sure that the top and bottom of the laminate remain traction free, an expression for the interlaminar normal stress is found to be:

$$\begin{aligned} \sigma_{zz}^i = & \frac{1}{2} F_{\alpha\beta,\alpha\beta}^i (z^i - t^i)^2 + \left(\sum_{j=1}^i F_{\alpha\beta,\alpha\beta}^j t^j \right) (z^i - t^i) \\ & - \frac{1}{2} \sum_{j=1}^i F_{\alpha\beta,\alpha\beta}^j (t^j)^2 + \sum_{j=1}^i \left(\sum_{k=1}^i F_{\alpha\beta,\alpha\beta}^k t^k \right) t^j \end{aligned} \quad (3.32)$$

The equations of equilibrium have now been solved in terms of the three functions represented by $F_{\alpha\beta}$ for each ply, which are, at this point, still unknown. The manner in which the best of all possible $F_{\alpha\beta}$ are chosen is through the use of the principle of minimum complementary energy. This principle states that the "best" solution obtainable for a given family of stresses

satisfying both equilibrium and the stress boundary conditions will be that which minimizes the complementary energy given by:

$$\Pi_c = \frac{1}{2} \int \int \int_V \sigma_{ij} S_{ijkl} \sigma_{kl} dV \quad (3.33)$$

Using the definition of the stresses given in Equations 3.30 and 3.32, the expression for Π_c is obtained in terms of the arbitrary function tensor F which, in this generic form, is still quite complicated. With the z -dependence of the stresses known explicitly, due to the assumption on the z -dependence of the in-plane stresses, Equation 3.33 can be integrated with respect to z , leaving an area integral in the in-plane coordinates. To apply the principle of minimum complementary energy, the variation of the integrated Equation 3.33 is taken and set equal to zero. The result of these manipulations are coupled fourth order partial differential equations which will be triple the number of plies in number. These equations, although still generally difficult to solve, are now greatly simplified as compared to the general three-dimensional elasticity problem.

Although a rather compact, generic solution was obtainable in the previous section for the interlaminar stresses in in-plane gradient stress fields, no such "clean" solution exists for this current problem. Equation 3.33 has, in general, 81 terms, and even if the solution were to be expressed in tensor form, six terms would be required, each involving cross products of terms like those given in Equations 3.30 and 3.32. Little insight would be gained by such an exercise. However, some suggestions can be made showing

how to choose the tensor $\mathbf{F}$ so as to maximize the utility of the resulting equations.

3.2.2 Choice of the Tensor $\mathbf{F}$

The most convenient choice for the assumed stress tensor $\mathbf{F}$ was first used by Lagace and Kassapoglou. Their technique assumed an explicit x_2 dependence, which is different than the current analysis. However, the form of their expressions is of great simplifying importance, and may be written as:

$$\sigma_{\alpha\beta}^i = a^i F_{\alpha\beta} \quad (3.34)$$

This assumption that the ratio of $\sigma_{\alpha\beta}$ of the i^{th} ply to $\sigma_{\alpha\beta}$ of the j^{th} ply is a constant everywhere in the x_1 - x_2 plane greatly simplifies Equations 3.30 and 3.32. This assumption is based on the fact that material property mismatches through the laminate gave rise to the original residual in-plane stress distribution, and, since these material properties do not change, this through-the-thickness stress distribution is characteristic for the laminate with the given applied loading. The validity of this assumption has been shown previously [34, 37, 57]. Using the definition of Equation 3.34, these equations become:

$$\sigma_{\alpha z}^i = - \left(a^i (z^i - t^i) + \sum_{j=1}^i a^j t^j \right) F_{\alpha\beta, \beta} \quad (3.35)$$

and

$$\sigma_{zz}^i = \left(\frac{1}{2} a^i (z^i - t^i)^2 + \left(\sum_{j=1}^i a^j t^j \right) (z^i - t^i) - \frac{1}{2} \sum_{j=1}^i a^j (t^j)^2 + \sum_{j=1}^i \left(\sum_{k=1}^j a^k t^k \right) t^j \right) F_{\alpha\beta, \alpha\beta} \quad (3.36)$$

Again, after defining quantities similar to Equations 3.25 and 3.26:

$$b^i = \sum_{j=1}^i a^j t^j \quad (3.37)$$

and

$$c^i = -\frac{1}{2} \sum_{j=1}^i a^j (t^j)^2 + \sum_{j=1}^i b^j t^j \quad (3.38)$$

Equations 3.35 and 3.36 simplify to:

$$\sigma_{\alpha z}^i = - \left(a^i (z^i - t^i) - b^i \right) F_{\alpha\beta, \beta} \quad (3.39)$$

and

$$\sigma_{zz}^i = \left(\frac{1}{2} a^i (z^i - t^i)^2 + b^i (z^i - t^i) + c^i \right) F_{\alpha\beta, \alpha\beta} \quad (3.40)$$

Using this assumed stress distribution, the number of equations to be solved has been reduced to three as compared with the three times the number of plies previously encountered. Because the resulting problems are eigenvalue type problems, this reduces the computation time by the cube of the number of plies. This methodology has been used for the case of a straight free edge, resulting in the program SAILS [57], with results presented by Lagace and Saeger [37]. Although this approximation may seem rather crude, results

obtained using the SAILS program compare quite well with published finite element results.

The more generic methodology of assuming different functions for each ply can also be simplified somewhat. The simplification comes in the way in which the separate functions are written. If the functions are expressed as:

$$\sigma_{\alpha\beta}^i = -\frac{1}{t^i} F_{\alpha\beta}^i + \frac{1}{t^i} F_{\alpha\beta}^{i-1} \quad (3.41)$$

then Equations 3.35 and 3.36 may be rewritten as:

$$\sigma_{\alpha z}^i = F_{\alpha\beta,\beta}^i \left(\frac{z^i}{t^i} \right) - F_{\alpha\beta,\beta}^{i-1} \left(\frac{z^i}{t^i} - 1 \right) \quad (3.42)$$

and

$$\sigma_{zz}^i = -\frac{1}{2} \left(\frac{1}{t^i} F_{\alpha\beta,\alpha\beta}^i \left(\frac{z^i}{t^i} \right)^2 - \frac{1}{t^i} F_{\alpha\beta,\alpha\beta}^{i-1} \left(\frac{z^i}{t^i} - 1 \right)^2 + \sum_{j=1}^{i-1} \left(\frac{t^{j+1} + t^j}{t^{j+1} t^j} \right) F_{\alpha\beta,\alpha\beta}^j \right) \quad (3.43)$$

The advantage of writing the in-plane stresses via the expression in Equation 3.41 is most obvious in Equation 3.42. In this equation, it may be seen that $F_{\alpha\beta,\beta}^i$ represents the interlaminar shear stress $\sigma_{\alpha z}^i$ evaluated at the top of the i^{th} ply, and the term $F_{\alpha\beta,\beta}^{i-1}$ represents the interlaminar shear stress $\sigma_{\alpha z}^i$ evaluated at the bottom of the i^{th} ply, or conversely, at the top of the $(i-1)^{\text{th}}$ ply. This also eliminates the summation term that was present in Equation 3.30 which was necessary to ensure traction continuity between the plies.

This definition has some obvious advantages over the definitions of the stresses in Equations 3.35 and 3.36. First, it is easily seen that what is actually being done is that the interlaminar shear stresses are being defined at each

interface via Equation 3.42, and the corresponding in-plane stresses and the interlaminar normal stress are found from this definition. Another obvious advantage is that the summations present in the previous equations have been greatly simplified. This will simplify both the resulting energy expression and the differential equations which are to be solved for the tensor $\mathbf{F}$ after the variation of the complementary energy is taken.

This latter choice for the assumed stress tensor $\mathbf{F}$ is used by Bhat [58]. It has a significant advantage over the choice for distribution of Equation 3.34 in that if a given ply is discretized into two plies of half the original thickness, then the solution will improve. This procedure could be continued *ad infinitum* until the exact solution was obtained. Also, this method allows the stresses in different plies to decay at different rates. This is a significant relaxation of the constraints imposed by Equation 3.34.

These advantages are not gained, however, without great cost. The previously mentioned computation time which is proportional to the cube of the number of plies is required for this solution technique. Also, this solution will produce three functions for each ply, all of which must be evaluated to get the stress at a given point. For a large number of plies, this method requires greater computation time than typical finite element methods which have computation times which are linearly proportional to the number of plies. However, the solution obtained using the eigenfunction technique will be more accurate for a given degree of discretization in the z-direction, since this solution is continuous, or infinitely discretized, in the x-direction and finite element solutions will still have some fixed discretization in this direction. A further disadvantage of this expanded eigenfunction technique is that the

results are similar to finite element results, in that a tremendous degree of post-processing is required in order to make sense of the results.

The method represented by the assumptions of Equation 3.34, although not as accurate as the previously discussed methodology, requires computation time which is linearly proportional to the number of plies. Also, the number of functions which must be evaluated in order to obtain the stress at a given point is fixed. This solution technique thus tends to be extremely fast and produces highly readable results. The drawback, of course, is a decrease in the accuracy of the solution.

This loss in accuracy has, in part, already been investigated by Lagace and Kassapoglou [34], and by Lagace and Saeger [37]. When the solutions obtained by Lagace and Kassapoglou were compared with existing finite element solutions, they found that their solution was in good agreement, especially for the case of the dominant stresses at the ply interface, such as σ_{1z} . The solution obtained by Lagace and Saeger, which has been described previously in this chapter, is more accurate than is that obtained by Lagace and Kassapoglou because of the reduced number of assumptions which have been used. This solution has also been shown to compare favorably with other analyses. For these reasons, limiting the solution to three assumed functions is not considered to affect the solution to an appreciable degree, especially not in the engineering sense.

Chapter Four

Solution for Circular Hole

In this chapter, the results for the preceding chapter are specialized for the case of a generally anisotropic symmetric laminate containing a circular hole. The formulation which follows is meant to maximize efficiency and accuracy for laminates with small ratios of laminate thickness to hole radius. Other formulations could be made for large values of this ratio, however, as has been shown by previous investigators, these cases are of less importance for delamination initiation because the peak values of the interlaminar stress are lower and these stresses decay more quickly [47].

Because it has been assumed that the hole is large compared to through the thickness length scales, certain simplifications may be made which add to the compactness of the solution. The first of these simplifications is that the laminate is in a state of plane stress. This assumption allows for the use of a previously obtained two-dimensional solution for the in-plane stress field caused by the hole. This assumption also makes the calculation of the interlaminar stresses due to gradients in the in-plane stress field easier because σ_{zz} need not enter into the equations for the interlaminar shear stresses, since the laminate is in a state of plane stress.

Another important assumption which follows for the case of a large hole, is that variations in the θ direction are small as compared to variations in the radial direction. In calculating the contribution to the interlaminar stresses caused by the presence of a free edge, this assumption allows the assumed in-plane stresses to be considered to be a function of the radial direction only, for a given value of angular position θ . In essence, this will

reduce the two-dimensional problem for the assumed functions as suggested in Chapter Three, into the solution of one-dimensional problems along spokes extending from the hole. Each of these problems is independent, and hence, the solution may be obtained along only one radial, if a single direction were of particular interest.

In order to begin this specific case of a laminate with a hole, a solution procedure which is generically valid for any geometry is used. This procedure is shown schematically in Figure 4.1. First, the in-plane stress field is calculated by assuming that the laminated material is homogeneous. After this solution has been obtained, the interlaminar stresses caused by the in-plane gradients in this stress field can be quickly calculated. Finally, the stresses which remain at what should be the free edge of the laminate will be removed via the principle of superposition. Again, because a large hole diameter is assumed, this may be performed on an angle-by-angle basis.

4.1 Analytic Expression for In-Plane Stresses in the Presence of a Circular Hole

The following solution for the in-plane stress field in an infinite anisotropic plate subjected to far-field loading, shown in Figure 4.2, is attributed to Savin [59]. Using rectangular Cartesian coordinates x_1 and x_2 , a coordinate transformation is made to the new complex coordinates z_1 and z_2 via:

$$z_1 = x_1 + s_1 x_2 \tag{4.1}$$

and

$$z_2 = x_1 + s_2 x_2 \tag{4.2}$$

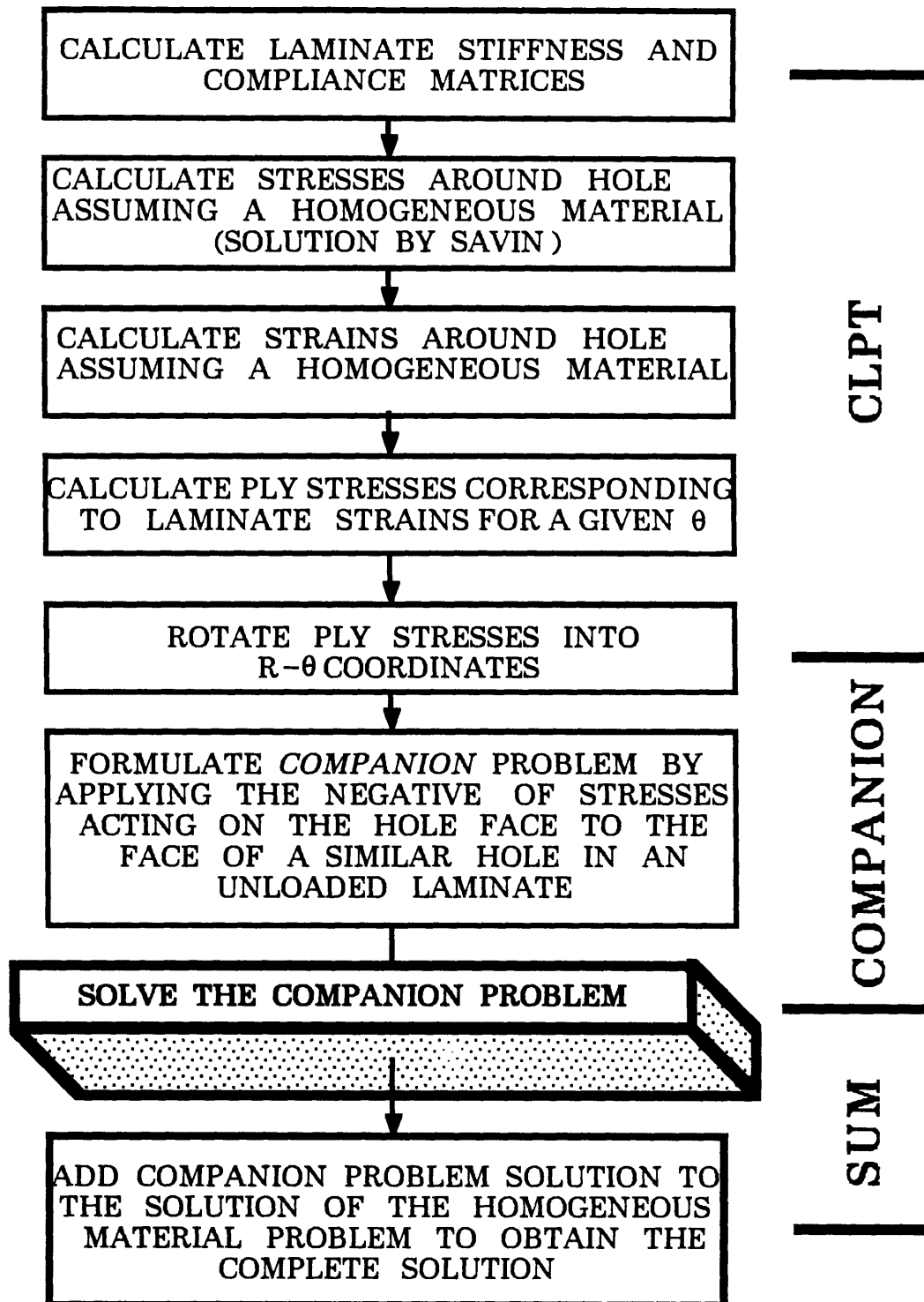


Figure 4.1 Solution methodology for interlaminar stresses due to a stress-free edge.

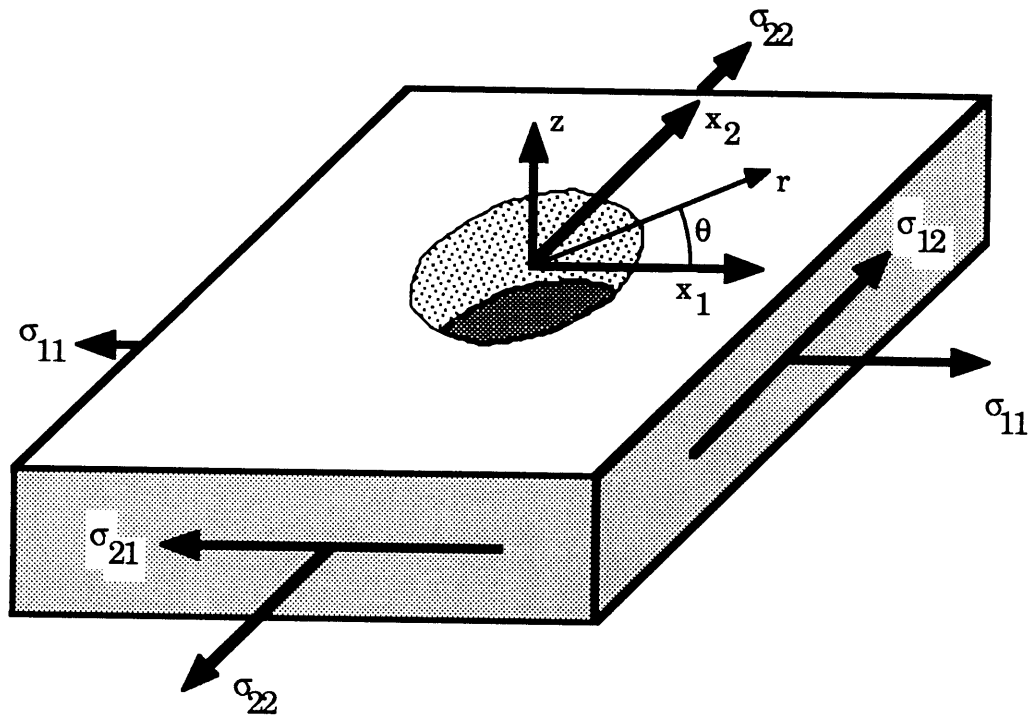


Figure 4.2 Geometry of typical hole problem.

where s_1 and s_2 are the roots of the equation given by:

$$S_{1111}^L s^4 - 4 S_{1112}^L s^3 + (2 S_{1122}^L + 4 S_{1212}^L) s^2 - 4 S_{2212}^L s + S_{2222}^L = 0 \quad (4.3)$$

Except in special cases, the four roots of this equation will appear as two sets of complex conjugates. For uniqueness, the roots s_1 and s_2 are taken to be the roots with positive imaginary components. Given a far-field loading of the form:

$$\begin{aligned} \sigma_{11}^{(\infty)} &= p \cos^2 \alpha \\ \sigma_{22}^{(\infty)} &= p \sin^2 \alpha \\ \sigma_{12}^{(\infty)} &= p \sin \alpha \cos \alpha \end{aligned} \quad (4.4)$$

Savin found the solution for the in-plane stresses to be:

$$\begin{aligned} \sigma_{11} &= \sigma_{11}^{(\infty)} + 2 \operatorname{Re} \left[s_1^2 \Phi'_0(z_1) + s_2^2 \Psi'_0(z_2) \right] \\ \sigma_{22} &= \sigma_{22}^{(\infty)} + 2 \operatorname{Re} \left[\Phi'_0(z_1) + \Psi'_0(z_2) \right] \\ \sigma_{12} &= \sigma_{12}^{(\infty)} - 2 \operatorname{Re} \left[s_1 \Phi'_0(z_1) + s_2 \Psi'_0(z_2) \right] \end{aligned} \quad (4.5)$$

where the functions Φ_0 and Ψ_0 for a circular hole are given by:

$$\Phi_0(z_1) = -\frac{ipR^2(1-is_1)}{4(s_1-s_2)} \left\{ \frac{s_2 \sin 2\alpha + 2\cos^2 \alpha}{z_1 + \sqrt{z_1^2 - R^2(1+s_1^2)}} + i \frac{2s_2 \sin^2 \alpha + \sin 2\alpha}{z_1 + \sqrt{z_1^2 - R^2(1+s_1^2)}} \right\} \quad (4.6)$$

and

$$\Psi_0(z_2) = \frac{ipR^2(1-is_2)}{4(s_1-s_2)} \left\{ \frac{s_1 \sin 2\alpha + 2\cos^2 \alpha}{z_2 + \sqrt{z_2^2 - R^2(1+s_2^2)}} + i \frac{2s_1 \sin^2 \alpha + \sin 2\alpha}{z_2 + \sqrt{z_2^2 - R^2(1+s_2^2)}} \right\} \quad (4.7)$$

where R is the radius of the circle. In Equation 4.5, the superscripted commas denote differentiation with respect to z_1 or z_2 .

4.2 Interlaminar Stresses Due to Gradient Stress Field

With an analytical solution for the in-plane stress field, obtaining the approximation for the interlaminar stresses due to in-plane gradients in the stress field is straightforward. Recalling Equations 3.27a and 3.27b,

$$\sigma_{\alpha z}^i(z^i = t^i) = J_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\beta}^L \quad (3.27 \text{ a})$$

and

$$\sigma_{zz}^i(z^i = t^i) = K_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma,\alpha\beta}^L \quad (3.27 \text{ b})$$

all that is required to calculate these interlaminar stresses is the differentiation of the stresses with respect to x_α and x_β . This is accomplished most expediently by noting that:

$$\frac{\partial}{\partial x_\beta} = \frac{\partial z_\alpha}{\partial x_\beta} \frac{\partial}{\partial z_\alpha} \quad (4.8)$$

From Equations 4.1 and 4.2, it may be seen that:

$$\frac{\partial z_\alpha}{\partial x_\beta} = (s_\alpha)^{\delta_{2\beta}} \quad (4.9)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta function shown in Equation 3.8. The required derivatives can be found by applying Equation 4.9 to Equation 4.5.

The solution of these equations, expressed analytically in terms of the in-plane location, the applied far-field stresses, and material parameters was programmed in FORTRAN. A listing of this program is included in Appendix B. As can be seen from the general formulation of Chapter Three and the specific formulation of this chapter, the stresses expressed in Equation 3.27 will satisfy stress-free boundary conditions on the top and bottom surfaces of the laminate, via the definitions of **J** and **K**. However, no mention has yet been made about the stresses which would be acting on the surface of the hole. For the remainder of the derivations, polar coordinates will be more useful, because it is easier to see the required stress-free boundary conditions in this coordinate system.

In addition to the usual components of the companion problems involved to eliminate the in-plane normal and tangential forces acting on the hole surface, the interlaminar shear force σ_{rz} as calculated above may also be present and thus add an additional component to this problem. To find this residual stress, the stresses represented in Cartesian coordinates in Equation 3.27 must be rotated into r - θ coordinates at the hole boundary. This procedure gives:

$$\sigma_{rz}^i = \cos \theta \sigma_{1z}^i + \sin \theta \sigma_{2z}^i \quad (4.10)$$

With the contribution to interlaminar stresses from in-plane gradients calculated, and with the additional residual stresses at the hole interface due to the presence of σ_{rz} computed, the companion problem may now be solved in order to obtain the complete approximation for the interlaminar stresses.

4.3 Interlaminar Stresses Due to Residual Stresses --- Companion Solution

To solve the companion problem for the case of a laminate containing a circular hole, the residual stresses acting on the hole face must first be computed. For the interlaminar shear stress σ_{rz}^i this was done in the previous section, with the results given by Equation 4.10. The two remaining residual stresses can be found on a ply-by-ply basis using the expressions:

$$\sigma_{rr}^i = \cos^2 \theta \sigma_{11}^i + \sin^2 \theta \sigma_{22}^i + 2 \sin \theta \cos \theta \sigma_{12}^i \quad (4.11)$$

and

$$\sigma_{r\theta}^i = -\sin\theta \cos\theta (\sigma_{11}^i - \sigma_{22}^i) + (\cos^2\theta - \sin^2\theta) \sigma_{12}^i \quad (4.12)$$

where the in-plane stresses are given via:

$$\sigma_{\alpha\beta}^i = I_{\alpha\beta\sigma\gamma}^i \sigma_{\sigma\gamma}^L \quad (4.13)$$

with I defined in Equation 3.6 and σ^L defined by Equation 4.5.

A residual stress vector is defined as:

$$R^i = -\sigma_{rr}^i, -\sigma_{r\theta}^i, \text{ or } -\sigma_{rz}^i \quad (4.14)$$

The choice of which of the three stresses to use in Equation 4.14 is determined by which stress is to be eliminated. The total solution is obtained by combining the resulting solutions obtained for each choice of R . In other words, the solution for the interlaminar stresses at a free edge is obtained by the superposition of the three components of the companion solution.

With the residual stress vectors now computed, the companion solution can be solved. The first step is to assume a distribution for the in-plane stresses. Because of the specifics of the problem to be solved, general tensor notation will no longer be as advantageous as it was previously, so the following is given in either matrix or expanded form for added clarity. At this point, two simplifying assumptions will be made. The first assumption is that the form for the in-plane stresses used by Lagace and Kassapoglou [34] and shown in Equation 3.34 will yield sufficient accuracy. This assumption was that the ratio of the in-plane stresses in any two plies remains constant in the

r - θ plane. The other assumption is that, for the problems of interest, variations with respect to θ are much smaller than variations in the r direction, and more importantly that:

$$\frac{1}{r} \frac{\partial}{\partial \theta} \approx 0$$

The first assumption, that the ratio of the in-plane stresses do not change, was based on the proposition that the integral of these stresses through the thickness of the laminate are always equal to zero. This is obviously true at the free edge where this must happen by definition, and at infinity where these stresses will decay identically to zero. For the case of a straight free edge, this assumption also holds for any arbitrary distance from the free edge, because, if the laminate were to be cut along an arbitrary x_2 -line, then another stress-free edge situation would result, thus requiring a zero integration through-the-thickness. This argument holds by analogy for the case of a circular hole. The current specification for the in-plane stresses guarantees that these stresses will always integrate through the thickness to zero. A practical consideration motivating this choice is that it is the simplest mode shape which satisfies the stress boundary conditions.

The assumption that terms of the form $\frac{1}{r} \frac{\partial}{\partial \theta}$ are negligible as compared to gradients in the radial direction is easily justifiable for laminates with a small ratio of ply thickness to hole diameter. It has been shown [34] that the distance over which interlaminar stresses due to a stress-free edge act is proportional to the ply/laminate thickness. This indicates that gradients in the r -direction are inversely proportional to the ply/laminate thickness. On the other hand, it is easily seen that the term $\frac{1}{r} \frac{\partial}{\partial \theta}$ has gradients which are

inversely proportional to the hole diameter. Thus, the terms which are being ignored are of the order of the ratio of the ply/laminate thickness to the hole diameter as compared to gradients in the radial direction.

Another valid procedure for dealing with the θ -variation would be to discretize the problem in the θ direction so that the problem resembled a pie cut into various arcs. Along the interface of each arc, the in-plane stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, as well as the interlaminar shear stress $\sigma_{\theta z}$, would have to be matched. Within a given arc, the assumption could be made that the variation of the in-plane stresses with respect to θ was linear. This procedure would improve the performance of the solution for small laminate thickness to hole radius ratios. However, the computational costs and complexity would also increase. For these reasons as well as others, this further discretization was not performed.

4.3.1 Formulation of Equations

The assumptions used greatly simplify the expressions for the in-plane stresses as well as the expressions for the interlaminar stresses. The resulting expressions for the in-plane stresses become:

$$\begin{aligned}\sigma_{rr}^i &= R^i F_1(r) \\ \sigma_{\theta\theta}^i &= R^i F_2(r) \\ \sigma_{r\theta}^i &= R^i F_3(r)\end{aligned}\tag{4.15}$$

Thus, each of the in-plane stresses in the i^{th} ply are proportional to the same constant R^i , and each has a unique shape function F which is yet to be determined. Using the equilibrium equations in polar coordinates (in the absence of body forces) which are given by:

$$\begin{aligned}\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0\end{aligned}\tag{4.16}$$

the interlaminar stresses may be found by inserting the definitions of Equation 4.15 into these equilibrium equations, resulting in:

$$\begin{aligned}\sigma_{rz}^i &= \left(\frac{\partial F_1}{\partial r} + \frac{F_1 - F_2}{r} \right) \{ -a^i(z^i - t^i) - b^i \} \\ \sigma_{\theta z}^i &= \left(\frac{\partial F_3}{\partial r} + \frac{2F_3}{r} \right) \{ -a^i(z^i - t^i) - b^i \} \\ \sigma_{zz}^i &= \left(\frac{\partial^2 F_1}{\partial r^2} + \frac{1}{r} \left(2 \frac{\partial F_1}{\partial r} - \frac{\partial F_2}{\partial r} \right) \right) \left\{ \frac{1}{2} a^i (z^i - t^i)^2 + b^i (z^i - t^i) + c^i \right\}\end{aligned}\tag{4.16}$$

where the constants a , b , and c are constants of integration which are required to satisfy traction continuity between plies and the condition that the top and

bottom surfaces of the laminate are stress-free. These constants were defined in Equations 3.37 and 3.38 to be:

$$a^i = -R^i$$

$$b^i = \sum_{j=1}^i a^j t^j \quad (3.37)$$

$$c^i = -\frac{1}{2} \sum_{j=1}^i a^j (t^j)^2 + \sum_{j=1}^i b^j t^j \quad (3.38)$$

With all the stresses now defined, at a particular θ location, the energy expression can now be formed. The complementary energy is given by the equation:

$$\Pi_c = \frac{1}{2} \iiint_V \sigma_{ij} S_{ijkl} \sigma_{kl} dV \quad (3.33)$$

Because the z dependence of the stresses has been explicitly assumed, the integration in z is performed to reduce this volume integral to an area integral. The z -distributions have been assumed to be piecewise continuous, so this integration is performed by summing the individual integrals in z^i . Additionally, because variations in the θ -direction have been assumed to be small, none of the stresses have been assumed to exhibit any θ -variation. For this reason, the integral in θ is performed over an arbitrary pie-shaped region in the r - θ plane. Because all properties are assumed to be piecewise independent of θ , the actual range over which the integration is performed is

immaterial. In this pie shaped region, the compliance tensor $\mathbf{S}$ is assumed to remain constant. This integration reduces the remaining area integral to a line integral. These simplifications lead to a simplified expression for the complementary energy as given by:

$$\Pi_c = \frac{\Delta\theta}{2} \int_R \left(\sum_{p=1}^n \int_0^{t^p} \sigma_{ij}^p S_{ijkl}^p \sigma_{kl}^p dz^p \right) r dr \quad (4.18)$$

After performing the designated integrations, the variation of Π_c may be taken and set equal to zero. This leads to three coupled differential equations for $\{F\}$ which may be written as:

$$\begin{aligned} & r^3(t^*)^4 [A_{34}] \{F\}'''' + r^2(t^*)^4 [A_{23}] \{F\}''' \\ & + \left[r^3(t^*)^2 [A_{32}] + r(t^*)^4 [A_{12}] \right] \{F\}'' \\ & + \left[r^2(t^*)^2 [A_{21}] + (t^*)^4 [A_{01}] \right] \{F\}' \\ & + \left[r^3 [A_{30}] + r(t^*)^2 [A_{10}] \right] \{F\} = 0 \end{aligned} \quad (4.19)$$

where t^* is the effective ply thickness. Normally, this effective ply thickness would be set equal to the thickness of a ply, or in the case of a laminate such as $[0_n/90_n]_s$, the effective ply thickness would be taken as n times the thickness of a single ply. In cases where plies of different thickness are used, the individual must chose some characteristic thickness dimension, such as the average ply thickness or the laminate thickness. The various matrices $[A_{xx}]$

represent thickness-weighted elastic constants resulting from the indicated integrations, and are given by:

$$\begin{aligned}
 [A_{34}] &= \begin{bmatrix} e_6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 [A_{32}] &= \begin{bmatrix} (2e_9 - e_4) & e_{11} & (e_{12} - e_{13}) \\ e_{11} & 0 & 0 \\ (e_{12} - e_{13}) & 0 & -e_5 \end{bmatrix} \\
 [A_{30}] &= \begin{bmatrix} e_1 & e_7 & e_8 \\ e_7 & e_2 & e_{10} \\ e_8 & e_{10} & e_3 \end{bmatrix} \\
 [A_{23}] &= \begin{bmatrix} 2e_6 & -e_6 & 0 \\ -e_6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 [A_{21}] &= \begin{bmatrix} (2e_9 - e_4) & (e_4 - e_9) & -2e_{13} \\ (2e_{11} - e_4 + e_9) & 0 & (e_{12} - e_{13}) \\ 2e_{12} & -(e_{12} - e_{13}) & -e_5 \end{bmatrix} \\
 [A_{12}] &= \begin{bmatrix} -4e_6 & 2e_6 & 0 \\ 2e_6 & -e_6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.20} \\
 [A_{10}] &= \begin{bmatrix} e_4 & -e_4 & 2e_{13} \\ -e_4 & e_4 & -2e_{13} \\ 2e_{13} & -2e_{13} & 4e_5 \end{bmatrix}
 \end{aligned}$$

$$[A_{01}] = \begin{bmatrix} 4e_6 & -2e_6 & 0 \\ -2e_6 & e_6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the constants e_i are given by:

$$\begin{aligned} e_1 &= \sum_{i=1}^n S_{1111}^i d_1^i & e_8 &= \sum_{i=1}^n 2 S_{1112}^i d_1^i \\ e_2 &= \sum_{i=1}^n S_{2222}^i d_1^i & e_9 &= \sum_{i=1}^n S_{1133}^i d_4^i \\ e_3 &= \sum_{i=1}^n 4 S_{1212}^i d_1^i & e_{10} &= \sum_{i=1}^n 2 S_{2212}^i d_1^i \\ e_4 &= \sum_{i=1}^n 4 S_{1313}^i d_2^i & e_{11} &= \sum_{i=1}^n S_{2233}^i d_4^i \\ e_5 &= \sum_{i=1}^n 4 S_{2323}^i d_2^i & e_{12} &= \sum_{i=1}^n 2 S_{3312}^i d_4^i \\ e_6 &= \sum_{i=1}^n S_{3333}^i d_3^i & e_{13} &= \sum_{i=1}^n 4 S_{1323}^i d_2^i \\ e_7 &= \sum_{i=1}^n S_{1122}^i d_1^i \end{aligned} \quad (4.21)$$

with the d_i representing the result of integrating through the thickness of a ply. The d_i are given by:

$$\begin{aligned}
 d_1^i &= (a^i)^2 \left(\frac{t^i}{t^*} \right) \\
 d_2^i &= \frac{1}{3} (a^i)^2 \left(\frac{t^i}{t^*} \right)^3 - a^i \left(\frac{b^i}{t^*} \right) \left(\frac{t^i}{t^*} \right)^2 + \left(\frac{b^i}{t^*} \right)^2 \left(\frac{t^i}{t^*} \right) \\
 d_3^i &= \frac{1}{20} (a^i)^2 \left(\frac{t^i}{t^*} \right)^5 - \frac{1}{4} a^i \left(\frac{b^i}{t^*} \right) \left(\frac{t^i}{t^*} \right)^4 \\
 &\quad + \frac{1}{3} \left(a^i \left(\frac{c^i}{(t^*)^2} \right) + \left(\frac{b^i}{t^*} \right)^2 \right) \left(\frac{t^i}{t^*} \right)^3 - \left(\frac{c^i}{(t^*)^2} \right) \left(\frac{b^i}{t^*} \right) \left(\frac{t^i}{t^*} \right)^2 \\
 &\quad + \left(\frac{c^i}{(t^*)^2} \right)^2 \left(\frac{t^i}{t^*} \right) \\
 d_4^i &= \frac{1}{6} (a^i)^2 \left(\frac{t^i}{t^*} \right)^3 - \frac{1}{2} a^i \left(\frac{b^i}{t^*} \right) \left(\frac{t^i}{t^*} \right)^2 + \left(\frac{c^i}{(t^*)^2} \right)^2 \left(\frac{t^i}{t^*} \right)
 \end{aligned} \tag{4.22}$$

Physically, the d_x represent the integrals through the thickness of the z-variations of the different types of stresses: d_1 corresponds to the integral for two in-plane stresses multiplied together; d_2 for the multiplication of two interlaminar shear stresses; d_3 for the square of the interlaminar normal stress; and d_4 for the multiplication of an in-plane stress with the interlaminar normal stress. The two remaining permutations, the multiplication of an in-plane stress with an interlaminar shear stress and the multiplication of the interlaminar normal stress with an interlaminar shear stress, do not appear because the compliance terms $S_{ii\alpha z}$ as well as $S_{12\alpha z}$ are identically equal to zero for an orthotropic ply rotated about the z-axis.

The e_x represent the effects of weighting these d 's by the appropriate material constants and then summing them through the thickness of the laminate. These e_x are analogous to the values of the A, B, and D matrices in classical laminated plate theory, in that in both cases the material constants are weighted by different z -integrals. The units of the e 's are [energy/unit volume] and, the F 's, by definition, are unitless. These F 's represent the shapes of the various stresses. After F and its various derivatives are multiplied together (representing the quadratic stress terms in the complementary energy expression), they can be integrated over the area of the plate. The e_x , which include not only the effects of the material properties and layup, but also the residual loading, can then be multiplied by these integrals in order to obtain the total complementary energy in the plate. The "best" set of F 's will be those which minimize this total energy.

4.3.2 Solution of System of Equations

The problem has now been reduced to solving Equation 4.19. In order to accomplish this, a new dimensionless variable, ρ , will be defined via:

$$\rho = \frac{r - R}{t^*} \quad (4.23)$$

where ρ represents a normalized distance from the hole surface, R is the radius of the hole, and as previously mentioned, t^* is the effective ply thickness which is used as a normalizing distance. Differentiation with respect to ρ ,

denoted by $(\dot{})$, may be used to replace differentiation with respect to r by using the relation:

$$\frac{\partial}{\partial r}(\bullet) = (\bullet)' = \frac{\partial \rho}{\partial r} \frac{\partial}{\partial \rho}(\bullet) = \frac{1}{t^*}(\dot{\bullet}) \quad (4.24)$$

where $(\bullet)$ represents a generic quantity which is to be differentiated. Another relation which will be useful is that which associates r with ρ :

$$r = \rho t^* + R \quad (4.25)$$

Using the definitions of Equations 4.24 and 4.25, Equation 4.19 may be rewritten as:

$$\begin{aligned} & (1 + 3\rho\epsilon + 3\rho^2\epsilon^2 + \rho^3\epsilon^3) [A_{34}] \{\ddot{\ddot{F}}\} \\ & + (\epsilon + 2\rho\epsilon^2 + \rho^2\epsilon^3) [A_{23}] \{\ddot{\ddot{F}}\} \\ & + (1 + 3\rho\epsilon + 3\rho^2\epsilon^2 + \rho^3\epsilon^3) [A_{32}] \{\ddot{\ddot{F}}\} \\ & + (\epsilon^2 + \rho\epsilon^3) [A_{12}] \{\ddot{\ddot{F}}\} \\ & + (\epsilon + 2\rho\epsilon^2 + \rho^2\epsilon^3) [A_{21}] \{\dot{\ddot{F}}\} \\ & + \epsilon^3 [A_{01}] \{\dot{\ddot{F}}\} \\ & + (1 + 3\rho\epsilon + 3\rho^2\epsilon^2 + \rho^3\epsilon^3) [A_{30}] \{F\} \\ & + (\epsilon^2 + \rho\epsilon^3) [A_{10}] \{F\} = 0 \end{aligned} \quad (4.26)$$

where ϵ is the ratio of the averaging dimension, t^* , to the hole radius R :

$$\epsilon = \frac{t^*}{R} \quad (4.27)$$

If it is assumed that ϵ is a small parameter, then a perturbation solution may be obtained for Equation 4.26. To obtain such a solution, the function F is assumed to be the sum of the dominant F solution, which will be called F_0 , or the zeroth order solution, and successively higher order correction terms which will each be multiplied by correspondingly higher order ϵ terms. The equation for F can be written as:

$$F = F_0 + F_1\epsilon + F_2\epsilon^2 + F_3\epsilon^3 + \dots \quad (4.28)$$

Equation 4.28 must then formally be substituted into Equation 4.26. After this has been done, all terms multiplying a common power of ϵ are grouped together. Each of these individual groupings are then set equal to zero independently in order to obtain expressions for each F_x . After performing this analysis, the following equations are obtained for the Zeroth Order Solution:

$$[A_{34}]\{\ddot{\ddot{F}}_0\} + [A_{32}]\{\dot{\ddot{F}}_0\} + [A_{30}]\{F_0\} = 0 \quad (4.29)$$

the First Order Solution:

$$\begin{aligned} [A_{34}]\{\ddot{\ddot{F}}_1\} + [A_{32}]\{\dot{\ddot{F}}_1\} + [A_{30}]\{F_1\} \\ = -[A_{23}]\{\ddot{\ddot{F}}_0\} - [A_{21}]\{\dot{\ddot{F}}_0\} \end{aligned} \quad (4.30)$$

and the Second Order Solution:

$$\begin{aligned}
 & [A_{34}]\{\ddot{\ddot{F}}_2\} + [A_{32}]\{\ddot{F}_2\} + [A_{30}]\{F_2\} \\
 & = \rho \left\{ [A_{23}]\{\ddot{F}_1\} + [A_{21}]\{\dot{F}_1\} \right\} \\
 & \quad - [A_{23}]\{\ddot{F}_1\} - [A_{21}]\{\dot{F}_1\} \\
 & \quad - [A_{12}]\{\ddot{F}_0\} - [A_{10}]\{F_0\}
 \end{aligned} \tag{4.31}$$

and so on for additional solutions. It can be seen from these equations that the left hand side of the equation does not change. What does change is the forcing function on the right hand side of the equation, which is, in general, a function of all of the previously found F 's. The method of solution for these equations is to first solve the zeroth order solution and apply the boundary conditions of the problem to this solution. Then using this solution, an expression for the first order solution is obtained, and so on. The boundary conditions for the first order solution and succeeding solutions are then all zero because all boundary conditions have been satisfied by the zeroth order solution.

The zeroth order solution is an easily solvable constant coefficient problem. The solution for such problems are known to be exponential, such that:

$$\{F_0\} = \{\phi\} e^{\lambda \rho} \tag{4.32}$$

where ϕ is a vector of constants. The parameters λ and ϕ are found by substituting Equation 4.32 into Equation 4.29 to obtain:

$$\left[\lambda^4 [A_{34}] + \lambda^2 [A_{32}] + [A_{30}] \right] \{\phi\} = 0 \tag{4.33}$$

which is a generalized eigenvalue problem, with λ being one of the eigenvalues and ϕ being the corresponding eigenvector. In order to solve Equation 4.33, if the matrices $\mathbf{A}$ were fully populated, an eighth order equation in terms of λ^2 would need to be solved. Because no equation of order higher than the fourth may be solved analytically, this equation would have to be solved iteratively. However, because of the many zero entries in the matrices $\mathbf{A}_{34}$ and $\mathbf{A}_{32}$, a third order equation for λ^2 is obtained which can be solved analytically.

Now that the zeroth order solution has been obtained, the first order correction can be computed. This solution may be obtained by assuming that F_1 has a form given by:

$$\{F_1\} = \{(a\rho + g)\{\phi\} + \{\psi\}\} e^{\lambda\rho} \quad (4.34)$$

where "a" is an, as yet, undetermined constant, and ψ is an undetermined vector. This form of the solution is similar to the solution which is used when an ordinary differential equation has repeated roots, and one of the solutions must be multiplied by the independent variable, which is, in this case, ρ . The values for "a" and ψ are chosen so that $\{F_1\}$ satisfies Equation 4.30. The constant "g" gives the magnitude of the homogeneous solution, $\phi e^{\lambda\rho}$, which will be required to satisfy boundary conditions. The function F_1 is then substituted into Equation 4.30 in order to determine values for "a" and ψ . The resulting equation is:

$$a \left[4\lambda^3 [\mathbf{A}_{34}] + 2\lambda [\mathbf{A}_{32}] \right] \{\phi\}$$

$$\begin{aligned}
 & + \left[\lambda^4 [A_{34}] + \lambda^2 [A_{32}] + [A_{30}] \right] \{\psi\} \\
 & = - \left[\lambda^3 [A_{23}] + \lambda [A_{21}] \right] \{\phi\}
 \end{aligned} \tag{4.35}$$

Equation 4.35 may be solved for "a", if the term containing the unknown vector ψ can be eliminated. The matrix multiplying this vector is the matrix from the eigenvalue problem of Equation 4.33. One of the eigenvectors, namely ϕ , is already known, as is its associated eigenvalue which is equal to zero. This is most easily seen by the fact that ϕ and λ satisfy Equation 4.33. Because this matrix is symmetric, the remaining eigenvectors will be orthogonal to the eigenvector ϕ . Thus, the vector resulting from the multiplication of ψ , which can be written as the sum of ϕ and the remaining two eigenvectors, by this matrix will be orthogonal to ϕ .

This is enough information to state that the vector obtained from the term containing ψ is orthogonal to the vector ϕ . Therefore, if Equation 4.35 is multiplied through by the vector ϕ^T , the term containing ψ in the equation will drop out. The solution for the constant "a" is then found to be:

$$a = - \frac{\{\phi\}^T \left[\lambda^2 [A_{23}] + [A_{21}] \right] \{\phi\}}{\{\phi\}^T \left[4\lambda^2 [A_{34}] + 2[A_{32}] \right] \{\phi\}} \tag{4.36}$$

Now, Equation 4.35 can be used to solve for ψ by using the following procedure. First assume that ψ may be written as the sum of two vectors which are orthogonal to ϕ , because any components in the direction of ϕ can be included in the $g\phi$ term in Equation 4.34. Thus, ψ will be defined as:

$$\{\psi\} = v_1 \{Y_1\} + v_2 \{Y_2\} \quad (4.37)$$

with $\mathbf{Y}$ defined by any normal vectors which span the plane which is orthogonal to ϕ . This requires that the vectors satisfy the relations:

$$\begin{aligned} \{Y_1\}^T \{Y_1\} &= 1 \\ \{Y_1\}^T \{\phi\} &= 0 \end{aligned} \quad (4.38)$$

and

$$\{Y_2\} = \{Y_1\} \times \{\phi\}$$

The procedure is to first chose any vector $\mathbf{Y}_1$, which is orthogonal to ϕ , and has a length of unity. Then $\mathbf{Y}_2$ is found by taking the cross product of $\mathbf{Y}_1$ with ϕ . All that remains to be found are the constants v_1 and v_2 . To help facilitate this, two auxiliary vectors $\mathbf{Z}$, which will be perpendicular to ϕ , can be defined in terms of the vectors $\mathbf{Y}$ as:

$$\{Z_1\} = [\lambda^4 [A_{34}] + \lambda^2 [A_{32}] + [A_{30}]] \{Y_1\}$$

and

$$\{Z_2\} = [\lambda^4 [A_{34}] + \lambda^2 [A_{32}] + [A_{30}]] \{Y_2\} \quad (4.39)$$

For convenience, one more vector, $\mathbf{W}$, is defined as:

$$\{W\} = -[a[4\lambda^3 [A_{34}] + 2\lambda [A_{32}]] + \lambda^3 [A_{23}] + \lambda [A_{21}]] \{\phi\} \quad (4.40)$$

With these definitions, Equation 4.33 may be rewritten as:

$$\{W\} = v_1 \{Z_1\} + v_2 \{Z_2\} \quad (4.41)$$

Two equations for v_1 and v_2 may be obtained by multiplying Equation 4.38 by Z_1 and Z_2 respectively. These two equations are

$$\{Z_1\}^T \{W\} = v_1 \{Z_1\}^T \{Z_1\} + v_2 \{Z_1\}^T \{Z_2\}$$

(4.42)

and

$$\{Z_2\}^T \{W\} = v_1 \{Z_2\}^T \{Z_1\} + v_2 \{Z_2\}^T \{Z_2\}$$

This represents two algebraic equations for the constants v_1 and v_2 . The resulting expressions for the constants "v", found by solving these two equations simultaneously, are then given by:

$$v_1 = \frac{(Z_2^T Z_2)(Z_1^T W) - (Z_1^T Z_2)(Z_2^T W)}{(Z_1^T Z_1)(Z_2^T Z_2) - (Z_1^T Z_2)(Z_2^T Z_1)}$$

and

$$v_2 = \frac{(Z_1^T Z_1)(Z_2^T W) - (Z_2^T Z_1)(Z_1^T W)}{(Z_1^T Z_1)(Z_2^T Z_2) - (Z_1^T Z_2)(Z_2^T Z_1)} \quad (4.43)$$

With the v 's now obtained, the vector ψ is found from Equation 4.37 and the solution for the first order correction has been obtained with it.

Only these two functions F_0 and F_1 were used in the current analysis. As can be seen, the complexity of the the higher order F 's begin to increase significantly. Also, correcting with higher order terms may not make sense in

light of other assumptions which have been made in the analysis. It was found previously that the terms corresponding to derivatives in the θ -direction, which have been dropped, are of the order of the parameter ε . Therefore, these two F terms will give an accuracy which is consistent with this aforementioned approximation.

To complete the solution of the companion solution, boundary conditions must be applied. As was stated previously, the solution of Equation 4.33 gives six eigenvectors and eigenvalues. Any complex eigenvalues and eigenvectors will appear as complex conjugate pairs. Because these six eigenvalues come from taking the positive and negative square roots of the three λ^2 which solve a cubic equation, three of these eigenvalues will have positive real parts. One set of boundary conditions require that the stresses from the companion solution decay to zero for large values of r , which corresponds to large values of ρ . Therefore, the coefficients multiplying these solutions must be set equal to zero. The resulting equation for the solution of F_0 may then be expressed as:

$$\{F_0\} = f_1\{\phi_1\}e^{\lambda_1\rho} + f_2\{\phi_2\}e^{\lambda_2\rho} + f_3\{\phi_3\}e^{\lambda_3\rho} \quad (4.45)$$

Boundary conditions at the hole edge must now be used to solve for the constants f_j . The stresses acting on the hole edge, in terms of the expression given in Equation 4.45 may be written as:

$$\sigma_{r\theta}^i \Big|_{\rho=0} = a^i \sum_{j=1}^3 f_j \phi_j^1$$

$$\sigma_{r\theta}^i \Big|_{\rho=0} = a^i \sum_{j=1}^3 f_j \phi_j^3 \quad (4.46)$$

$$\sigma_{rz}^i \Big|_{\rho=0} = b^i \left(\sum_{j=1}^3 f_j \lambda_j \phi_j^3 + \epsilon \sum_{j=1}^3 f_j (\phi_j^1 - \phi_j^2) \right)$$

In these equations, the subscript on the vector ϕ specifies which eigenvector is being used, and the superscript on the vector indicates which component of the vector is being used. Thus, ϕ_j^i is a scalar which represents the i^{th} component of the j^{th} eigenvector. The resulting equations for the f s can then be written as:

$$\begin{bmatrix} \phi_1^1 & \phi_2^1 & \phi_3^1 \\ \phi_1^3 & \phi_2^3 & \phi_3^3 \\ \lambda_1 \phi_1^1 - \epsilon (\phi_1^1 - \phi_1^2) & \lambda_2 \phi_2^1 - \epsilon (\phi_2^1 - \phi_2^2) & \lambda_3 \phi_3^1 - \epsilon (\phi_3^1 - \phi_3^2) \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \quad (4.47)$$

The vector $\mathbf{r}$ depends on which component of residual stress is being removed. If the component to be removed is the σ_{rr} component, then $\mathbf{r}^T = [1 \ 0 \ 0]$; if $\sigma_{r\theta}$ is the component to be removed, then $\mathbf{r}^T = [0 \ 1 \ 0]$; and, finally, the vector corresponding to the removal of the σ_{rz} component is $\mathbf{r}^T = [0 \ 0 \ 1]$.

This complete solution for the zeroth order problem must now be substituted into Equations 4.36 through 4.43 to obtain the ψ . The final equation for the g 's in Equation 4.36 is given by:

$$\begin{bmatrix} \phi_1^1 & \phi_2^1 & \phi_3^1 \\ \phi_1^3 & \phi_2^3 & \phi_3^3 \\ \lambda_1 \phi_1^1 - \varepsilon(\phi_1^1 - \phi_1^2) & \lambda_2 \phi_2^1 - \varepsilon(\phi_2^1 - \phi_2^2) & \lambda_3 \phi_3^1 - \varepsilon(\phi_3^1 - \phi_3^2) \end{bmatrix} \begin{Bmatrix} f_1 g_1 \\ f_2 g_2 \\ f_3 g_3 \end{Bmatrix} = - \begin{Bmatrix} \sum_{j=1}^3 \psi_j^1 \\ \sum_{j=1}^3 \psi_j^3 \\ \sum_{j=1}^3 a_j f_j \phi_j^1 + \lambda_j \phi_j^1 + \varepsilon(\psi_j^1 - \psi_j^3) \end{Bmatrix} \quad (4.48)$$

Now that all of the constants have been defined, the total solution for the function F may be approximated by:

$$\{F\} = \sum_{i=1}^3 f_i \{ (1 + \varepsilon(a_i \rho + g_i)) (\phi_i^1) + \varepsilon(\psi_i^1) \} e^{\lambda_i \rho} \quad (4.49)$$

This completes the derivation of the necessary equations for the solution for the approximate interlaminar stresses around circular holes.

4.4 Computer Implementation

The equations of this chapter have been implemented in a FORTRAN program called S.L.A.S.H., Software for Laminate Analysis of Stresses about Holes. The listing for this program is given in Appendix B. The structure of the program falls into several logical units which are shown in the flow chart in Figure 4.3. First, the input data is read into the program. Next, the three-dimensional material constants are found on a ply-by-ply basis for the specific layup which has been input. In this step, the classical laminated plate theory material constants are also found. Following this, the Savin solution for the stresses about holes in an anisotropic plate is used to calculate the stresses acting at the hole surface for a given θ location, as well as the stress derivatives along this radial direction.

These stress derivatives are used in conjunction with Equations 3.27a and 3.27b in order to calculate the interlaminar stresses due to in-plane gradients in the stress field. During this step, the residual σ_{rz} is also calculated.

The calculation of the interlaminar stresses due to the straight free edge is conducted three times, once for each of the component stresses which must be eliminated. The first step in this process is to calculate the zeroth order solution. When this has been obtained, the first order correction term can be calculated. The equations of this chapter can be implemented "as is" to perform these calculations.

The final unit of this program outputs the results of the analysis. The full output of this program gives the three eigenvalues associated with each component of the free edge solution as well as the various constants, i.e.

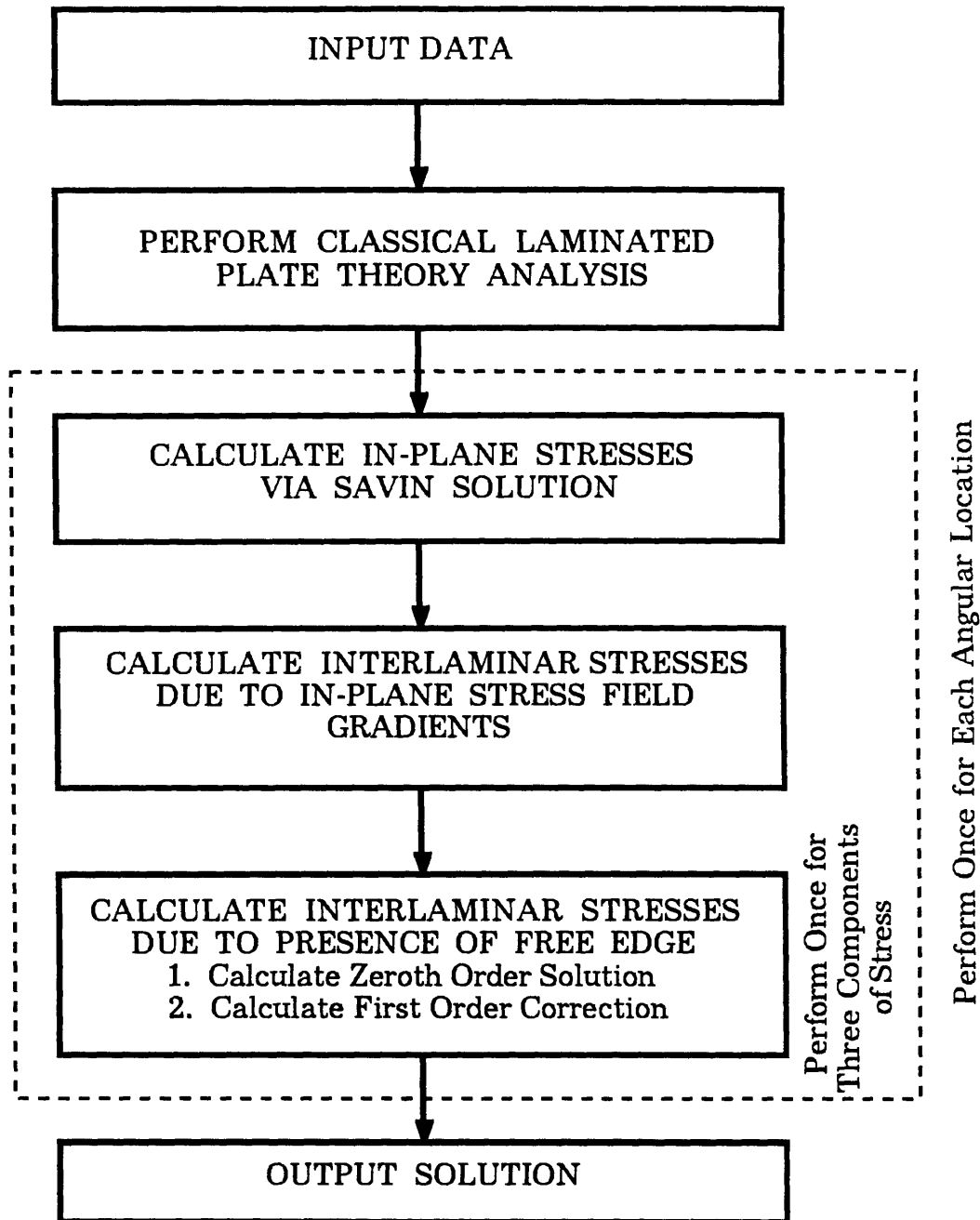


Figure 4.3 S.L.A.S.H. program overview.

components of the eigenvectors multiplied by their corresponding magnitudes. Also included is a summary which gives the values of the stresses $\sigma_{\theta z}$ and σ_{zz} at the free edge, for each ply interface. The program also calculates the average of all three of the interlaminar stresses over a user specified averaging distance, and finally, the program uses these averages in conjunction with the Quadratic Delamination Initiation Criterion in order to predict the onset of delamination. This is further described in Section 6.3. The ratio of the stress predicted for delamination initiation to the applied stress is also outputted for each interface. In the full output mode, files are also created for each angular location investigated which may be used with the program PLOTTER, given in Appendix C, to generate plotting files for the various stresses.

In the condensed output mode, only the summary giving the values of the stresses at the free edge, the average of the interlaminar stresses, and the failure data are stored.

This program has been found to execute very quickly on an IBM-PC[®], allowing for the rapid completion of parametric studies. For example, the run time required for a ten-ply laminate, examining the laminate at ten different angular locations, requires 18 seconds, when the program is run in the condensed output mode.

Chapter Five

Solution Verification

The solution which was obtained in Chapter Four has many built in assumptions and approximations. As different parts of these assumptions and approximations break down, the solution will lose validity. The major assumptions, in the order in which they were made, are one, that variations in the θ direction are small; two, that the approximate equations obtained via the minimization of complementary energy accurately describe the case for which these θ variations are small; and three, that the perturbation solution obtained for the approximate equations is valid. These three assumptions are not, in fact, mutually exclusive, and are also not necessarily a complete list of all assumptions made. However, pinpointing these particular assumptions does allow for a systematic investigation into the accuracy of the solution.

The most straightforward assumption to check is that concerning the validity of the perturbation solution. This perturbation solution will be very accurate if the perturbation parameter is small, and will become less accurate as this parameter increases. The goal here is to find for which values of the perturbation parameter the solution is accurate.

Next, the assumption that the approximate equations accurately model the case for which θ variations are small is analyzed. An idealized problem, for which an exact elasticity solution exists, is formulated, and the approximate equations are compared with this solution. To isolate this assumption, an exact solution to the approximate equations (i.e. not the perturbation solution) is used.

Finally, the assumption that the modelled problem actually represents the case of a loaded laminate will be investigated by comparing it to an existing finite element solution which has been carefully modelled.

5.1 Validity of Perturbation Solution

The first step in the verification process is to determine the range of values of the perturbation parameter ϵ , the ratio of the ply thickness to the hole radius, for which the perturbation solution to Equation 4.19 is valid. In order to perform this investigation, an exact solution for Equation 4.19 must be obtained. This is done numerically with the use of the finite difference method.

The problem chosen for this investigation is the case of a $[0/90]_s$ laminate made of AS4/3501-6 graphite/epoxy and subjected to a far-field loading of a unit σ_{11} . The properties for this material are given in Appendix A, and are repeated in Table 5.1 for convenience. For this investigation, the hole radius is changed, thus changing ϵ , and the solution obtained using the finite difference method is compared to that obtained using the perturbation solution. Neither of these solutions includes the effects of the in-plane gradients because the goal here is only to determine the validity of the solution of Equation 4.19. What is being checked here is the accuracy of the solution of the companion problem, not the general interlaminar stress problem. The loading and the laminate were chosen so that numerical values could be given to the matrices in this equation.

For this laminate, with this loading, an angle of approximately 75° from the loading direction is found to be the location for the maximum of the

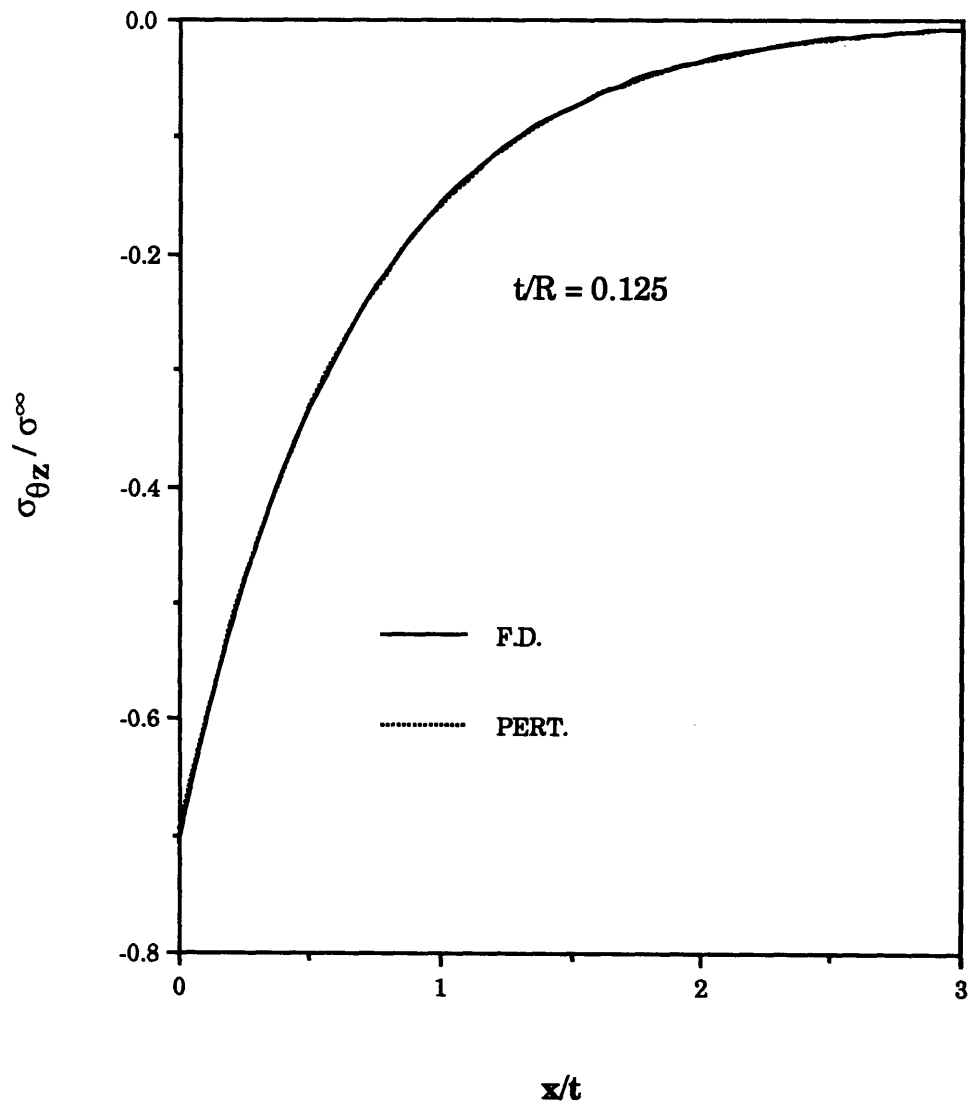


Figure 5.1 Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.125.

interlaminar shear stress, $\sigma_{\theta z}$. The solutions presented here are given, therefore, for this angle. Also, only the out-of-plane stresses will be shown here. This is done because the formulation which was used for the finite difference problem was achieved by assuming that the two interlaminar shear stresses were the primary unknowns. This led to a more stable set of equations which had to be solved, and in this new system of equations, the in-plane stresses were not calculated. The interlaminar shear stresses, being a secondary unknown, are also less likely to be predicted accurately for the perturbation solution. These stresses, as well as the interlaminar normal stress, will thus give a magnified view of the breakdown of the perturbation solution.

The plot of the stress $\sigma_{\theta z}$ for various values of the perturbation parameter ϵ is given in Figures 5.1 through 5.4. It can be seen that for ϵ of $1/8$, the case shown in Figure 5.1, the finite difference solution and the perturbation solution of the approximate equations are practically identical. As this parameter is doubled, some slight variation can be observed for values of x/t less than approximately 0.5. As the perturbation parameter is again doubled, this time to a value of $1/2$, a measurable discrepancy can be seen between the solutions. This case is shown in Figure 5.3. As ϵ is increased to a value of 1, representing the case where the hole radius is equal to a single ply thickness, the perturbation solution has diverged. This is evidenced in Figure 5.4 where it may be seen that at the free edge, the perturbation solution has the wrong sign, and then oscillates to compensate. This is typical of the behavior of perturbation solutions with a large perturbation parameter.

Some general observations can be made concerning the interlaminar shear stress, $\sigma_{\theta z}$, from these plots. The general behavior of this stress as a

TABLE 5.1
AS4/3501-6 MATERIAL PROPERTIES

$E_L = 142 \text{ GPa}$	$G_{LT} = 6.0 \text{ GPa}$	$\nu_{LT} = 0.3$
$E_T = 9.81 \text{ GPa}$	$G_{LZ} = 6.0 \text{ GPa}$	$\nu_{LZ} = 0.3$
$E_Z = 9.81 \text{ GPa}$	$G_{TZ} = 3.8 \text{ GPa}$	$\nu_{TZ} = 0.3$
	$t = 0.134 \text{ mm}$	

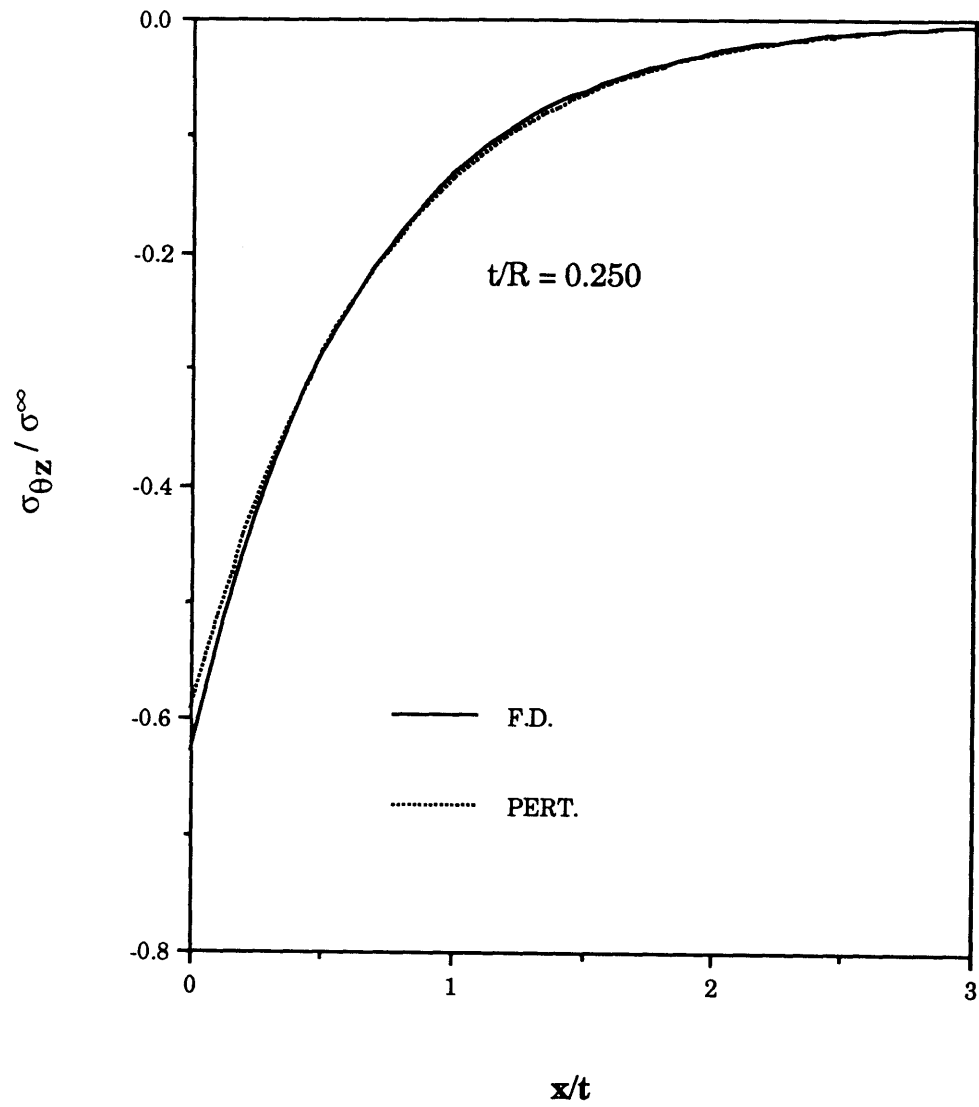


Figure 5.2 Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.250.

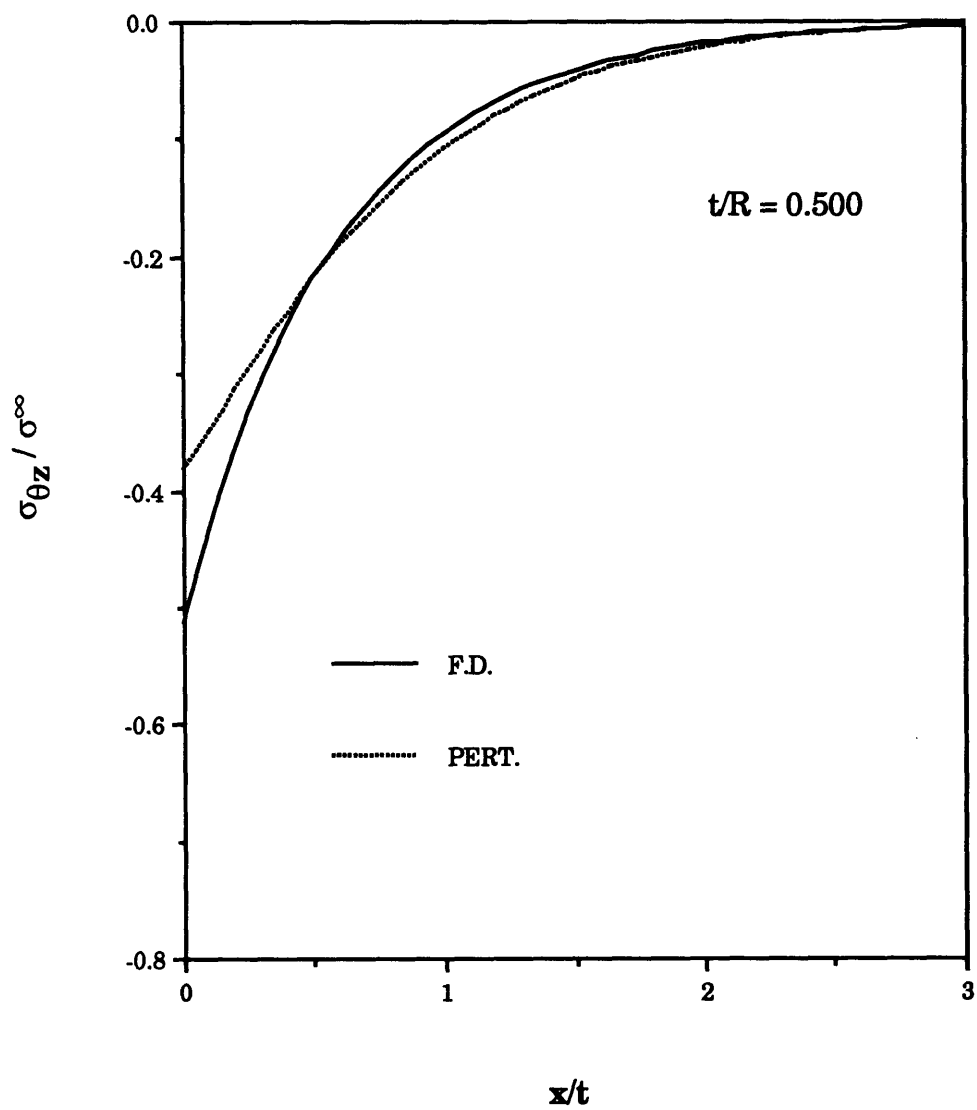


Figure 5.3 Comparison of computed values for σ_{θ_z} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.500.

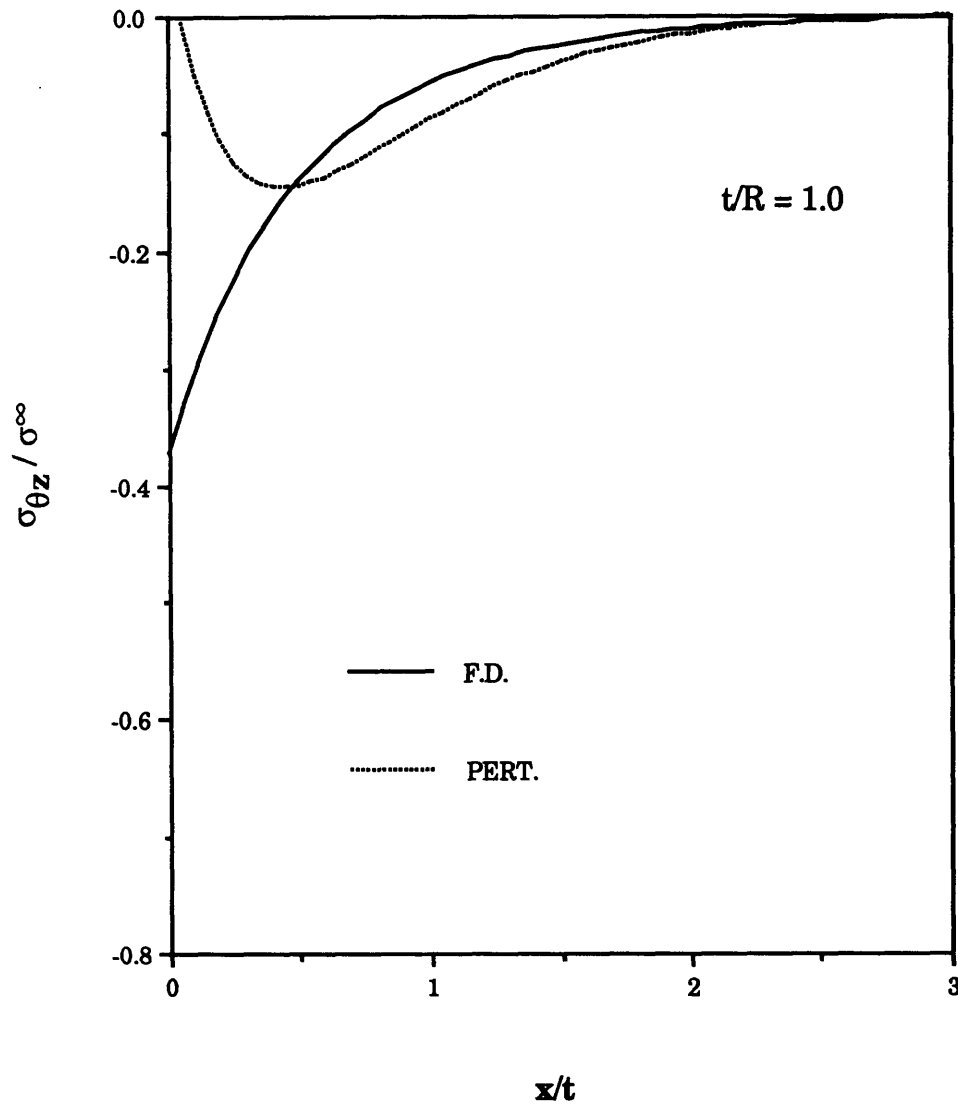


Figure 5.4 Comparison of computed values for $\sigma_{\theta z}$ for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 1.000.

function of the perturbation parameter can be seen in the progression from Figure 5.1 to Figure 5.4, corresponding to a continual doubling of the parameter ϵ , beginning from an ϵ of 1/8 and ending with an ϵ of 1. As ϵ is increased each time, the magnitude of $\sigma_{\theta z}$ can be seen to decrease. For ϵ of 1/8, the magnitude of this stress at the free edge is approximately 0.7, or 70% of the far-field applied stress. The corresponding value of this stress for an ϵ of 1 is 0.36, representing a 50% decrease. Although the magnitude of this stress is decreasing, the region of influence of the stress appears to remain unchanged. In all cases, this stress acts over a region of approximately three ply thicknesses from the free edge.

The behavior of the solutions for the interlaminar shear stress σ_{rz} are depicted in Figures 5.5 through 5.8. In Figure 5.5, it can be seen that both solution techniques, finite difference and perturbation, give identical values for this interlaminar shear stress for ϵ equal to 1/8. This correlation begins to degrade as ϵ is increased to 1/4, as portrayed in Figure 5.6. For the still larger value of ϵ equal to 1/2, the perturbation solution has begun to diverge. In Figure 5.8, the catastrophic divergence of the perturbation solution for the case with ϵ equal to 1 may be observed. The perturbation solution in this case gives a solution which has an initial slope in the wrong direction, and which then oscillates wildly.

The progression from Figure 5.5 to 5.8 reveals the same general behavior for the stress σ_{rz} as was observed for $\sigma_{\theta z}$. As the parameter ϵ is increased, the magnitude of the stress decreases. In this case, for an ϵ of 1/8, the peak value of this stress is 0.026, and for an ϵ of 1, this peak value has diminished to 0.01. Also the trend displayed previously which indicated that the zone of influence did not change with ϵ is again evidenced here. However,

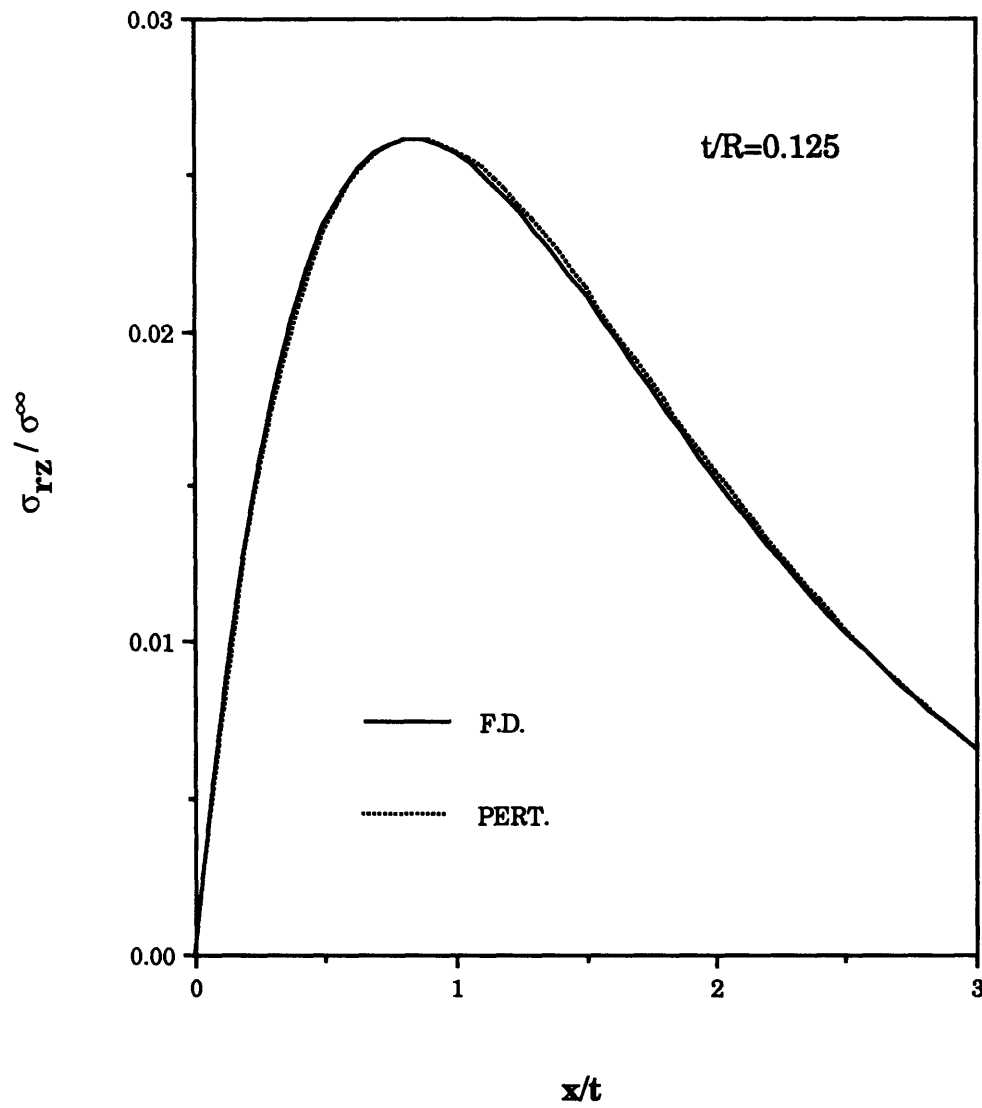


Figure 5.5 Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.125.

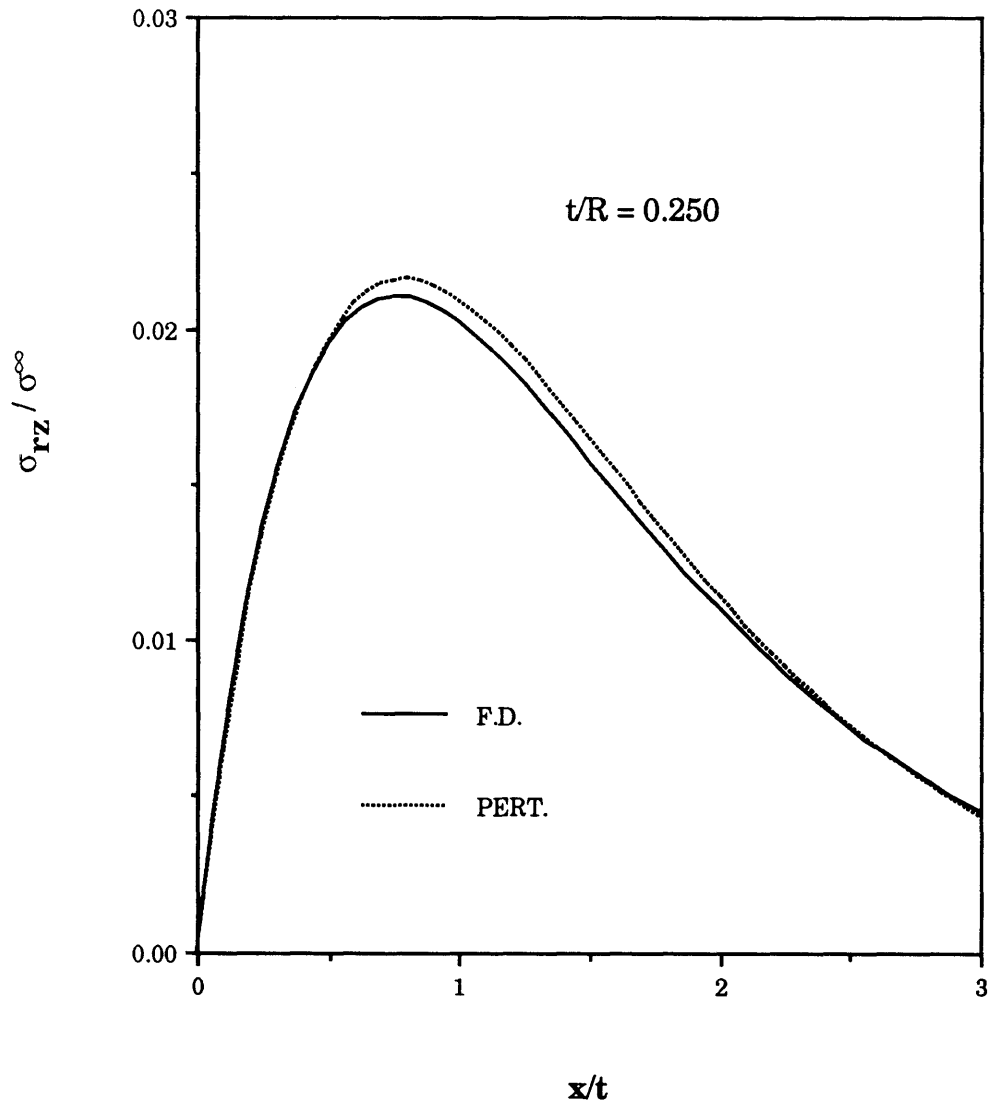


Figure 5.6 Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.250.

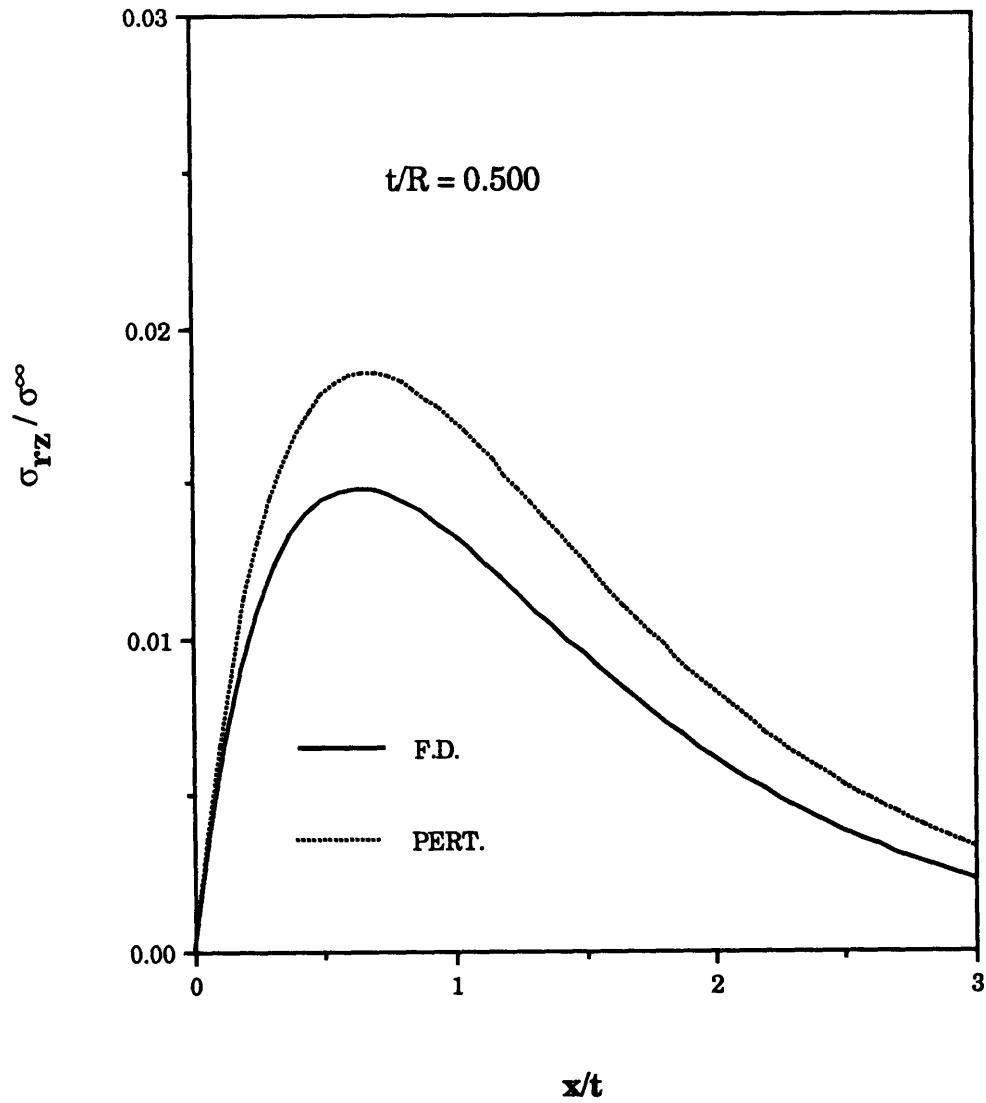


Figure 5.7 Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.500.

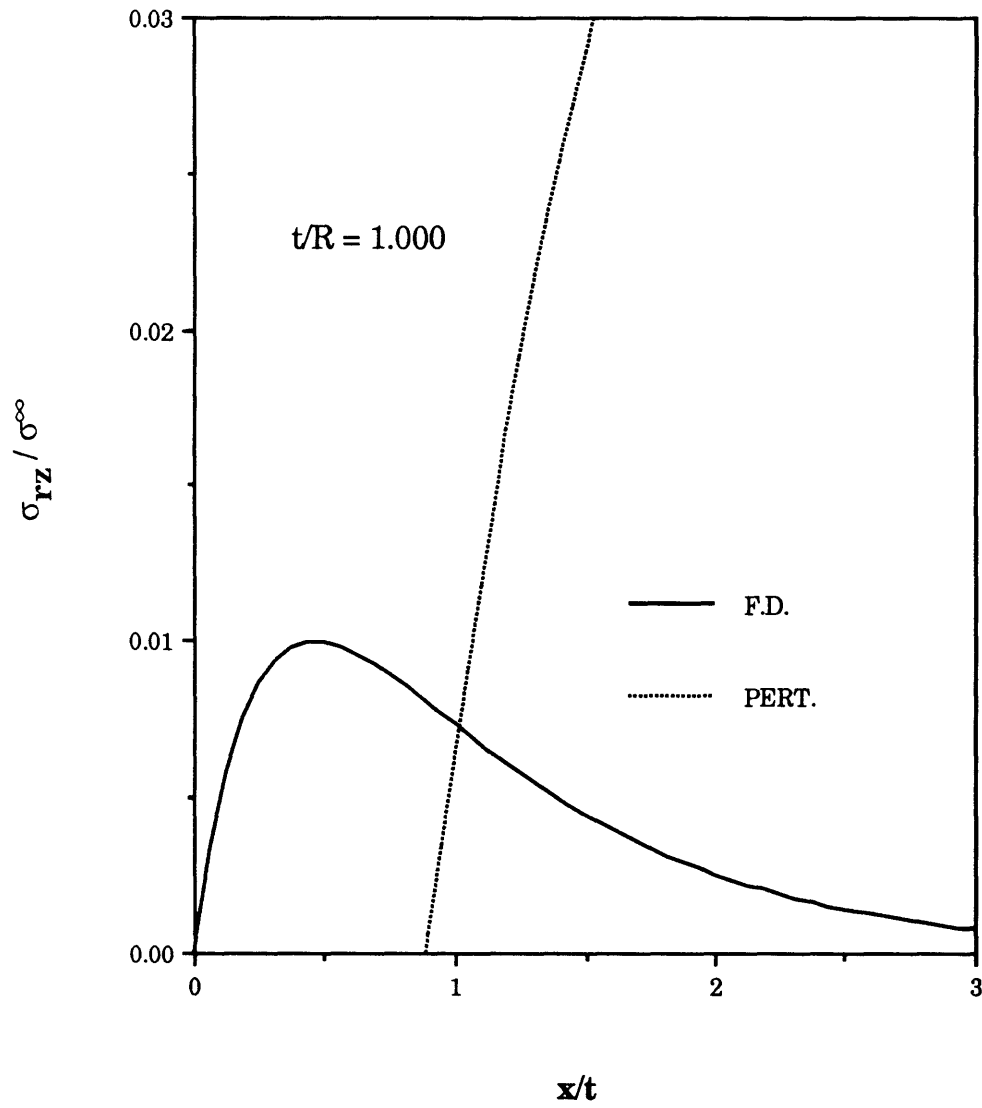


Figure 5.8 Comparison of computed values for σ_{rz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 1.000.

the location at which the peak stress value occurs shifts from a value of x/t equal to 0.77 for ϵ of 1/8 to an x/t of 0.45 for an ϵ of unity.

The final step in the comparison of these solutions is the evaluation of the predictions for the interlaminar normal stress σ_{zz} . In Figures 5.9 through 5.12, the correlation between these solution techniques is portrayed. Again, the parameter ϵ takes on the values of 1/8, 1/4, 1/2, and 1. Also, as previously noted, the two solutions correlate well for ϵ less than or equal to 1/4. As ϵ is further increased to 1/2, the solutions are seen to diverge, and for an ϵ of unity, as shown in Figure 5.12, the perturbation solution has failed.

These plots reveal that the magnitude of the interlaminar normal stress also decreases with increasing ϵ . For the values of the perturbation parameter given in the figures, the interlaminar normal stress has a maximum magnitude of 0.04 for an ϵ of 1/8. This maximum magnitude decreases to a value of 0.032 for an ϵ of 1. The point at which the solution passes through zero for the case of ϵ of 1/8 is for x/t equal to 0.95. For an ϵ of 1, this occurs for an x/t of 0.75. However, the shapes of these plots indicate that the region of influence of this stress is not greatly affected by the parameter ϵ .

From these plots, the conclusion can be drawn that for small values of the perturbation parameter, the perturbation solution to Equation 4.19 is quite accurate. As this perturbation parameter increases, the solution deteriorates, dramatically so for ϵ of 1. These results indicate that this perturbation solution should be used only for values of ϵ greater than or equal to roughly 0.25. For the material used, this corresponds to a hole diameter of approximately 1.072 mm.

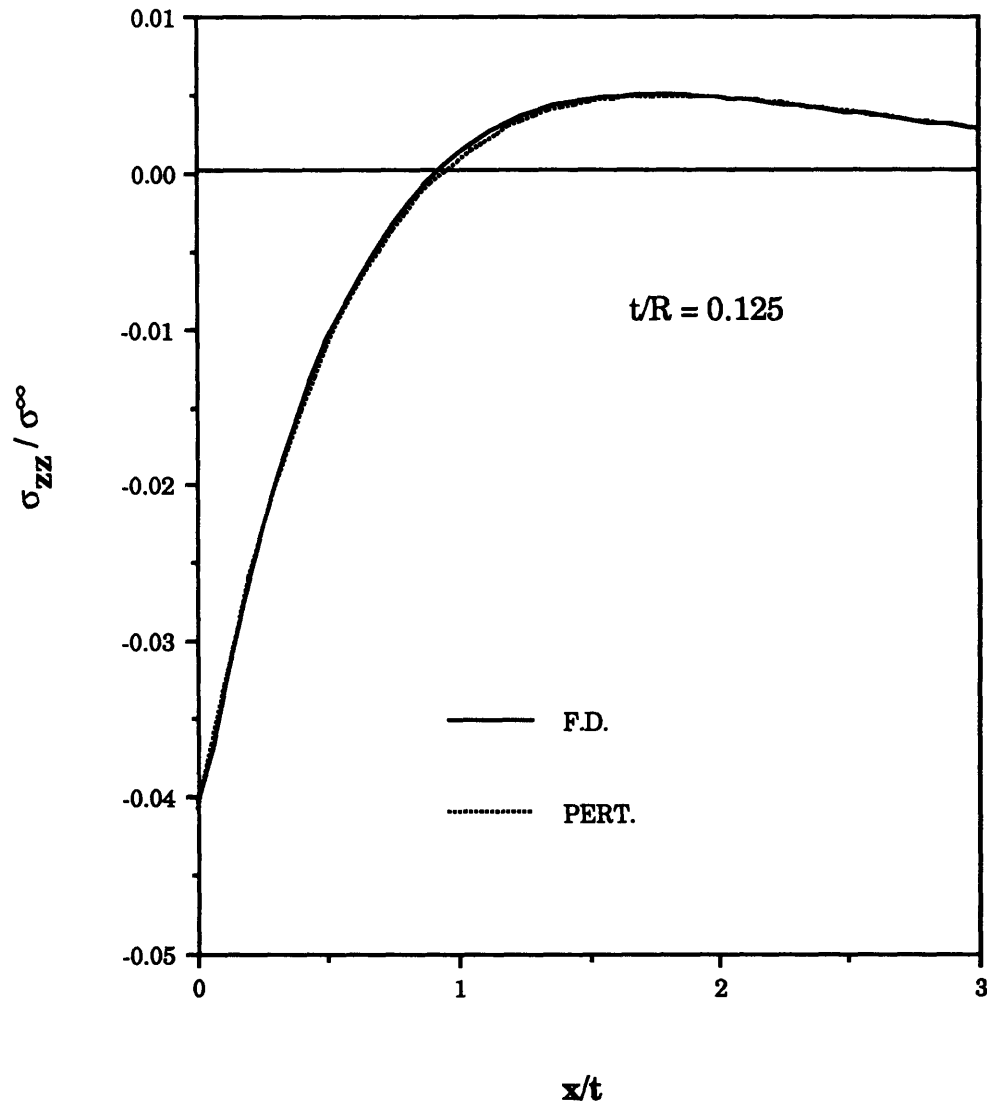


Figure 5.9 Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.125.

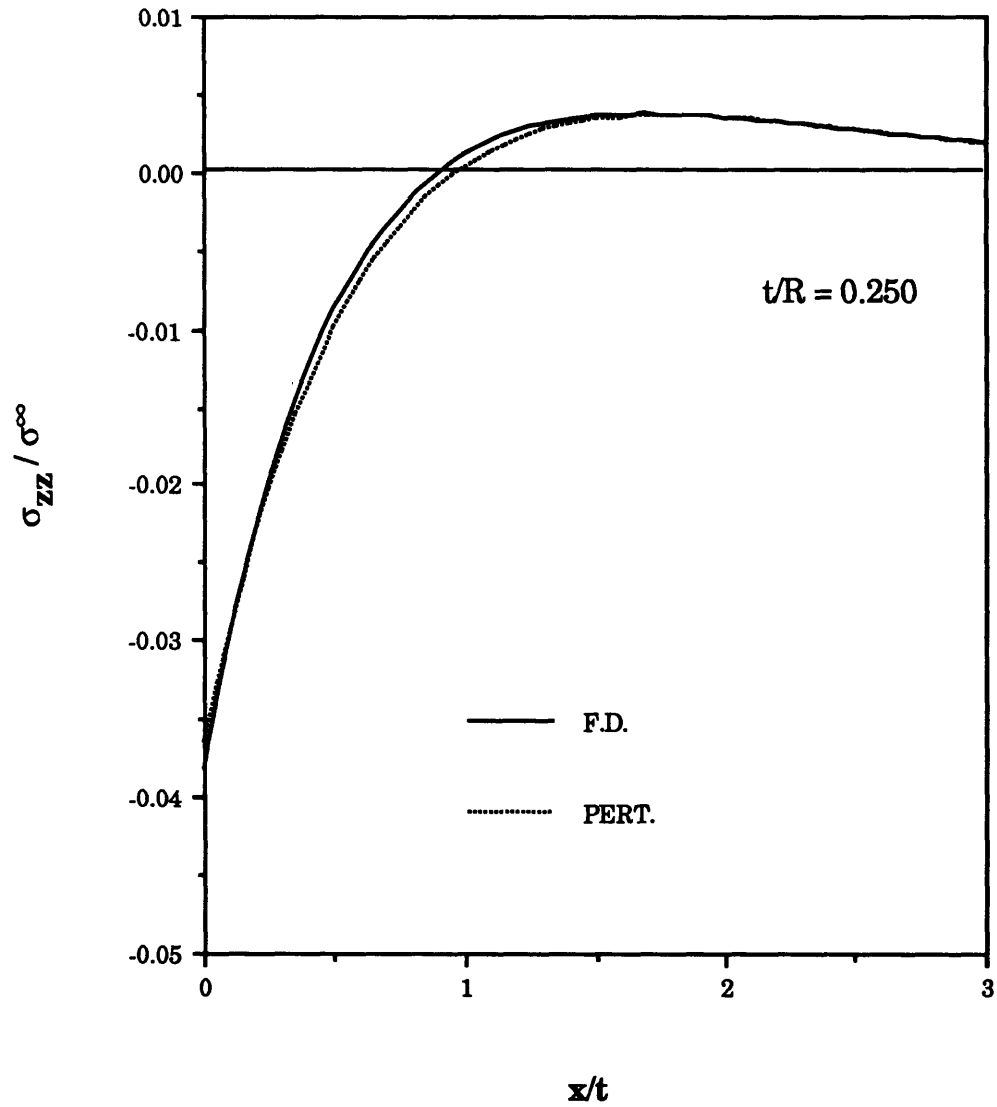


Figure 5.10 Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.250.

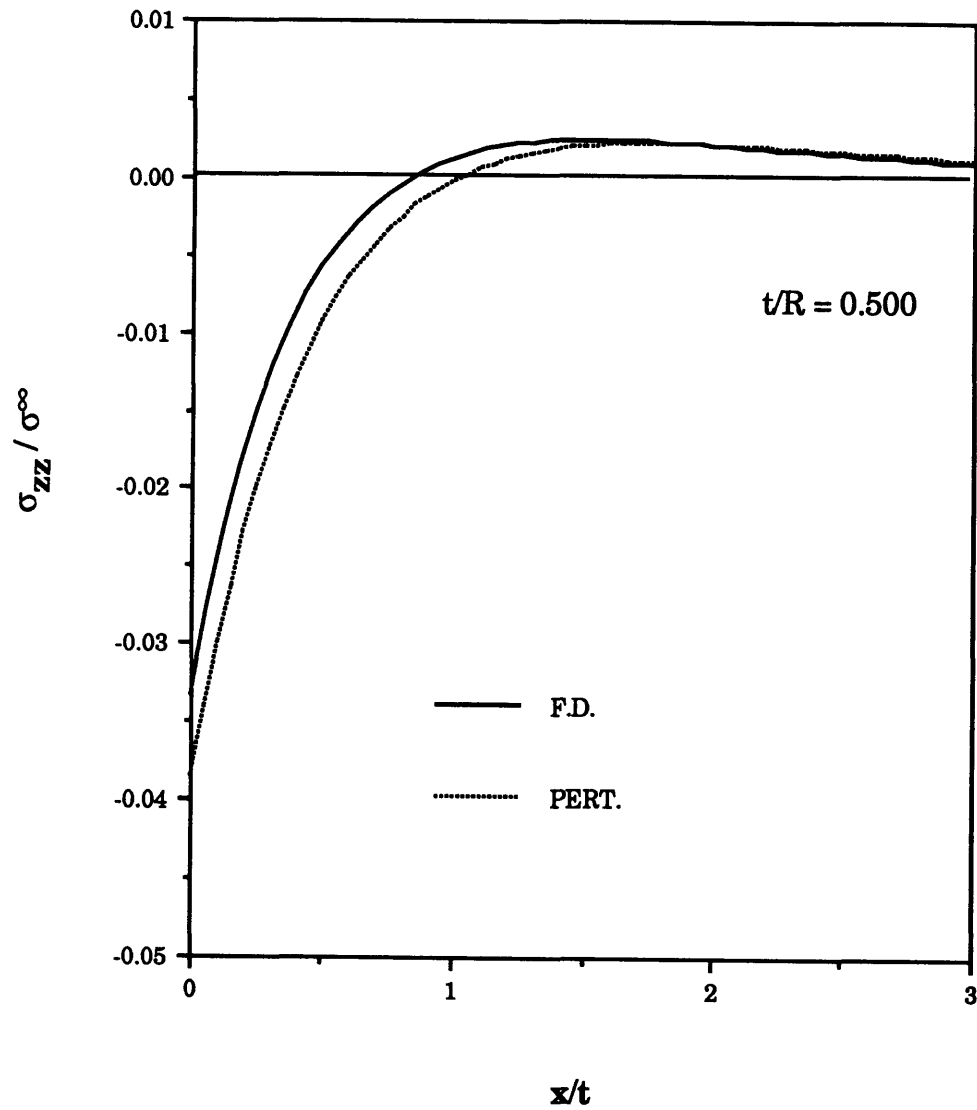


Figure 5.11 Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 0.500.

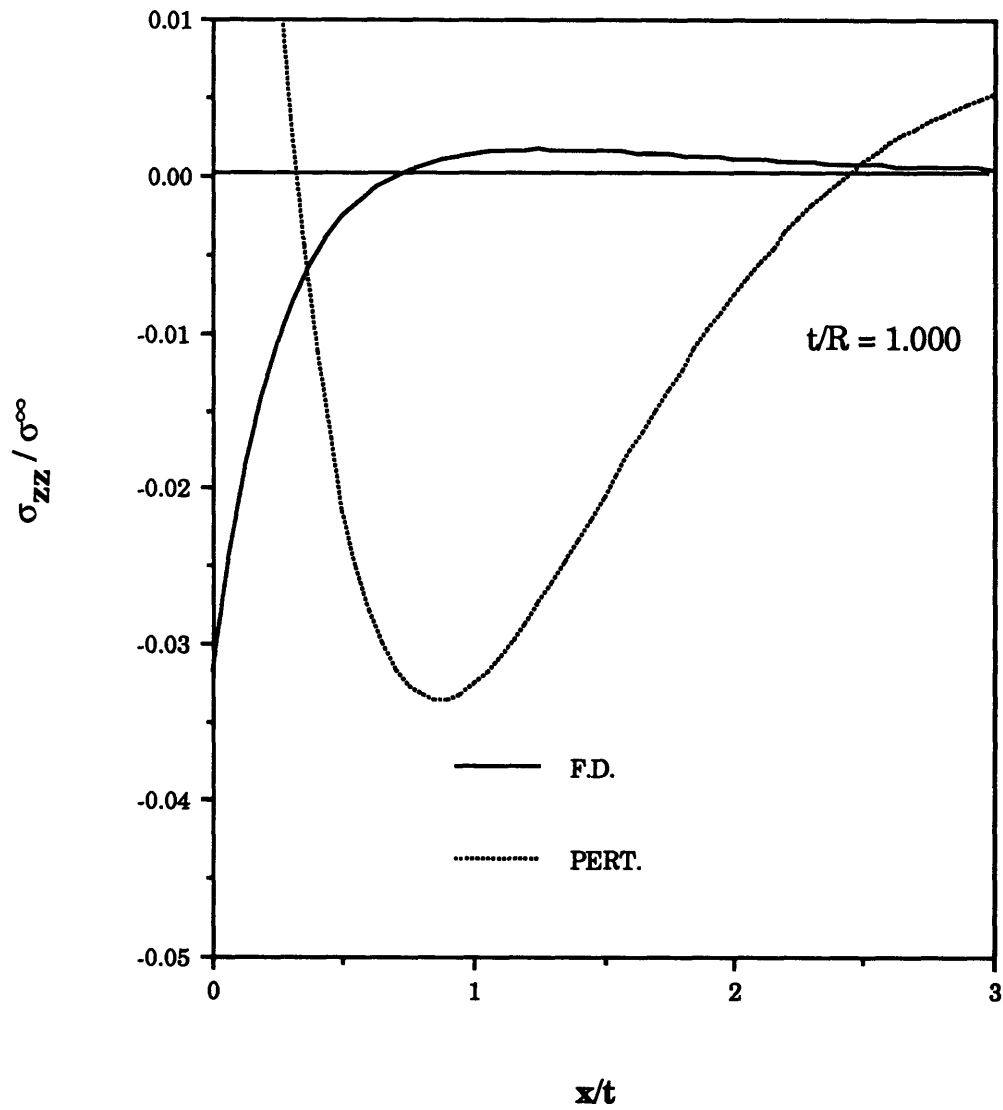


Figure 5.12 Comparison of computed values for σ_{zz} for the finite difference solution and the perturbation solution of Equation 4.17 for t/R equal to 1.000.

5.2 Comparison with Idealized Problem

The next step in the verification process is to compare the solution given by the approximate equations to the solution for an idealized problem for which an exact elasticity solution exists. Such a problem is illustrated in Figure 5.13. This problem represents an isotropic material which is infinite in extent in both the z and the r directions. This structure, with a hole of radius R , is subjected to a square wave shear loading, $\sigma_{r\theta}$, with a period of $2h$. Here, h may be considered as an equivalent ply thickness. The reason this is a good measure of the capabilities of the solution is because this again represents the companion problem, however, this time the material is isotropic. The fact that the material is isotropic allows for a relatively simple closed form solution.

This problem is solved by assuming that displacements in the r and the z directions are equal to zero, and that the displacement in the θ direction is a function of r and z only. These assumptions may be written as:

$$\begin{aligned} u_r &= 0 \\ u_z &= 0 \\ u_\theta &= u_\theta(r, z) \end{aligned} \tag{5.1}$$

Under these assumptions, the only two nonzero strains are given by:

$$\begin{aligned} \epsilon_{\theta z} &= \frac{\partial u_\theta}{\partial z} \\ \epsilon_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \end{aligned} \tag{5.2}$$

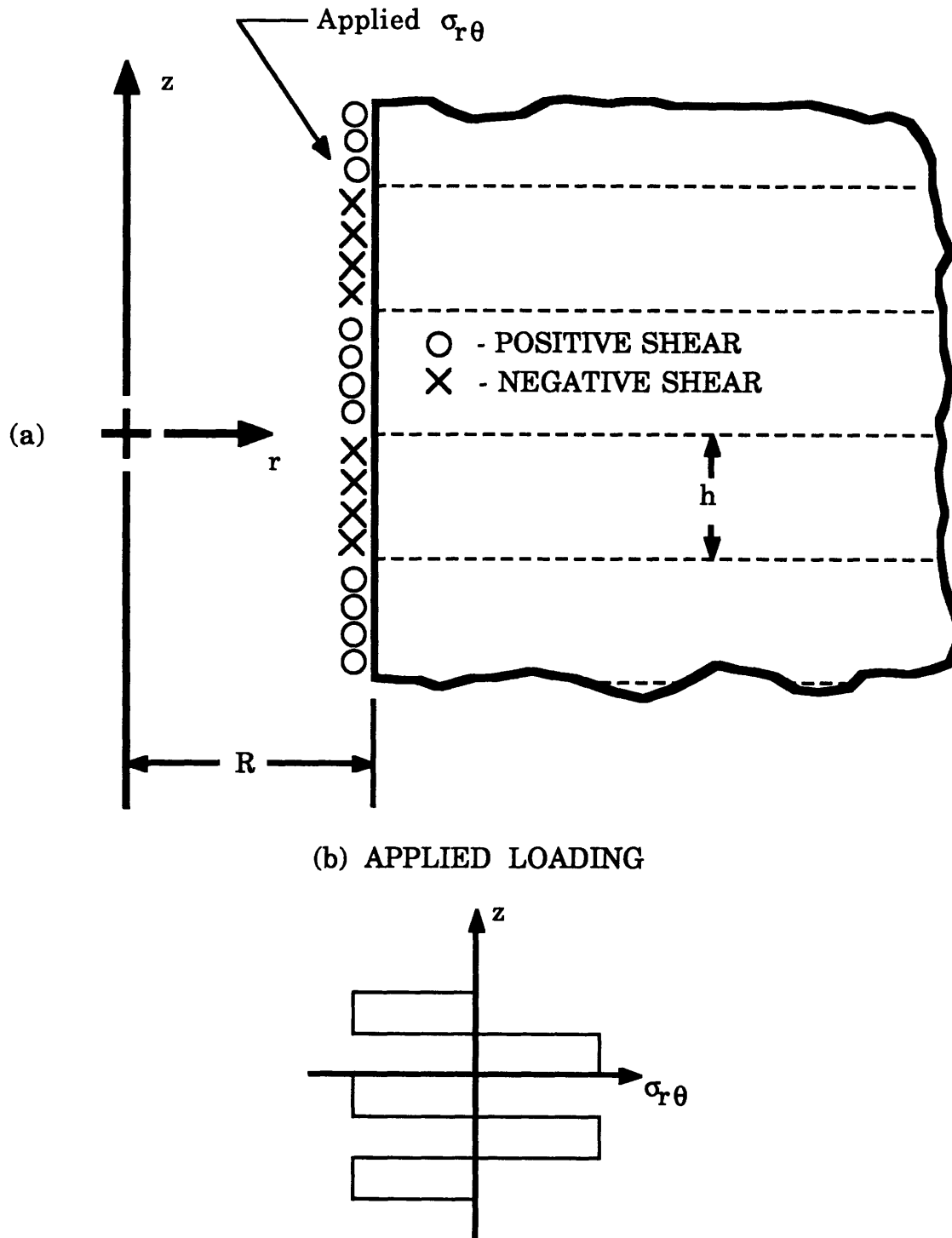


Figure 5.13 Illustration of (a) the geometry and (b) the applied loading of the idealized isotropic problem.

These strains will yield only two nonzero stresses for an isotropic material, $\sigma_{r\theta}$ and $\sigma_{\theta z}$. Because all of the remaining stresses are zero, the equilibrium equations in polar coordinates given in Chapter Four simplify to:

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0 \quad (5.3)$$

These equations can be written in terms of the strains given in Equation 5.2 as:

$$\left(\frac{\partial \epsilon_{r\theta}}{\partial r} + \frac{\partial \epsilon_{\theta z}}{\partial z} + \frac{2\epsilon_{r\theta}}{r} \right) G = 0$$

where the shear modulus is the shear modulus of the material. An equation in terms of the displacement u_θ can be obtained by dividing the above equation by G and then substituting the expressions given in Equation 5.2 for the strains. This results in:

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = 0 \quad (5.4)$$

This equation may be solved by assuming a separable solution in the form:

$$u_\theta = f(r) \sin(\omega_n z) \quad (5.5)$$

After this expression is formally substituted in Equation 5.4, the following equation for the function $f(r)$ results:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \left(\omega_n^2 + \frac{1}{r^2} \right) f = 0 \quad (5.6)$$

The solution for this equation is given by:

$$f(r) = c_n I_1(\omega_n r) + d_n K_1(\omega_n r) \quad (5.7)$$

Here, I_1 is the modified Bessel function of the first kind, of order 1, and K_1 is the modified Bessel function of the second kind, also of order 1. The coefficients c_n must all be zero because of the fact that I_1 goes to infinity for large values r . The values for d_n and ω_n are found by applying the boundary conditions at the hole edge. After the boundary conditions have been applied, the two stresses can be written as:

$$\sigma_{r\theta} = \frac{4}{h} \sum_{n=0}^{\infty} \frac{1}{\omega_n} \frac{K_2(\omega_n r)}{K_2(\omega_n R)} \sin(\omega_n z) \quad (5.8)$$

$$\sigma_{\theta z} = -\frac{4}{h} \sum_{n=0}^{\infty} \frac{1}{\omega_n} \frac{K_1(\omega_n r)}{K_2(\omega_n R)} \cos(\omega_n z) \quad (5.9)$$

with

$$\omega_n = \frac{(2n+1)\pi}{h} \quad (5.10)$$

The corresponding problem using Equation 4.19 results in an equation for $\sigma_{r\theta}$ given by:

$$\frac{\partial^2 \sigma_{r\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial r} - \left(\frac{12}{h^2} + \frac{4}{r^2} \right) \sigma_{r\theta} = 0 \quad (5.11)$$

This equation was obtained by using the properties of an isotropic material, along with the given loading at the hole surface to calculate the constants in Equation 4.19. In general, this equation fully couples all of the components of the assumed shape functions {F}. However, in this special case, the equation for the component F_3 decouples, resulting in an ordinary differential equation for this component. The component F_3 represents the shape of the stress $\sigma_{r\theta}$, and when this substitution is made, Equation 5.11 results.

This equation is very similar to Equation 5.6 and has the solution given in Equation 5.7. For the interval of z between 0 and h , the stress $\sigma_{r\theta}$ is given by:

$$\sigma_{r\theta} = \frac{K_2 \left(2\sqrt{3} \frac{r}{h} \right)}{K_2 \left(2\sqrt{3} \frac{R}{h} \right)} \quad (5.12)$$

Using Equation 5.3, the interlaminar shear stress is found to be:

$$\sigma_{\theta z} = 2\sqrt{3} \frac{K_1 \left(2\sqrt{3} \frac{r}{h} \right)}{K_2 \left(2\sqrt{3} \frac{R}{h} \right)} \left(\frac{1}{2} - \frac{z}{h} \right) \quad (5.13)$$

Functionally, these two different solutions to this idealized problem are very similar. The difference being, of course, that the elasticity solution is in terms of an infinite series, and the solution of the approximate equation only

gives one term. The frequency of the term in the approximate equation solution is approximately 3.4641, which is very near the first frequency in the series for the elasticity solution which is π , approximately given by 3.1415.

These stresses have been evaluated at the location z equal to zero for various values of the perturbation parameter ϵ . Plots showing comparisons for the two solution procedures are given in Figure 5.14 for an ϵ of 1, and in Figure 5.15 for an ϵ of 0.125. As can be seen, in both cases, fairly good agreement is obtained. The correlation between the two solutions improves slightly as ϵ decreases, as can be seen in Figure 5.15, where the maximum value of the interlaminar shear stress has increased by over 40% as compared to the case in Figure 5.14.

From these plots, and from a comparison of the forms of the equations, it appears that the approximate equation does an adequate job in predicting the stress state for this idealized problem.

5.3 Comparison with Finite Element Model

It has been determined that the approximate solution adequately models a specialized isotropic case in which no θ variations exist, as was shown in Section 5.2, for a fairly wide range of values for ϵ , as was found in Section 5.1. The next and final step in the current verification process is to compare the solution with an existing finite element solution for a general laminate with a hole.

The finite element work chosen for comparison was the work of Lucking, et al, [47]. This work was chosen because of the accurate modelling and the relatively large hole size used. These investigators analyzed a $[0/90]_s$

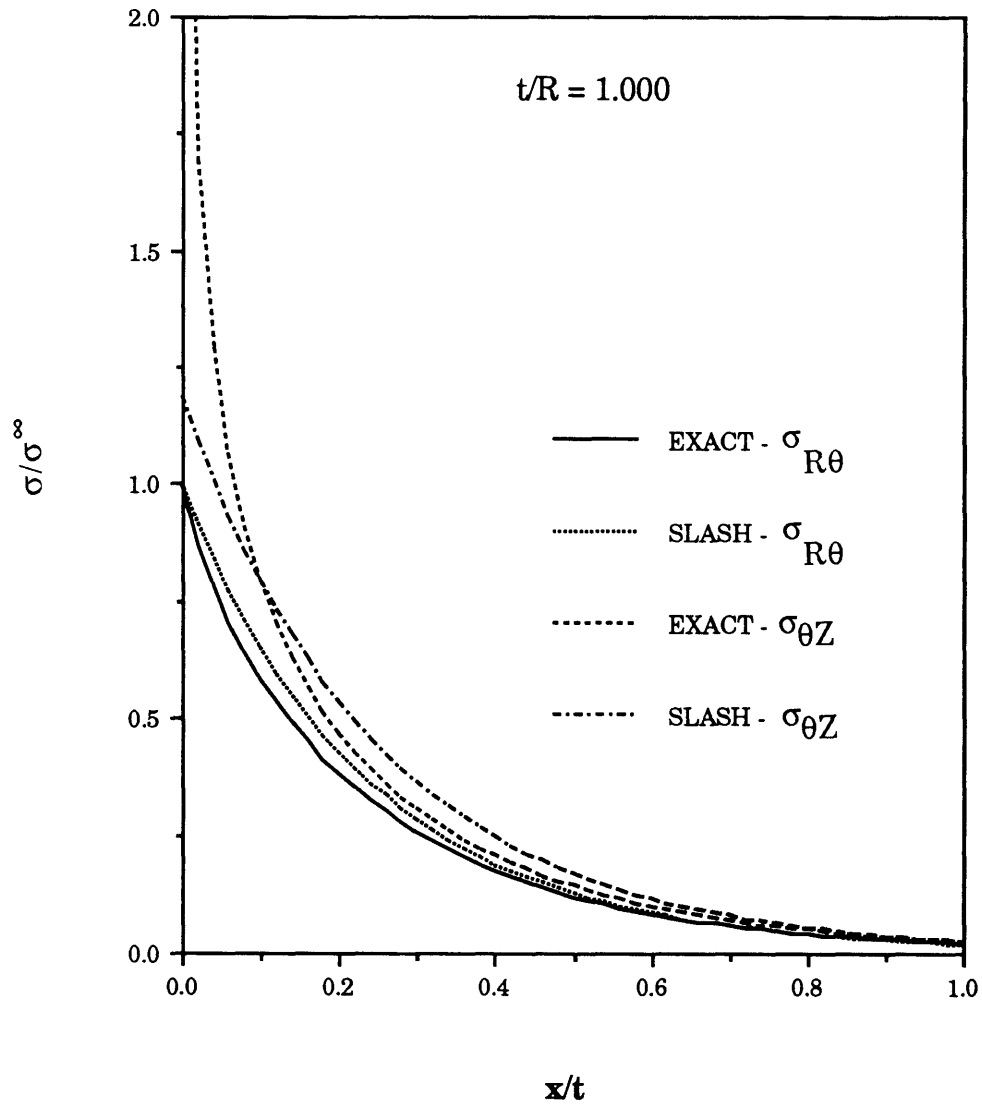


Figure 5.14 Comparison of exact solution and solution obtained via SLASH for an idealized problem with ϵ equal to 1.000.

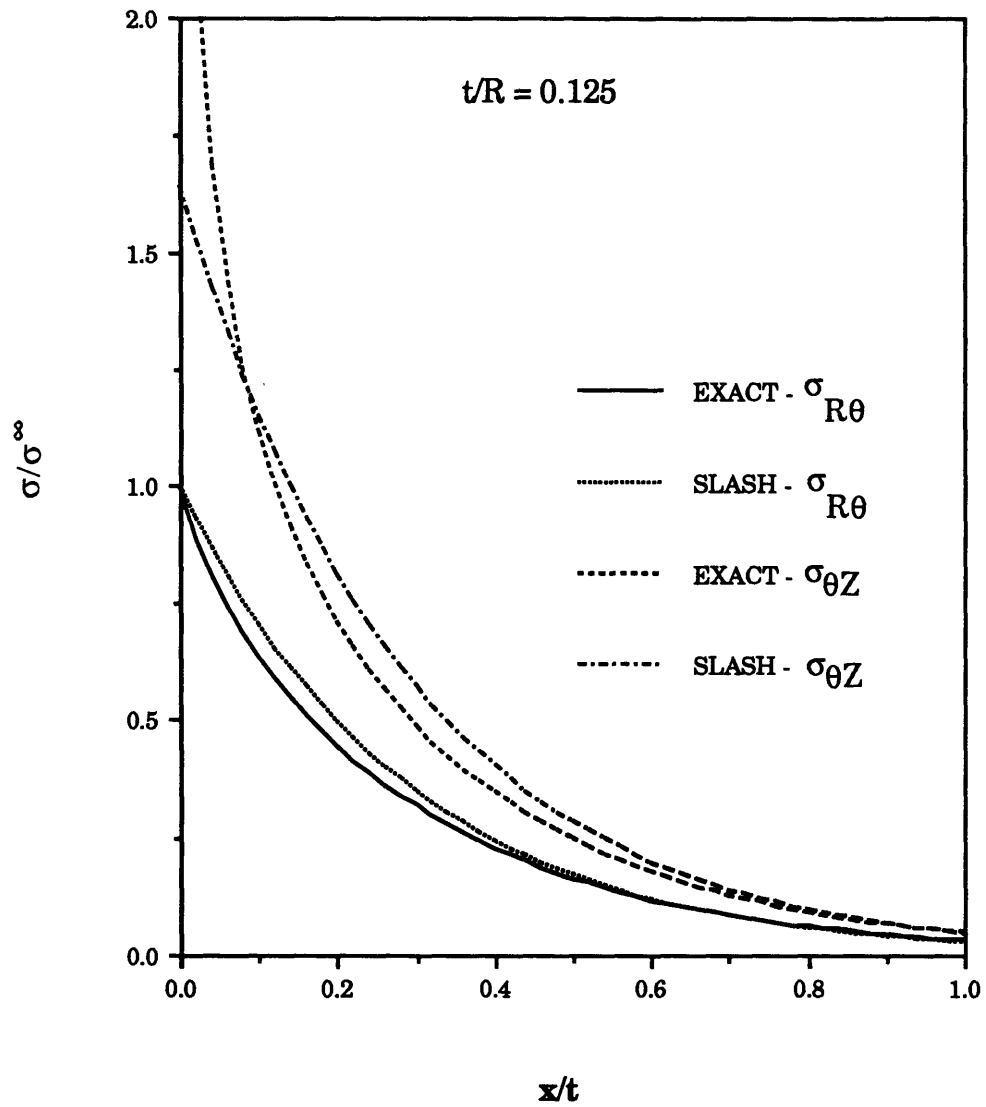


Figure 5.15 Comparison of exact solution and solution obtained via SLASH for an idealized problem with ϵ equal to 0.125.

plate made of T300/5208 graphite epoxy. The elastic properties for this material are given in Table 5.2 as well as in Appendix A. The model which they used had a plate width which was sixteen times the radius of the included hole. The plate length was taken to be twenty times the hole radius. The geometry of this problem is shown in Figure 5.16. The applied loading was imposed as a uniform pressure applied to each ply group. The magnitude of these pressures were the classical laminated plate theory values for the case of uniform strain in an infinite laminate under uniaxial tension. The plate was modelled using 20 node isoparametric finite elements. The program used was SAP IV. Because of limited computer memory, solutions were obtained in stages using a technique similar to substructuring.

Two angles corresponding to the angles reported in the reference which had the highest interlaminar shear stresses will be used here for comparison. These angles are 63° (Figures 5.17 through 5.19) and 81° (Figures 5.20 through 5.23). All stresses shown here have been normalized with respect to the applied far-field load.

The excellent agreement between the two solutions for the stress $\sigma_{\theta z}$ can be seen in Figure 5.17. The plot of the stress σ_{rz} , given in Figure 5.18, does not compare as well. However, as can be seen, the shapes of these two curves match fairly well, and it should be noted that the value of this stress at the free edge does not return to zero, as it should, for the finite element solution. Also, the maximum of this stress, in the case of the finite element solution, is an order of magnitude smaller than for the interlaminar shear stress $\sigma_{\theta z}$.

The plot of σ_{zz} is given in Figure 5.19. As can be seen, this stress, especially in the integral sense, is fairly well predicted. Because of differences between the current solution methodology and the methodology used in the

TABLE 5.2
T300/5208 MATERIAL PROPERTIES

$E_L = 145 \text{ GPa}$	$G_{LT} = 4.5 \text{ GPa}$	$\nu_{LT} = 0.31$
$E_T = 10.7 \text{ GPa}$	$G_{LZ} = 4.5 \text{ GPa}$	$\nu_{LZ} = 0.31$
$E_Z = 10.7 \text{ GPa}$	$G_{TZ} = 3.6 \text{ GPa}$	$\nu_{TZ} = 0.49$
	$t = 0.254 \text{ mm}$	

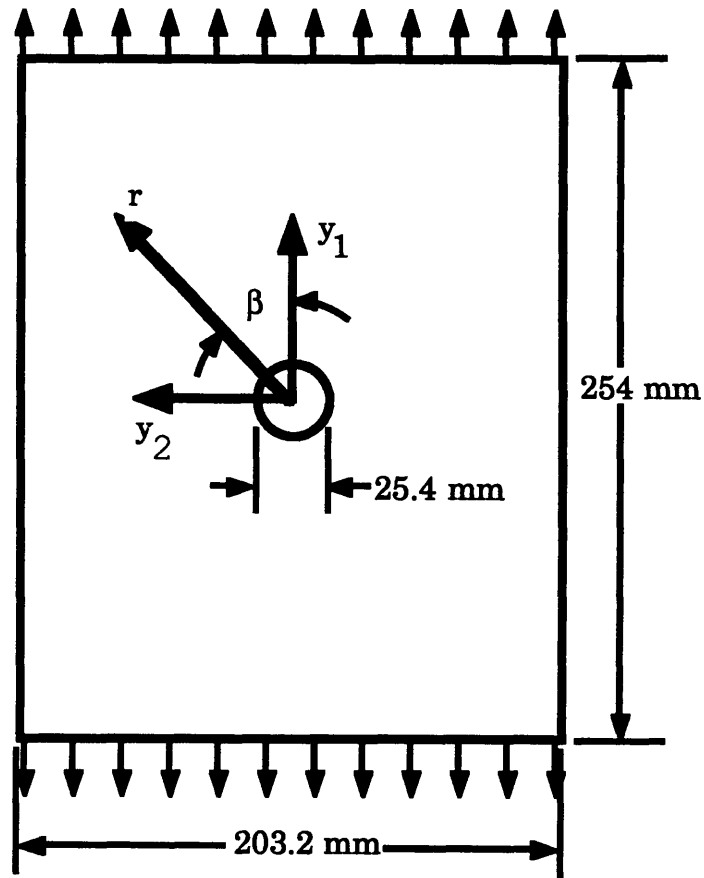


Figure 5.16 Geometry of problem solved via Finite Element analysis by Lucking, Hoa, and Sankar [47].

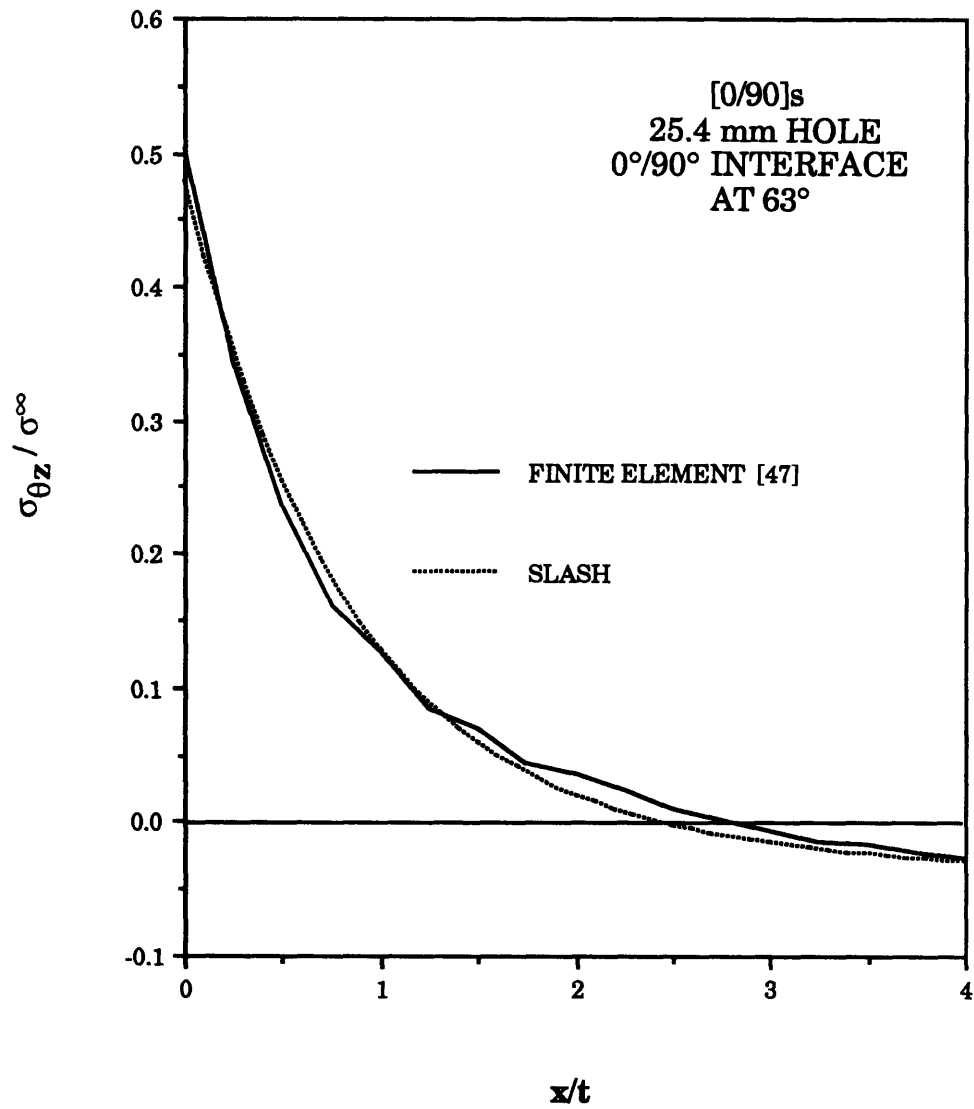


Figure 5.17 Comparison of finite element solution and SLASH solution for $\sigma_{\theta z}$ along a radial at 63° from the load axis.

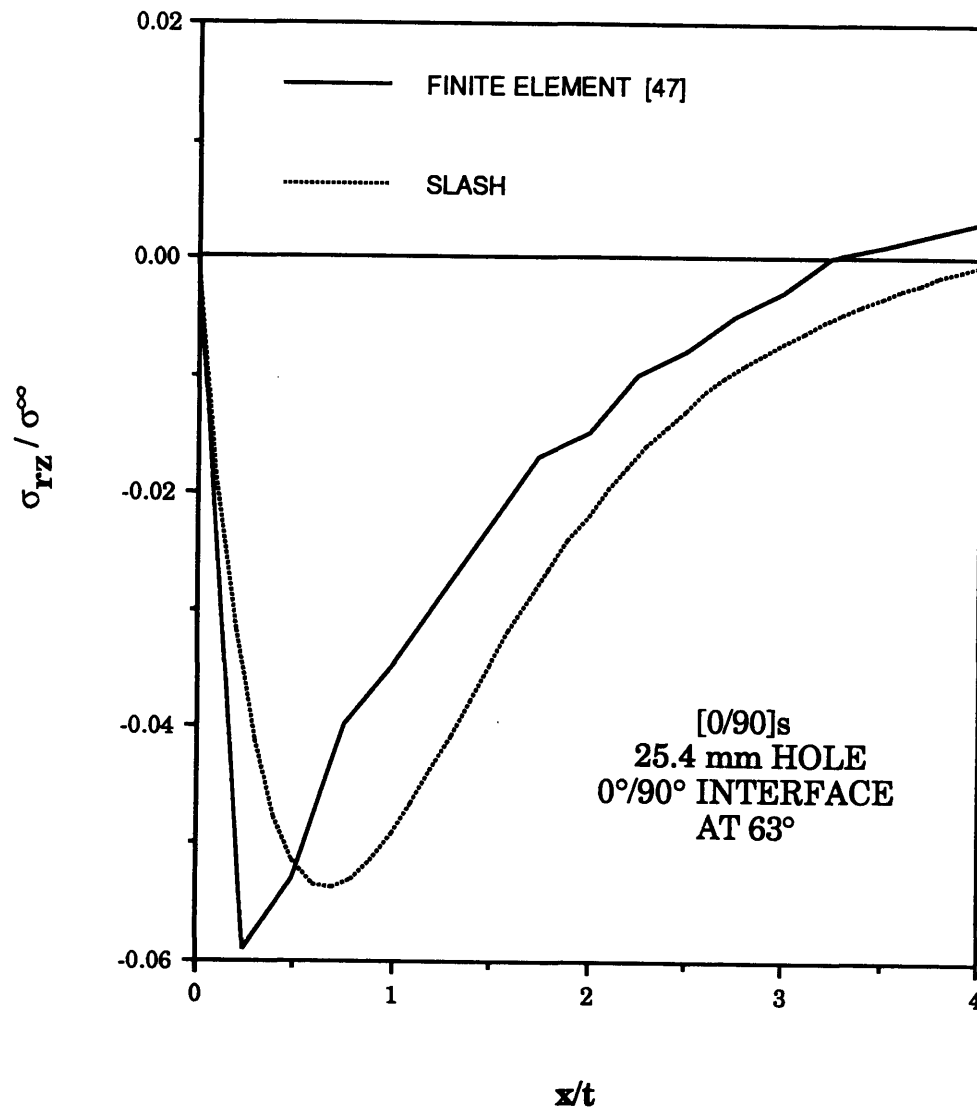


Figure 5.18 Comparison of finite element solution and SLASH solution for σ_{rz} along a radial at 63° from the load axis.

finite element analysis, these interlaminar normal stresses are not shown at identical locations. In the finite element solution, an accurate value for this stress was not obtainable at the midplane, so this stress was given at 0.25 ply thicknesses above the midplane. Because the SLASH program was formulated to aid in the prediction of delamination, stresses from this solution are always given at ply interfaces. Thus, the curve for the SLASH solution corresponds to the midplane of the laminate. The fact that the interlaminar normal stress is a maximum at the midplane explains some of the discrepancy between the solutions given in Figure 5.19.

At the angle 81° , similar results are obtained. In this case, the interlaminar shear stress, $\sigma_{\theta z}$, as predicted by the perturbation solution, is in quite good agreement with that predicted by the finite element analysis, as can be seen in Figure 5.20. As before, the interlaminar shear stress σ_{rz} is not predicted as well, as shown in Figure 5.21. But it should be noted that this stress is also an order of magnitude smaller than is the stress $\sigma_{\theta z}$. At this new angular location, the interlaminar normal stress is predicted to a higher degree of accuracy, as shown in Figure 5.22. Again, some of the discrepancies in the magnitudes of these stresses may be explained by the different z locations at which the solutions are given.

This comparison with a finite element solution not only verifies the effectiveness of the approximate equations, but also the accuracy of the solution for the interlaminar stresses due to the in-plane stress gradients. At these two angular locations, the contributions of the in-plane stress gradients are significant and as has been shown, the current solution is able to predict this behavior.

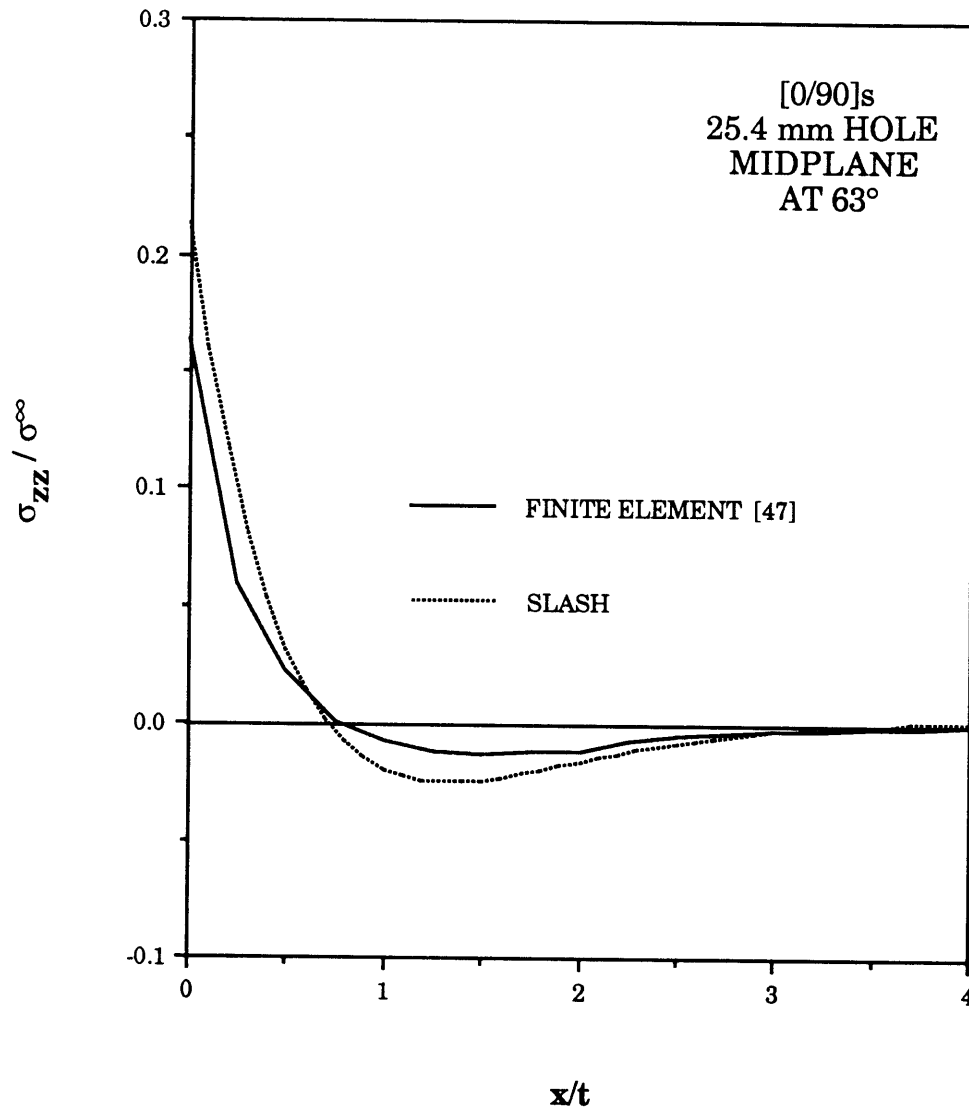


Figure 5.19 Comparison of finite element solution and SLASH solution for σ_{zz} along a radial at 63° from the load axis.

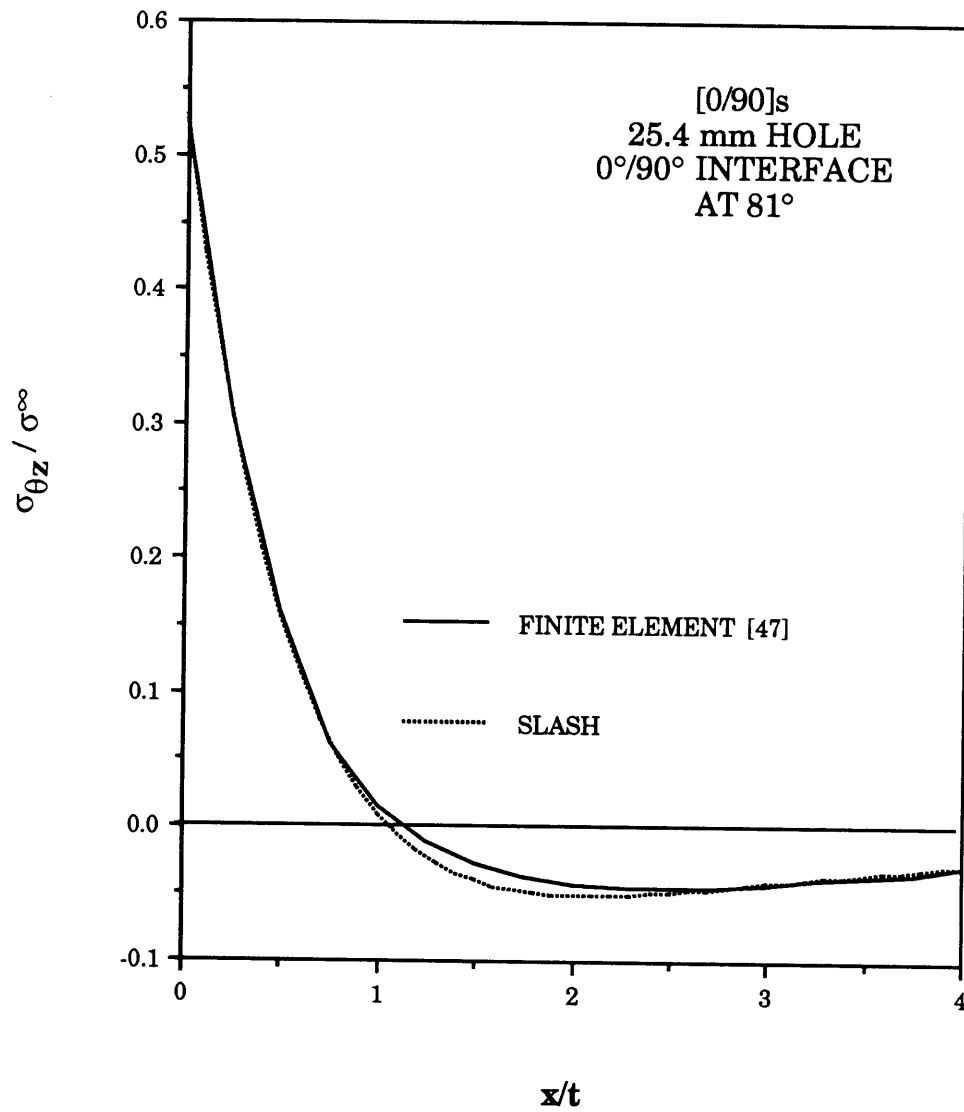


Figure 5.20 Comparison of finite element solution and SLASH solution for $\sigma_{\theta z}$ along a radial at 81° from the load axis.

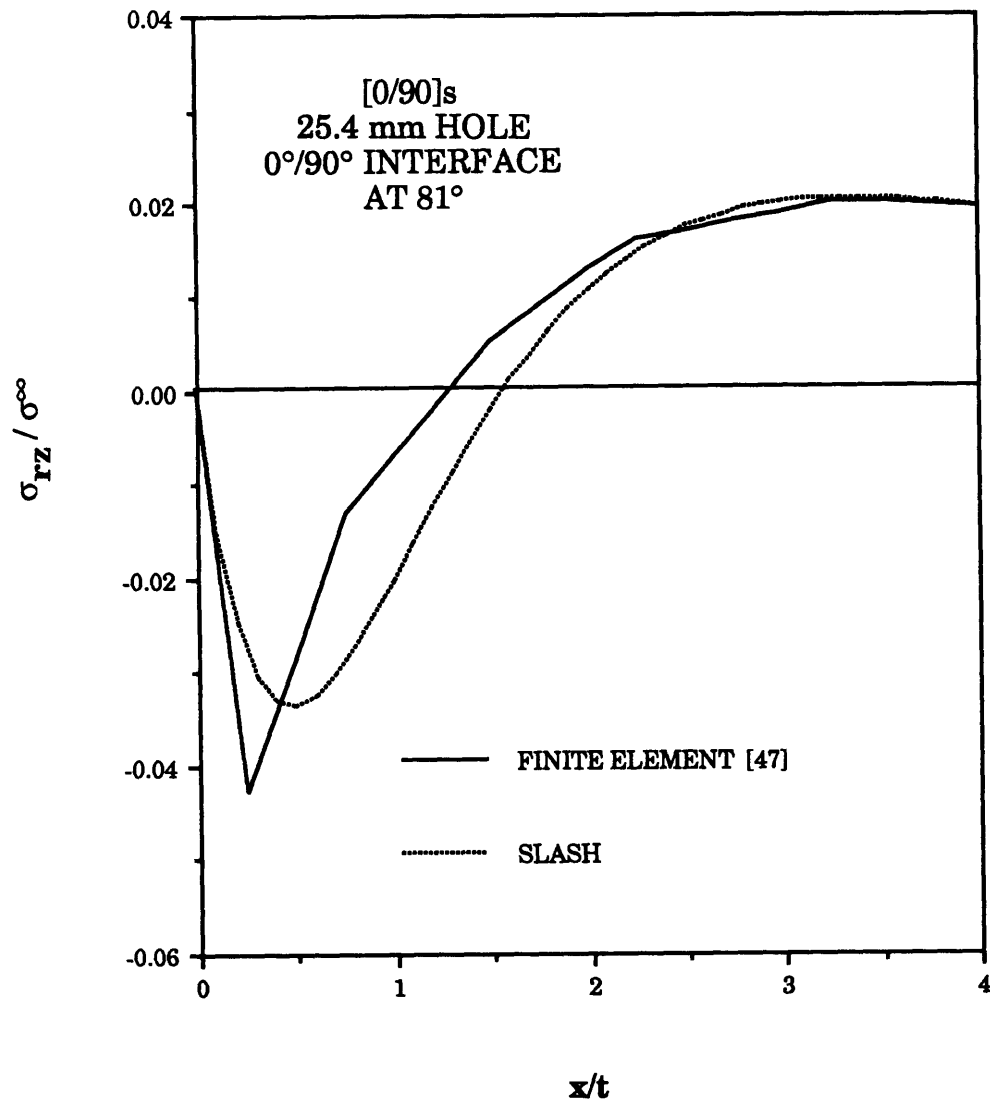


Figure 5.21 Comparison of finite element solution and SLASH solution for σ_{rz} along a radial at 81° from the load axis.

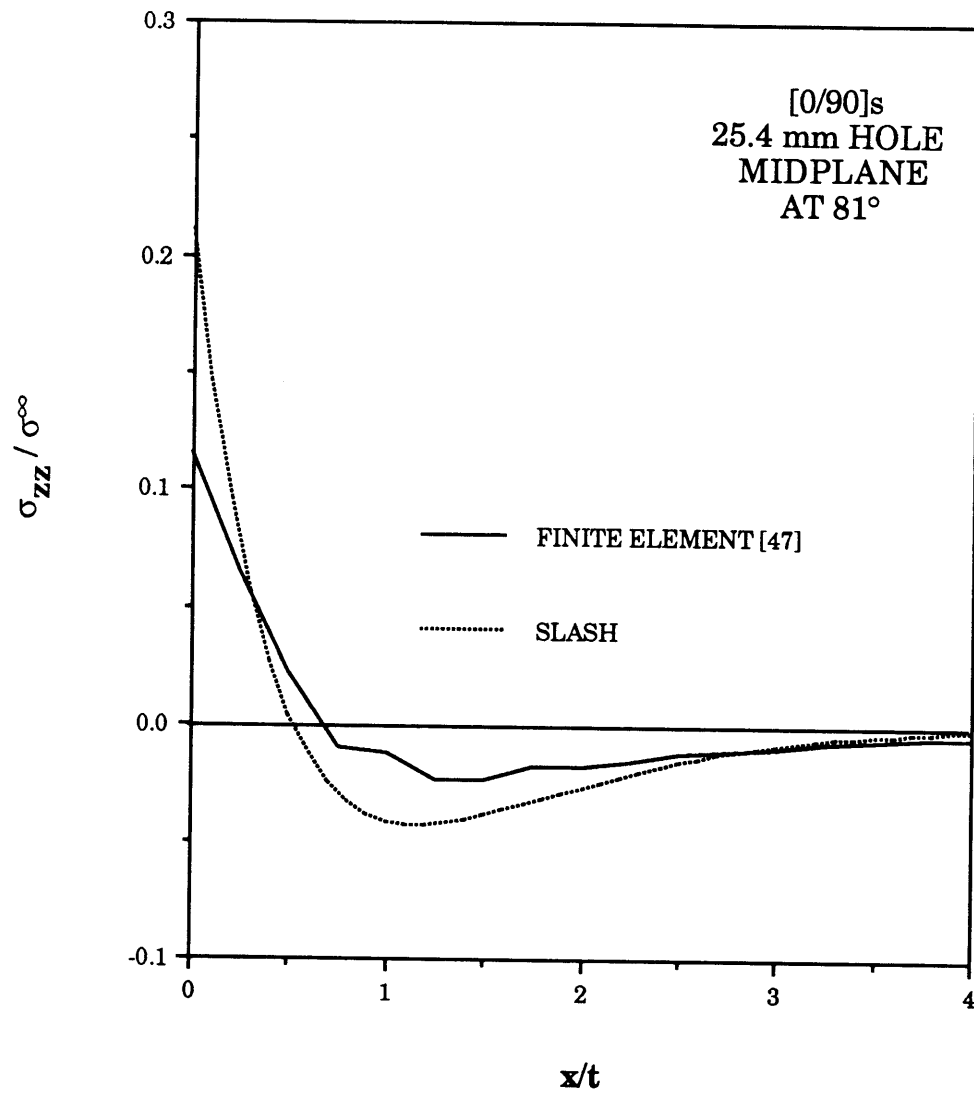


Figure 5.22 Comparison of finite element solution and SLASH solution for σ_{zz} along a radial at 81° from the load axis.

5.4 Summary

The three steps taken in the verification process have revealed that the approximate equations in general, and the SLASH program in particular, adequately solve the problems at hand. From the first verification step, it was found that the perturbation solution adequately solves the approximate equations for values of the perturbation parameter greater than or equal to approximately 0.25. The second verification step showed that the approximate equations can represent the solution to an idealized problem for which an exact solution has been obtained. Finally, comparison with an existing finite element solution has shown that the solution for the interlaminar stresses due to both in-plane stress gradients and due to the presence of a stress-free edge are fairly accurate, especially for the interlaminar shear stress which is often most important for delamination initiation, $\sigma_{\theta z}$.

Chapter Six

Parametric Studies

The equations and solution procedures developed in the previous chapters have been incorporated in the computer program SLASH (Software for Laminate Analysis of Stresses around Holes). This program uses the geometry of the laminate and the laminate material properties to calculate the interlaminar stress state near holes. This stress state is very complicated, and finding a meaningful way in which to represent the data can be difficult. The program SLASH has been used to study a number of cases in order to locate and understand important trends in the stress state, as determined by geometry and layup. Also, the relative importance of the two contributors to the interlaminar stress state have been investigated. Because an engineer is primarily interested in when a certain piece will fail, and therefore only indirectly interested in the actual values of the interlaminar stresses, delamination failure aspects of laminates are also considered.

6.1 Importance of In-Plane Gradient Stress Field Contribution

The comparison of the SLASH solution with the Finite Element solution of Lucking, et al, in Chapter Five, showed only the net result of the SLASH solution. As was shown in the general derivation in Chapter Three and the specific derivation in Chapter Four, interlaminar stresses may be thought of as arising from two different phenomenon, namely free edge boundary conditions and in-plane gradient stress fields. Now, the relative importance of these two contributors is examined.

The problem investigated here is that of a $[0/90]_g$ laminate made of T300/5208 graphite/epoxy subjected to a uniform uniaxial stress at infinity. The properties of this material are given in Appendix A, and the geometry of the problem has been shown in Figure 5.16. The ratio of the thickness of a ply to the radius of the hole is 0.02 with the hole radius being 25.4 mm. This particular problem has been chosen to demonstrate the relative magnitudes of the two contributors to the interlaminar stress state because it has already been shown that the net SLASH solution compares well with the finite element solution in this case. As was done previously, angular locations 63° and 81° from the load axis will be examined because the interlaminar shear stress $\sigma_{\theta z}$ is near maximum at these locations.

The two components of the interlaminar shear stress $\sigma_{\theta z}$ are represented graphically in Figure 6.1 along a radial line which is 63° from the load axis. A coordinate x is used to measure the distance from the hole edge. Shown in this plot are the very different natures of these two components. It may first be seen that, near the free edge, the contribution due to the free edge is much larger than the contribution from the in-plane gradients. However, each solution tends to dominate a different regime. In this plot, x denotes the distance from the hole edge, and t is the ply thickness. The free edge contribution dominates here for small values of x/t , whereas for values of x/t greater than approximately 2.0, the contribution from the in-plane gradients dominates. Also depicted in this figure is the fact that these two contributors may have vastly different scale factors. In general, the portion of the solution arising from the free edge condition will have decay lengths which will be proportional to the effective ply thickness. The portion of the solution which arises from the in-plane gradient will have decay lengths which are

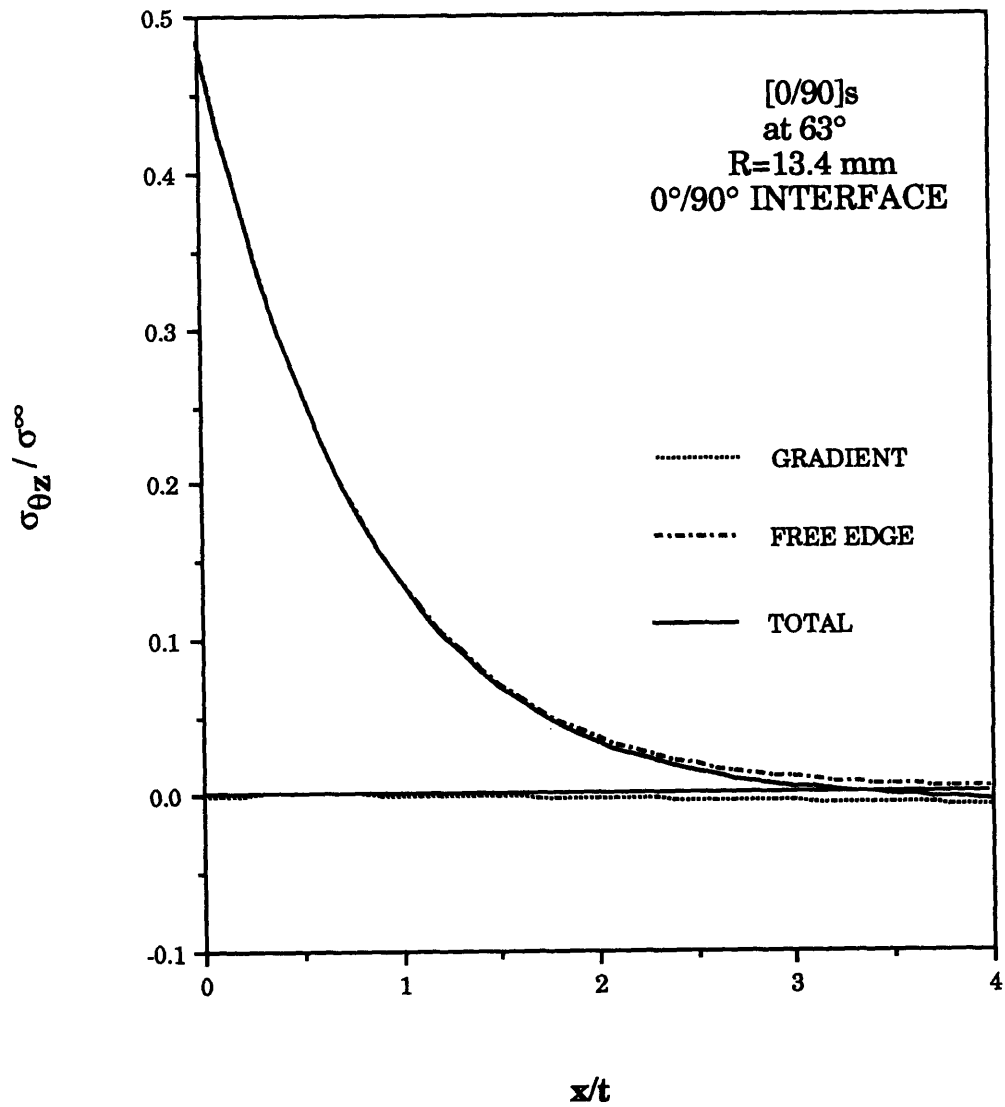


Figure 6.1 Components of $\sigma_{\theta z}$ at 0°/90° interface for [0/90]_s laminate with a 26.8 mm hole at 63° from load axis.

proportional to in-plane characteristic dimensions, in this case, the hole radius.

The two contributors to the interlaminar stresses along a line 81° from the load axis are given in Figure 6.2. In this figure, it may be seen that in this case the decay rates of the two contributions appear again to be different. More important, however, is the fact that in this case the magnitude of the contribution from the in-plane gradients is much greater than is the case along the 63° line. Here, the contribution from the in-plane gradients is nearly 15% of the free edge contribution. Also different in this case is the fact that the in-plane gradient contribution has a maximum at the free edge.

These two figures help to illustrate the importance of including the contribution to interlaminar stresses from in-plane gradient stress fields. In the first case, it was seen that the characteristic dimensions involved may be very different for the different contributors. In the second case, it was found that the magnitude of the two contributions may be roughly of the same order. Both of these factors are significant and can play important roles when determining delamination initiation.

The remaining two interlaminar stresses, σ_{rz} and σ_{zz} , are shown in their component form in Figures 6.3 through Figure 6.6. In Figure 6.3, the interlaminar shear stress σ_{rz} is shown along a 63° radial, and in Figure 6.4, it is given along an 81° radial. In these two figures, it can be seen that the magnitudes of the in-plane gradient contributions at these two angular locations are comparable, but the sign of these stresses has been changed. A similar trend is seen for the case of the interlaminar normal stress shown in Figures 6.5 and 6.6.

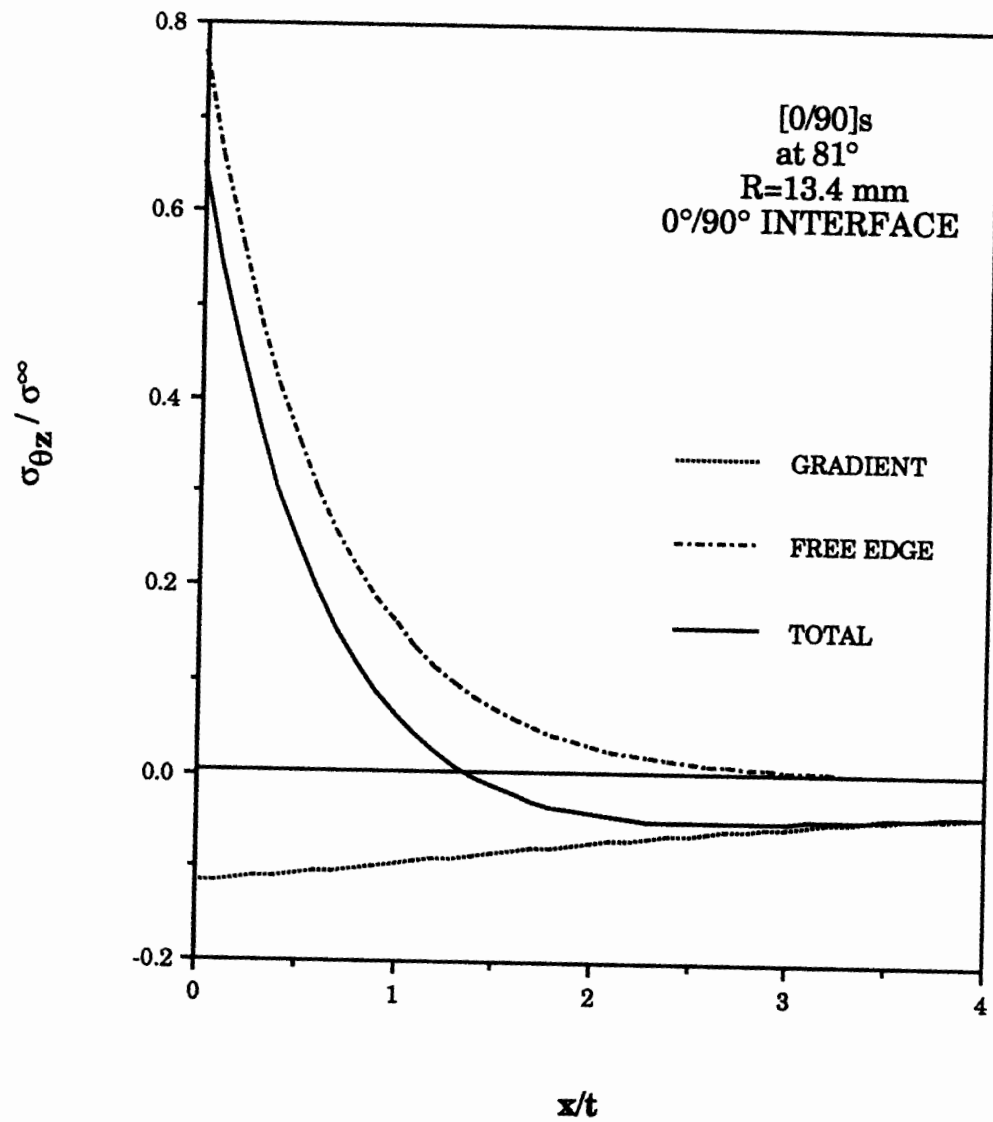


Figure 6.2 Components of $\sigma_{\theta z}$ at 0°/90° interface for [0/90]_s laminate with a 26.8 mm hole at 81° from load axis.

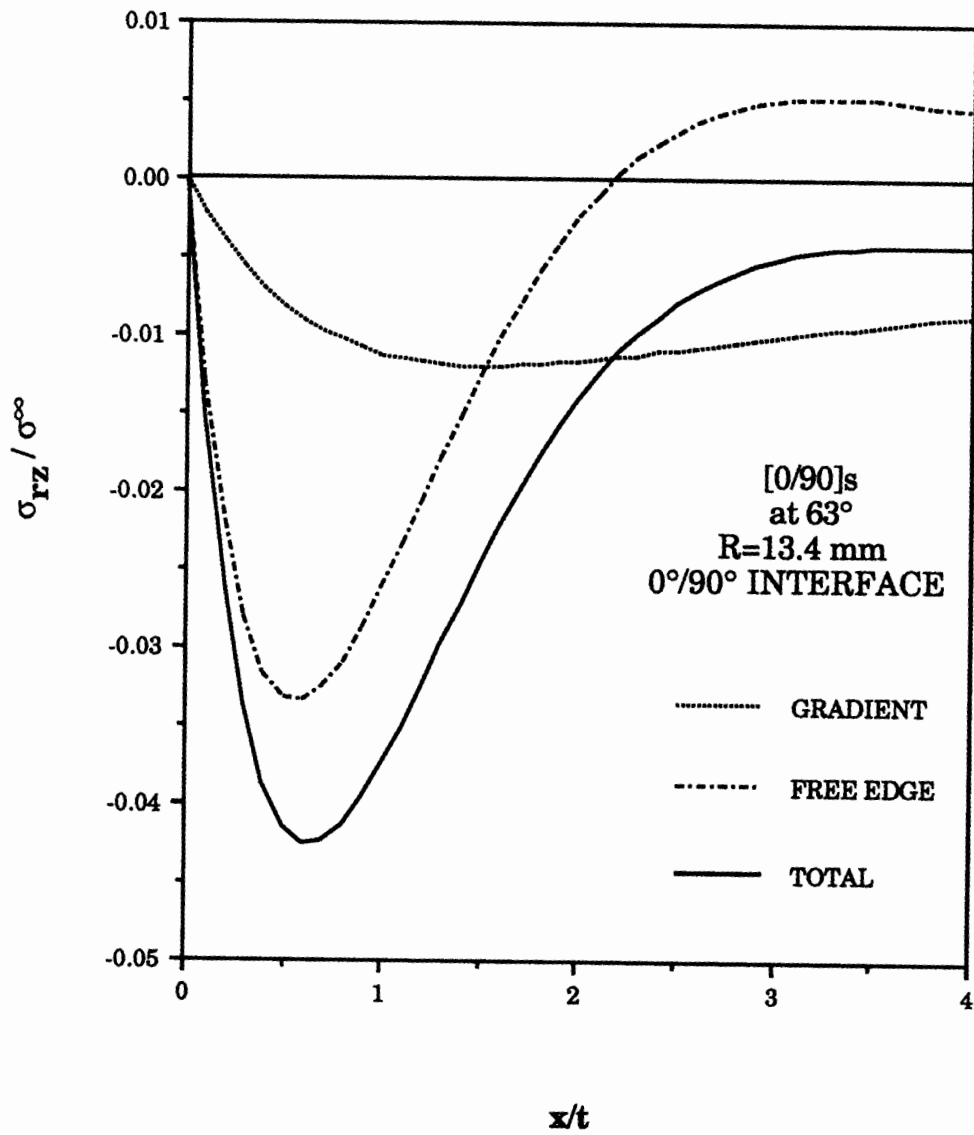


Figure 6.3 Components of σ_{rz} at $0^\circ/90^\circ$ interface for $[0/90]_s$ laminate with a 26.8 mm hole at 63° from load axis.

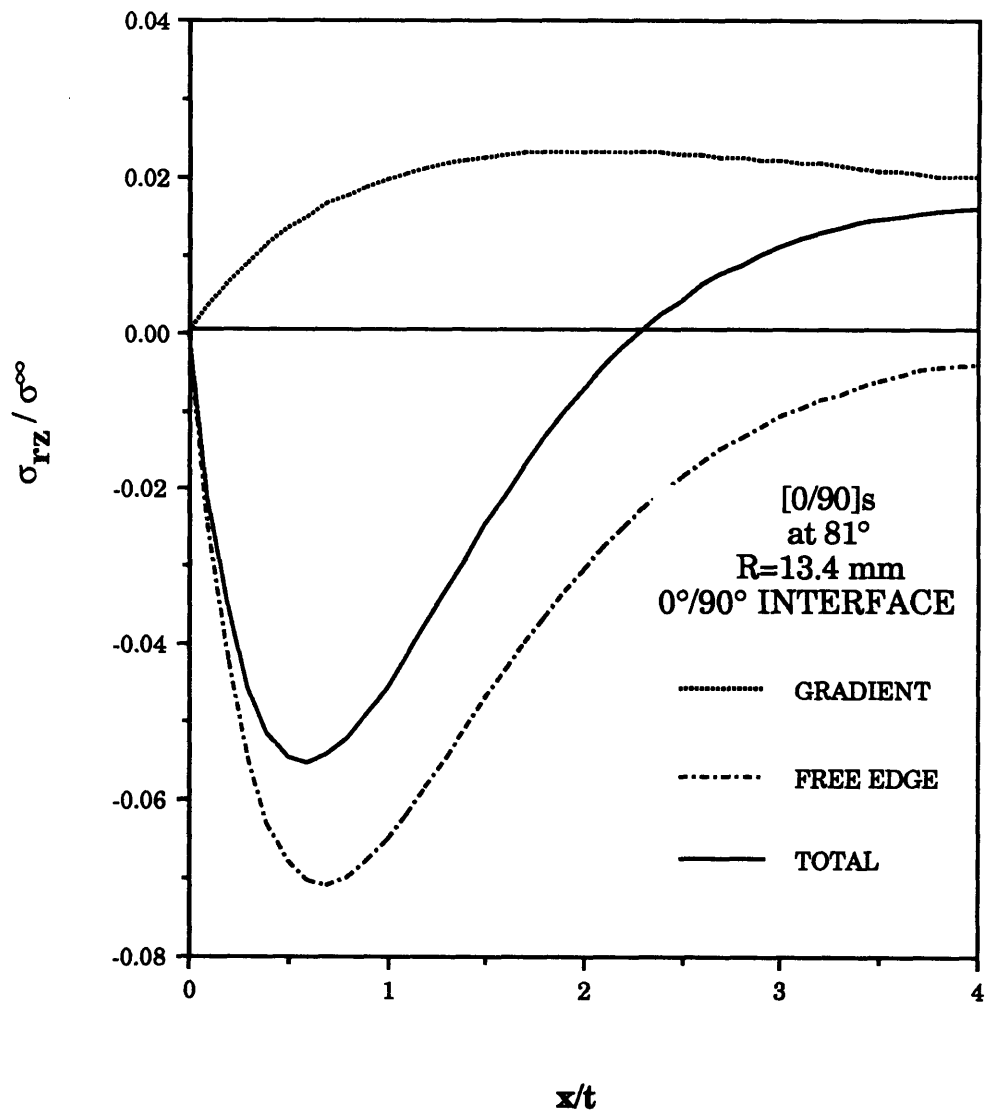


Figure 6.4 Components of σ_{rz} at 0°/90° interface for [0/90]_s laminate with a 26.8 mm hole at 81° from load axis.

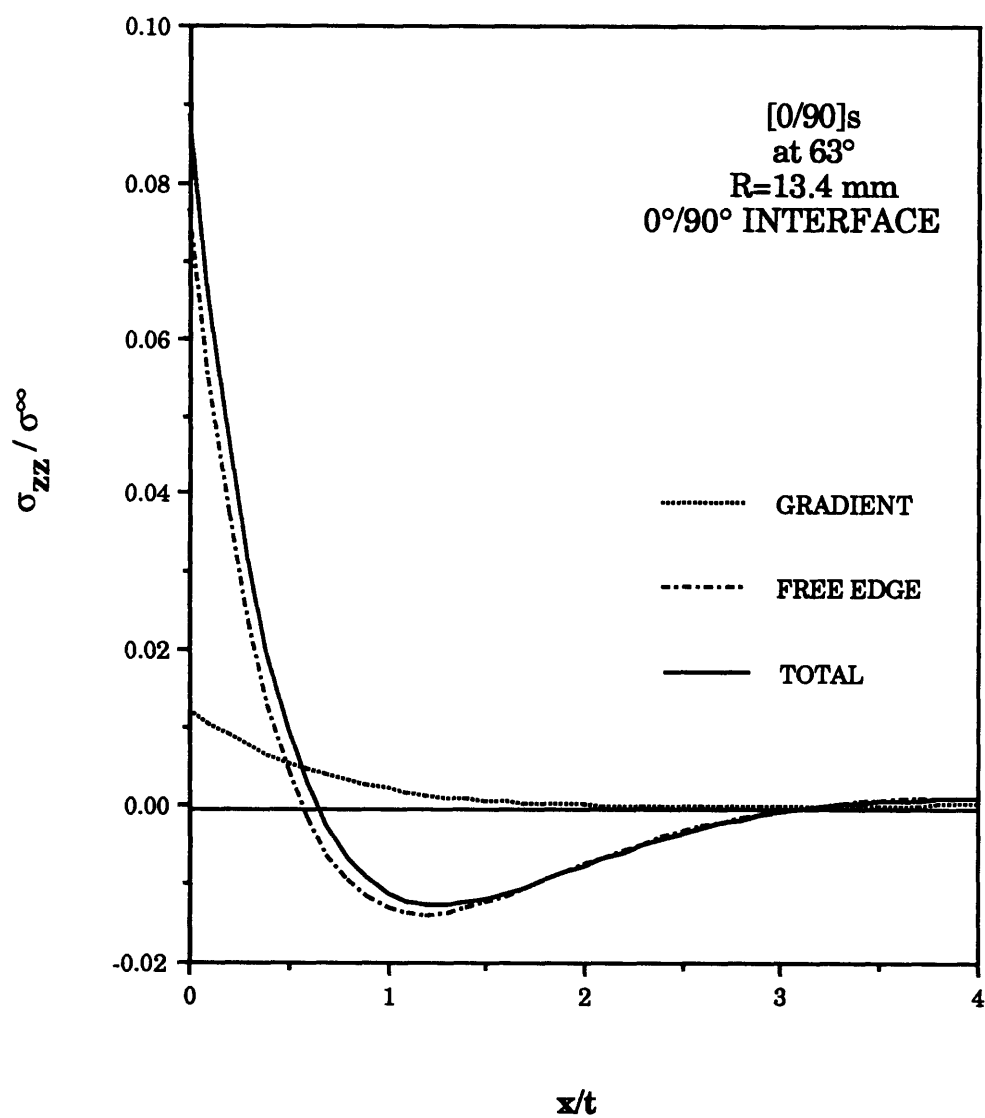


Figure 6.5 Components of σ_{zz} at 0°/90° interface for [0/90]_s laminate with a 26.8 mm hole at 63° from load axis.

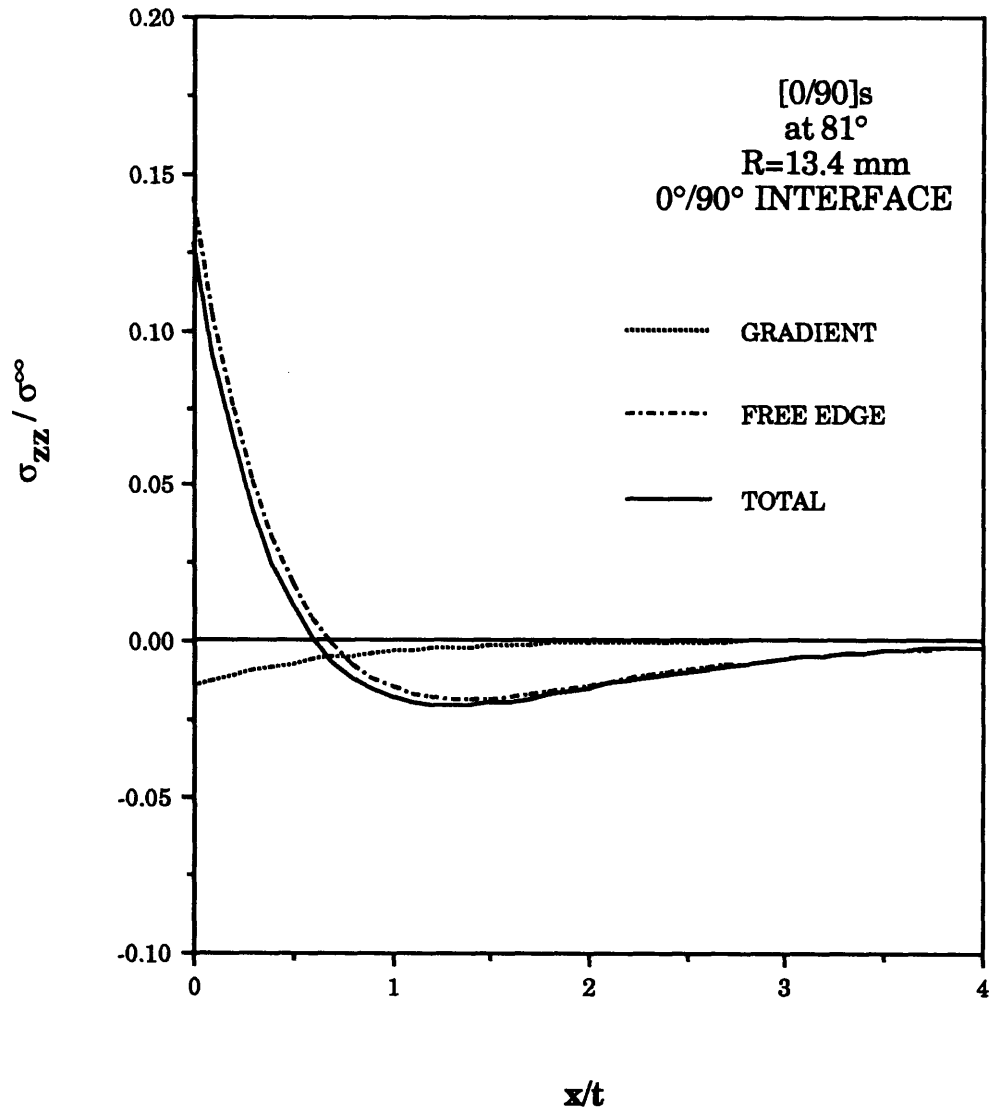


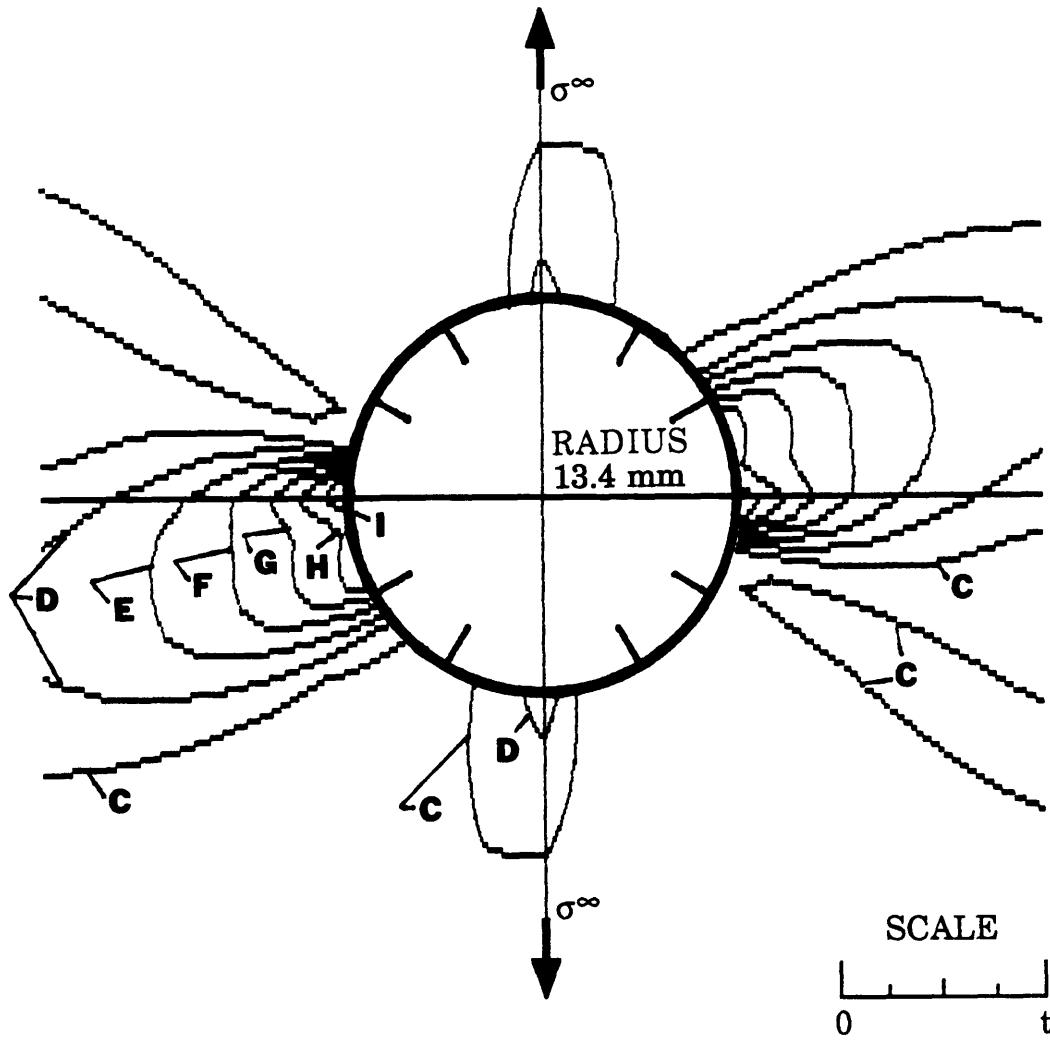
Figure 6.6 Components of σ_{zz} at 0°/90° interface for [0/90]_s laminate with a 26.8 mm hole at 81° from load axis.

6.2 Stress State Characterization

In order to gain some insight into the general nature of the interlaminar stress state around holes, a quasi-isotropic $[\pm 45/0/90]_s$ laminate made of AS4/3501-6 graphite/epoxy subjected to a uniform uniaxial stress is investigated. Three different hole radii are investigated so that geometric effects can be determined. The properties of AS4/3501-6 graphite/epoxy, as utilized, are given in Appendix A.

6.2.1 General Characteristics

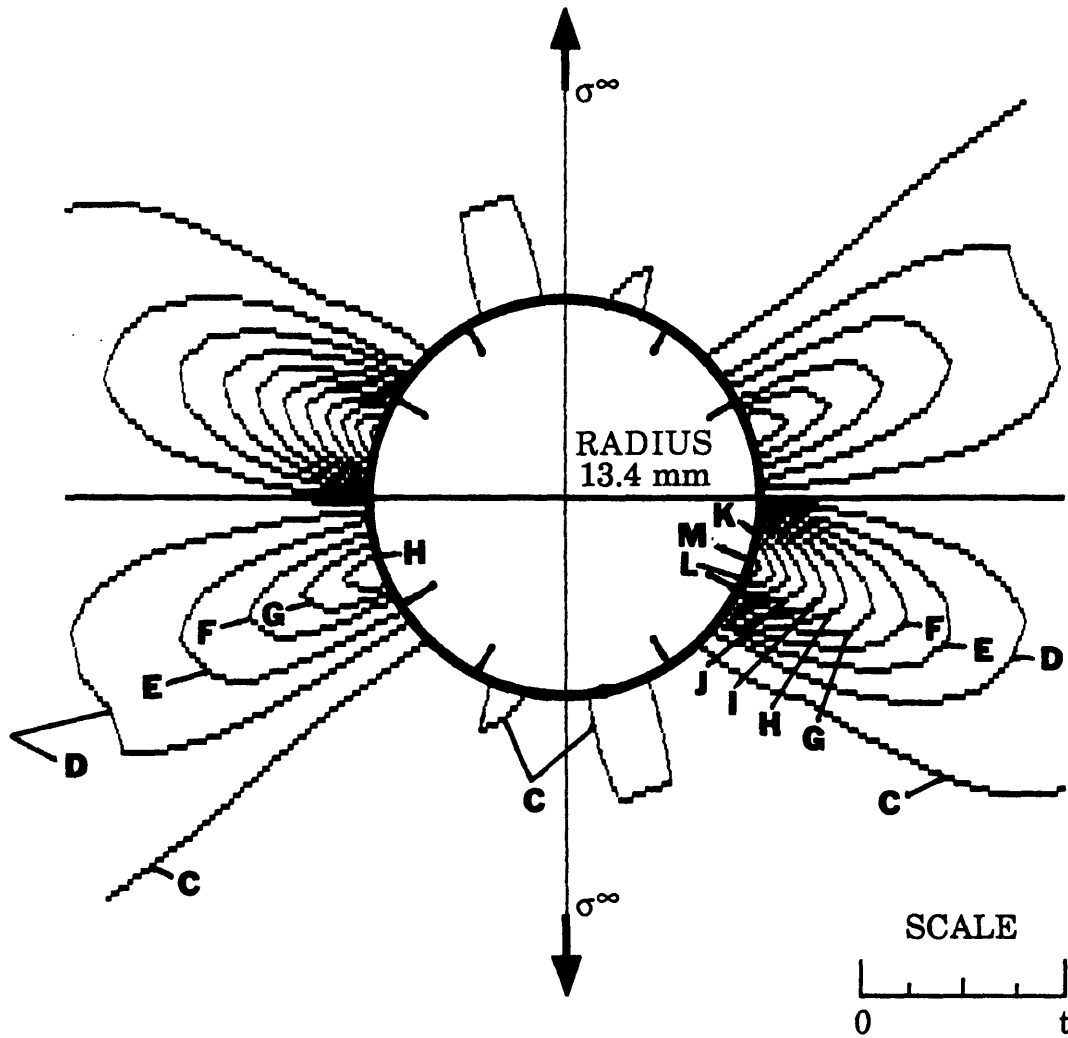
First, in order to gain a general feel for the interlaminar stress state, contour plots at each of the different interfaces for different stresses are provided. For the initial characterization of the stress state, a ply thickness to hole radius ratio of 0.01 is selected. This corresponds to a hole with a 26.8 mm diameter for the material being used. In Figures 6.7 through 6.9, the resultant shear contours are shown at the $+45^\circ/-45^\circ$ interface, the $-45^\circ/0^\circ$ interface, and the $0^\circ/90^\circ$ interface, respectively. The resultant shear is defined by the magnitude of the vector addition of the interlaminar shear stresses $\sigma_{\theta z}$ and σ_{rz} . This will be shown later to be an important quantity for delamination initiation prediction. Because these stresses decay very rapidly, it would be impractical to show these stresses on a scale where the hole dimensions were represented at full scale. The entire figure would be represented by a hole with a very narrow strip of laminate surrounding it. Therefore, the holes for the following plots will always be represented as having a radius of one effective ply thickness. This radius has no physical significance whatsoever. However,



$\tau_{IL} / \sigma^\infty$ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

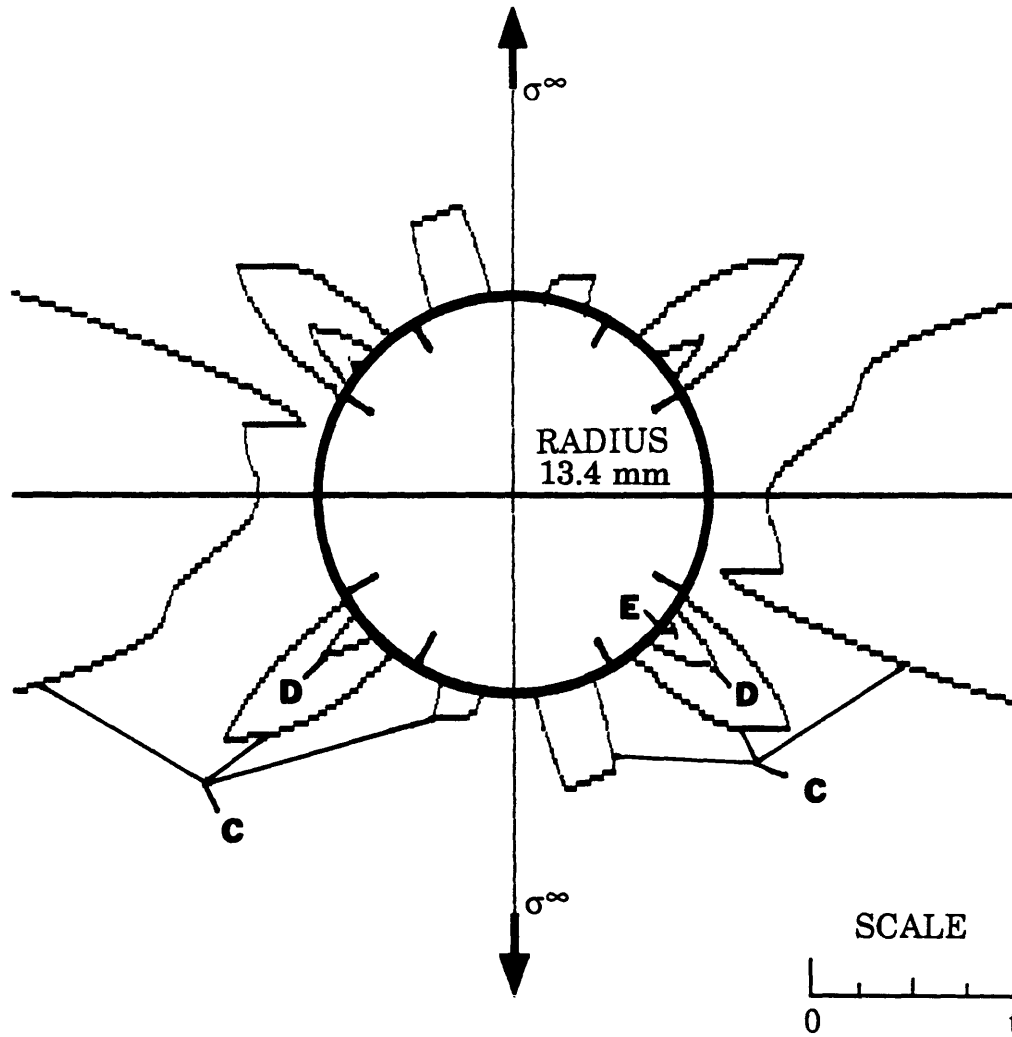
Figure 6.7 Constant resultant shear contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $+45^\circ/-45^\circ$ interface.



τ_{IL}/σ^∞ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

Figure 6.8 Constant resultant shear contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $-45^\circ/0^\circ$ interface.



$\tau_{IL} / \sigma^\infty$ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

Figure 6.9 Constant resultant shear contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface.

all radial distances measured from the hole edge are given by the scale shown in each figure. This method of representing the data has the effect of condensing distances in the θ direction. The hole circumference which is given by $2\pi R$ will be given instead by $2\pi t$. Because of this, gradients in this condensed direction will appear to be greater than they actually are. At the hole boundary, lengths in the θ direction are condensed by a factor of t/R . These contour plots are generated by examining the stresses at 10° intervals around the laminate and interpolating between radials.

In Figure 6.7, it may be seen that at the $+45^\circ/-45^\circ$ interface, the maximum value of the resultant shear, which is 1.4 times the applied far-field stress, occurs at an angle of 90° from the load axis. This corresponds to the location of maximum in-plane tangential stress for this laminate. Because this laminate is quasi-isotropic, the maximum tangential stress is three times the applied far-field stress. Also evident in this figure is the fact that the shear contours are skewed towards the second and fourth quadrants. In this figure, the required symmetry that the solution at θ is identical to the solution at $\theta+\pi$ is evident. This symmetry is required because a rotation of the coordinate system by π produces an equivalent problem in both laminate loading and laminate layup.

The resultant shear contours at the $-45^\circ/0^\circ$ interface are shown in Figure 6.8. These contours show a much steeper gradient as well as a greater value of the maximum stress. The value of this maximum stress is 2.4 times the applied far-field stress and occurs at an angular location 70° from the load axis. Again, only the $\theta+\pi$ symmetry is present.

In Figure 6.9, the resultant shear contours at the $0^\circ/90^\circ$ interface are shown to be much smaller than was the case for the previous two interfaces.

The maximum value here is 0.8 times the far-field stress and occurs at 50° from the load axis. In this figure, a mirror symmetry about the axes can be seen to be developing. The shear resultants at the midplane are identically equal to zero, and are, therefore, not shown.

A summary of the interlaminar shear stress at the free edge as a function of angular location around the hole is shown in Figure 6.10. It must be emphasized that, mathematically, a singularity exists for this stress at the free edge, so what is important in this figure is the relative magnitudes of this stress at different angular locations, and not the absolute magnitude of the stress itself. Noteworthy in this figure is the development of an anti-symmetry about the 90° location except in the case of the $+45^\circ/-45^\circ$ interface where no such symmetry exists. However, moving deeper into the laminate to the $-45^\circ/0^\circ$ interface, a symmetry can be seen to begin developing. This symmetry is even more pronounced at the $0^\circ/90^\circ$ interface. Also interesting here is the virtual coincidence of the curves for the $-45^\circ/0^\circ$ interface and the $0^\circ/90^\circ$ interface for angles from the load axis less than approximately 45° .

The interlaminar normal stress σ_{zz} is next examined. Contours of this stress at each of the four interfaces are given in Figures 6.11 through 6.14. In Figure 6.11, the interlaminar normal stress at the $+45^\circ/-45^\circ$ is seen to be quite small in comparison to the interlaminar shear stresses. The maximum value of this stress is only 0.3 times the applied far-field stress and occurs at an angle which is 70° from the load axis. In Figure 6.12, the interlaminar normal stress at the $-45^\circ/0^\circ$ interface has a maximum which now coincides with the location of maximum tangential stress, a location 90° from the load axis. The maximum value of this stress now has a value which is 0.4 times the applied far-field stress.

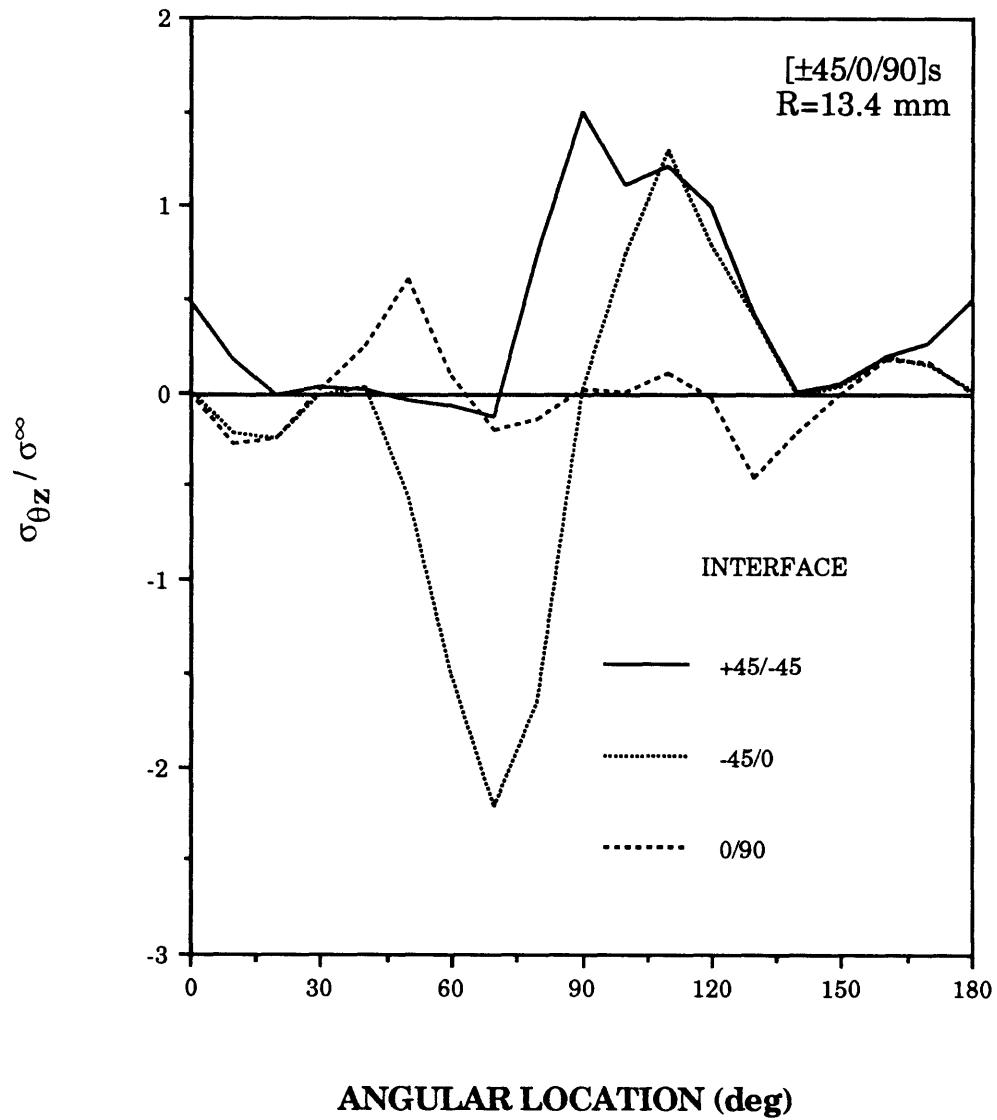
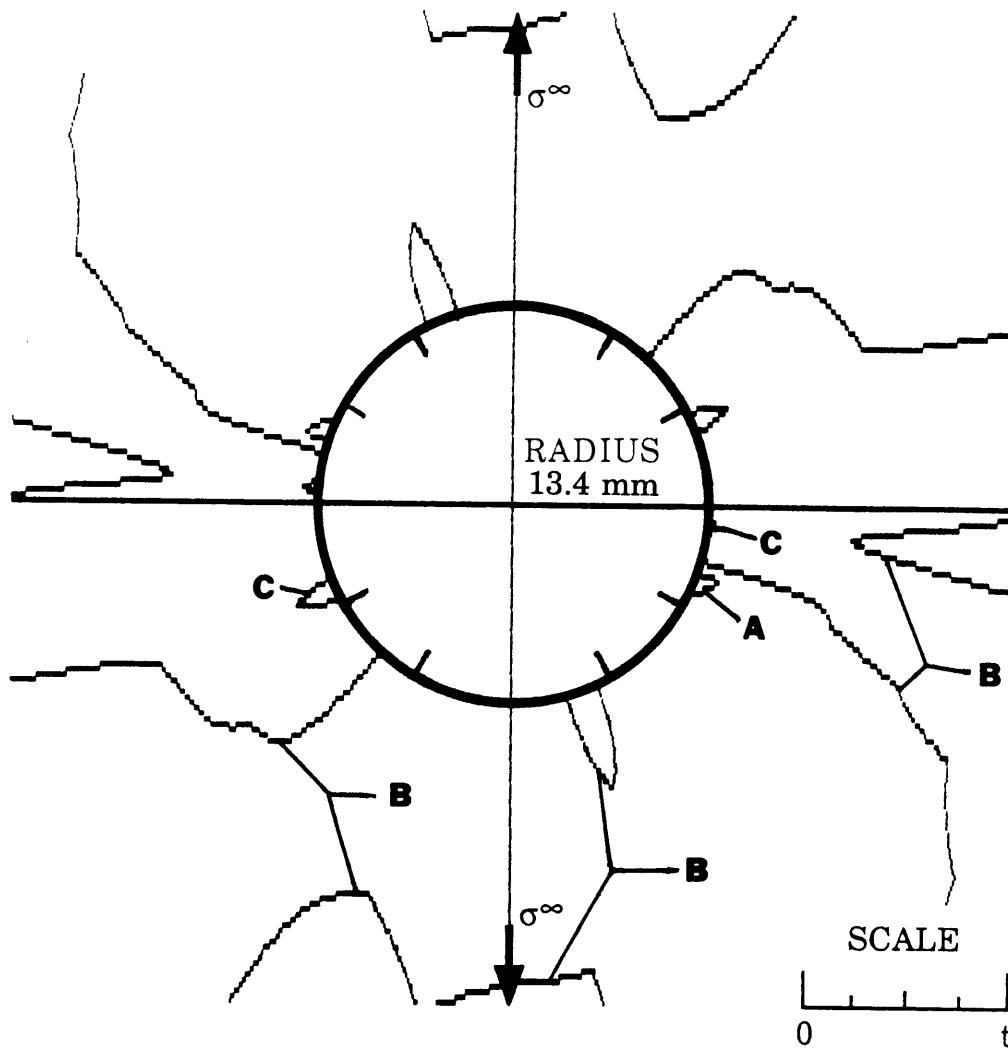


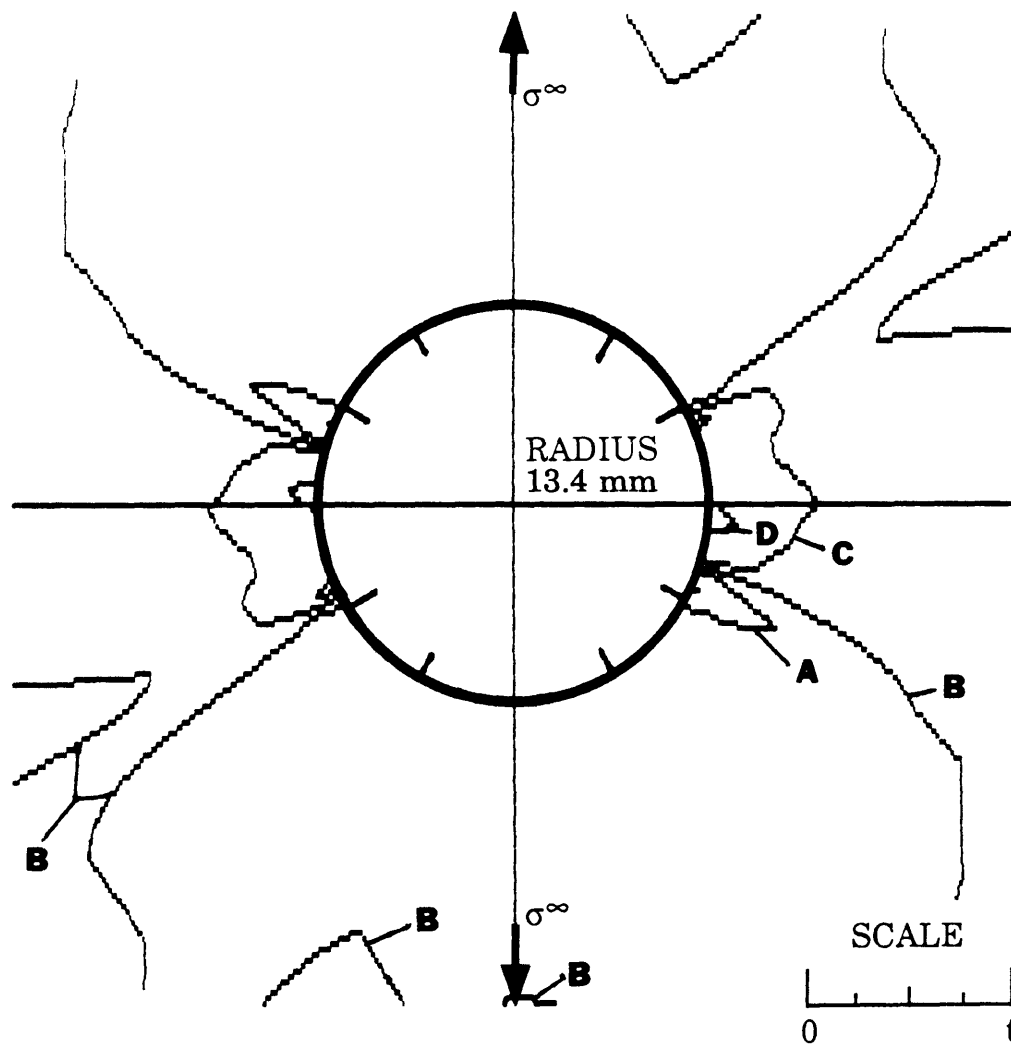
Figure 6.10 Variation of σ_{θ_z} with angular location for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.



$\sigma_{zz} / \sigma^{\infty}$ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

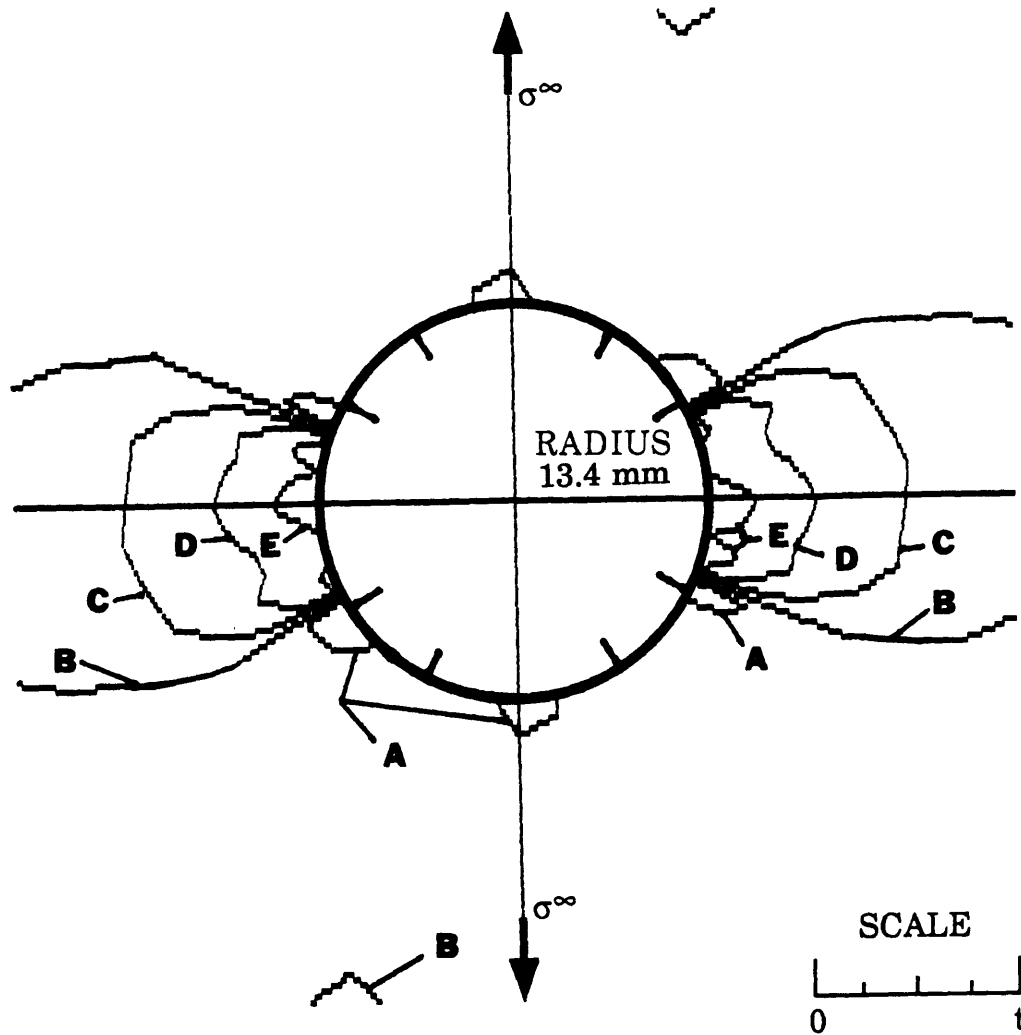
Figure 6.11 Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $+45^\circ/-45^\circ$ interface.



$\sigma_{zz} / \sigma^\infty$ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

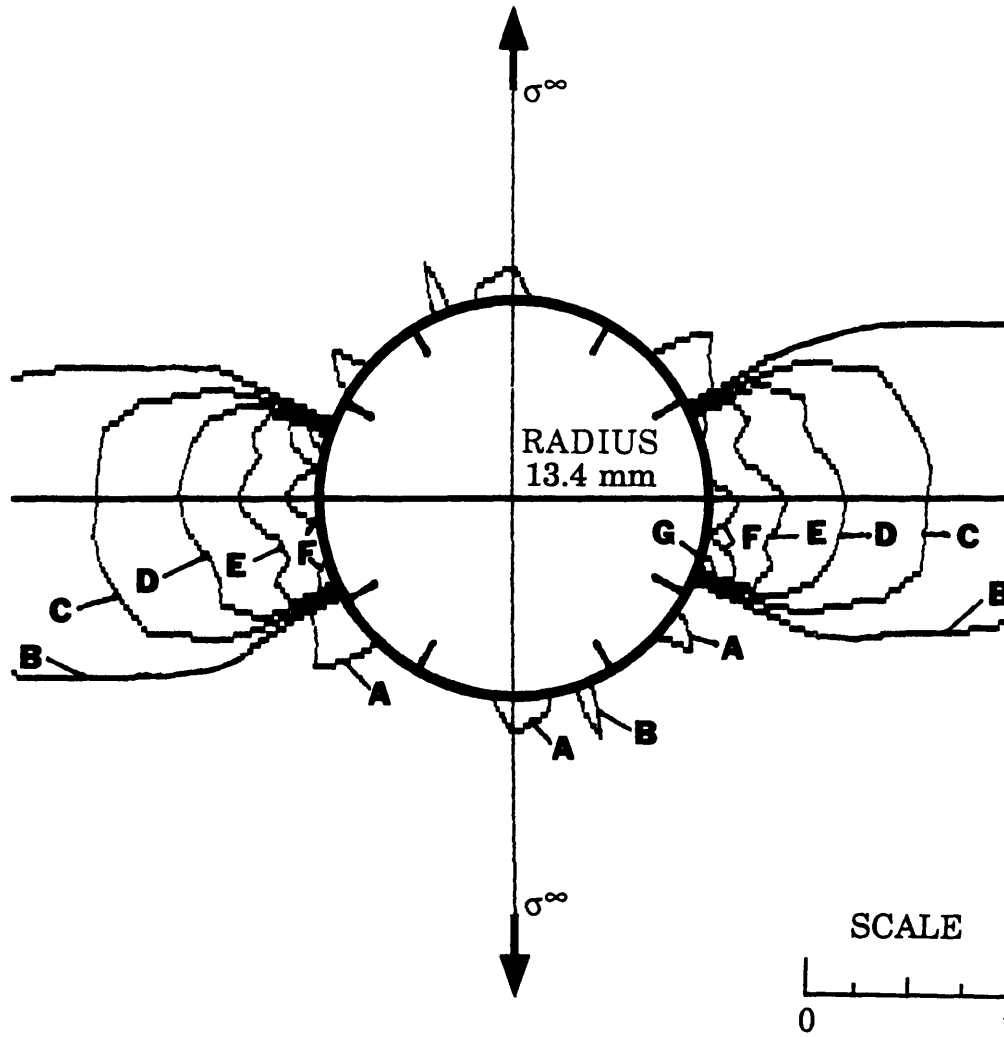
Figure 6.12 Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $-45^\circ/0^\circ$ interface.



$\sigma_{zz} / \sigma^\infty$ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

Figure 6.13 Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface.



$\sigma_{zz} / \sigma^\infty$ CONTOUR LEVELS

A: -0.2	F: 0.8	K: 1.8
B: 0.0	G: 1.0	L: 2.0
C: 0.2	H: 1.2	M: 2.2
D: 0.4	I: 1.4	
E: 0.6	J: 1.6	

Figure 6.14 Constant interlaminar normal stress contours for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole at the midplane.

At the $0^\circ/90^\circ$ interface, a more significant interlaminar normal stress state is seen to develop, as is depicted in Figure 6.13. The maximum of the stress has now doubled to 0.8 times the far-field stress, and the location of this maximum has remained at 90° from the load axis. Also, a near symmetry about the laminate axis is seen to be developing. At the midplane of the laminate, the interlaminar normal stress reaches an overall maximum value which is 1.2 times the applied stress as can be seen in Figure 6.14. This maximum has shifted to a location which is 70° from the load axis. The gradients in the θ -direction in this region about 70° are quite high.

The value of the interlaminar normal stress at each of the interfaces, as a function of angular location, is depicted in Figure 6.15. Unlike the case of the interlaminar shear stress, the variations of the interlaminar normal stresses with angular location, at each of the different interfaces, are quite similar. The majority of the change from interface to interface is merely the increasing of the magnitude of the stress as the distance to the midplane decreases. Also dissimilar to the case of the interlaminar shear stress is the fact that no additional symmetries appear to develop for this case.

These studies of the interlaminar stresses at each of the interfaces, and for all possible angular locations aids in the location of areas of particular interest for further study. In the case of the interlaminar shear stress, this laminate displayed a maximum at the second interface at a location 70° from the load axis. Plots depicting the decay of the stress $\sigma_{\theta z}$ near this location are given in Figure 6.16. In this figure it may be seen that the stresses at 50° , 60° , 70° , 80° and 90° all die away in approximately two ply thicknesses. However, at 50° , this stress has obtained a small negative value, and at 80° and 90° it has obtained similarly small positive values. These nonzero values represent

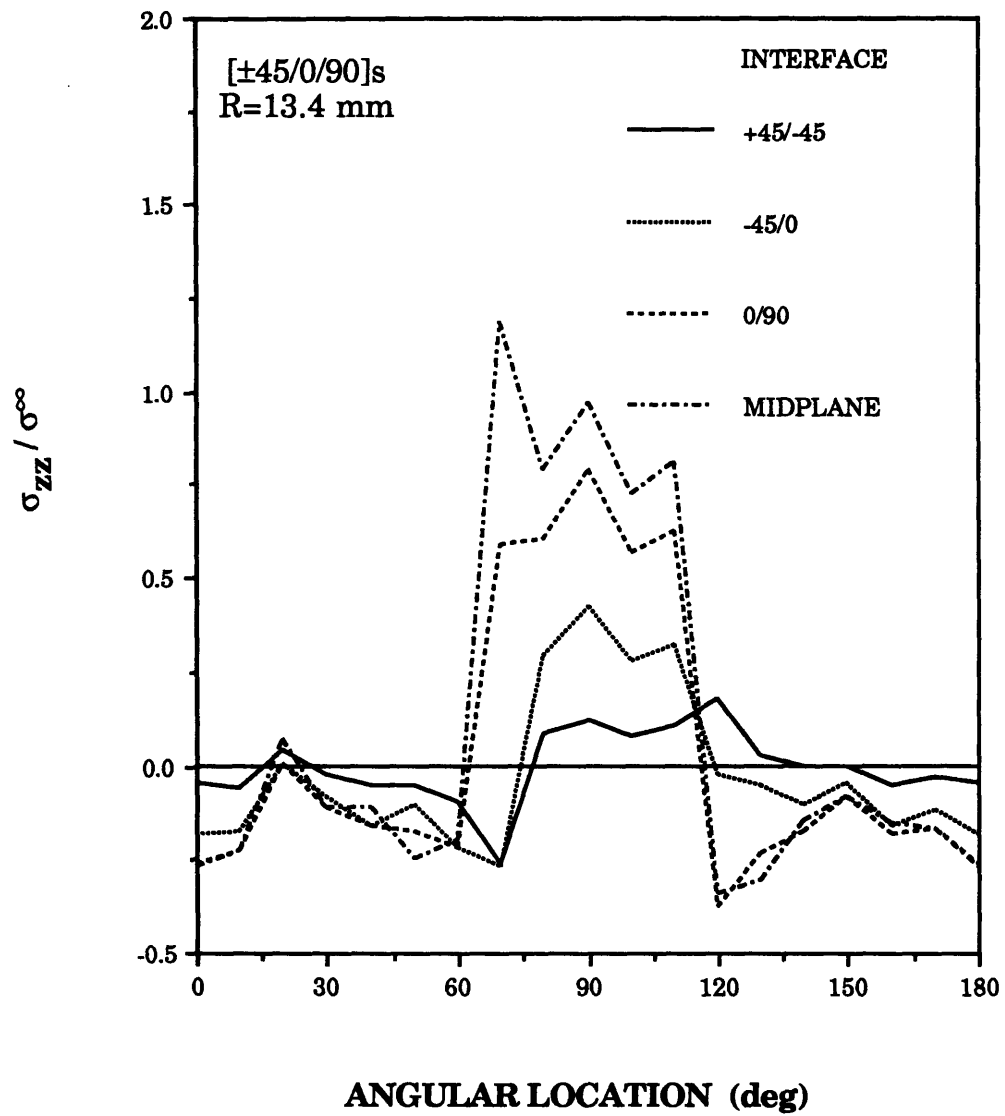


Figure 6.15 Variation of σ_{zz} with angular location for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.

regions where the solution due to in-plane gradients in the stress field are still measurable. These stresses eventually decay to zero. However, because of the much slower decay rates, these stresses maintain these small values for eight to twelve effective ply thicknesses from the hole edge. It is not practical to show this phenomenon on these plots. As was shown in the section on the contributions of the two components, the two contributors can have vastly different decay rates, and this fact is evidenced here.

A similar groups of plots has been made for the interlaminar normal stress at the midplane. Here, as shown in Figure 6.17, five angular locations centered about a position 90° from the load axis, have been chosen. It is seen that the behavior at all angles, except for 70° , are essentially identical. However, at 70° , the location of the maximum interlaminar normal stress, a much steeper gradient occurs. Also, the overall decay rates depicted in this figure are much slower than was the case for the interlaminar shear stress.

Another effect which may be important is the effect of the companion solution on the in-plane stresses in the laminate. Because of the assumed form of the in-plane stresses, the through-the-thickness integral of the in-plane stresses remains unchanged with the addition of the companion problem solution. However, the distribution of the in-plane stresses does change. This effect is shown for the in-plane tangential stress, $\sigma_{\theta\theta}$, at a location 90° from the laminate axis at the hole edge in Figure 6.18. As can be seen in this figure, the net effect on the stresses is minimal, except for the case of the 90° ply. The stress in this ply has approximately doubled. This may have important ramifications in failure analyses because of the fact that this ply is relatively weak in this load direction.

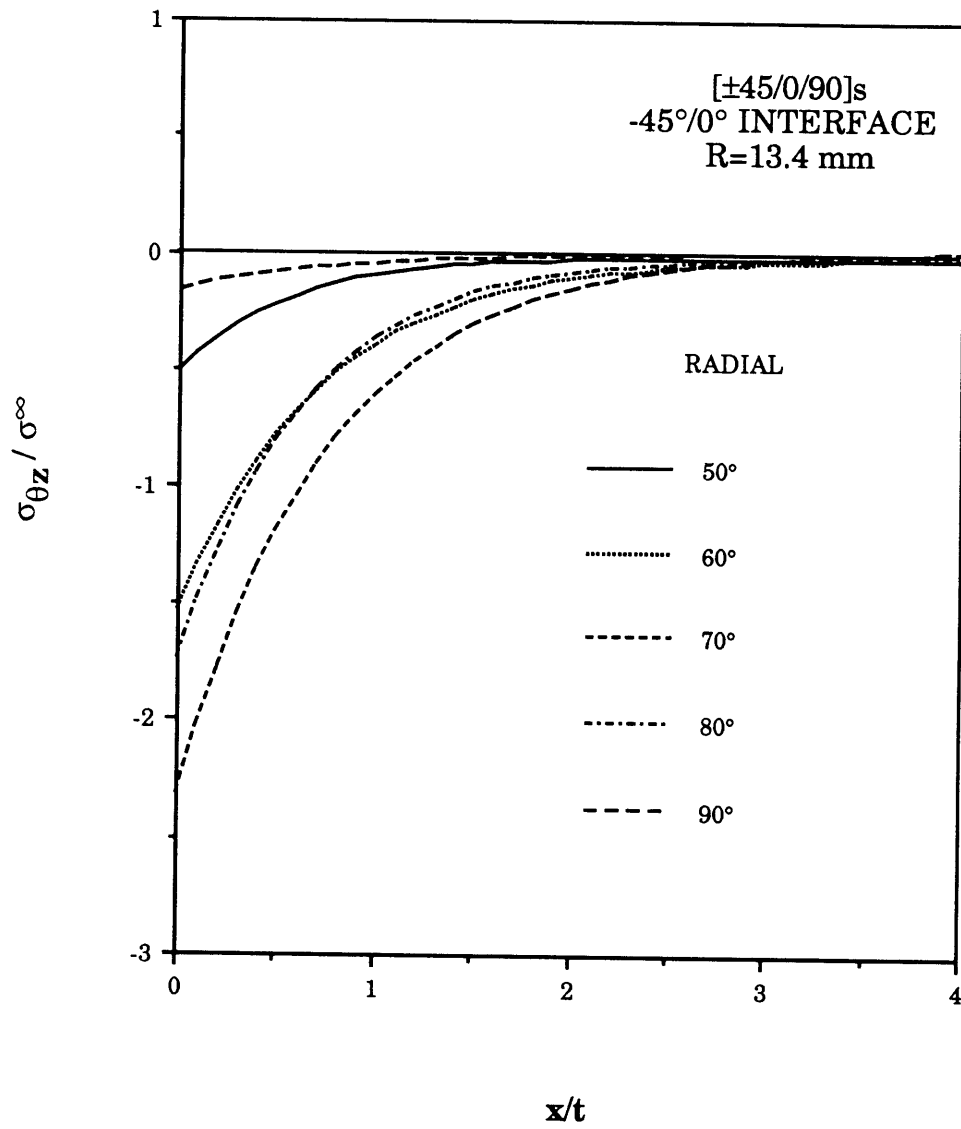


Figure 6.16 Plot of $\sigma_{\theta z}$ at various radial positions at $-45^\circ/0^\circ$ interface for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.

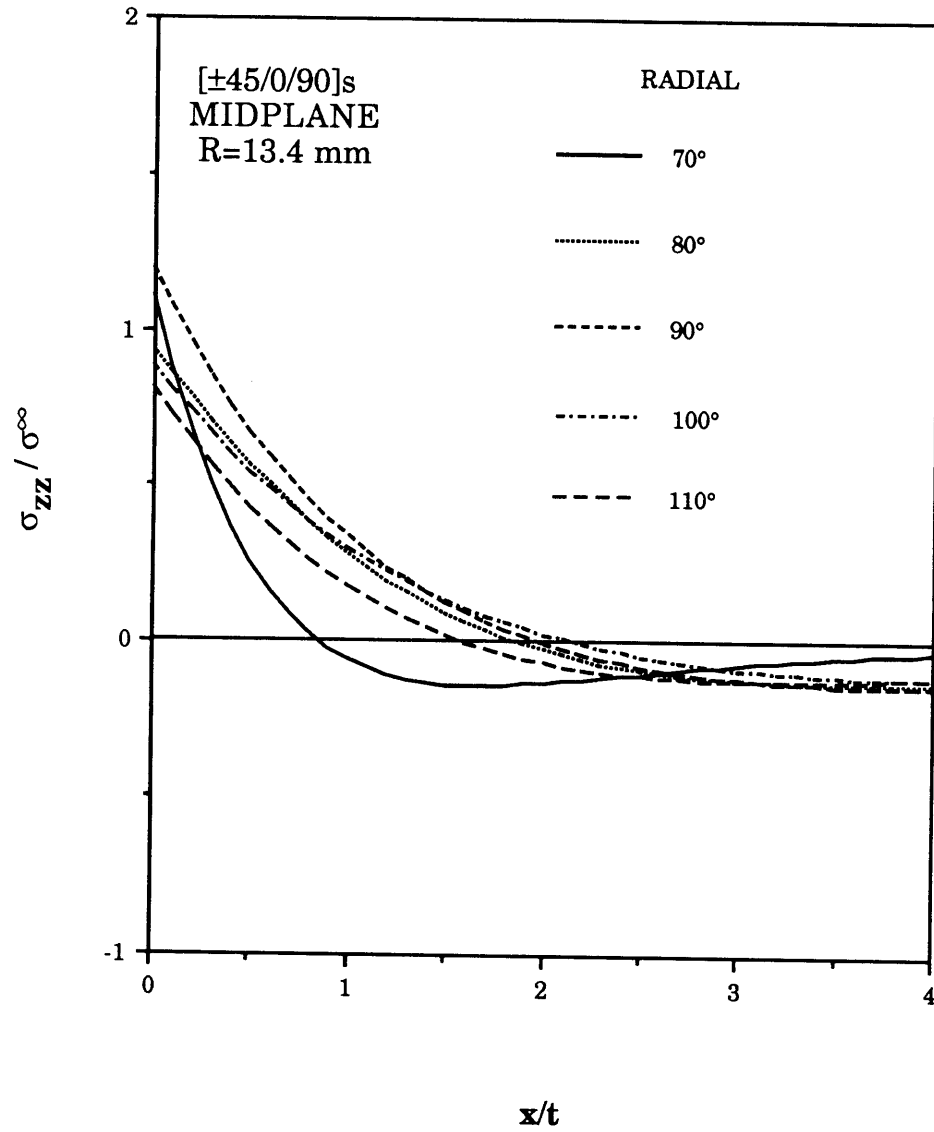


Figure 6.17 Plot of σ_{zz} at various radial positions at the midplane for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.

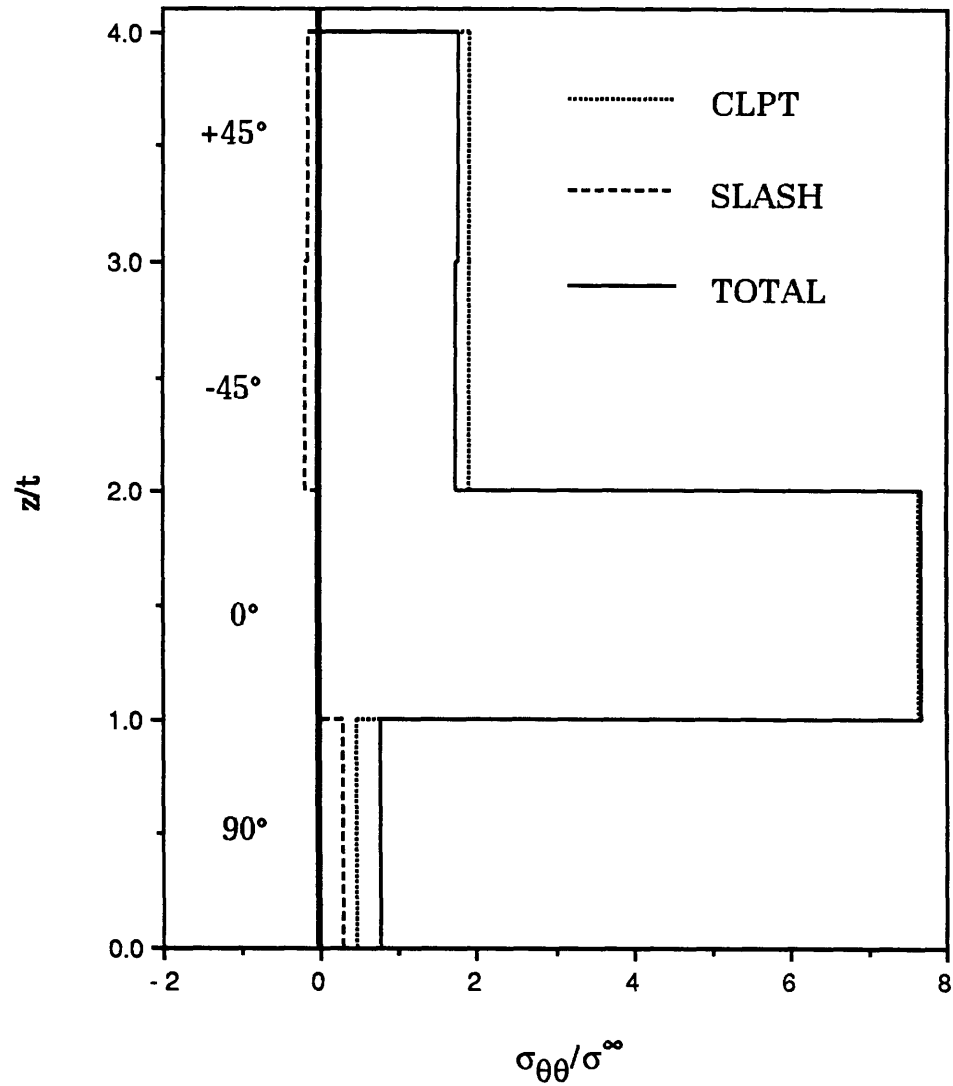


Figure 6.18 Plot of $\sigma_{\theta\theta}$, at a position on the 90° radial corresponding to the hole edge, as a function of the through-the-thickness location for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with a 26.8 mm hole.

6.2.2 Effect of Geometry

Making use of the results from the previous section the effects of changing the ratio of the ply thickness to the hole radius was examined. To compare with the work done for a ratio of ply thickness to hole radius of 0.01, cases for hole diameters of 5.36 mm and 134.0 mm are also considered. These result in t/R values of 0.05 and 0.002, respectively.

The first comparison, which is made, examines the variation of the interlaminar shear stress $\sigma_{\theta z}$ at the hole boundary as a function of angular location. The plots showing this stress at the $+45^\circ/-45^\circ$ interface for the three values of t/R used are given in Figure 6.19. As can be seen, the general trend as the ratio of t/R increases is for the magnitudes of this stress at the hole boundary to increase. A plot showing the corresponding curves for the $-45^\circ/0^\circ$ interface is shown in Figure 6.20 and for the $0^\circ/90^\circ$ interface in Figure 6.21. In all cases, the general shapes of these curves remains the same, with the magnitude decreasing. An interesting observation which can be made here is that the decrease in the relative magnitude of the curve for a t/R of 0.010 from the curve for a t/R of 0.002 is much less than the decrease experienced in the curve for a t/R of 0.050 as compared to the curve for a t/R of 0.010. This suggests that no simple linear relation exists between the magnitude of these curves and the ply thickness to hole radius ratio, both because of the different nature of the two contributors to the solution, and due to the nature of the exponential solution for the contribution from the free edge solution.

The magnitude of the stress at the free edge is only a rough indication of the stress state. In order to better describe this state, the decay shape of these stresses as a function of the distance from the hole edge must also be

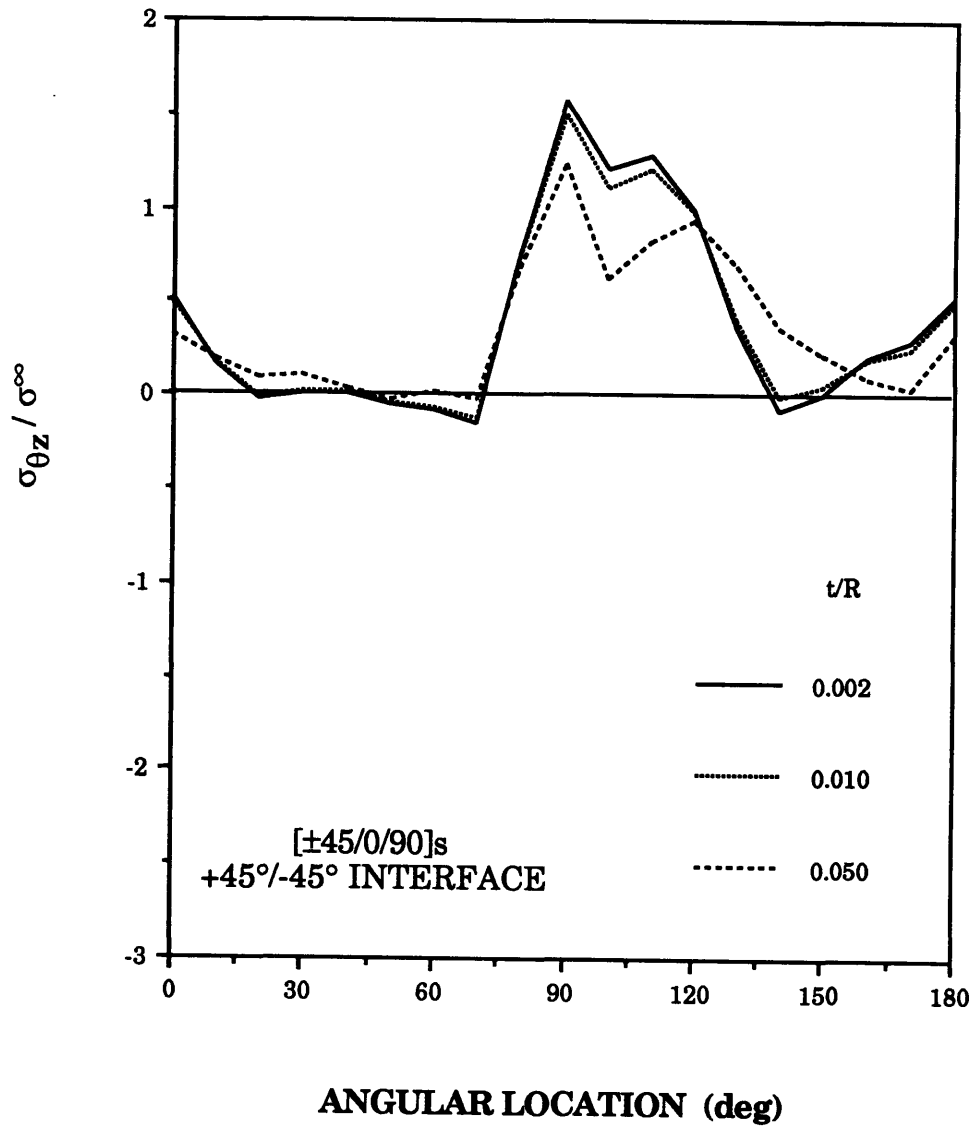


Figure 6.19 Variation of $\sigma_{\theta z}$ at hole edge at $+45^\circ/-45^\circ$ interface for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.

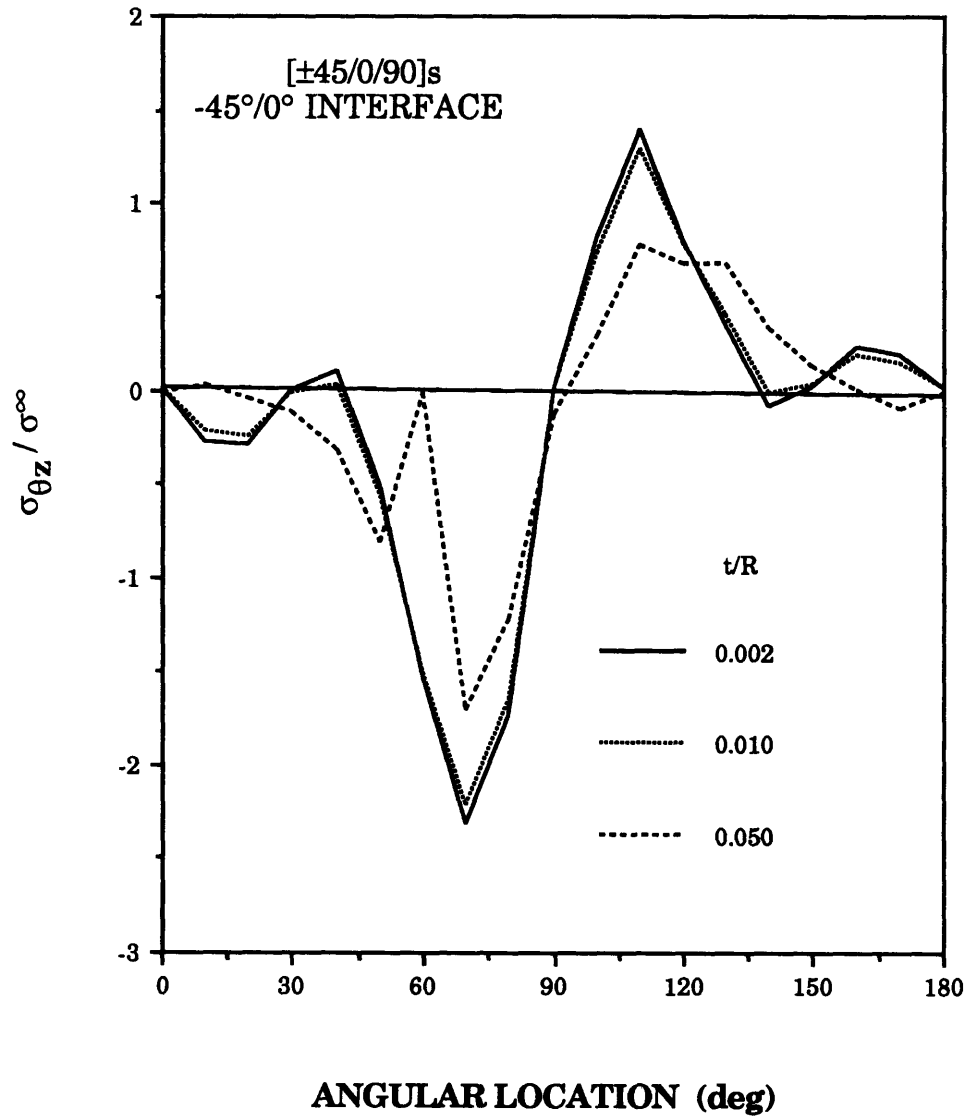


Figure 6.20 Variation of $\sigma_{\theta z}$ at hole edge at $-45^\circ/0^\circ$ interface for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.

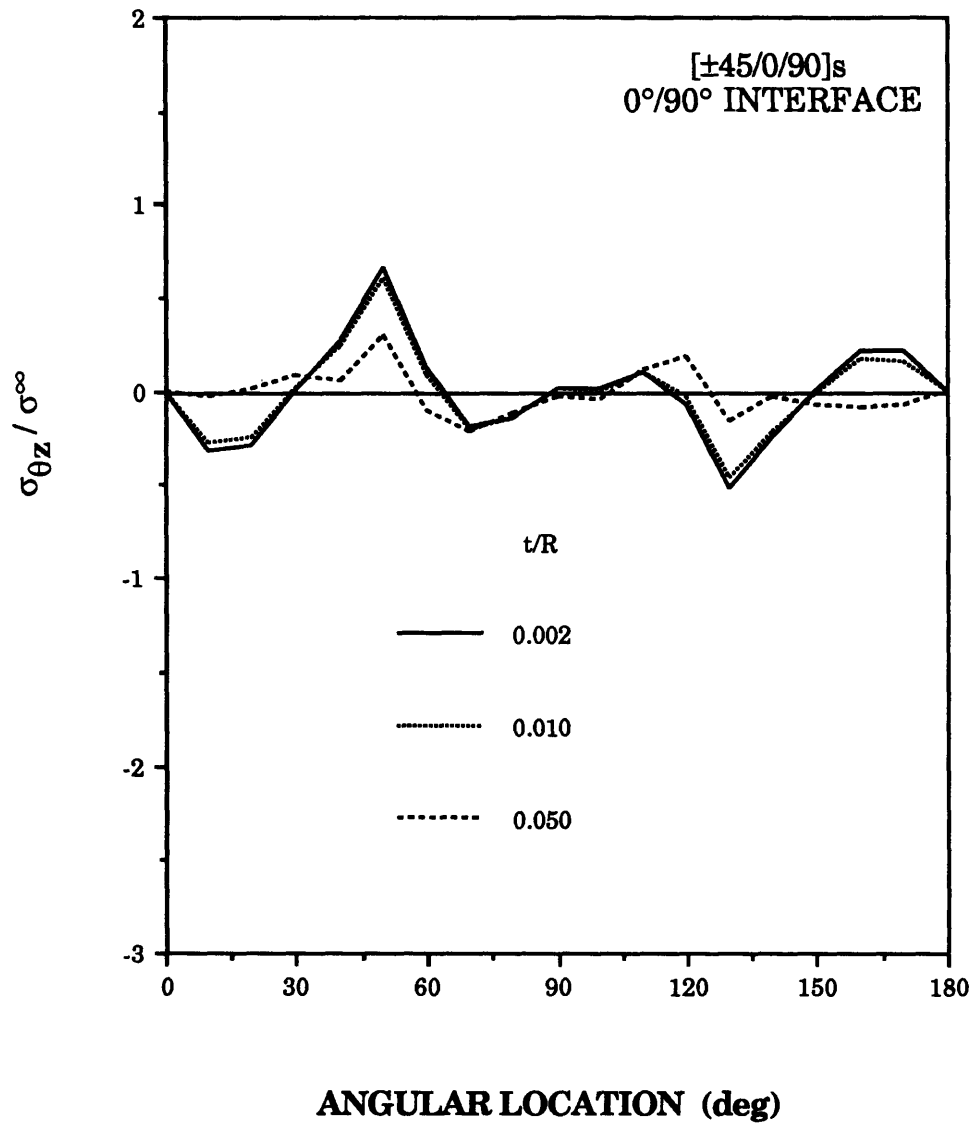


Figure 6.21 Variation of $\sigma_{\theta z}$ at hole edge at $0^\circ/90^\circ$ interface for uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.

considered. A plot showing the variation of this decay shape with the thickness to radius ratio is shown in Figure 6.22 for a angular location which is 50° from the load axis at the $-45^\circ/0^\circ$ interface. This location is unusual because here the stress increases with increasing t/R . Such a situation has been found to arise in regions where the contribution from the in-plane gradient solution dominates the total interlaminar stress field solution. As can be seen in this figure, the decay rate for the largest value of t/R is somewhat faster than for the other two values of this parameter.

A more typical example of the behavior of these stresses is given in Figure 6.23. This figure shows the stresses at the $-45^\circ/0^\circ$ interface at an angle of 70° from the load axis. In the previous section this was found to be the location for which this stress is a maximum. In this figure, it is seen that the magnitude of the stresses decrease with increasing t/R . This decrease again becomes more dramatic as t/R becomes smaller.

The interlaminar normal stress, σ_{zz} , shows a behavior similar to that for the interlaminar shear stress $\sigma_{\theta z}$. The variations of the interlaminar normal stress as a function of the angular location at the midplane is shown for the three values of t/R in Figure 6.24. As had been found for the interlaminar shear stress, the magnitude of the interlaminar normal stress at the free edge decreases, in general, with increasing t/R . A notable exception to this rule occurs at a location 70° from the load axis. At this location, the magnitude of the interlaminar normal stress has increased, and, as can be seen in Figure 6.25, the decay rates are similar for the three different hole radii. A more typical plot is shown in Figure 6.26. In this figure, the stresses are given along a radial which is at 90° from the load axis. Here the overall

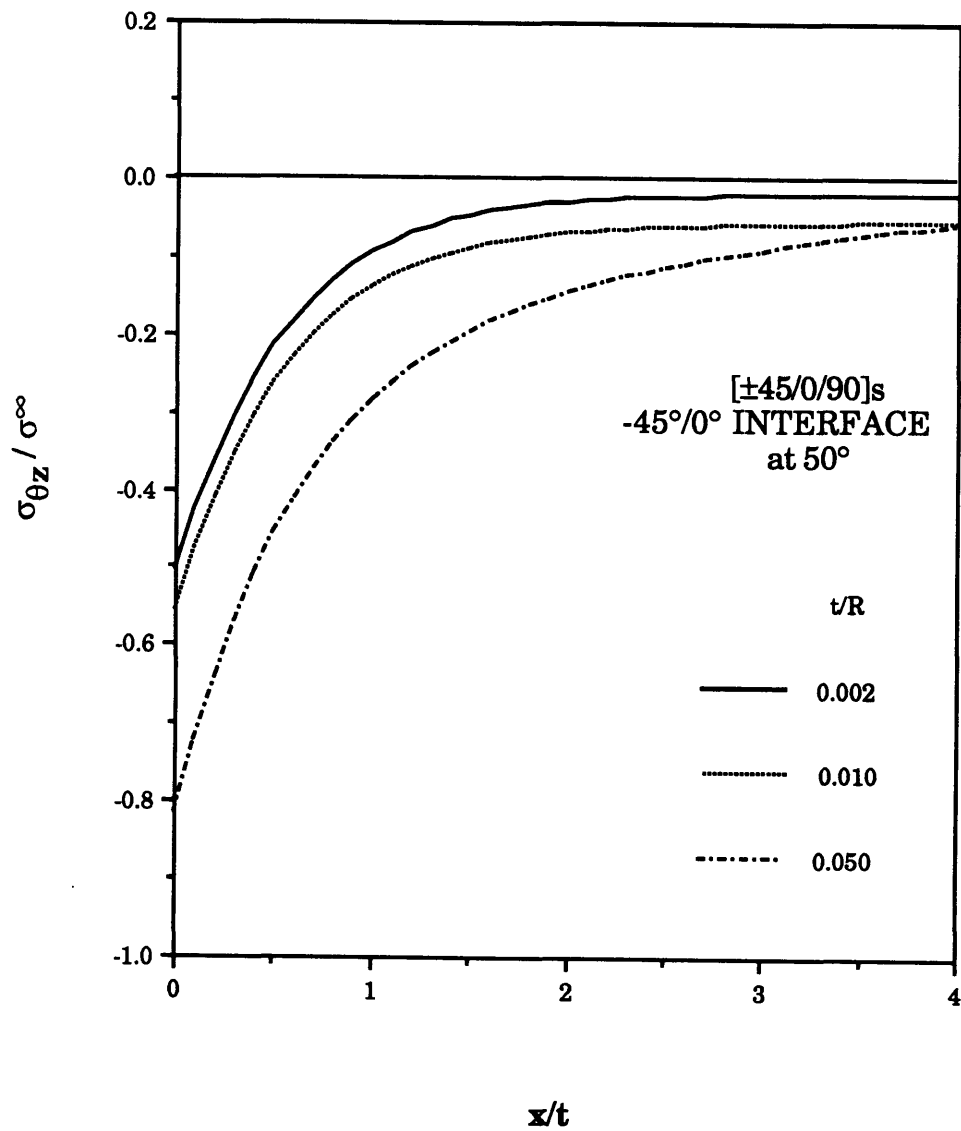


Figure 6.22 Plot of $\sigma_{\theta z}$ along 50° radial at the $-45^\circ/0^\circ$ interface of a uniaxially-loaded $[\pm 45/0/90]_s$ laminate for various hole sizes.

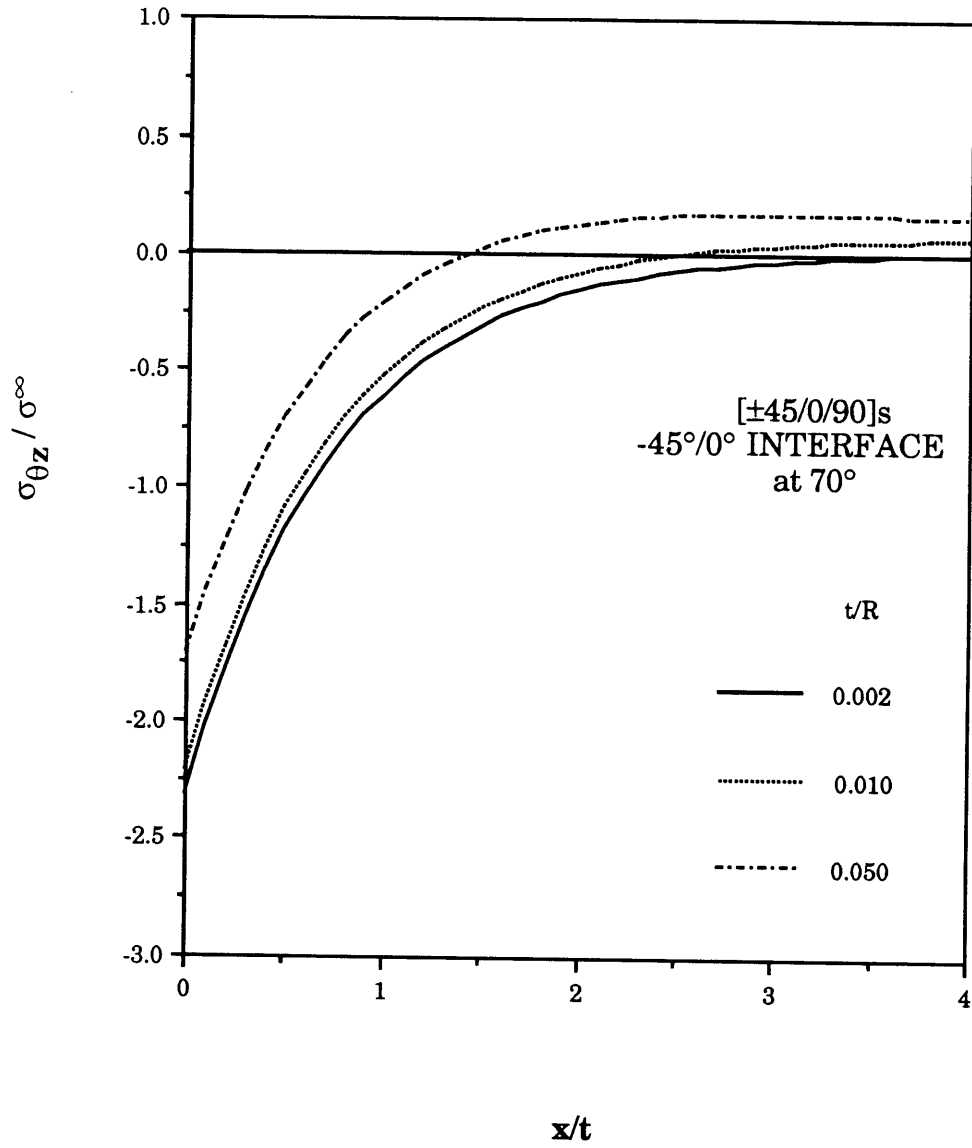


Figure 6.23 Plot of $\sigma_{\theta z}$ along 70° radial at the -45°/0° interface for a $[\pm 45/0/90]_s$ laminate for various hole sizes.

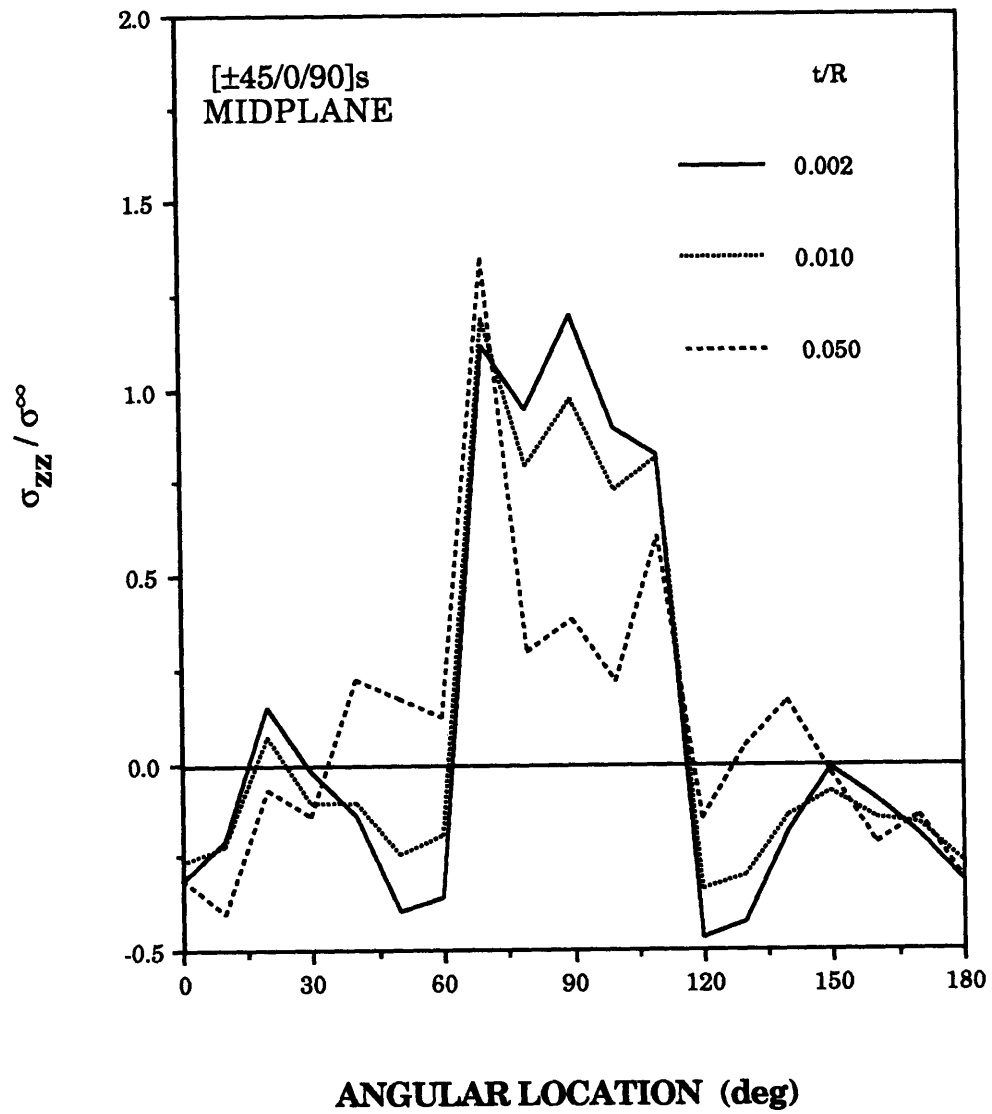


Figure 6.24 Variation of σ_{zz} at the hole edge at the midplane for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate with various hole sizes.

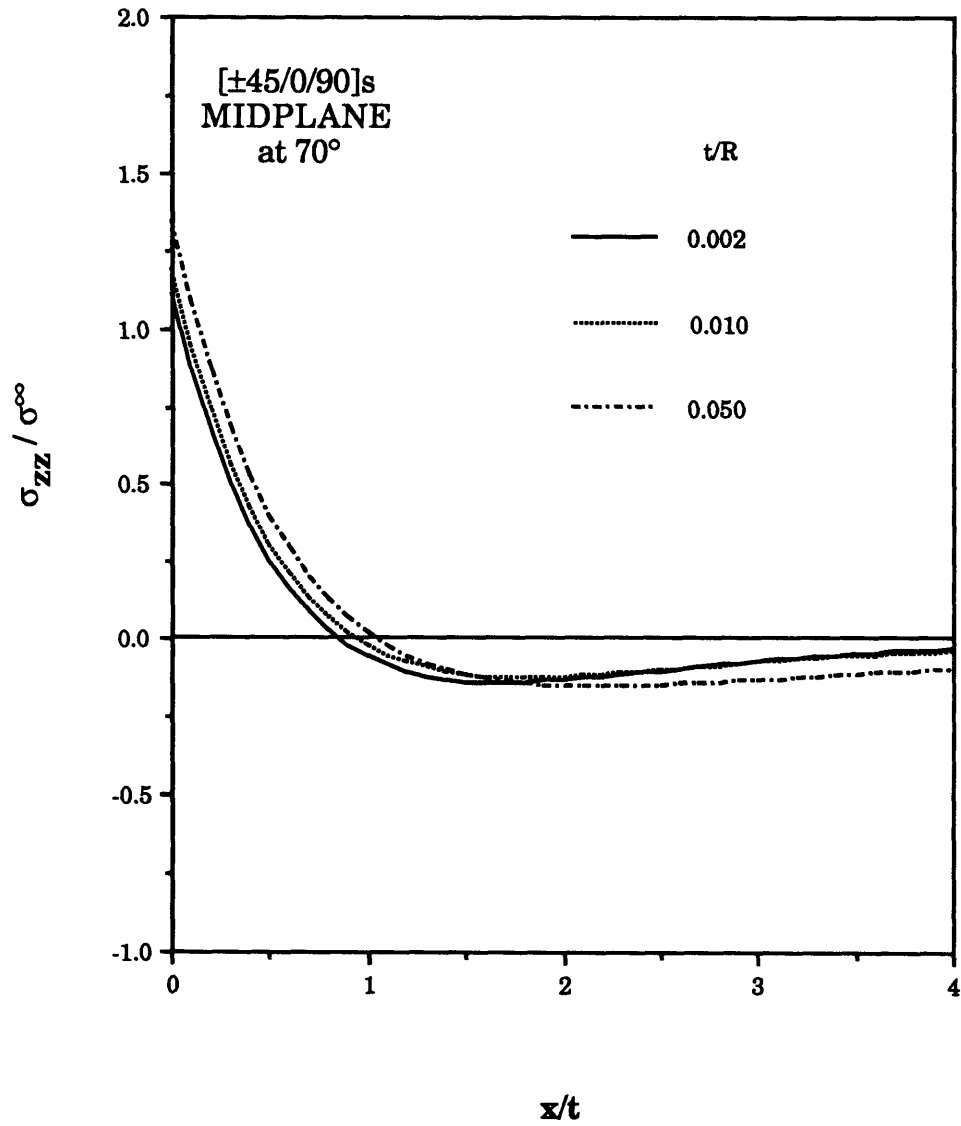


Figure 6.25 Plot of σ_{zz} along 70° radial at the midplane for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate for various hole sizes.

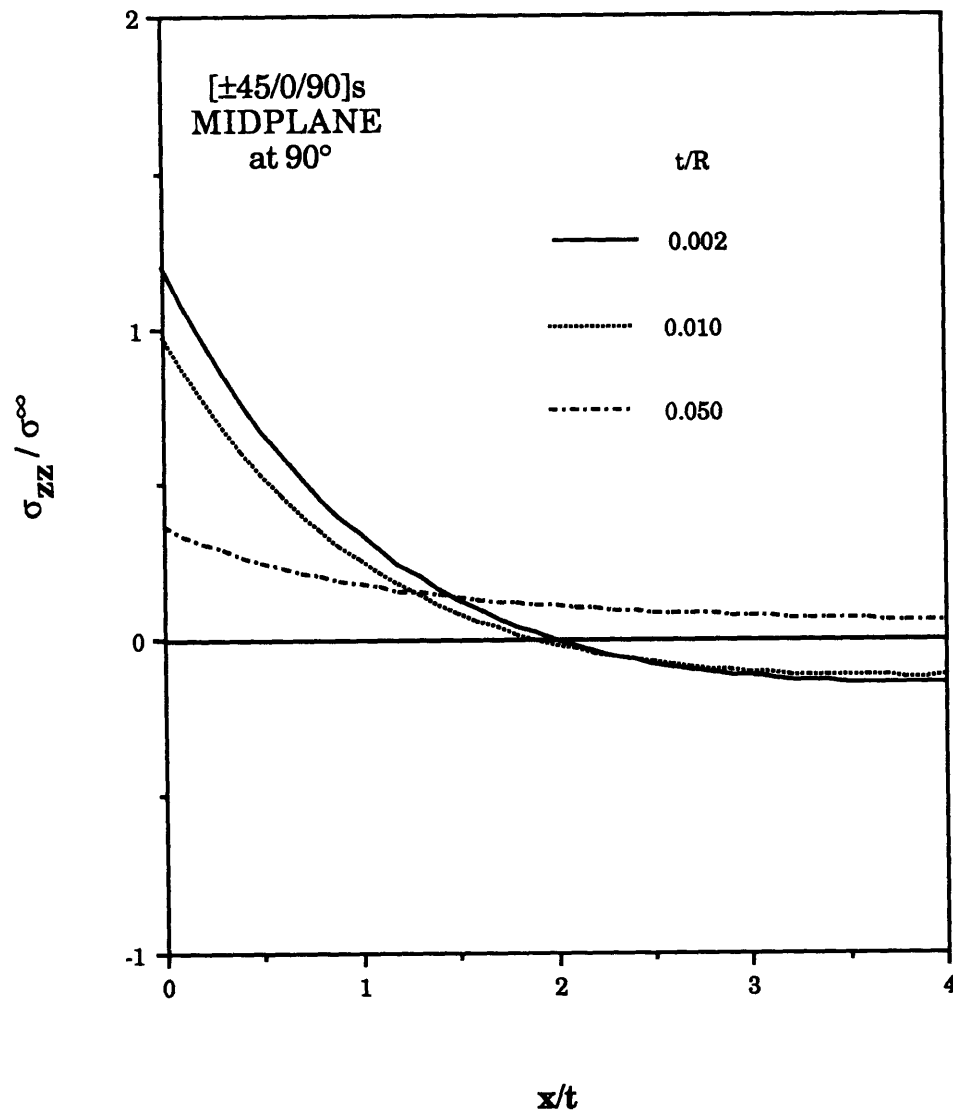


Figure 6.26 Plot of σ_{zz} at the midplane along 90° radial for a uniaxially-loaded $[\pm 45/0/90]_s$ laminate for various hole sizes.

magnitude of the curve has decreased, and the shape of the curve for the largest value of t/R is much more gradual than it is for the other two.

From this investigation, the general conclusion which may be drawn is that the general magnitude of the interlaminar stresses decreases as the ratio of the ply thickness to the hole radius increases. Other than this, trends are difficult to quantify. This difficulty arises in part because of the vastly different length scales present in this problem. Decreasing the radius of the hole will linearly compress the radial variation of the in-plane gradient contribution. Also, as the radius of the hole decreases, the magnitude of this contribution increases. These combined effects tend to blur the contributions due to the in-plane gradient solution with that due to the free edge solution. Other than the fact that the free edge contribution tends to decrease with increasing t/R , little additional can be stated about this contribution. Note should be taken that increasing the effective thickness of the ply for a constant hole diameter is equivalent to decreasing the diameter of the hole for a constant effective ply thickness. The solution depends only on the ratio of these two parameters, when the solutions are presented as a function of the non-dimensional coordinate x/t .

6.3 Quadratic Delamination Initiation Criterion

In order to investigate the failure characteristics of a laminate, "failure" must first be defined and then a criterion must be found which accurately predicts this failure. For the purposes of this investigation, failure will be defined as the load at which delamination would be expected to initiate

anywhere along the hole face, ignoring the effect of any in-plane failures which might occur before or after this load is reached.

A method for determining the load at which delamination is expected to initiate has been developed by Lagace and Brewer [55]. In this work, these investigators found that they could accurately predict the onset of delamination by using the average values of the interlaminar stresses. These average values were then used in a von Mises type quadratic failure criterion which has the form:

$$\left(\frac{\bar{\sigma}_{1z}}{S}\right)^2 + \left(\frac{\bar{\sigma}_{2z}}{S}\right)^2 + \left(\frac{\bar{\sigma}_{zz}^t}{Z^T}\right)^2 + \left(\frac{\bar{\sigma}_{zz}^c}{Z^C}\right)^2 = 1.0 \quad (6.1)$$

In this equation, the barred stresses represent the average value of the stress over some averaging distance. This averaging distance is assumed to be a material constant, and, therefore, is independent of the layup. The other parameters in this equation are S , which represents the interlaminar shear strength of the material, Z^T , which is the tensile interlaminar normal strength, and Z^C , which is the compressive interlaminar normal strength. No directionality is assumed for the shear strength because it is assumed that delamination initiates in a thin matrix rich region between the plies. In this region, the properties of the matrix are of prime importance. The strength parameters, S , Z^T , and Z^C , are given for AS4/3501-6 in Appendix A and are repeated in Table 6.1 for convenience.

This equation also suggests that perhaps a more important parameter than either of the individual interlaminar shear stresses alone, is the

TABLE 6.1

Interlaminar strength properties for AS4/3501-6

S = 105 MPa	ZT = 49.4 MPa	ZC = 186 MPa
Averaging Distance = 1.78 mm		

resultant interlaminar shear stress, hereafter referred to as τ_{IL} . Because of the nature of the interlaminar shear stresses, it is always possible to perform a rotation about the z-axis in order to obtain a case for which the shear stress is a maximum. This maximum shear stress is merely the vector sum of the two interlaminar shear stresses, and its magnitude is given by:

$$\tau_{IL} = \sqrt{\sigma_{\theta z}^2 + \sigma_{rz}^2} \quad (6.2)$$

It can be noted that the square of this quantity is similar to the first two terms given in Equation 6.1. Making use of this relation will allow for a more concise output for the analytical results.

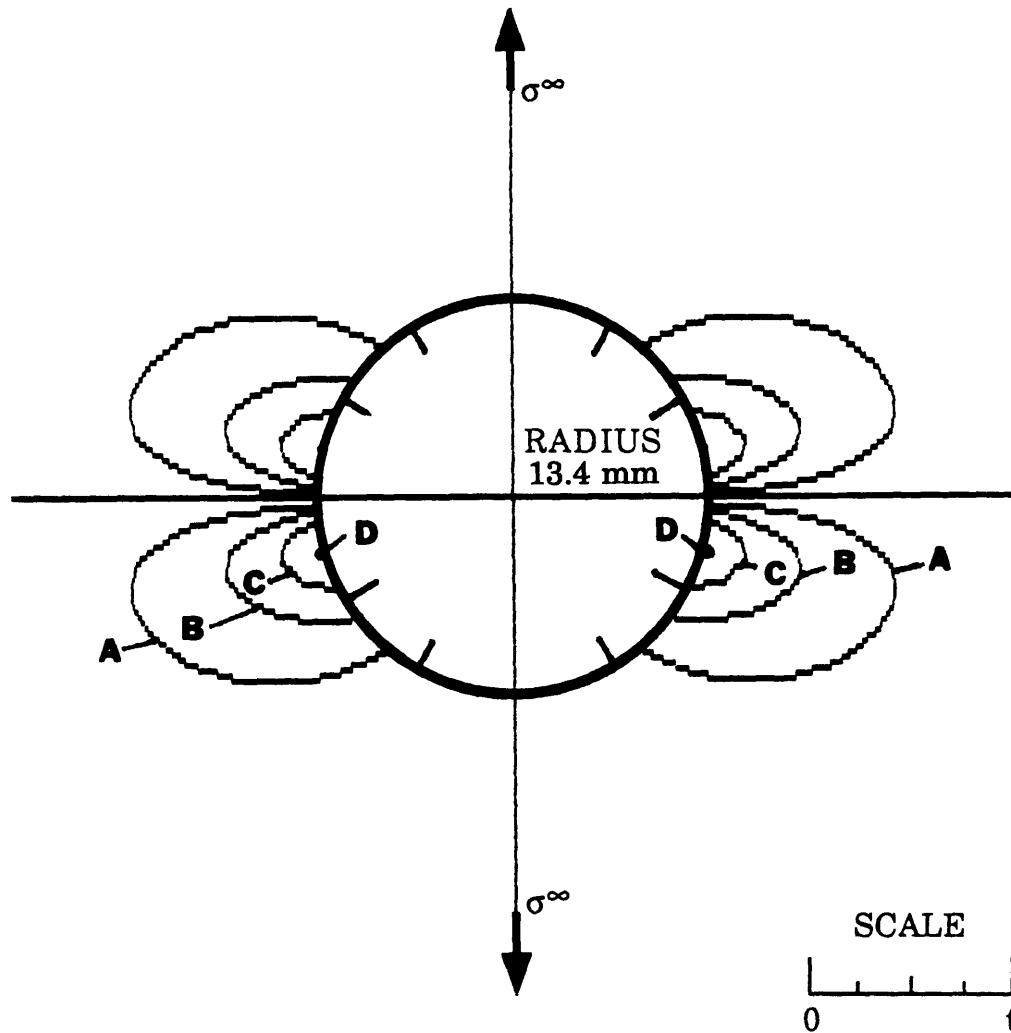
The fact that an averaging distance which is a material constant exists modifies the conclusion from Section 6.2.2 that only the ratio of the effective ply thickness to the hole radius is important. When determining delamination initiation, the average of the stress is taken over a fixed distance from the free edge. This distance is independent of effective ply thickness. Although the plot of a given interlaminar stress as a function of the normalized coordinate x/t is a function of ϵ only, the x/t distance over which this stress is averaged is a function of the effective ply thickness. For example, a $[0/90]_s$ laminate made of AS4/3501-6 graphite/epoxy containing a 26.8 mm diameter hole has an ϵ of 0.01. The averaging distance for this material is 0.178 mm, which correlates to an x/t value of 1.33. If the laminate were instead a $[0_2/90_2]_s$ laminate made of the same material, but containing a 53.6 mm hole, the ϵ value remains unchanged. Because ϵ is unchanged, the plot of a given interlaminar stress as a function of x/t also remains unchanged. However, the averaging distance now corresponds to an x/t value of 0.67. This is important for failure because

the largest of the interlaminar stresses, $\sigma_{\theta z}$ and σ_{zz} , is greatest in magnitude at the free edge, and decays rapidly as the distance from the free edge increases. Thus, the average of these stresses will be increased as the x/t distance over which they are integrated decreases, thereby decreasing the delamination initiation stress. For the current example, the $[0_2/90_2]_s$ laminate would delaminate at a lower applied far-field stress than would the $[0/90]_s$ laminate. This dependence of the delamination initiation stress on the effective ply thickness has been previously reported for the case of a straight free edge [60].

6.4 $[0/90]_s$ Laminates

In order to demonstrate both the utility of the developed methodology as well as the complexity of the problem of interlaminar stresses around holes, a number of studies of different laminate families have been performed. The next group of studies which will be discussed are those that were calculated for the $[0/90]_s$ laminate. For these studies, the laminates were taken to be made of AS4/3501-6 graphite/epoxy, whose properties are given in Appendix A. The ratio of effective ply thickness to hole radius was taken to be 0.01. For a typical AS4/3501-6 ply of 0.134 mm thickness, this corresponds to a hole whose diameter is equal to 26.8 mm. In these studies, the angle at which the load is applied, called β , is varied so that the sensitivity of the delamination initiation stress to various loadings can be studied.

In Figure 6.27, the resultant shear contours for a $[0/90]_s$ laminate which has been loaded with a uniaxial stress along the 0° direction are given. It may be seen here that although the maximum tangential stress in the laminate



$\tau_{IL} / \sigma^\infty$
CONTOUR LEVELS

A: 0.2	D: 0.8
B: 0.4	E: 1.0
C: 0.6	

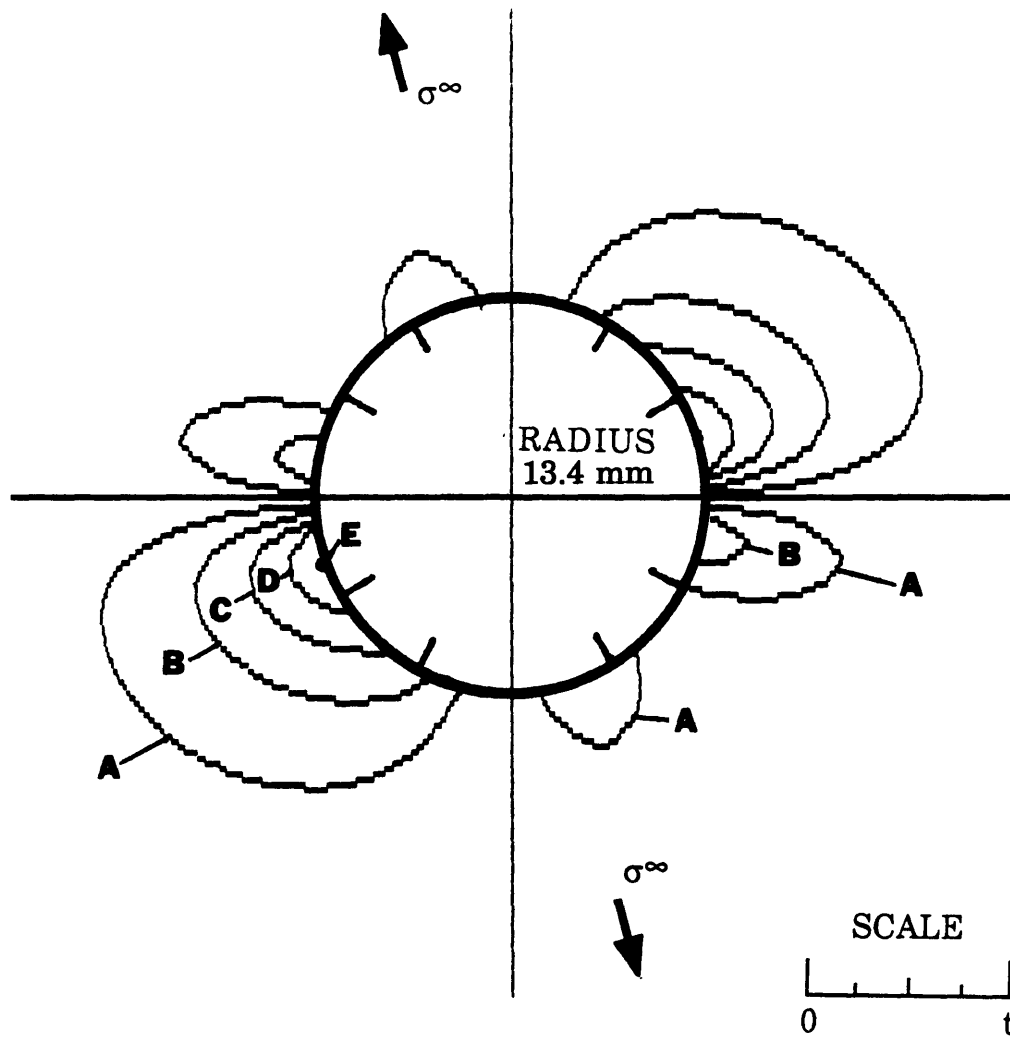
Figure 6.27 Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied along the laminate axis.

arises at 90° from the load direction, the location for maximum resultant shear is at 75° from the load and laminate axes. In all such plots as this, it should be remembered that they will all be symmetric in the r -direction, that is a stress at a given (r, θ) location is identical to the stress at $(-r, \theta)$, or alternatively, if r is required to be positive, at the location $(r, \theta + \pi)$. For the case of this cross-ply laminate, an additional symmetry about the laminate axis is also observed.

The plot in Figure 6.28 represents the resultant shear contours corresponding to a uniaxial stress applied at 15° to the laminate axis. These contours for the case of uniaxial applied stresses at 30° and 45° to the laminate axis are given in Figures 6.29 and 6.30 respectively. The four different loading conditions may be viewed as four different angular locations on the Mohr's circle shown in Figure 6.31.

Whereas little shear existed along the load axis for the first case with β equal to 0° , with the increase of β to 15° , such stresses are found to appear. Also, as the load axis was changed, the location of the maximum shear relative to laminate axes also moved, staying at nearly 90° to the load direction. For the case where the loading is applied at 15° to the laminate axis, the location of the maximum resultant shear stress is at 108° from the laminate axis and 93° from the load axis. These developments are even more pronounced in Figure 6.29 where β has been set to 30° . Here, the location of the maximum shear resultant has rotated to 115° from the laminate axis which is 85° from the load axis.

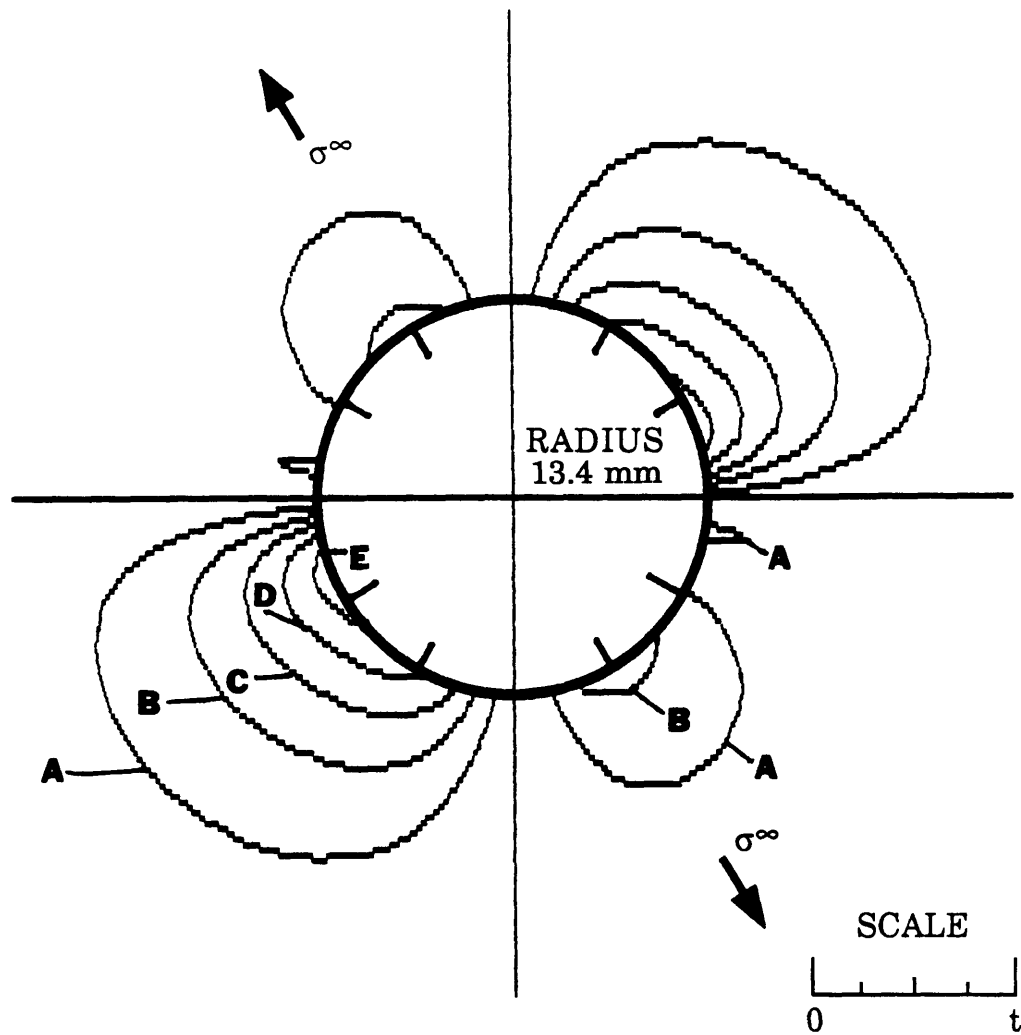
When the load is applied at a β of 45° , a new symmetry occurs, as shown in Figure 6.30. This situation is identical to a $[-45/+45]_s$ laminate loaded along the laminate axis. Here the two different pockets of shear resultants are



$\tau_{IL} / \sigma^\infty$
CONTOUR LEVELS

A: 0.2	D: 0.8
B: 0.4	E: 1.0
C: 0.6	

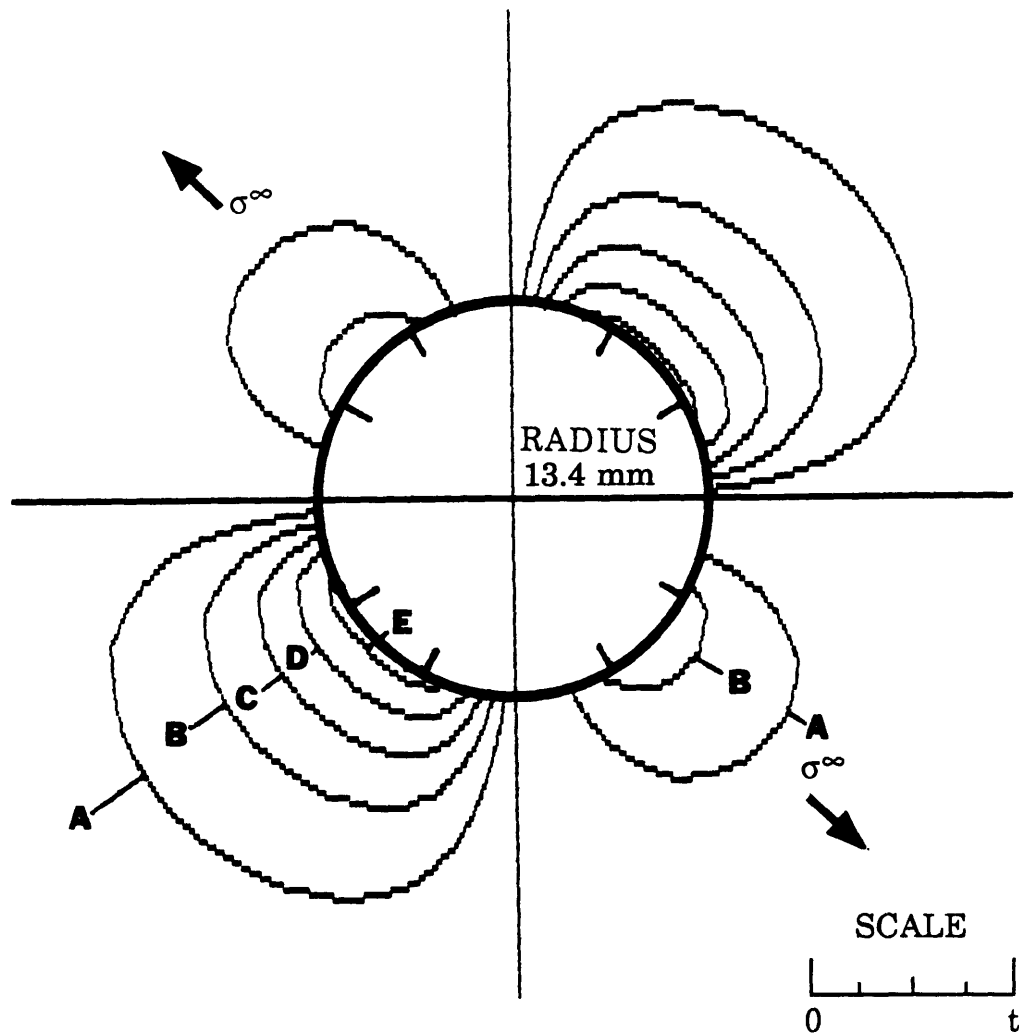
Figure 6.28 Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied at 15° from the laminate axis.



$\tau_{IL} / \sigma^\infty$
CONTOUR LEVELS

A: 0.2	D: 0.8
B: 0.4	E: 1.0
C: 0.6	

Figure 6.29 Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied at 30° from the laminate axis.



$\tau_{IL} / \sigma^\infty$
CONTOUR LEVELS

A: 0.2	D: 0.8
B: 0.4	E: 1.0
C: 0.6	

Figure 6.30 Resultant shear contours for a $[0/90]_s$ laminate with a 26.8 mm hole at the $0^\circ/90^\circ$ interface for a load applied at 45° from the laminate axis.

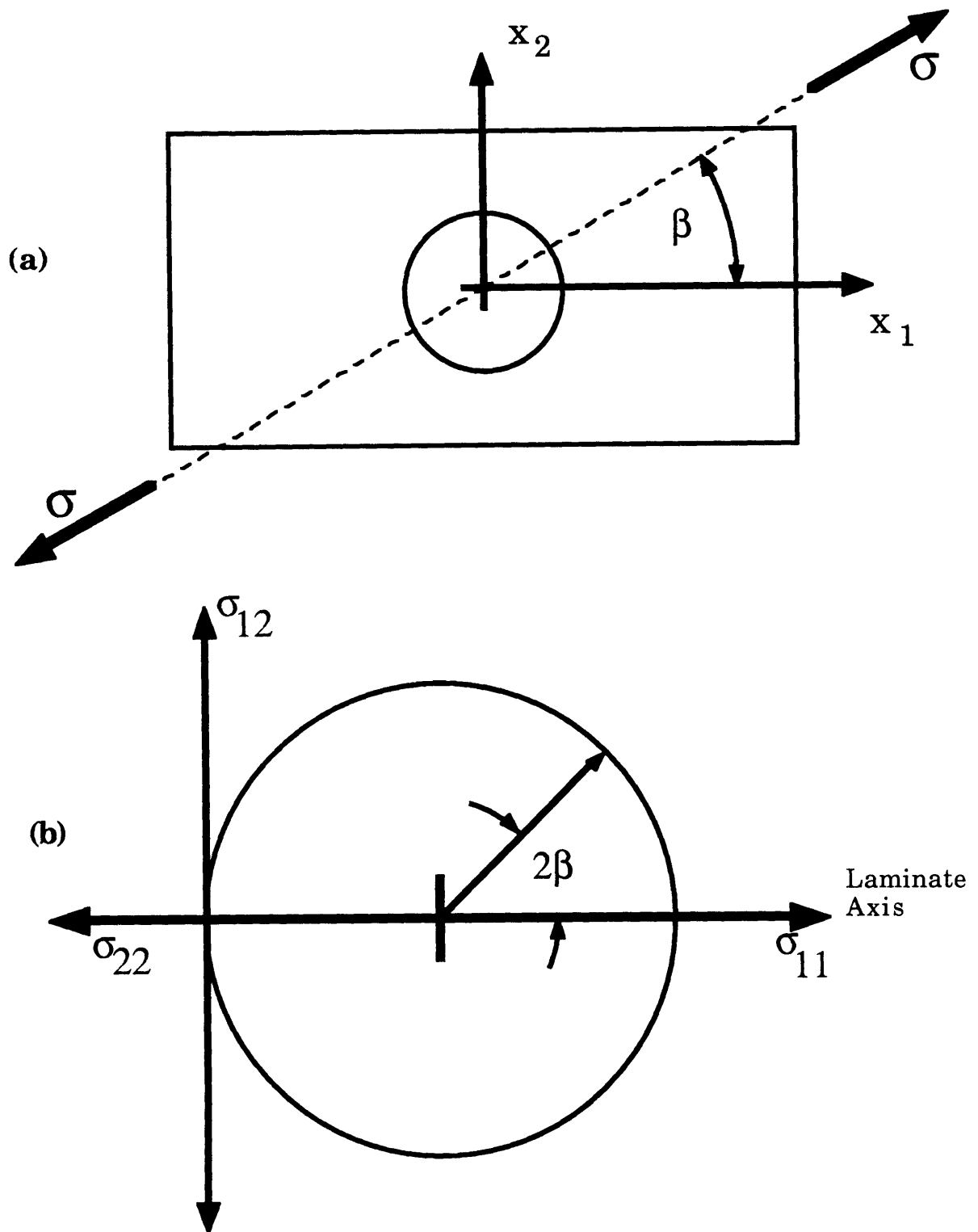


Figure 6.31 Illustration of a) the geometry of the laminate loading and b) the laminate stresses as a function of the load angle.

symmetric about the load direction axis as well as an axis which is 90° from the load axis. These figures showing the evolution of the resultant shear stress state as the loading direction is changed are given to help illustrate how many different phenomena are occurring, even in the case of this very simple four-ply laminate. The interlaminar normal stresses are small in comparison to the interlaminar shear stresses for these laminates and, therefore, are not shown.

A simple way in which the severity of these different loading conditions can be evaluated is by examining the predicted delamination initiation stresses as the load angle changes. A plot showing the results of such an analysis is shown in Figure 6.32. In this figure, it can be seen that the predicted delamination initiation stresses change dramatically with the load direction. Also, the angular location at the hole edge at which this delamination is expected to initiate is constantly changing, although this initiation location does remain at roughly 90° to the load axis. As a measure of how sensitive this layup is to load direction, the coefficient of variation of the delamination initiation stress for all load angles was calculated. This coefficient of variation was found to be 17%, when normalized by the average initiation stress.

As can be seen in these plots, the predicted delamination initiation stress is greatly dependent on the load direction as is the location of the failure. These plots represent a series of what might be tremendously lengthy calculations. First, for each load angle, the stresses at a suitably large number of angular locations must be manipulated for use with an initiation criterion. Then this must be performed at each interface, and the critical location and critical stress must be determined. For the current analysis, once the load angle is given (actually, the loading in laminate coordinates has been

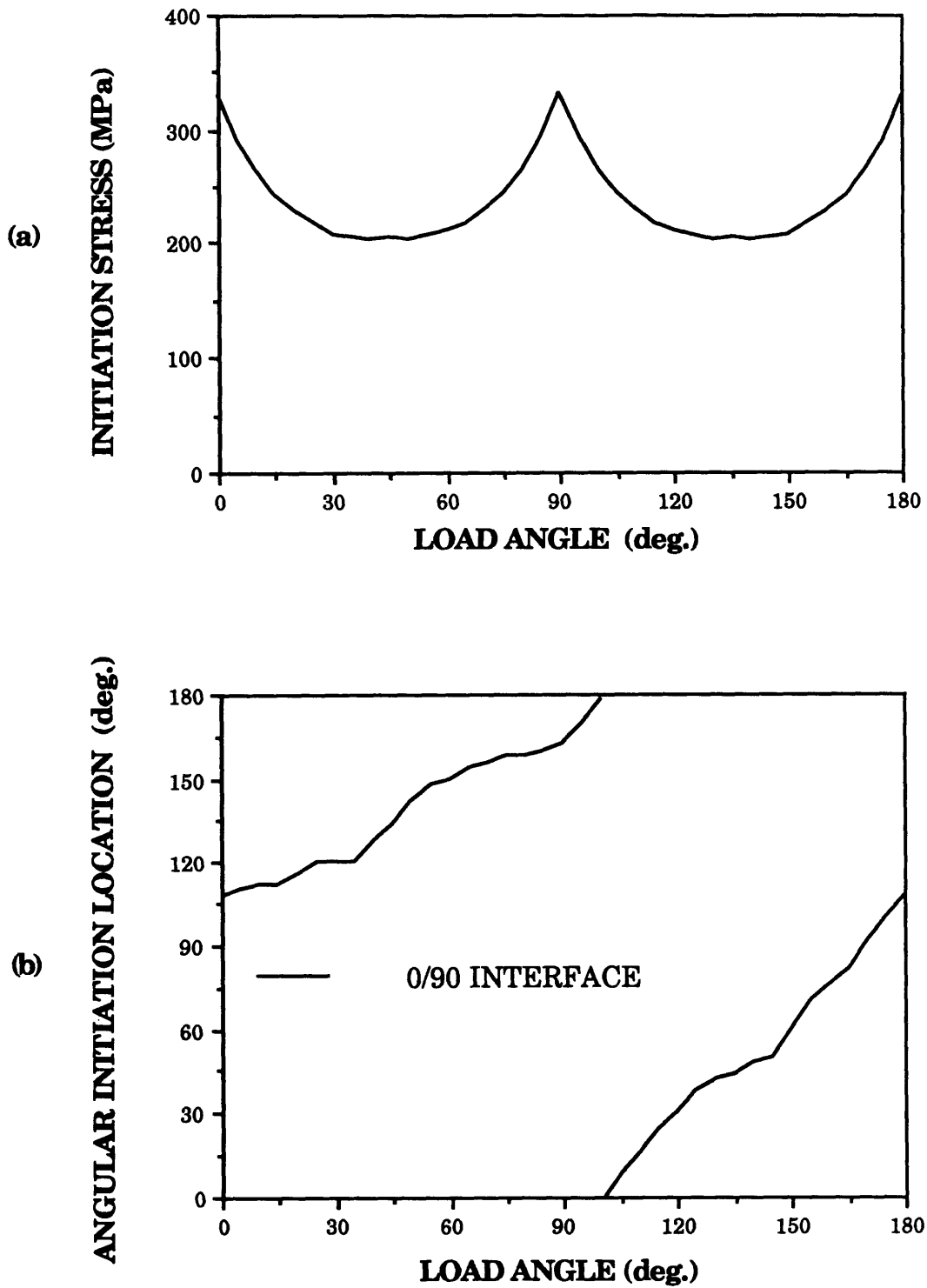


Figure 6.32 Delamination initiation prediction of (a) load and (b) angular location for $[0/90]_s$ laminates with 26.8 mm holes.

entered), the calculation of the integrals of the interlaminar stresses for use with an initiation criterion, (in the present case, averaging the interlaminar stresses for use with the Quadratic Delamination Initiation Criterion) and searching for the location for the minimum initiation stress requires approximately 7 seconds on an IBM PC, if the stresses are calculated at every 10° . It is believed that this is considerably faster than what might be achieved using Finite Element methods for calculating the stresses.

6.5 Quasi-Isotropic Laminates

Of more practical interest than the cross-ply $[0/90]_s$ laminate are the quasi-isotropic laminates. The quasi-isotropic laminates which will be considered here are the $[0/\pm 45/90]_s$, the $[0/90/\pm 45]_s$, and the $[\pm 45/0/90]_s$ laminates. Again, these laminates are made of AS4/3501-6 graphite/epoxy, and have a hole diameter of 26.8 mm, resulting in a ply thickness to hole radius ratio of 0.01.

As was done for the case of the $[0/90]_s$ laminate, these laminates were subjected to different loading angles. Although only three of the possible 24 permutations of the eight-ply quasi-isotropic laminate family appear to be represented, the information for twelve different laminates may be obtained from these laminates due to the ability to view a laminate in any arbitrary coordinate system. In this case, each coordinate system is taken at 45° from the preceding system. This fact is depicted in Table 6.2.

The same format as was used for the $[0/90]_s$ laminates was also used here. Two plots are given in each figure, one each for the Initiation Stress and the Initiation Location. The case of a $[0/90/+45/-45]_s$ laminate is shown in

TABLE 6.2

Equivalent Laminates for Various Load Angles

LOAD ANGLE	LAYUP		
0°	[0/+45/-45/ 90] _s	[0/ 90/+45/-45] _s	[+45/-45/ 0/ 90] _s
45°	[-45/ 0/ 90/+45] _s	[-45/+45/ 0/ 90] _s	[0/ 90/-45/+45] _s
90°	[90/-45/+45/ 0] _s	[90/ 0/-45/+45] _s	[-45/+45/ 90/ 0] _s
135°	[+45/ 90/ 0/-45] _s	[+45/-45/ 90/ 0] _s	[90/ 0/+45/-45] _s

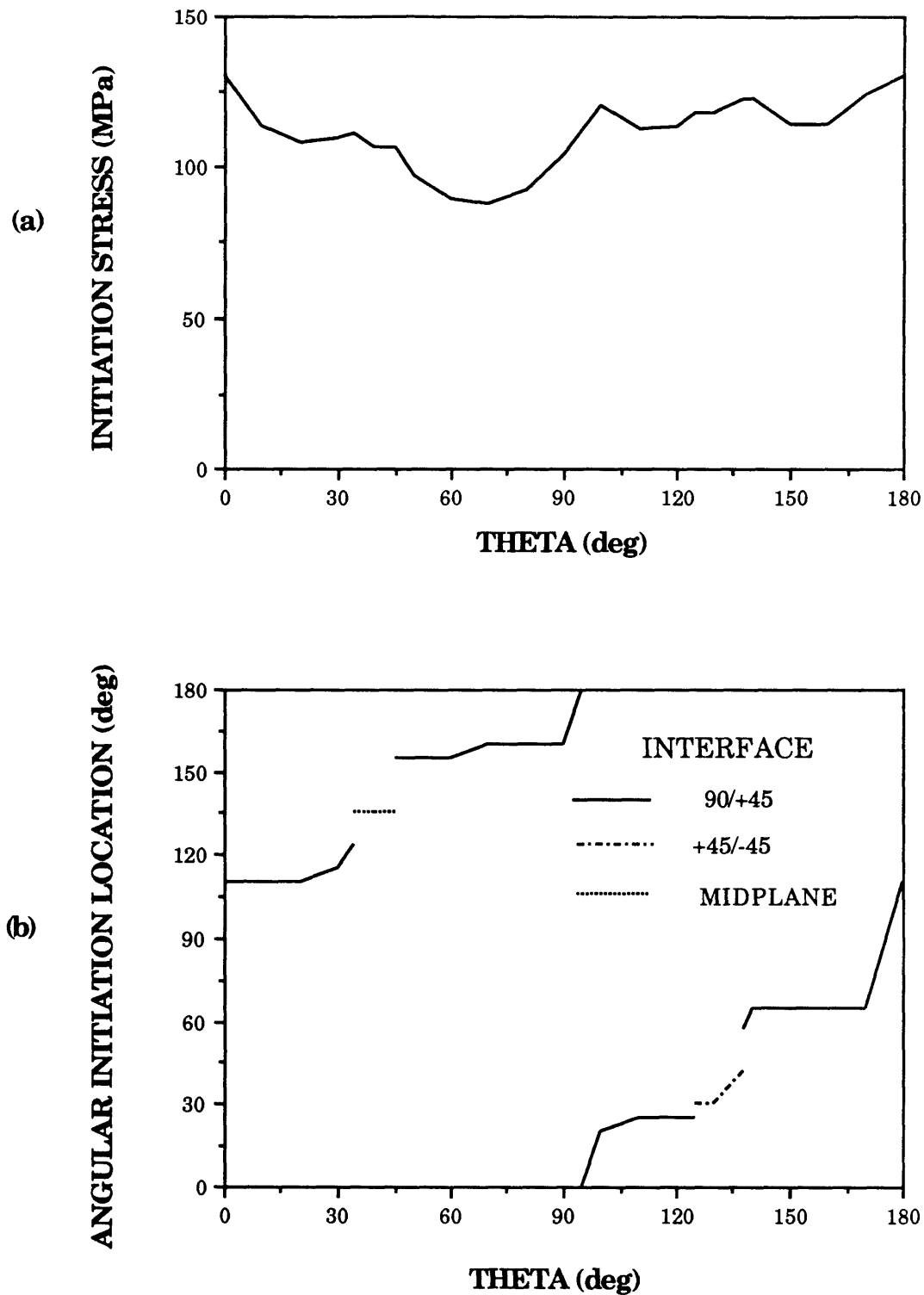


Figure 6.33 Delamination initiation prediction of (a) load and (b) angular location for $[0/90/\pm 45]_s$ laminates with 26.8 mm holes.

Figure 6.33. In this figure, it may be seen that the variation in the predicted delamination initiation stress is much less than it was for the case for the $[0/90]_s$ laminate. Again, the location at which delamination is predicted to initiate remains at roughly 90° to the load axis. In this case, however, the interface at which initiation is predicted is found to change. The coefficient of variation of these initiation stresses is found to be 11%. This is less sensitive to loading direction than was the $[0/90]_s$ laminate.

The case of the $[+45/-45/0/90]_s$ laminate is given in Figure 6.34. Of special interest here is that this laminate displays behavior which is identical to that shown in the $[0/90/+45/-45]_s$ laminate. The correlation found here can be expressed as:

$$\sigma_{[\pm 45/0/90]_s}^{\text{initiation}}(\theta) = \sigma_{[0/90/\pm 45]_s}^{\text{initiation}}\left(\frac{\pi}{4} - \theta\right) \quad (6.3)$$

This is expected because, as shown in Table 6.2, the $[0/90/+45/-45]_s$ laminate with load at 45° is equivalent to a $[-45/45/0/90]_s$ laminate with the load at 0° . Thus, only the signs on the 45° plies has changed between these laminates. This change in sign is the reason behind the change in sign in θ in Equation 6.3.

Because of this additional symmetry, results from the $[0/+45/-45/90]_s$ can be used to predict the behavior of the $[90/+45/-45/0]_s$ laminate. The behavior of the $[0/+45/-45/90]_s$ laminate is shown in Figure 6.35. Here, the predicted initiation load is again relatively insensitive to load angle, and the initiation location again roughly follows at 90° from the load axis. For this laminate, the coefficient of variation of the delamination initiation stress is 10%.

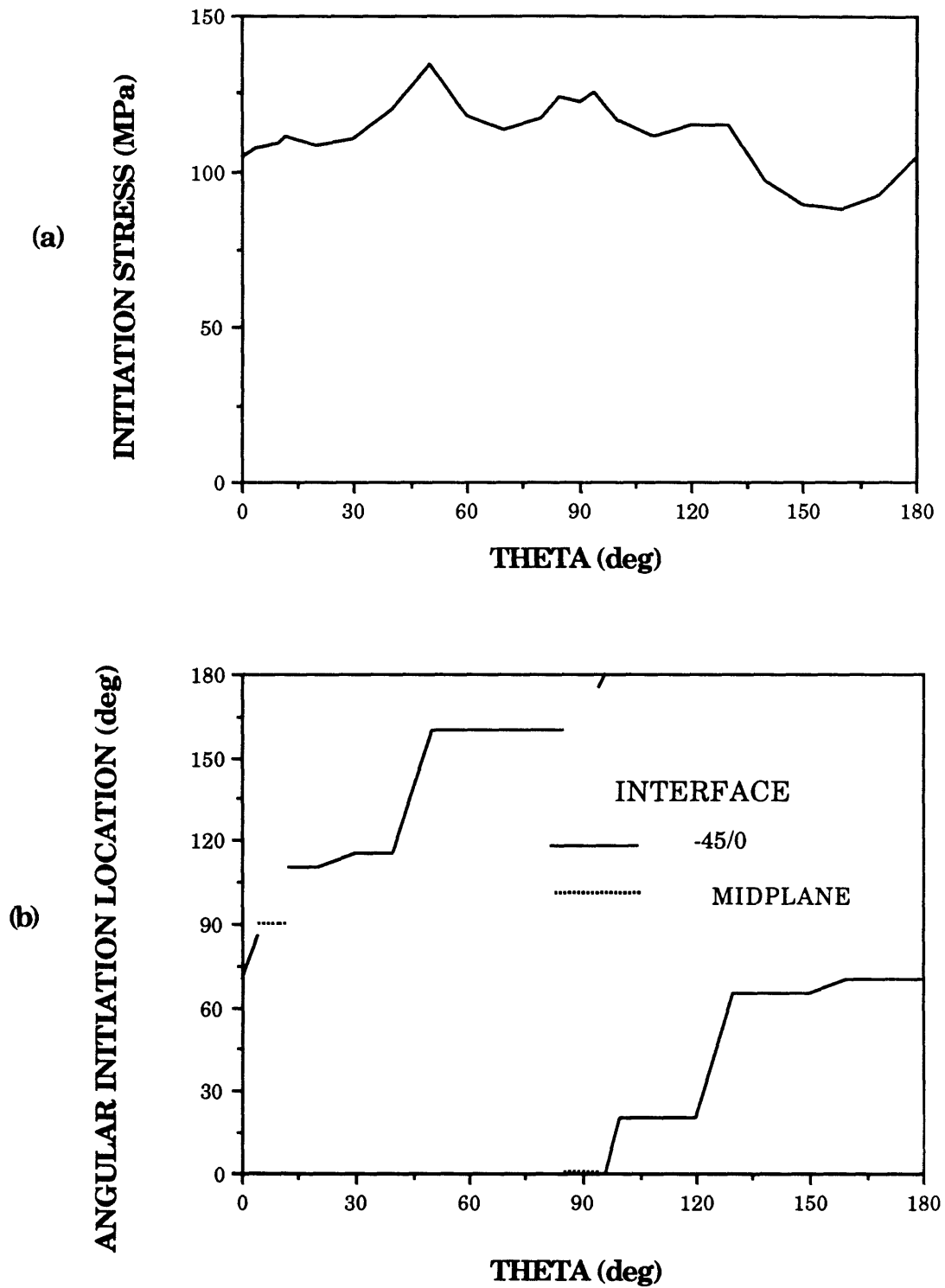


Figure 6.34 Delamination initiation prediction of (a) load and (b) angular location for $[\pm 45/0/90]_s$ laminates with 26.8 mm holes.

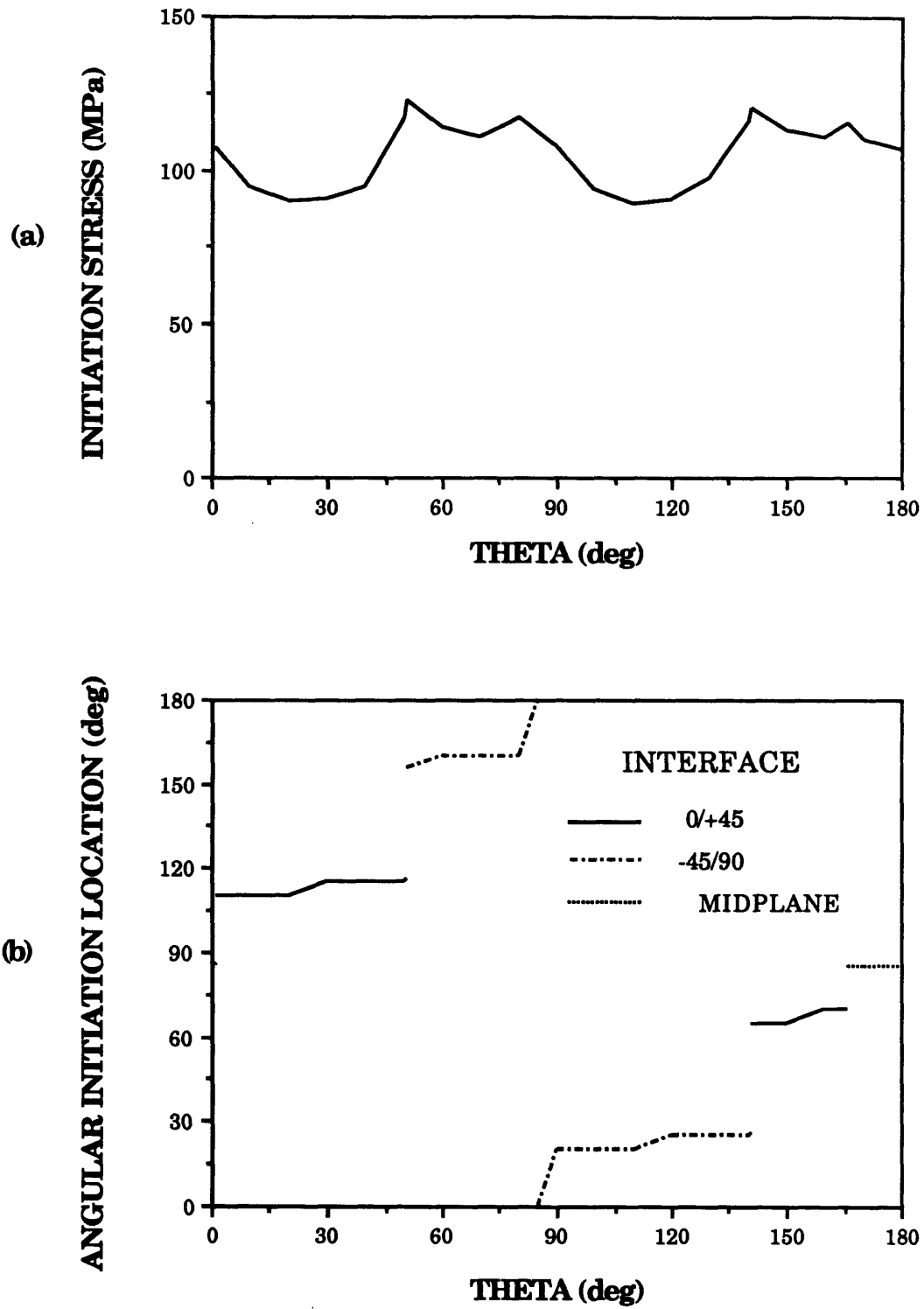


Figure 6.35 Delamination initiation prediction of (a) load and (b) angular location for $[0/\pm 45/90]_s$ laminates with 26.8 mm holes.

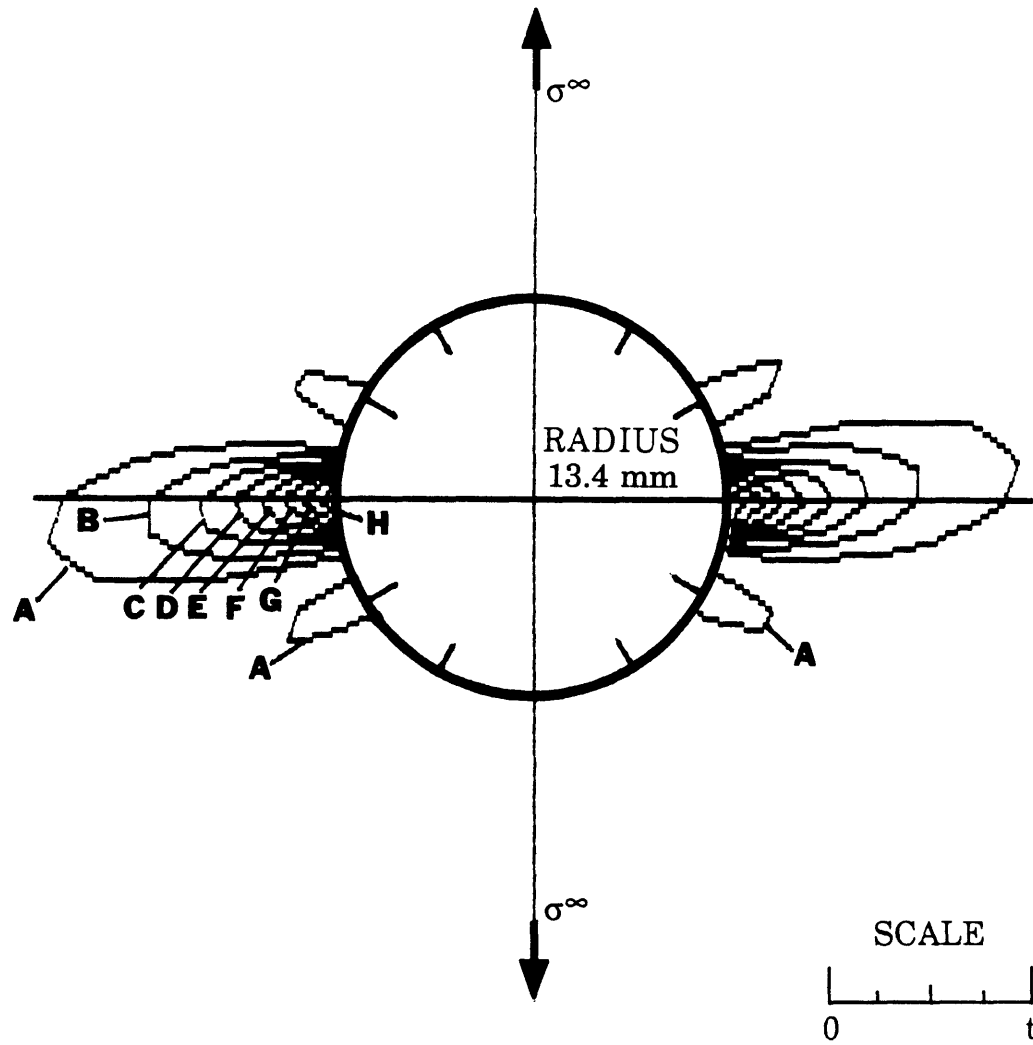
The study of these laminates has helped to demonstrate a somewhat obscure symmetry between the $[0/\pm 45/90]_s$ laminate and the $[90/\pm 45/0]_s$ laminate which might have otherwise gone unnoticed. This study also demonstrates that the delamination initiation stress, which, for a given material system and hole radius, is effected by the relative orthotropy of the laminate. These quasi-isotropic layups which have laminate material constants which do not change with in-plane rotation show much less variation in the delamination initiation stress than does the orthotropic $[0/90]_s$ laminate.

6.6 $[\pm\theta/0]_s$ Laminates

Another interesting laminate group is the $[\pm\theta/0]_s$ family. This laminate family is interesting because, for certain values of θ , straight-edged coupons made using this ply orientation have been found to fail due to delaminations arising at the stress free edge [55]. Now, the effect of holes in these laminates is analytically investigated. As has been done for the previous two laminate types, these laminates are made of AS4/3501-6 graphite/epoxy and have a ply thickness to hole radius ratio of 0.01.

In Figure 6.36, the resultant shear contours for the +15/-15 interface in a $[+15/-15/0]_s$ laminate are presented. A very high gradient exists at a location 90° from the load direction for this first interface. In contrast, the stresses at the -15/0 interface are quite small, as is shown in Figure 6.37. This behavior is typical for values of θ less than 45° .

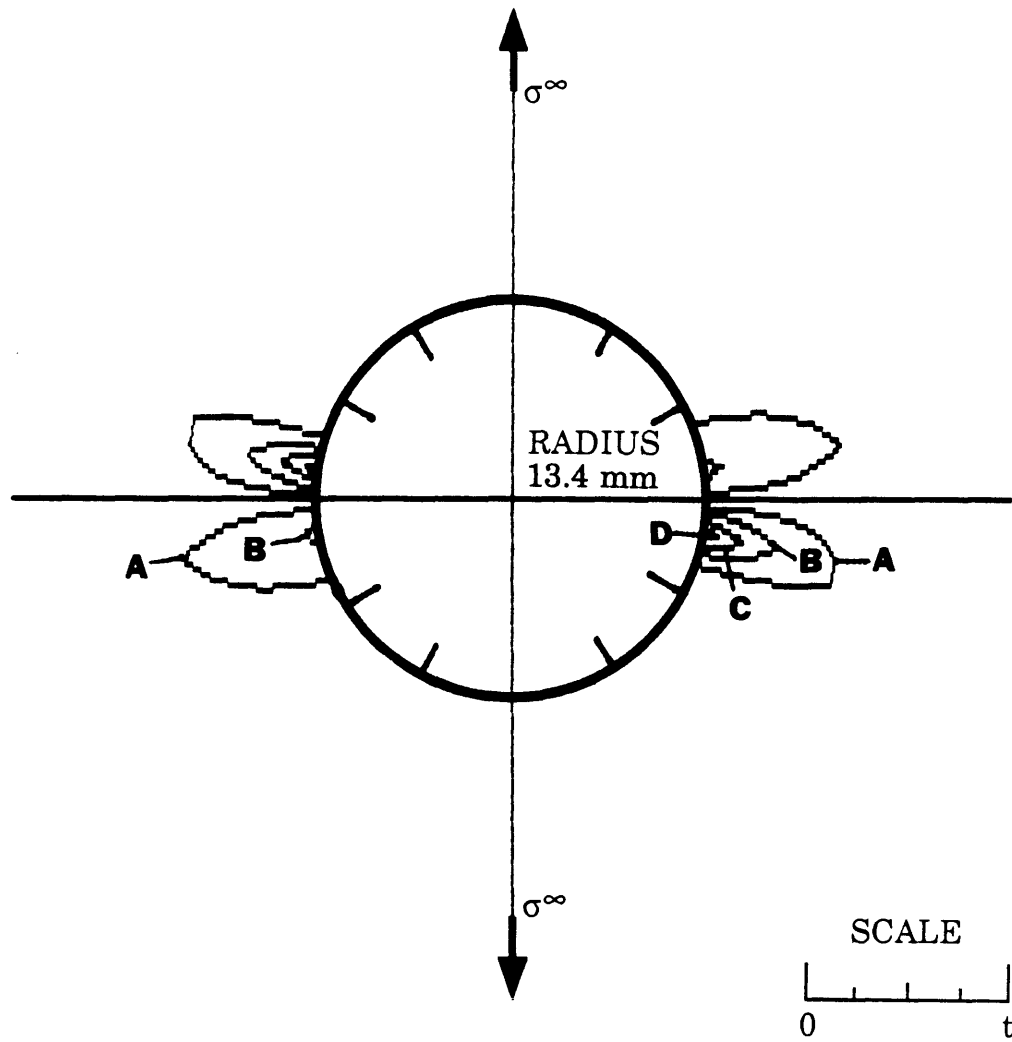
As θ is increased to 45° , the nature of the shear stress resultants changes in character, as is seen in Figures 6.38 and 6.39. The location of the



$\tau_{IL} / \sigma^\infty$ CONTOUR LEVELS

A: 0.2	D: 0.8	G: 1.4
B: 0.4	E: 1.0	H: 1.6
C: 0.6	F: 1.2	I: 1.8

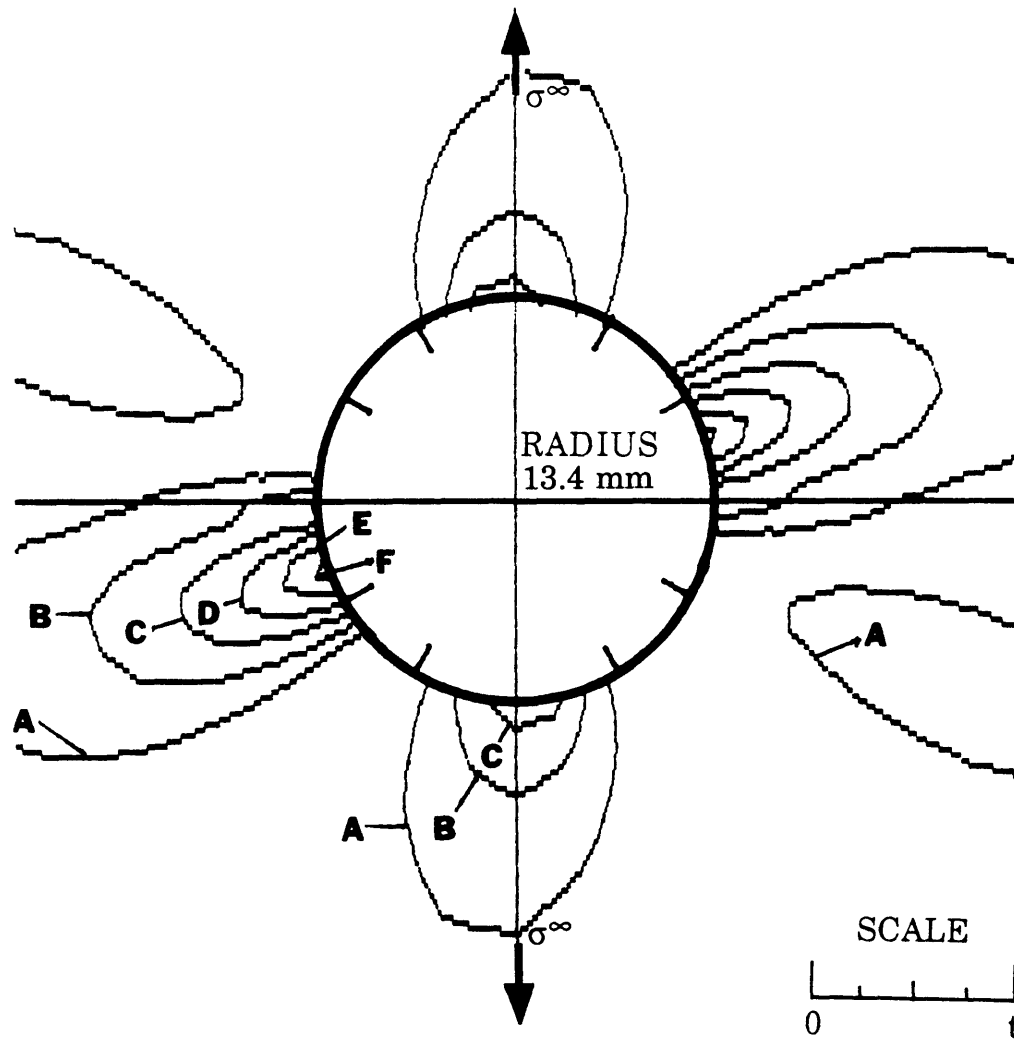
Figure 6.36 Resultant shear contours for a uniaxially-loaded $[\pm 15/0]_s$ laminate with a 26.8 mm hole at the $+15^\circ/-15^\circ$ interface.



$\tau_{IL} / \sigma^\infty$ CONTOUR LEVELS

A: 0.2	D: 0.8	G: 1.4
B: 0.4	E: 1.0	H: 1.6
C: 0.6	F: 1.2	I: 1.8

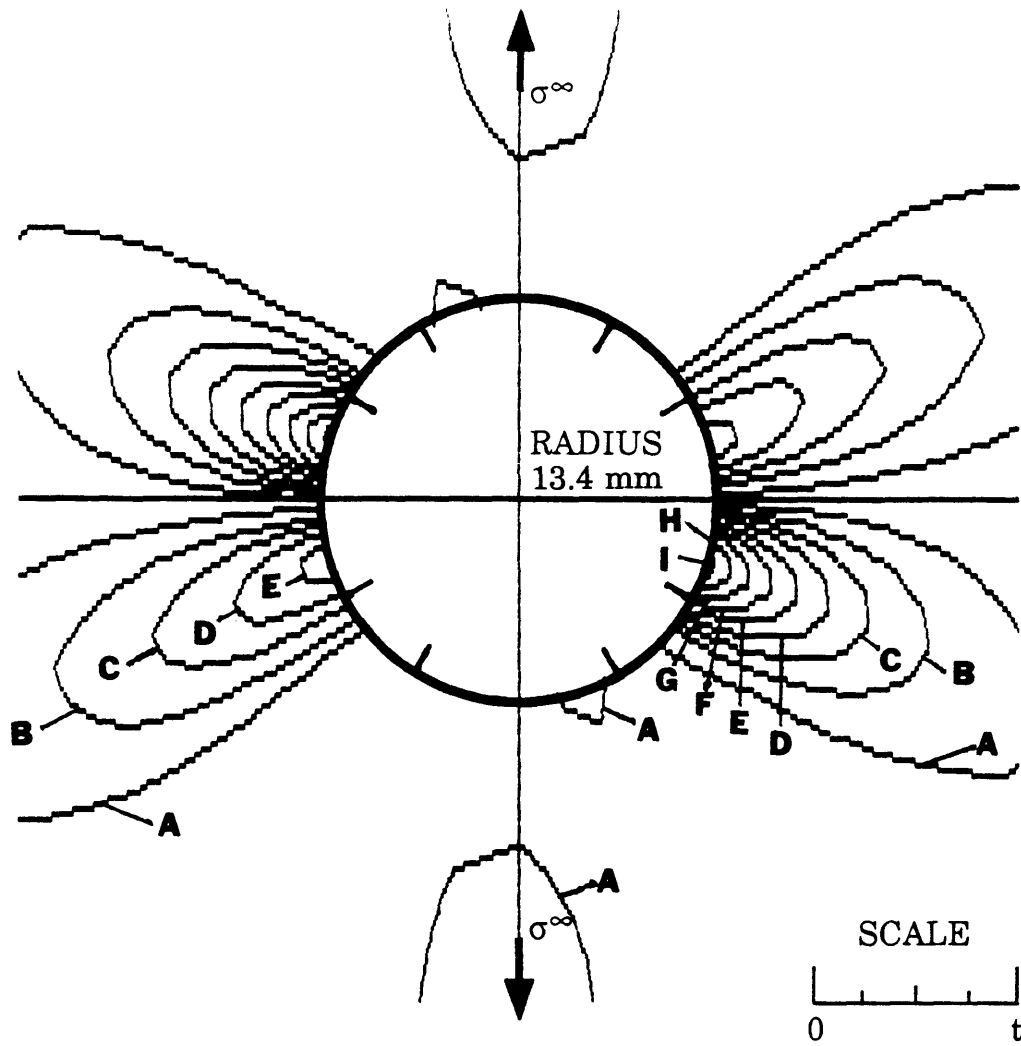
Figure 6.37 Resultant shear contours for a uniaxially-loaded $[\pm 15/0]_s$ laminate with a 26.8 mm hole at the $-15^\circ/0^\circ$ interface.



$\tau_{IL} / \sigma^{\infty}$ CONTOUR LEVELS

A: 0.2	D: 0.8	G: 1.4
B: 0.4	E: 1.0	H: 1.6
C: 0.6	F: 1.2	I: 1.8

Figure 6.38 Resultant shear contours for a uniaxially-loaded $[\pm 45/0]_s$ laminate with a 26.8 mm hole at the $+45^\circ/-45^\circ$ interface.



$\tau_{IL} / \sigma^{\infty}$ CONTOUR LEVELS

A: 0.2	D: 0.8	G: 1.4
B: 0.4	E: 1.0	H: 1.6
C: 0.6	F: 1.2	I: 1.8

Figure 6.39 Resultant shear contours for a uniaxially-loaded $[\pm 45/0]_s$ laminate with a 26.8 mm hole at the $-45^\circ/0^\circ$ interface.

maximum resultant shear at the first interface has moved from the 90° position, and the resultant shear at the second interface is now comparable in magnitude to that found in the first interface. This dominance of the second interface remains for θ greater than 40° .

The calculated delamination initiation stresses for these laminates as a function of the lamination angle, θ , are shown in Figure 6.40. As can be seen, for lamination angles less than 40° , delamination is predicted to initiate at the $\pm\theta$ interface. For lamination angles less than 30° , the location of initiation is predicted to occur at the point of maximum tangential stress, i.e. at 90° from the load direction. For lamination angles between 40° and 60° , the delamination is predicted to initiate at the $-\theta/0^\circ$ interface, at a location 70° from the load axis. The predicted initiation stress reaches a minimum value of 110 MPa for a lamination angle of 75° .

For all of these laminates, the interlaminar shear stress $\sigma_{\theta z}$ is the stress which dominates the failure. Even for the case of the $[\pm 90/0]_s$ laminate the average value of this stress at the delamination initiation stress is approximately 95% of the allowable value. The interlaminar normal stress plays a minor role in these laminates both because of its relatively small magnitude and because it has compressive values in regions where its magnitude is greatest.

6.7 Summary

The results which have been presented in this chapter are given to supply an insight into the scope and complexity of the problem of interlaminar stresses arising in laminates containing holes. As has been shown, the

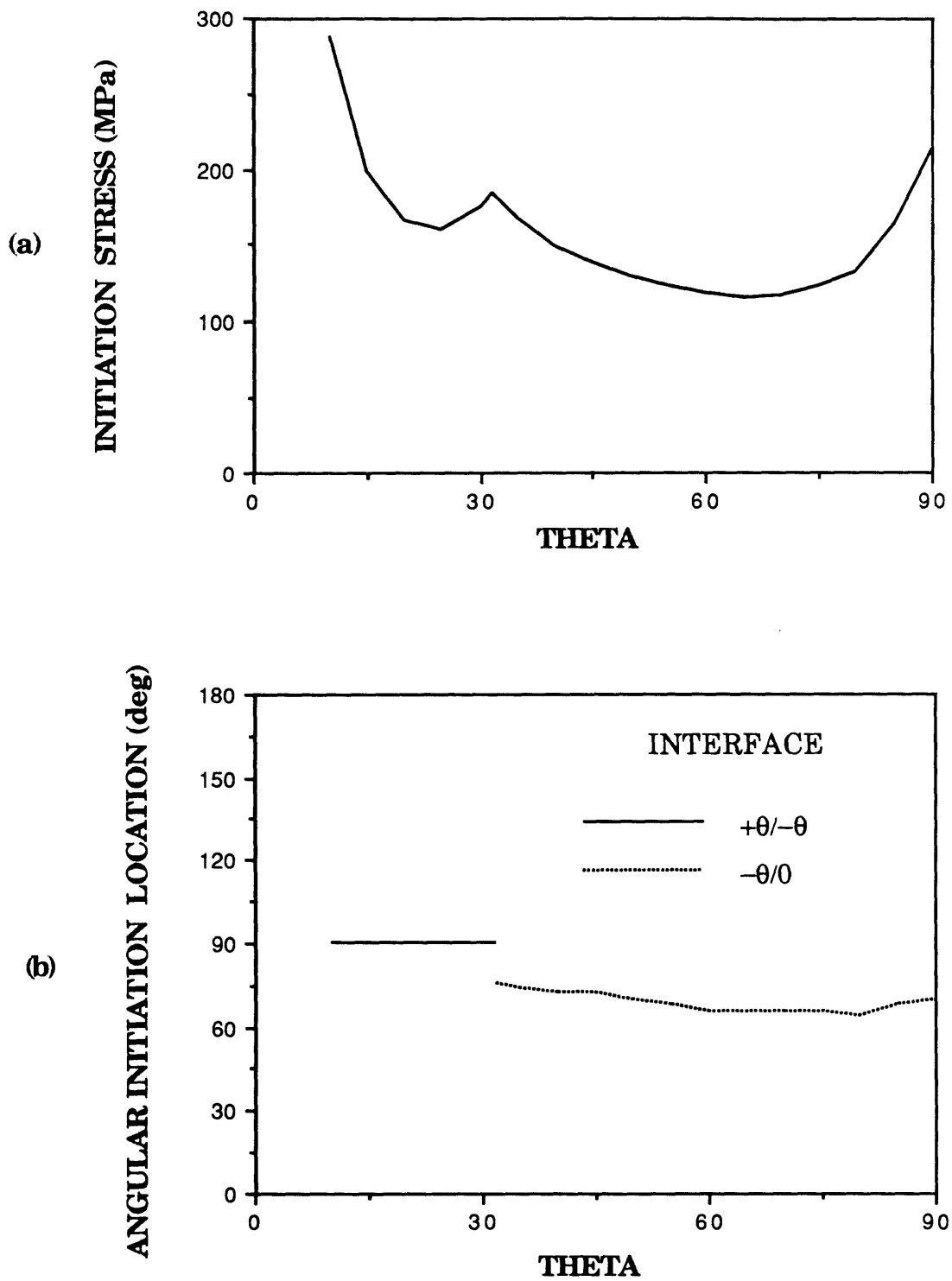


Figure 6.40 Delamination initiation prediction of (a) load and (b) angular location for $[\pm\theta/0]_s$ laminates with 26.8 mm holes.

predicted locations for delamination initiation do not always coincide with the point of maximum in-plane tangential stress, as one might suspect. The relatively wide range of angular positions from the load axis at which delamination is predicted to initiate reveals that a knowledge of the straight free edge delamination behavior of a laminate is not sufficient for the prediction of initiation of a similar laminate with a hole. In other words, the predicted delamination initiation stress of a $[\pm 45/0/90]_s$ laminate with a hole is not simply one third of the straight free edge delamination initiation stress of this laminate. Another complicating factor in these analyses is that no one normalizing parameter exists, with the total solution depending on both the effective ply thickness and the hole radius. This factor alone makes the need for an efficient solution more acute since a given solution cannot be rescaled to solve other cases.

Also found in these studies is that the general trend is for both the magnitude of the interlaminar stresses as well as the region of influence for these stresses to decrease as the ratio of the effective ply thickness to the hole radius increases. However, in regions in which the contribution from the in-plane gradient solution dominates, the magnitude of the interlaminar shear stresses is found to increase, although the zone of influence still decreases. This phenomena shows the importance of accurately calculating this contribution to the in-plane gradients, as well as the difficulty in predicting a general behavior of the laminate for all angular locations.

The current analysis allows the user to quickly scan through these complexities and locate probable points of interest as well as laminates which may be particularly prone to delamination. This analysis, because of its speed and relative brevity of the output, is particularly well suited for preliminary design.

Chapter 7

Conclusions and Recommendations

7.1 Assessment of Methodology

The procedure for calculating interlaminar stresses given in Chapter Three is completely generic. This procedure can be used for an arbitrary problem by simply following the suggested steps. The first step in this procedure requires the solution of the in-plane problem using material constants obtained via Classical Laminated Plate Theory. For sufficiently simple geometries and loadings, analytical solutions obtained by previous investigators are often available. If this is the case, the solution procedure progresses in a straightforward manner. The solution for the in-plane stresses which are obtained can be differentiated analytically and used directly with the derived expressions for calculating the interlaminar stresses due to gradient stress fields. At all free edges which exist, the residual stresses are calculated via Classical Laminated Plate Theory. These residual stresses are then used to form a companion problem which must be solved.

In cases where an analytic solution for the in-plane stress field cannot be obtained due to complexities in the problem to be solved, the above procedure is changed only slightly. In such a case, the in-plane stresses can be obtained by any of the widely available numerical techniques for the solution of two-dimensional stress problems, such as the Finite Element method. If such a numerical method is used, the stress derivatives required for the solution of the stresses due to gradients in the in-plane stress field are calculated numerically instead of analytically. The remainder of the process for finding

this contribution to the interlaminar stress state remains unchanged. In order to find the contribution to the interlaminar stress state from the companion problem, very little change in procedure is required. All that is required from the in-plane solution is the strain state at the free edge. This strain state is then used along with Classical Laminated Plate Theory to formulate the companion problem. The solution of the companion problem depends only on specimen geometry and the residual loading. It does not depend on the remainder of the in-plane stress state for the given problem.

The major benefit of this methodology is that only two-dimensional elasticity problems ever need to be solved. The solution of such problems, either analytically or numerically, is much easier than the solution of a three-dimensional problem. The solution for the contribution to the interlaminar stresses due to gradients in the in-plane stress field obviously only requires the solution for the in-plane stress field. The solution for the contribution due to the companion problem requires only the residual stress state, which is again a two-dimensional phenomenon. The solution of the companion problem, once the form of the in-plane stresses has been assumed, always results in, at worst, a two-dimensional problem. Thus, the current methodology is easily used to greatly simplify the solution of complicated three-dimensional stress fields. The ease of use of this methodology has been demonstrated with the derivations for the specific example of the interlaminar stresses around circular holes given in Chapter Four.

7.2 Conclusions

A simple approximate method to determine interlaminar stresses in symmetrically laminated materials containing circular holes was presented as an example of a more general methodology. The method is based on the principle of minimum complementary potential energy. The boundary conditions are exactly satisfied. Far from the free edge, the solution for holes in a smeared anisotropic plate is recovered. Due to the fact that strain compatibility is satisfied only in an average sense, displacement compatibility is not achieved. However, for stress-based failure criteria this is not a major concern. What is of prime importance is that the stress fields are calculated to a sufficient degree of accuracy to allow reliable predictions based on these criteria. Particular conclusions from this investigation are:

1. The present method compares well with other methods of analysis. It can deal efficiently with thick laminates which the other methods cannot do. Hybrid or variable ply thickness laminates can be analyzed very effectively.
2. The method is more efficient than other analytical methods proposed in the past. Typical computer run times for this analysis are similar to those required to obtain classical laminated plate theory constants, with 18 seconds of computing time required for the analysis of a twenty-ply laminate at ten different angular locations on an IBM-PC[®] personal computer.
3. Interlaminar stresses are shown to arise whenever gradients in the in-plane stress field exist.

4. The interlaminar stresses due to gradients in the in-plane stress field are an important component of the total interlaminar stress field.
5. The two contributors to the interlaminar stress field, in-plane gradient stress fields and a stress-free edge, have vastly different decay rates, thereby making a simple scaling relationship impossible to obtain in the general case.
6. It was found that for quasi-isotropic laminates, such as $[0/\pm 45/90]_s$ and its permutations, the dominant factor for delamination initiation stress was the relative position of the 45° plies to the 0° ply and the 90° ply.
7. Delamination is not always predicted to initiate at the location of maximum in-plane tangential stress.
8. Laminates with a greater polar variation of in-plane stiffness have a stronger dependence of delamination initiation stress on the angle of the applied load.
9. The interface as well as the angular location at which delamination is predicted to occur can change abruptly as the angle at which the laminate is loaded is changed.
10. Because of the limited accuracy of the solution for the contribution due to gradients in the in-plane stress field, the current solution should be used for

values of t/R less than 0.05. This corresponds to a 5.68 mm diameter hole for a laminate made of AS4/3501-6 graphite/epoxy.

11. The current solution methodology provides a quick preliminary assessment tool for finding regions of interest for refined studies in delamination initiation around holes.

7.3 Recommendations

Based on the investigations used in the solution verification process and on the parametric studies which have been performed, the following recommendations are made:

1. The solution of the contribution due to in-plane gradients in the stress field is accurate only for small values of the perturbation parameter ϵ equal to t/R . This solution should be improved so that it has errors on the order of ϵ^2 , as does the perturbation solution for the contribution from the free edge.
2. Ignoring the variations of the stresses in the θ -direction has introduced errors on the order of the perturbation parameter, ϵ . A straightforward method for including variations in θ has been given in Chapter Four and should be included if higher accuracy is required.
3. The analysis should be extended to the more general case of an ellipse, and, more importantly, to the case of a loaded hole.

4. Experimental studies should be performed to determine what effects stress gradients in two directions has on the use of the Quadratic Delamination Initiation Criterion.
5. Experiments should be performed to investigate the delamination initiation behavior around holes in composite laminates and compare these results with the predictions obtained with the current methodology.

REFERENCES

1. Pagano, N.J., "Exact Solutions for Composite Laminates in Cylindrical Bending," Journal of Composite Materials, Vol. 3, May 1969. pp. 398-411.
2. Whitney, J.M., "Stress Analysis of Thick Laminated Composite and Sandwich Plates," Journal of Composite Materials, Vol. 6, October 1972. pp. 426-440.
3. Pipes, R.B., and Pagano, N.J., "Interlaminar Stresses in Composite Laminates Under Uniform Axial Extension," Journal of Composite Materials, Vol. 4, 1970. pp. 538-548.
4. Pipes, R.B., and Pagano, N.J., "Some Observations on the Interlaminar Strength of Composite Laminates," International Journal of Mechanical Sciences, Vol. 15, 1973. p. 679.
5. Rybicki, E.F., "Approximate Three-Dimensional Solutions for Symmetric Laminates Under In-plane Loading," Journal of Composite Materials, Vol. 5, 1971. pp. 354-360.
6. Foye, R.L., and Baker, D.J., "Design/Analysis Methods for Advanced Composite Structures," AFML-TR-299, 1971.
7. Renerieri, G.D., and Herakovich, C.T., "Non-Linear Analysis of Laminated Fibrous Composites," VPI-E-76-10, Virginia Polytechnic Institute and State University, 1976.
8. Pagano, N.J., "On the Calculation of Interlaminar Normal Stress in Composite Laminate," Journal of Composite Materials, Vol. 8, January 1974. pp. 65-81.
9. Pipes, R.B., and Pagano, N.J., "Interlaminar Stresses in Composite Laminates - An Approximate Elasticity Solution," Journal of Applied Mechanics, Vol. 41, 1974. pp. 668-672.
10. Tang, S., and Levy, A., "A Boundary Layer Theory - Part II: Extension of Laminated Finite Strip," Journal of Composite Materials, Vol. 9, January 1975. pp. 42-52.
11. Hsu, P.W., and Herakovich, C.T., "Edge Effects in Angle-Ply Composite Laminates," Journal of Composite Materials, Vol. 11, October 1977. pp. 422-428.

12. Wang, A.S.D., and Crossman, F.W., "Some New Results on Edge Effect in Symmetric Composite Laminates," Journal of Composite Materials, Vol. 11, January 1977. pp. 92-106.
13. Wang, J.T.S., "Interlaminar Stresses in Symmetric Composite Laminates," Journal of Composite Materials, Vol. 12, October 1978. pp. 390-402.
14. Pagano, N.J., "Stress Fields in Composite Laminates," International Journal of Solids and Structures, Vol. 14, 1978. pp. 385-400.
15. Pagano, N.J., "Free Edge Fields in Composite Laminates," International Journal of Solids and Structures, Vol. 14, 1978. pp. 401-406.
16. Soni, S.R., and Chu, D.K., "An Improved Procedure for Free Edge Stress Analysis in Composite Laminates," Proceedings of the American Society for Composites Third Technical Conference, Seattle, WA, September, 1988. pp. 80-92.
17. Wang, S.S., and Choi, I., "Boundary Layer Thermal Stresses in Angle-Ply Composite Laminates," Modern Developments in Composite Materials and Structures, J.L. Vinson, Ed., American Society of Mechanical Engineering, 1979. pp. 315-341.
18. Dempsey, J.P., and Sinclair, G.B., "On the Stress Singularities in the Plane Elasticity of the Composite Wedge," Journal of Elasticity, Vol. 9, No. 4, October 1979. pp. 1287-1313.
19. Ting, T.C.T., and Chou, S.C., "Edge Singularities in Anisotropic Composites," International Journal of Solids and Structures, Vol. 17, 1981. pp. 317-327.
20. Zwierys, R.I., and Ting, T.C.T., and Spilker, R.L., "On the Logarithmic Singularity of Free-Edge Stress in Laminated Composites Under Uniform Extension," Journal of Applied Mechanics, Vol. 49, No. 3, September 1982. pp. 561-569.
21. Zwierys, R.I., and Ting, T.C.T., "Singularity of Contact-Edge Stress in Laminated Composites Under Uniform Extension," Journal of Composite Materials, Vol. 17, January 1982. pp. 49-63.
22. Kanwal, R.P., "Boundary Value Problems of Composite Media," Computers & Structures, Vol. 16, No. 1-4, 1983. pp. 471-478.
23. Whitcomb, J.D., Raju, I.S., and Goree, J.G., "Reliability of the Finite Element Method For Calculating Free Edge Stresses in Composite Laminates," Computers & Structures, Vol. 15, No. 1, 1982. pp. 23-37.

24. Whitcomb, J.D., and Raju, I.S., "Superposition Method for Analysis of Free-Edge Stresses," Journal of Composite Materials, Vol. 17, November 1983. pp. 492-507.
25. Altus, E., Rotem, A., and Shmueli, M., "Free Edge Effect in Angle Ply Laminates - A New Three Dimensional Finite Difference Solution," Journal of Composite Materials, Vol. 14, January 1980. pp. 21-30.
26. Chang, C., Sandhu, R.S., Sierakowski, R.L., and Wolfe, W.E., "Continuous Strain Finite Element Analysis of Free-Edge Effect in Laminated Composite Specimens," Journal of Composites Technology & Research, Vol. 10, No. 2, Summer 1988. pp. 54-64.
27. Laschet, G., and Marechal, E., "Finite Element Analysis of Interlaminar Stresses in Laminated Composites," Report SA-116 Aerospace Laboratory of the University of Liege, March 1984.
28. Lin, K.Y., Hwang, I.H., and Ilcewicz, L.B., "History Dependent Free Edge Stresses in Composites," Proceedings of the American Society for Composites Third Technical Conference, Seattle, WA, September 1988. pp. 517-526.
29. Fish, J.C., and Lee, S.W., "Strength of Glass-Epoxy Laminates Based on Interply Resin Failure," Proceedings of the American Society for Composites Third Technical Conference, Seattle, WA, September 1988. pp. 242-252.
30. Bar-Yoseph, P., "On the Accuracy of Interlaminar Stress Calculation in Laminated Plates," Computer Methods in Applied Mechanics and Engineering, Vol. 36, No. 3, March 1983. pp. 309-329.
31. Bar-Yoseph, P., and Siton, G., "The Effect of Material Non-Linearity on the Interlaminar Stress Field in Composite Laminates," Computers & Structures, Vol. 21, No. 6, 1985. pp. 1105-1118.
32. Bar-Yoseph, P., and Avrashi, J., "New Variational-Asymptotic Formulations for Interlaminar Stress Analysis in Laminate Plates," Journal of Applied Mathematics and Physics, Vol. 37, May 1986. pp. 305-321.
33. Ueng, C.E.S., and Zhang, K., "A Simplified Approach for Interlaminar Stresses in Orthotropic Laminated Strips," Journal of Reinforced Plastics and Composites, Vol. 4, July 1985. pp. 273-286.
34. Kassapoglou, C., and Lagace, P.A., "An Efficient Method for the Calculation of Interlaminar Stresses in Composite Materials," Journal of Applied Mechanics, Vol. 53, No. 4, December 1986. pp. 744-750.

35. Kassapoglou, C., "Determination of Interlaminar Stresses in Composite Laminates under Combined Loads," Submitted to Journal of Reinforced Plastics
36. Kassapoglou, C., "The Effect of Combined Loading and Stacking Sequence on the Interlaminar Stress Field at Free Edges of Composite Laminates," AIAA/ASME/ASCE/AHS/ASC 30th Structures, Structural Dynamics and Materials Conference, Part 4, Mobile, AL, April 3-5, 1989. pp. 2089-2097.
37. Lagace, P.A., and Saeger, K.J., "Approaches for Preliminary Design Assessment of Delamination Potential in Composite Laminates," presented at the 1st USSR-US Symposium on Mechanics of Composite Materials, Riga, Latvian SSR, May 1989.
38. Dana, J.R., and Barker, R.M., "Three-Dimensional Analysis for the Stress Distribution Near Circular Holes in Laminated Composites," Report VPI-E-74-18 from Virginia Polytechnic Institute and State University to Department of Defense, U.S. Army, Contract No. DAA-FO7-69-C-0444 with Watervliet Arsenal, Watervliet, New York, August, 1974.
39. Rybicki, E.F., and Schmueser, D.W., "Three-Dimensional Stress Analysis of a Laminated Plate Containing an Elliptical Cavity," AFML-TR-76-32, April 1976.
40. Rybicki, E.F., and Schmueser, D.W., "Effect of Stacking Sequence and Lay-Up Angle on Free Edge Stresses Around a Hole in a Laminated Plate Under Tension," Journal of Composite Materials, Vol. 12, July 1978. pp. 300-313.
41. Tang, S., "Interlaminar Stresses Around Circular Cutouts in Composite Plates Under Tension," AIAA Journal, November 1977. p. 1631.
42. Rybicki, E.F., and Hopper, A.T., "Analytical Investigation of Stress Concentrations Due to Holes in Fiber Reinforced Plastic Laminated Plates, Three-Dimensional Models," Technical Report AFML-TR-73-100, Battelle, Columbus Laboratories, Columbus, OH, June 1973.
43. Altus, E., and Bar-Yoseph, P., "A 3-D Finite Difference Solution for Orthotropic Laminated Composites using Curvilinear Coordinates," Computers & Structures, Vol. 17, No. 4., 1983. pp. 573-577.
44. Barsoum, R.S., and Freese, C.E., "An Iterative Approach for the Evaluation of Delamination Stresses in Laminated Composites,"

- International Journal for Numerical Methods in Engineering, Vol. 20, 1984. pp. 1415-1431.
45. Oplinger, D.W., "Edge Effects in Angle Ply Composites," Army Material and Mechanics Research Center, Watertown, MA, AMMRC TR 71-62.
 46. Carlson, L., "Interlaminar Stresses at a Hole in a Composite Member Subjected to In-Plane Loading," Journal of Composite Materials, Vol. 17, May 1983. pp. 238-249.
 47. Lucking, W.M., Hoa, S.V., and Sankar, T.S., "The Effect of Geometry on Interlaminar Stresses of (0/90)s Composite Laminates with Circular Holes," Journal of Composite Materials, Vol. 18, March 1984. pp. 188-198.
 48. Bar-Yoseph, P., and Avrashi, J., "Interlaminar Stress Analysis for Laminated Plates Containing a Curvilinear Hole," Computers & Structures, Vol. 21, No. 5, 1985. pp. 917-932.
 49. Klang, E.C., and Hyer, M.W., "Damage Initiation at Curved Free Edges: Application to Uniaxially Loaded Plates Containing Holes and Notches," Recent Advances in Composites in the United States and Japan, ASTM STP 864, J.R. Vinson and M. Taya Eds., American Society for Testing and Materials, Philadelphia, PA, 1985. pp. 62-90.
 50. Ericson, K., Persson, M., Carlsson, L., and Gustavsson, A., "On the Prediction of the Initiation of Delamination in a [0/90]s Laminate With a Circular Hole," Journal of Composite Materials, Vol. 18, September 1984. pp. 495-506.
 51. Zhang, K., and Ueng, C.E.S., "A Simplified Approach for Interlaminar Stresses Around a Hole in [0/90]s Laminates," Journal of Composite Materials, Vol. 22, February 1988. pp. 192-202.
 52. Kassapoglou, C., "Three-Dimensional Stress Fields in Angle-Plied Laminates with Unloaded Circular Holes," AIAA/ASME/ASCE/AHS 28th Structures, Structural Dynamics and Materials Conference, Part 1, Monterey, California, April 6-8, 1987. pp. 36-43.
 53. O'Brien, T.K., "Interlaminar Fracture of Composites," NASA TM 85768, June 1984.
 54. Kim, R.Y., and Soni, S.R., "Experimental and Analytical Studies On the Onset of Delamination in Laminated Composites," Journal of Composite Materials, Vol. 18., January 1984. pp. 70-80.
 55. Brewer, J.C., and Lagace, P.A., "Quadratic Stress Criterion for Initiation of Delamination," Journal of Composite Materials, Vol. 22. pp. 1141-1155.

56. Lagace, P.A., and Cairns, D.S., "Tensile Response of Laminates to Implanted Delaminations," **Advanced Materials Technology '87**, SAMPE, April 1987. pp. 720-729.
57. Lagace, P.A., and Saeger, K.J., "STRAFE, The Reduced Eigenfunction Solution Technique for the Solution of Interlaminar Stresses for a Straight Free Edge," **Technology Laboratory for Advanced Composites Report 88-8**, Massachusetts Institute of Technology, June 1988.
58. Bhat, N., "Delamination Suppression in Graphite/Epoxy Composites via Efficient Use of Film Adhesive Layers," S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, September 1989.
59. Savin, G.N., **Stress Concentration Around Holes**, translated by E. Gros, Pergamon Press, New York, 1961.
60. Lagace, P.A., Brewer, J.C., and Kassapoglou, C., "The Effect of Thickness on Interlaminar Stresses and Delamination," **Journal of Composites Technology & Research**, Fall 1987. pp. 81-87.

Appendix A

Summary of Material Properties Used

Included in the following tables are the material properties used for the examples run in this investigation. These tables contain the elastic and strength parameters for the materials used.

TABLE A.1

AS4/3501-6 MATERIAL PROPERTIES

ELASTIC PROPERTIES		
$E_L = 142 \text{ GPa}$	$G_{LT} = 6.0 \text{ GPa}$	$\nu_{LT} = 0.3$
$E_T = 9.81 \text{ GPa}$	$G_{LZ} = 6.0 \text{ GPa}$	$\nu_{LZ} = 0.3$
$E_Z = 9.81 \text{ GPa}$	$G_{TZ} = 3.8 \text{ GPa}$	$\nu_{TZ} = 0.3$
	$t_{\text{ply}} = 0.134 \text{ mm}$	
FAILURE PROPERTIES		
$S = 105 \text{ MPa}$	$Z^T = 49.4 \text{ MPa}$	$Z^C = 186 \text{ MPa}$
Averaging Distance = .178 mm		

TABLE A.2

T300/5208 MATERIAL PROPERTIES

ELASTIC PROPERTIES		
$E_L = 145 \text{ GPa}$	$G_{LT} = 4.5 \text{ GPa}$	$\nu_{LT} = 0.31$
$E_T = 10.7 \text{ GPa}$	$G_{LZ} = 4.5 \text{ GPa}$	$\nu_{LZ} = 0.31$
$E_Z = 10.7 \text{ GPa}$	$G_{TZ} = 3.6 \text{ GPa}$	$\nu_{TZ} = 0.49$
	$t_{\text{ply}} = 0.127 \text{ mm}$	

(Note: No failure analyses performed for this material.)

Appendix B

Computer Listing for S.L.A.S.H.

This appendix contains a complete listing of the FORTRAN source code used in the analysis for the interlaminar stresses around holes as described in Chapter Five. The source code is in Fortran 77 and was designed to run efficiently on a personal computer. The code, in its current form, can be implemented on any personal computer with a minimum of 512K (kilobytes) of memory. Comment statements are included so as to add to the program readability.

```
C*****
C
C Software for Laminate Analysis of Stresses about Holes
C      (SLASH)
C      Written by K.J. Saeger
C
C COPYRIGHT ©1989 MASSACHUSETTS INSTITUTE OF TECHNOLOGY
C
C Permission to use, copy and modify this software and its documentation
C for internal purposes only and without fee is hereby granted provided that
C the above copyright notice and this permission appear on all copies of the
C code and supporting documentation. For any other use of this software,
C in original or modified form, including but not limited to, adaptation as
C the basis of a commercial software or hardware product, or distribution in
C whole or in part, specific prior permission and/or the appropriate license
C must be obtained from MIT. This software is provided "as is" without any
C warranties whatsoever, either express or implied, including but not limited
C to the implied warranties of merchantability and fitness for a particular
C purpose. This software is a research program, and MIT does not represent
C that it is free of errors or bugs or suitable for any particular task.
C
C*****
C
C 3-D CONSTANTS          2-D CONSTANTS
C
C   PLY          PLY          LAMINATE
C COMPLIANCES          STIFFNESSES          STIFFNESSES
C (P=PLY NO.)          (P=PLY NO.)
C
C SP(P,1)=S1111      EP(P,1,1)=E1111      EL(1,1)=E1111
C SP(P,2)=S1122      EP(P,1,2)=E1122      EL(1,2)=E1122
C SP(P,3)=S1133      EP(P,1,3)=E1112      EL(1,3)=E1112
C SP(P,4)=S1112      EP(P,2,2)=E2222      EL(2,2)=E2222
C SP(P,5)=S2222      EP(P,2,3)=E2212      EL(2,3)=E2212
C SP(P,6)=S2233      EP(P,3,3)=E1212      EL(3,3)=E1212
C SP(P,7)=S2212                      EL(4,4)=1
C SP(P,8)=S3333                      EL(5,5)=1
C SP(P,9)=S3312                      EL(6,6)=1
C SP(P,10)=S2323          (USED IN INVERSION)
C SP(P,11)=S2313          ALL OTHERS = 0.0
C SP(P,12)=S1313
C SP(P,13)=S1212  LAMINATE COMPLIANCES (ENGINEERING)
C
C          SL(1,1)=S1111      SL(4,4)=1
C          SL(1,2)=S1122      SL(5,5)=1
C          SL(1,3)=2.*S1112  SL(6,6)=1
C          SL(2,2)=S2222  (USED IN INVERSION)
C          SL(2,3)=2.*S2212  ALL OTHERS = 0.0
C          SL(3,3)=4.*S1212
C
C
C*****
C PROGRAM SLASH
C implicit real*8(a-h,o-z)
```

```

CHARACTER*50 PROBN
CHARACTER*30 MATNAM(50)
COMMON FULL
COMMON /FAIL2/ BB,BETMIN(100),FSMIN(100)
LOGICAL DONE,FULL
DIMENSION MATNA(100),BETAA(50),SHOUT(10,3)
DIMENSION FFSTR(3),BETA(50),X(10),Y(10),SOUT(4,3)
DIMENSION RPLYP(50,10),THETA(100),MATNO(100),
&SPOL(4,3)
DIMENSION SP(100,13),EP(100,3,3),EL(6,6),SL(6,6),SHL(6)
DIMENSION EOUT(4,3),SRR(100),SRT(100),SRZ(100),T(100)
DIMENSION RI(100,3,3),A(100,4),B(100,4),C(100,6)
DIMENSION RINT(100,3),STZ(100),SZZ(100),RH(100,2)
DIMENSION RINTA(100,3),STZA(100),SZZA(100),SLOUT(10,10)
DIMENSION RIOUT(10)
DONE=.FALSE.
DTR=1.745329252E-2
OPEN(UNIT=3,FILE='OUTPUT.OUT')
C*****
C  WRITE HEADER
C*****
  DO 10 I=1,30
10 WRITE(5,*) ' '
    WRITE(5,11)
11 FORMAT('*****',
& '*****')
    WRITE(5,*) ' '
    WRITE(5,12)
12 FORMAT('          S.L.A.S.H.')
    WRITE(5,*) ' '
    WRITE(5,13)
13 FORMAT('          SOFTWARE FOR LAMINATE ANALYSIS',
& ' OF STRESSES ABOUT HOLES')
    WRITE(5,*) ' '
    WRITE(5,14)
14 FORMAT('*****',
& '*****')
    DO 15 I=1,10
15 WRITE(5,*) ' '
C*****
C  CALL SUBROUTINE TO OPEN INPUT FILE
C*****
  CALL READF(RPLYP,MATNAM)
C*****
C  CASE BY CASE CALL SUBROUTINE TO DETERMINE ELASTIC
C  CONSTANTS AND COMPLIANCES
C*****
20 CALL READG(THETA,MATNO,FFSTR,PROBN,NPL,AVDIM,XI,DIAM,
& KASNO,BETA,DONE,FULL)
  NUMBER=1
  DO 25 KL=1,KASNO
    BETAA(KL)=BETA(KL)
25 CONTINUE
  IF(DONE) GOTO 101

```

```
CALL SLOUT0(PROBN)
CALL CONSTANTS(KASNO,RPLYP,THETA,MATNO,NPL,SP,EP,EL,SL)
DO 16 I=1,NPL
  T(I)=RPLYP(MATNO(I),10)
  FSMIN(I)=1.D30
16 CONTINUE
  SANG=0.0
  AR=DIAM/2.
  NPOINTS=1
  DO 30 MN=1,KASNO
    BETA(MN)=BETAA(MN)
30 CONTINUE
  CALL SLOUT3(PROBN,DIAM,NPL,FFSTR,SHL,EP,SL,T,RI,A,B,C)
  DO 1 MN=1,NPL
    T(MN)=T(MN)/AVDIM
1 CONTINUE
  DO 100 IJ=1,KASNO
C
C CALCULATE IN-PLANE CONTRIBUTION
C
    BETAN=DTR*BETA(IJ)
    C1=COS(BETAN)
    S1=SIN(BETAN)
    C2=C1*C1
    S2=S1*S1
    X(1)=AR*C1
    Y(1)=AR*S1
    NPTS=1
    CALL SLADON(FFSTR,SHL,AR,NPTS,X,Y,SLOUT)
    CALL ADONAV(FFSTR,SHL,AR,XI*AVDIM,BETAN,RIOUT)
    DO 40 I=1,NPL
      S1Z=0.
      S2Z=0.
      SI1Z=0.
      SI2Z=0.
      RINTA(I,1)=0.
      RINTA(I,2)=0.
      RINTA(I,3)=0.
      DO 41 J=1,4
        S1Z=S1Z+A(I,J)*SLOUT(J,1)
        S2Z=S2Z+B(I,J)*SLOUT(J,1)
        SI1Z=SI1Z+A(I,J)*RIOUT(J)
        SI2Z=SI2Z+B(I,J)*RIOUT(J)
41 CONTINUE
      STZ(I) = C1*S1Z+S1*S2Z
      STZA(I)=-S1*S1Z+C1*S2Z
      RINTA(I,1)= C1*SI1Z+S1*SI2Z
      RINTA(I,2)=-S1*SI1Z+C1*SI2Z
      SZZA(I)=0.
      DO 42 J=1,6
        SZZA(I)=SZZA(I)+C(I,J)*SLOUT(J+4,1)
        RINTA(I,3)=RINTA(I,3)+C(I,J)*RIOUT(J+4)
42 CONTINUE
40 CONTINUE
```

```

      SRZ(1)=STZ(1)/T(1)
      DO 43 I=2,NPL
        SRZ(I)=(STZ(I)-STZ(I-1))/T(I)
43    CONTINUE
C
C CALCULATE FREE EDGE CONTRIBUTIONS
C
      CALL SAVIN(FFSTR,SHL,SANG,AR,AR,NPOINTS,X,Y,SHOUT)
C
C CALCULATE PLY STRESSES IN LAMINATE AXES
C
      DO 250 N=1,NPL
        DO 230 J=1,3
          SOUT(1,J)=0.0
          DO 230 K=1,3
            SOUT(1,J)=SOUT(1,J)+RI(N,J,K)*SHOUT(1,K)
230    CONTINUE
C
C ROTATE STRESSES INTO R - THETA COORDINATES
C
      SPOL(1,1)=C2*SOUT(1,1)+S2*SOUT(1,2)+2.*C1*S1*SOUT(1,3)
      SPOL(1,2)=S2*SOUT(1,1)+C2*SOUT(1,2)-2.*C1*S1*SOUT(1,3)
      SPOL(1,3)=C1*S1*(SOUT(1,2)-SOUT(1,1))+(C2-S2)*SOUT(1,3)
      SRR(N)=SPOL(1,1)
      SRT(N)=SPOL(1,3)
      STZ(N)=0.
      SZZ(N)=0.
250  CONTINUE
      BB=BETA(IJ)
      CALL SLOUT1(IJ,AVDIM,DIAM,BB)
      CALL SPLY(RPLYP,THETA,MATNO,NPL,SP,BETAN)
      CALL SLASH1(NPL,DIAM,AVDIM,T,SP,SRR,SRT,SRZ,STZ,SZZ,XI,RH,RINT,
&      BETAN)
      CALL SLOUT4(NPL,XI,RINT,RINTA,STZ,SZZ,STZA,SZZA)
100  CONTINUE
      GOTO 20
101  CONTINUE
      WRITE(*,1000)
      DO 875 I=1,NPL
        WRITE(*,1001) I,BETMIN(I),FSMIN(I)
875  CONTINUE
1000 FORMAT(' INTERFACE      ANGLE      STRESS')
1001 FORMAT(I7,F15.1,E15.3)
      CLOSE(UNIT=1)
      CLOSE(UNIT=3)
      STOP
      END

```

```
SUBROUTINE SLASH1(NPL,DIAM,TAVG,T,SP,SRR,SRT,SRZ,STZ,SZZ,XI,RH,
& RINT,ANGL)
C*****
C
C SUBROUTINE CALCULATES THE ZEROth ORDER PERTURBATION SOLUTION
C
C*****
  implicit real*8(a-h,o-y)
  IMPLICIT COMPLEX(Z)
  COMMON /GS/G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13
C*****
C NPL = NUMBER OF PLIES
C DIAM = DIAMETER OF HOLE
C T = VECTOR OF THICKNESSES
C SP = COMPLIANCE MATRIX
C   SP(I,1)=S1111 SP(I,6)=S2233 SP(I,10)=S2323
C   SP(I,2)=S1122 SP(I,7)=S2212 SP(I,11)=S2313
C   SP(I,3)=S1133 SP(I,8)=S3333 SP(I,12)=S1313
C   SP(I,4)=S1112 SP(I,9)=S3312 SP(I,13)=S1212
C   SP(I,5)=S2222
C SRR = VECTOR OF RESIDUAL SIGRR STRESSES
C SRT = VECTOR OF RESIDUAL SIGRT STRESSES
C SRZ = VECTOR OF RESIDUAL SIGRZ STRESSES
C*****
  DIMENSION T(100),SP(100,13),SRR(100),SRT(100),SRZ(100)
  DIMENSION RH(100,2),S(100)
  DIMENSION ZA(3),ZX(3),ZLAM(3,3),ZPHI(3,3,3),ZMAT(3,4)
  DIMENSION ZPSI(3,3,3)
  DIMENSION STZ(100),SZZ(100),RINT(100,3)
  LOGICAL NOSTRESS(3)
C
C NORMALIZE LOAD VECTORS AND THICKNESSES
C
  SIGRR=0.0
  SIGRT=0.0
  SIGRZ=0.0
  TTOT=0.0
  NOSTRESS(1)=.FALSE.
  NOSTRESS(2)=.FALSE.
  NOSTRESS(3)=.FALSE.
  DO 15 I=1,NPL
    RINT(I,1)=0.
    RINT(I,2)=0.
    RINT(I,3)=0.
    SIGRR=SIGRR+SRR(I)*T(I)
    SIGRT=SIGRT+SRT(I)*T(I)
    SIGRZ=SIGRZ+SRZ(I)*T(I)
    TTOT=TTOT+T(I)
  15 CONTINUE
  SIGRR=SIGRR/TTOT
  SIGRT=SIGRT/TTOT
  SIGRZ=SIGRZ/TTOT
  DO 16 I=1,NPL
    SRR(I)=SRR(I)-SIGRR
```

```
      SRT(I)=SRT(I)-SIGRT
      SRZ(I)=SRZ(I)-SIGRZ
16  CONTINUE
      SIGRR=0.0
      SIGRT=0.0
      SIGRZ=0.0
      DO 1 I=1,NPL
        SIGRR=SIGRR+SRR(I)*SRR(I)
        SIGRT=SIGRT+SRT(I)*SRT(I)
        SIGRZ=SIGRZ+SRZ(I)*SRZ(I)
1  CONTINUE
      SIGRR=SQRT(SIGRR)
      SIGRT=SQRT(SIGRT)
      SIGRZ=SQRT(SIGRZ)
      IF(SIGRR.EQ.0.) NOSTRESS(1)=.TRUE.
      IF(SIGRT.EQ.0.) NOSTRESS(2)=.TRUE.
      IF(SIGRZ.EQ.0.) NOSTRESS(3)=.TRUE.
      DO 2 I=1,NPL
        IF (.NOT.NOSTRESS(1)) SRR(I)=SRR(I)/SIGRR
        IF (.NOT.NOSTRESS(2)) SRT(I)=SRT(I)/SIGRT
        IF (.NOT.NOSTRESS(3)) SRZ(I)=SRZ(I)/SIGRZ
2  CONTINUE
C
C SOLVE THE FOLLOWING ONCE EACH FOR SIGMA RR, RT, AND RZ
C
      DO 100 KEVIN=1,3
        DO 3 I=1,NPL
          IF(KEVIN.EQ.1) S(I)=-SRR(I)
          IF(KEVIN.EQ.2) S(I)=-SRT(I)
          IF(KEVIN.EQ.3) S(I)=-SRZ(I)
3  CONTINUE

C
C CALCULATE THE RH VECTOR
C
      RH(1,1)=S(1)*T(1)
      RH(1,2)=RH(1,1)*T(1)-0.5*S(1)*T(1)*T(1)
      DO 4 I=2,NPL
        RH(I,1)=RH(I-1,1)+S(I)*T(I)
        RH(I,2)=RH(I-1,2)+RH(I,1)*T(I)-0.5*S(I)*T(I)*T(I)
4  CONTINUE

C
C CALCULATE THE G VECTOR
C
      G1 = 0.0
      G2 = 0.0
      G3 = 0.0
      G4 = 0.0
      G5 = 0.0
      G6 = 0.0
      G7 = 0.0
      G8 = 0.0
      G9 = 0.0
      G10= 0.0
```

```

G11=0.0
G12=0.0
G13=0.0
DO 6 I=1,NPL
C
C FIRST CALCULATE THE F's (INTEGRALS IN Z DIRECTION)
C
  F1=S(I)*S(I)*T(I)
  F2=T(I)*(RH(I,1)*RH(I,1)-T(I)*(RH(I,1)*S(I)-T(I)*S(I)*S(I)/3.)
&    )
  F3=T(I)*(RH(I,2)*RH(I,2)-T(I)*(RH(I,1)*RH(I,2)-T(I)*((RH(I,2)*
&    S(I)+RH(I,1)*RH(I,1))/3.-T(I)*(RH(I,1)*S(I)/4.-T(I)*S(I)*
&    S(I)/20.))))
  F4=T(I)*(RH(I,2)*S(I)-T(I)*(RH(I,1)*S(I)/2.-T(I)*S(I)*S(I)/6.)
&    )
  F5=T(I)*(RH(I,1)*RH(I,1)-T(I)*(RH(I,1)*S(I)-T(I)*S(I)*S(I)/3.)
&    )
C
C NOW CALCULATE THE G(i)'s
C
  G1 =G1 + SP(I, 1)*F1
  G2 =G2 + SP(I, 5)*F1
  G3 =G3 +4.*SP(I,13)*F1
  G4 =G4 +4.*SP(I,12)*F2
  G5 =G5 +4.*SP(I,10)*F2
  G6 =G6 + SP(I, 8)*F3
  G7 =G7 + SP(I, 2)*F1
  G8 =G8 +2.*SP(I, 4)*F1
  G9 =G9 + SP(I, 3)*F4
  G10=G10+2.*SP(I, 7)*F1
  G11=G11+ SP(I, 6)*F4
  G12=G12+2.*SP(I, 9)*F4
  G13=G13+4.*SP(I,11)*F5
6 CONTINUE
C
C CALCULATE COEFFICIENTS FOR CUBIC EQUATION
C
  A0=G5*(G11*G11-G2*G6)
  AT=G6*(G2*G3-G10*G10)-G2*G5*(2.*G9-G4)+G5*G7*G11-G11*(G3*G11-
&    G5*G7-G10*G12+G10*G13)+(G12-G13)*(G10*G11-G2*G12+G2*G13)
  ZA(1)=CMPLX(AT,0.d0)/A0
  AT=(2.*G9-G4)*(G2*G3-G10*G10)-G1*G2*G5-G11*(G3*G7-G8*G10)-G7*
&    (G3*G11-G5*G7-G10*G12+G10*G13)+(G12-G13)*(G7*G10-G2*G8)+
&    G8*(G10*G11-G2*G12+G2*G13)
  ZA(2)=CMPLX(AT,0.d0)/A0
  AT=G1*(G2*G3-G10*G10)-G7*(G3*G7-G8*G10)+G8*(G7*G10-G2*G8)
  ZA(3)=CMPLX(AT,0.d0)/A0
C
C CALCULATE THE EIGENVALUES
C
  IF(.NOT.NOSTRESS(KEVIN)) THEN
    CALL CUBIC(ZA,ZX)
  ENDIF
C

```

C ZX ARE THE SQUARE OF THE EIGENVALUES, TAKE THE SQUARE ROOT
C AND KEEP ONLY THE ROOT WITH A NEGATIVE REAL PART.
C

```
DO 7 I=1,3
  ZT=SQRT(ZX(I))
  IF(REAL(ZT).GT.0.0) ZT=-ZT
  ZX(I)=ZT
  IF(NOSTRESS(KEVIN)) ZX(I)=0.0
```

7 CONTINUE

C
C SORT THESE EIGENVALUES
C

```
8 IFLAG=0
DO 9 I=2,3
  IF(REAL(ZX(I)).LT.REAL(ZX(I-1))) THEN
    ZT=ZX(I)
    ZX(I)=ZX(I-1)
    ZX(I-1)=ZT
    IFLAG=1
  ENDIF
```

```
9 CONTINUE
IF(IFLAG.EQ.1) GOTO 8
IF(ABS(AIMAG(ZX(1))).GT.1.E-6) THEN
  ZT=ZX(3)
  ZX(3)=ZX(1)
  ZX(1)=ZT
  IF(AIMAG(ZX(2)).LT.0.) THEN
    ZT=ZX(2)
    ZX(2)=ZX(3)
    ZX(3)=ZT
  ENDIF
ENDIF
```

C
C CALCULATE THE EIGENVECTORS
C

```
DO 12 JAY=1,3
  ZL2=ZX(JAY)*ZX(JAY)
  ZMAT(1,1)=G1+ZL2*(-G4+2.*G9+ZL2*G6)
  ZMAT(1,2)=G7+ZL2*G11
  ZMAT(1,3)=G8+ZL2*(G12-G13)
  ZMAT(2,2)=G2
  ZMAT(2,3)=G10
  ZMAT(3,3)=G3-ZL2*G5
  ZMAT(2,1)=ZMAT(1,2)
  ZMAT(3,1)=ZMAT(1,3)
  ZMAT(3,2)=ZMAT(2,3)
  IF(.NOT.NOSTRESS(KEVIN)) THEN
    CALL EVECTOR(ZMAT)
  ELSE
    ZMAT(1,4)=0.0
    ZMAT(2,4)=0.0
    ZMAT(3,4)=0.0
  ENDIF
  ZPHI(KEVIN,JAY,3)=ZMAT(3,4)
```

```
      ZPHI(KEVIN,JAY,2)=ZMAT(2,4)
      ZPHI(KEVIN,JAY,1)=ZMAT(1,4)
      ZNORM=CMPLX(0.d0,0.d0)
      DO 10 I=1,3
        ZNORM=ZNORM+ZPHI(KEVIN,JAY,I)*CONJG(ZPHI(KEVIN,JAY,I))
10    CONTINUE
      ZNORM=SQRT(ZNORM)
      IF(CABS(ZNORM).EQ.0.) ZNORM=1.
      DO 11 I=1,3
        ZPHI(KEVIN,JAY,I)=ZPHI(KEVIN,JAY,I)/ZNORM
11    CONTINUE
      ZLAM(KEVIN,JAY)=ZX(JAY)
12    CONTINUE
C
C CALCULATE THE CONSTANTS
C
C FIRST LOAD MATRIX
C
      EPS=2.*TAVG/DIAM
      ZMAT(1,1)=ZPHI(KEVIN,1,1)
      ZMAT(1,2)=ZPHI(KEVIN,2,1)
      ZMAT(1,3)=ZPHI(KEVIN,3,1)
      ZMAT(2,1)=ZPHI(KEVIN,1,3)
      ZMAT(2,2)=ZPHI(KEVIN,2,3)
      ZMAT(2,3)=ZPHI(KEVIN,3,3)
      ZMAT(3,1)=(ZX(1)+EPS)*ZPHI(KEVIN,1,1)-EPS*ZPHI(KEVIN,1,2)
      ZMAT(3,2)=(ZX(2)+EPS)*ZPHI(KEVIN,2,1)-EPS*ZPHI(KEVIN,2,2)
      ZMAT(3,3)=(ZX(3)+EPS)*ZPHI(KEVIN,3,1)-EPS*ZPHI(KEVIN,3,2)
      ZMAT(1,4)=0.0
      ZMAT(2,4)=0.0
      ZMAT(3,4)=0.0
      IF(KEVIN.EQ.1) ZMAT(1,4)=1.0
      IF(KEVIN.EQ.2) ZMAT(2,4)=1.0
      IF(KEVIN.EQ.3) ZMAT(3,4)=-1.0
      IF(.NOT.NOSTRESS(KEVIN)) THEN
        CALL TXTSOLV(ZMAT,ZX)
        CALL SLASH2(KEVIN,EPS,ZLAM,ZX,ZPHI,ZPSI)
      ENDIF
      DO 13 I=1,3
        DO 13 J=1,3
          ZPHI(KEVIN,I,J)=ZPHI(KEVIN,I,J)*ZX(I)
          ZPSI(1,I,J)=EPS*ZPSI(1,I,J)+ZPHI(KEVIN,I,J)
          ZPSI(2,I,J)=EPS*ZPSI(2,I,J)
13    CONTINUE
      IF(KEVIN.EQ.1) FACTOR=SIGRR
      IF(KEVIN.EQ.2) FACTOR=SIGRT
      IF(KEVIN.EQ.3) FACTOR=SIGRZ
      CALL SLOUT2(KEVIN,NPL,EPS,FACTOR,S,ZLAM,ZPSI,RH,STZ,SZZ,XI,
        & ANGL,RINT)
100 CONTINUE
      RETURN
      END
```

SUBROUTINE TXTSOLV(ZMAT,ZX)

```

C*****
C
C SUBROUTINE PERFORMS COMPLEX GAUSSIAN ELIMINATION
C
C*****
  implicit real*8 (a-h,o-y)
  IMPLICIT COMPLEX(Z)
  LOGICAL ROTATE
  DIMENSION ZMAT(3,4),ZX(3)
  DO 4 I=1,2
1  ROTATE=.FALSE.
    DO 3 J=I+1,3
      IF(CABS(ZMAT(J,I)).GT.CABS(ZMAT(J-1,I))) THEN
        ROTATE=.TRUE.
      DO 2 K=1,4
        ZTEMP=ZMAT(J,K)
        ZMAT(J,K)=ZMAT(J-1,K)
        ZMAT(J-1,K)=ZTEMP
      2  CONTINUE
    ENDIF
  3  CONTINUE
    IF(ROTATE) GOTO 1
    DO 4 J=I+1,3
      ZMULT=ZMAT(J,I)/ZMAT(I,I)
      DO 4 K=I,4
        ZMAT(J,K)=ZMAT(J,K)-ZMULT*ZMAT(I,K)
      4 CONTINUE
    ZX(3)=ZMAT(3,4)/ZMAT(3,3)
    ZX(2)=(ZMAT(2,4)-ZMAT(2,3)*ZX(3))/ZMAT(2,2)
    ZX(1)=(ZMAT(1,4)-ZMAT(1,2)*ZX(2)-ZMAT(1,3)*ZX(3))/ZMAT(1,1)
    RETURN
  END

```

SUBROUTINE EVECTOR(ZMAT)

```

C*****
C
C SUBROUTINE GETS EIGENVECTORS VIA INVERSE ITERATION
C
C*****
  implicit real*8 (a-h,o-y)
  IMPLICIT COMPLEX(Z)
  LOGICAL ROTATE
  DIMENSION ZMAT(3,4),ZX(3,3)
  DO 6 I=1,3
    DO 5 J=1,3
      ZX(I,J)=0.0
    5  CONTINUE
    ZX(I,I)=1.0
  6  CONTINUE
  DO 4 I=1,2
1  ROTATE=.FALSE.
    DO 3 J=I+1,3

```

```
IF(CABS(ZMAT(J,I)).GT.CABS(ZMAT(J-1,I))) THEN
  ROTATE=.TRUE.
DO 2 K=1,3
  ZTEMP=ZMAT(J,K)
  ZMAT(J,K)=ZMAT(J-1,K)
  ZMAT(J-1,K)=ZTEMP
  ZTEMP=ZX(J,K)
  ZX(J,K)=ZX(J-1,K)
  ZX(J-1,K)=ZTEMP
2  CONTINUE
  .ENDIF
3  CONTINUE
  IF(ROTATE) GOTO 1
DO 4 J=I+1,3
  TEST=CABS(ZMAT(I,I))
  IF(TEST.EQ.0.D0) THEN
    IF(I.EQ.1) THEN
      ZMAT(1,4)=1.D0
      ZMAT(2,4)=0.D0
      ZMAT(3,4)=0.D0
      GOTO 13
    ELSE
      ZDET=ZMAT(1,1)*CONJG(ZMAT(1,1))+ZMAT(1,2)*CONJG(ZMAT(1,2))
      ZDET=SQRT(ZDET)
      ZMAT(1,4)=-ZMAT(1,2)/ZDET
      ZMAT(2,4)= ZMAT(1,1)/ZDET
      ZMAT(3,4)= 0.D0
      GOTO 13
    ENDIF
  ENDIF
  ZMULT=ZMAT(J,I)/ZMAT(I,I)
DO 4 K=1,3
  ZMAT(J,K)=ZMAT(J,K)-ZMULT*ZMAT(I,K)
  ZX(J,K)=ZX(J,K)-ZMULT*ZX(I,K)
4  CONTINUE
  TEST=CABS(ZMAT(3,3))
  IF(TEST.LE.1.D-6) THEN
    ZMAT(1,4)=(ZMAT(1,3)*ZMAT(2,2)-ZMAT(1,2)*ZMAT(2,3))/ZMAT(1,1)
    ZMAT(2,4)= ZMAT(2,3)
    ZMAT(3,4)=-ZMAT(2,2)
    ZDET=0.D0
    DO 14 I=1,3
      ZDET=ZDET+ZMAT(I,4)*CONJG(ZMAT(I,4))
14  CONTINUE
    ZDET=SQRT(ZDET)
    DO 15 I=1,3
      ZMAT(I,4)=ZMAT(I,4)/ZDET
15  CONTINUE
    GOTO 13
  ENDIF
DO 7 I=3,2,-1
DO 7 J=1,I-1
  ZMULT=ZMAT(J,I)/ZMAT(I,I)
DO 7 K=1,3
```

```

      ZMAT(J,K)=ZMAT(J,K)-ZMULT*ZMAT(I,K)
      ZX(J,K)=ZX(J,K)-ZMULT*ZX(I,K)
7  CONTINUE
      DO 9 I=1,3
        DO 8 J=1,3
          ZX(I,J)=ZX(I,J)/ZMAT(I,I)
6  CONTINUE
      ZMAT(I,4)=1.0
9  CONTINUE
      DO 12 II=1,4
        ZDET=0.0
        DO 11 I=1,3
          ZMAT(I,3)=0.0
          DO 10 J=1,3
            ZMAT(I,3)=ZMAT(I,3)+ZX(I,J)*ZMAT(J,4)
10  CONTINUE
        ZDET=ZDET+ZMAT(I,3)*CONJG(ZMAT(I,3))
11  CONTINUE
      DO 12 I=1,3
        ZMAT(I,4)=ZMAT(I,3)/SQRT(ZDET)
12  CONTINUE
13 CONTINUE
      RETURN
      END

```

SUBROUTINE SLASH2(K,EPS,ZLAM,ZX,ZPHI,ZPSI)

```

C *****
C
C SUBROUTINE CALCULATES FIRST ORDER CORRECTION FOR
C PERTURBATION SOLUTION
C
C *****

```

```

      implicit real*8 (a-h,o-y)
      IMPLICIT COMPLEX(Z)
      COMMON /GS/G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13
      DIMENSION ZLAM(3,3),ZPHI(3,3,3),ZA(3),ZPSI(3,3,3)
      DIMENSION ZB(3,3),ZC(3,3),ZD(3,3),ZX(3)
      DIMENSION A0(3,3),C0(3,3),E0(3,3),B1(3,3),D1(3,3)
      DIMENSION ZVEC1(3),ZVEC2(3),ZY1(3),ZY2(3),Z1(3),Z2(3)
      DIMENSION ZMAT(3,4),ZBB(3),ZQ(3,3)

```

```

C
C LOAD A0,C0,E0,B1, AND D1 MATRICES
C

```

```

      A0(1,1)=G6

```

```

      C0(1,1)=-G4+2.*G9
      C0(1,2)= G11
      C0(1,3)=G12-G13
      C0(2,1)=G11
      C0(3,1)=G12-G13
      C0(3,3)=-G5

```