WAVE-CURRENT INTERACTION

OVER UNEVEN BOTTOM

by

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ABSTRACT

Wave-current interaction over slowly varying depth is studied by considering two extreme cases. In the first case the current is assumed to affect the waves without being affected by them. Using the method of Multiple Scales, the wave envelope is found to satisfy a cubic Schrodinger equation with variable coefficients. Benjamin-Feir's side band instability criterion is extended. By numerical integration, the evolution of a sech profile over variable depth and current is found and, Djordjevic and Redekopp's assumption of conservation of the shape of a soliton descending a shelf is assessed. In the second case, a second order mean current is assumed to be generated by a slowly varying wave group. The method of Multiple Scales and the method of Averaging are used to find the governing equations for the mean current and mean free surface displacement. They are shown to yield equivalent results. The free surface displacement is decomposed into an averaged mean set down, plus two different waves: one locked to the wave envelope and the second forced by the depth variation. Depending on the

angle of incidence and the water depth, the later can either be trapped in the region of varying depth or radiate progressive long free waves outside that region. A numerical computation using a 'Hybrid Element Method' is performed to find the forced waves.

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PART I :

EVOLUTION OF SURFACE WAVE PACKETS OVER SLOWLY VARYING DEPTH AND CURRENT

1. Introduction

Wave propagation in near shore waters is often affected by depth variation and current. This interaction is of practical interest in some coastal engineering problems. At a harbor entrance, strong currents can be induced by tides, by a river flow or during storms by local wind. Waves propagating in the same direction as a current are lengthened and flattened while waves propagating in the opposite direction are shortened and steepened. These effects can be enhanced by the existence of submarine ridges and therefore can present serious hazards to navigation.

The evolution of progressive gravity waves on large scale currents was studied first by Longuet-Higgins and Stewart (1960-1961) who introduced the concept of radiation stress. The use of Whitham's approach based on averaging a Lagrangian and the introduction of the concept of wave action(Bretherton and Garrett (1968)) have further improved the understanding of wave-current interaction. A general survey by Peregrine (1976) gives more details of previous work. So far the study of waves moving over a current has been made within the limits of the linear theory, our purpose is to derive a nonlinear extension of the equation of conservation of wave action for the case of waves propagating on a stong current over variable depth. We will focus on strong currents which affect waves but are not affected by waves, and assume , as is frequently the case in nature , that the characteristic time and distance of the current are much greater than those of the waves . The method used to solve the problem is an extension of the method used by Djordjevic and Redekopp

(1978) who studied nonlinear evolution of a wave packet moving over variable depth, and the method used by Mei (1982) who showed that the results of linear theory for waves-current interaction, obtained by using Whitham's method can be derived by making use of the method of multiple scales. This method has the advantage of showing explicitly the small parameter ϵ and the assumptions made.

Turpin began this work in his M.S.thesis(1981).Because of some missing terms in the third order wave, the governing equation he found was not consistent with the linearized limit. Therefore, he assumed the value of some of the coefficients of the governing equation, in order to avoid the inconsistency. Using this equation, he then computed numerically, the evolution of a soliton over variable depth and current, for some cases. The purpose of this work is to derive the correct governing equation and to perform a comprehensive numerical study.

A nonlinear cubic Schrodinger equation with variable coefficients is shown to govern the evolution of waves amplitude. In the case of a stationary current a slight generation of Stokes wave train propagating on variable depth is found. These waves are shown to be unstable in the region where $k(\xi+h)$ is larger than 1.363 (k is the wave number, h the depth, ξ the free surface displacement due to the current). This result corresponds to the well known Benjamin-Feir (1967) instability. The evolution of a soliton propagating in the same direction or in the opposite direction of a stationary current, over variable depth is then computed numerically, using the finite difference method with an implicit scheme of the Crank-Nicholson type as applied by Yue (1980).

2. Governing Equations

We consider a unidirectional progressive wave packet moving on the free surface of a colinear current along the x-direction. The depth $h = h(x)$, is measured from the still water level and is allowed to vary along the direction of propagation. The vertical z-axis is oriented upward. We restrict the study to weakly nonlinear waves propagating over an intermediate depth i.e. the amplitude of the waves is assumed to be much smaller than their wavelength, while the depth is assumed to be of the same order:

$$\varepsilon = ka << 1 \qquad \mu = kh = O(1) \tag{2.1}$$

where k and a are the characteristic wavenumber and wave amplitude of the waves. It is further assumed that the current could be much stronger than the waves and hence the current is not affected by the waves while the waves are affected by the current at the leading order i.e.:

$$U_c = 0\ (\sqrt{gh}) \qquad \text{and} \qquad (u,w) = O(\varepsilon\sqrt{gh}) \tag{2.2}$$

where U_c is the horizontal current velocity, u and w the horizontal and vertical components of the wave velocity.

As is frequently the case in nature, the current U_c, the depth h, the wavenumber k and the wave period $2\pi/\omega$ are assumed to vary much more slowly than the wave induced velocity, pressure or displacement. Let introduce the slow coordinates

$$x_1 = \varepsilon x \qquad x_2 = \varepsilon^2 x \qquad t_1 = \varepsilon t \qquad t_2 = \varepsilon^2 t \tag{2.3}$$

we have

$$\vec{U}_c = \vec{U}_c\,(x_2,z,t_2) \quad h = h(x_2)\ k = k(x_2,t_2)\ \omega = \omega(x_2,t_2) \tag{2.4}$$

while the wave related quantities depend on both fast and slow coordinates:

$$\vec{u} = \vec{u}(x,x_1,x_2,z,t,t_1,t_2) \tag{2.5}$$

2.a. Governing Equations for the Current

We first consider the current $\vec{U}_c = (U_c,W_c)$ without waves. Following Mei (1982) we show that if the current is weakly rotational then the horizontal component U_c, and the free surface displacement ξ_c are governed, at the order $O(\varepsilon^4)$, by Airy's equations.

Considering the fluid to be inviscid and incompressible, the current then satisfies Euler's equations:

$$\frac{\partial U_c}{\partial x} + \frac{\partial W_c}{\partial z} = 0 \tag{2.6}$$

$$\frac{\partial U_c}{\partial t} + U_c \frac{\partial U_c}{\partial x} + W_c \frac{\partial U_c}{\partial z} = -\frac{1}{\rho}\frac{\partial P_c}{\partial x} \tag{2.7}$$

$$\frac{\partial W_c}{\partial t} + U_c \frac{\partial W_c}{\partial x} + W_c \frac{\partial W_c}{\partial z} = -\frac{1}{\rho}\frac{\partial P_c}{\partial z} - g \tag{2.8}$$

Since the current depends on the coordinates x_2, z, t_2 the operators $\frac{\partial}{\partial t}.$ and $\frac{\partial}{\partial x}.$ are $O(\varepsilon^2)$ while the operator $\frac{\partial}{\partial z}.$ is $O(1)$ when acting on a current related quantity i.e.

$$\frac{1}{kU_c}\frac{\partial U_c}{\partial x} = \frac{1}{\omega U_c}\frac{\partial U_c}{\partial t} = \frac{1}{kh}\frac{dh}{dx} = O(\varepsilon^2),\ \frac{1}{kU_c}\frac{\partial U_c}{\partial z} = O(1) \tag{2.9}$$

Using (2.9), the continuity equation (2.6) gives:

$$W_c(x_2,z,t_2) = O(\varepsilon^2) \tag{2.10}$$

Integrating with respect to z the vertical momentum equation (2.8) we find that, at the order $0(\varepsilon^4)$, the pressure is hydrostatic:

$$P_c = \rho g(\xi - z) + 0(\varepsilon^4) \tag{2.11}$$

where $\xi = \xi(x_2, t_2)$ is the elevation of the free surface above the water level due to the current. If we further assume that the current is weakly rotational in the horizontal direction

$$\Omega = \frac{\partial U_c}{\partial z} - \frac{\partial W_c}{\partial x} = 0(\varepsilon^4) \tag{2.12}$$

we then deduce after using (2.10) that the horizontal component of the current could be considered to the fourth order to be independent on depth:

$$U_c = U(x_2, t_2) + 0(\varepsilon^4) \tag{2.13}$$

Plugging (2.13) and (2.11) into the horizontal momentum equation yields:

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial \xi}{\partial x} = 0(\varepsilon^4) \tag{2.14}$$

The kinematic boundary conditions on the free surface and at the sea bottom are:

$$W_c = \frac{\partial \xi}{\partial t} + U_c \frac{\partial \xi}{\partial x} \qquad z = \xi(x,t) \tag{2.15}$$

$$W_c = - U_c \frac{\partial h}{\partial x} \qquad z = - h(x) \tag{2.16}$$

Integrating then the continuity equation (2.6) and using the boundary conditions (2.15) and (2.16) we obtain:

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} [U(\xi+h)] = 0(\varepsilon^4) \tag{2.17}$$

Equations (2.11), (2.14) and (2.17) are the so called Airy's equations.

They can be written in terms of the slow coordinates

$$\frac{\partial \xi}{\partial t_2} + \frac{\partial}{\partial x_2} (U(\xi+h)) = O(\varepsilon^2) \tag{2.18}$$

$$\frac{\partial U}{\partial t_2} + U \frac{\partial U}{\partial x_2} + g \frac{\partial \xi}{\partial x_2} = O(\varepsilon^2) \tag{2.19}$$

$$P_c = \rho g(\xi - z) + O(\varepsilon^4) \tag{2.20}$$

The vertical current component can be found by integrating the continuity equation from $z = -h$ to z

$$W_c(x_2, z, t_2) = \varepsilon^2((z+h) \frac{\partial U}{\partial x_2} - \frac{\partial h}{\partial x_2} U) + O(\varepsilon^4) \tag{2.21}$$

2.b. Governing Equations for the Waves

When we superpose waves on the current, the total velocity $\vec{U}_T = (U_c + u, W_c + w)$, the total pressure $P_T = P_c + p$ and the total free surface elevation $\xi_T = \xi + \eta$ are governed by Euler's equations with appropriate boundary conditions. [The lower cases variables are related to waves, the upper cases variables are related to the current]. The wave related quantities (u,w,p,η) are $O(\varepsilon)$ and depends on the fast coordinates (x,z,t) so that the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial t}$ are $O(1)$ when applied to these quantities.

i. Continuity and Momentum Equations

Continuity equation:

$$\frac{\partial(U_c+u)}{\partial x} + \frac{\partial(W_c+w)}{\partial z} = 0 \tag{2.22}$$

Since the current satisfies a similar equation (2.6), we have

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.23}$$

x-momentum equation:

$$\frac{\partial(U_c+u)}{\partial t} + (U_c+u)\frac{\partial(U_c+u)}{\partial x} + (W_c+w)\frac{\partial(U_c+u)}{\partial z} = -\frac{1}{\rho}\frac{\partial(P_c+p)}{\partial x} \tag{2.24}$$

using the x-momentum equation satisfied by the current and the approximations (2.10) and (2.13) we obtain:

$$\frac{\partial u}{\partial t} + u\frac{\partial U}{\partial x} + U\frac{\partial u}{\partial x} + W\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + O(\varepsilon^5) \tag{2.25}$$

z-momentum equation:

$$\frac{\partial(W_c+w)}{\partial t} + (U_c+u)\frac{\partial(W_c+w)}{\partial x} + (W_c+w)\frac{\partial(W_c+w)}{\partial z} = -\frac{1}{\rho}\frac{\partial(P_c+p)}{\partial z} - g \tag{2.26}$$

using the z-momentum equation satisfied by the current we have:

$$\frac{\partial w}{\partial t} + w\frac{\partial W}{\partial z} + U\frac{\partial w}{\partial z} + W\frac{\partial w}{\partial x} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + O(\varepsilon^5) \tag{2.27}$$

If we take the second derivative with respect to z of the vertical component of the current given by (2.21), we find that

$$\frac{\partial^2 W}{\partial z^2} = O(\varepsilon^4) \tag{2.28}$$

Finally taking $\frac{\partial}{\partial x}$ (2.25) + $\frac{\partial}{\partial z}$ (2.27) and using the approximations for the current (2.10), (2.13) and (2.28) and the continuity equation for the waves we get the governing equation for the wave pressure

$$\nabla^2 p = -2\rho\{(\frac{\partial u}{\partial x})^2 + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial U}{\partial x}\} \quad -h < z < \xi + \eta \tag{2.29}$$

ii. Boundary Conditions

The kinematic condition at the sea bottom is:

$$(\vec{U}_c + \vec{u})\cdot \vec{n} = 0 \tag{2.30}$$

where $\vec{n}$ is the normal at the depth profile at the point x. Using the fact that the current satisfies a similar condition (2.16), the waves should satisfy the following equation:

$$w = -u\frac{dh}{dx} \quad \text{on} \quad z = -h(x) \tag{2.31}$$

It should be noted that since u is $0(\varepsilon)$ and $\frac{\partial h}{\partial x}$ is $0(\varepsilon^2)$, the vertical velocity at the sea bottom is (ε^3)

$$w = 0(\varepsilon^3) \quad \text{on} \quad z = -h(x) \tag{2.32}$$

From the kinematic condition (2.31) we want to infer a boundary condition for the pressure. For this purpose we differentiate (2.32) with respect to x:

$$\frac{\partial w}{\partial x} - \frac{dh}{dx}\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}\frac{dh}{dx} + 0(\varepsilon^5) \quad \text{on} \quad z = -h(x) \tag{2.33}$$

The x-momentum equation at the order $0(\varepsilon^2)$ reduces to:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + 0(\varepsilon^2) \tag{2.34}$$

The z-momentum equation (2.27) applied at the sea bottom with the conditions (2.10) and (2.32) gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + W\frac{\partial w}{\partial z} + O(\varepsilon^4) \quad \text{on} \quad z = -h(x) \tag{2.35}$$

Using the kinematic condition at the bottom of the sea, (2.31), (2.33) and (2.16) and the continuity equation (2.6) and (2.23) into (2.35) gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{dh}{dx}\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right) + O(\varepsilon^4) \quad \text{on} \quad z = -h(x) \tag{2.36}$$

we finally combine (2.34) and (2.36) to obtain the boundary condition for the pressure

$$\frac{\partial p}{\partial z} = -\frac{dh}{dx}\frac{\partial p}{\partial x} + O(\varepsilon^4) \quad \text{on} \quad z = -h(x) \tag{2.37}$$

The kinematic condition at the free surface $(\xi+\eta)$ is

$$\frac{\partial(\xi+\eta)}{\partial t} + (U_c+u)\frac{\partial(\xi+\eta)}{\partial x} = W_c + w \quad \text{on} \quad z = \xi + \eta \tag{2.38}$$

Since the waves induced displacement η is $O(\varepsilon)$ we expand $\vec{U}_c$, $\vec{u}$ and p in Taylor series about $z = \xi$

$$U_c(x, \xi+\eta, t) = U(x,t) + O(\varepsilon^4)$$

$$W_c(x, \xi+\eta, t) = W(x,\xi,t) + \eta\left.\frac{\partial W}{\partial z}\right|_{z=\xi} + O(\varepsilon^4) \tag{2.39}$$

$$u(x, \xi+\eta, t) = u(x,\xi,t) + \eta\left.\frac{\partial u}{\partial z}\right|_{z=\xi} + \frac{\eta^2}{2}\left.\frac{\partial^2 u}{\partial z^2}\right|_{z=\xi} + O(\varepsilon^4)$$

similar equations to (2.39.c) are obtained for w and p. Substituting these expansions into equation (2.38) and using the kinematic condition (2.15) and the continuity equation (2.6) gives

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \xi}{\partial x} + U\frac{\partial \eta}{\partial x} = -\eta\frac{\partial U}{\partial x} + w + \eta\frac{\partial w}{\partial z} - u\frac{\partial \eta}{\partial x} - \eta\frac{\partial \eta}{\partial x}\frac{\partial u}{\partial z} + \frac{\eta^2}{2}\frac{\partial^2 w}{\partial z^2} + O(\varepsilon^4)$$

$$\text{on } z = \xi \tag{2.40}$$

The dynamic boundary conditions at the free surface $(\xi+\eta)$ is that the total pressure doesnot vary following the fluid motion along the free surface i.e.

$$\frac{\partial (P_c+p)}{\partial t} + (U_c+u)\frac{\partial (P_c+p)}{\partial x} + (W_c+w)\frac{\partial (P_c+p)}{\partial z} = 0 \quad \text{on } z = \xi+\eta \tag{2.41}$$

plugging the previous expansions (2.39) into (2.41) and using the expression for the pressure P_c given in (2.11) and the continuity equation (2.6) we have:

$$\frac{\partial p}{\partial t} + \eta\frac{\partial^2 p}{\partial t \partial z} + \frac{\eta^2}{2}\frac{\partial^3 p}{\partial t \partial z^2} + U\frac{\partial p}{\partial x} + U\eta\frac{\partial^2 p}{\partial z \partial x} + \frac{U\eta^2}{2}\frac{\partial^3 p}{\partial z^2 \partial x} + u\frac{\partial p}{\partial x} +$$

$$\eta\frac{\partial p}{\partial x}\frac{\partial u}{\partial z} + u\eta\frac{\partial^2 p}{\partial x \partial z} + w\frac{\partial p}{\partial z} + \eta\frac{\partial p}{\partial z}\frac{\partial w}{\partial z} + w\eta\frac{\partial^2 p}{\partial z^2} + \rho g u\frac{\partial \xi}{\partial x} + W\frac{\partial p}{\partial z} +$$

$$\rho g\eta\frac{\partial U}{\partial x} - \rho g(w+\eta\frac{\partial w}{\partial z} + \frac{\eta^2}{2}\frac{\partial^2 w}{\partial z^2}) = O(\varepsilon^4)$$

$$\text{on } z = \xi \tag{2.42}$$

In conclusion, the governing equations for the waves are given by (2.23), (2.25) and (2.29) with the boundary conditions (2.37), (2.40) and (2.42).

3. Derivation of the Evolution Equation

3.a Multiple Scale Procedure

The depth and the current vary slowly in space and time so the parameters characterizing the waves will change over the same scales, i.e.

$$k = k(X,T) \quad \omega = \omega(X,T) \tag{3.1}$$

where $X = x_2 = \varepsilon^2 x$, $T = t_2 = \varepsilon^2 t$. The variation of the wavenumber k and the frequency ω are related by the law of conservation of wave crests:

$$\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0 \tag{3.2}$$

We shall restrict our attention to slowly varying wave packet of length $1/\varepsilon$, propagating in a fixed frame at the group velocity $C_g(X,T)$, therefore, by following Djordjevic and Redekopp (1978) we introduce the multiple scale variables:

$$\tau = \varepsilon\left(\int \frac{dx}{C_g(X,T)} - t\right), \quad X \ , \quad T \tag{3.3}$$

since we are dealing with progressive waves, for small enough ε we can expand the wave pressure, velocity and free surface elevation in the following form:

$$p(x,z,t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{n} P_{nm}(X,T,\tau,z)\, e^{im\phi} \tag{3.4.a}$$

$$\eta(x,t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n} \eta_{nm}(X,T,\tau) e^{im\phi} \tag{3.4.b}$$

where ϕ is defined by:

$$\frac{\partial \phi}{\partial x} = k \qquad \frac{\partial \phi}{\partial t} = -\omega \tag{3.5}$$

and since all the left hand side quantities are real we have:

$$P_{n,-m} = P^*_{n,m} \qquad \eta_{n,-m} = \eta^*_{n,m} \tag{3.6}$$

where $(\cdot)^*$ denotes the complex conjugate. The expansions for u and w are similar to Eq. (3.4.a).

We substitute the expansions (3.4) into the governing equations (2.23), (2.25) and (2.29) and the boundary conditions (2.37), (2.40) and (2.42) and then separate terms of different orders and different harmonics. It should be noted that the multiple scale method implies that the derivatives with respect to space and time are

$$\frac{\partial \cdot}{\partial x} \to \varepsilon^2 \frac{\partial \cdot}{\partial X} \qquad \frac{\partial \cdot}{\partial z} \to \frac{\partial \cdot}{\partial z} \qquad \frac{\partial \cdot}{\partial t} \to \varepsilon^2 \frac{\partial \cdot}{\partial T} \tag{3.7}$$

when applied to quantities related to the current, but they become:

$$\frac{\partial}{\partial x} \cdot \to (imk)\cdot + \frac{\varepsilon}{C_g}\frac{\partial}{\partial \tau}\cdot + \varepsilon^2 \frac{\partial}{\partial X}\cdot \tag{3.8}$$

$$\frac{\partial^2}{\partial X^2}\cdot \to (-m^2k^2)\cdot + \varepsilon \frac{2imk}{C_g}\frac{\partial}{\partial \tau}\cdot + \varepsilon^2 \left(2imk \frac{\partial}{\partial X}\cdot + \frac{1}{C_g^{\,2}}\frac{\partial^2}{\partial \tau^2}\cdot + im \frac{\partial k}{\partial X}\cdot\right) \tag{3.9.a}$$

$$\frac{\partial \cdot}{\partial z} \to \frac{\partial \cdot}{\partial z} \tag{3.9.b}$$

$$\frac{\partial}{\partial t} \to (-im\omega)\cdot - \frac{\varepsilon}{C_g}\frac{\partial}{\partial \tau}\cdot + \varepsilon^2 \frac{\partial}{\partial T}\cdot \tag{3.9.c}$$

when applied to quantities related to the waves .

For the n^{th} order and m^{th} harmonic we then obtain the following set of equations:

$$(2.29) \rightarrow \left(\frac{\partial^2}{\partial z^2} - m^2k^2\right) P_{nm} = I_{nm} \qquad -h < z < \xi \qquad (3.10.a)$$

$$(2.37) \rightarrow \frac{\partial P_{nm}}{\partial z} = J_{nm} \qquad z = -h(x) \qquad (3.10.b)$$

$$(2.25) \rightarrow -im\sigma\, U_{nm} + \frac{imk}{\rho} P_{nm} = K_{nm} \qquad -h < z < \xi \qquad (3.10.c)$$

$$(2.23) \rightarrow \frac{\partial W_{nm}}{\partial z} + imk\, U_{nm} = L_{nm} \qquad -h < z < \xi \qquad (3.10.d)$$

$$(2.40) \rightarrow -im\sigma\, \eta_{nm} - W_{nm} = M_{nm} \qquad z = \xi \qquad (3.10.e)$$

$$(2.42) \rightarrow -im\sigma\, P_{nm} - \rho g W_{nm} = N_{nm} \qquad z = \xi \qquad (3.10.f)$$

where the right hand side terms are lower order terms and are listed in Appendix A1 and σ is the intrinsic frequency of the waves: $\sigma = \omega - kU$.

The procedure of solution is as follows. Once the problem at orders 1,2,...n-1 are solved, we use (3.10.a) and (3.10.b) to find the pressure P_{nm} for $m = 0,1,...n$, then (3.10.c) and (3.10.d) to find the horizontal and vertical velocity U_{nm}, W_{nm} and (3.10.e) to find the surface displacement η_{nm}. Finally by substituting the previous results into (3.10.f) we find a solvability condition which determines the arbitrary functions introduced at lower orders.

3.b First Order Solution

Solving the set of equations (3.10) for the first order and zeroth harmonic wave, it is found that the pressure P_{10} is not dependent on z i.e. P_{10} (X,T ,τ) and that there is no vertical velocity i.e.

$$W_{10} = 0 \tag{3.11}$$

At the first harmonic the usual linear wave solution is found. In particular the pressure, wave velocity and surface elevation are given respectively by:

$$P_{11} = A \frac{\cosh k(z+h)}{\cosh kd} \tag{3.12}$$

$$U_{11} = \frac{k}{\rho\sigma} A \frac{\cosh k(z+h)}{\cosh kd} \tag{3.13}$$

$$W_{11} = -\frac{ikA}{\rho\sigma} \frac{\sinh k(z+h)}{\cosh kd} \tag{3.14}$$

$$\eta_{11} = \frac{A}{\rho g} \tag{3.15}$$

where

$$d = \xi + h \tag{3.16}$$

denotes the mean water depth including the current set-up. The wavelength is also found to satisfy the following dispersion relation

$$\sigma^2 \equiv (\omega - Uk)^2 = gk \tanh kd \tag{3.17}$$

At this point the amplitude of the waves $\frac{A}{\rho g}(X,T,\tau)$ the zeroth harmonic component of the pressure $P_{10}(X,T,\tau)$ and of the horizontal velocity $U_{10}(X,T,\tau)$ are still unknown. We need to go to higher order to find their governing equations.

3.c Second Order Solution

At the second order zeroth-harmonic, Eqs. (3.10.a) and (3.10.b) can be solved to yield the pressure

$$P_{20} = -\frac{k^2|A|^2}{\rho\sigma^2}\frac{\cosh 2k(z+h)}{\cosh^2 kd} + \underset{\sim}{P}_{20}(X,T,\tau) \tag{3.18}$$

where $\underset{\sim}{P}_{20}$ is an arbitrary function of X,T,τ which will contribute to the mean pressure. At the zeroth harmonic the left hand side of Eq. (3.10.c) is zero, and implies that

$$\frac{\partial U_{10}}{\partial \tau} = \frac{1}{\rho(C_g-U)}\frac{\partial P_{10}}{\partial \tau} \tag{3.19}$$

This equation also shows that the horizontal, first order velocity has no z dependence. Integrating Eq. (3.10.d) with respect to z we then obtain:

$$W_{20} = -\frac{(z+h)}{C_g}\frac{\partial U_{10}}{\partial \tau} \tag{3.20}$$

The remaining two equations (3.10.e) and (3.10.f) give :

$$W_{20} = -\frac{(Cg-U)}{C_g}\frac{\partial \eta_{10}}{\partial \tau} \quad \text{on} \quad z = \xi \tag{3.21}$$

$$-\rho g W_{20} = \frac{(Cg-U)}{C_g}\frac{\partial P_{10}}{\partial \tau} \quad \text{on} \quad z = \xi \tag{3.22}$$

Eqs. (3.19) - (3.22) imply that

$$\frac{\partial U_{10}}{\partial \tau} = 0 \quad \frac{\partial \eta_{10}}{\partial \tau} = 0 \quad \frac{\partial P_{10}}{\partial \tau} = 0 \quad W_{20} = 0 \tag{3.23}$$

U_{10}, η_{10}, P_{10} are therefore three arbitrary functions of X,T which represent a current of order ε. By assuming that far downstream ($\tau \to +\infty$) their values are zero, we can take these arbitrary functions to be zero, i.e.

$$U_{10} = \eta_{10} = P_{10} = 0 \tag{3.24}$$

Similarly the first harmonic of the second order wave is obtained by setting n = 2, m = 1. The solution of (3.10.a) and (3.10.b) gives

$$P_{21} = D(X,T,\tau) \frac{\cosh k(z+h)}{\cosh kd} - \frac{iA_\tau}{C_g} (z+h) \frac{\sinh k(z+h)}{\cosh kd} \tag{3.25}$$

where D is an arbitrary function associated with an homogeneous solution. From Eq. (3.10.c) after using (3.25) we have:

$$U_{21} = \frac{k}{\rho\sigma} D \frac{\cosh k(z+h)}{\cosh kd} - \frac{ik A_\tau}{\rho\sigma C_g \cosh kd} [(z+h)\sinh k(z+h) + (\frac{1}{k} - \frac{C_g - U}{\sigma}) \cosh k(z+h)] \tag{3.26}$$

which gives after plugging it into (3.10.d) and integrating with respect to z

$$W_{21} = -\frac{ik}{\rho\sigma} D \frac{\sinh k(z+h)}{\cosh kd} - \frac{k A_\tau}{\rho\sigma C_g \cosh kd} [(z+h) \cosh k(z+h) + (\frac{1}{k} - \frac{C_g - U}{\sigma}) \sinh k(z+h)] \tag{3.27}$$

The free surface elevation at the second order and first harmonic is then obtained from (3.10.e)

$$\eta_{21} = \frac{D}{\rho g} - \frac{ikA_\tau}{\rho\sigma^2 C_g}\left[d + \left(\frac{1}{k} - 2\left(\frac{C_g - U}{\sigma}\right)\right)\tanh kd\right] \tag{3.28}$$

The dynamic free surface condition (3.10.f) gives after using (3.25) and (3.27)

$$C_g = U + \frac{\sigma}{2k}\left(1 + \frac{2kd}{\sinh 2kd}\right) \tag{3.29}$$

which is just the usual formula for the group velocity in a fixed frame i.e. $C_g = \frac{\partial\omega}{\partial k}$.

The second harmonic of the second order wave is obtained in a similar way.

$$P_{22} = F(X,T,\tau)\frac{\cosh 2k(z+h)}{\cosh 2kd} - \frac{k^2A^2}{2\rho\sigma^2\cosh^2 kd} \tag{3.30}$$

where F is a function which would be determined by the dynamic condition (3.10.f). We notice that, as usual, the second harmonic does not introduce any homogeneous solution. The horizontal and vertical velocities and the free surface elevation are:

$$U_{22} = \frac{k}{\rho\sigma} F \frac{\cosh 2k(z+h)}{\cosh 2kd} \tag{3.31}$$

$$W_{22} = -\frac{ik}{\rho\sigma} F \frac{\sinh 2k(z+h)}{\cosh 2kd} \tag{3.32}$$

$$\eta_{22} = \frac{kF}{2\rho\sigma^2}\tanh 2kd + \frac{A^2k^2}{\rho^2 g\,\sigma^2} \tag{3.33}$$

The condition (3.10.f) with the results (3.30) and (3.32) give as expected the value of the function F

$$F = \frac{3k^2\cosh 2kd}{2\rho\sigma^2\cosh^2 kd\,\sinh^2 kd}A^2 \tag{3.34}$$

At this point we have only three arbitrary functions left: A, $\underset{\sim}{P}_{20}$, D. The dynamic boundary condition (2.12) will give for the zeroth and first harmonic of the third order two equations which relate these functions. (It should be noted that as in the case of constant depth and no current, D is expected not to appear in these equations).

3.d Third Order Solution

The dynamic boundary condition at the zeroth harmonic can be determined without calculating P_{30}. From Eq. (3.10.c) we find that K_{30} is zero which yields

$$\frac{\partial U_{20}}{\partial \tau} = \frac{1}{\rho(C_g - U)} \frac{\partial \underset{\sim}{P}_{20}}{\partial \tau} \tag{3.35}$$

or by integration with respect to τ

$$U_{20} = \frac{1}{\rho(C_g - U)} \underset{\sim}{P}_{20} + F_1(X,T) \tag{3.36}$$

where F_1 is an arbitrary real function. It should be noted that since $\underset{\sim}{P}_{20}$ has no z-dependence, U_{20} will not be dependent on z. Using (3.36) to calculate L_{30} and integrating (3.10.d) with respect to z we obtain:

$$W_{30} = - \frac{(z+h)}{\rho g(C_g - U)} \frac{\partial \underset{\sim}{P}_{20}}{\partial \tau} \tag{3.37}$$

Equation (3.10.e) gives a relation between W_{30} and M_{30} which is

$$(C_g - U)\frac{\partial \eta_{20}}{\partial \tau} - \frac{d}{\rho(C_g - U)}\frac{\partial P_{20}}{\partial \tau} - \frac{2k|A|_\tau^2}{\rho^2 g\sigma} = 0 \tag{3.38}$$

after using (3.37). By integrating with respect to τ, we get

$$\eta_{20} = \frac{d\, P_{20}}{\rho(C_g - U)^2} + \frac{2k\,|A|^2}{\rho^2 \sigma C_g (C_g - U)} + F_2(X,T) \tag{3.39}$$

where F_2 is another arbitrary function. Equation (3.10.f) is then used with (3.37) to obtain a relation between A and P_{20}

$$\left(1 - \frac{(C_g - U)^2}{gd}\right)\frac{\partial P_{20}}{\partial \tau} = -\frac{C_g (C_g - U)k^2}{gd\,\rho\sigma^2}\left(\frac{2\sigma}{kC_g} + \frac{(C_g - U)}{C_g}\frac{1}{\cosh^2 kd}\right)|A|_\tau^2 \tag{3.40}$$

or integrating it with respect to τ

$$\left(1 - \frac{(C_g - U)^2}{gd}\right) P_{20} = \frac{-k^2}{\rho\sigma^2}\frac{C_g(C_g - U)}{gd}\left(\frac{2\sigma}{kC_g} + \frac{C_g - U}{C_g}\frac{1}{\cosh^2 kd}\right)|A|^2 + F_3(X,T) \tag{3.41}$$

where F_3 is an arbitrary real function. We note that $(C_g - U)^2 < gd$ always. In very shallow water $(C_g - U)^2 \to gd$, the assumption of intermediate depth i.e. $kh = O(1)$ breaksdown and our weakly nonlinear theory is no longer valid. P_{20} is the pressure due to the mean flow which is produced by the wave modulation. In the limiting case of nocurrent, its expression has been given by Longuet-Higgins and Stewart (1962).

To get the governing equation for A we need to apply the dynamic boundary condition at the first harmonic. For this purpose P_{31} and W_{31} must be calculated. From (3.10.a) and (3.10.b)

$$P_{31} = G(X,T,\tau)\frac{\cosh k(z+h)}{\cosh kd} - \frac{i}{C_g} D_\tau \frac{\sinh k(z+h)}{\cosh kd} - \frac{i(z+h)\sinh k(z+h)}{\cosh kd} A_X$$

$$- \frac{3}{2}\frac{k^4}{\rho^2\sigma^4}\frac{\cosh 3k(z+h)}{\cosh^3 kd \sinh^2 kd}|A|^2 A - (z+h)^2 \frac{\cosh k(z+h)}{2C_g^2 \cosh kd} A_{\tau\tau}\frac{-iA}{\cosh kd}$$

$$\{[\frac{k(z+h)^2}{2}]_X \cosh k(z+d) + (\frac{2k}{\sigma} U_X - (kd)_X \tanh kd)(z+h)\sinh k(z+h)\} \qquad (3.42)$$

where G is a new arbitrary function which correponds to an homogeneous solution. To obtain W_{31} it is easier to use directly the z-momentum equation (2.27) at the order $O(\varepsilon^3)$ for m = 1. This equation can be put in the form

$$-i\sigma W_{31} + \frac{1}{\rho}\frac{\partial P_{31}}{\partial z} = Q_{31} \qquad (3.43)$$

where Q_{31} is a sum of lower order terms and is listed in Appendix A1. Using (3.42) in (3.43) and replacing the lower order term by their respective values we get:

$$W_{31} = -\frac{ik^2 \sinh k(z+h)}{\rho^2\sigma^2(C_g - U)\cosh kd} P_{20} A - \frac{ik}{\rho\sigma}\frac{\sinh k(z+h)}{\cosh kd} G + \underline{0}\cdot|A|^2\underline{A}$$

$$- \frac{k\ D_\tau}{\rho\sigma C_g \cosh kd}[(z+h)\cosh k(z+h) + (\frac{1}{k} - \frac{(C_g - U)}{\sigma}\sinh k(z+h)]$$

$$- \frac{k\ A_X}{\rho\sigma \cosh kd}[(z+h)\cosh k(z+h) + (\frac{1}{k} + \frac{U}{\sigma})\sinh k(z+h)]$$

$$- \frac{k \sinh k(z+h)}{\rho\sigma^2 \cosh kd} A_T + \frac{ikA_{\tau\tau}}{\rho\sigma C_g^2 \cosh kd}[\frac{(z+h)^2}{2}\sinh k(z+h) + (\frac{1}{k} - \frac{(C_g - U)}{\sigma})$$

$$(z+h)\coh\, k(z+h) - \frac{(C_g-U)}{\sigma}\left(\frac{1}{k} - \frac{C_g-U}{\sigma}\right)\sinh\, k(z+h)] - \frac{A}{\rho\sigma\, \cosh\, kd}$$

$$[\frac{(k^2)}{4}X\,(z+h)^2\,\sinh\, k(z+h) + (z+h)\,\cosh\, k(z+h)(k_X + kk_X\,\frac{U}{\sigma} + k^2\frac{U_X}{\sigma}$$

$$- k(kd)_X\,\tanh\, kd) + kh_X\,\cosh\, k(z+h) + k^2h_X(z+h)\sinh\, k(z+h)$$

$$+ \sin\, k(z+h)\underline{((\frac{kU}{\sigma})_X - (kd)_X}\,\tanh\, kd(1 + \frac{kU}{\sigma})) + \cosh\, k(z+h)$$

$$\underline{(\frac{k}{\sigma}\,\frac{\sinh\, k(z+h)}{\cosh\, kd})_T}] - \frac{i\,k^2\sinh\, k(z+h)}{\rho\sigma^2\cosh\, kd}\,AF_1 \qquad (3.44)$$

This expression is similar to Turpin's (1981) except for the terms which are underlined. This difference in the two results is due to the fact that a term is missing in his expression for Q_{31} and that an error has been made in the evaluation of two other terms (see Appendix A1).

Finally using (3.42) and (3.44) in (3.10.f) gives the governing equation for A:

$$\alpha_1 A + \frac{1}{C_g}\,A_T + A_X + i\alpha_2 A_{\tau\tau} + i\alpha_3|A|^2A + iQA + \alpha_4 G + \alpha_5 D_\tau = 0 \qquad (3.45)$$

After a lengthy algebra the coefficients of this equation are found to be:

$$\alpha_1(X,T) = \frac{1}{2C_g}\{(\frac{\cosh\, k(z+h)}{\cosh\, kd})_T\Big|_{z=\xi} + \frac{\sigma}{k\beta}\,(\frac{k}{\sigma}\,\frac{\sinh\, k(z+h)}{\cosh\, kd})_T\Big|_{z=\xi}$$

$$+ U_X(2 + \frac{kd}{\beta}\,(1-3\beta^2)) + \sigma d_X\,(kd(\beta^2-1) - \frac{(\beta^2-1)}{\beta} - \frac{2k\beta U}{\sigma})$$

$$+ \sigma\, k_X(d^2(\beta^2-1) + \frac{d}{k}\,\frac{(1-\beta^2)}{\beta}(1+\frac{kU}{\sigma}) + \frac{U}{\sigma k}) - \frac{U\sigma_X}{\sigma}\} \qquad (3.46)$$

$$\alpha_2(X,T) = \frac{1}{2C_g\sigma(1-\beta^2)}\left[\left(\frac{C_g-U}{C_g}\right)^2(1+3\beta^2) + \left(\frac{C_p}{C_g}\right)^2 - \frac{2C_p(C_g-U)}{C_g^2}(1+\beta^2)\right] \quad (3.47)$$

$$\alpha_3(X,T) = \frac{k^4}{4\rho^2\sigma^3\beta^2C_g}\left[9 - 12\,\beta^2 + 13\,\beta^4 - 2\,\beta^6 - \frac{2\beta^2(C_g-U)^2}{gd - (C_g-U)^2}\right.$$

$$\left.\left(\frac{2C_p}{C_g-U} + 1 - \beta^2\right)^2\right] \quad (3.48)$$

$$Q(X,T) = \frac{2k}{C_g}F_1 + \frac{k\sigma}{\beta C_g}(1-\beta^2)F_2 + \left[\frac{k^2}{\rho\sigma C_g}\left(\frac{2C_p}{(C_g-U)} + 1 - \beta^2\right)\frac{gd}{gd-(C_g-U)^2}\right]F_3 \quad (3.49)$$

$$\alpha_4 = \alpha_5 = 0 \quad (3.50)$$

where $\beta = \tanh kd$ and $C_p = \frac{\sigma}{k}$ (relative phase velocity). It should be noted that the arbitrary functions G and D representing the second and third order homogeneous solutions disappear from (3.45) because of (3.50) and that Q is a real function of (X,T) depending on the unknown functions F_1, F_2, F_3.

3.e. Evolution Equation

So far the evolution equation depends on the unknown function Q(X,T). Lets consider a wave group started from rest, at a finite distance, the wave group will be still localized i.e. the motion becomes zero as $T \to +\infty$. Therefore, from Eqs. (3.36), (3.39) and (3.41) we deduce that

$$F_1 = F_2 = F_3 = 0 \quad (3.51)$$

so Q will be identically equal to zero. In the particular case of a stationary current i.e. with no dependence on T, we can get rid of Q without saying that there is no motion at $\tau = \infty$. Since Q is then a function of X only we introduce

$$B = A \exp i \int^{X} Q(u)du \tag{3.52}$$

Because Q is real B and A have the same amplitude, however B satisfies an equation similar to (3.45) without the Q term. In this case Q affects the phase of A only. For these reasons Q will not be considered in what follows. The governing equation of the first order wave amplitude becomes

$$\alpha_1 A + \frac{1}{C_g} A_T + A_X + i\alpha_2 A_{\tau\tau} + i\alpha_3 |A|^2 A = 0 \tag{3.53}$$

It is a nonlinear cubic Schrodinger equation with variable coefficients. These coefficients can be further simplified by using the governing equation for the current (2.18, 2.15), the conservation of wave crest (3.2), the dispersion relation (3.17) and the expression of the group velocity (3.29).

$$\alpha_1(X,T) = -\frac{1}{2\sigma C_g}\frac{\partial\sigma}{\partial T} + \frac{\sigma}{2C_g}\frac{\partial(C_g/\sigma)}{\partial X} \tag{3.54.a}$$

$$\alpha_2(X,T) = \frac{1}{2\sigma}\frac{(C_g-U)^2}{C_g^3}\left[1 - \frac{gd}{(C_g-U)^2}(1-\beta^2)(1-\beta kd)\right] = -\frac{1}{2C_g^3}\frac{\partial^2\omega}{\partial k^2} \tag{3.54.b}$$

$$\alpha_3(X,T) = \frac{k^4}{4C_g\rho^2\sigma^3\beta^2}\left\{9-10\beta^2+9\beta^4 - \frac{2\beta^2(C_g-U)^2}{gd-(C_g-U)^2}\left[4\left(\frac{C_p}{C_g-U}\right)^2 + 4\left(\frac{C_p}{C_g-U}\right)(1-\beta^2) + \frac{gd}{(C_g-U)^2}(1-\beta^2)^2\right]\right\} \tag{3.54.c}$$

From the linear theory Turpin (1981) guessed that α_1 should have the form (3.54.a), however he was unable to deduce it because of the errors made in the third order velocity. Our derivation shows the validity of this assumption.

In the particular case where there is no current (U = 0) and no dependence on T we recover the evolution equation found by Djordjevic and Redekopp (1978):

$$\alpha_1' A + A_X + i\alpha_2' A_{\tau\tau} + i\alpha_3' |A|^2 A = 0 \tag{3.55}$$

where α_1' can be put in the same form as Djordjevic and Redekopp's parameter μ by using the dispersion relation and the expression of the group velocity.

$$\alpha_1'(X) = \frac{\omega}{2C_g}\frac{\partial(C_g/\omega)}{\partial X} = \frac{\omega}{2k\beta C_g}(1-\beta^2)(1-kh\beta)\frac{dkh}{dX} \tag{3.56.a}$$

$$\alpha_2'(X) = \frac{1}{2\omega C_g}\left[1 - \frac{gh}{C_g^2}(1-\beta^2)(1-\beta kh)\right] \tag{3.56.b}$$

$$\alpha_3'(X) = \frac{k^4}{4C_g\omega^3\rho^2\beta^2}\{9-10\beta^2+9\beta^4 - \frac{2\beta^2C_g^2}{gh-C_g^2}(\frac{4C_p^2}{C_g^2} + 4\frac{C_p}{C_g}(1-\beta^2)$$

$$+ \frac{gh}{C_g^2}(1-\beta^2)^2\} \tag{3.56.c}$$

where $\beta = \tanh kh$ and $C_p = \frac{\omega}{k}$

As a limiting case, the result of linear theory can be found by neglecting the nonlinear term $|A|^2A$ and taking A to be a function of the very slow variables (X,T) only. (3.53) becomes then:

$$\frac{1}{C_g} A_T + \frac{\sigma}{2C_g} \frac{\partial (C_g/\sigma)}{\partial X} A - \frac{1}{2\sigma C_g} A + A_X = 0 \qquad (3.57)$$

Let's take (357)* $\cdot \frac{C_g A}{\sigma}$ + (3.57) $\cdot \frac{C_g A^*}{\sigma T}$ we obtain:

$$\frac{\partial}{\partial T} \left(\frac{|A|^2}{\sigma}\right) + \frac{\partial}{\partial X} \left(\frac{C_g |A|^2}{\sigma}\right) = 0 \qquad (3.58)$$

which is just the conservation of wave action. The evolution equation (3.53) is therefore an extended law of conservation of wave action.

The coefficient α_2 of the evolution equation is a measure of the wave dispersion. From (3.54.b) it can be seen that α_2 remains positive for any value of the wave period, the current velocity and depth. The coefficient α_3 is a measure of the nonlinearity. Let us introduce.

$$Y = \frac{4C_g \rho^2 \sigma^3 \beta^2}{k^4} \alpha_3(X,T)$$

Y can be easily seen to be a function of kd only. Y is plotted in fig. (3.1). It shows that Y increases from -9 when kd $\to$ 0 to the assymptotic value 8 when kd $\to +\infty$. Its sign changes from negative to positive across kd $\simeq$ 1.363. So even if the value of α_3 depends on several parameters as ω, U, h, its sign only depends on the value of kd. It should be noted that Y (kd) is the same as X(K) defined by Eq. (30) in Benjamin's paper (1967) where he studied the stability of Stokes waves.

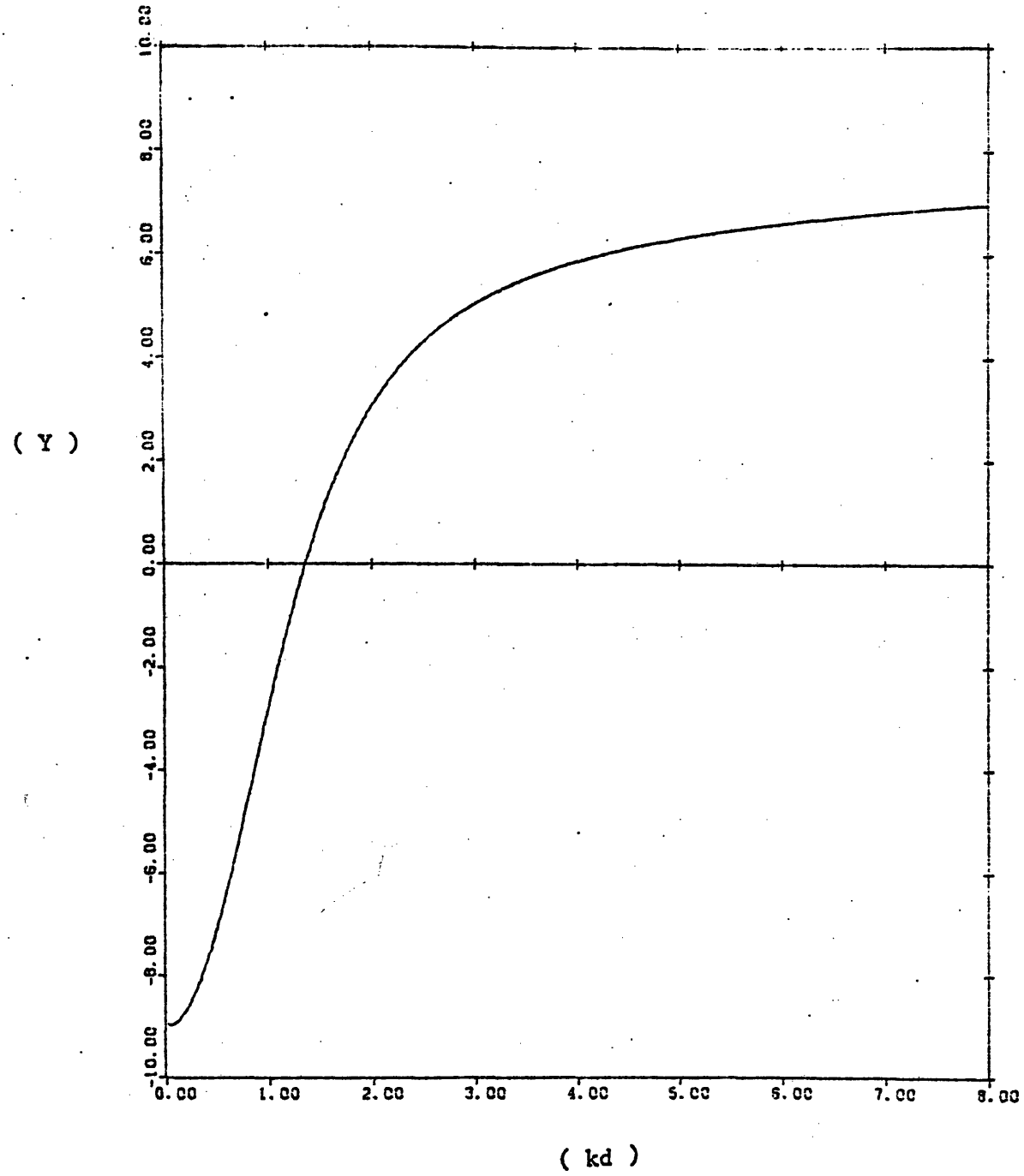


Figure 3.1 Non Linearity Coefficient

4-Wave Packets Over an Uneven Bottom Profile and a Stationary Current

To solve the evolution equation of the first order wave amplitude, we first need to solve Airy's equations for the current, which is in itself a complicated task. However the problem may be simplified in the special case of a stationary current on variable depth i.e. the current does not vary with time. By further assuming that the wave number has no time dependence the conservation of crests Eq. (3.2) implies that ω is constant. Therefore if the boundary conditions of A do not involve T, the problem is reduced to a 2-dimensional one in (X,τ).

4.a. Properties of the Stationary Current

In the case where $\frac{\partial}{\partial T} = 0$, Airy's equations (2.18) and (2.19) can be integrated to give:

$$(\xi+h)U = \text{constant} = U_1 H_1 \tag{4.1}$$

$$\frac{1}{2} U^2 + g\xi = \text{constant} = \frac{U_1^2}{2} \tag{4.2}$$

where U_1 and H_1 are the velocity of the current and the depth as $X \to -\infty$. Following the open channel theory we introduce the total head H and the specific head H_s defined by

$$H = \frac{1}{2}\frac{U^2}{g} + \xi \tag{4.3}$$

$$H_s = \frac{1}{2}\frac{U^2}{g} + d \tag{4.4}$$

H_s represents the height of the total head line above the sea floor. Eq. (4.2) states that the total head is constant i.e. there is a

balance between the kinematic and potential energy. Substituting the expression of U obtained from (4.1) into H_s we get:

$$H_s = \frac{1}{2}\frac{(U_1 H_1)^2}{g d^2} + d \qquad (4.5)$$

The specific head diagram in Fig. (4.1), shows that for a given specific head H_{so} larger than a minimum value H_{sc}, two different flows are possible: The flow will be either at high velocity and small depth (i.e. $U > U_c$, $d < d_c$) or at low velocity and large depth (i.e. $U < U_c$, $d > d_c$). If the specific head is below the critical value H_{sc}, steady flow cannot occur. The magnitude of the total critical depth d_c, at which the minimum value of H_s occurs can be determined by setting to zero the derivative of H_s with respect to d.

$$\frac{dH_s}{d(d)} = -\frac{(U_1 H_1)^2}{g d_c^3} + 1 = 0 \quad \text{or} \quad dc = \sqrt[3]{\frac{(U_1 H_1)^2}{g}} \qquad (4.6)$$

The critical specific head can then be deduced from (4.6) to be

$$H_{sc} = \frac{3}{2}\sqrt[3]{\frac{(U_1 H_1)^2}{g}} \qquad (4.7)$$

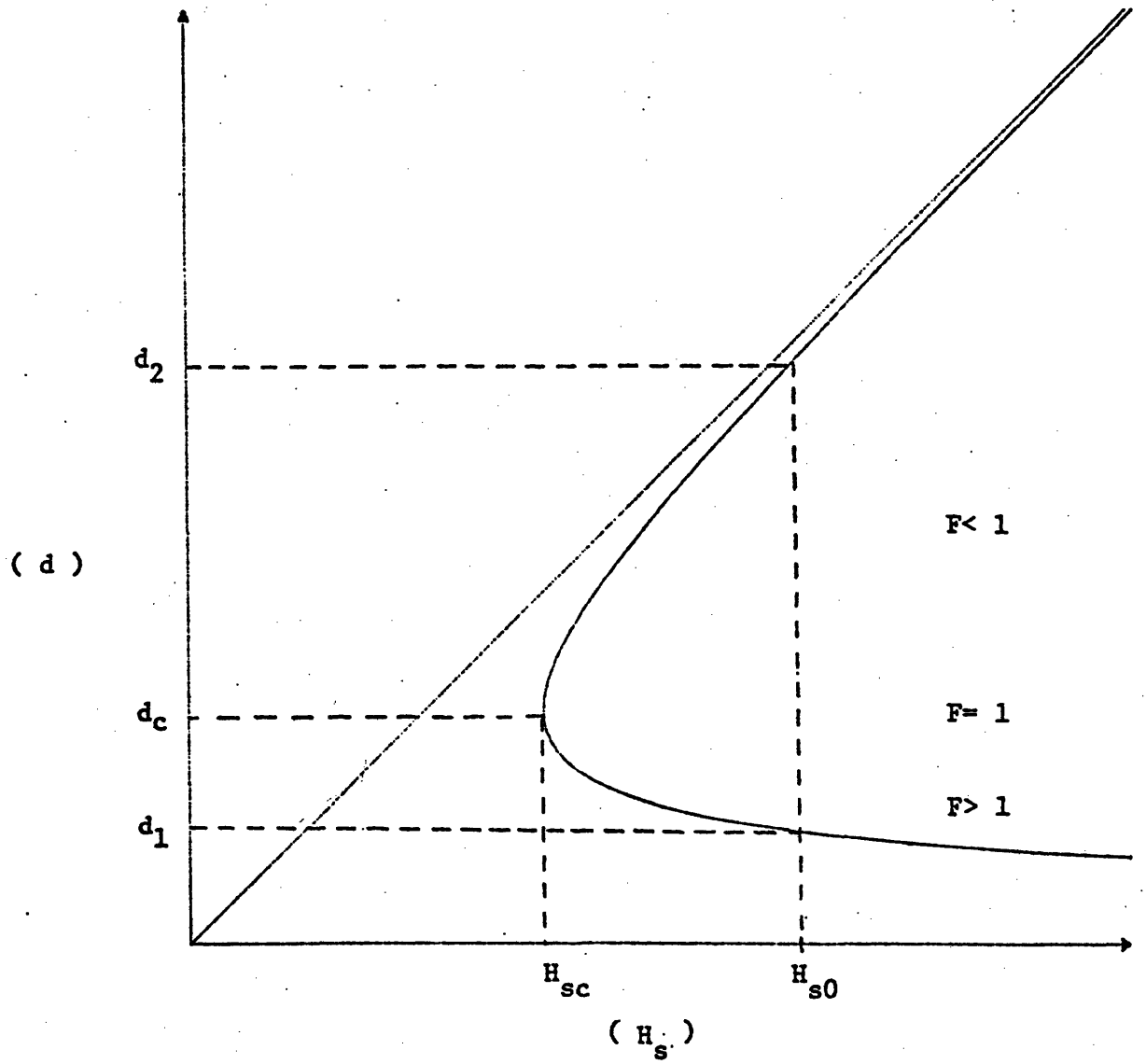


Figure 4.1 Specific Head Diagram

while the total head is found from (4.2) and (4.3):

$$H = H_s - h = \frac{U_1^2}{2g} \tag{4.8}$$

(4.7) can be used in (4.8) to obtain the critical depth h_c

$$h_c = \frac{3}{2}\sqrt[3]{\frac{(U_1 H_1)^2}{g}} - \frac{U_1^2}{2g} \tag{4.9}$$

$\frac{h_c}{H_1}$ is plotted in fig. (4.2) as a function of $\frac{|U_1|}{\sqrt{gH_1}}$. We notice that $h_c \leq H_1$. Therefore if $h \geq H_1$, $H_s > H_{sc}$ always and steady current is possible. When $U_1^2 \geq 3\sqrt{3}\ gH_1$, h_c becomes negative i.e. for such a current the flow is possible for any depth variation. When $U_1^2 \leq 3\sqrt{3}\ gH_1$ the flow is possible only if h is larger than the critical value h_c. At the critical depth , (4.1) and (4.6) determine the critical velocity U_c :

$$U_c = \sqrt{gd_c} \tag{4.10}$$

The Froude number of the flow may be defined as $F = \frac{U}{\sqrt{gH_1}}$.

$F < 1$ corresponds to a subcritical flow, while $F > 1$ corresponds to a supercritical flow. Depending on whether the flow is originally subcritical or supercritical, Fig. (4.1) shows that two totally different surface profiles can be produced: If orginally the flow is subcritical (i.e. $U_1 < \sqrt{gH_1}$) an increase in depth h implies an increase of the total depth $\xi+h$. However if the orginal flow is supercritical (i.e. $U_1 > \sqrt{gH_1}$) an increase in depth implies an increase of the specific

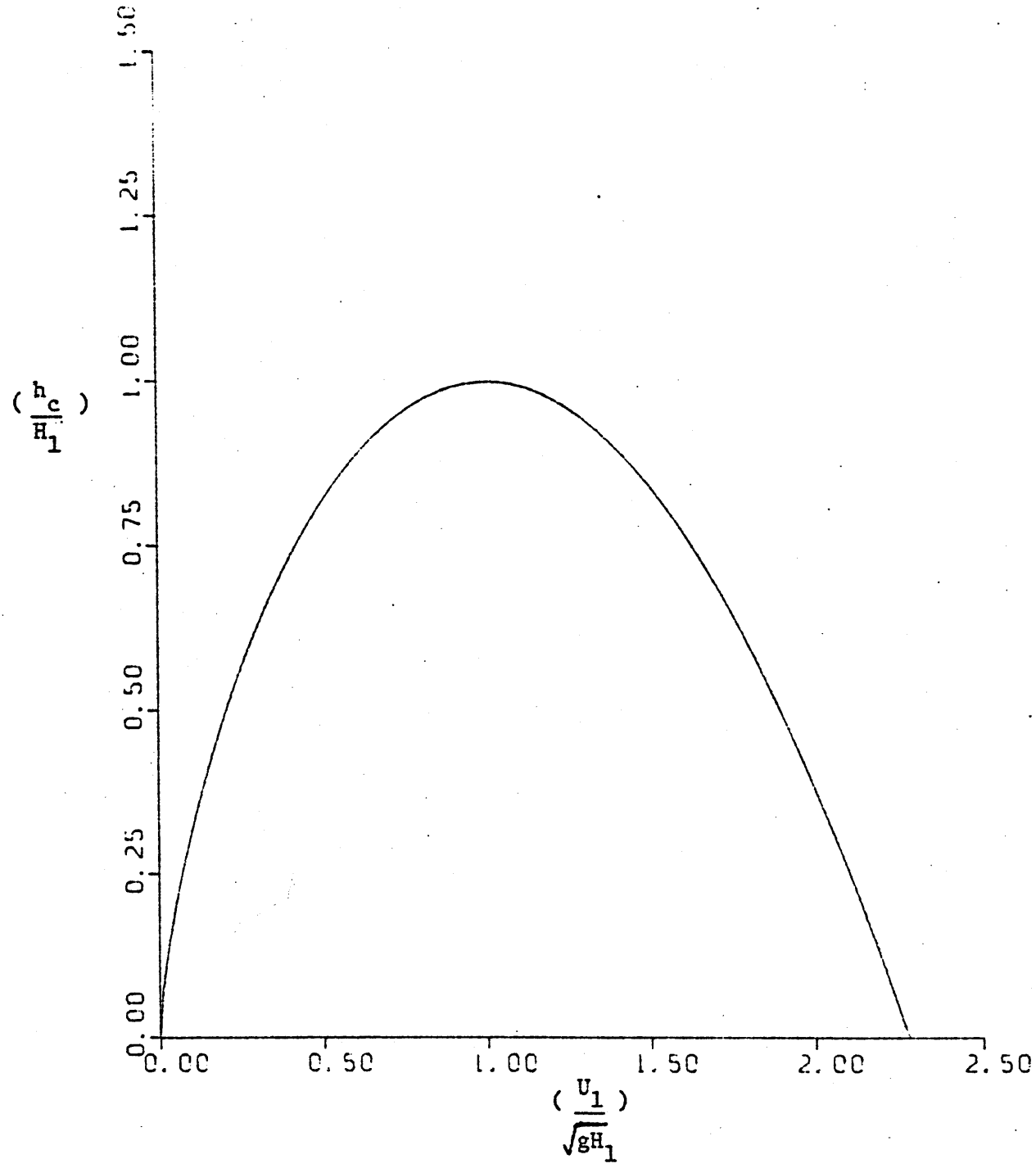


Figure 4.2 Critical Depth

head which will result in a decrease of the total depth d. If the total flow is critical i.e. $U_1 = \sqrt{gH_1}$ when the depth increases the flow will be subcritical or supercritical depending on the conditions downstream. The flow is then unstable i.e. a small perturbation will produce a large variation of the surface profile.

These qualitative results can be illustrated by solving the problem numerically in the different cases. By eliminating U from (4.1) and (4.2) we obtain the governing equation for the total depth d

$$(\frac{d}{H_1})^3 - (\frac{h}{H_1} + \frac{U_1^2}{2gH_1})(\frac{d}{H_1})^2 + \frac{U_1^2}{2gH_1} = 0 \qquad (4.11)$$

$\frac{d}{H_1} = \frac{\zeta+h}{H_1}$ is the root of this equation which satisfies $(\frac{d}{H_1}) \to 1$ when $h \to H_1$. Fig. (4.3) represents the variation of the dimensionless total depth $\frac{d}{H_1}$ as the depth varies from the critical value h_c to a given value $h = 2H_1$ for different values of the current at $X = -\infty$. As expected the plot shows that for a subcritical current $U_1' = \frac{U_1}{\sqrt{gH_1}} = 0.5$ the total depth increases with depth, for a supercritical current $U_1' = \frac{U_1}{\sqrt{gH_1}} = 2$, the total depth decreases as the depth increases and for a critical current $U_1 = \sqrt{gH_1}$, there are two possible solutions which behave very differently.

It should be noted that when the flow is supercritical, the assumptions made in the derivation of the governing equations of the current could break down in particular viscocity and turbulence could become important.

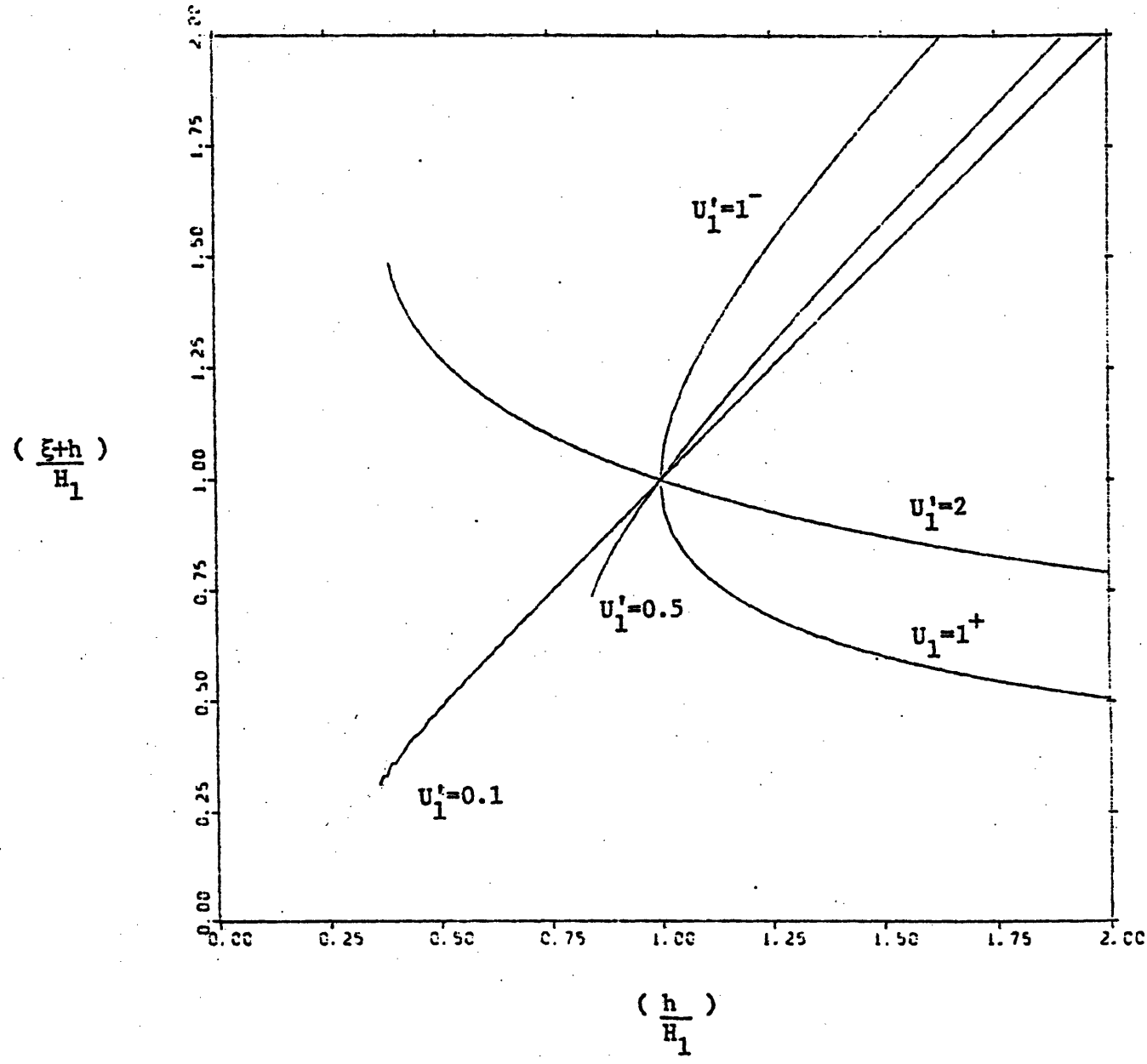


Figure 4.3 Set-up Due to the Current

4.b Wave kinematics

A second step in solving the problem is to solve the dispersion relation.

$$\sigma^2 \equiv (\omega - kU)^2 = gk \tanh kd \tag{4.12}$$

Following Peregrine (1976), we consider the intersection of the two curves:

$$\sigma(k) = \omega - kU \quad \text{and} \quad W(k) = \pm \sqrt{gk \tanh kd} \tag{4.13}$$

These curves are plotted in Fig. (4.4).

If the waves propagate in the same direction as the current i.e. $U > 0$, there are always two solution points A and B. A corresponds to waves whose phase velocity relative to the current is in the same direction as the current (i.e. $C_p = \frac{\sigma}{k} > 0$). Since the frequency of these waves in a fixed frame ω is larger than their frequency relative to the water σ, the effect of the current is to increase the wavelength. B corresponds to waves whose phase velocity and group velocity in a moving frame propagate in the opposite direction of the current (i.e. $C_p = \frac{\sigma}{k} < 0$, $(C_g)_{rel} = \frac{\partial \sigma}{\partial k} < 0$) but a fixed observer sees the waves propagating downstream (i.e. $C_p + U > 0$).

If the waves propagate against the current, i.e. $U < 0$, the solution points, when they exist, are C and D. C represents waves whose crests and energy in a fixed frame propagate upstream (i.e. $\frac{\omega}{k} = C_p + U > 0$, $C_g = \frac{\partial \omega}{\partial k} = (C_g)_{rel} + U > 0$). Since the frequency ω is less than the relative frequency σ, the effect of the current is to shorten the wavelength. D represents waves whose crests in a fixed frame propagate upstream (i.e.

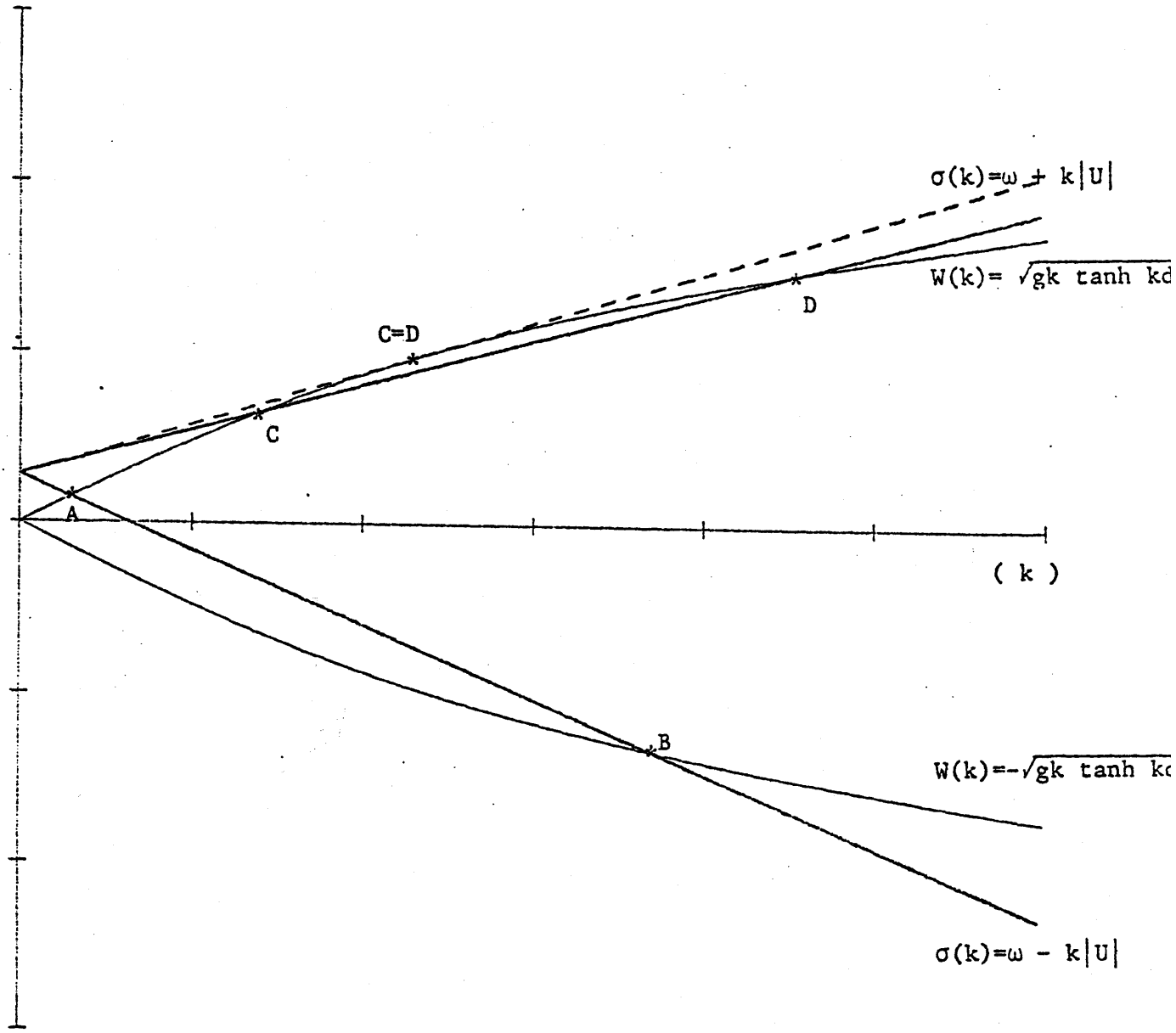


Figure 4.4 Dispersion Relation Diagram

$C_p + U > 0$) but their energy for a fixed observer propagates downstream (i.e. $C_g < 0$). When the opposite current becomes very large the solution C and D may not exist. The limiting case is when the two points C and D coalesce. This case occurs when

$$C_g = \frac{\partial \omega}{\partial k} = (C_g)_{rel} + U = 0 \tag{4.14}$$

The wave energy in a fixed frame is, then at rest. Such a current velocity is called *stopping velocity*. For this velocity the wave number and frequency are both finite but the wave energy density increases and the amplitude becomes very large. Therefore our small amplitude theory is not valid near a stopping velocity. A more refined theory has been given by Stiassnie and Dagan (1979). The stopping velocity is defined by (4.12) and (4.14) which can be combined and put in a dimensionless form. If we introduce

$$U'' = \frac{U}{\sqrt{gd}} \qquad \omega'' = \omega\sqrt{\frac{d}{g}} \qquad k'' = kd \tag{4.15}$$

we have:

$$U'' + \frac{1}{2}\sqrt{\frac{\tanh k''}{k''}}\left(1 + \frac{2k''}{\sinh 2k''}\right) = 0 \tag{4.16}$$

$$\omega'' = k''U'' + \sqrt{k''\tanh k''} \tag{4.17}$$

(4.16) and (4.17) define implicitly ω'' as a function of U' or if we use the dimensionless period $T'' = \frac{2\pi}{\omega''}$, T'' as a function of U''. This function is plotted in figure (4.5.a). It can be easily seen from (4.16) that when $U'' \to -1$, $k'' \to 0$ so $T'' \to \infty$. This means that for an opposite current $U'' \leq -1$, waves cannot propagate no matter how large

their period is. On the other hand when the current tends to zero, waves of any period can always propagate.

In the case of a stationary currents propagating over variable depth the velocity U is dependent on its value U_1 at $-\infty$ and on the depth variation (cf. Eq. 4.1). For given values of $\frac{U_1}{\sqrt{gH_1}} = U_1'$ and $\frac{h}{H_1} = h'$ waves of period $T_s\sqrt{\frac{g}{H_1}} = T_s'$ will be stopped. From U_1' and h', $\frac{d}{H_1}$ and $\frac{U}{U_1}$ can be found by using (4.11) and (4.1) therefore from $T''(U'')$ we can deduce $T_s'(U_1', h')$. Fig. (4.5.b) shows in dimensionless form how the stopping period T_s' varies with h' for different currents $U_1' = -0.1, -0.3, -0.6, -0.9$. Alternatively this plot also shows the depth at which stopping occurs for given T_s' and U_1'.

In subsequent numerical computation we shall only consider the solution A and C of the dispersive relation which corresponds to the cases where the phase and group velocity are positive whether in a moving or in a fixed frame.

4.c. Evolution Equation

In the case of a steady current the evolution equation (3.53) can be further simplified. Let us introduce the following normalized variables:

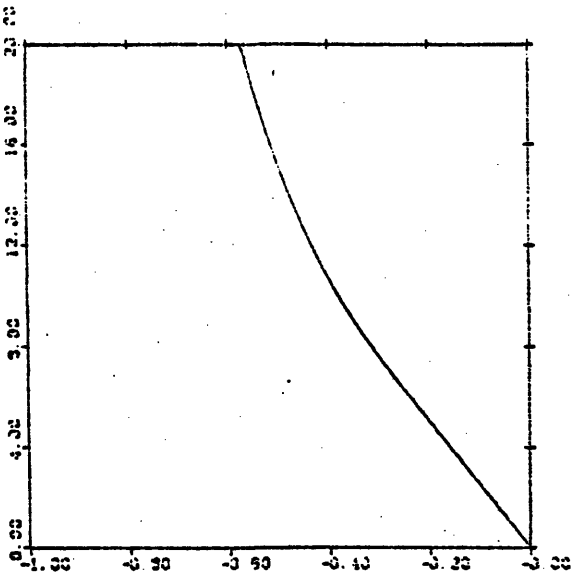
$$(U', C_g', C_p') = (U, C_g, C_p)(gH_1)^{-1/2}$$

$$X' = \frac{X}{H_1}\left(\frac{a}{H_1}\right)^2 \qquad \tau' = \tau\sqrt{\frac{g}{H_1}}\left(\frac{a}{H_1}\right) \tag{4.18}$$

$$A' = A/\rho g a \qquad k' = kH_1$$

$$d' = \frac{d}{H_1} = \frac{(\xi+h)}{H_1} \qquad \sigma' = \sqrt{\frac{g}{H_1}}\,\sigma \qquad T' = \frac{2\pi}{\omega}\sqrt{\frac{H_1}{g}}$$

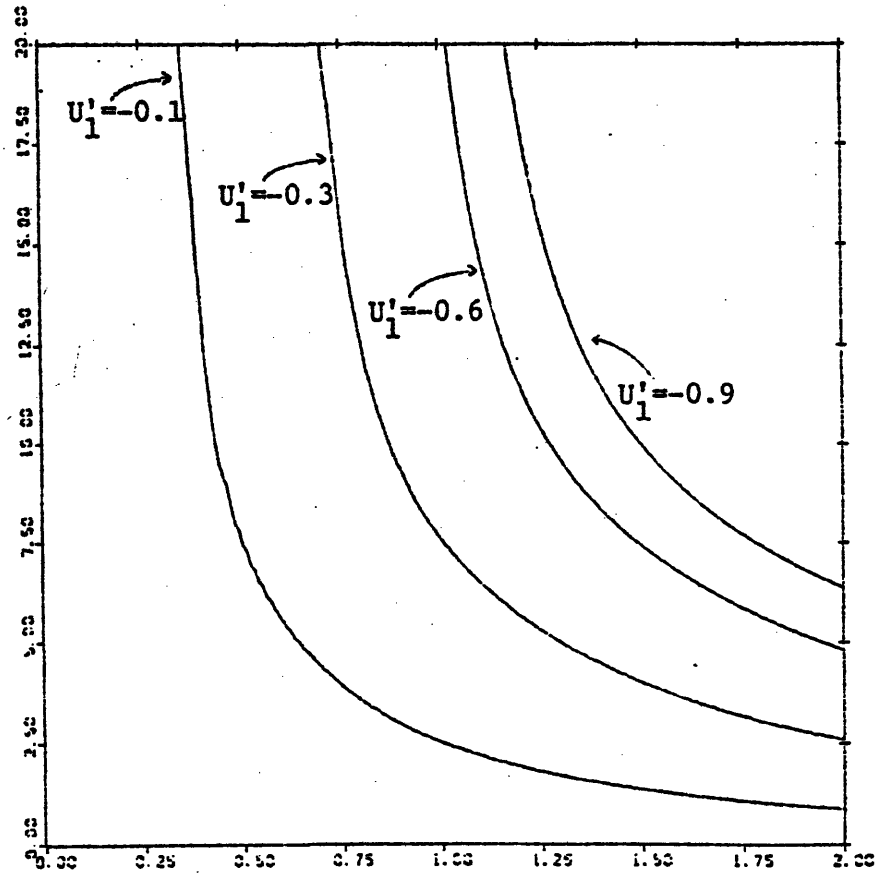
(a)



(T")

(U")

(b)



(T')

(h / H1)

Figure 4.5 Stopping Periods

where a represents the wave amplitude and H_1 represents a reference depth. The evolution equation then becomes after omitting the primes

$$\alpha_1 A + A_X + i\alpha_2 A_{\tau\tau} + i\alpha_3 |A|^2 A = 0 \tag{4.19}$$

$$\alpha_1(X) = \frac{\sigma}{2C_g} \frac{\partial(C_g/\sigma)}{\partial X} \tag{4.20}$$

$$\alpha_2(X) = \frac{1}{2\sigma C_g} \left(\frac{C_g - U}{C_g}\right)^2 \left[1 - \frac{d}{(C_g-U)^2} (1-\beta^2)(1-\beta kd)\right] \tag{4.21}$$

$$\alpha_3(X) = \frac{k^4}{4\sigma^3 \beta^2 C_g} \{9 - 10\,\beta^2 + 9\,\beta^4 - 2\,\beta^2 \frac{(C_g-U)}{d-(C_g-U)^2} \left(4\left(\frac{C_p}{C_g-U}\right)^2 + \frac{4C_p}{C_g-U}(1-\beta^2) + \frac{d(1-\beta^2)^2}{(C_g-U)^2}\right)\} \tag{4.22}$$

For later purposes we introduce the following transformation (Djordjevic and Redekopp (1978))

$$\bar{X} = \int_o^X \alpha_2(u)\,du \tag{4.23}$$

$$A = B \exp\left(-\int_o^X \alpha_1(u)\,du\right) = sB \tag{4.24}$$

Equation (4.19) can be put in its canonical form:

$$-iB_{\bar{X}} + B_{\tau\tau} + K|B|^2 B = 0 \tag{4.25}$$

where $K = \frac{\alpha_3}{\alpha_2} s^2$

and $$s = \exp\left(-\int_o^X \alpha_1(u)\,du\right) = \left(\frac{(C_g/\sigma)_1}{(C_g/\sigma)}\right)^{1/2} \tag{4.26}$$

The subscript 1 denotes the value at $X = 0$. The quantity s depends only on X or $\bar{X}$ and is known to be the shoaling factor of infinitesimal waves.

4.d Generalization of Stokes Waves

In the case of constant depth, the coefficients α_2, α_3 are constants and $\alpha_1 = 0$. The evolution equation (4.19) is just a nonlinear-cubic Schrodinger equation with constant coefficients. Such an equation was first obtained by Zakharov and Shabat (1968) in the case of deep water and then generalized to the case of intermediate depth by Hasimoto and Ono (1972). Its general solution has been obtained by Zakharov and Shabat (1972 - 1973) for both cases $\alpha_2\alpha_3 > 0$ and $\alpha_2\alpha_3 < 0$, by using the Inverse Scattering method. General or particular solutions could also be found in Tappert and Zakusky (1971), Hasimoto and Ono (1972), Satsuma and Yajima (1975).

In the case of variable coefficients, no general solution has been reported and one has to solve the evolution equation numerically. However if the initial and boundary conditions are such that a Stokes wave train is umodulated on the length scale and time scale $O(\varepsilon^{-1})$, then A depends only on X but not on τ and so does B (Eq. 4.24). The governing equation for B becomes

$$B_{\bar{X}} + iK|B|^2B = 0 \tag{4.27}$$

Let us take (4.27)* B - (4.11)B* we deduce that

$$|B| = \text{constant} = B(0) \tag{4.28}$$

Substituting into (4.24) we have:

$$B(\bar{X}) = B(0)\ \exp(-i\ B^2(0) \int_0^{\bar{X}} K\ dv + \phi) \tag{4.29}$$

where ϕ is an arbitrary constant. In terms of the amplitude A, (4.29) can be written as

$$A(X) = B(0)s\ \exp(-i\ B^2(0) \int_0^{X} K(v)dv + \phi) \tag{4.30}$$

Since A(0) = B(0) we have:

$$\left|\frac{A(X)}{A(0)}\right| = s = ((C_g/\sigma)_1/(C_g/\sigma))^{1/2} \tag{4.31}$$

which is nothing but the shoaling law of infinitesimal waves. Thus at the leading order the amplitude of Stokes waves on a variable medium varies with X according to the shoaling law.

4.e Side Band Instability

In the case of constant depth, (4.30) just gives the classical Stokes wave with spatial modulation. It is known that a Stokes wave train is stable to side band disturbance if $K < 0$ or $kh < 1.36$ and is unstable if $K > 0$ or $kh > 1.36$. This result can be extended to the solution (4.30), (4.31). We let B have a dependence on τ:

$$B(X,\tau) = b(X,\tau)\ \exp if\ (X,\tau) \tag{4.32}$$

where b and f are real functions of X and τ. Substituting into the evolution equation (4.27) and taking the real and imaginary part we obtain:

$$\begin{aligned} &\frac{\partial b^2}{\partial X} - \frac{\partial}{\partial \tau}\ (b^2 W) = 0 \\ &\frac{\partial W}{\partial X} + \frac{\partial}{\partial \tau}\ (\frac{b_{\tau\tau}}{b} - W^2 + Kb^2) = 0 \end{aligned} \tag{4.33}$$

where $W = f_\tau$. With reference to (4.30) Stokes wave solution is obtained for b constant and f independent of τ, i.e. $b = B(0) = b_o$, $W = 0$. To consider the linear stability of the Stokes waves we superpose on them a small disturbance $b' << b_o$ and $W' << 1$. Since b' and W' are very small the equation for $b = b_o$ and $W = W'$ is obtained from (4.33) after linearization.

$$\begin{aligned} b'_X - b_o W'_\tau &= 0 \\ W'_X + \frac{b'_{\tau\tau}}{b_o} + b_o b'_\tau &= 0 \end{aligned} \tag{4.34}$$

We assume a wavelike disturbance

$$\begin{pmatrix} b' \\ W' \end{pmatrix} = \begin{pmatrix} \hat{b} \\ \hat{W} \end{pmatrix} \exp i(\rho(x) - \Omega\tau) \tag{4.35}$$

Plugging these expressions into (4.34) we find the eigenvalue condition

$$\frac{d\rho}{dX} = \Omega b_o \sqrt{\frac{\Omega^2}{2b_o^2} - K} \tag{4.36}$$

From the definition of K (4.26), it can be noticed that K has the same sign as $\frac{\alpha_3}{\alpha_2}(X)$. But as shown in section 3, α_2 is always positive and the sign of α_3 is just dependent on the value of $kd = k(\xi+h)$ with respect to 1.363. When $kd < 1.363$, K is negative. Therefore (4.36) shows that $\frac{d\rho}{dx}$ is real and the disturbance will not be amplified. When $kd > 1.363$ K is positive. For small Ω, $\frac{d\rho}{dX}$ is imaginary so '$i\rho$' will be real and monotonic and the disturbance will be unstable i.e. it will decrease or grow exponentially with X while the wave propagates. The quantity kd varies with the current velocity at $X = -\infty$, the wave period and the depth variation.

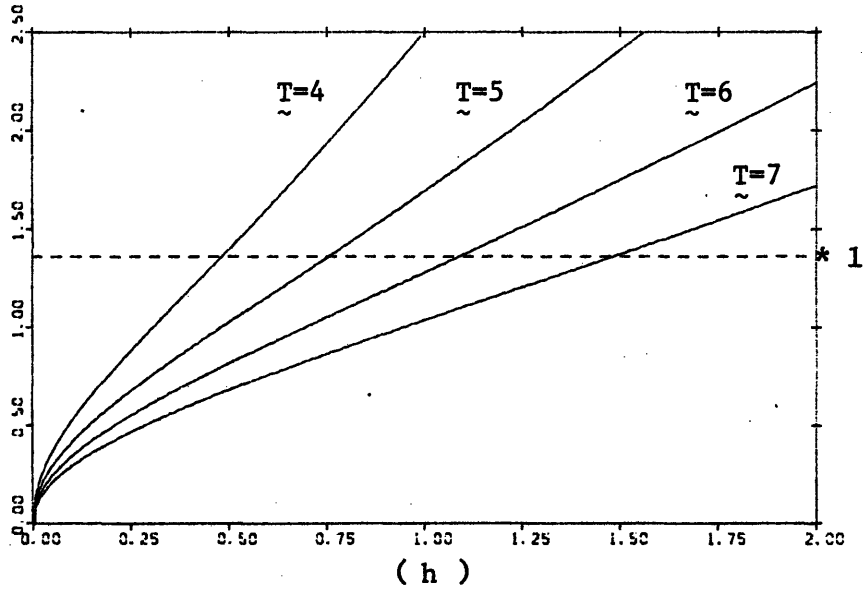
When the waves propagate over variable depth, as long as $k(\xi+h)$ is larger than 1.363 the disturbance will grow but if $k(\xi+h)$ becomes smaller than 1.363 the disturbance will stop growing. So we can have a situation where a periodic wavetrain is unstable at a given depth but becomes stable while it propagates into a region of different depth. Similarly a wavetrain which is stable in one region could become unstable in a region of different depth. To illustrate the effect of variable depth and current on the solution found previously, we have plotted in figures 4.6, 4.7, 4.8 and 4.9 for different values of the current velocity U_1, the variation with depth of the dimentionless wave number kd, of the instability parameter $K = \frac{\alpha_3}{\alpha_2} s^2$ and of the inverse of the shoaling coefficient $s^{-2} = \frac{(C_g/\sigma)}{(C_g/\sigma)_1}$. All the quantities are normalized according to (4.18). Figure (4.6) corresponds to the case without current (i.e. $U_1 = 0$). It shows the variation of kh, K, and s^{-2} as the depth increases from zero to a given value h = 2, for different wave periods $\underset{\sim}{T} = 4,5,6,7$. From the dispersion relationship, we have $kh = 0(\omega\sqrt{h})$ in very shallow water (i.e. $h \to 0$) and $kh = 0(\omega^2 h)$ in deep water (i.e. $h \to \infty$). Therefore as shown in figure (4.6.a) kh increases with depth and passes through the value 1.363. Figure (4.6.b) shows that K will then increase from negative to positive values and cross zero at the point where $kh \tilde{=} 1.363$. This is the well known result of Benjamin-Feir (1967): Stokes wave train are stable in a region where the depth is such that $kh < 1.363$. The effect of increasing the wave period is to move the unstability zone toward deeper water. Figure (4.6.c) shows that except for very shallow water, the shoaling coefficient does not

Figure 4.6 No Current $U_1=0$

(a)

Wavenumber

(kh)

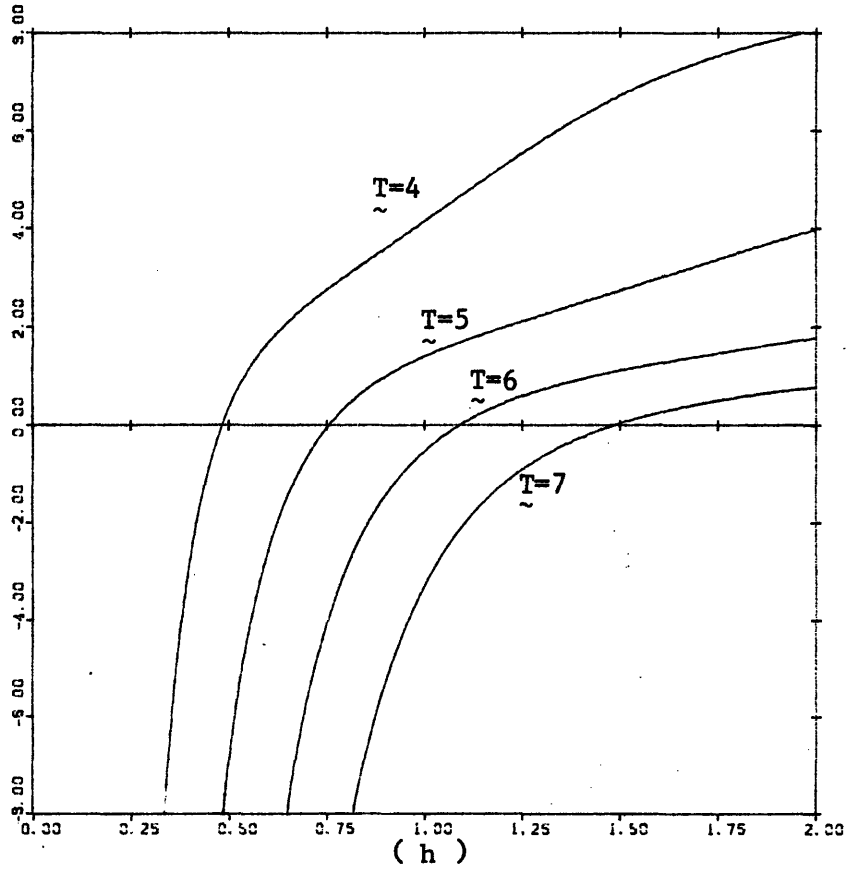


(b)

Instability

Parameter

(K)

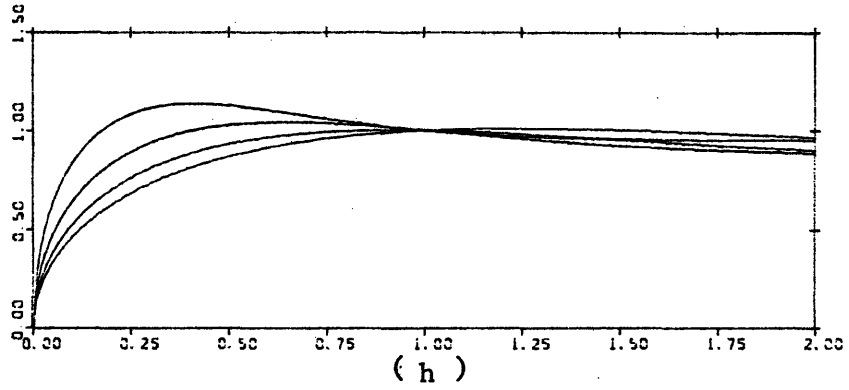


(c)

Inverse of

Shoaling Coef.

(s^{-2})



vary rapidly. So outside the very shallow region where our solution breaks down, Stokes waves amplitude will vary very slowly with depth.

In the case where waves propagate in the same direction as a sub-critical current, the variation with depth of kd, K is similar to that of figure (4.6). Figure (4.7) illustrates this case for a current $U_1 = 0.5$ and wave periods $\underset{\sim}{T} = 3,4,5,6$. The depth varies from the critical depth $h_c = 0.82$ (see fig. 4.2) to a given depth $h = 2$. By comparison with figure (4.6), the effect of the current is to increase the wave length. Therefore, for a given wave period $k(\xi+h)$ crosses 1.363 in deeper water i.e. the instability region moves toward deeper water when the current velocity increases. Figure (4.7.b) shows that K crosses zero at the point where $k(\xi+h) = 1.363$ as predicted in section(4.7). The variations of K are more rapid than in the case without current. Figure (4.7.c) shows that the shoaling coefficient has larger amplitude variation with depth than in the case without current. Stokes waves amplitude will increase substantially with increasing depth.

When the current velocity U_1, becomes supercritical, the total depth $(\xi+h)$ decreases drastically with increasing depth (see Fig. 4.3). Such a behavior implies that $k(\xi+h)$ will decrease as the depth increases. In this case, Stokes waves will be unstable in shallow water and stable in deep water, since K decreases from positive to negative values. Figure (4.8) illustrates these features for a current $U_1 = 2$ and wave periods $\underset{\sim}{T} = 1, 1.5, 2, 2.5$. The depth varies from a critical depth h_c to $h = 2$. Comparing figure (4.8) with figure (4.7) we notice that the variation of $k(\xi+h)$, K and s^{-2} are opposite in each figure. For $U_1 = 2$,

Figure 4.7 Subcritical Current U_1=0.5

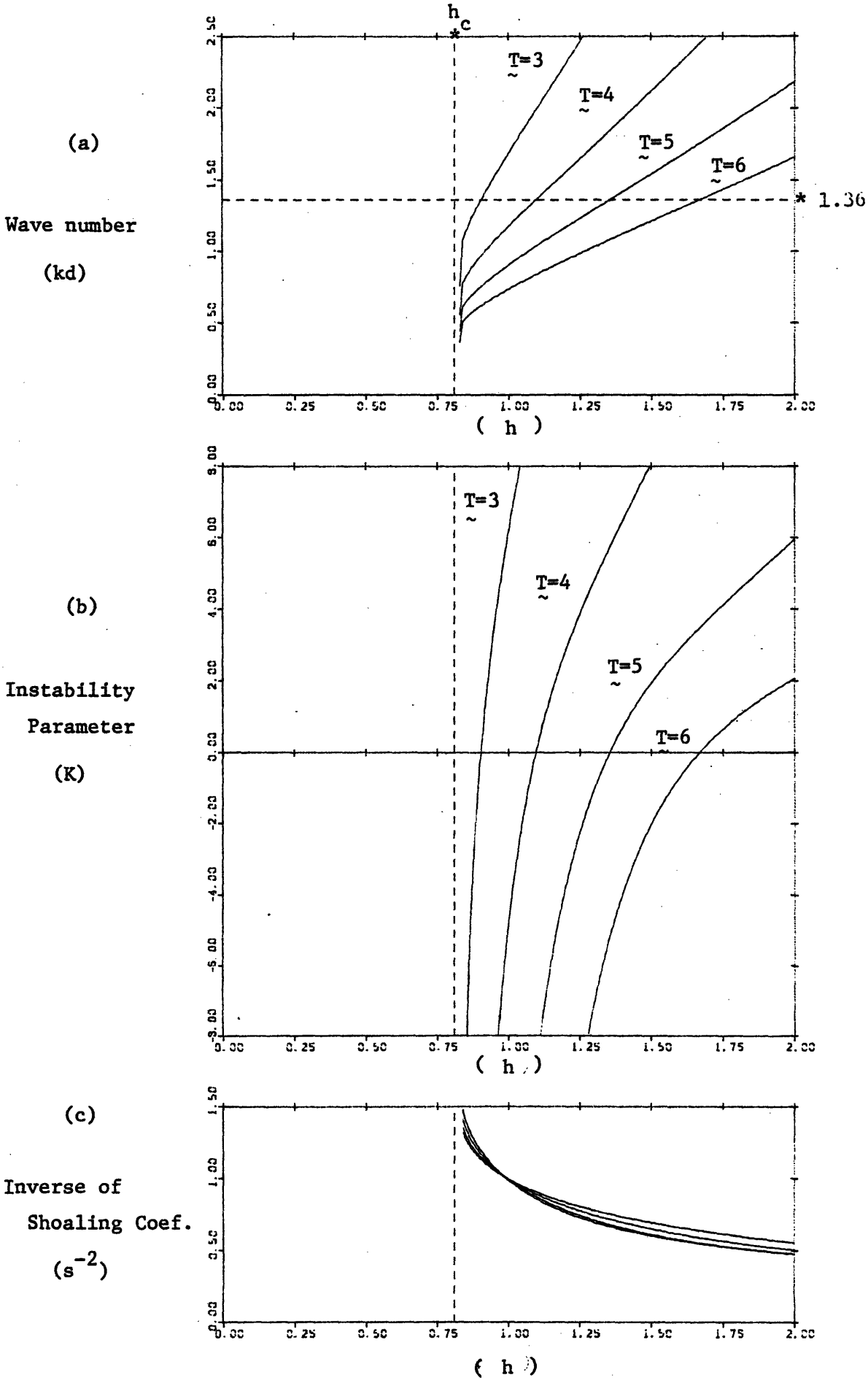
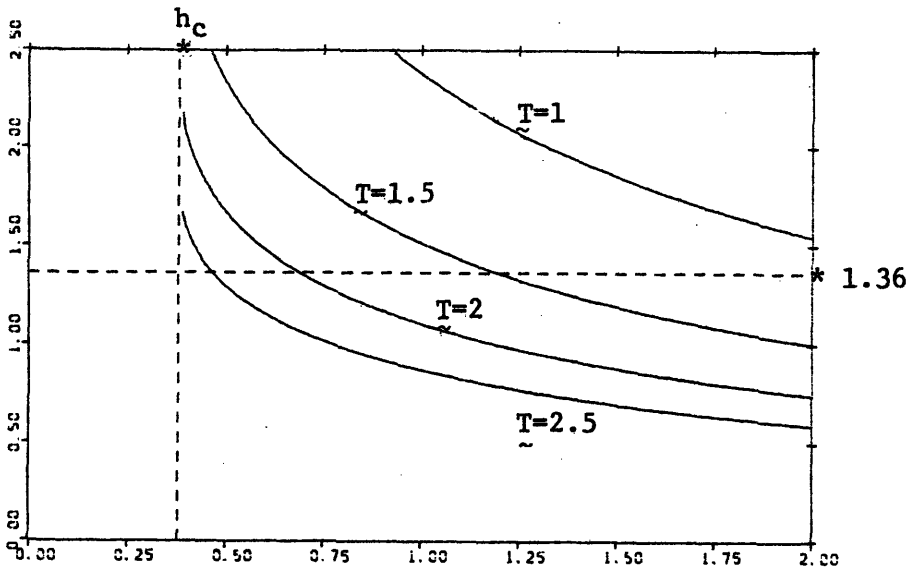


Figure 4.8 Supercritical Current $U_1 = 2.0$

(a)

Wavenumber

(kd)

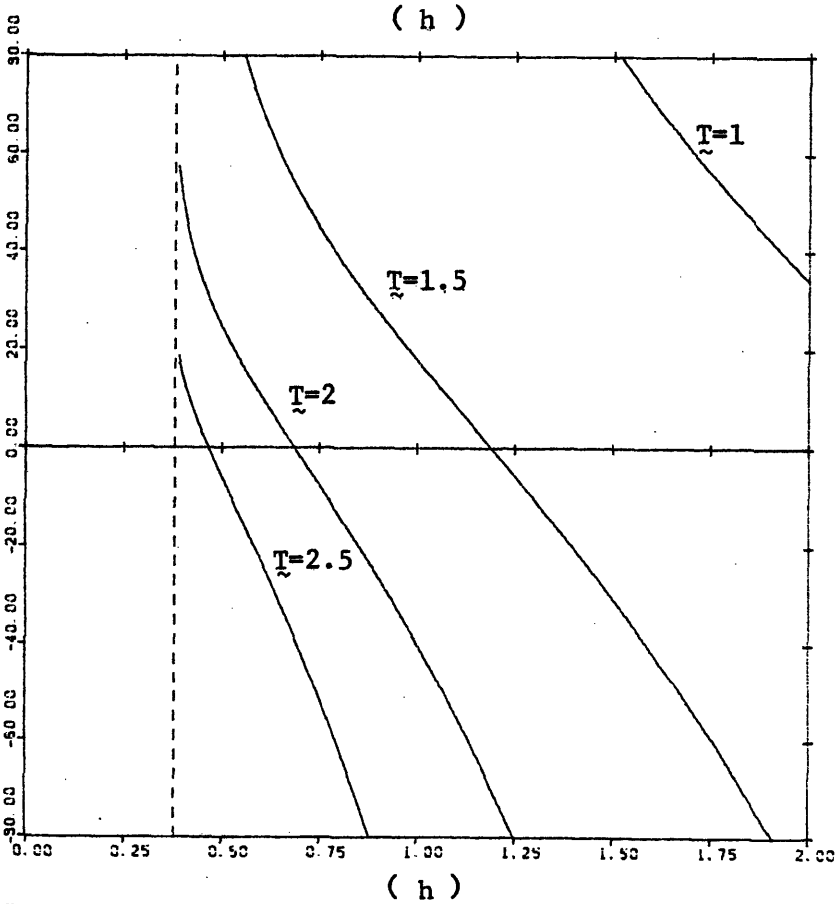


(b)

Instability

Parameter

(K)

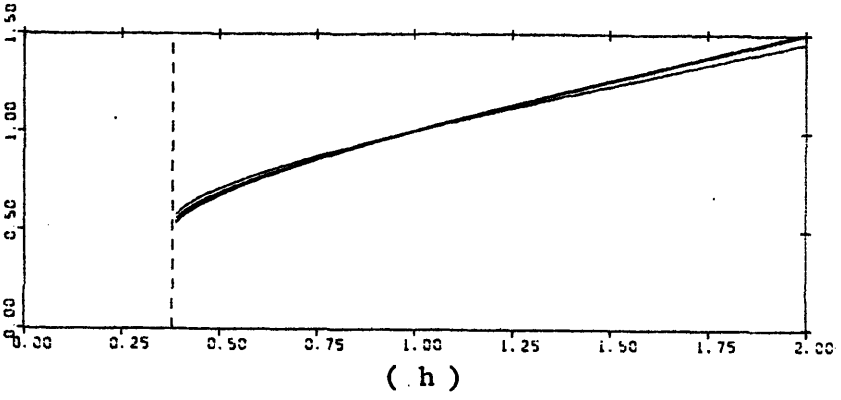


(c)

Inverse of

Shoaling Coef.

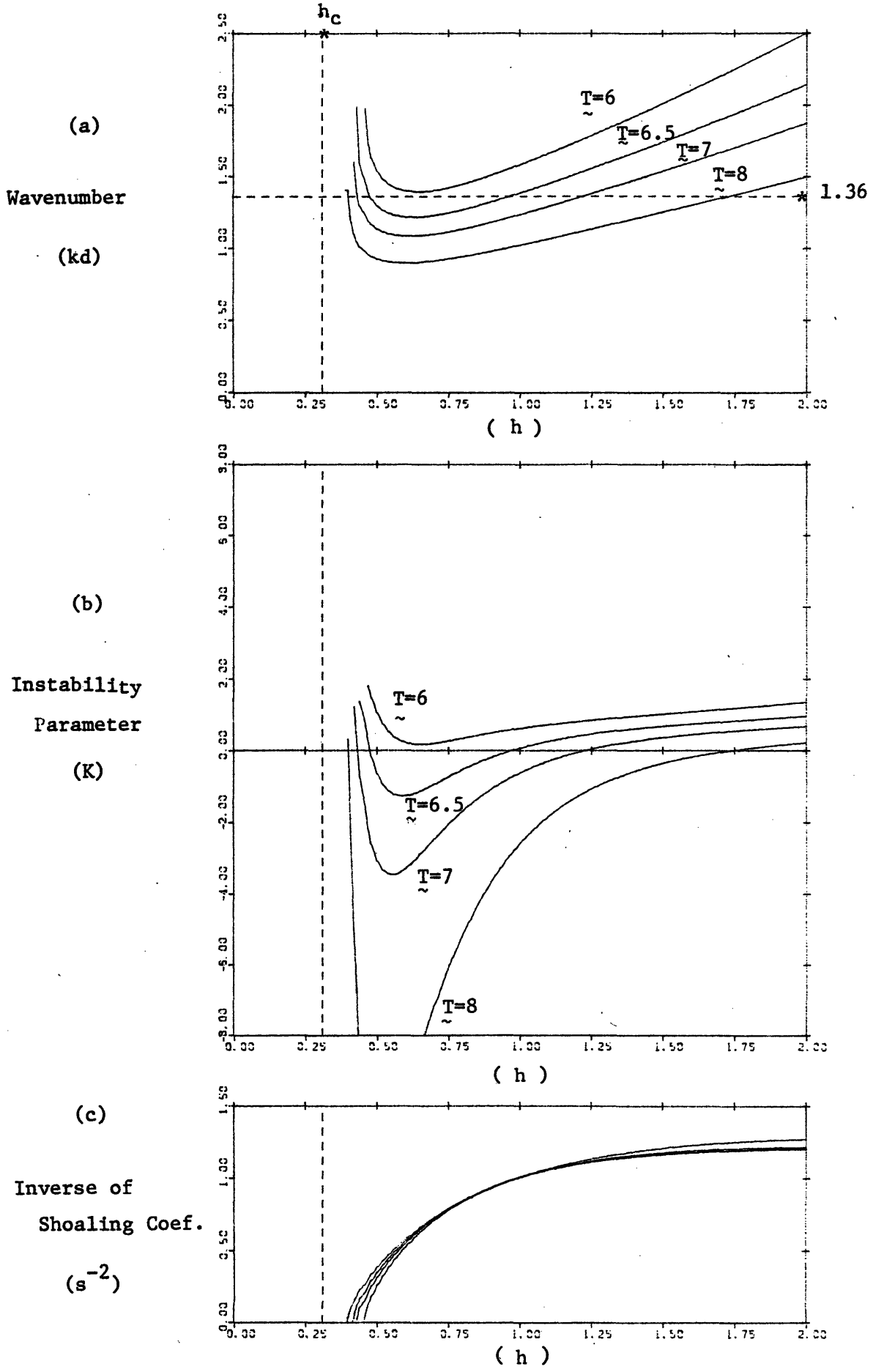
(s^{-2})



K varies more rapidly (For this range of period $\underset{\sim}{T}$ = 1, 2.5, α_2 is very small i.e. $\alpha_2 = 0(10^{-2})$ and hence $K = \frac{\alpha_3}{\alpha_2} s^2$ varies very rapidly).

In the case where waves propagate against a current, for a given velocity and wave period there is a depth h_o for which waves are stopped i.e. $C_g \to 0$ (cf. figure 4.5.b). This depth which could be called the stopping depth corresponds to a cautic. In the neighborhood of this point our nonlinear theory breaks down. Figure (4.9) illustrates this case for a current $U_1 = -0.1$. The wave periods considered in the plot are $\underset{\sim}{T}$ = 6, 6.5, 7, 8 and the depth varies from the stopping point h_o to h = 2. For large depth the variation of $k(\xi+h)$, K, s^{-2} is similar to the variation of these quantities in the case without current, but in the neighborhood of the stopping point $k(\xi+h)$ and K admit a minimum value. This minimum value proceeds from the fact that near the stopping depth the wavelength becomes shorter (i.e. k is larger). When we move away from h_o towards deep water, k decreases rapidly and so does $k(\xi+h)$; but far enough from h_o, the increase in depth becomes larger than the decrease in the wave number, so the trend is reversed and $k(\xi+h)$ increases. For short wave period K can be positive everywhere implying that Stokes waves with a small period will be unstable at any depth, but as the wave period increases K decreases and becomes negative in shallow water. For a given wave period and depth, when the opposite current increases in magnitude the wavelength decreases and so does $k(\xi+h)$. The point where $k(\xi+h)$ crosses the value 1.363 moves toward deeper water so Stokes wave train becomes stable in shallow and intermediate depth.

Figure 4.9 Opposing Current $U_1 = -0.1$



5. Numerical Approach

While the non-linear cubic Schrodinger equation with constant coefficients, has been solved analyically using inverse scattering transform method (Zakharov and Shabat), very few results have been obtained in the case of variable coefficients. Djordjevic and Redekopp (1978) studied analytically the problem of the evolution of a soliton moving over a slowly varying depth, without current. To solve the problem, they made very strong assumptions about the shape of the soliton on the shelf. In order to check these assumptions, it is necessary to solve numerically Eq. (4.19). Furthermore in the case of waves propagating over stationary current and varying depth from a region where $K = \alpha_3/\alpha_2\ s^2 < 0$ into a region where $K > 0$ or inversely, no analytical results are available. One must then resort to numerical solutions. In the following numerical study we restrict ourselves to the case of a compact wave packet propagating over variable depth in the same direction as a stationary current or in the opposite direction.

5.a Numerical Method

The problem we have to solve is:

$$A_X + \alpha_1(X)A + i\,\alpha_2(X)\,A_{\tau\tau} + i\,\alpha_3(X)\,|A|^2A = 0 \qquad (5.1.a)$$

on $$D = \{0 < X < \infty,\ +\infty < \tau < -\infty\} \qquad (5.1.b)$$

$$A(X = 0,\ \tau) = f(\tau) \qquad (5.1.c)$$

$$A(X, \tau) \to 0 \quad \text{as} \quad |\tau| \to \infty \qquad (5.1.d)$$

where f is a prescribed initial wave envelope. The assumption that f is compact (i.e. $f \to 0$ as $|\tau| \to +\infty$) implies that at any finite X the wave envelope is still compact. For our numerical computation, we restrict the region D to a finite region.

$$D_F = \{0 < X < X_2 \; ; -\tau_o < \tau < \tau_o\} \tag{5.2}$$

where τ_o is choosen sufficiently large to avoid any boundary effect when the condition (5.1.d) is applied at $\tau = \pm \tau_o$.

To solve numerically Eq. (5.1.a) an implicit scheme of Crank-Nicholson type for integration in X, and centered second order for differentiation in τ, is used. This scheme is known to be inconditionally stable for the linear case with constant coefficients. It has been used by Yue (1980) for the nonlinear case with constant coefficients and extended by Turpin (1981) for the case with variable coefficients and is found to be stable for reasonable choice of δX and $\delta\tau$. The error due to discretisation is $O((\delta X)^2, (\delta\tau)^2)$. Eq. (5.1.a) then becomes

$$\frac{A_j^{n+1} - A_j^n}{\delta X} + \frac{\alpha_1(n+1) + \alpha_1(n)}{2} A_j^n + i \frac{\alpha_2(n+1) + \alpha_2(n)}{2} \left(\frac{A_{j+1}^{n+1} - 2A_j^{n+1} + A_{j-1}^{n-1}}{2(\delta\tau)^2} \right.$$

$$\left. + \frac{A_{j+1}^{n} - 2A_j^{n} + A_{j-1}^{n}}{2(\delta\tau)^2} \right) + i \frac{\alpha_3(n+1) + \alpha_3(n)}{2} \left(\frac{|\tilde{A}_j^{n+1}|^2 A_j^{n+1} + |A_j^n|^2 A_j^n}{2} \right) = 0 \tag{5.3}$$

where

$$\tilde{A}_j^{n+1} = A_j^n - \delta X \left\{ \alpha_1(n) A_j^n + i\alpha_2(n) \frac{(A_{j+1}^n - 2A_j^n + A_{j-1}^n)}{(\delta\tau)^2} + i\alpha_3(n) |A_j^n|^2 A_j^n \right\} \tag{5.4}$$

$$A_j^n = A((n-1)\delta X,\ (j - J + 1)\delta\tau)\quad n = 1,2,\ldots N;\ j = 1,\ldots 2J+1 \qquad (5.5)$$

N and J are defined by

$$(N-1)\ \delta X = X_2 \quad \text{and} \quad J\delta\tau = \tau_o \qquad (5.6)$$

If the initial data $f(\tau)$ is even in τ, then, since τ appears in (5.1.a) through $A_{\tau\tau}$, the solution $A(X,\tau)$ will be even in τ. By imposing at $\tau=0$ the condition $\frac{\partial A}{\partial X} = 0$ the problem can be solved for $\tau \geq 0$ only. In this case A_n^j will be defined by the following:

$$A_j^n = A((n-1)\delta X,\ (j-2)\delta\tau)\quad n = 1,\ldots N;\ j = 1,\ldots,\ 2J+1 \qquad (5.7)$$

$$\text{where } (N-1)\delta X = X_2 \quad \text{and} \quad (2J-1)\delta\tau = \tau_o \qquad (5.8)$$

Applying (5.3) for a fixed n and $j = 2,\ldots 2J$ we obtain a linear system in A_j^{n+1}:

$$\begin{aligned}
&\gamma_1^n\ A_3^{n+1} + \beta_1^n A_2^{n+1} + \alpha_1^n\ A_1^{n+1} = W_1^n \\
&\gamma_j^n\ A_{j+2}^{n+1} + \beta_j^n A_{j+1}^{n+1} + \alpha_j^n\ A_j^{n+1} = W_j^n \qquad (5.9) \\
&\gamma_{2J-1}^n\ A_{2J+1}^{n+1} + \beta_{2J-1}^n\ A_{2J}^{n+1} + \alpha_{2J-1}^n A_{2J-1}^{n+1} = W_{2J-1}^n
\end{aligned}$$

where

$$\gamma_j^n = \alpha_j^n = i\,\frac{\delta X}{2(\delta\tau)^2}\,\frac{(\alpha_2(n+1)+\alpha_2(n))}{2}$$

$$\beta_j^n = 1 + \frac{\delta X}{2}\left[\frac{\alpha_1(n+1)+\alpha_1(n)}{2} - i\,\frac{\alpha_2(n+1)+\alpha_2(n)}{(\delta\tau)^2} + i\,\frac{\alpha_3(n+1)+\alpha_3(n)}{2}\,|\tilde{A}_j^{n+1}|^2\right]$$

$$W_j^n = A_j^n\left[1 - \frac{\delta X}{2}\left(\frac{\alpha_1(n+1)+\alpha_1(n)}{2} - i\,\frac{\alpha_2(n+1)+\alpha_2(n)}{(\delta\tau)^2} + i\,\frac{\alpha_3(n+1)+\alpha_3(n)}{2}\,|A_j^n|^2\right)\right]$$

$$-i\,\frac{\delta X}{2(\delta\tau)^2}\,\frac{\alpha_2(n+1)+\alpha_2(n)}{2}\,(A_{j+1}^n + A_{j-1}^n) \qquad (5.10)$$

Following D. Potter, the solution of the linear system (5.9) can be found by introducing:

$$A_{j+1}^{n+1} = X_j^{n+1}\,A_j^{n+1} + y_j^{n+1} \qquad j = 1,\ldots 2J \qquad (5.11)$$

By plugging (5.11) in the set of equations(5.9) we find the general recurrence relations:

$$X_j^{n+1} = \frac{-\alpha_1^n}{\gamma_i^n X_{j+1}^{n+1} + \beta_j^n} \qquad y_j^{n+1} = \frac{W_j^n - y_{j+1}^{n+1}}{\gamma_j^n X_{j+1}^{n+1} + \beta_j^n} \qquad j = 2J-2,\ldots,1 \qquad (5.12)$$

Since $A \to 0$ as $\tau \to +\infty$ we have $A_{2J+1}^{n+1} = 0$ which implies from (5.9.c)

$$X_{2J-1}^{n+1} = \frac{-\alpha_{2J-1}^n}{\beta_{2J-1}^n} \qquad y_{2J-1}^{n+1} = \frac{W_{2J-1}^n}{\beta_{2J-1}^n} \qquad (5.13)$$

If the initial perturbation $A(0,\tau)$ is not even in τ, the condition $A \to 0$ as $\tau \to -\infty$ gives $A_1^n = 0$ which implies from (5.9.a)

$$A_2^{n+1} = \frac{W_1^n - \gamma_1^n y_2^{n+1}}{\gamma_1^n X_2^{n+1} + \beta_i^n} \tag{5.14}$$

If the initial perturbation $A(0,\tau)$ is even in τ, we solve the problem for $\tau \geq 0$ only by imposing the condition $\frac{\partial A}{\partial \tau}(X, \tau) = 0$. Using the notations (5.7) and (5.8), this condition becomes:

$$\frac{A_3^{n+1} - A_1^{n+1}}{2(\delta\tau)} = 0 \qquad \text{or} \quad A_3^{n+1} = A_1^{n+1} \tag{5.15}$$

which gives after using (5.11):

$$A_1^{n+1} = \frac{X_2^{n+1} y_1^{n+1} + y_2^{n+1}}{1 - X_2^{n+1} X_1^{n+1}} \tag{5.16}$$

The numerical procedure can be summarized as follows:

A_j^1 $j = 1 \ldots 2J+1$ is the initial data $A(0,\tau) = f(\tau)$ which is given. The coefficients (5.10) can then be calculated for $j = 1, \ldots 2J-1$. Using (5.12) and (5.13) X_j^2 and y_j^2 $j = 1, \ldots 2J-1$ are found by recurrence. If the initial data is not even in τ, A_2^2 is given by (5.14) and A_j^2 $j = 3 \ldots 2J$ can be determined using (5.11). If the initial data is even in τ, A_1^2 is given by (5.16) and A_j^2 $j = 2 \ldots 2J$ can be determined using (5.11). Once A_j^2 is known, $A_j^3 \ldots A_j^N$ $j = 1 \ldots 2J+1$ can be solved using the same procedure.

5.b Check of the Numerical Results

i. Case of Constant Depth

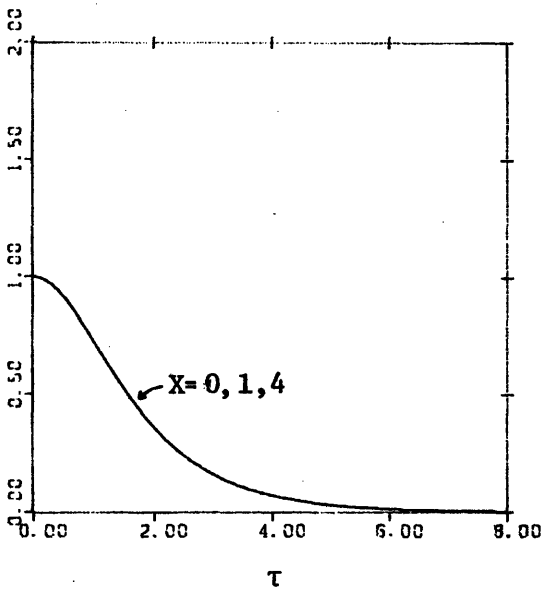
The program has been checked in the case of constant depth, where some exact solutions are known. In particular for $K > 0$ it is known that an initial solution

$$A(X = 0, \tau) = \text{sech}\left(\sqrt{\frac{\alpha_3}{2\alpha_2}}\,\tau\right) \qquad (5.17)$$

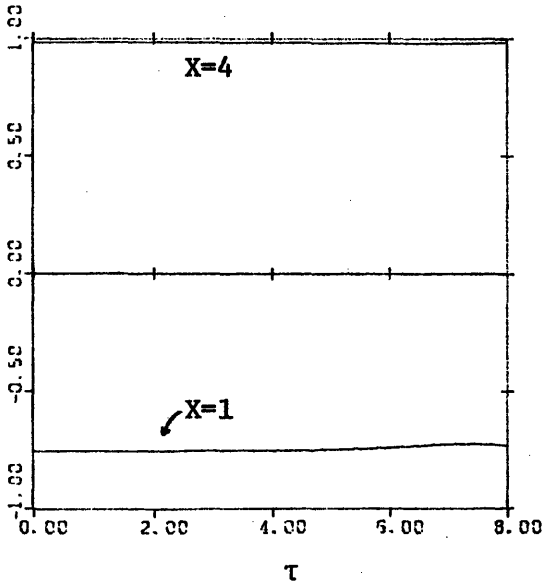
preserve the same shape while it propagates on constant depth (See Hasimoto and Ono (1972)). Fig. (5.1) shows the wave envelope as a function of X and τ. The current velocity at $X = -\infty$, the wave period and the depth have been choosen to be $U_1 = 0$, $\underset{\sim}{T} = 5$, $h = 1$ such that $K = (1.4)$. Since $A(X,\tau)$ is even in τ, the wave envelope has been plotted for $\tau \geq 0$ only. Fig. (5.1.a) is a general view of the wave envelope during its propagation from $X = 0$ to $X = 4$ and for $\tau = 0$ to $\tau_o = 15$. In Fig. (5.1.b) the wave envelope profiles at $X = 0$, $X = 1$ and $X = 4$ have been superposed. It shows that the initial soliton profile is conserved for a distance of propagation larger than $\underset{\sim}{X}_2 = 4$. Fig. (5.1.c) gives the phase of the wave at $X = 1$ and $X = 4$. We can notice that, as expected the phase is independent of τ but varies with X.

ii. Evolution Law

The cubic Schrodinger equation with constant coefficients admits an infinite number of conservation laws, some of them express the conservation of physical quantities as the mass and the energy conservation. When the coefficients are not constants, laws allowing to follow the evolution of certain quantities can be found but they are not of the type of conservaton laws (see Turpin, 1981). Here we derive the first evolution law and use it to check our numerical result. For this purpose we use the canonical evolution equation (4.25). Let us take (4.25)* B- (4.25) B*:



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{Arg(A(X,\tau))}{\pi}$

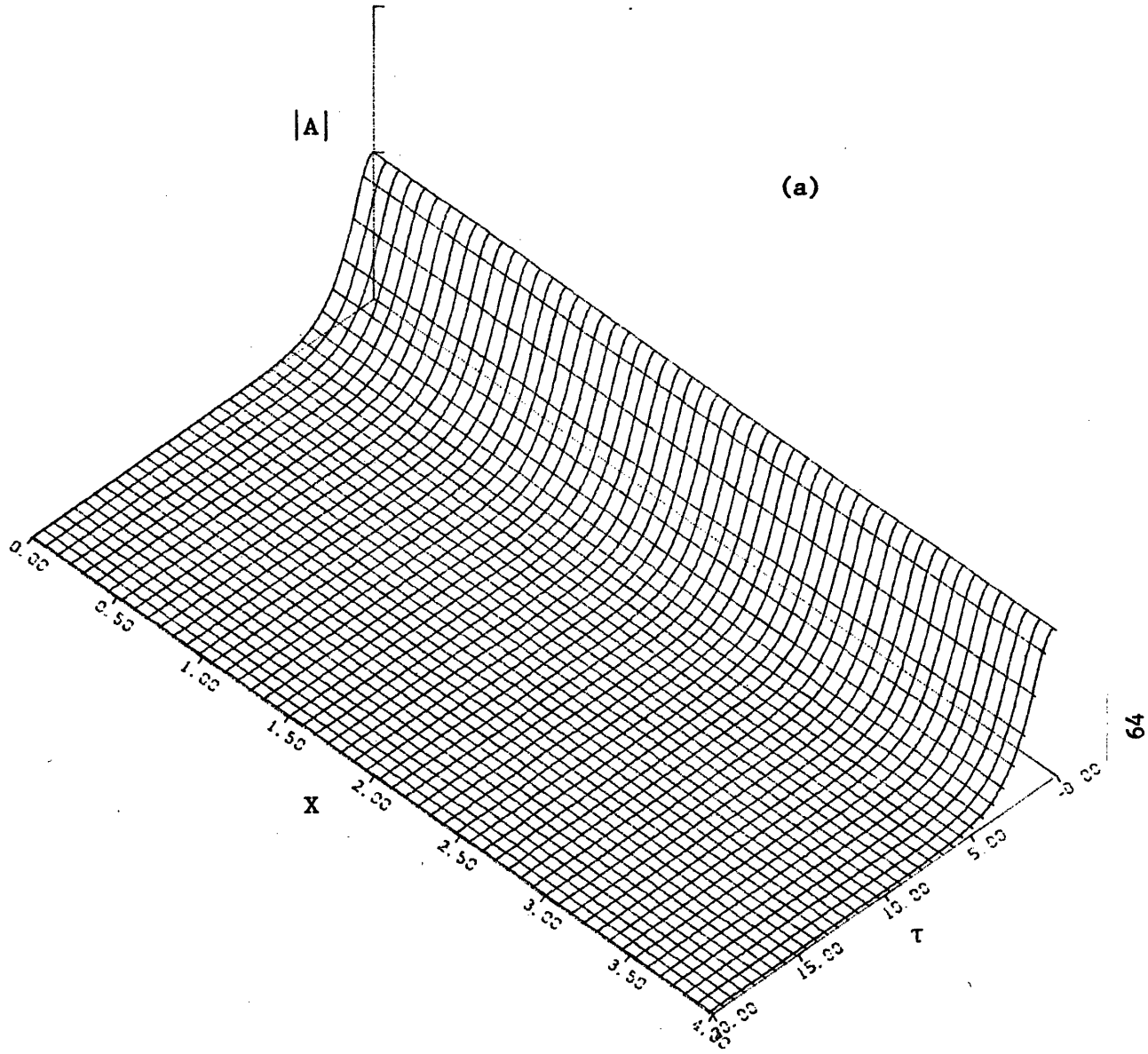


Figure 5.1 Evolution of a Sech Profile over Constant Depth, No Current, K Positive

$$i \frac{\partial |B|^2}{\partial X} + \frac{\partial}{\partial \tau} (B_\tau^* B - B_\tau B^*) = 0 \qquad (5.18)$$

If we assume B and $B_\tau \to 0$ as $\tau \to \pm \infty$, (5.18) gives by integration with respect to τ:

$$\frac{\partial}{\partial X} \int_{-\infty}^{+\infty} |B|^2 \, d\tau = 0 \qquad (5.19)$$

which implies by using (4.24)

$$(C_g/\sigma)(X) \int_{-\infty}^{+\infty} |A(X,\tau)|^2 \, d\tau = \text{constant} \qquad (5.20)$$

This result shows that for waves propagating against a current, when $C_g \to 0$ $\int_{-\infty}^{+\infty} |A(X,\tau)|^2 \, d\tau \to +\infty$. This is just an extension of the linear result $|A(X,\tau)| \to \infty$ in the neighborhood of a caustic. The evolution law (5.20) is used as a measure of the total error due to discretization, round-off and truncation of τ and is satisfied within few percents (less than 5%) for all computed cases in the following sections.

5.c. Geometry of the Problem

In our numerical computation we consider a more specific problem which is summarized in Fig. (5.2). The depth is considered to be constant and equal to 1 for $X \leq 0$, it varies as a cosine curve in the transition zone $0 \leq X \leq L$ i.e. $h = 1 + \frac{dh}{2} (\cos (\frac{\pi X}{L}) - 1)$ and becomes constant and equal to $1 - dh$ in the region $X \geq L$. dh is the height of the step. It is positive when the depth decreases and negative when

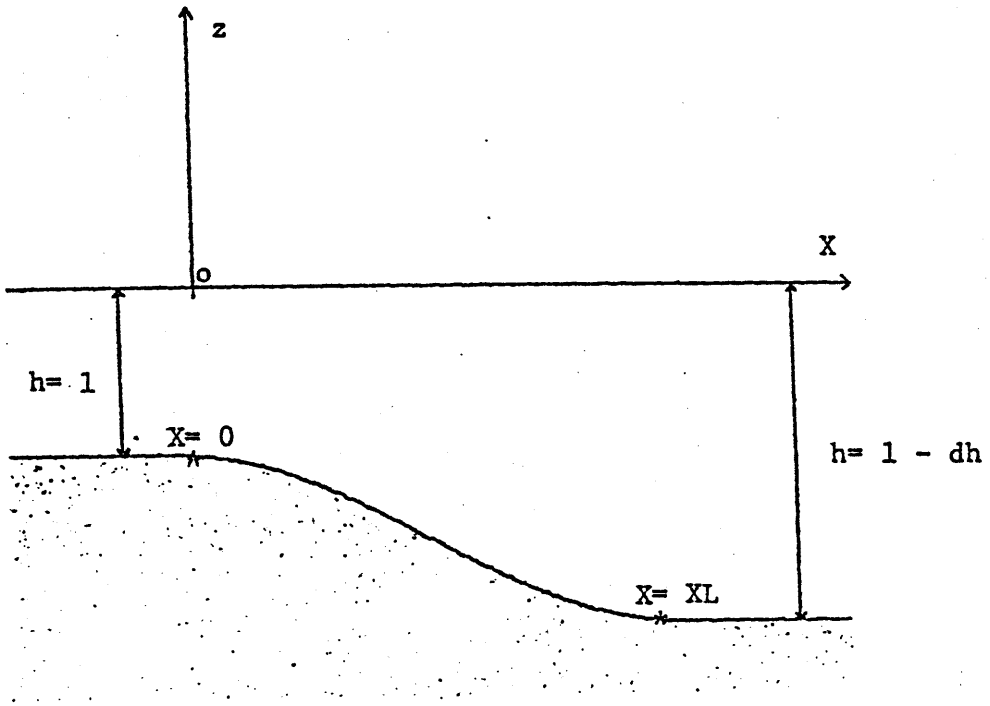


Figure 5.2 Geometry of the Problem

the depth increases. The transition zone L, has been taken equal to 1 in all the computed cases. It should be noted that the depth is normalized with respect to H_1 while the distance L is normalized as X, so the assumption of slowly varying depth is satisfied. U_1 is the current in the region $X \leq 0$. If U_1 is positive waves propagate in the same direction as the current, if U_1 is negative waves propagate in the opposite direction. The current induces a free water displacement $\xi(X)$ which can be found as shown in section (4.a). The original of the vertical axis has been choosen such that $\xi = 0$ in the region $X \leq 0$. $\underset{\sim}{T}$ is the wave period. $\underset{\sim}{T}$ is constant during the propation of the waves. $A(0,\tau)$ is the initial wave envelope profile. It has been choosen in all the cases to be of the form

$$A(0,\tau) = \operatorname{sech}\left(\sqrt{\frac{\alpha_3}{2\alpha_2}\,(X=0)}\;\tau\right) \tag{5.21}$$

where α_3 and α_2 are the coefficients of the nonlinear cubic Schrodinger equation as defined in (4.21) and (4.22). This initial profile represents a soliton solution which is appropriate for the depth h = 1 i.e. if the depth remains constant the wave envelope profile would not change. Since the initial profile is even in τ, we only consider the wave envelope profile for $\tau \geq 0$. τ_o must be choosen sufficiently large so that the condition $A(X,\tau_o) = 0$ does not induce any noise effect in the solution. From (5.21) we can see that τ_o depends on $\frac{\alpha_3}{\alpha_2}$. It also depends on X_2, the distance of propagation considered: the larger X_2 is the larger

τ_o should be. X_2 is the distance of propagation over which the wave envelope is studied. To limit the computation time, in most of the cases this distance has not been choosen to be very large. The program used for the numerical computation has been developed by Turpin (1981) and has been slightly modified to include the case of waves propagating against current. This program is given in appendix A2 .

6. Propagation of a Wave Packet Over Varying Depth

6.a Djordevic - Redekopp's Solution

One purpose of the numerical study is to check the validity of the assumptions made by Djordevic and Redekopp (1978) in solving the problem of a soliton moving over a slowly varying depth in a region where K is positive everywhere. In this section, we present the method they used. It should be noted that their method can be generalized without modifications to the case of a stationary current. In what follows, we refer to Djordevic and Redekopp's paper as D-R. We use the same transformation as in section (4.c):

$$s = \exp - \int_0^{\bar{X}} \alpha_1 \, du \quad \text{where} \quad \bar{X} = \int_0^X \alpha_2(u)du \tag{6.1}$$

$$A(X,\tau) = s(X)\, P(X,\tau)$$

By substituting into (4.19) P is shown to be solution of the equation

$$-i\, P_{\bar{X}} + P_{\tau\tau} + K|P|^2 P = 0 \tag{6.2}$$

We consider an initial perturbation in region (1) ($X \leq 0$) which is a soliton, whose amplitude length ratio is adapted to this region of constant depth i.e.

$$P = P_o \text{ sech } \tilde{K}\tau \quad \text{and} \quad \tilde{K} = P_o \sqrt{\frac{\alpha_3}{2\alpha_2}(1)} \tag{6.3}$$

Djordjevic and Redekopp made the assumption that at the end of the depth variation $X = L$, P conserves its original shape and can be considered as

the initial condition for propagation in region (2). This assumption means that if in region (1):

$$A(0,\tau) = P_o \text{ sech } (P_o\sqrt{\frac{\alpha_3}{2\alpha_2}(1)}\ \tau) \tag{6.4}$$

At the end of the depth variation it is assumed to be:

$$A(L,\tau) = s(L)\ P_o \text{ sech}(P_o \sqrt{\frac{\alpha_3}{2\alpha_2}(1)}\ (\tau-\tau_0)\ ) \tag{6.5}$$

where $s(L) = \sqrt{\frac{(C_g/\sigma)(1)}{(C_g/\sigma)(2)}}$

In region (2) $X \geq L$, this initial condition will in general evolve into N solitons where N is determined from the Zakharov - Shabat eigen-value problem (cf Zakharov and Shabat (1972)). If the potential q of the eigen value problem is of the form $q(\tau) = a \text{ sech } \tilde{K}(\tau-\tau_o)$ then, the number of discrete eigen values with positive real parts i.e. the number of solitons which will emage as $X \to \infty$ is the largest integer matter than $\frac{a}{K} + \frac{1}{2}$. Using the wave envelope (6.5), the associated Zakharov and Shabat eigen value problem has the potential

$$q(z) = \sqrt{\frac{1}{2}\frac{\alpha_3}{\alpha_2}(2)}\ s\ P_o \text{ sech } \tilde{K}(\tau-\tau_o) \tag{6.6}$$

So the assymptotic number of solitons as $X \to +\infty$ is:

$$N = \text{largest integer smaller than } [(\frac{\alpha_3}{\alpha_2}(2)\ \frac{\alpha_2}{\alpha_3}(1)\ \frac{C_g}{\sigma}(1)\ \frac{\sigma}{C_g}(2))^{1/2} + \frac{1}{2}] \tag{6.7}$$

This result is given by Eqs. (5.5), (5.7) and (5.8) in Djordjevic and Redekopp's paper (1978). Let us introduce $\mathcal{A}$ the area under the wave profile at the end of the transition zone.

$$\mathcal{A} = \int_{-\infty}^{+\infty} A(L,\tau)\, d\tau \qquad (6.8)$$

From Djordjevic - Redekopp's theory, this area can be calculated by using (6.5). It gives

$$\mathcal{A}_s = \pi \left(\frac{C_g}{\sigma}(1)\, \frac{\sigma}{C_g}(2)\, \frac{2\alpha_2}{\alpha_3}(1)\right)^{1/2} \qquad (6.9)$$

To check the validity of the assumptions which have been made to find (6.7) the assymptotic number of solitons N, the value of $\mathcal{A}_s$ expected by D-R theory is compared to the value $\mathcal{A}$ obtained from the numerical computation. We define

$$\Delta = 1 - \frac{\mathcal{A}_s}{\mathcal{A}} \qquad (6.10)$$

Δ measures the relative error between the numerical computation and D-R theory.

6.b Numerical Results

i . Constant Depth

We first consider the evolution of an initial sech profile on constant depth. The results of this case are known both analitycally (Zakharov and Shabat, Satsuma and Yajima) and numerically (Yue, 1980). However it is important to present here some of these results in order to compare them with those of the case of variable depth. Let us consider the initial profile to be :

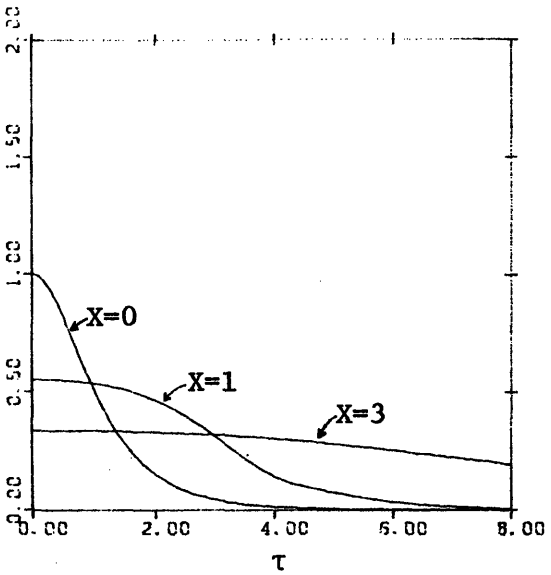
$$A = a \operatorname{sech}(\tau/\lambda) \tag{6.11}$$

Then, we know that in a region where $K < 0$, the initial wave packet flattens and spreads toward large τ. The decay of the envelope is as $\frac{1}{\sqrt{X}}$ and no soliton emerges. Fig. (6.1) illustrates this case, the parameter λ is 1 and the wave period is $\underset{\sim}{T} = 7$ so that $K = \frac{\alpha_3}{\alpha_2} = -3.33 < 0$. It shows the steadily decrease of the maximum height of the envelope which is associated with the radiation of energy towards large τ.

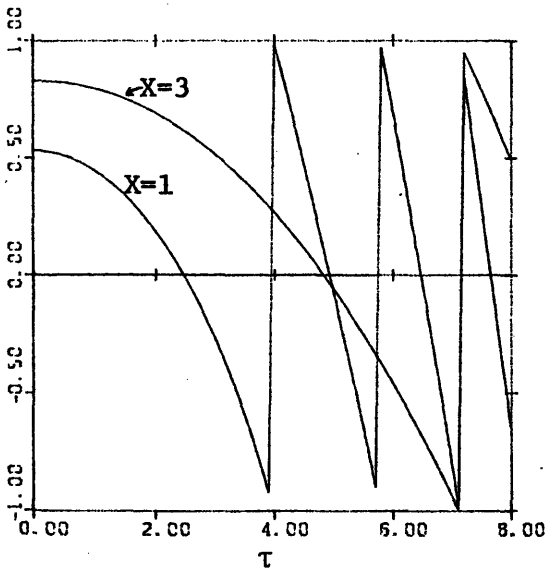
In region where $K > 0$, the assymptotic state consists of N solitons plus decaying oscillations where N is the largest integer less than

$$m = \sqrt{\frac{\alpha_3}{2\alpha_2}}\ \frac{\mathcal{A}}{\pi} + \frac{1}{2} \tag{6.12}$$

where $\mathcal{A} = \pi a\lambda$ is the area of the initial profile (6.11). If $1 < m < 1.5$, the assymptotic state consists of one soliton. The evolution picture is shown in Fig. (6.2) in the case of $\underset{\sim}{T} = 5, K = 1.4, m = 1.25$. It shows a dispersive decay of the wave envelope and does not reveal any recurrence period. If $1.5 < m \leq 2$ only one soliton emerges for large X while if $2 < m < 2.5$ two soliton emerge. However the evolution pictures of these two cases are similar. There is a single 'recurrence period' after which the profile repeats itself with a slow decay of the maximum amplitude and a slight spreading towards large τ, signalling radiation. This result is shown in Fig. (6.3) in the case of $\underset{\sim}{T} = 5$, $K = 1.4$ and $m = 2$. It should



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

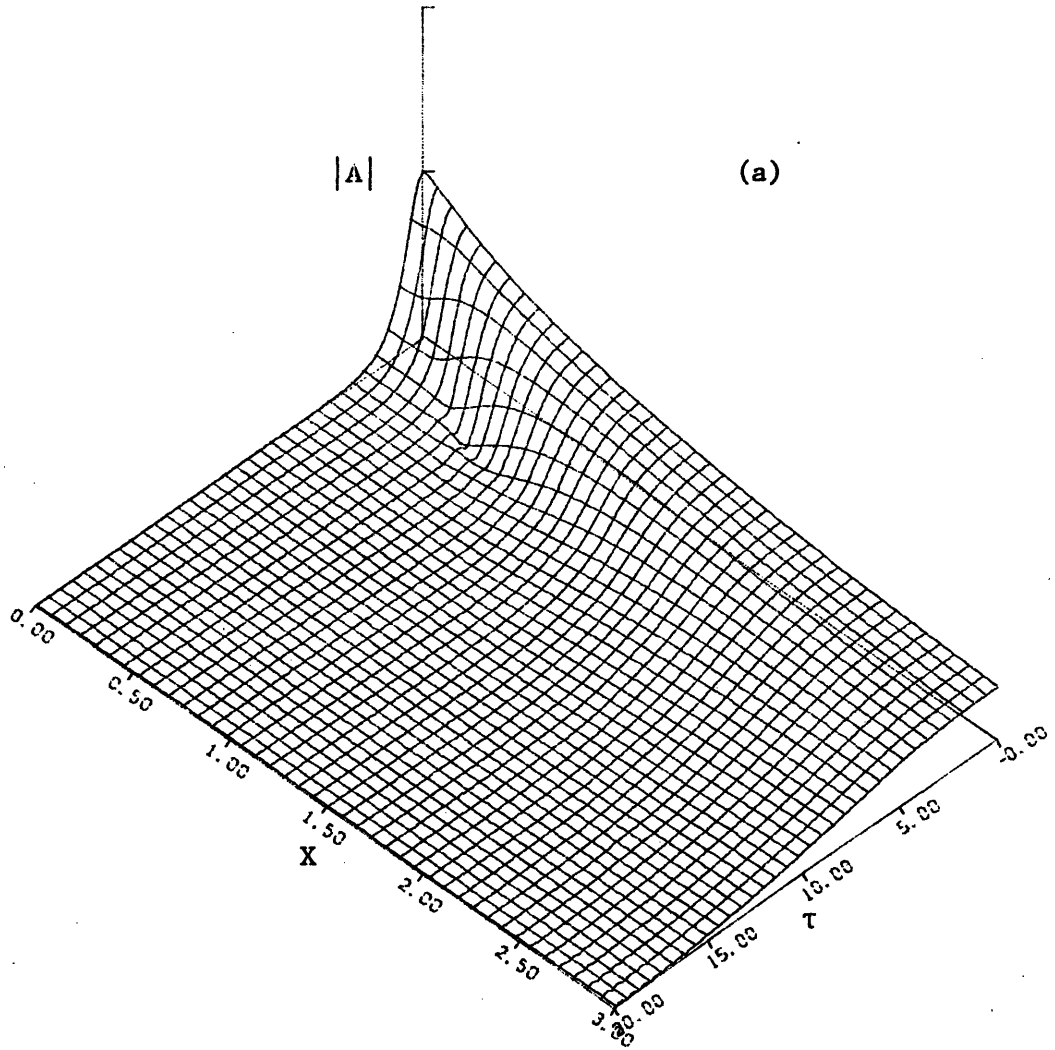
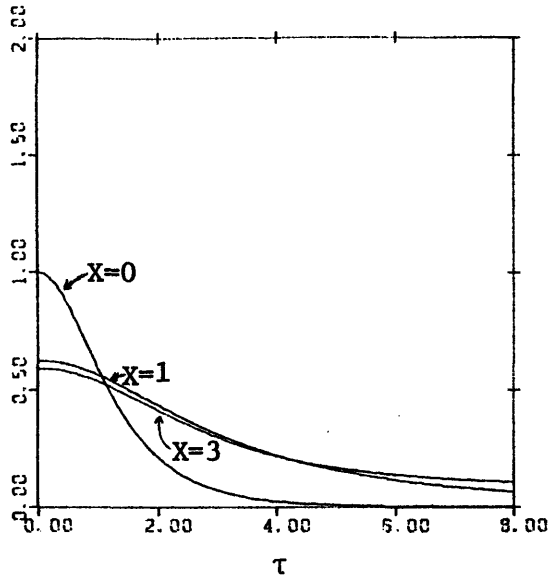
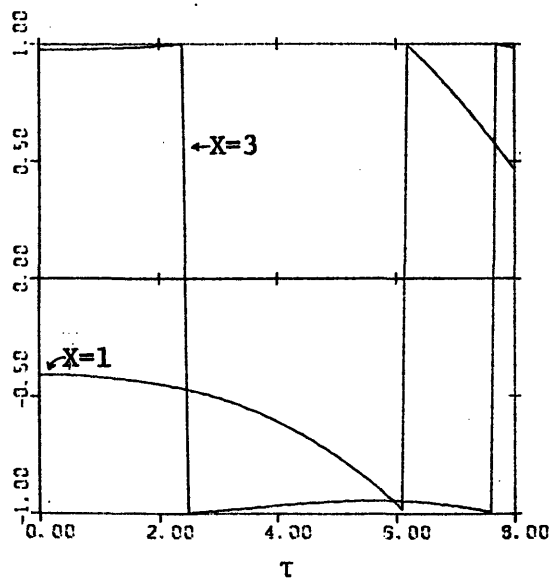


Figure 6.1 Evolution of a Sech Profile over Constant Depth, No Current, K Negative



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

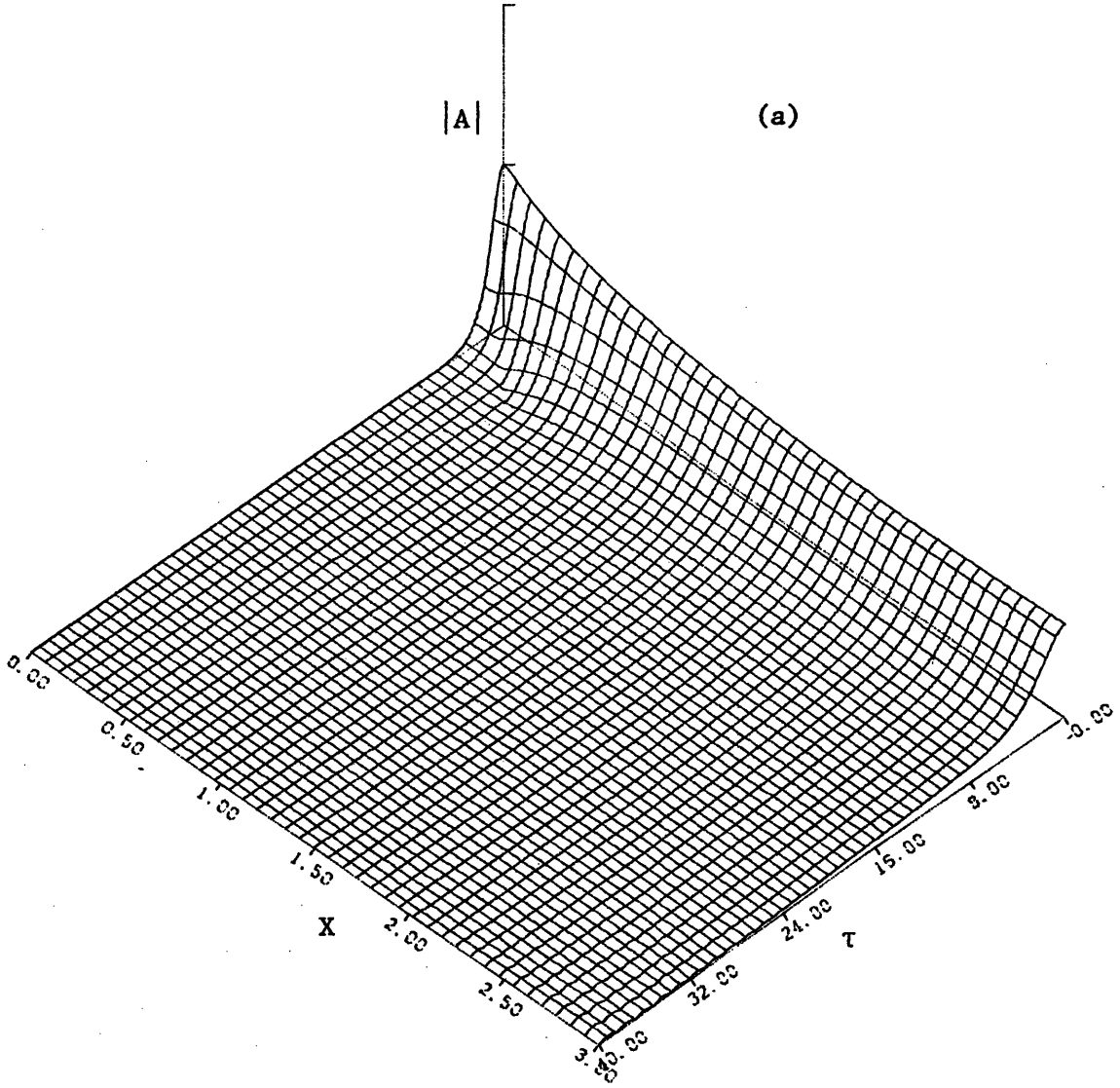
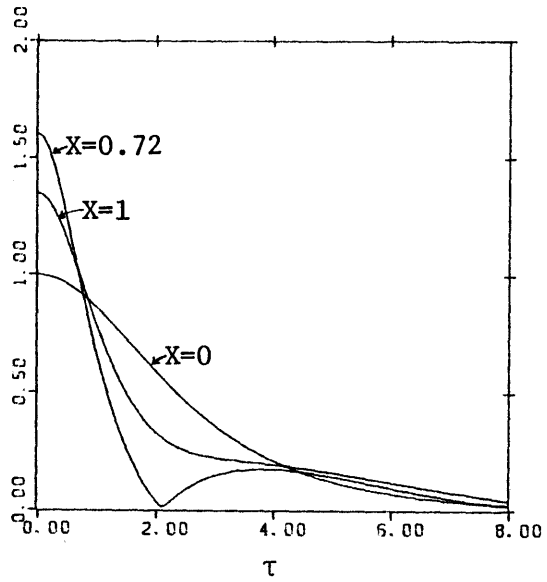
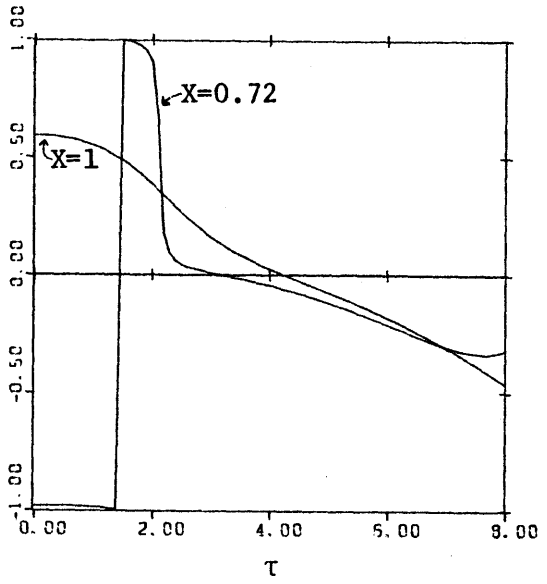


Figure 6.2 Evolution of a Sech profile over Constant Depth,

No Current, K Positive, m=1.25



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{Arg(A(X,\tau))}{\pi}$

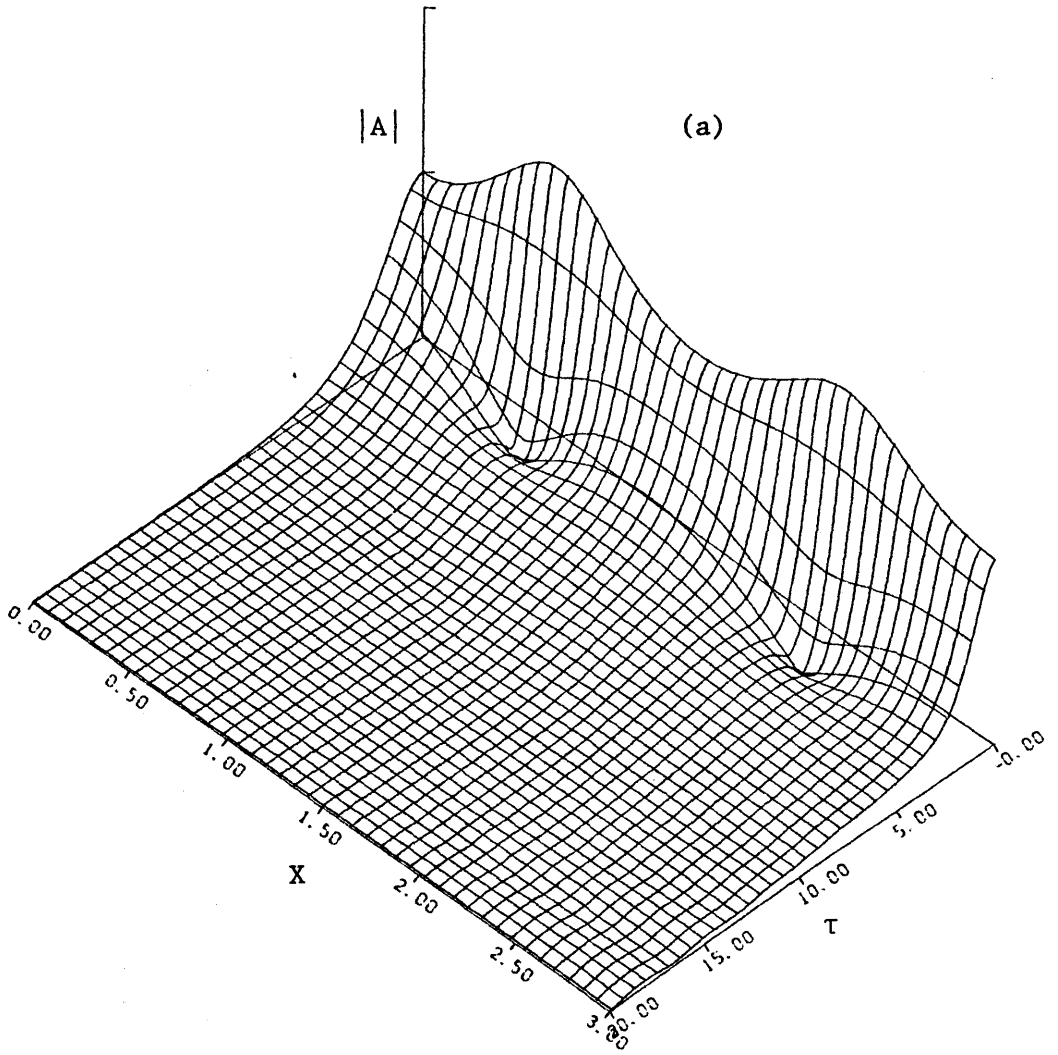


Figure 6.3 Evolution of a Sech Profile over Constant Depth,

No current, K Positive, m=2.0

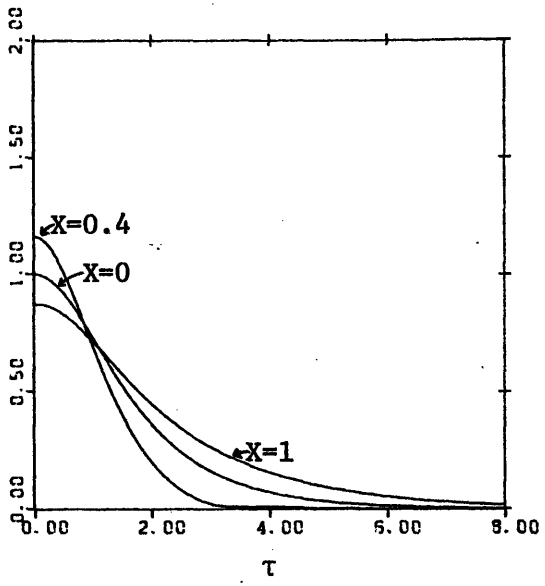
be noted that for two initial envelopes of the same area and height, the evolution towards the final state can be quite different if their shapes or phase distributions are different. This statement is illustrated in Fig. (6.4) where the propagation of the initial profile:

$$A(0,\tau) = \text{sech}\left(\sqrt{\frac{\alpha_3}{2\alpha_2}}(1)\,\tau\right)e^{i0.1\tau^2} \tag{6.13}$$

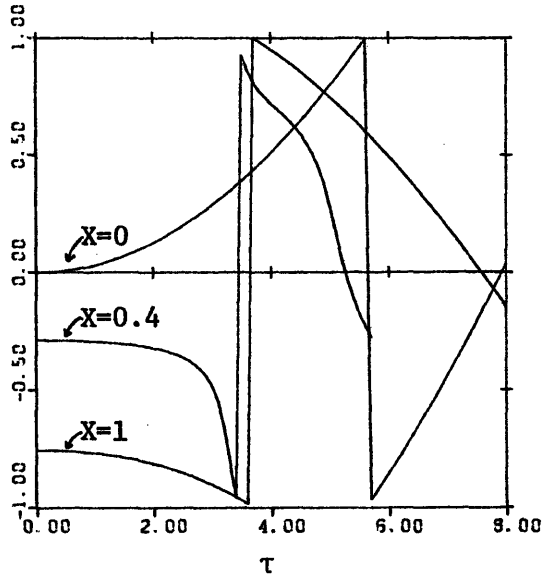
is shown. Comparing Fig. (6.4) and Fig. (5.1) we note that a difference in the phase distribution of the initial profiles induces a qualitative difference in the subsequent evolution of the wave envelope.

ii. Variable Depth

In what follows we consider the initial profile (5.21) such that if the nonlinearity parameter K is positive everywhere and the depth constant the wave envelope will evolve without change in amplitude. However when the depth varies, K changes according to Fig. (4.6.b) and the initial profile must adapt to its new environment: its shape is not conserved. In Fig. (6.5) we present the evolution of an initial sech profile, corresponding to the envelope of waves with a period $\underset{\sim}{T} = 5$, during its propagation into deeper water (dh = - 0.5). We know from Fig. (4.6.b) that K is positive and increases from $K_1 = 1.4$ to $K_2 = 2.8$. The 3-dimensional plot gives the amplitude of the waves as a function of τ and X. As the depth increases the initial data becomes steeper and steeper. At X = 1 the envelope is no longer a soliton, it has two side groups separated by nodes as could be seen from Fig. (6.5.b). The



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

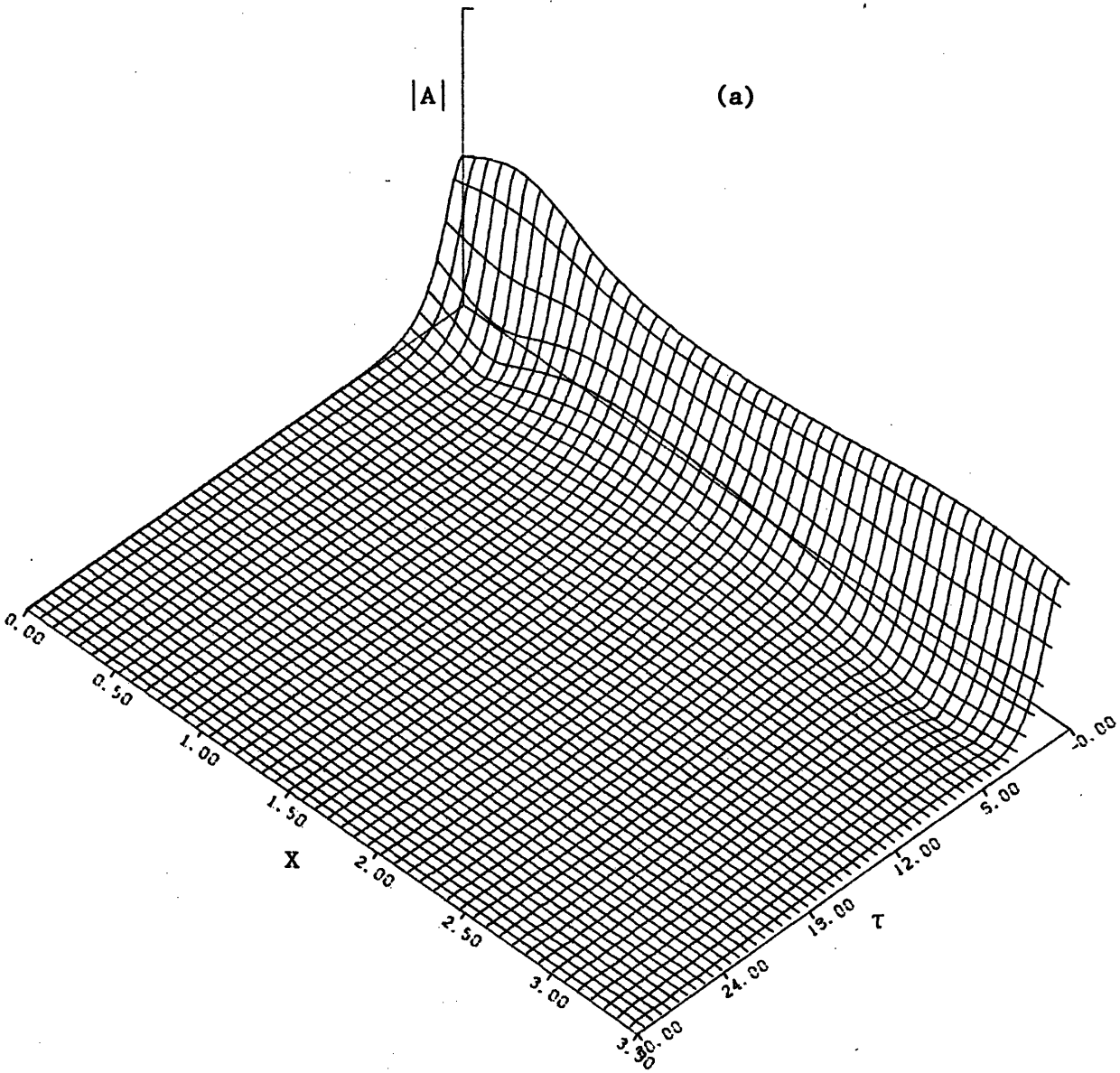
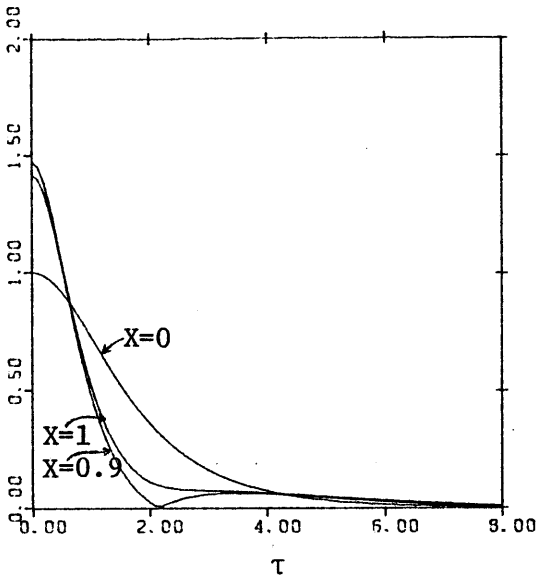
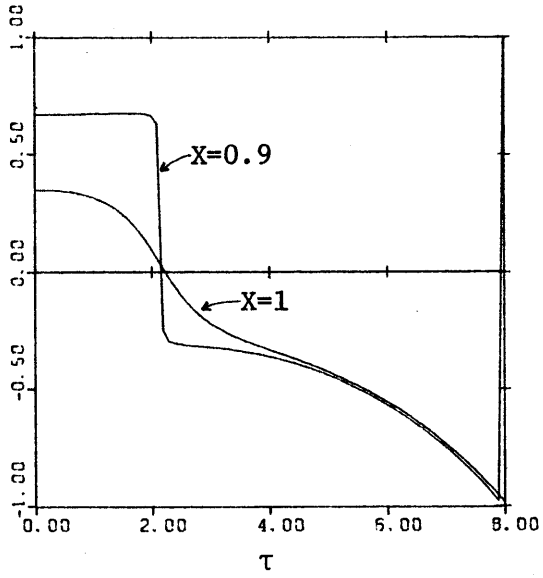


Figure 6.4 Evolution of a Sech Profile with Parabolic Phase Distribution over Constant Depth,No Current, K Positive



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

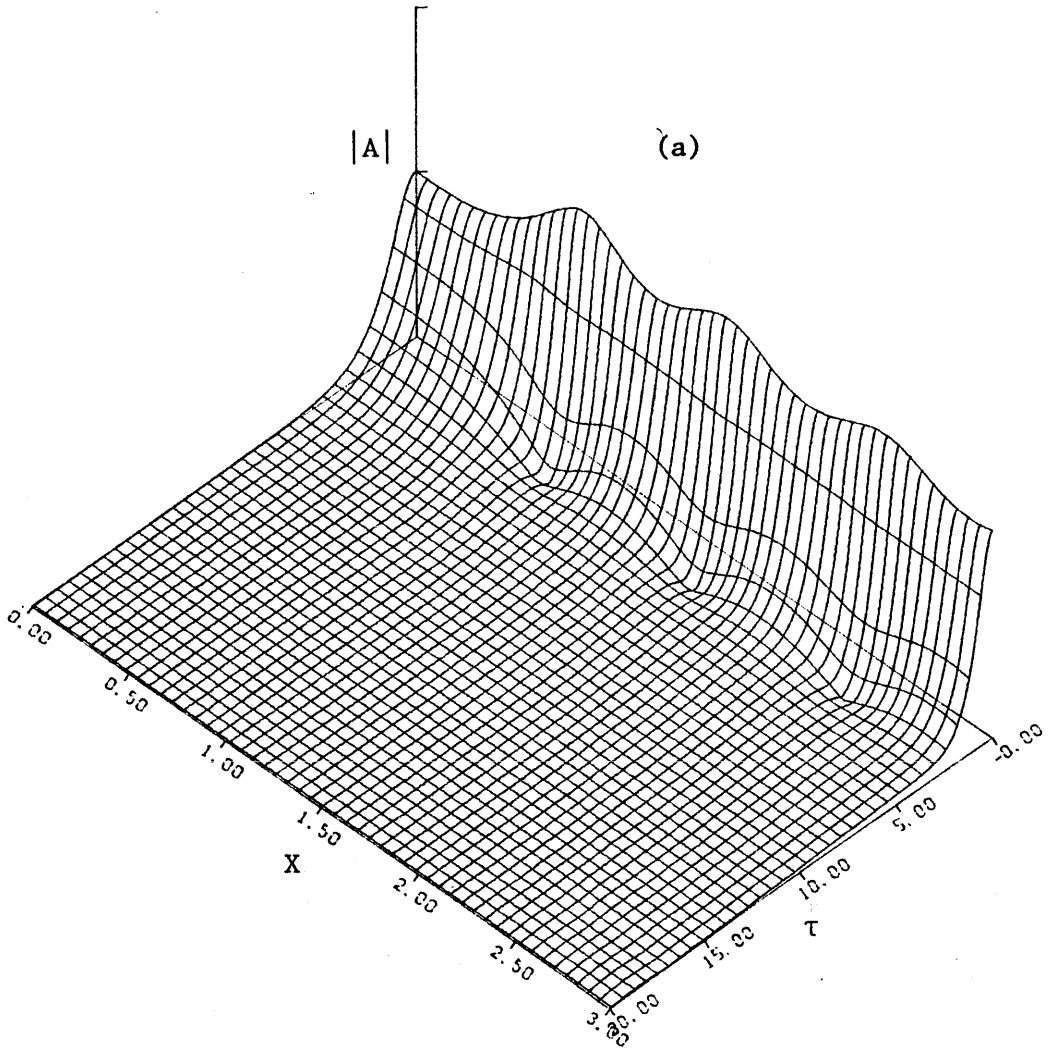


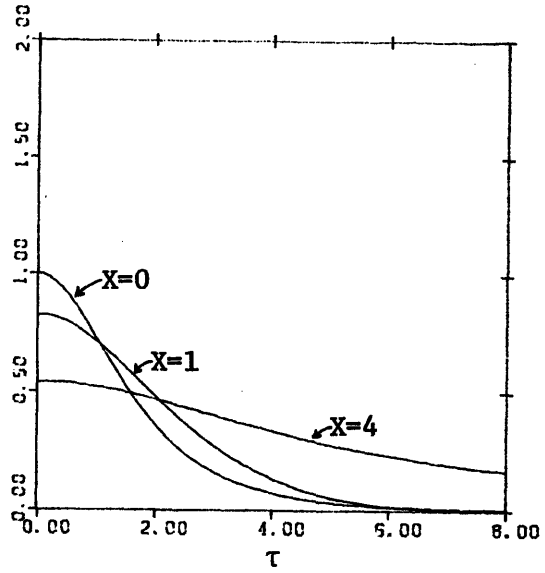
Figure 6.5 Evolution of a Sech Profile over Increasing Depth

No Current, K Positive

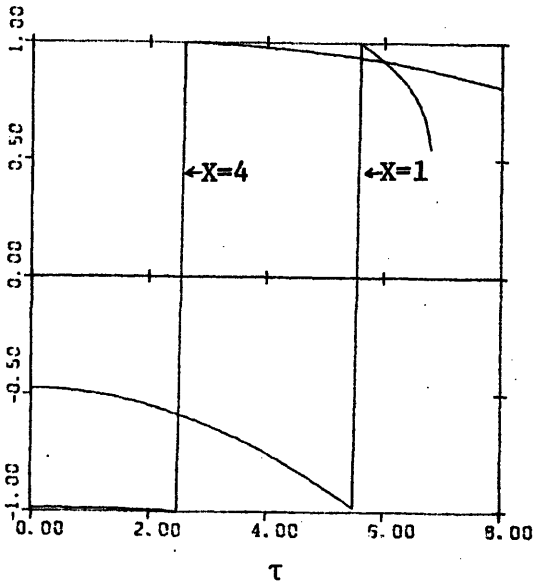
evolution picture is qualitatively similar to Fig. (6.3). It shows that the profile repeats itself with some decay after a recurrence period which is approximately $X \simeq 0.9$. As pointed by Yue (1980) it can be noted that the node of the wave amplitude is accompanied by a sudden jump by π of the phase (see Fig. (6.5.c)). In the complex plane it means that A crosses zero without discontinuity of its derivatives. It also implies that the wave frequency has a peak at the node.

Fig. (6.6) presents the evolution of the same initial profile during its propagation into shallower water (dh = 0.15). The depth variation has been choosen such that K is positive everywhere. As shown in Fig. (4.6.b) K decreases with decreasing depth: $K_1 = 1.4$, $K_2 = 0.72$. The 3-D plot shows that as the depth decreases the initial profile becomes flatter and flatter. A more quantitative picture is obtained by superposing the wave envelope profile at different locations X = 0, 1, 2, 4 as a function of τ (Fig. 6.6.b). It shows the continuous decrease in the profile steepness during the evolution. Fig. (6.6.c) shows that, at each location X, the phase decreases slightly as a function of τ.

Since K is positive everywhere in the two previous cases, their results can be used to check Djorjevic and Redekopp's assumptions (see section 6.a). The relative difference between the numerical results and the D-R results as defined by (6.10) is for the case presented in Fig. (6.5) $\Delta = -22\%$ and for the case presented in Fig. (6.6) $\Delta = 11\%$. It should be recalled that in all the cases the evolution law (5.20) is satisfied within 5%. Therefore these differences are larger than the numerical errors and hence could affect the prediction of the number of solitons emitted. We note that $\Delta \lesseqgtr 0$ when $dh \gtreqless 0$, this result has been found to be valid in all



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

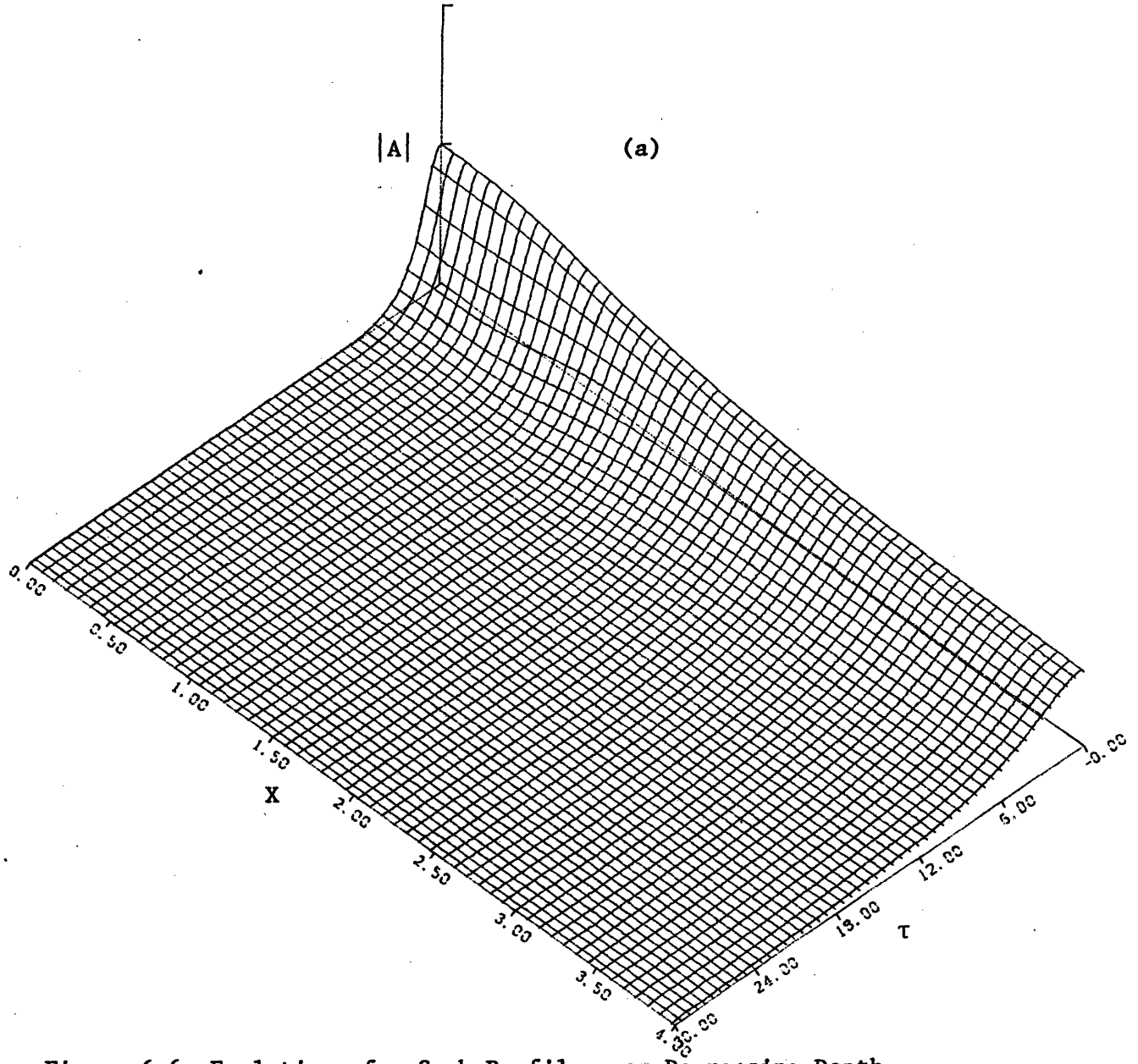


Figure 6.6 Evolution of a Sech Profile over Decreasing Depth,

No Current, K Positive

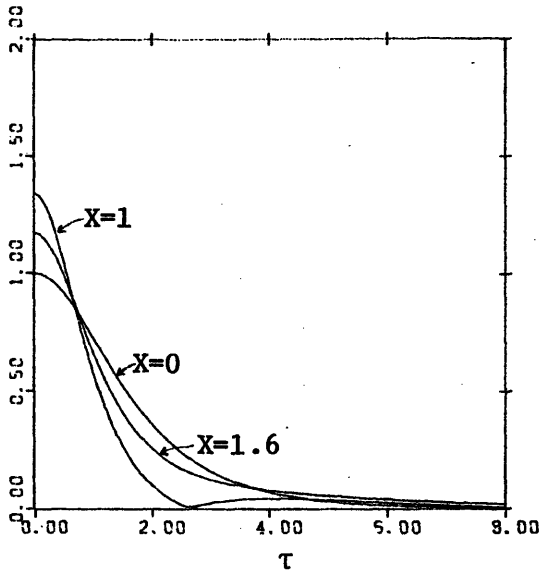
the cases treated. It implies that for waves propagating into deeper water, the area under the wave envelope becomes smaller than expected by D-R while fore waves propagating into shallower water the area under the wave envelope becomes larger than expected by D-R. Furthermore D-R assumes that the profile at the end of the depth variation is still a sech profile (Eq. 6.5) but a comparison between the initial profile and the profile at X = 1 in Figs. (6.5) and (6.6) shows that this assumption is not satisfied. Based on D-R theory the number of solitons which will appear as $X \to \infty$ is N the largest integer smaller than m_s = 1.87 in the first case and m_s = 1.22 in the second. So in both cases the initial profile would evolve into one soliton plus residual oscillations which decay as $(X)^{-1/2}$. Since our computation has been limited to a small distance of propagation (X_2 = 3, X_2 = 4 respectively) these predictions could not be checked numerically. However based on the numerical value of the area of the wave envelope at X = 1 one can find a more accurate value for m (Eq. 6.12): m = 1.5 for the case of Fig. (6.5). This result implies that if the profile at X = 1 was a sech profile then it would evolve without shape variation. The fact that we see in Fig. (6.5) both recurrence and radiation is due to the initial departure from (6.11). In the case of Fig. (6.6) m = 1.3 and the evolution of the wave envelope is qualitatively similar to the one presented in Fig. (6.2).

It should be noted that in the case where K is positive everywhere and where the depth increases the effect of increasing the magnitude of

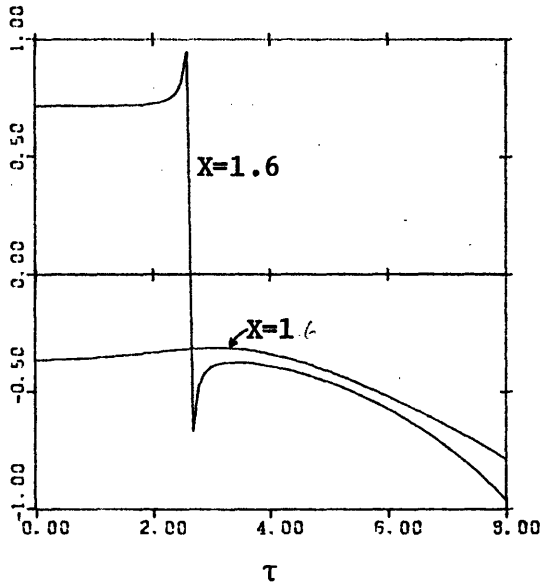
the depth variation dh, letting all the other parameters fixed is to increase the maximum amplitude and to shorten the inter-nodal distance. This result can be seen by comparing Fig. (6.7) where dh = -0.3 and Fig. (6.5) where dh = -0.5.

In the case where K < 0 everywhere, we know that the initial sech profile does not conserve its shape even on constant depth (see Fig. 6.1). We present the evolution of a sech profile corresponding to waves of period $\underset{\sim}{T}$ = 7 which are propagating into deeper water (Fig.(6.8),dh= -0.3) and shallower water (Fig.(6.9),dh= 0.3) . These figures are qualitatively very similar to Fig. (6.1). The three of them can be superposed with a good agreement at least during the initial stage of propagation. It shows that in the case where K< 0 depth variation has very little effect on the qualitative features of the evolution which is dominated by radiation.

Fig. (6.10) illustrates the case where K crosses zero during the propagation of the wave. It shows the evolution of an initial sech profile corresponding to waves with period $\underset{\sim}{T}$ = 6, which propagates into deeper water, dh = - 0.5. Fig. (4.6.b) shows that K is negative initially but becomes positive as the depth increases: K_1 = - 0.56, K_2 = 0.73. The initial profile first decays and radiates as in the case K < 0, then begins to steepen as it propagates into deeper water. Soliton emission is clear in the deeper water, numerical computation of m at X = 1 gives m = 1.7. In the case where K decreases from positive to negative values an initial soliton flattens. The evolution of a wave



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

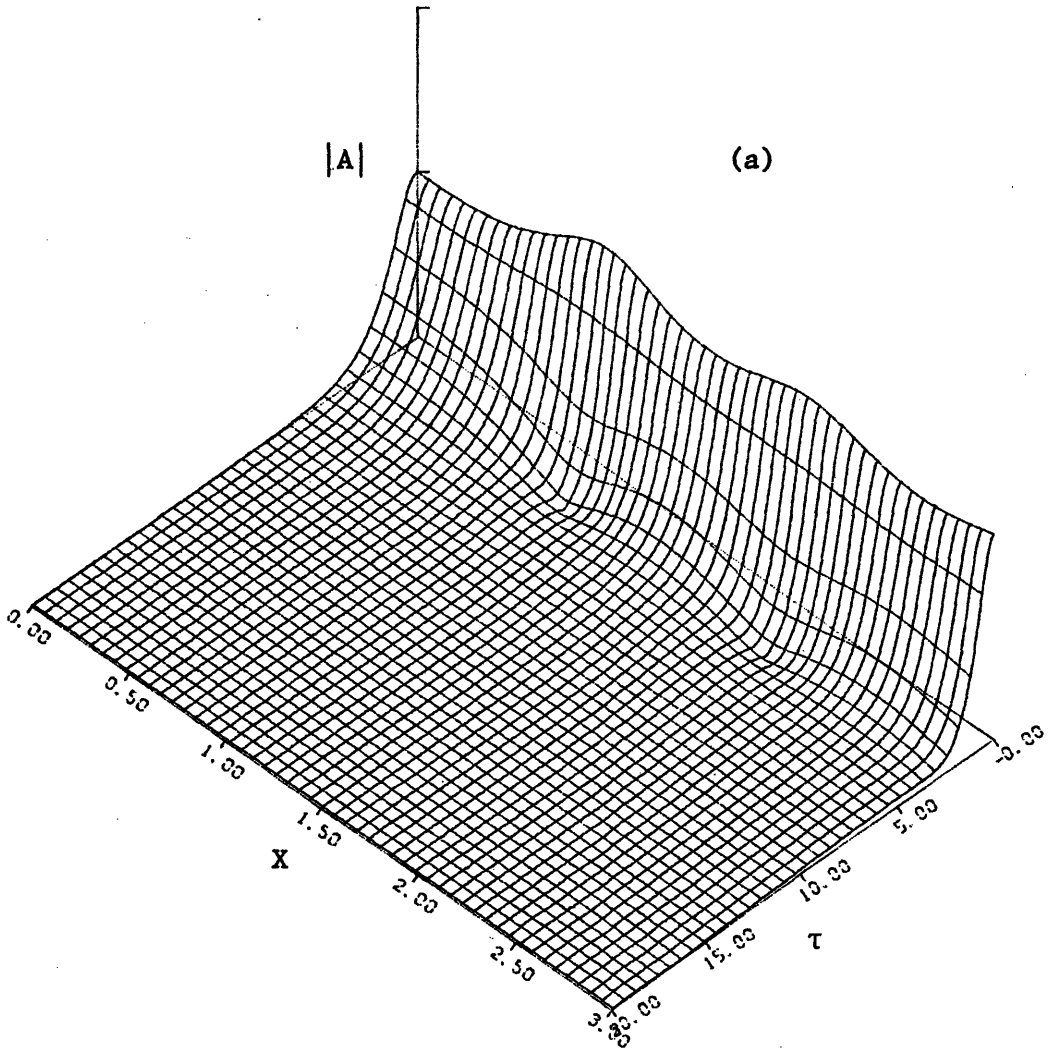
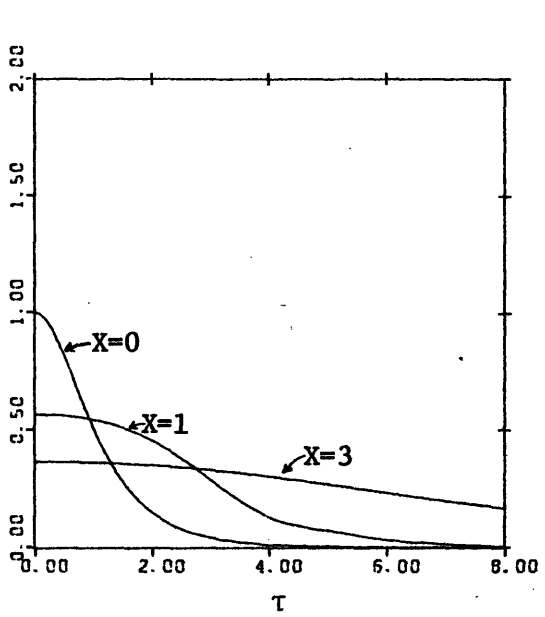
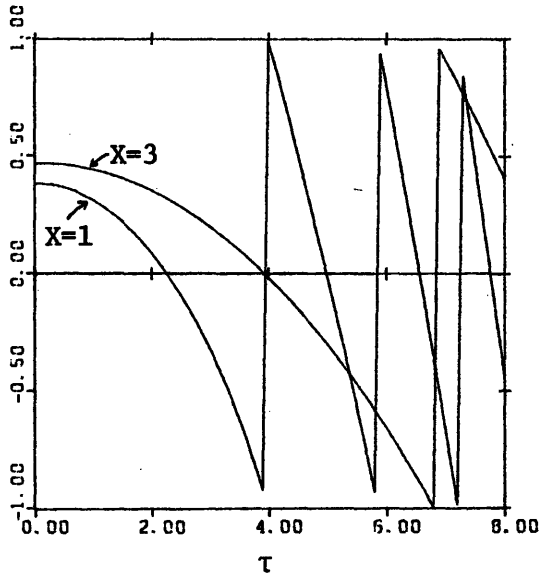


Figure 6.7 Evolution of a Sech Profile over Increasing Depth, No Current, K Positive



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

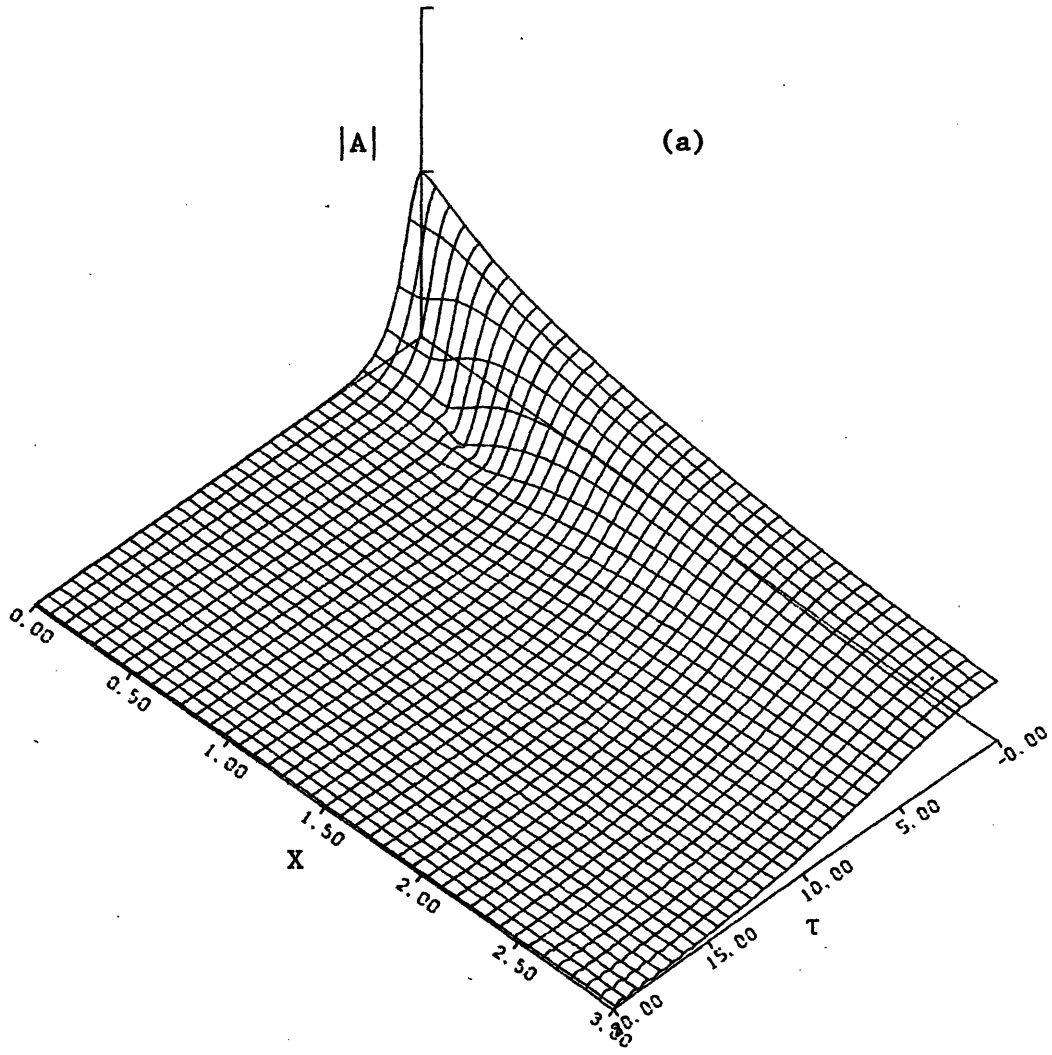
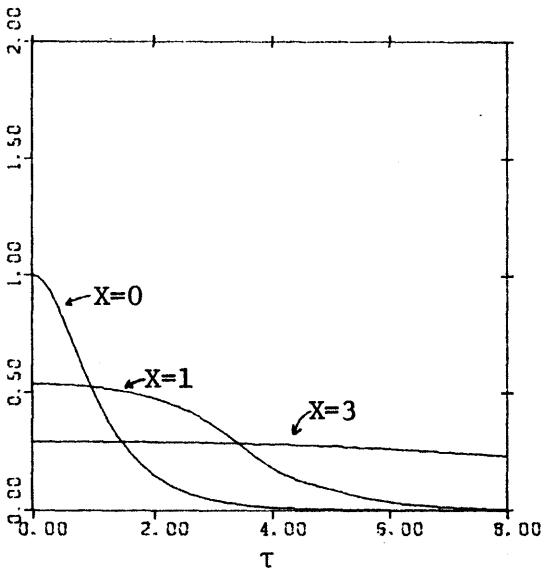
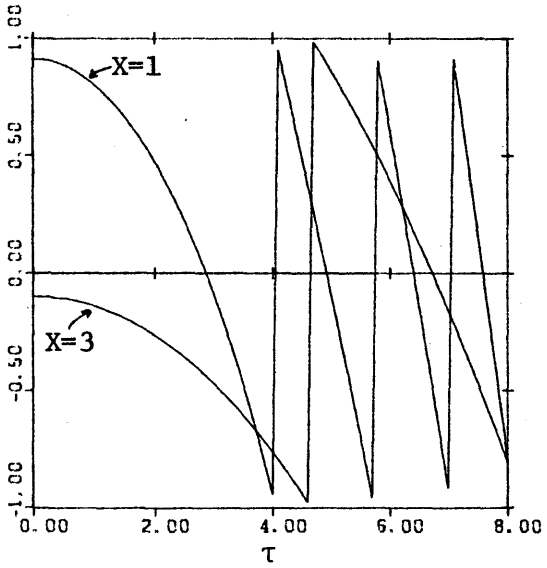


Figure 6.8 Evolution of a Sech Profile over Increasing Depth, No Current, K Negative



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{Arg(A(X,\tau))}{\pi}$

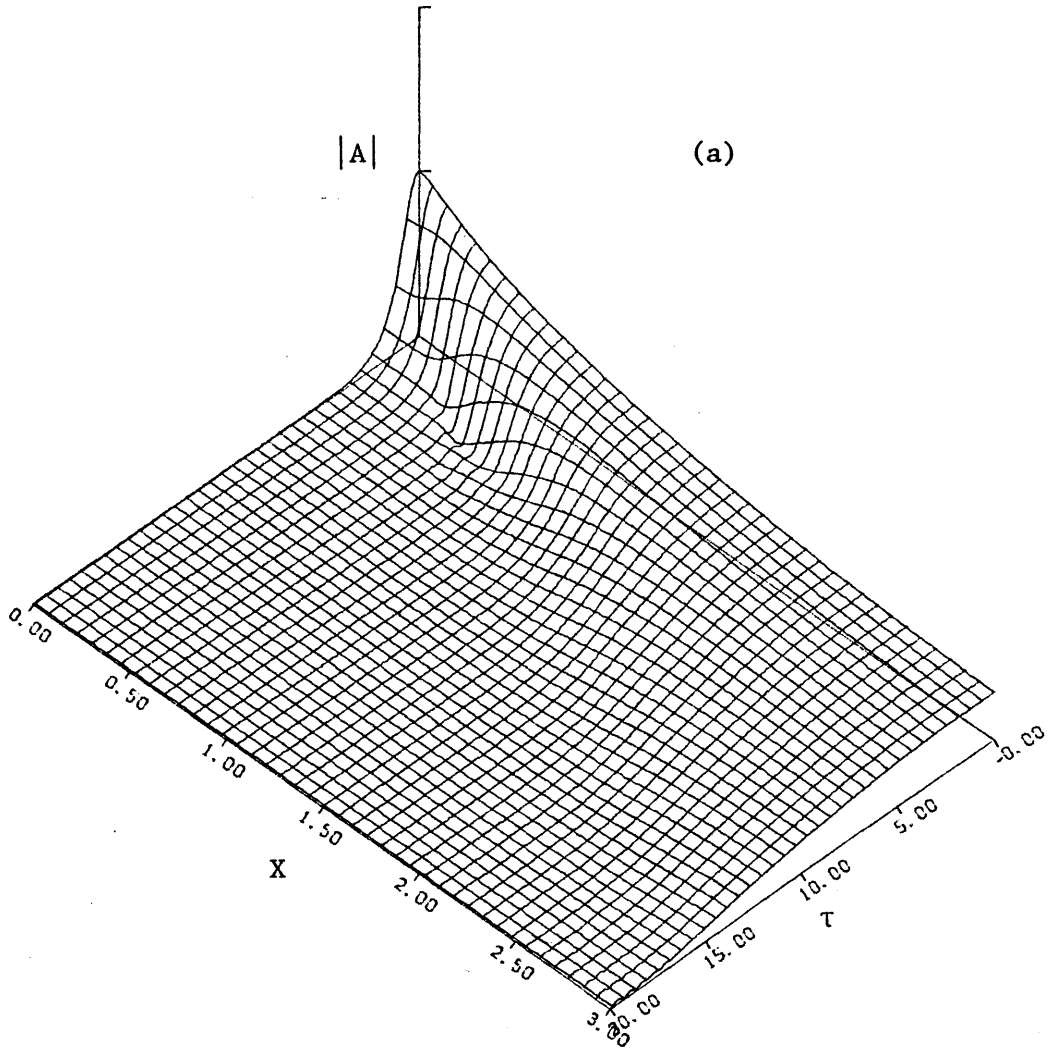
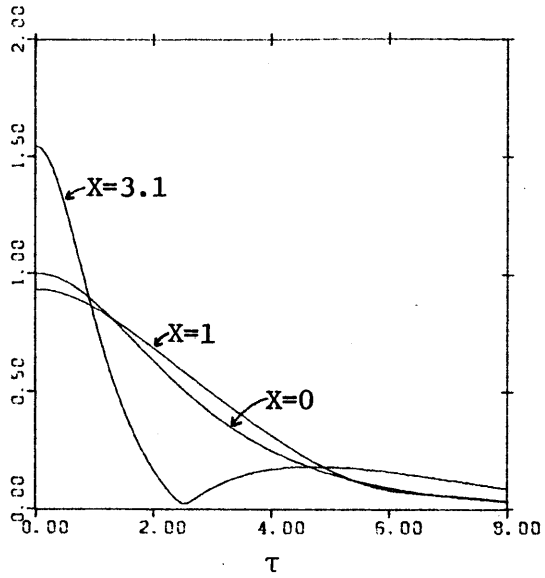
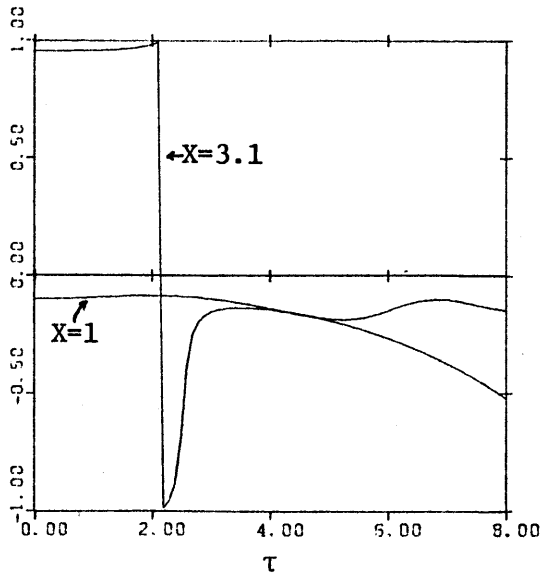


Figure 6.9 Evolution of a Sech Profile over Decreasing Depth,

No Current, K Negative



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

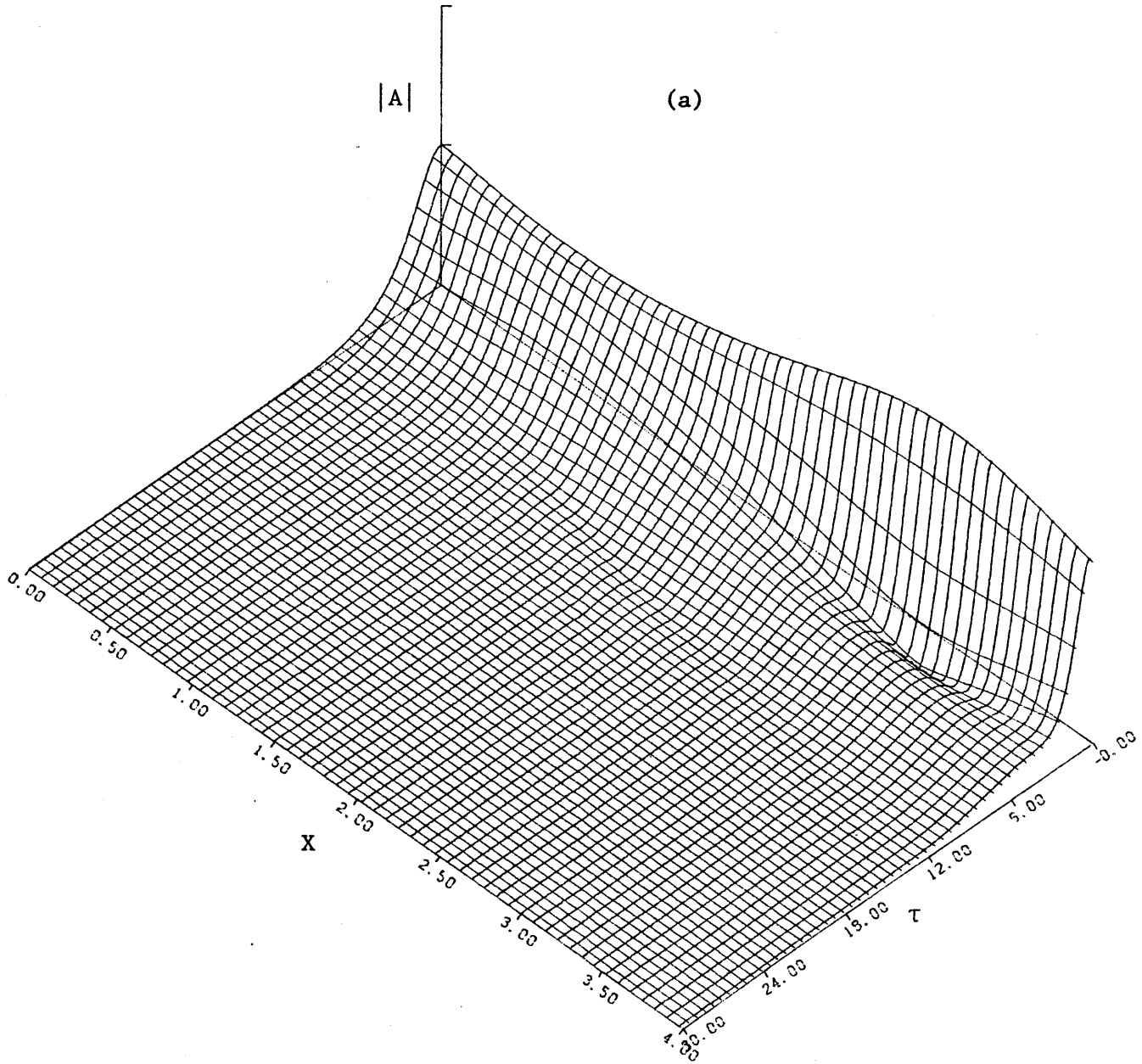
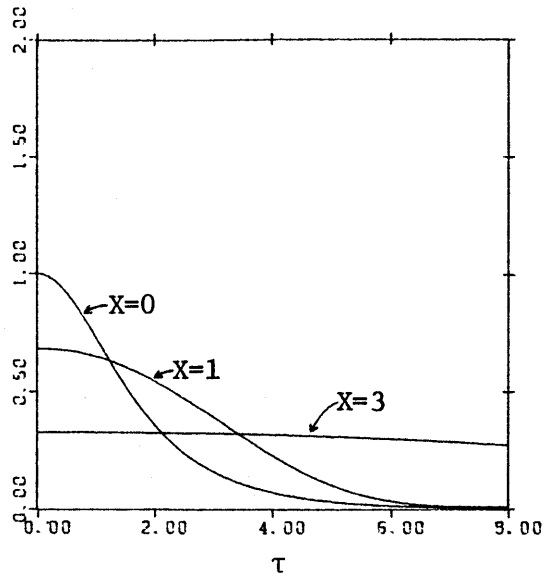
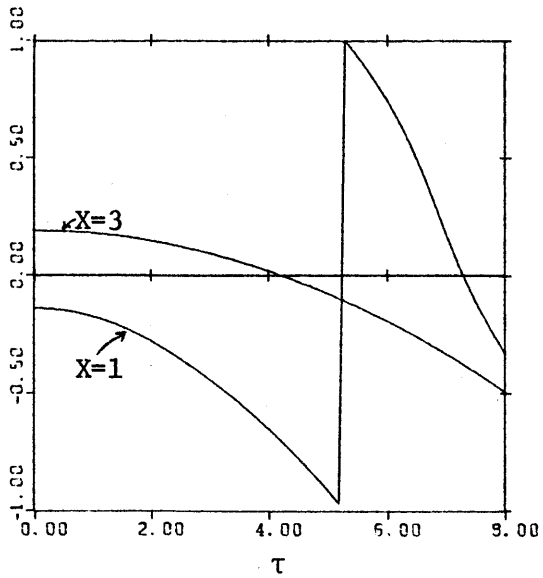


Figure 6.10 Evolution of a Sech Profile over Increasing Depth, No Current, K Increases from Negative to Positive Values

envelope (see Fig. 6.11) is a combination of an initial slow decay similar to Fig. (6.6), followed by a very strong radiation similar to Fig. (6.8) or (6.9).



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{Arg(A(X,\tau))}{\pi}$

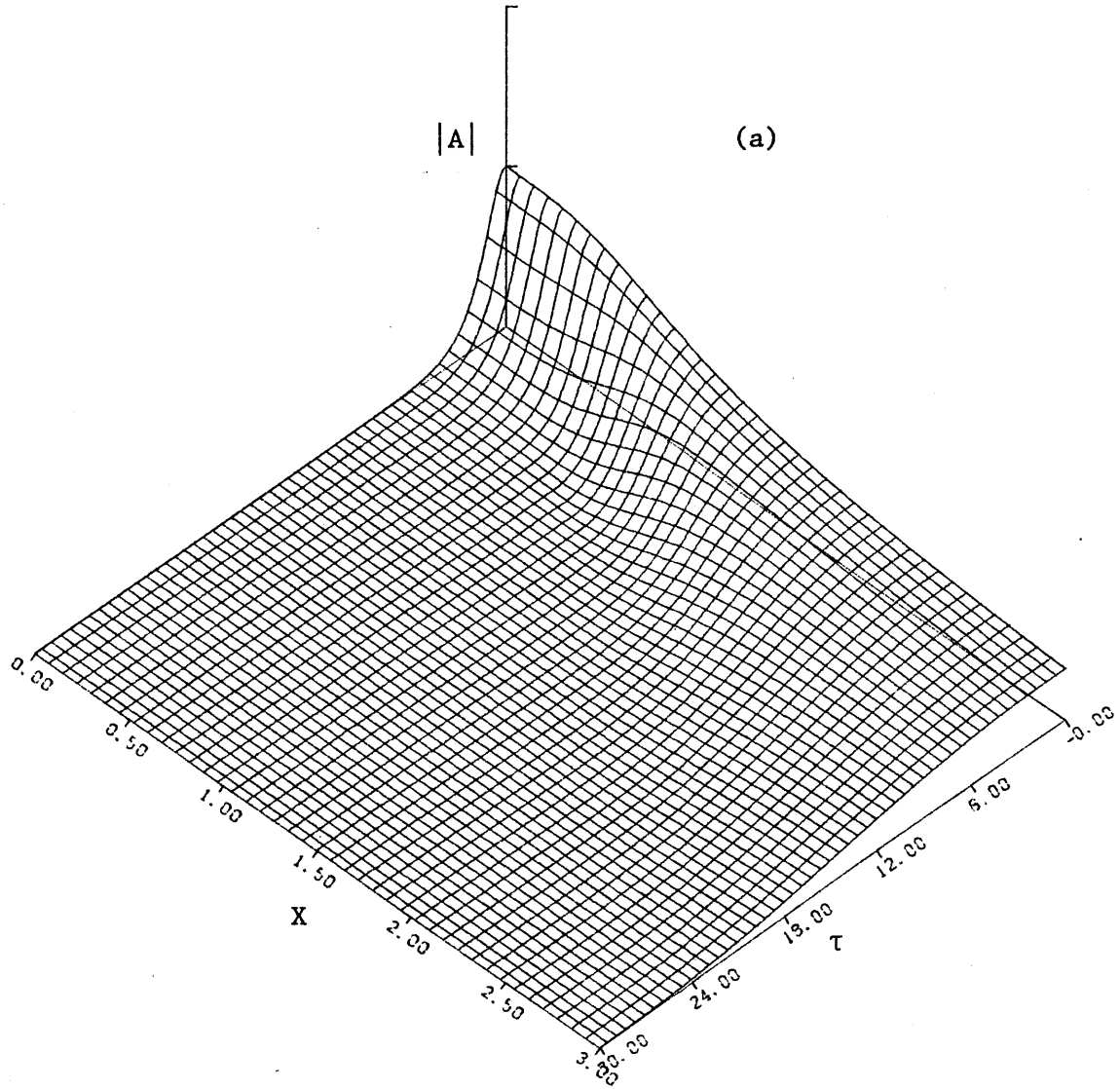
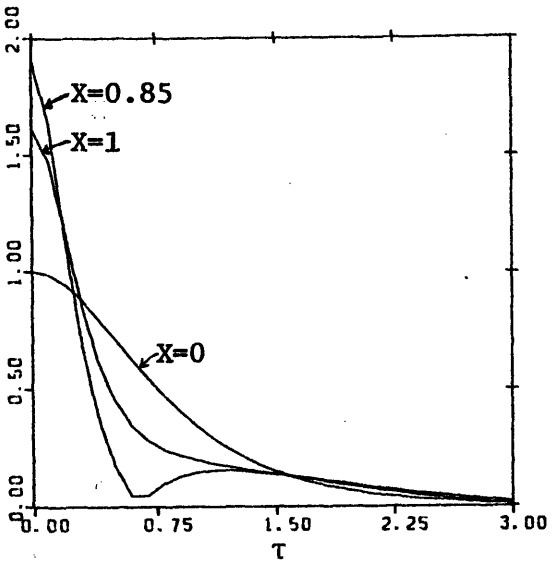


Figure 6.11 Evolution of a Sech Profile over Decreasing Depth, No Current, K Decreases from Positive to Negative Values

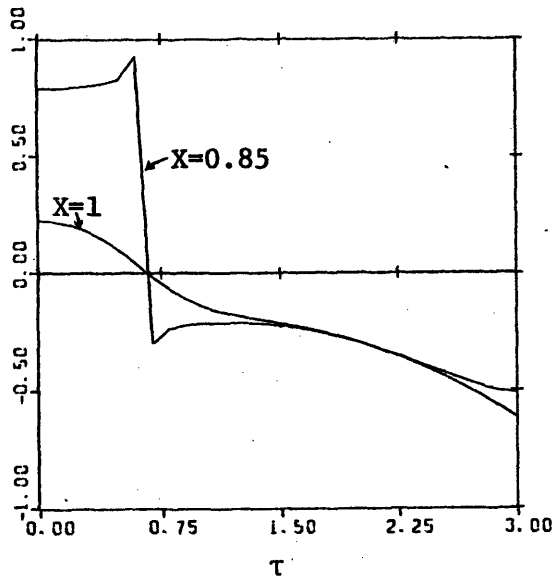
7. Propagation of a Wave Packet Over Varying Depth and Current

7.a Subcritical Current

As discussed at the end of section (4.e), one effect of the current is to increase the wavelength and hence to move the point where K crosses zero towards deeper water. Assuming the geometry of the problem to be fixed as defined in section (5.c) then at h = 1 (i.e. X = 0) K is positive for small periods only (see Fig. 4.7). For such periods the initial sech profile evolves into N solitons during its propagation over increasing depth, while for larger periods K is negative at h = 1 and the initial profile tends to radiate during its evolution. Figs. (7.1) and (7.2) illustrate the effect of a subcritical current $U_1 = 0.5$ propagating in the same direction as the waves. Fig. (7.1) corresponds to waves of period $\underset{\sim}{T} = 3$ propagating into deeper water dh = - 0.3. As it can be seen in Fig. (4.7.b) K is positive everywhere and increases with depth. The initial profile becomes steeper during its propagation into deeper water. It desintegrates and then rebuilds with smaller amplitude with a recurrence distance of the order of 0.6. The qualitative features of the evolution of the initial profile in Fig.(7.1) are similar to those presented in Fig. (6.5) in the case of no currents. The numerical computation gives m = 1.9 so that assymptotically only one soliton evolves. However applying D-R's assumption to this case gives $m_s = 2.1$ i.e. two solitons instead of one. It should be noted that this difference becomes important only when X is large. Fig. (7.2) presents the case where K < 0



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

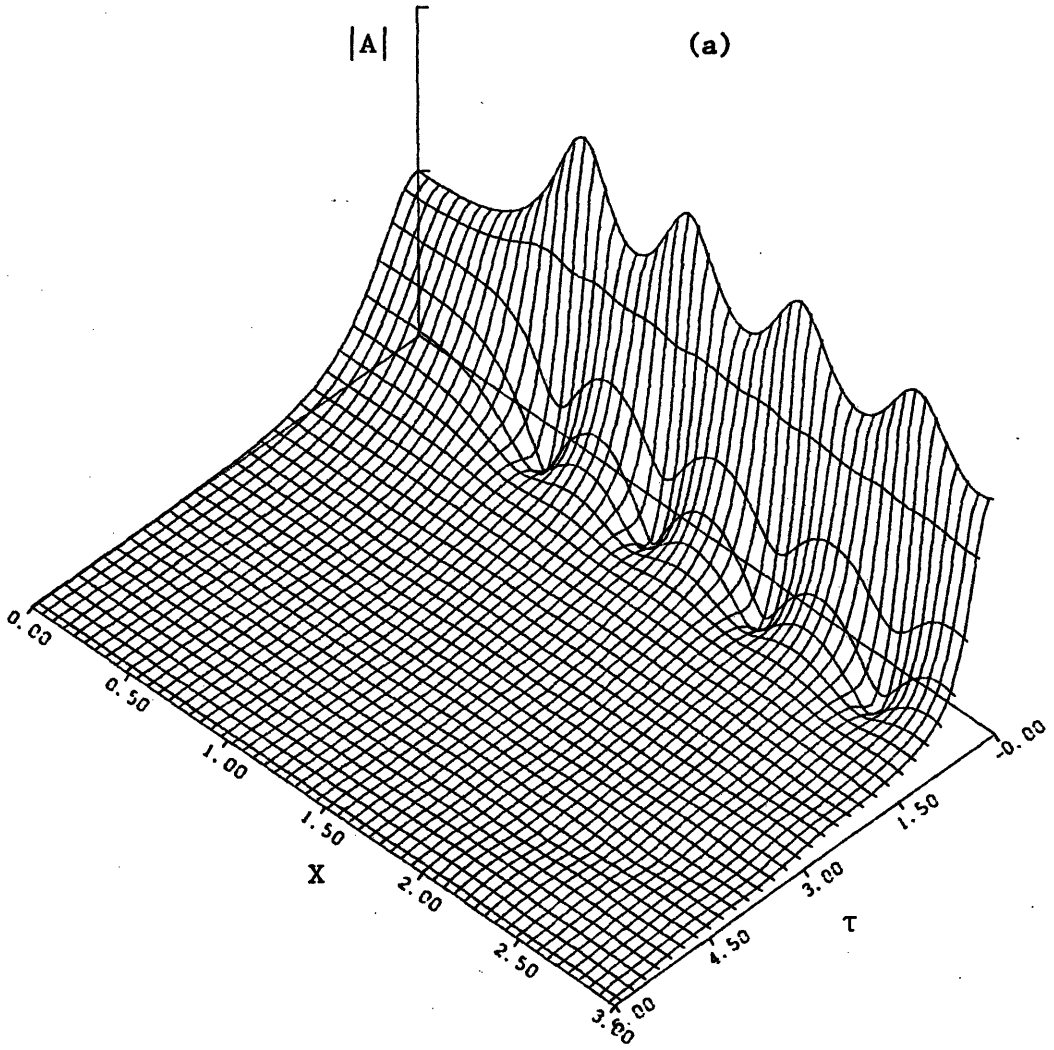
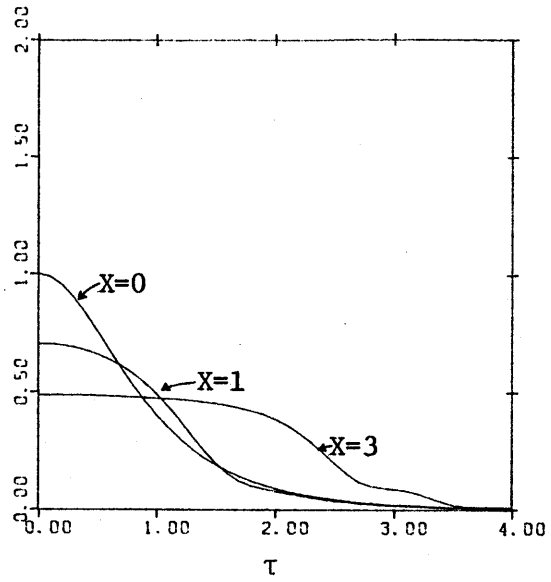
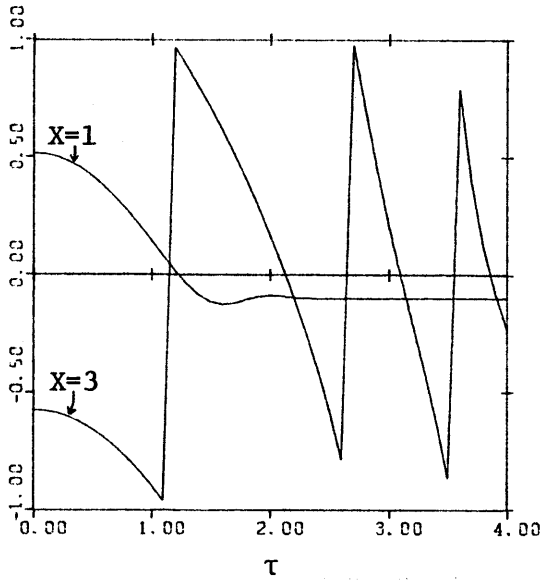


Figure 7.1 Evolution of a Sech Profile over Increasing Depth, Subcritical Current, K Positive



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

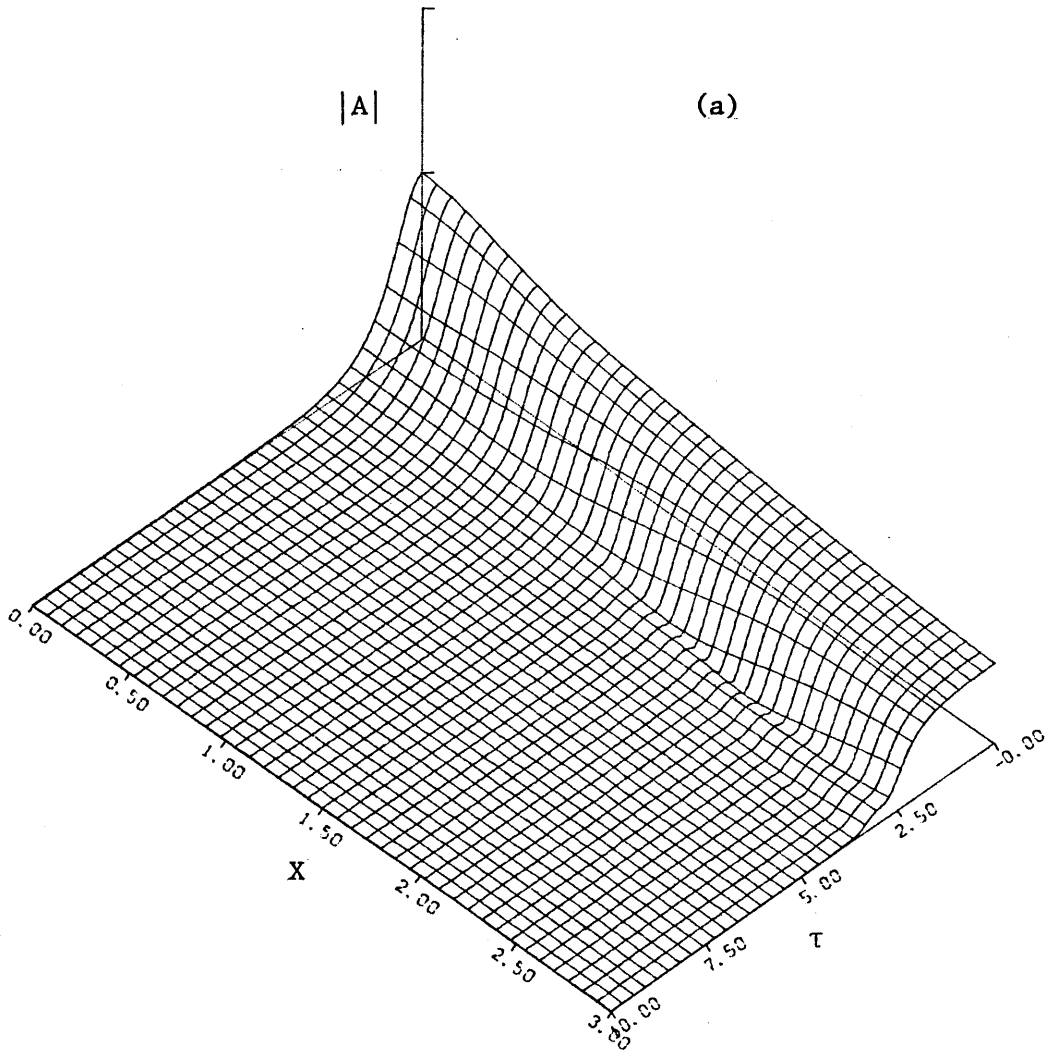
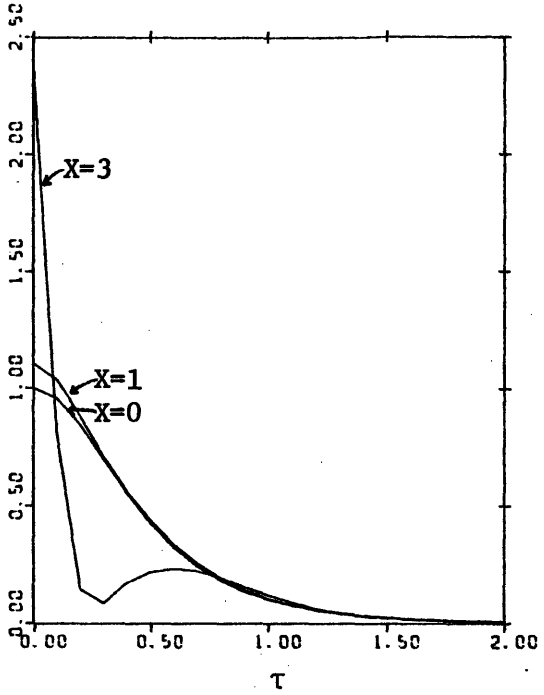


Figure 7.2 Evolution of a Sech Profile over Decreasing Depth, Subcritical Current, K Negative

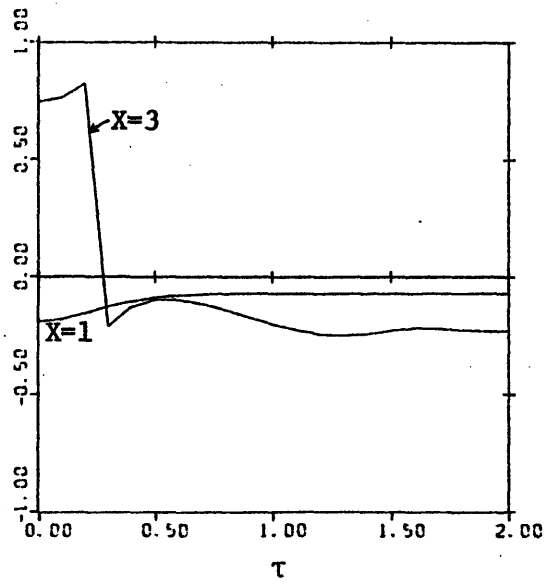
everywhere. Waves of period $\underset{\sim}{T} = 4$ are propagating towards shallower water dh = 0.15. The depth variation has been limited to 0.15 to avoid the critical depth $h_c = 0.82$ (see Fig. 4.7). For such period and depth variation K is negative and decreases with decreasing depth. Fig. (7.2) shows that there is a radiation of the initial profile which becomes flatter and spread towards large τ. Nevertheless the rate of decay is much smaller than in the case of no current see Fig. (6.9).

7.b Supercritical Current

As noted in section (4.e), when the current becomes supercritical turbulence cannot be neglected and our theory is incomplete. Nevertheless in order to show that the evolution of an initial profile depends mainly on the sign of K, we have plotted in Fig. (7.3) and Fig. (7.4) this evolution in the case of a current $U_1 = 2$. It should be recalled that for a supercritical current the variations of $k(\xi+h)$, K and s^{-2} with depth are the opposite of those presented in the case of subcritical current (see Fig. 4.8). Therefore we expect the features of the evolution of an initiate profile to be reversed: for example if $K > 0$ the initial profile should steepen with decreasing depth and flatten with increasing depth. We also note that for $U_1 = 2$, $\alpha_2 = 0$ (10^{-2}) so the dispersion has no effect but after a very long distance of propagation. Finally Fig. (4.8) suggests that at the depth h = 1, K is negative except for very small periods (say $\underset{\sim}{T} \lesssim 1.75$). Fig. (7.3) illustrates the case of waves of period $\underset{\sim}{T} = 1.5$ propagating into shallower water dh = 0.3. It shows the appearance of the first node after a long distance of propagation X = 3. The result is opposite to that obtained in the case of subcritical current where the initial sech profile flattened when propagating into shallower water. The numerical computation of m gives m = 2.1 which



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

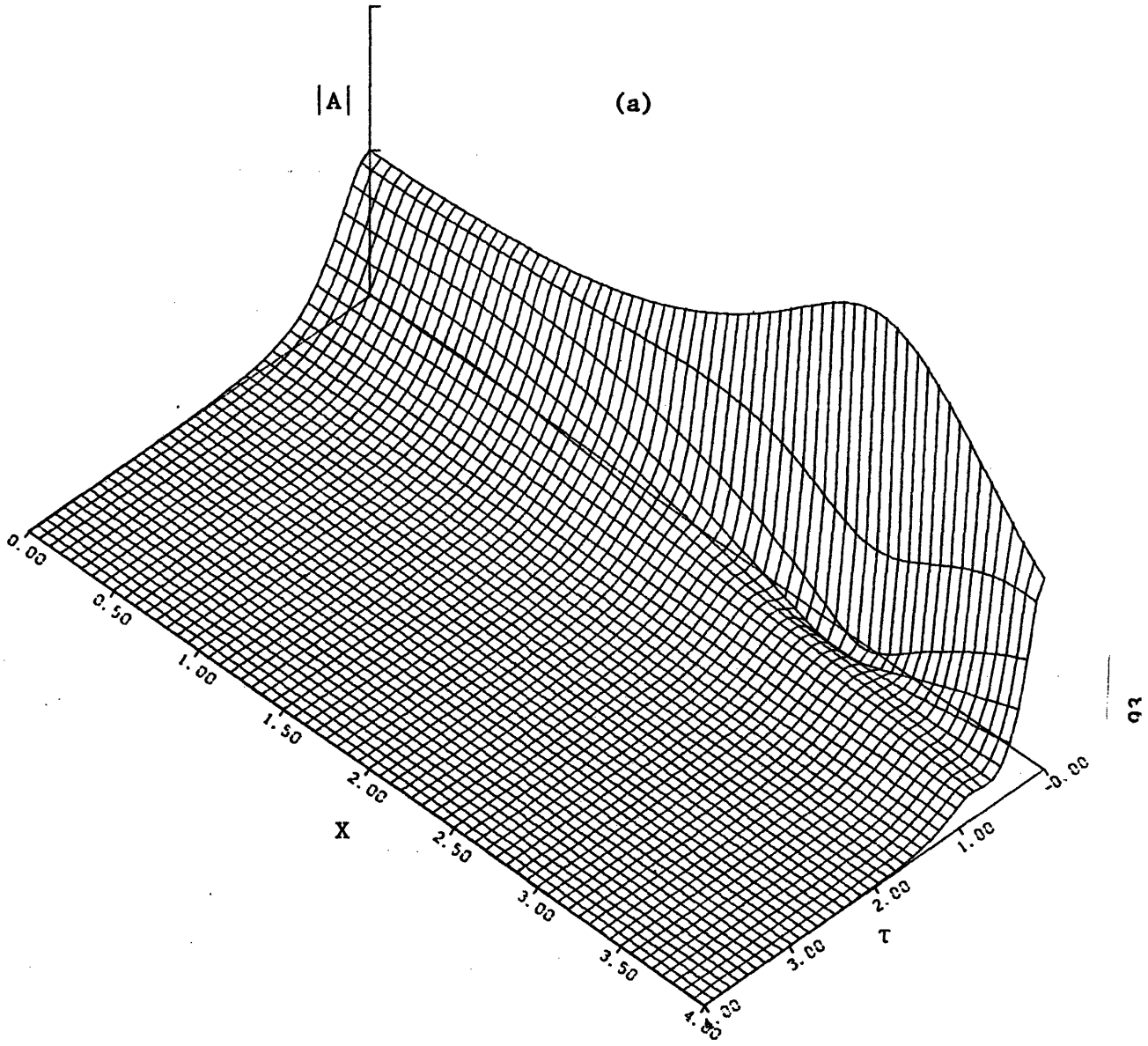
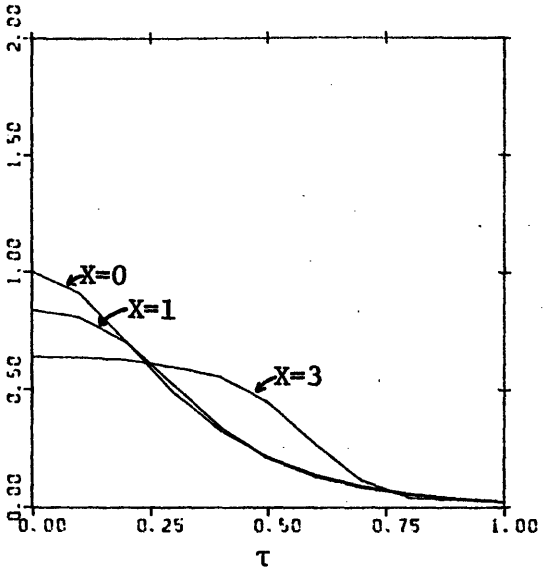
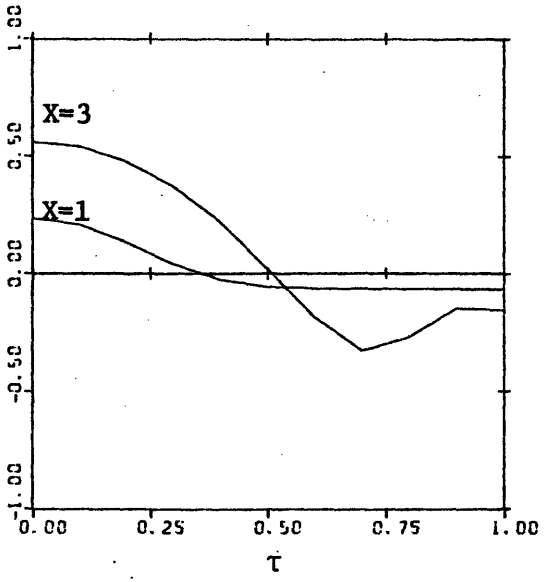


Figure 7.3 Evolution of a Sech Profile over Decreasing Depth, Supercritical Current, K Positive



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

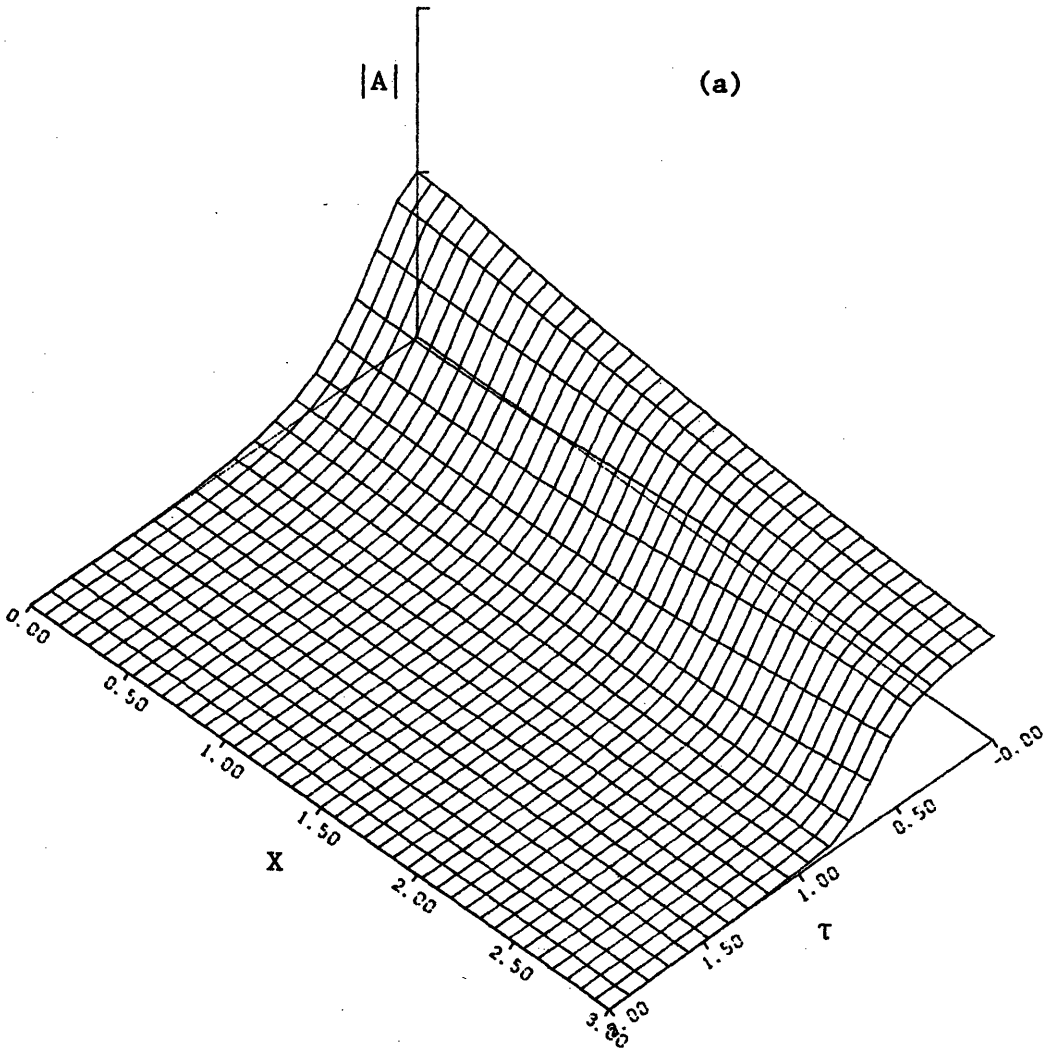
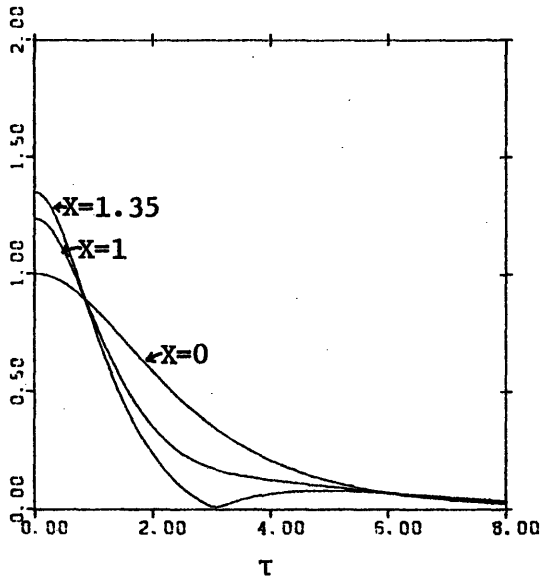


Figure 7.4 Evolution of a Sech Profile over Increasing Depth, Supercritical Current, K Negative

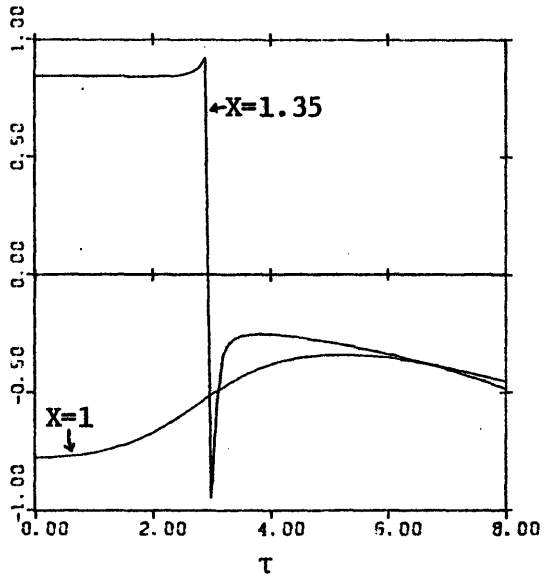
suggests that the wave envelope evolves into two solitons plus some oscillations which dies for large X. Fig. (7.4) corresponds to waves of period T = 2 propagating into deeper water dh = - 0.3. In this case K is negative everywhere and the initial profile decays. The radiation is very small. At this point the sign of K and its variations with depth appear to be essential in determining the qualitative features of the evolution of an initial sech profile.

7.c Subcritical Opposing Current

A current propagating against waves tends to decrease their wavelength and hence moves the point where K crosses zero towards shallower water (see Fig. (4.9)). But as shown in Fig. (4.5) for any given current velocity and depth there exist a critical period such that waves with smaller period can not propagate against that current. For example waves of period less than $\sim$ 2.5 cannot propagate at h = 1 against a current U_1 = - 0.1. Near the stopping point, our assumption of weakly nonlinear wave breakdown; the dispersion effect represented by α_2 becomes very large and hence the initial profile will change over a very short distance of propagation. Several cases have been studied to illustrate the effect of an opposite current on the evolution of waves. Fig. (7.5) - (7.6) present the case where K is positive everywhere. Fig. (7.5) represents waves of period $\underset{\sim}{T}$ = 6 propagating into deeper water dh = 0.5, against a current U_1 = - 0.1. The initial soliton profile increases in steepness during its propagation, disintegrates and then rebuilds itself after a near recurrence period of order 2. The evolution picture is very



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{Arg(A(X,\tau))}{\pi}$

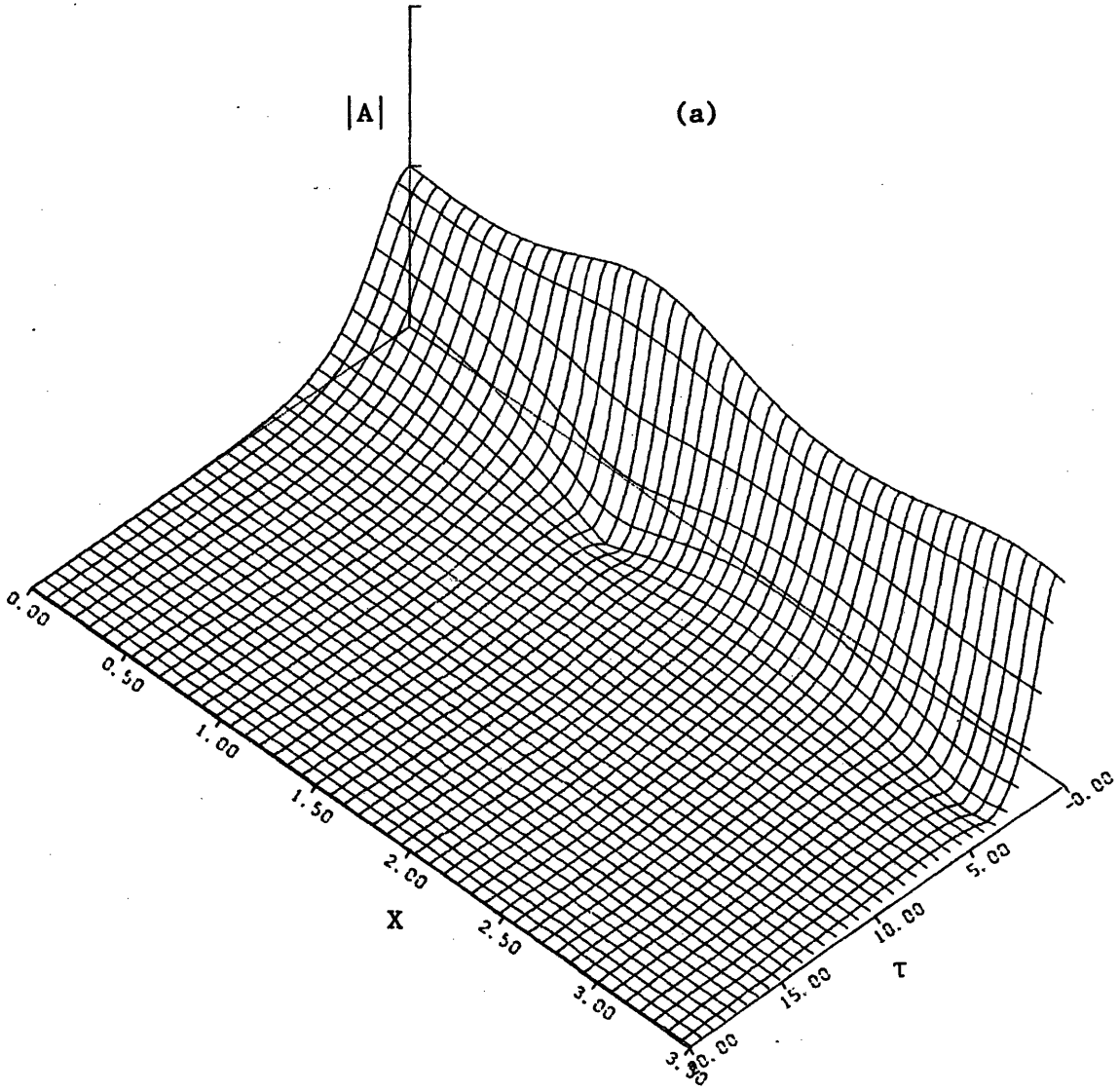
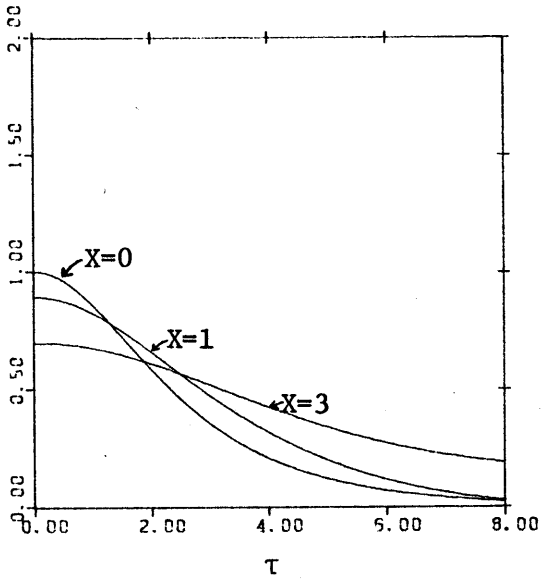
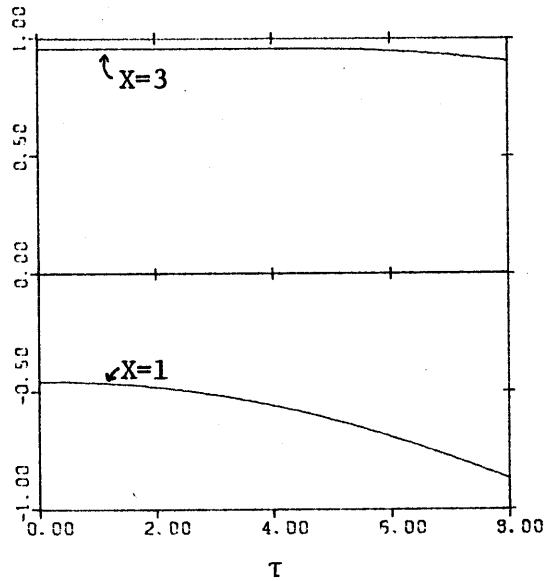


Figure 7.5 Evolution of a Sech Profile over Increasing Depth, Opposing Current, K Positive

similar to the one obtained in Fig. (6.5). In Fig. (7.6) the wave period $\underset{\sim}{T}$ is still $\underset{\sim}{T} = 6$, however the waves propagate towards decreasing depth dh = 0.15 and against a current $U_1 = -0.1$. In this case $K > 0$ but decreases. The plot shows a small decay in the wave envelope maximum amplitude very similar to the one obtained in Fig. (6.6). Since K is positive everywhere in the two previous cases we have calculated Δ, the relative difference between the area under the wave envelope at X = L as expected by D-R theory and as computed numerically. In the case of Fig. (7.5) $\Delta = 8.4\%$. The large values obtained for Δ shows once again that the D-R assumption is not reliable. According to the numerical results m = 1.61 for Fig. (7.5) 1 soliton would evolve for large X. To point out the effect of the magnitude of the depth variation we have plotted in Fig. (7.7) the propagation of a soliton corresponding to waves of period $\underset{\sim}{T} = 6$ over an increasing depth dh = -0.3. By comparing this figure to the previous one, we can see that the effect of decreasing the magnitude of dh is to lengthen the near recurrence period and to decrease the maximum amplitude. As the opposite current increases, the only waves which can propagate at h = 1 without being stopped are those having a large period. For such waves the parameter K tends to become negative in the neighborhood of h = 1 (see Fig. 4.9) and therefore an initial sech profile will decay during its propagation over variable depth. This effect is illustrated in Fig. (7.8) for waves with period $\underset{\sim}{T} = 7$ propagating into deeper water dh = - 0.15, against a current $U_1 = -0.1$. The initial profile radiates during its propagation qualitatively as in the case of no current, K < 0 (see Fig. 6.8).



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{Arg(A(X,\tau))}{\pi}$

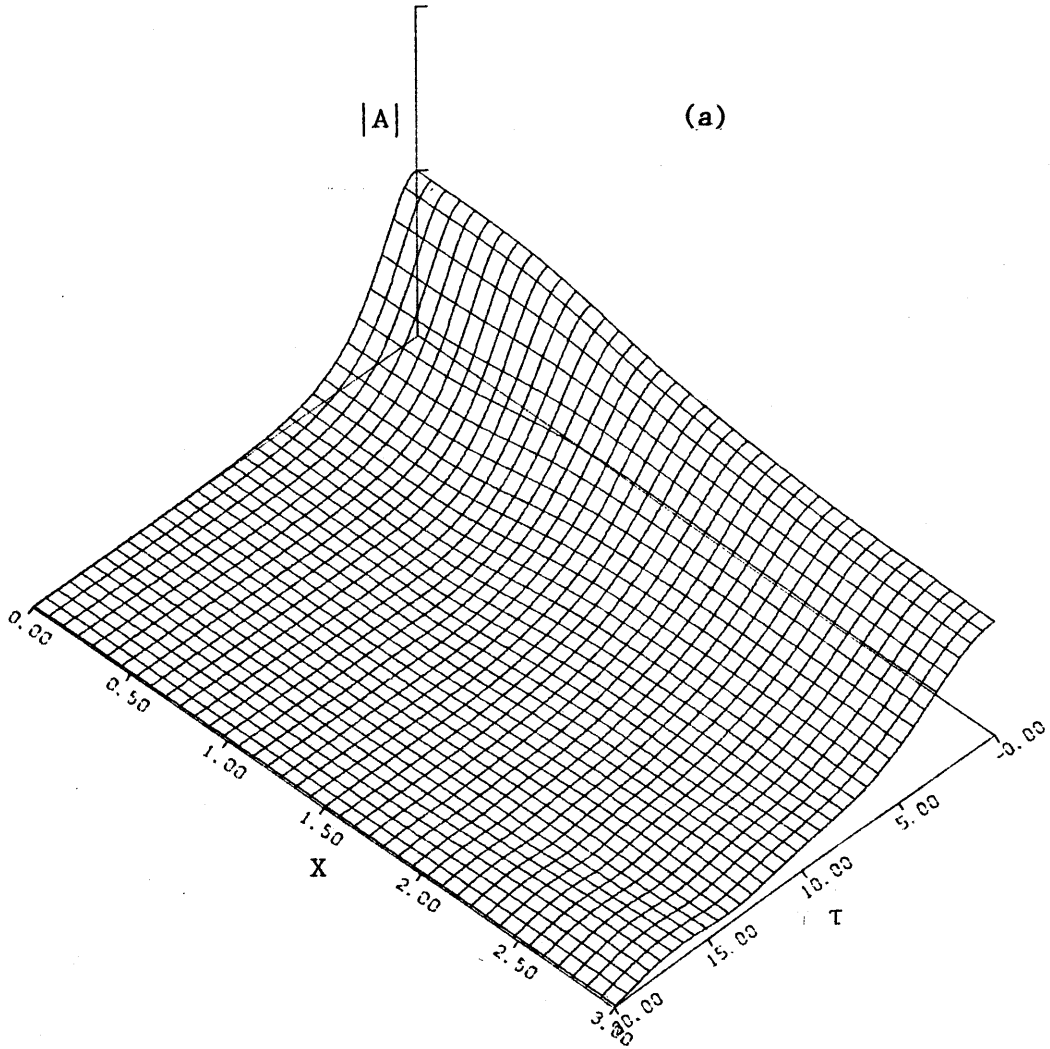
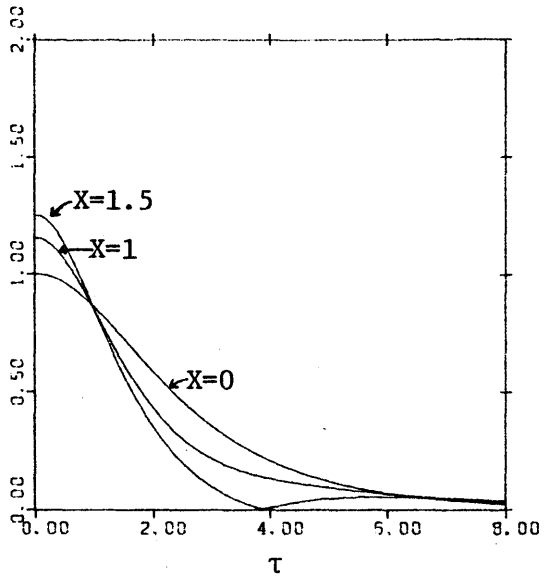
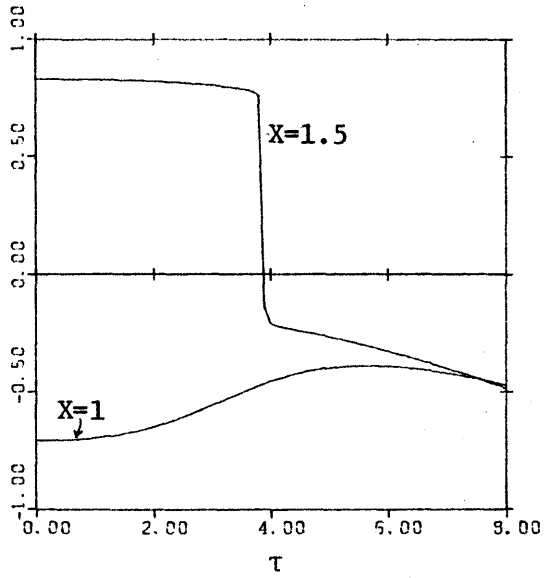


Figure 7.6 Evolution of a Sech Profile over Decreasing Depth, Opposing Current, K Positive



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

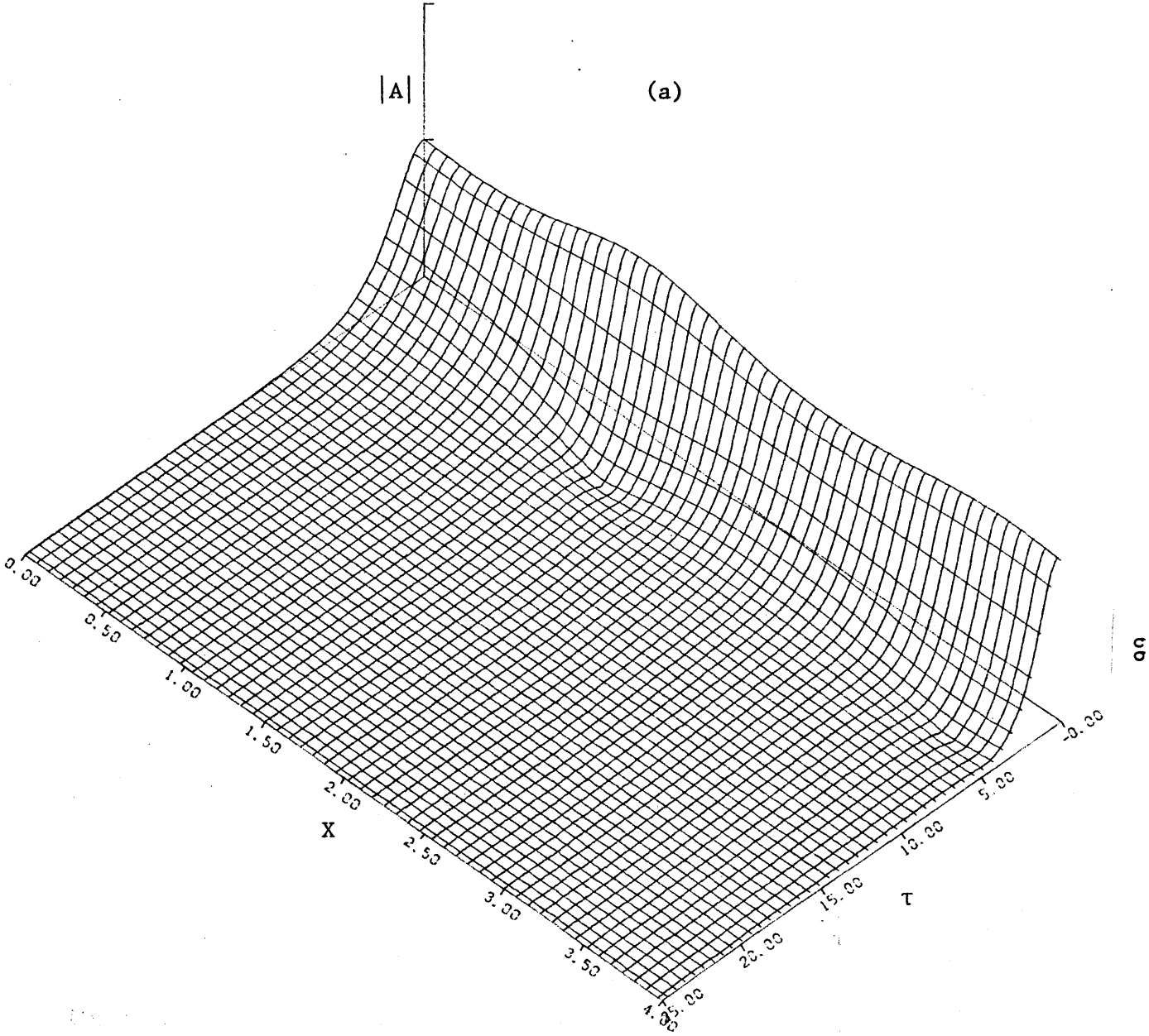
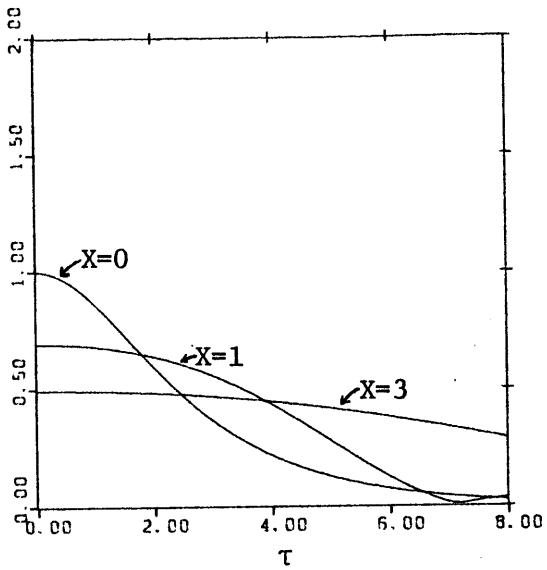
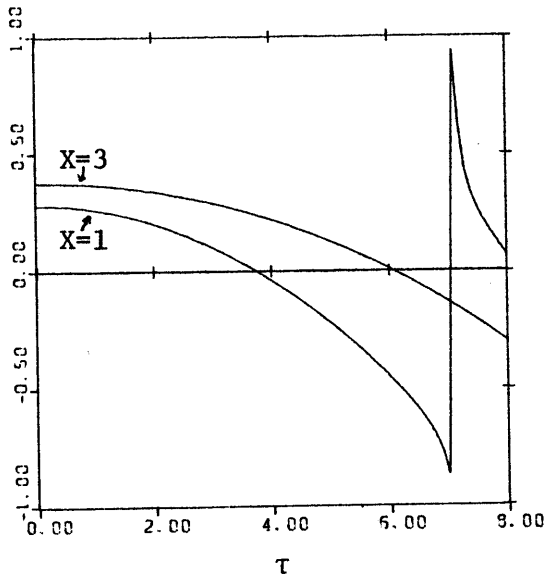


Figure 7.7 Evolution of a Sech Profile over Increasing Depth, Opposing Current, K Positive



(b) Wave Amplitude $|A(X,\tau)|$



(c) Wave Phase $\frac{\text{Arg}(A(X,\tau))}{\pi}$

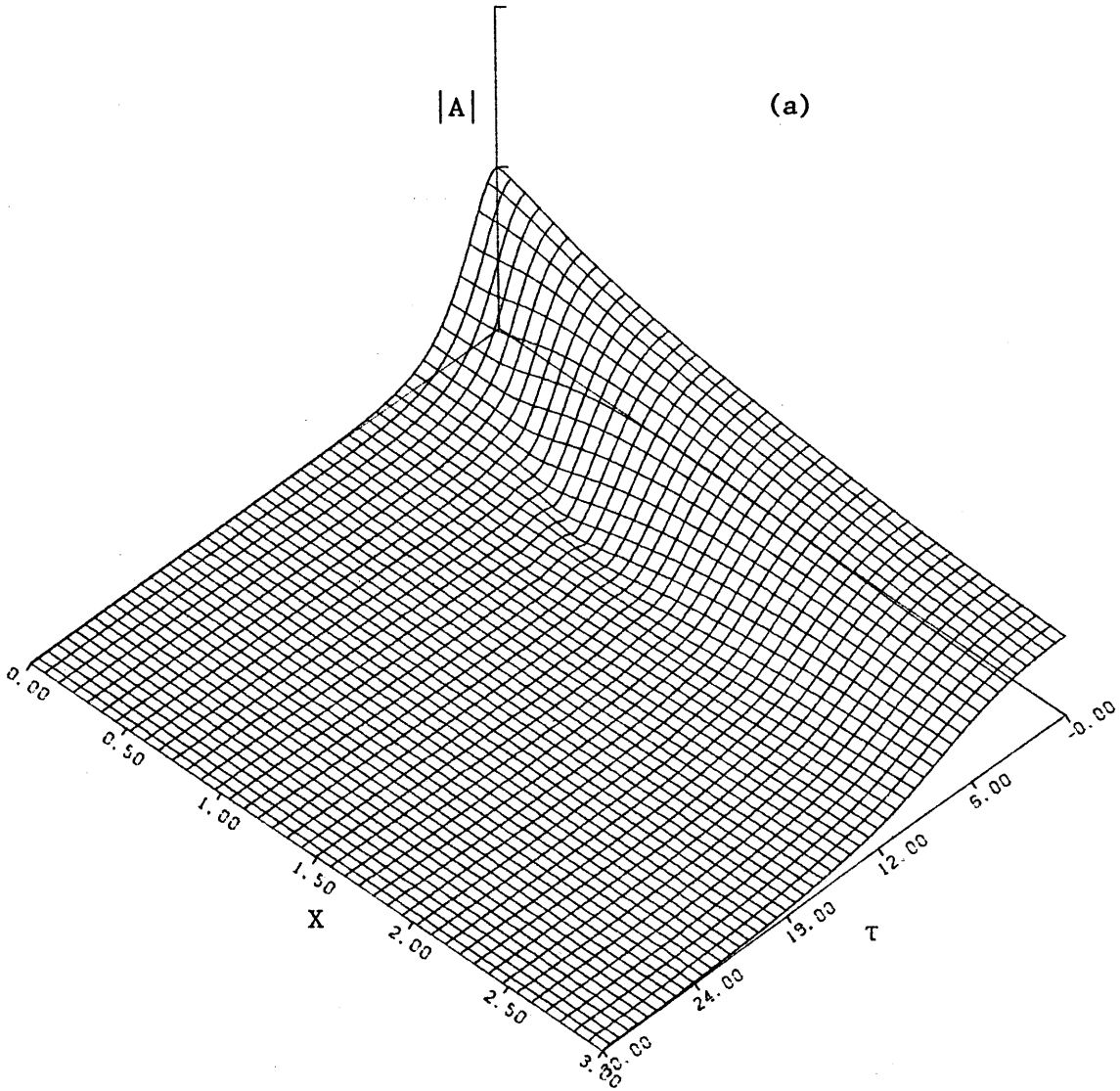


Figure 7.8 Evolution of a Sech Profile over Increasing Depth, Opposing Current, K Negative

8. Conclusion

The previous results are a general survey of the evolution of a slowly modulated wave packet propagating over varying depth, in the same direction as a strong current or in the opposite direction. They show that for any subcritical current the qualitative features of the evolution depend only on the sign of K and its variations: K increases with increasing depth and decreases with decreasing depth.

In the case where K is positive everywhere, an initial soliton profile propagating into deeper water becomes steeper and steeper. At the end of the depth change, the wave envelope is more peaked and evolves into a soliton or a multisoliton appropriate for the new depth. Depending on the characteristics of the initial soliton, of the depth variation and of the current, the profile can fission into 2-bounded solitons with near recurrence periods. On the other hand when the initial soliton profile propagates into shallower water, it becomes flatter and flatter. At the end of the depth change, the wave envelope can either evolves into a soliton adapted to the new depth or desintegrates as $X \to \infty$.

In the case where K is negative everywhere, whatever the depth variation is, the initial profile desintegrates during its propagation. The width in τ of the wave packet increases with X while its maximum amplitude decays. The rate of decay of the maximum amplitude with X is dependent on the characteristics of the waves, the current and the depth variation.

When K changes sign as the depth varies, two cases can occur. The first one corresponds to waves propagating into shallower water, in this case K is initially positive and becomes negative as the depth decreases. The initial profile, first begins to flatten but as the sign of K changes its rate of decay increases and finally it desintegrates. The second case corresponds to waves propagating into deeper water and hence K increases from an initial negative value to a positive one. In the region where K is negative the waves envelope begins to desintegrate. If the profile of the waves envelope is very flat at the point where K becomes negative, it would just pursue its desintegration while propagating into the region where K is positive, otherwise it would evolve into a soliton or a multi-soliton.

The previous discussion pointed out the importance of the sign of K for the evolution of an initial wave packet. As shown in Figure (4.6), (4.9) this sign depends on the wave period, the depth and the current. For a given period a strong current in the same direction as the waves tends to decrease the value of K at a given depth and hence moves the point where K changes its sign towards deeper water. Therefore in the presence of the current, except for short wave packets, any initial wave period whose envelope is a sech profile desintegrates during its propagation over variable depth. Depending on the depth variation a short wave packet can desintegrate if the depth decreases or evolves into a soliton or a multisoliton if the depth increases. For a current propagating against waves the effect is to move the point where K crosses zero towards shallower water and therefore long wave packets only, desintegrates during their propagation over variable depth. (The opposite current also affects the

propagation of waves since it can stop the shortest of them; our weakly nonlinear theory is not valid in the neighborhood of this stopping depth. Therefore as the current increases only long waves can propagate at the initial depth h = 1 and the envelope of such a long wave desintegrates during its propagation over varying depth.

Through the numerical study we have shown that the assumption of 'shape conservation' made by Djordjevic and Redekopp to derive their analytical solution for the propagation of an initial soliton over slowly varying depth, is not valid. The difference between the areas under the wave envelope profile at the end of the depth change as predicted using D-R's assumption and as calculated numerically is in some cases about 22%. It has been checked that this large difference is not due to the numerical method.

Finally this study shows the very drastic effect of depth variation and strong current on the propagation of a wave packet. It is based on the assumption that the current is strong enough to influence the waves without being influenced by them. This assumption holds in some cases, for example a harbor entrance with a river flow; our results can then be applied directly. In nature most currents are weak i.e. of the same order of magnitude as waves. A new study is then needed to include the effect of waves on the current itself and our results should be considered as a first step to improve our understanding of this problem.

Figure Captions: Part I

Figure 3.1: Nonlinearity coefficient Y

Figure 4.1: Specific Head Diagram

Figure 4.2: Critical depth

Figure 4.3: Set-up due to the current

Figure 4.4: Dispersion Relation Diagram

Figure 4.5: (a) Stopping periods $T'' = \frac{2\pi}{\omega}\sqrt{\frac{g}{d}}$ for negative currents

$$U'' = \frac{U}{\sqrt{gd}}$$

(b) Stopping periods $T' = \frac{2\pi}{\omega}\sqrt{\frac{g}{H_1}}$ as a function of depth $\frac{h}{H_1}$.

for different current $\frac{U_1}{\sqrt{gH_1}} = -0.1, -0.3, -0.6, -0.9$

Figure 4.6: (a) Wave number kh, (b) Instability parameter K, (c) Inverse of the shoaling coefficient s^{-2}. (No current). $\underset{\sim}{T} = 4,5,6,7$

Figure 4.7: (a) Wave number kd, (b) Instability parameter K, (c) Inverse of the shoaling coefficient s^{-2} for a subcritical current $U_1 = 0.5$. $\underset{\sim}{T} = 3, 4, 5, 6$

Figure 4.8: (a) Wave number kd, (b) Instability parameter K, (c) Inverse of the shoaling coefficient s^{-2} for a supercritical current $U_1 = 2$. $\underset{\sim}{T} = 1, 1.5, 2, 2.5$

Figure 4.9: (a) Wave number kd, (b) Instability parameter K, (c) Inverse of the shoaling coefficient s^{-2} for an opposing current $U_1 = -0.1$. $\underset{\sim}{T} = 6, 6.5, 7, 8$

Figure 5.1: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$;

(b) Wave amplitude $|A(X,\tau)|$, and (c) phase $\mathrm{Arg}(A(X,\tau))$ for $X = 0, 1, 4$. No current, constant depth $h = 1$, wave period $\underset{\sim}{T} = 5$, $K = 1.4$, $(kh) = 1.69$, $m = 1.5$

Figure 5.2: Geometry of the problem

Figure 6.1: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\text{Arg}(A(X,\tau))$ for $X = 0, 1, 3$, $U_1 = 0$, $\underset{\sim}{T} = 7$, $dh = 0$, $K = -3.33$, $(kh) = 1.04$

Figure 6.2: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\text{Arg}(A(X,\tau))$ for $X = 0, 1, 3$, $U_1 = 0$, $\underset{\sim}{T} = 5$, $dh = 0$, $K = 1.4$, $(kh) = 1.69$, $m = 1.25$

Figure 6.3: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\text{Arg}(A(X,\tau))$ for $X = 0, 0.72, 1$, $U_1 = 0$, $\underset{\sim}{T} = 5$, $dh = 0$, $K = 1.4$, $(kh) = 1.69$, $m = 2.0$

Figure 6.4: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) wave amplitude $|A(X,\tau)|$, and (c) Phase $\text{Arg}(A(X,\tau))$. for $X = 0, 0.4, 1$, $U_1 = 0$, $T = 5$, $dh = 0$, $T = 5$, $dh = 0$, $K = 1.4$, $(kh) = 1.69$, $m = 1.5$

Figure 6.5: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\text{Arg}(A(X,\tau))$ for $X = 0, 0.9, 1$, $U_1 = 0$, $\underset{\sim}{T} = 5$, $dh = -0.5$, $K_1 = 1.4$, $(kh)_1 = 1.69$, $K_2 = 2.6$, $(kh)_2 = 2.4$, $m = 1.62$, $\Delta = -22\%$, global energy error = 4.2%

Figure 6.6: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\text{Arg}(A(X,\tau))$ for $X = 0\ 1, 4$, $U_1 = 0$, $\underset{\sim}{T} = 5$, $dh = 0.15$, $K_1 = 1.4$, $(kh)_1 = 1.69$, $K_2 = 0.72$, $(kh)_2 = 1.49$, $m = 1.3$, $\Delta = 11\%$, global energy error = 0.1%

Figure 6.7: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for $X = 0.1, 1.6, U_1 = 0, \underset{\sim}{T} = 5$, $dh = -0.3$, $K_1 = 1.4$, $(kh)_1 = 1.69$, $K_2 = 2.19$, $(kh)_2 = 2.11$, $m = 1.5$, $\Delta = -26.6\%$, global energy error = 2.5%

Figure 6.8: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for $X = 0, 1, 3, U_1 = 0, \underset{\sim}{T} = 7$, $dh = -0.3$, $K_1 = -3.3$, $(kh)_1 = 1.04$, $K_2 = -0.65$, $(kh)_2 = 1.24$, global energy error $< 0.1\%$

Figure 6.9: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for $X = 0, 1, 3, U_1 = 0, \underset{\sim}{T} = 7$, $dh = 0.3$, $K_1 = -3.30$, $(kh)_1 = 1.04$, $K_2 = -14.6$, $(kh)_2 = 0.83$, global energy error = 0.1%

Figure 6.10: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for $X = 0, 1, 3$, $U_1 = 0, \underset{\sim}{T} = 6$, $dh = -0.5$, $K_1 = -0.54$, $(kh)_1 = 1.28$, $K_2 = 1.10$, $(kh)_2 = 1.75$, $m = 1.98$, global energy error = 0.05%

Figure 6.11: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for $X = 0, 1, 3$, $U_1 = 0, \underset{\sim}{T} = 5$, $dh = 0.3$, $K_1 = 1.4$ $(kh)_1 = 1.69$, $K_2 = -0.68$, $(kh)_2 = 1.29$, global energy error: 0.1%

Figure 7.1: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for X= 0,0.85,1, $U_1 = 0.5$, $\underset{\sim}{T} = 3$, dh = - 0.3, K_1 = 6.0, $(kd)_1 = 1.68$, $K_2 = 15.7$, $(kd)_2 = 2.62$, m = 1.9, Δ = - 15.2% global energy error = 5%

Figure 7.2: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for X = 0,1, 3, $U_1 = 0.5$, $\underset{\sim}{T} = 4$, dh = 0.15, K_1 = -4.8, $(dk)_1 = 1.17$, $K_2 = -42$, $(kd)_2 = 0.82$, global energy error.

Figure 7.3: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for X = 0, 1, 3, $U_1 = 2$, $\underset{\sim}{T} = 1.5$, dh = 0.3, $K_1 = 17.9$, $(kd)_1 = 1.51$, $K_2 = 48.0$, $(kd)_2 = 1.86$, m = 2.1, Δ = -0.9% global energy error: 0.2%

Figure 7.4: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for X = 0, 1, 3, $U_1 = 2$, $\underset{\sim}{T} = 2$, dh = - 0.3, $K_1 = -39.9$ $(kd)_1 = 1.1$, $K_2 = -91.0$, $(kd)_2 = 0.95$, global energy error: 0.05%

Figure 7.5: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $Arg(A(X,\tau))$ for X = 0, 1, 1.35, $U_1 = -0.1$, $\underset{\sim}{T} = 6$, dh = - 0.5, $K_1 = 0.63$, $(kd)_1 = 1.58$, $K_2 = 1.1$, $(kd)_2 = 2.01$, m = 1.61, Δ = - 13.4% global energy error = 0.5%

Figure 7.6: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\mathrm{Arg}(A(X,\tau))$ for $X = 0, 1, 3$, $U_1 = -0.1$, $\underset{\sim}{T} = 6$, $dh = 0.15$, $K_1 = 0.63$, $(kd)_1 = 1.58$, $K_2 = 0.41$, $(kd)_2 = 1.48$, $m = 1.3$, $\Delta = 8.4\%$ global energy error $< 0.1\%$

Figure 7.7: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\mathrm{Arg}(A(X,\tau))$ for $X = 0, 1, 1.5$, $U_1 = -0.1$, $\underset{\sim}{T} = 6$, $dh = -0.3$, $K_1 = 0.63$, $(kd)_1 = 1.58$, $K_2 = 0.94$, $(kd)_2 = 1.83$, $m = 1.6$, $\Delta = 9.4\%$ global energy error $= 0.3\%$

Figure 7.8: (a) 3-dimensional plot of the wave amplitude $|A(X,\tau)|$, (b) Wave amplitude $|A(X,\tau)|$, and (c) Phase $\mathrm{Arg}(A(X,\tau))$ for $X = 0, 1, 3$, $U_1 = -0.1$, $\underset{\sim}{T} = 7$, $dh = -0.15$, $K_1 = -0.61$, $(kd)_1 = 1.23$, $K_2 = -0.15$, $(kd)_2 = 1.32$, global energy error $= 0.05\%$

PART II :

GENERATION OF SECOND ORDER LONG WAVES

IN WATER OF VARYING DEPTH

1. Introduction

The interaction between waves and strong current was studied in the first part. The current is found to affect strongly the waves by changing their wavelength, their evolution properties and their stability regions while the waves do not affect the current. A second extreme case is studied in this part: the current is assumed to be weaker than the waves. The waves then do affect the current but are not affected by it, at the leading order. As an example, a slowly modulated wave evolving on constant depth can generate a mean current and a mean set-down (see Longuet-Higgins and Stewart (1962)). The mean free surface displacement, induced by the wave group envelope, is a second order long wave propagating with the group velocity and hence is bound to the wave group. As noted by Longuet-Higgins and Stewart, this locked wave is the result of the variations of the radiation stress which are induced by the slowly varying envelope. If in addition the depth varies slowly (on the same scale as the envelope), the locked wave alone will not compensate the effect of the radiation stress change due to the depth variation and a forced wave is generated in this region. After being created in the region where the depth varies, the forced wave could radiate energy towards the farfield in the form of progressive long waves traveling at the phase velocity $\sqrt{gh}$. In contrast to the locked waves, these long waves are not bound to the short waves. Since their period is of the order of few minutes,they can be of practical interest in some engineerings problems: Harbor resonance, deep water structure dynamics...In coastal regions they have also been related to surf-beats.

Such mechanism for the generation of second order free long waves was studied by Molin (1982) in the case where the first order short waves are considered to be deep water waves while the generated second order waves are considered to be shallow water waves. He found that the waves radiated away from the region of depth variation have very small amplitude (i.e. third order) and thattheir amplitude increases as the water depth becomes intermediate, i.e. $kh = O(1)$. In this part we extend Molin's study to the case of plane wave in water of intermediate depth. We first derive the governing equation for the mean free surface displacement induced by a plane wave group with slow modulations in both directions x and y. For this derivation we use two different methods and show that they are equivalent; the first one is the multiple scale method which is more formal but straightforward, the second one is an averaging method which is much less straightforward but whose results can be easily interpreted physically using the radiation stress concept. Then limiting ourselves to the case where the depth variation is in the x direction only and where the wave envelope is periodic in time with constant frequency, we derive a governing equation for the forced waves which is solved numerically by using a hybrid element method.

2. Governing Equations for the Mean Current and Set-Down

We consider a weakly nonlinear wave group propagating into water of variable but intermediate depth h i.e.

$$\varepsilon = ka << 1 \qquad \mu = kh = O(1) \tag{2.1}$$

where k is the wave number and a is the wave amplitude. The wave envelope and the depth are assumed to vary slowly . Introducing the slowly varying time and horizontal space coordinates

$$T = \varepsilon t \qquad X = \varepsilon x \qquad Y = \varepsilon y \tag{2.2}$$

we have

$$h = h(X,Y) \quad \text{and} \quad A = A(X,Y,T) \tag{2.3}$$

where h is the depth of the water measured from the still water level and A is the wave amplitude. Therefore the length of the wave envelope is $O(1/\varepsilon)$ and is of the same order as the distance over which the depth varies. The wave envelope modulation is known to induce a second order free surface displacement and current, which vary on the same scales as the first order envelope (i.e. X,Y,T scales). The governing equations of the induced current and free surface displacement can be found either by using the multiple scale method or by averaging the equation of the flow over the short scales.

2.a. The Multiple Scale Method

This method has been applied to a slowly modulated Stokes wave train by Chu and Mei (1970). Through their derivation of the nonlinear governing equation of the wave amplitude, they obtained the governing equation

of the second-order mean free surface displacement and current (Eqs. (4.5) and (4.6)). Let us summarize the main steps of this derivation. In terms of the potential velocity ϕ, the basic governing equations are

$$\nabla_2^2\phi + \phi_{zz} = 0 \quad -h(\vec{x}) \leq z \leq \eta(\vec{x},t) \tag{2.4}$$

$$\phi_{tt} + g\,\phi_z + \left(\frac{\partial}{\partial t} + \frac{1}{2}\,\vec{u}\cdot\nabla_3\right)|\vec{u}|^2 = 0 \quad z = \eta(\vec{x},t) \tag{2.5}$$

$$\phi_z + \nabla_2\phi\cdot\nabla h = 0 \qquad z = -h(\vec{x}) \tag{2.6}$$

$$\eta = -\frac{1}{g}\left(\phi_t + \frac{1}{2}\,|\vec{u}|^2\right) \quad z = \eta(\vec{x},t) \tag{2.7}$$

where $\vec{x} = (x,y)$ denotes the horizontal spatial coordinates, $\vec{u} = (\phi_x, \phi_y, \phi_z)$ is the velocity of a fluid particle, $\nabla_2 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\nabla_3 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ represent respectively the horizontal and total gradient operators, and η is the free surface elevation. Since $\eta = O(\varepsilon)$, the free surface conditions (2.5) and (2.7) can be expanded about $z = 0$. Now we assume the following expansions

$$\phi(\vec{x},z,t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \phi_{nm}(\vec{X},T)\, e^{imS} \tag{2.8}$$

$$\eta(\vec{x},t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^{+n} \eta_{nm}(\vec{X},T)\, e^{imS} \tag{2.9}$$

where $\vec{X} = (X,Y) = (\varepsilon x, \varepsilon y)$ and $T = \varepsilon t$. The phase function $S(\vec{x},t)$ then defines the wavenumber $\vec{k}$ and frequency ω.

$$\vec{k} = \nabla_2 S \tag{2.10.a}$$

$$\omega = - S_t \tag{2.10.b}$$

If ω is assumed to be constant, then the previous equations imply that $\vec{k}$ is independent of t and satisfies

$$\nabla_2 x\ k = 0 \tag{2.11}$$

Substituting the expansions (2.8) and (2.9) into the governing equations (2.4), (2.6) and the expanded form of (2.5) and (2.7) about $z = 0$, and separating different orders and harmonics give a set of equations:

$$\phi_{nmzz} - m^2 k^2 \phi_{nm} = R_{nm}(\vec{X}, z, T) \qquad -h \leq z \leq 0 \tag{2.12.a}$$

$$g\phi_{nmz} - m^2 \omega^2 \phi_{nm} = G_{nm}(\vec{X}, T) \qquad z = 0 \tag{2.12.b}$$

$$\phi_{nmz} = F_{nm}(\vec{X}, T) \qquad z = -h \tag{2.12.c}$$

$$\eta_{nm} = \frac{1}{g} (im\omega\ \phi_{nm} - H_{nm}(\vec{X}, T)) \qquad z = 0 \tag{2.12.d}$$

where R_{nm}, G_{nm}, F_{nm} and H_{nm} depend on lower order terms and are given in Appendix (B1). At the first order n = 1 the system of equations (2.12) yields that ϕ_{10} is independent of depth and ϕ_{11} is the classical linear wave solution

$$\phi_{10} = \phi_{10}(\vec{X},T) \tag{2.13.a}$$

$$\phi_{11} = - iA(\vec{X},T) \frac{g}{2\omega} \frac{\text{coh } k(z+h)}{\cosh kh} \tag{2.13.b}$$

$$\eta_{11} = \frac{A}{2} (\vec{X},T) \tag{2.13.c}$$

where $\omega^2 = kg \tanh kh$ (2.14)

At the second order zeroth harmonic Eq. (2.12.d) gives an equation relating the mean quantities ϕ_{10} and η_{20}:

$$\eta_{20} = - \frac{1}{4} A^2 \frac{\omega^2}{g} \frac{1}{\sinh^2(kh)} - \frac{1}{g} \phi_{10t} \tag{2.15}$$

From the set of equation (2.12) it can be seen by using Green's formula that a necessary condition for the second order first harmonic solution to exist is that the following solvability condition is satisfied.

$$\frac{1}{g} G_{21} = \int_{-h}^{0} dz R_{21} \frac{\cosh k(z+h)}{\cosh kh} + \frac{F_{21}}{\cosh kh} \tag{2.16}$$

which gives

$$\frac{\partial A^2}{\partial T} + \nabla \cdot (\vec{C}_g A^2) = 0 \tag{2.17}$$

after using the expressions for G_{21} and R_{21}. Where $\nabla = (\frac{\partial}{\partial X}, \frac{\partial}{\partial Y})$ and $\vec{C}_g$ is the group velocity defined as:

$$\vec{C}_g = \frac{\vec{k}\omega}{2k^2} (1 + \frac{2kh}{\sinh 2kh}) \tag{2.18}$$

Equation (2.17) represents the conservation of energy.

Similarly at the third order zeroth harmonic, using the Green's formula we find the solvability condition:

$$\frac{1}{g} G_{30} = \int_{-h}^{0} dz\, R_{30} + F_{30} \tag{2.19}$$

which gives a second equation relating η_{20} and ϕ_{10}

$$\frac{\partial \eta_{20}}{\partial T} + \nabla\cdot(h\nabla\phi_{10} + \frac{\vec{k}}{\omega}\frac{1}{2} gA^2) = 0 \tag{2.20}$$

Taking the derivative with respect to time of (2.20) and using (2.15), we obtain the governing equation for the mean free surface displacement

$$\nabla\cdot(gh\nabla\eta_{20}) - \frac{\partial^2 \eta_{20}}{\partial T} = \nabla\cdot(\frac{\vec{k}g}{2\omega}(A^2)_T) - \nabla.(h\nabla(\frac{\omega^2 A^2}{4\sinh^2 kh})) \tag{2.21}$$

Eq. (2.21) can be written in a slightly different form. Let us differentiate the dispersion relation (2.14)

$$\nabla k = -\frac{k\omega}{C_g \sinh 2kh}\nabla h \tag{2.22}$$

Using (2.17) and (2.22) in the right hand side of Eq. (2.21), this equation becomes:

$$\nabla\cdot(gh\nabla\eta_{20}) - \frac{\partial^2 \eta_{20}}{\partial T} = -\nabla\cdot[\nabla(\frac{gA^2 kh}{2\sinh 2kh}) + \frac{g}{2\omega}(\vec{k}\nabla\cdot(\vec{C}_g A^2) + A^2 C_g \nabla k)] \tag{2.23.a}$$

Using the irrotationality of $\vec{k}$ Eq. (2.23.a) reduces to

$$\nabla\cdot(gh\nabla\eta_{20}) - \frac{\partial^2 \eta_{20}}{\partial T} = -\frac{\partial^2}{\partial X_i^2}(\frac{gA^2}{2}(\frac{C_g}{C} - \frac{1}{2})) - \frac{\partial}{\partial X_i}\frac{\partial}{\partial X_j}(\frac{gA^2 C_g}{2C}\frac{k_i k_j}{k^2}) \tag{2.23.b}$$

2.b. The Averaging Method

Since we are interested in the governing equations of mean quantities, arguments similar to those leading to the Reynolds equations for the mean turbulent flow, can be applied. Therefore by averaging the Euler's equations over the rapidly varying scales, we would expect to obtain the conservation laws of mass and horizontal momentum for the mean current . This method have been applied by Longuet-Higgins and Stewart (1962). They introduced the radiation stress concept which is the equivalent of the Reynold stress in the case of turbulent flow. In what follows we rederive their results in the context of our problem.

The basic equations are Euler's equations with appropriate boundary conditions

$$\nabla\cdot\vec{u} = 0 \tag{2.24}$$

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla(P + \rho g z) \tag{2.25}$$

$$w + \vec{u}\cdot\nabla h = 0 \qquad z = -h(x,y) \tag{2.26}$$

$$\frac{\partial\xi}{\partial t} + \vec{u}\cdot\nabla\xi = w \qquad z = \xi(x,y,t) \tag{2.27}$$

$$P = 0 \qquad z = \xi(x,y,t) \tag{2.28}$$

where $\vec{u} = (u_1, u_2, w)$ is the flow velocity, P the pressure and ξ the free surface displacement. We now define the mean velocity U_i ($i = 1,2$) by integrating u_i over the water depth and then over the short time scale T ($T = \frac{2\pi}{\omega}$ represents the period of the short waves)

$$U_i = \frac{1}{\bar{\xi}+h} \int_{-h}^{\overline{\xi}} u_i \, dz \qquad i = 1,2 \tag{2.29}$$

where the time averaging of a quantity (Q) is defined as:

$$\bar{Q} = \frac{1}{\tilde{T}} \int_{t}^{t+\tilde{T}} Q \, dt \tag{2.30}$$

Physically $\rho(\bar{\xi}+h)U_i$ represents the mean rate of mass flux across a vertical plane of unit width along x_i = constant and is called the mass flux velocity (see Mei (1982)). Let $\tilde{u}_i$ be the deviation of the velocity from its mean i.e.

$$u_i = U_i + \tilde{u}_i \tag{2.31}$$

then it follows from the definition (2.29) that

$$\int_{-h}^{\overline{\xi}} \tilde{u}_i \, dz = 0 \tag{2.32}$$

The continuity equation (2.24) integrated with respect to depth gives

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x_i} \left(\int_{-h}^{\xi} u_i \, dx \right) = 0 \tag{2.33}$$

after using the boundary conditions (2.25) and (2.26). Substituting (2.31) into (2.33) and averaging over the short time scale we get

$$\frac{\partial \bar{\xi}}{\partial t} + \frac{\partial}{\partial x_i} \left((\bar{\xi} + h) U_i \right) = 0 \tag{2.34}$$

where Eq. (2.32) has been used.

Since the flow is inviscid, the equations of momentum conservation can be written as follows:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_i} + \frac{\partial u_j w}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial x_i} (- P\delta_{ij}) \qquad i = 1,2 \tag{2.35}$$

$$\frac{\partial w}{\partial t} + \frac{\partial u_i w}{\partial x_i} + \frac{\partial w^2}{\partial z} = - \frac{1}{\rho} \frac{\partial}{\partial z} (P + \rho g z) \tag{2.36}$$

Let us now integrate (2.35) with respect to depth, use Leibnitz rule to inverse the respective position of the derivative with respect to t or x_i and the integration with respect to z and use the boundary conditions (2.26) - (2.28):

$$\frac{\partial}{\partial t} \left(\int_{-h}^{\xi} u_j dz\right) + \frac{\partial}{\partial x_i} \int_{-h}^{\xi} (u_i u_j + \frac{P}{\rho} \delta_{ij}) dz - \frac{P|_{-h}}{\rho} \frac{\partial h}{\partial x_j} = 0 \tag{2.37}$$

Using (2.31) and averaging with respect to time gives:

$$\frac{\partial}{\partial t} ((\bar{\xi} + h)U_j) + \frac{\partial}{\partial x_i} ((h+\bar{\xi})U_i U_j) = \overline{\frac{P|_{-h}}{\rho}} \frac{\partial h}{\partial x_j} - \frac{\partial}{\partial x_i} \overline{\left(\int_{-h}^{\xi} (\tilde{u}_i \tilde{u}_j + \frac{P}{\rho} \delta_{ij}) dz\right)} \tag{2.38}$$

Introducing $\bar{p}$, the mean dynamic pressure at the bottom

$$\bar{p} = \overline{P|_{-h}} - \rho g(\bar{\xi}+h) \tag{2.39}$$

Eq. (2.38) can be written in a slightly different way:

$$\frac{\partial}{\partial t} ((\bar{\xi} + h)U_j) + \frac{\partial}{\partial x_i} ((h + \bar{\xi})U_i U_j) = \frac{\bar{p}}{\rho} \frac{\partial h}{\partial x_j} - g(\bar{\xi} + h) \frac{\partial \bar{\xi}}{\partial x_j} -$$

$$\frac{1}{\rho} \frac{\partial (S_{ij})}{\partial x_i} \tag{2.40}$$

where $S_{ij} = \int_{-h}^{\xi} dz[\rho\tilde{u}_i\tilde{u}_j + (\tilde{P} - \rho g(\bar{\xi}-z))\ \delta_{ij}]$ (2.41)

S_{ij} represents the sum of the i^{th} component of the net momentum flux across and the excess hydrodynamic pressure on, a surface normal to the j-th direction. This tensor representing the excess momentum fluxes is called the radiation stress (see Longuet-Higgins and Stewart, 1960). In order to obtain the average pressure we integrate Eq. (2.36) from z to ξ

$$P(z) = \rho g(\xi - z) + \frac{\partial}{\partial x_i}\left(\int_z^{\xi} \rho u_i w dz\right) - \rho w^2 + \rho \frac{\partial}{\partial t}\left(\int_z^{\xi} w dz\right) \qquad (2.42)$$

where the Leibnitz rule has been used to inverse derivation and integration and the boundary conditions (2.27) and (2.28) have been applied. So far the equations found are very general but to go further we need to make some assumptions. First we assume that the depth is slowly varying, i.e.

$$\nabla h = O(\varepsilon) \qquad (2.43)$$

we also assume that at the leading order the flow corresponds to the propagation of an infinitesimal wave with slowly varying amplitude over the variable depth. Therefore the wave velocity and the free surface displacement are:

$$u_j = \frac{k_j g A}{2\omega} \frac{\cosh k(z+h)}{\cosh kh} e^{iS} + * + O(\varepsilon^2) \quad j = 1,2 \qquad (2.44)$$

$$w = -\frac{ikgA}{2\omega}\frac{\sinh k(z+h)e^{iS}}{\cosh kh} + * \quad + O(\varepsilon^2) \tag{2.45}$$

$$\xi = \frac{A}{2}e^{iS} + * \quad + O(\varepsilon^2) \tag{2.46}$$

where $S = \int k_1 dx + \int k_2 dy - \omega t$ (2.47)

and * denotes the complex conjugate. Eqs. (2.44) - (2.47) imply that the derivative with respect to time or horizontal space coordinate of an averaged quantity is of higher order i.e.

$$\frac{\partial}{\partial t}(\bar{\cdot}),\ \frac{\partial}{\partial x_i}(\bar{\cdot}) = O(\varepsilon) \tag{2.48}$$

we also deduce that

$$\bar{u}_i,\ \bar{\xi},\ \bar{w} \lesssim O(\varepsilon^2) \tag{2.49}$$

As a comment, we note from Eq. (2.26) that $w\big|_{-h} = O(\varepsilon^3)$. From Eqs. (2.44) it can be seen that u_j and $\frac{\partial \xi}{\partial x_j}$ are out of phase therefore $\overline{u_j \frac{\partial \xi}{\partial x_j}} = O(\varepsilon^3)$ and Eq. (2.27) implies that $\bar{w}\big|_{\xi} = O(\varepsilon^3)$. Averaging Eq. (2.49) with respect to time and using the previous assumptions give

$$\bar{P}(z) = \rho g(\bar{\xi} - z) - \rho\overline{w^2} + O(\varepsilon^3) \tag{2.50}$$

The dynamical pressure at the bottom is

$$\bar{p} \lesssim O(\varepsilon^3) \tag{2.51}$$

Similarly, the radiation stress tensor can be estimated to the second order, by plugging the expression for the pressure (2.42) into (2.40)

$$S_{ij} = \overline{\int_{-h}^{\xi} \tilde{u}_i \tilde{u}_j dz} + \overline{\left(\int_{-h}^{\xi} dz \left(\frac{\partial}{\partial t} \int_{z}^{\xi} w dz'\right) + \int_{-h}^{\xi} dz \left(\frac{\partial}{\partial x_i} \int_{z}^{\xi} u_i w dz'\right) - \int_{-h}^{\xi} w^2 dz\right.}$$

$$\left. + \frac{1}{2} g \tilde{\xi}^2 \right) \delta_{ij} \qquad (2.52)$$

where $\tilde{\xi} = \xi - \bar{\xi}$. Applying the rules (2.48) and (2.49) we obtain:

$$S_{ij} = \int_{-h}^{o} \overline{\tilde{u}_i \tilde{u}_j} \, dz + \int_{-h}^{o} \overline{w^2} \, dz + \frac{1}{2} g \overline{\tilde{\xi}^2} + O(\varepsilon^3) \qquad (2.53)$$

To have S_{ij} to the second order we only need the first order approximation of the velocities and free surface displacement as given by Eqs. (2.44) - (2.47)

$$S_{ij} = \frac{gA^2}{2} \left(\frac{C_g}{C} - 1/2\right) \delta_{ij} + \frac{gA^2}{2} \frac{C_g}{C} \frac{k_i k_j}{k^2} + O(\varepsilon^3) \qquad (2.54)$$

where $C_g = \frac{C}{2} \left(1 + \frac{2kh}{\sinh 2kh}\right)$ is the group velocity and $C = \frac{\omega}{k}$ is the phase velocity. Since U_i and $\bar{\xi}$ are $O(\varepsilon^2)$ Eqs. (2.34) and (2.38) become

$$\frac{\partial \bar{\xi}}{\partial t} + \frac{\partial}{\partial x_i} (h \, U_i) = O(\varepsilon^5) \qquad (2.55)$$

$$h \frac{\partial U_j}{\partial t} + gh \frac{\partial \bar{\xi}}{\partial x_j} = - \frac{1}{\rho} \frac{\partial}{\partial x_i} S_{ij} + O(\varepsilon^4) \qquad (2.56)$$

Taking $- \frac{\partial}{\partial t}$ (2.55) $+ \frac{\partial}{\partial x_j}$ (2.56) we finally find the governing equation for the mean free surface displacement

$$- \frac{\partial^2 \bar{\xi}}{\partial t^2} + \frac{\partial}{\partial x_j} \left(gh \frac{\partial \bar{\xi}}{\partial x_j}\right) = - \frac{1}{\rho} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} (S_{ij}) \qquad (2.57)$$

which is identical to Eq. (2.23.b). This equation shows that $\bar{\xi}$ is equivalent to the surface elevation of a long wave when a horizontal force $-\frac{\partial S_{ij}}{\partial x_i}$ is applied to the fluid i.e. gradients of the excess momentum fluxes tend to change the mean free surface level.

3. Generation of Forced Waves

The governing equation for the mean free surface displacement is Eq. (2.57) or Eq. (2.21). However to solve it, one needs to find the variations of the wavenumber, the wave direction and the wave amplitude with respect to the slow coordinates X_i, T. These quantities are governed by the dispersion relation (2.14), the irrotationality condition (2.11) and the equation of conservation of energy (2.17). Using the following normalization

$$X_i' = k_\infty X_i \qquad h' = k_\infty h \qquad k' = \frac{k}{k_\infty}$$

$$A' = \frac{A}{a_o} \qquad \bar{\xi}' = \frac{\bar{\xi}}{k_\infty a_o^2} \qquad C_g' = \frac{k_\infty}{\omega} C_g \qquad T' = \frac{T}{\omega} \tag{3.1}$$

where $k_\infty = \frac{\omega^2}{g}$ is the wavenumber in deep water and a_o is the wave amplitude at some reference depth h_o; these equations become

$$\frac{\partial}{\partial X_i}\left(h \frac{\partial \bar{\xi}}{\partial X_i}\right) - \frac{\partial^2 \bar{\xi}}{\partial T^2} = -\frac{\partial}{\partial X_i}\left(h \frac{\partial}{\partial X_i}\left(\frac{A^2}{4 \sinh^2 kh}\right)\right) + \frac{\partial}{\partial X_i}\left(\frac{k_i}{2} \frac{\partial A^2}{\partial T}\right) \tag{3.2}$$

$$\frac{\partial A^2}{\partial T} + \frac{\partial}{\partial X_i}\left(A^2 C_{gi}\right) = 0 \tag{3.3}$$

$$1 = k \tanh kh \tag{3.4}$$

$$\frac{\partial k_1}{\partial X_2} = \frac{\partial k_2}{\partial X_1} \tag{3.5}$$

In the most general case these two dimensional equations need to be solved numerically. To simplify the analysis we assume the depth to vary in one direction only i.e. $h = h(X_1)$. Then Eq. (3.4) and (3.5) imply:

$$k = k(X_1) \qquad k_1 = k_1(X_1) \qquad k_2 = \text{constant} \tag{3.6}$$

or by introducing $\alpha(X_1)$ the angle between the direction of propagation of the wave and the X_1 - axis.

$$k_1 = k \cos \alpha \qquad k_2 = k \sin \alpha = k_o \sin \alpha_o \tag{3.7}$$

where the subscript 'o' refers to the values at the reference depth h_o. In what follows we will use (X,Y) as representing (X_1,X_2). We further assume the wave envelope to be periodic in T with a constant frequency $\nu\Omega$.

$$A(X,Y,T) = \frac{1}{2} (\tilde{A}(X,Y)e^{-i\nu\Omega T} + *) \tag{3.8}$$

and to satisfy the boundary condition:

$$\tilde{A}(X,Y) = e^{i\nu k_2 Y} \qquad \text{at} \qquad X = \underset{\sim}{X}_o \tag{3.9}$$

where $X = \underset{\sim}{X}_o$ is a reference line in the region of constant depth $h = h_o$. Then Eq. (3.3) becomes

$$k_1 \frac{\partial \tilde{A}^2}{\partial X} + k_2 \frac{\partial \tilde{A}^2}{\partial Y} + \left(\frac{k}{C_g} \frac{\partial C_{g_1}}{\partial X} - \frac{2i\nu\Omega k}{C_g}\right) \tilde{A}^2 = 0$$

$$k_1 \frac{\partial |\tilde{A}|^2}{\partial X} + k_2 \frac{\partial |\tilde{A}|^2}{\partial Y} + \frac{k}{C_g} \frac{\partial C_{g_1}}{\partial X} |\tilde{A}|^2 = 0 \qquad (3.10)$$

where C_{g_1} is the group velocity in the X direction. The solution of Eqs. (3.10) which satisfies the boundary conditon (3.9) is:

$$\tilde{A} = \sqrt{\frac{C_{g_{10}}}{C_{g_1}}}\, e^{i\nu\left(\int_{X_0}^{X} K dX + k_2 Y\right)} \qquad (3.11)$$

where $K = \frac{\Omega}{C_{g_1}} - \frac{k_2^2}{k_1}$. The wave envelope is then a progressive wave whose direction of propagation is at the angle β with respect to the X-axis

$$\tan \beta = \frac{k_2}{K} \qquad (3.12)$$

and whose phase velocity is

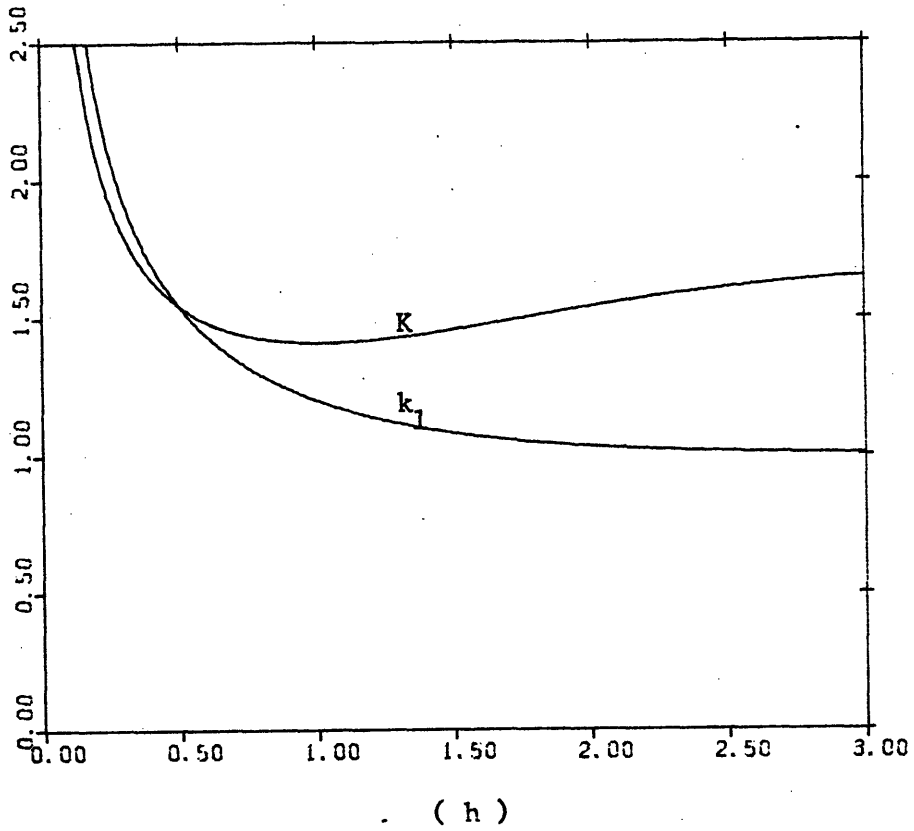
$$C = \frac{\Omega}{\sqrt{K^2 + k_2^2}} \qquad (3.13)$$

We choose $\Omega = C_{g_o} k_o$ such that at the reference depth h_o, $K = k_{10}$, $C = C_{g_o}$ i.e. the wave envelope is propagating at the group velocity in the same direction as that of the short waves. We have plotted in fig. (3.1) the variation of K and k_1 with depth for different values $\alpha_o = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. The reference depth is taken to be $h_o = 0.5$. As expected

(a)

Normal incidence

$\alpha_0 = 0$



(b)

Oblique Incidence

$\alpha_0 = \frac{\pi}{6}$

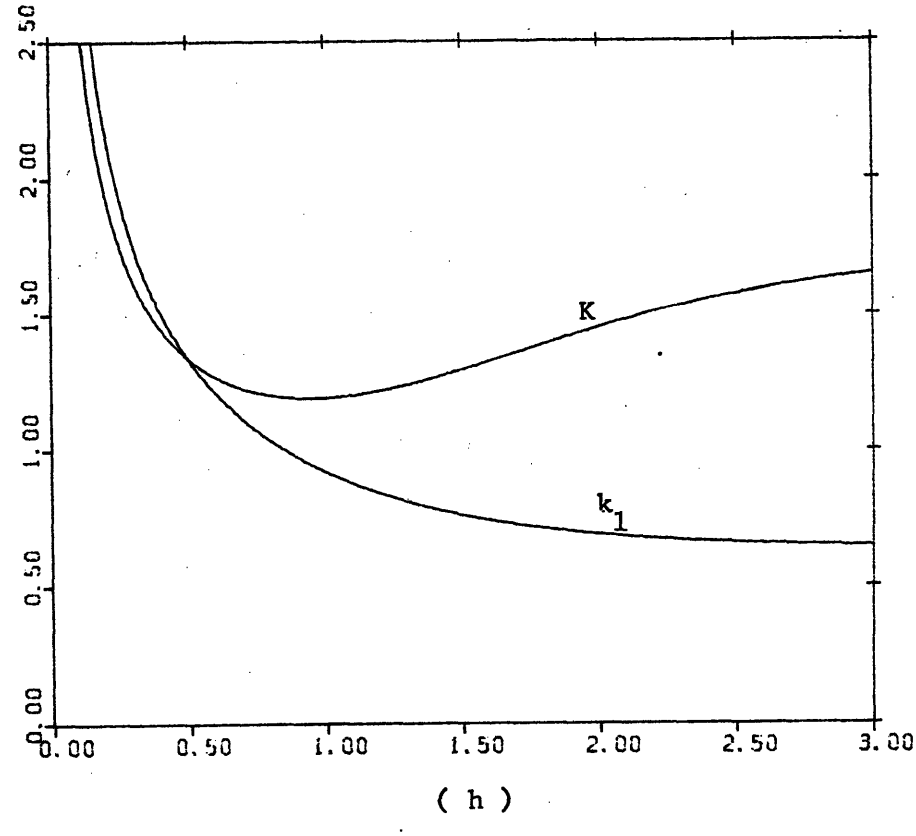
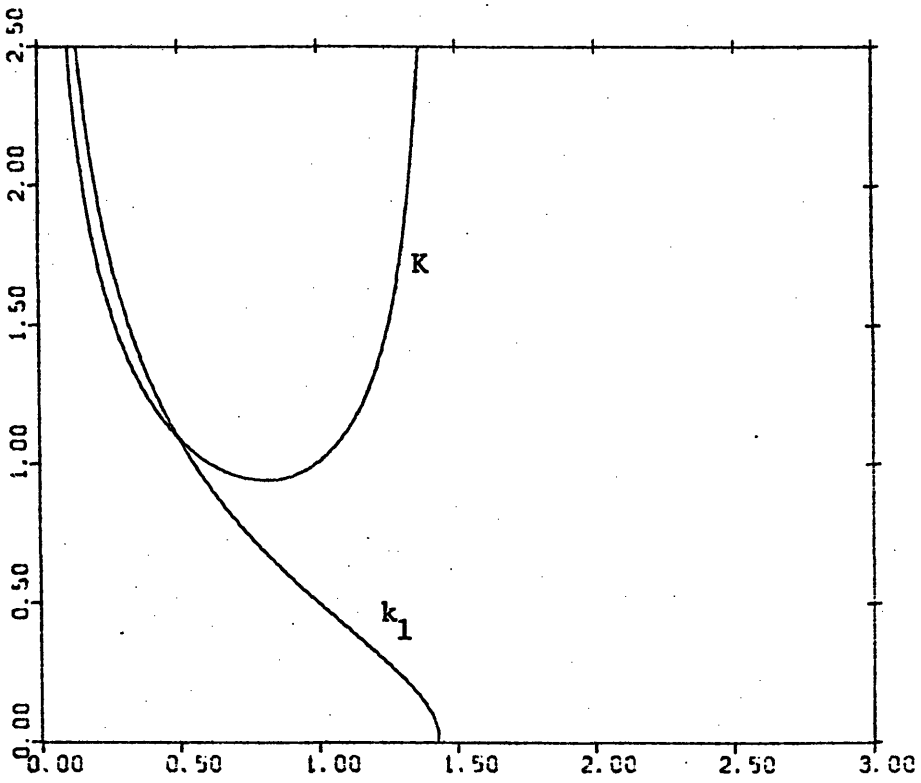


Figure 3.1 Wavenumbers in X-Direction, Reference Depth $h_0 = 0.5$

(c)

Oblique Incidence

$\alpha_0 = \frac{\pi}{4}$



(d)

Oblique Incidence

$\alpha_0 = \frac{\pi}{3}$

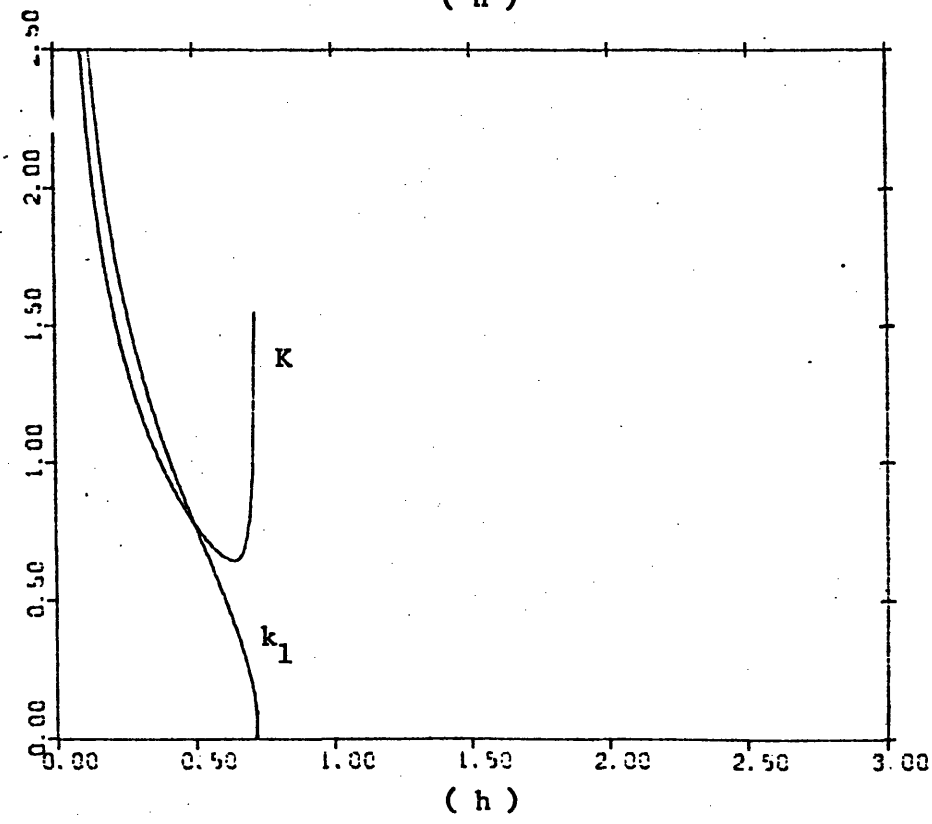


Figure 3.1 (Cont.)

at $h = h_o$ we have $K = k_1$, $\beta = \alpha$. In shallower water $K < k_1$ and therefore $\beta > \alpha$. As $h \to 0$, k_1 and $K \to \infty$ i.e. α and $\beta \to 0$. In deeper water $K > k_1$ and $\beta < \alpha$. Depending on the reference depth and the initial angle of incidence α_o, short waves propagating into deeper water can be reflected back by a Caustic (A line corresponding to the envelope of all the rays). The Caustic line is defined by :

$$k_2 = k \quad \text{or} \quad \frac{k_o \sin \alpha_o}{k} = 1 \tag{3.14}$$

In the neighborhood of the caustic $k_1 \to 0$ but $K \to \infty$: The short waves reach the caustic with an angle of incidence $\alpha \to \pi/2$ (i.e., glancing incidence) while their envelope reaches it with an angle $\beta \to 0$ (i.e. normal incidence). In the neighborhood of the caustic, Eq. (3.11) shows that $|A| \to \infty$, our small amplitude theory ceases to be valid and a more refined treatment is needed. In fig. (3.2), K and k_1 are shown for $h_o = 1$, before the caustic is approached. The variations of K and k_1 with depth are qualitatively similar to those presented in fig. (3.1).

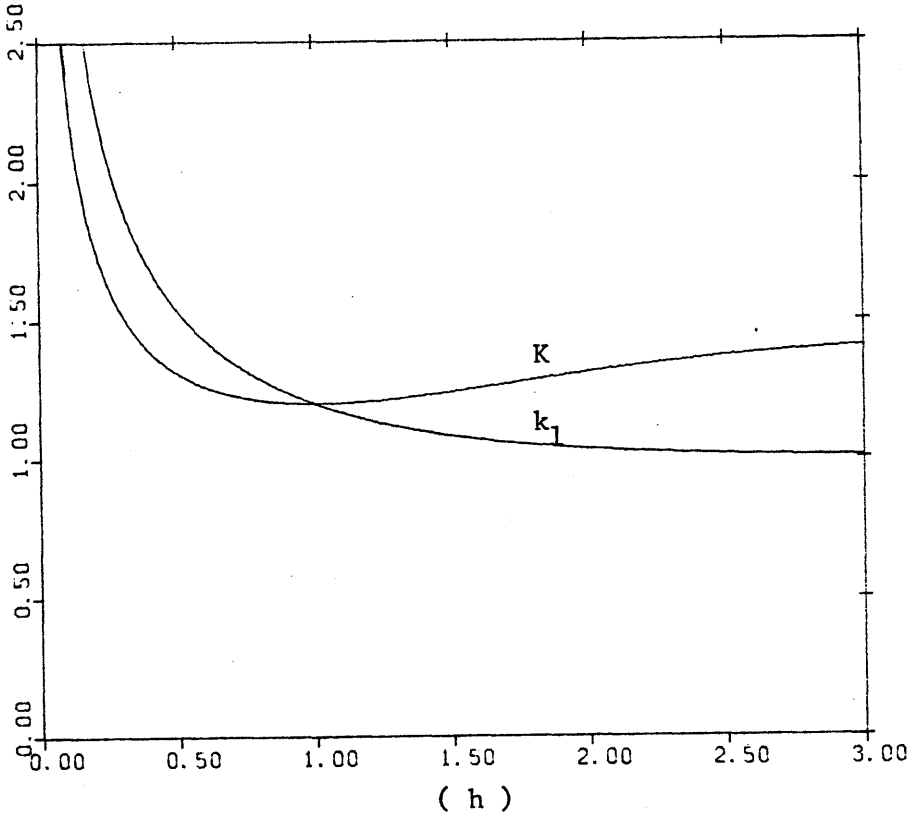
From Eq. (3.8) and (3.11), we note that A^2 consists of zeroth and second harmonic so the right hand side of Eq. (3.2) consists of the same harmonics, therefore we expect $\bar{\xi}$ to be of the form:

$$\bar{\xi} = \xi_o(X) + \frac{1}{2} \left(\tilde{\xi}(X) e^{2i\nu(k_2 Y - \Omega T)} + *\right) \tag{3.15}$$

(a)

Normal Incidence

$\alpha_0 = 0$



(b)

Oblique Incidence

$\alpha_0 = \frac{\pi}{6}$

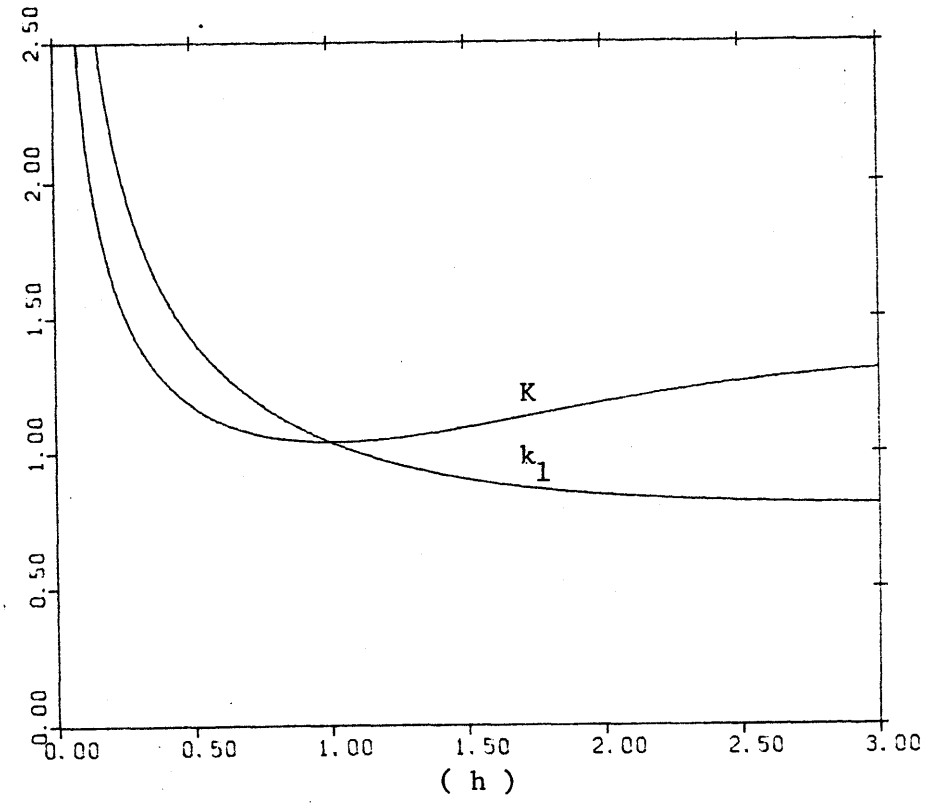
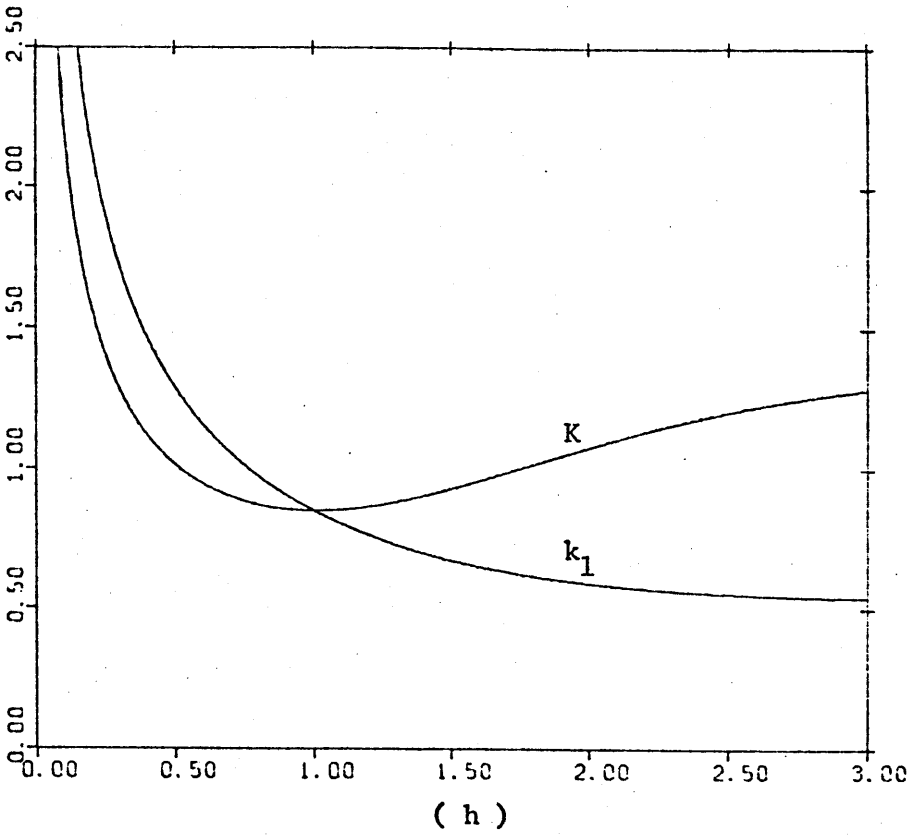


Figure 3.2 Wavenumbers in X-Direction, Reference Depth $h_0 = 1$

(c)

Oblique Incidence

$\alpha_0 = \frac{\pi}{4}$



(d)

Oblique Incidence

$\alpha_0 = \frac{\pi}{3}$

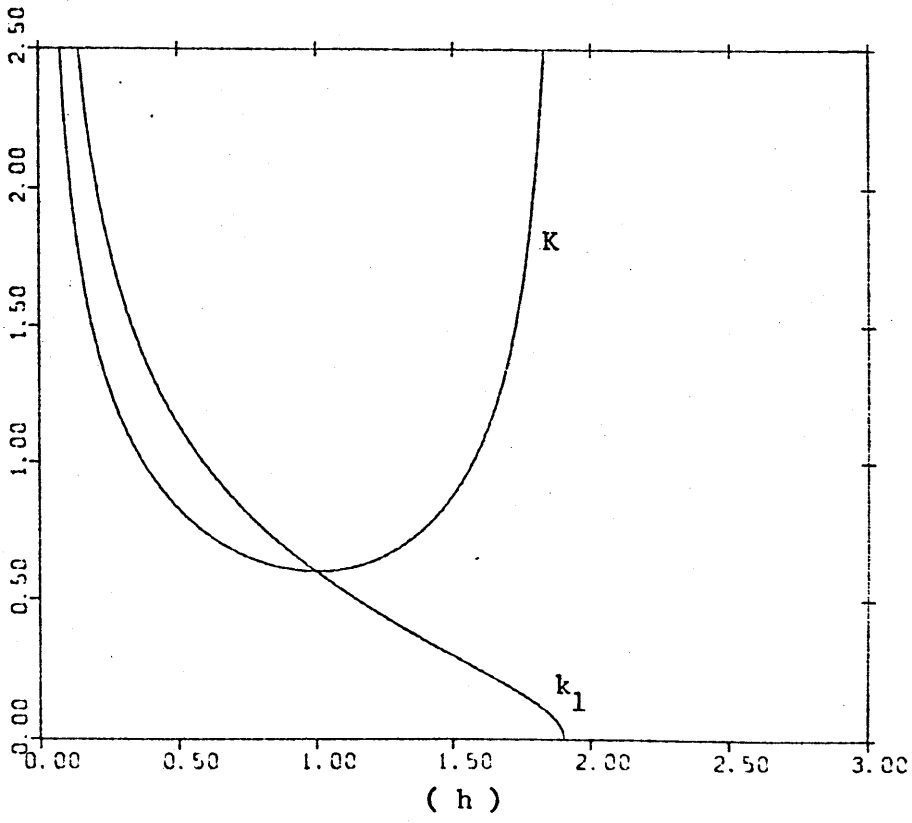


Figure 3.2 (Cont.)

Plugging (3.15) into (3.2) we obtain the expression for ξ_o and the governing equation for $\tilde{\xi}$:

$$\xi_o(X) = -\frac{C_{g_{10}}}{8C_{g_1}\sinh^2 kh} = -\frac{|\tilde{A}|^2}{8\sinh^2 kh} \tag{3.16}$$

$$\frac{d}{dX}\left(h\frac{d\tilde{\xi}}{dX}\right) + 4\nu^2(\Omega^2 - k_2^2 h)\tilde{\xi} = \left[\frac{kh\,(K^2+k_2^2)}{\sinh 2kh} + \frac{\Omega^2 k}{C_g}\right]\nu^2\frac{C_{g_{10}}}{C_{g_1}} e^{2i\nu\int_{\tilde{X}_o}^{X} K dX} +$$

$$\left[\frac{d}{dX}\left(h\frac{d\xi_o}{dX}\right) + i\nu\left(4Kh\frac{d\xi_o}{dX} + 2\xi_o\frac{d}{dX}(Kh) - \frac{\Omega C_{g_{10}}}{2}\frac{d}{dX}\left(\frac{k}{C_g}\right)\right]e^{2i\nu\int_{\tilde{X}_o}^{X} K\,dX} \tag{3.17}$$

Eq. (3.17) is then an ordinary differential equation in $\tilde{X}$.

For normal incidence $\alpha_o = 0$, a limiting case was studied by Molin (1982) by considering the first order waves to be deep water waves and the second order waves to be long waves. In this limit $kh = 0(\frac{1}{\sqrt{\varepsilon}})$, $C_g \to \frac{1}{2}$, $k \to 1$

$$\frac{d}{dX}\left(h\frac{d\tilde{\xi}}{dX}\right) + 4\nu^2\Omega^2\tilde{\xi} = 2\,\Omega^2\nu^2\, e^{\frac{2i\nu\Omega X}{C_g}} \tag{3.18}$$

Taking $\Omega\nu = 1$, Eq. (3.18) is equivalent to the governing equation obtained by Molin.

For $kh = 0(\,1\,)$, the steady state set-down, $\xi_o(X)$ is plotted in fig. (3.3) for different angles of incidence $\alpha_o = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. The

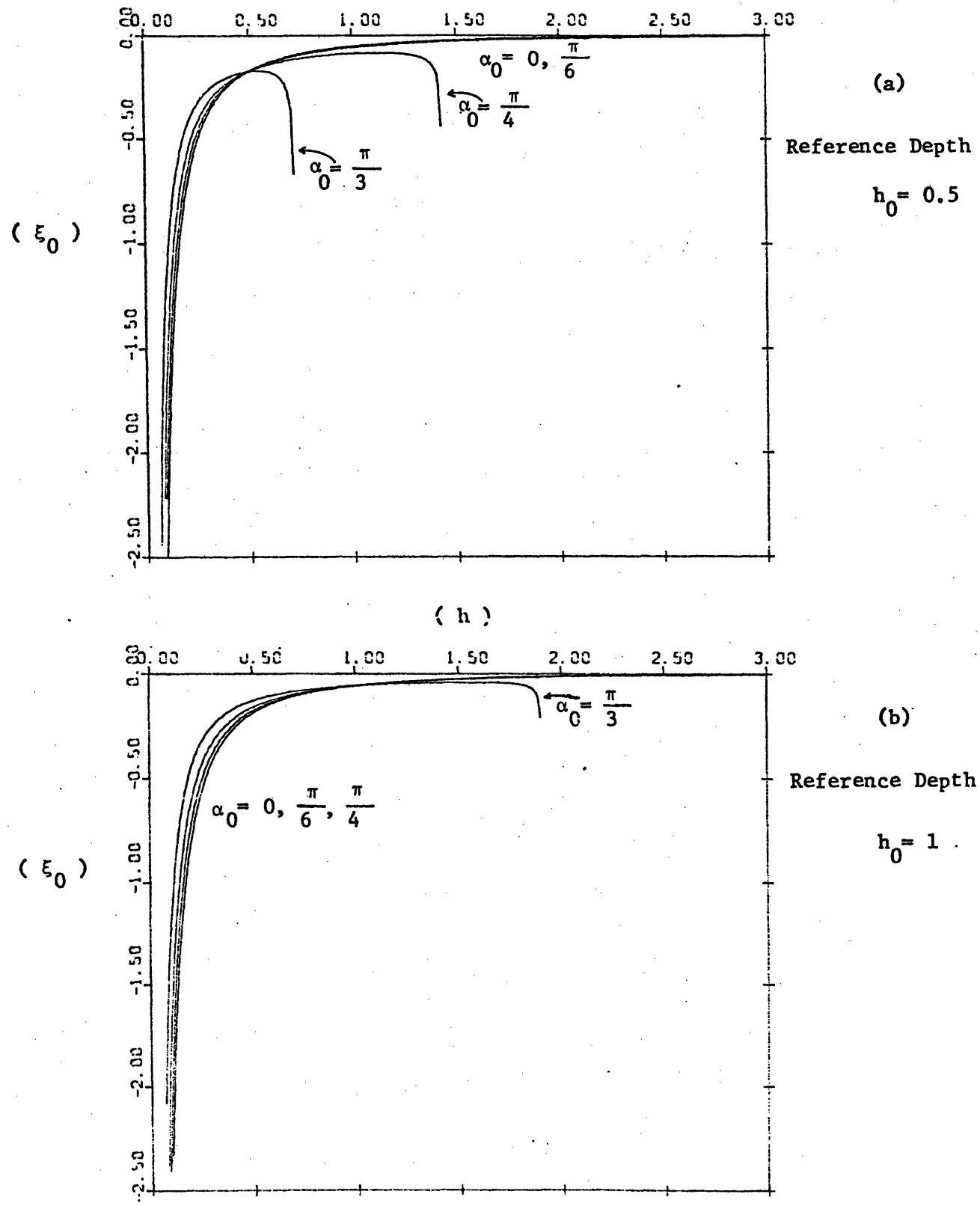


Figure 3.3 Mean Set-Down

reference depths in figs. (3.3.a) and (3.3.b) are respectively $h_o = 0.5$ and $h_o = 1$. As espected from Eq. (3.16), ξ_o is always negative and increases from $-\infty$ in very shallow water to zero in deep water. Near a caustic $k_1 \to 0$ and $\xi_o \to -\infty$. The effect of increasing angle of incidence is very small before the caustic is approached.

3.a Locked Waves on Constant Depth

In the case of constant depth, the second term on the right hand side of (3.17) is zero. The equation admits a solution of the form

$$\tilde{\xi} = \xi_L \, e^{2i\nu \int_{X_{-o}}^{X} K \, dX}$$

$$\text{with} \quad \xi_L = \frac{-C_{g_{10}} \left(\frac{2kh\,(K^2+k^2)}{\sinh 2kh} + \frac{2\Omega^2 k}{C_g} \right)}{8C_{g_1}\left(h(K^2+k_2^2) - \Omega^2\right)} \tag{3.19}$$

ξ_L represents the amplitude of a wave travelling in the same direction and with the same velocity as those of the envelope of the short waves. It is therefore locked to the wave group. Note that the directions of the short waves and their envelopes are different. In the case of normal incidence $\alpha_o = \beta = 0$ and at a depth $h = h_o$, ξ_L reduces to the expression of the mean free surface displacement given by Longuet-Higgins and Stewart (1962):

$$\xi_L = -\frac{(4k_o C_{g_o} - 1)}{8C_{g_o}(h_o - C_{g_o}^2)} \tag{3.20}$$

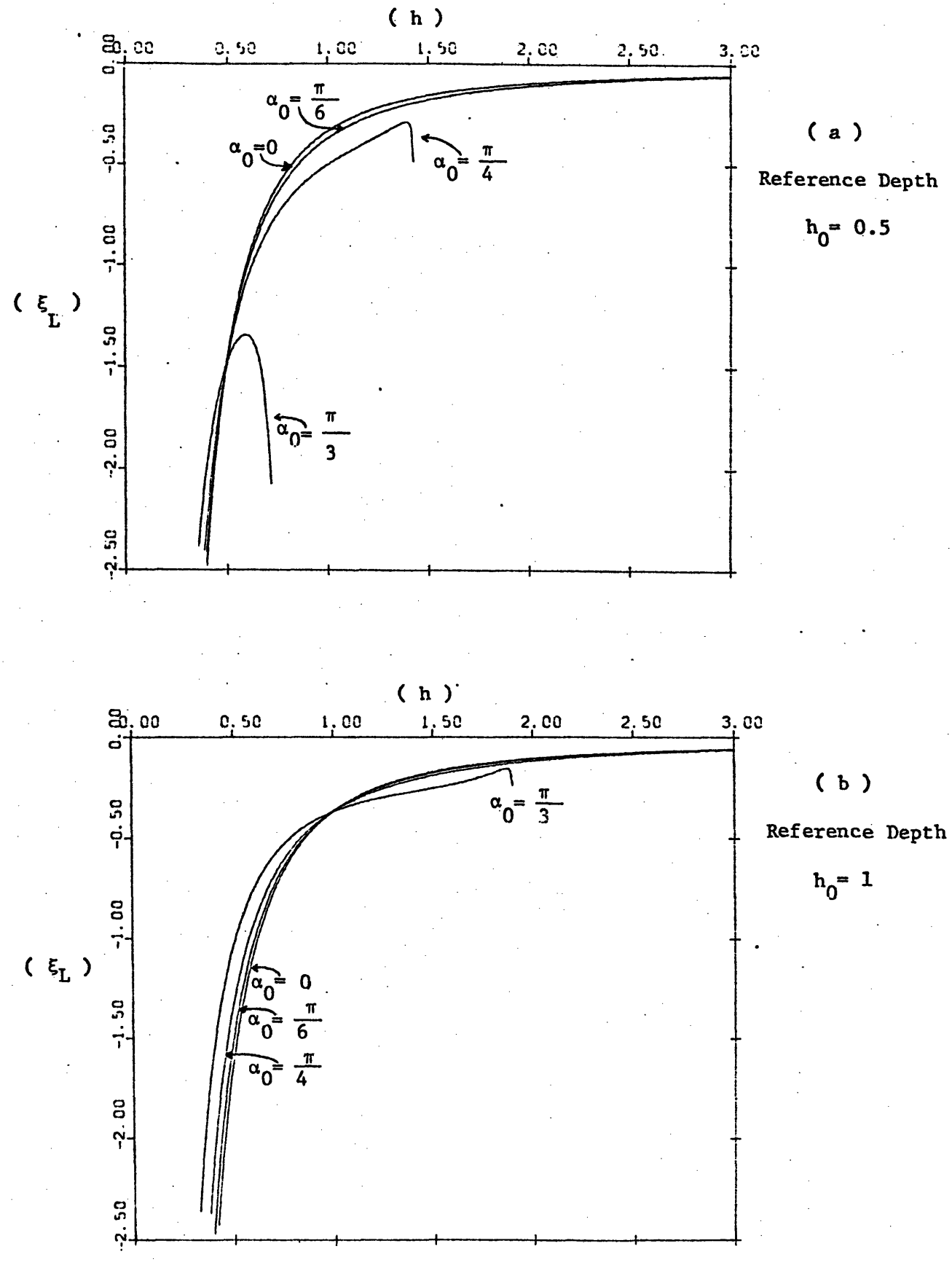


Figure 3.4 Set-Down Associated with the Locked Waves

ξ_L is plotted in Fig . (3.4) as a function of depth for different angle of incidence $\alpha_o = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. The reference depths in figs. (3.4.a) and (3.4.b) are respectively $h_o = 0.5$ and $h_o = 1$. ξ_L is always negative. It corresponds therefore to a set down of the mean free surface elevation. $|\xi_L|$ is larger than $|\xi_o|$ but their variation with depth are similar. The effect of increasing the angle of incidence remains very small except in the neighborhood of a caustic.

3.b. Forced Waves on Variable Depth

In the case of variable depth, the second term in the right hand side of Eq. (3.17) is non zero and contribute to the forcing of $\tilde{\xi}$. This equation has then to be solved numerically in the region where the depth varies. We assume the depth to vary from h_o in the region $X < \underset{\sim}{X}_o$ to h_1 in the region $X > \underset{\sim}{X}_1$. In the regions of constant depth, the solutions are of the form

$$\tilde{\xi} = \xi_{L0} e^{2i\nu \int_{\underset{\sim}{X}_o}^{X} K_o dX} + \xi_{F0} \qquad \text{for} \quad X < \underset{\sim}{X}_o \tag{3.21}$$

$$\tilde{\xi} = \xi_{L1} e^{2i\nu \int_{\underset{\sim}{X}_o}^{X} K_1 dX} + \xi_{F1} \qquad \text{for} \quad X > \underset{\sim}{X}_1 \tag{3.22}$$

where ξ_{L0} and ξ_{L1} are respectively the values of the locked wave amplitude at depths h_o and h_1. ξ_{F0} and ξ_{F1} are homogeneous solutions of Eq. (3.17), in regions of constant depth. These solutions depend on the sign of λ^2

$$\lambda^2 = \frac{1}{h}(\Omega^2 - k_2^2 h) = \frac{k_o^2}{h}\,(C_{g_o}^2 - \sin^2\alpha_o h) \qquad (3.23)$$

In the case where $\lambda_1^2 > 0$ at $h = h_1$, the solutions of (3.17) are of the form $e^{\pm 2i\nu|\lambda_1|X}$. By imposing the radiation condition we deduce that

$$\xi_{F1} = A_1\, e^{2i\nu|\lambda_1|X} \quad \text{for} \quad X > \underset{\sim}{X}_1 \qquad (3.24.a)$$

In the case where λ_1^2 is negative at $h = h_1$, the solution of (3.17) are of the form $e^{\pm 2\nu|\lambda_1|X}$. By imposing the condition that $\xi_{F1} \to 0$ as $X \to +\infty$ we deduce

$$\xi_{F1} = A_1\, e^{-2\nu|\lambda_1|X} \quad \text{for} \quad X > \underset{\sim}{X}_1 \qquad (3.24.b)$$

Similarly in the region $X < \underset{\sim}{X}_o$, depending on the sign of λ^2 we have

$$\xi_{F0} = A_o\, e^{2i\nu\lambda_o X} \quad \text{for} \quad X < \underset{\sim}{X}_o \qquad (3.25)$$

where $\lambda_o = -|\lambda_o|$ if $\lambda_o^2 > 0$ and $\lambda_o = -i|\lambda_o|$ if $\lambda_o^2 < 0$

A_o and A_1 are complex constants which can be found by matching together the solutions in regions of constant and variable depth. Outside the region of varying depth the free surface displacement is the sum of three terms: a mean set down ξ_o, a progressive locked wave

$$\xi_L\, e^{2i\nu\left(\int_{\underset{\sim}{X}_o}^{X} K\,dX + k_2 Y - \Omega T\right)}$$

and a progressive '*Forced Wave*'

$$A_j\, e^{2i\nu(\lambda_j X + k_2 Y - \Omega T)} \qquad j = 0 \text{ or } 1$$

If λ_j^2 is positive the forced waves are radiated towards large $|X|$ and propagate in a direction defined by

$$\tan \theta = \frac{k_2}{\lambda_j} \tag{3.26}$$

with the phase velocity

$$C_F = \frac{\Omega}{\sqrt{\lambda_j^2 + k_2^2}} = \frac{\Omega}{\sqrt{\frac{\Omega^2 + k_2^2(1-h_j)}{h_j}}} \tag{3.27}$$

The phase velocity and the direction of propagation of these radiated waves are constants which depend on the depths h_j. In general they are different from those of the locked waves. As imposed by the radiation condition θ_o is negative and θ_1 is positive.

If λ_j^2 negative the forced waves are propagating in Y direction with a phase velocity $\frac{\Omega}{k_2}$. Their amplitude is not constant as in the case $\lambda_j^2 > 0$ but decreases exponentially as we move away from the zone of variable depth. The forced waves are then trapped in this zone.

Depending on the values of the parameters h_o, α_o, h_1, any combination of the two previous cases can occur. We can have either radiated waves in both regions '0' and '1' or radiated waves in one region and trapped waves in the other or trapped waves in both of them. Table (3.1) illustrates these cases. For given values of h_o and α_o it gives the depth h^* at which λ^2 is zero. The value X at which h^* is reached, is called a turning point in the theory of differential equations. If h is smaller than h^*, λ^2 is positive but if h larger than h^*, λ^2 is negative.

h_o \ α_o	0^o	30^o	45^o	60^o
0.5	∞	1.20	0.60	0.40
1.0	∞	1.44	0.72	0.48
1.5	∞	1.34	0.77	0.45
2.0	∞	1.20	0.60	0.40
2.5	∞	1.10	0.55	0.38

Table 3.1 Critical Values of h^* at which $\lambda = 0$ (Turning Point)

	h	k_2	k_1	K	λ	α^o	β^o	θ^o
$h_o = 0.5$	0.50	0.77	1.33	1.33	1.05	30.0	30.0	143.8
	1.00	0.77	0.92	1.19	0.50	39.9	32.9	57.0
$h_o = 1.0$	0.50	0.60	1.42	1.17	1.04	22.9	27.1	30.0
	1.00	0.60	1.04	1.04	0.60	30.0	30.0	135.0

Table 3.2 , Direction of Propagation of Short Waves α, Wave Envelope β, and Forced Waves θ

For oblique incidence, the direction of propagation of the short waves α, that of the locked waves β and that of the radiated waves θ can be different. At some constant depth , the values of these angles are only dependent on h_o and α_o and not on the profile of the depth variation. They are given in table (3.2) for $h_o = 0.5$, $h_1 = 1.0$, and $h_o = 1.0$, $h_1 = 0.5$. In both cases $\alpha_o = 30^o$. As expected $\alpha_o = \beta_o$ but $\beta_1 < \alpha_1$ when $h_1 > h_o$ and $\beta_1 > \alpha_1$ when $h_1 < h_o$. The magnitude of θ increases with depth. When $h = h^*$, $|\theta| \to 90^o$: glancing incidence ($h^* = 1.20$ for $h_o = 0.5$ and $h^* = 1.44$ for $h_o = 1.0$, according to table (3.1)).

As mentioned above, in the region of varying depth the free surface has to be found by solving numerically Eq. (3.17) with the appropriate matching conditions. By analogy with the form of the solutions in the regions of constant depth (see Eqs. (3.21) and (3.22)) we decompose $\tilde{\xi}$ in two parts:

$$\tilde{\xi} = \xi_L(X)\, e^{2i\nu \int_{\tilde{X}_o}^{X} K dX} + \xi_F(X) \qquad (3.28)$$

where ξ_L is the locked wave amplitude as defined by Eq. (3.19) and $\xi_F(X)$ is the discrepancy of the free surface displacement from the locked wave due to the depth variation. ξ_F is a wave which is forced in varying depth and will be called the forced wave. This decomposition is mainly for numerical convenience. The governing equation for ξ_F is obtained from Eq. (3.17)

$$\frac{d}{dX}\left(h\frac{d\xi_F}{dX}\right) + 4\nu^2(\Omega^2 - k_2^2 h)\xi_F = F(X) \tag{3.29}$$

where $F(X) = \left[h\frac{d^2Z}{dX^2} + \frac{dh}{dX}\frac{dZ}{dX} + i\nu\left(4hK\frac{dZ}{dX} + 2Z\frac{d}{dX}(Kh)\right.\right.$

$$\left.\left. - \frac{(C_{g_o})^2 k_o \cos\alpha_o}{2}\frac{d}{dX}\left(\frac{k}{C_g}\right)\right] e^{2i\nu\int_{X_o}^{X} K dX}$$

$$Z(X) = \xi_o - \xi_L$$

The boundary conditions are:

$$\xi_F = \xi_{F0} = A_o e^{2i\nu\lambda_o X} \quad \text{and} \quad \frac{\partial\xi_F}{\partial X} = 2i\nu\lambda_o\xi_{F0} \quad \text{at} \quad X = X_o \tag{3.30}$$

$$\xi_F = \xi_{F1} = A_1 e^{2i\nu\lambda_1 X} \quad \text{and} \quad \frac{\partial\xi_F}{\partial X} = 2i\nu\lambda_1\xi_{F1} \quad \text{at} \quad X = X_1$$

where $\lambda_j = (-1)^{j+1}|\lambda_j|$ if $\lambda_j^2 > 0$, $\lambda_j = (-i)^{j+1}|\lambda_j|$ if $\lambda_j^2 < 0$

It should be noted that $|F|$ is zero in regions of constant depth and increases with increasing slope of the bottom profile. However this increase is very slow in deep water and very rapid in intermediate depth $h \simeq 0(0.5)$ (Note from figs. (3.3) and (3.4) that Z and $\frac{dZ}{dh}$ are very small in deep water but increases sharply in shallower water). Therefore $|\xi_F|$ is expected to be small (i.e. $|\xi_F| << 0(\varepsilon^2)$) if h_o and h_1 correspond both to deep water or if the slope of the bottom profile is small.

4. Numerical Procedure

4.a. Numerical Method

Eq. (3.29) with the boundary conditions (3.30) correspond to a boundary value problem in one dimension and can be solved easily by using a finite difference scheme. However in the most general case the governing equation is two-dimensional and the finite difference method becomes inappropriate. A different numerical method which can be easily extended to the 2-D case is the so called 'Hybrid element method' (see Mei, 1978). This method is an extension of the usual finite element method to the case where the domaine is infinite but an analytical form of the solution is known in some outer region. The solution is obtained by using finite elements in the inner domain and an analytical representation in the outer domain. By choosing an adequate variational form for the finite elements problem the matching conditions can be transformed into 'stationarity conditions' for the functional at the extreme nodal points. In our case the functional can be deduced from Mei's results (Eq. 4.4 p. 407)

$$J(\xi) = \int_{X_o}^{X_1} \left(-\frac{h}{2}(\xi')^2 + 2\lambda\tilde{\xi}^2 - F\xi\right)dX - \left[h\left(\frac{\xi_{F1}}{2} - \xi\right)\frac{\partial \xi_{F1}}{\partial X}\right]_{X_1}$$

$$+ \left[h\left(\frac{\xi_{F0}}{2} - \xi\right)\frac{\partial \xi_{F0}}{\partial X}\right]_{X_o} \qquad (4.1)$$

where $\tilde{\lambda} = \nu^2(\Omega^2 - k_2^2 h)$, ξ_{F0} and ξ_{F1} are given in Eqs. (3.30). The Variational form of (4.1) becomes after some manipulations:

$$\delta J = \int_{\underset{\sim}{X}_o}^{\underset{\sim}{X}_1} ((h\xi')' + 4\tilde{\lambda}\xi - F)\,\delta\xi dX + [\delta\xi h(\frac{\partial \xi_{F1}}{\partial X} - \frac{\partial \xi}{\partial X}) - \frac{\partial(\delta\xi_{F1})}{\partial X} h(\xi_{F1} - \xi)$$

$$- \frac{h}{2}\,\delta\xi_{F1}\frac{\partial \xi_{F1}}{\partial X} - \xi_{F1}\frac{\partial(\delta\xi_{F1})}{\partial X}]_{\underset{\sim}{X}_1} - [\delta\xi h\,(\frac{\partial \xi_{F0}}{\partial X} - \frac{\partial \xi}{\partial X}) - \frac{\partial(\delta\xi_{F0})}{\partial X}$$

$$h(\xi_{F0} - \xi) - \frac{h}{2}\,\delta\xi_{F0}\frac{\partial \xi_{F0}}{\partial X} - \xi_{F0}\frac{\partial \delta\xi_{F0}}{\partial X}]_{\underset{\sim}{X}_o} \qquad (4.2)$$

Eq. (4.2) shows that any solution of Eqs. (3.29) - (3.30) corresponds to a stationary point for the functional J i.e. $\delta J = 0$. We subdivise the interval $[\underset{\sim}{X}_o, \underset{\sim}{X}_1]$ into (N-1) elements of equal length $\delta x = \frac{(\underset{\sim}{X}_1 - \underset{\sim}{X}_o)}{N - 1}$. For each element (X_i, X_{i+1}) $i = 1 \ldots N-1$, we use a linear interpolating function:

$$\xi_i(X) = \frac{(X_{i+1} - X)}{\delta x}\bar{\xi}_i + (\frac{X - X_i}{\delta x})\bar{\xi}_{i+1} \qquad X_i < X < X_{i+1}$$

$$\xi_i = 0 \qquad X < X_i \text{ or } X > X_{i+1} \qquad (4.3)$$

where $\bar{\xi}_i$, $\bar{\xi}_{i+1}$ are the nodal unknowns at points X_i and X_{i+1}. The contribution of each element to the functional J is

$$E_i = \int_{X_i}^{X_{i+1}} (-\frac{h}{2}(\xi_i')^2 + 2\tilde{\lambda}\xi_i^2 - F\xi_i)dX \quad i = 2,\ldots, N-2 \tag{4.4}$$

which gives after integration:

$$E_i = \frac{a_i}{2}\bar{\xi}_i^2 + \frac{b_i}{2}\bar{\xi}_{i+1}^2 + c_i\bar{\xi}_i\bar{\xi}_{i+1} - d_i\bar{\xi}_i - e_i\bar{\xi}_{i+1} \tag{4.5}$$

The contributions of the end intervals, E_1 and E_{N-1} have an extra term coming from the boundary conditions. E_1, E_{N-1} are respectively the sum of Eq. (4.5) for i = 1 or i = N-1 and the terms $-i\nu h_o\lambda_o\bar{\xi}_1^2$ or $i\nu h_1\lambda_1\bar{\xi}_N^2$ where λ_j is defined in Eq. (3.30)

$$a_i = b_i = \frac{1}{(\delta x)^2}(-\int_{X_i}^{X_{i+1}} h\,dX + 4\int_{X_i}^{X_{i+1}} \tilde{\lambda}(X)(X_{i+1}-X)^2dX)$$

$$c_i = \frac{1}{(\delta x)^2}(\int_{X_i}^{X_{i+1}} h\,dX + 4\int_{X_i}^{X_{i+1}} \tilde{\lambda}(X_{i+1}-X)(X-X_i)dX$$

$$d_i = \frac{1}{\delta x}(\int_{X_i}^{X_{i+1}} F(X)(X_{i+1}-X)dX \tag{4.6}$$

$$e_i = \frac{1}{\delta x}(\int_{X_i}^{X_{i+1}} F(X)(X-X_i)dX)$$

The stationarity conditions at the nodal points (i.e. $\frac{\partial}{\partial\bar{\xi}_j}(\sum_{1=1}^{N-1} E_i) = 0$) then leads to a set of N equations for N unknowns:

$$\begin{cases} \gamma_1\bar{\xi}_2 + \beta_1\bar{\xi}_1 = W_1 \\ \gamma_i\bar{\xi}_{i+1} + \beta_i\bar{\xi}_i + \alpha_i\bar{\xi}_{i-1} = W_i \qquad i = 2,\ldots,N-1 \\ \beta_N\bar{\xi}_N + \alpha_N\bar{\xi}_N = W_N \end{cases} \tag{4.7}$$

where

$$\gamma_i = c_i = \frac{2}{3}(\delta x)\left(\frac{\tilde{\lambda}_i+\tilde{\lambda}_{i+1}}{2}\right) + \left(\frac{h_i + h_{i+1}}{2\delta x}\right) \qquad i = 1,\ldots,N-1$$

$$\beta_i = a_i + b_{i-1} = \frac{4}{3}(\delta x)\left(\frac{\tilde{\lambda}_{i+1} + 2\tilde{\lambda}_i + \tilde{\lambda}_{i-1}}{2}\right) - \left(\frac{h_{i+1} + 2h_i + h_{i-1}}{2\delta x}\right)$$

$$i = 2,\ldots,N-1$$

$$\alpha_i = c_{i-1} = \frac{2}{3}\,\delta x\left(\frac{\tilde{\lambda}_i + \tilde{\lambda}_{i-1}}{2}\right) + \frac{h_i + h_{i-1}}{2\delta x} \qquad i = 2,\ldots,N$$

$$W_i = d_i + e_{i-1} = F_i \delta x \qquad i = 2,\ldots,N-1$$

The extreme values for β and W are:

$$\beta_1 = \frac{4}{3}(\delta x)\left(\frac{\tilde{\lambda}_1+\tilde{\lambda}_2}{2}\right) - \left(\frac{h_1+h_2}{2\delta x}\right) - 2i\nu h_o \lambda_o$$

$$\beta_N = \frac{4}{3}(\delta x)\left(\frac{\tilde{\lambda}_{N-1}+\tilde{\lambda}_N}{2}\right) - \left(\frac{h_{N-1}+h_N}{2\delta x}\right) + 2i\nu h_1 \lambda_1 \qquad (4.9)$$

$$W_1 = \frac{F_1}{2}\,\delta x$$

$$W_N = \frac{F_N}{2}\,\delta x$$

The linear system (4.7) is solved as in section (I.5.a). Introducing the intermediate variables x_i and y_i such that:

$$\bar{\xi}_{i+1} = x_i \bar{\xi}_i + y_i \qquad i = 1,\ldots,N-1 \qquad (4.10)$$

They are found to obey the following recurrence relations:

$$x_i = \frac{-\alpha_{i+1}}{\gamma_{i+1}x_{i+1} + \beta_{i+1}} \qquad y_i = \frac{-\gamma_{i+1}y_{i+1} + W_i}{\gamma_{i+1}x_{i+1} + \beta_{i+1}} \qquad i = 1,\ldots,N-2$$

$$x_{N-1} = \frac{-\alpha_N}{\beta_N} \qquad y_{N-1} = \frac{W_N}{B_N} \qquad (4.11)$$

Therefore having the initial value

$$\bar{\xi}_1 = \frac{W_1 - \gamma_1 y_1}{\gamma_1 x_1 + \beta_1} \tag{4.12}$$

all the other unknowns can be calculated by using Eq. (4.10). Finally, from the values of $\bar{\xi}_i$, by using the linear relation (4.3) one can deduce the forced wave solution $\xi(X)$ for $\underset{\sim}{X}_o < X < \underset{\sim}{X}_1$. The listing of the program is given in Appendix (B2). It should be noted that the computation is done in double precision so that the forcing term which involves second derivatives of Z can be found with sufficient accuracy. The parameter ν has been taken to be $1/\Omega$ in all the computed cases. This means that in dimensional variables, ε also represents the ratio between the short wave frequency and the envelope frequency i.e. $\varepsilon = \frac{\Omega}{\omega}$

4.b. Check of the Numerical Results

The numerical solution was checked with a 'quasi analytical solution', in the case of normal incidence ($\alpha_o = 0$) and linear increasing depth. It was found to give the same results within 0.1 % (see Appendix (B2)).

The 'analytical solution' is found by solving Eq. (3.29) with the boundary conditions (3.30) in the case of a linear depth variation. Let us introduce:

$$h(X) = \gamma X \qquad \underset{\sim}{X}_o < X < \underset{\sim}{X} \tag{4.13}$$

Two cases have two be considered depending on whether the depth increases ($\gamma > 0$) or decreases ($\gamma < 0$). We first consider the case $\gamma > 0$. By using the transformation $\nu = 4\sqrt{\frac{X}{\gamma}}$, Eq. (3.29) becomes:

$$\frac{d^2\xi}{d\nu^2} + \frac{1}{\nu}\frac{d\xi}{d\nu} + \xi = \frac{F}{4} \tag{4.14}$$

whose solutions can be found in terms of Bessel's functions Y_o and J_o. After some manipulations, $\xi(X)$ can be written as follows:

$$\xi(X) = \frac{\pi}{\gamma}\left[(-E_1(X) + A)J_o(4\sqrt{\frac{X}{\gamma}}) + (E_2(X) + B)Y_o(4\sqrt{\frac{X}{\gamma}})\right] \tag{4.15}$$

where $E_1(X) = \int_{\underset{\sim}{X}_o}^{X} F(X')Y_o(4\sqrt{\frac{X'}{\gamma}})dX'$ $\quad E_2(X) = \int_{\underset{\sim}{X}_o}^{X} F(X')J_o(4\sqrt{\frac{X'}{\gamma}})dX'$

A,B are two constants which are found by using the boundary conditions (4.4).

$$A = \frac{c_2(c_3E_1(\underset{\sim}{X}_1) - c_4E_2(\underset{\sim}{X}_1))}{c_2c_3 - c_1c_4} \qquad B = -\frac{c_1}{c_2}A \tag{4.16}$$

with $c_1 = -J_1(4\sqrt{\frac{\underset{\sim}{X}_o}{\gamma}}) + i\,J_o(4\sqrt{\frac{\underset{\sim}{X}_o}{\gamma}})$

$c_2 = -Y_1(4\sqrt{\frac{\underset{\sim}{X}_o}{\gamma}}) + i\,Y_o(4\sqrt{\frac{\underset{\sim}{X}_o}{\gamma}})$

$c_3 = J_1(4\sqrt{\frac{\underset{\sim}{X}_1}{\gamma}}) + i\,J_o(4\sqrt{\frac{\underset{\sim}{X}_1}{\gamma}})$

$c_4 = Y_1(4\sqrt{\frac{\underset{\sim}{X}_1}{\gamma}}) + i\,Y_o(4\sqrt{\frac{\underset{\sim}{X}_1}{\gamma}})$

Actually this solution is not totally analytical in the sense that the forcing term F is rather complicated and has to be computed numerically, and similarly for $E_1(\chi)$ and $E_2(X)$. The listing of the program is given in Appendix (B2).

In the case where $\gamma < 0$, the solution of Eq. (3.29) can be similarly expressed in terms of confluent hypergeometric functions

$$\rho_1(X) = e^{-\nu} M(1/2, 1, 2\nu) \text{ and } \rho_2(X) = e^{-\nu} U(1/2, 1, 2\nu)$$

where $\nu = 4 \frac{X}{|\gamma|}$. Since the computation of these functions is in itself a long task, no attempt was made to use this analytical solution to check the numerical results.

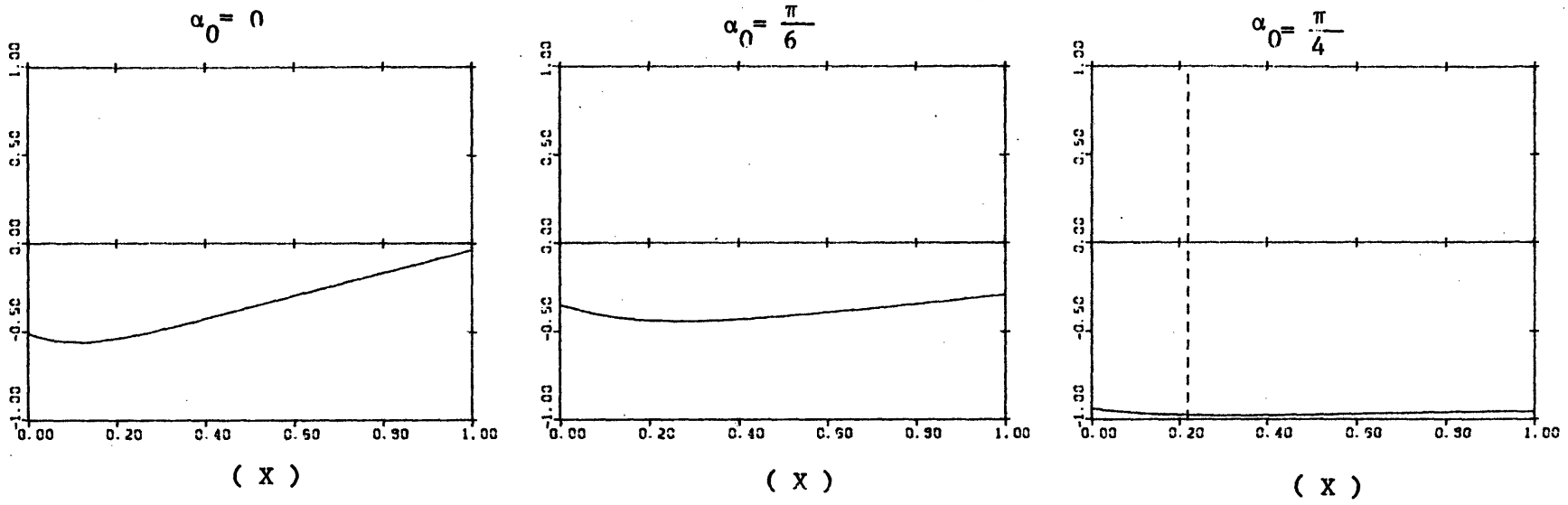
5. Numerical Results

The second order free surface displacement depends on the bottom profile. In the region of varying depth it can be decomposed into locked and forced waves. The magnitude of the forcing function depends on the water depth, the slope of the bottom profile and the initial angle of incidence. It vanishes in the region of constant depth. Depending on the wave parameters and the water depth, the forced waves can either be radiated towards large $|X|$ in the form of free waves with constant amplitude (case $\lambda^2 > 0$) or be trapped along the zone of depth variation with an amplitude decreasing exponentially with $|X|$ (case $\lambda^2 < 0$). For normally incident waves, the second order forced waves are always radiated in the X direction towards both infinities. In this section the forced wave amplitude $|\xi_F|$ and the corresponding phase are plotted for different bottom profiles. Note that $\frac{d}{dX}(\text{Arg}(\xi_F))$ gives the wave number of ξ_F in the X direction. In all the computed cases, the parameters are choosen such that there is no caustic.

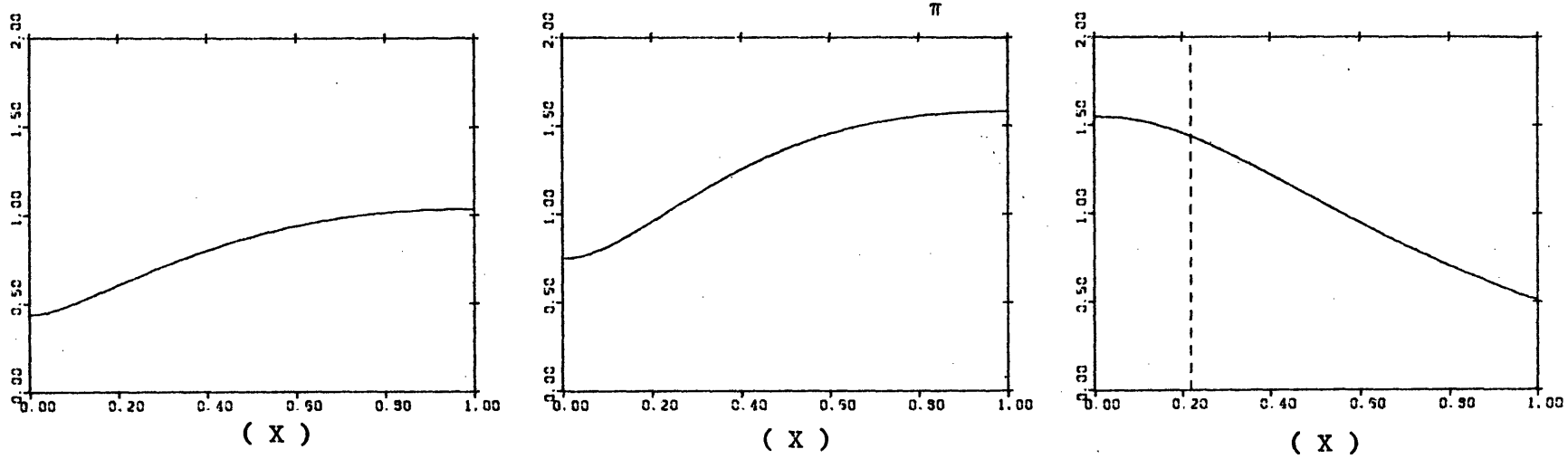
5.a. Linear Depth Variation

Fig. (5.1) presents the case where the depth increases from $h_o = 0.5$ to $h_o = 1$ over a distance XL = 1. The slope is then $\gamma = \frac{dh}{XL} = 0.5$. The forced waves are plotted for different angles of incidence $\alpha_o = 0$ (normal incidence), $\alpha_o = \pi/6$, $\alpha_o = \pi/4$. For $\alpha_o = 0$ and $\pi/6$ their amplitude is larger on the right hand side than on the left hand side. As expected their wavenumber in the X direction is positive on the right and negative

Figure 5.1 Forced Waves over Linearly Increasing Depth,
h_0=0.5, dh=0.5, XL=1



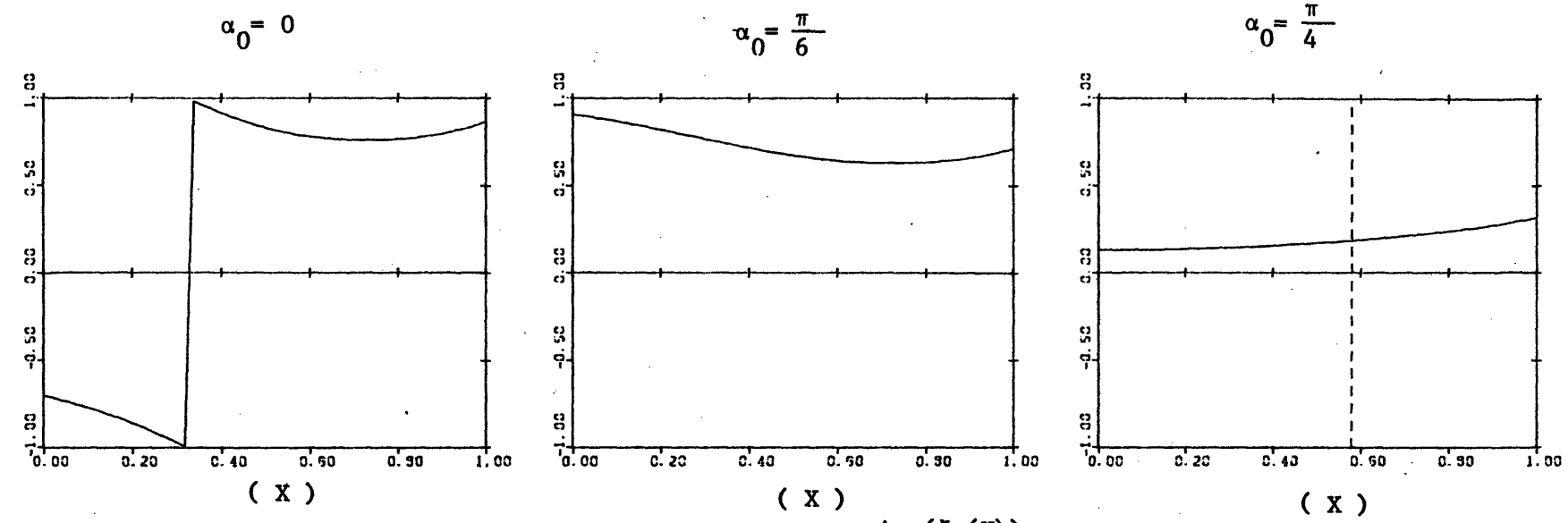
(a) Forced Wave Phase $\frac{\text{Arg}(\xi_F(X))}{\pi}$



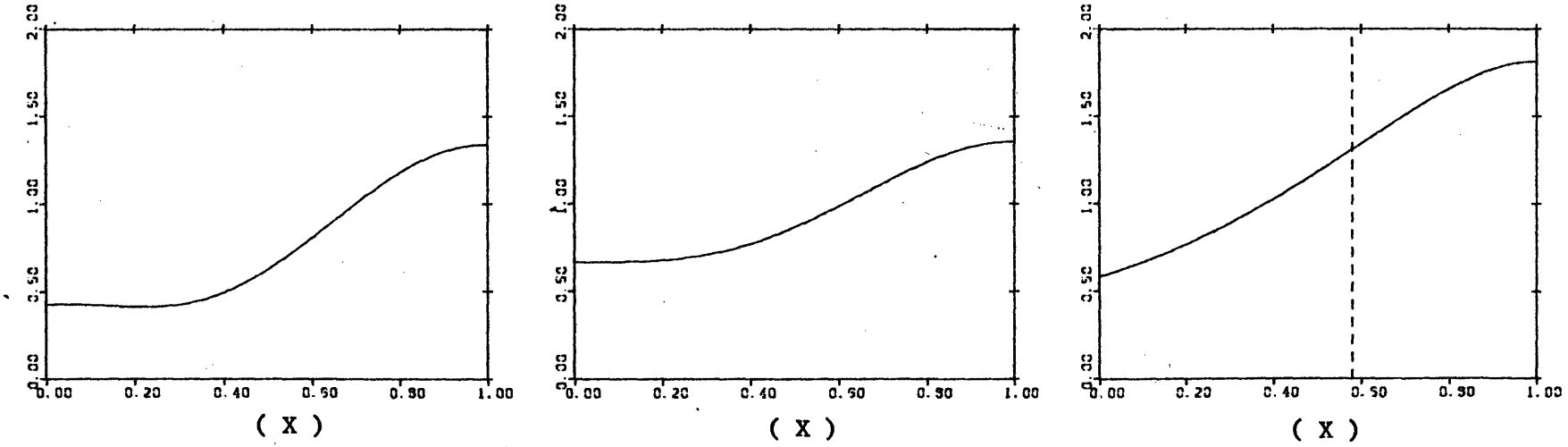
(b) Forced Wave Amplitude $|\xi_F(X)|$

Figure 5.2 Forced Waves over Linearly Decreasing Depth,

$h_0 = 1$, dh=-0.5, YL=1



(a) Forced Wave Phase $\frac{\mathrm{Arg}(\xi_F(X))}{\pi}$



(b) Forced Wave Amplitude $|\xi_F(X)|$

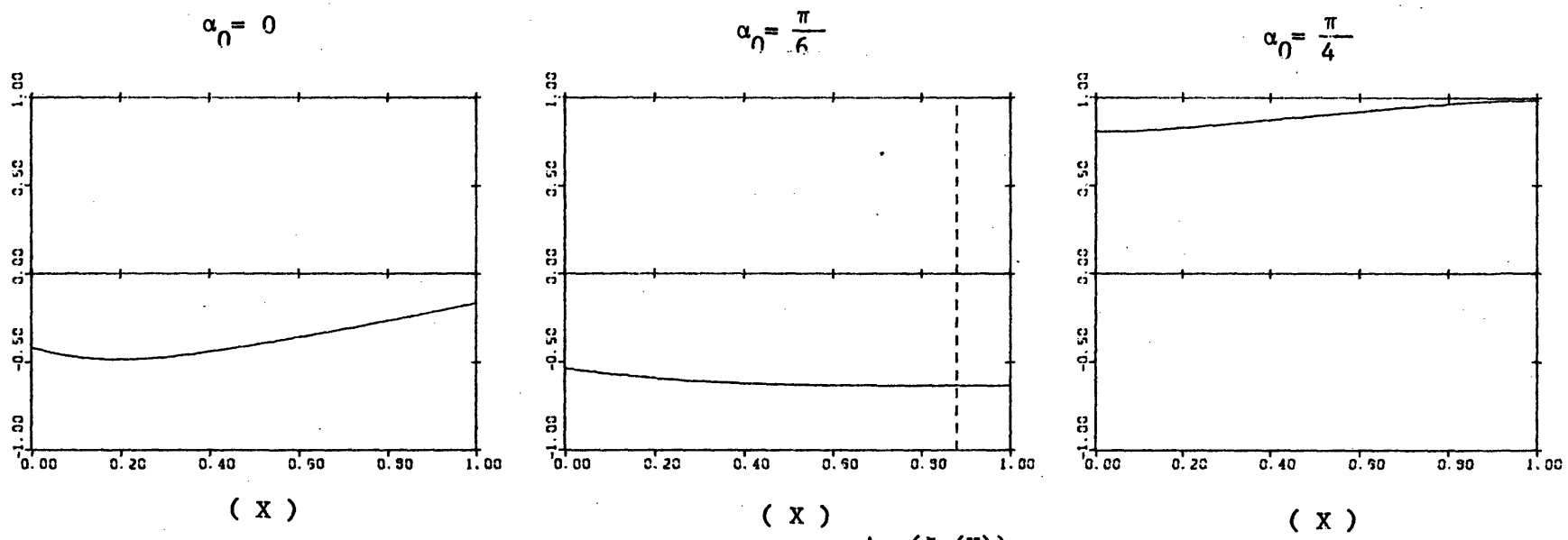
on the left, i.e. the forced waves are outgoing. For $\alpha_o = \pi/4$, the forced waves on the right hand side are trapped. The turning point is at $h^* = 0.6$, according to table (3.1), and is marked on the plot by the dash-line. The amplitude of the trapped waves is smaller than that of the outgoing waves, and decreases exponentially as X increases. Similarly fig. (5.2) illustrates the case where the depth decreases from $h_o = 0.5$ to $h_1 = 1.0$ over a distance XL= 1. For $\alpha_o = \pi/4$, the left going forced waves are trapped but the right going forced waves propagate towards increasing X with large amplitude.

From Figs. (3.3) and (3.4) we know that $|Z|$ and $|\frac{dZ}{dX}|$ become smaller as the depth increases. The forcing is then smaller and we expect the magnitude of $|\xi_F|$ to decrease with increasing depth. Fig. (5.3) shows the forced waves in the case where the depth increases from 1 to 1.5 with a slope $\gamma = 0.5$. By comparing Fig. (5.1) and Fig. (5.3) we note that the amplitude of the forced waves is much larger in the first case than in the second. For a more complete illustration of the effect of increasing depth, we have plotted for the case of normal incidence the right going and left going radiated wave amplitudes as a function of the initial depth h_o (see Fig. 5.4). h_o varies from 0.5 to 3, h_1 is taken equal to $(h_o + 0.5)$ and XL = 1 such that $dh = 0.5$ and $\gamma = 0.5$ always. The figure shows the rapid decrease in the amplitude of the radiated waves.

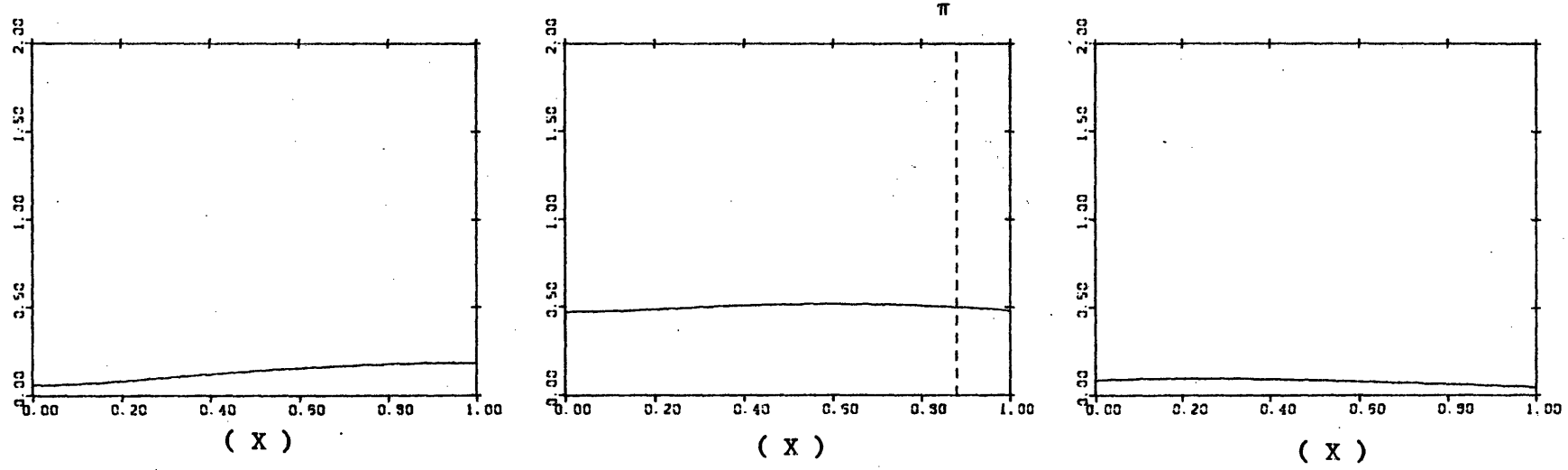
The forced waves are also determined by the slope of the bottom profile. In fig. (5.5) we consider the depth to vary from $h_o = 0.5$ to $h_1 = 1$ but

Figure 5.3 Forced Waves over Linearly Increasing Depth,

$h_0 = 1$, $dh = 0.5$, $XL = 1$



(a) Forced Wave Phase $\frac{\mathrm{Arg}(\xi_F(X))}{\pi}$



(b) Forced Wave Amplitude $|\xi_F(X)|$

Figure 5.4 Forced Wave Amplitudes on the Left and Right Hand Sides of a Linearly Increasing Depth region

Normal Incidence, dh= 0.5, XL= 1

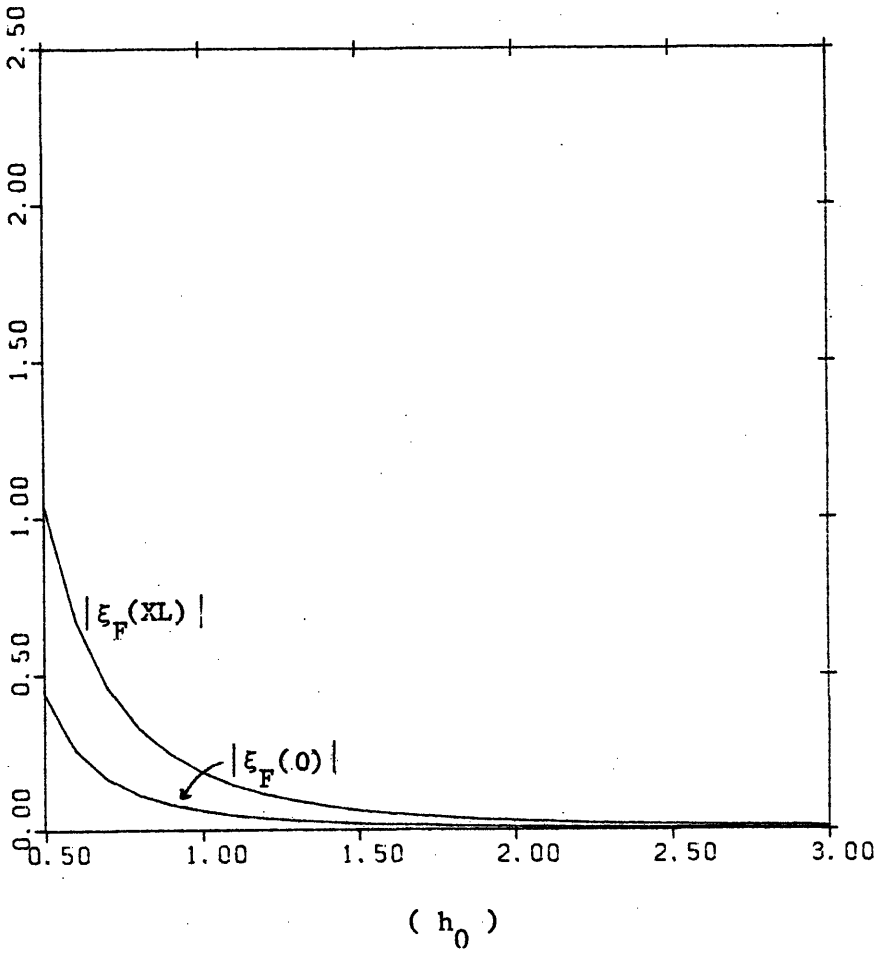
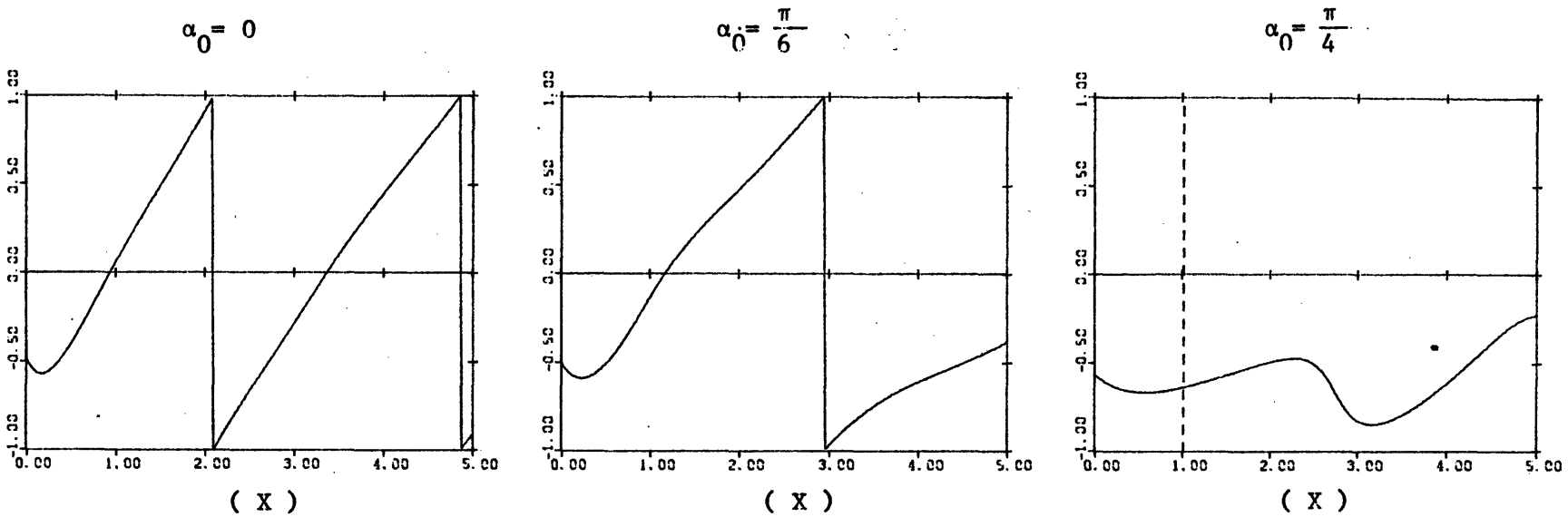
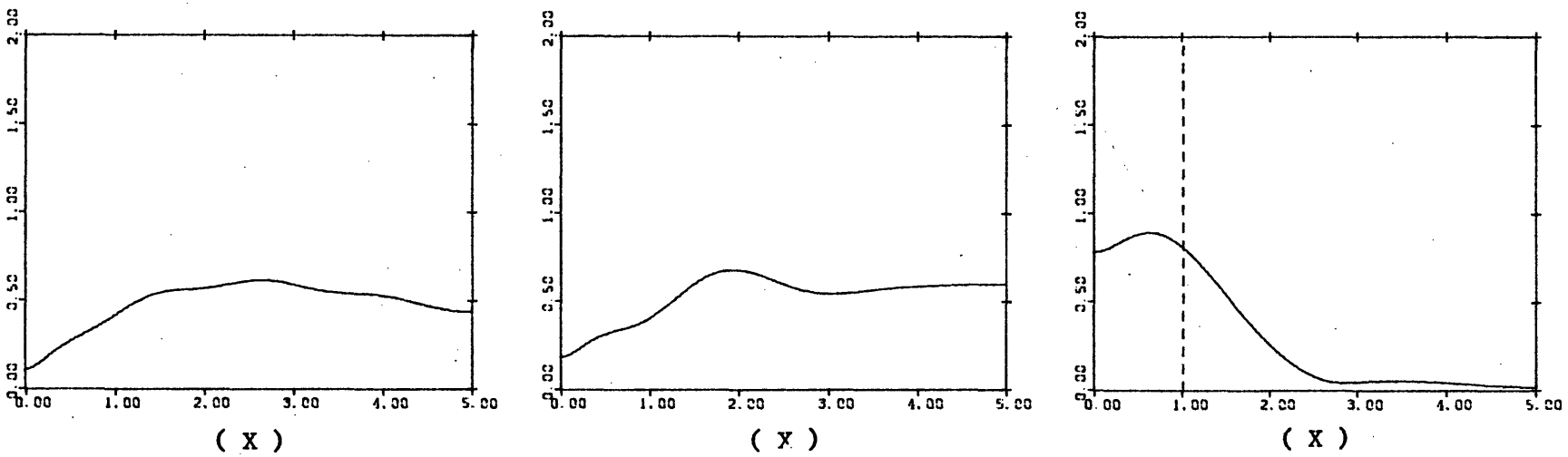


Figure 5.5 Forced Waves over Linearly Increasing Depth,

$h_0 = 0.5, dh = 0.5, XL = 5$



(a) Forced Wave Phase $\frac{\mathrm{Arg}(\xi_F(X))}{\pi}$

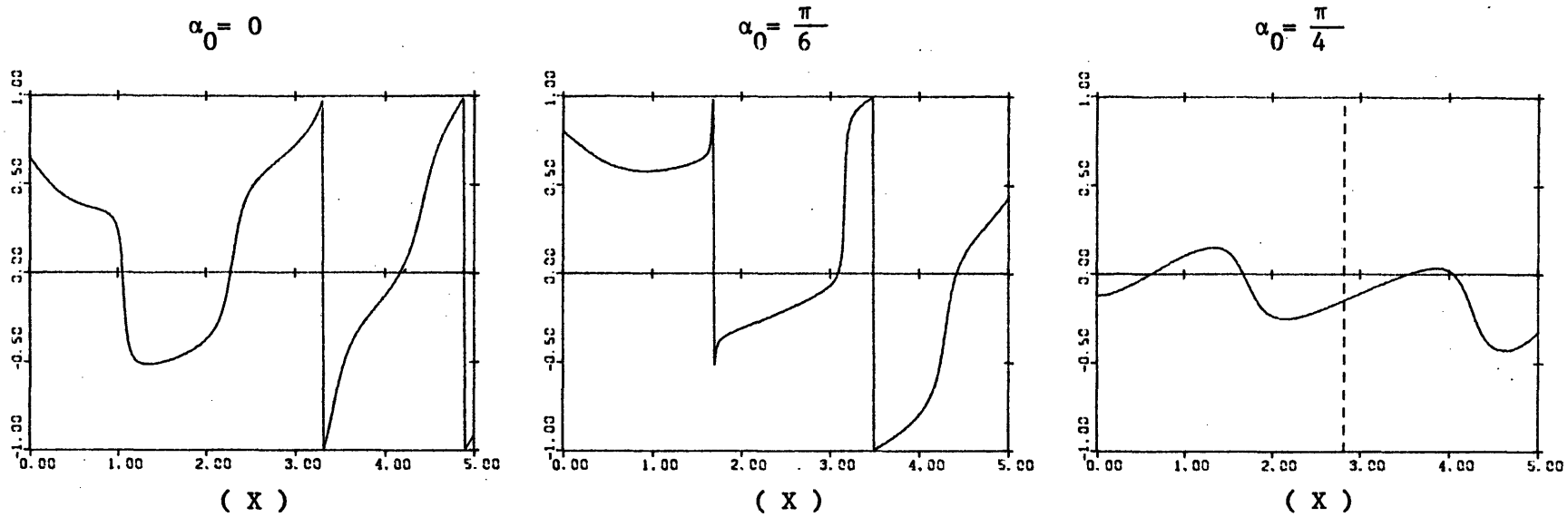


(b) Forced Wave Amplitude $|\xi_F(X)|$

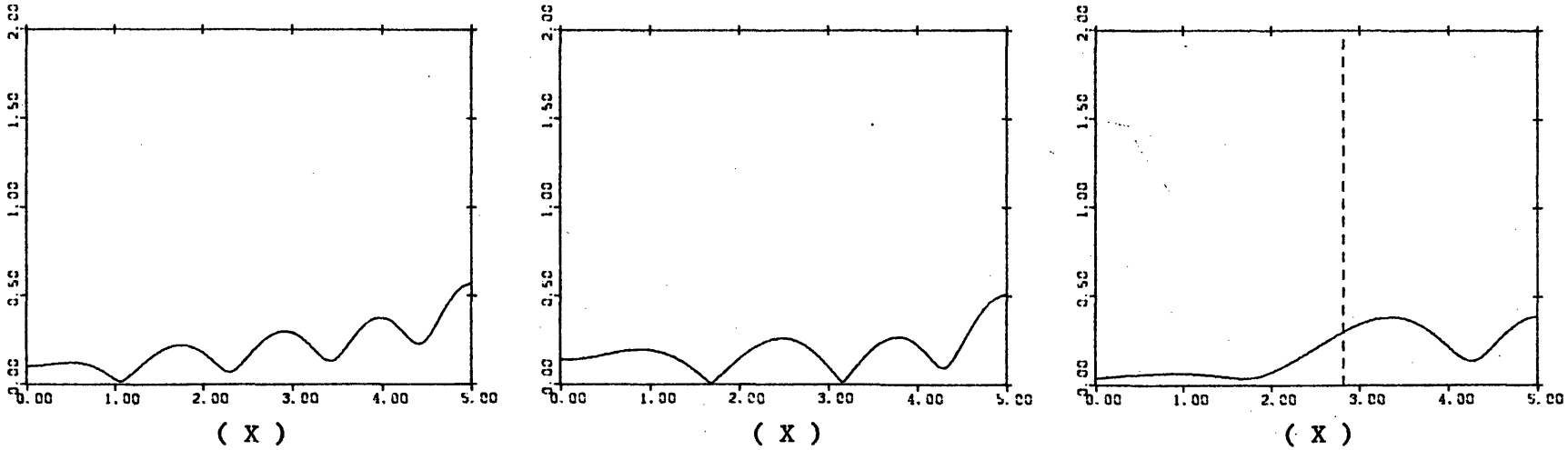
the length of the transition zone is taken to be XL = 5, and therefore $\gamma = 0.1$. Comparing Fig. (5.1) with Fig. (5.5) we note that the forced wave amplitude is smaller in the latter. Similarly in the case where the depth varies from $h_o = 1$ to $h_1 = 0.5$ over a distance XL = 5 , Fig. (5.6) shows smaller wave amplitude than those in Fig. (5.2). However in this case there is in addition oscillations of the forced wave amplitude with a period increasing with the angle of incidence. The existence of these oscillations depends on the length of the transition zone. At each node of $|\xi_F|$, there is a phase jump of π. In the complex plane it means that ξ_F crosses zero and that its derivative is continuous. For a more general picture about the effect of decreasing slope we display in Figs. (5.7) and (5.8) the variations with XL of the right hand side $|\xi_F|_R$ and left hand side $|\xi_F|_R$ forced wave amplitudes. Fig. (5.7) corresponds to an increasing depth $h_o = 0.5$, $h_1 = 1$ while Fig. (5.8) corresponds to a decreasing depth $h_o = 1$, $h_1 = 0.5$. In both figures normal incidence $\alpha_o = 0$, and oblique incidence $\alpha_o = \pi/4$, are considered. We note that in the case of normal incidence the left going wave amplitude is always smaller than the right going wave amplitude. In the case of oblique incidence $\alpha_o = \pi/4$, there is a turning point at $h^* = 0.6$, the forced wave amplitude $|\xi_F|$ is smaller in the regions where $h > h^*$ i.e. regions where forced waves are trapped. The effect of decreasing the slope is to decrease the amplitude of the forced waves on both sides. For normal incidence the decrease in the amplitude of the left going wave is accompanied with small oscillations.

Figure 5.6 Forced Waves over Linearly Decreasing Depth,

$h_0 = 1$, dh= -0.5, XL= 5



(a) Forced Wave Phase $\frac{\mathrm{Arg}(\xi_F(X))}{\pi}$



(b) Forced Wave Amplitude $|\xi_F(X)|$

Figure 5.7 Forced Wave Amplitudes on the Left and Right Hand Sides of the Linearly Decreasing Depth Region

$h_0 = 1$, $dh = -0.5$

(a) Normal Incidence $\alpha_0 = 0$

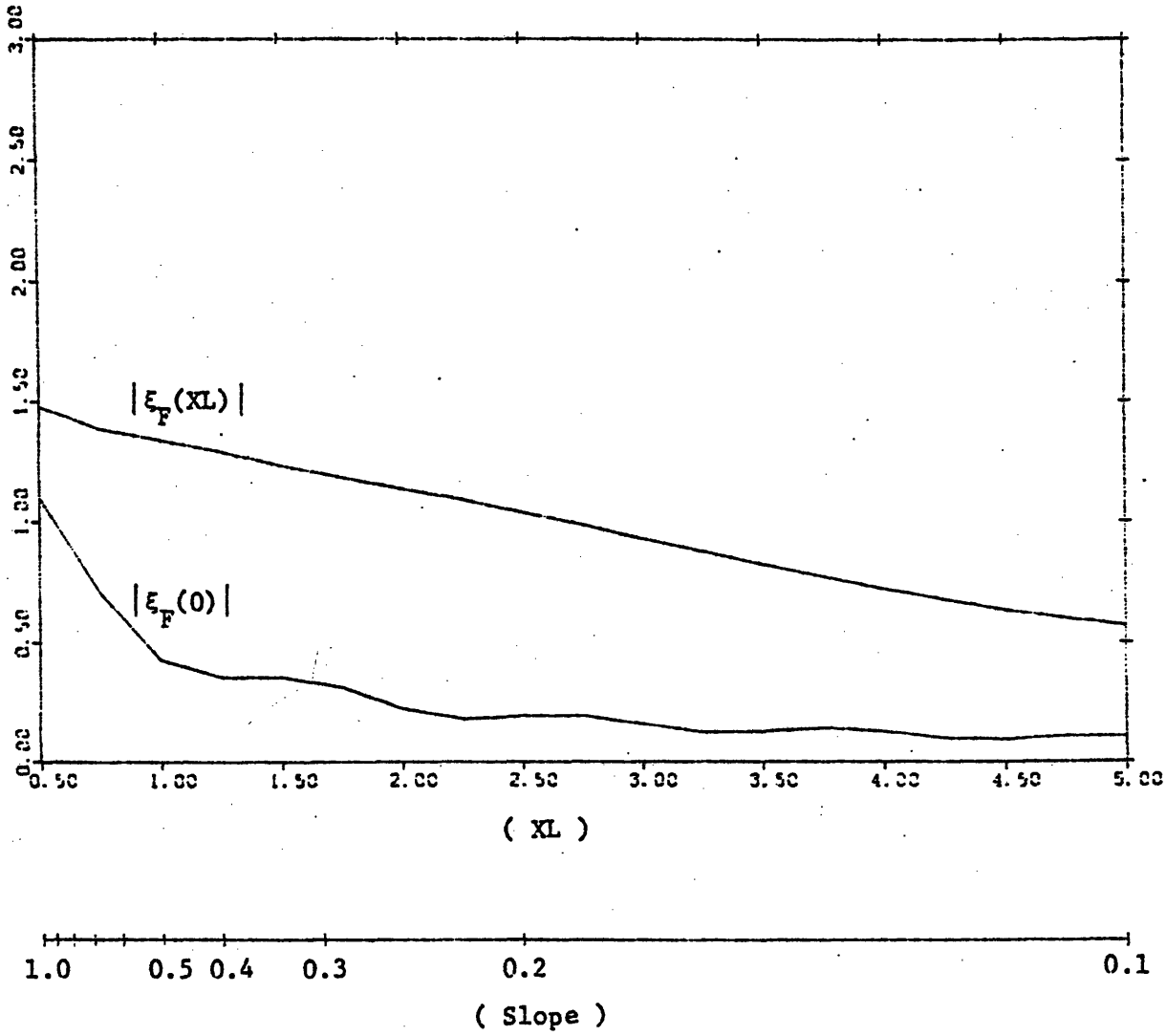


Figure 5.7 (Cont.)

(b) Oblique Incidence $\alpha_0 = \frac{\pi}{4}$

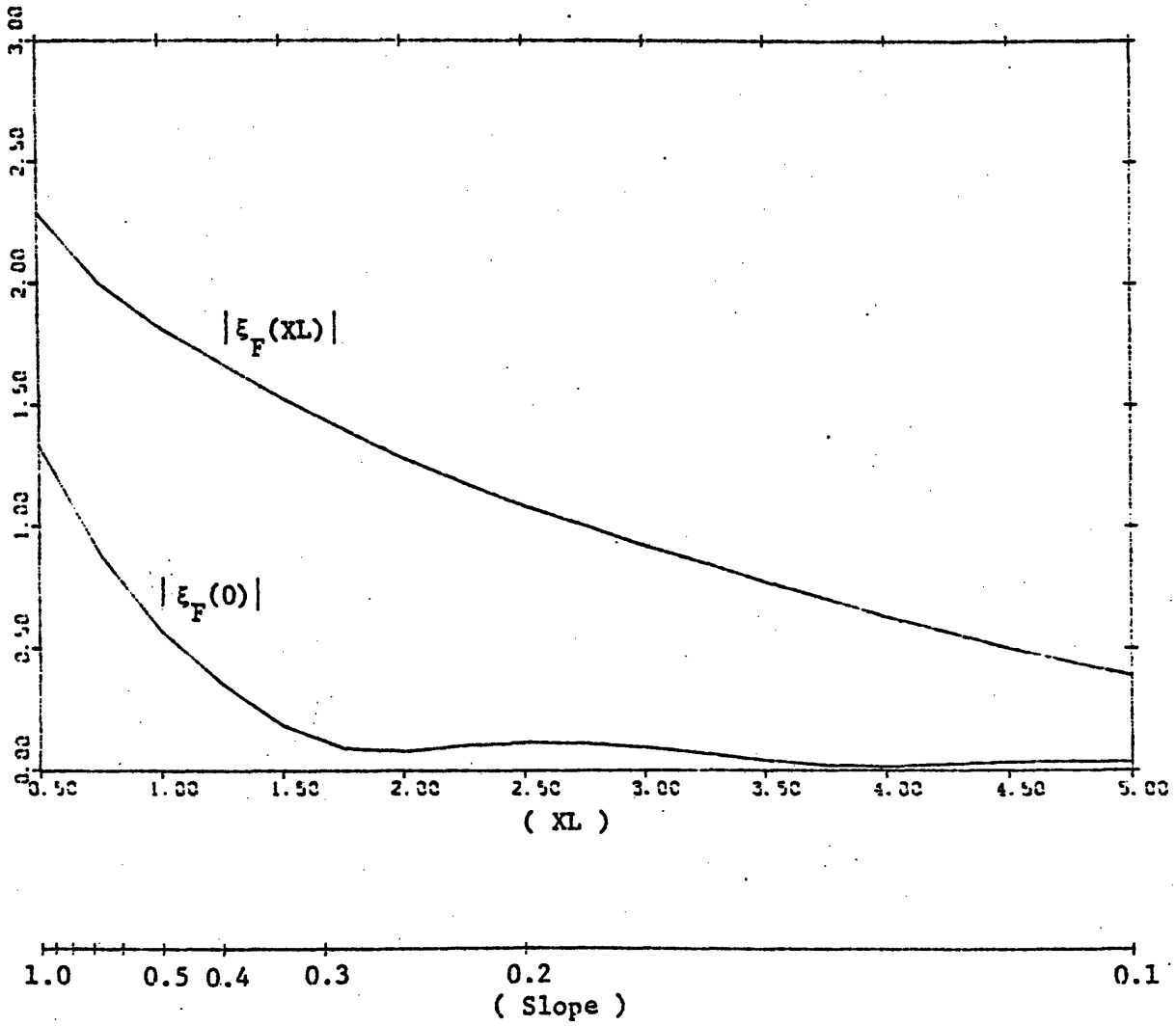


Figure 5.8 Forced Wave Amplitude onthe Left and Right Hand Sides of a Linearly Increasing Depth Region

h_0= 0.5, dh= 0.5

(a) Normal Incidence α_0= 0

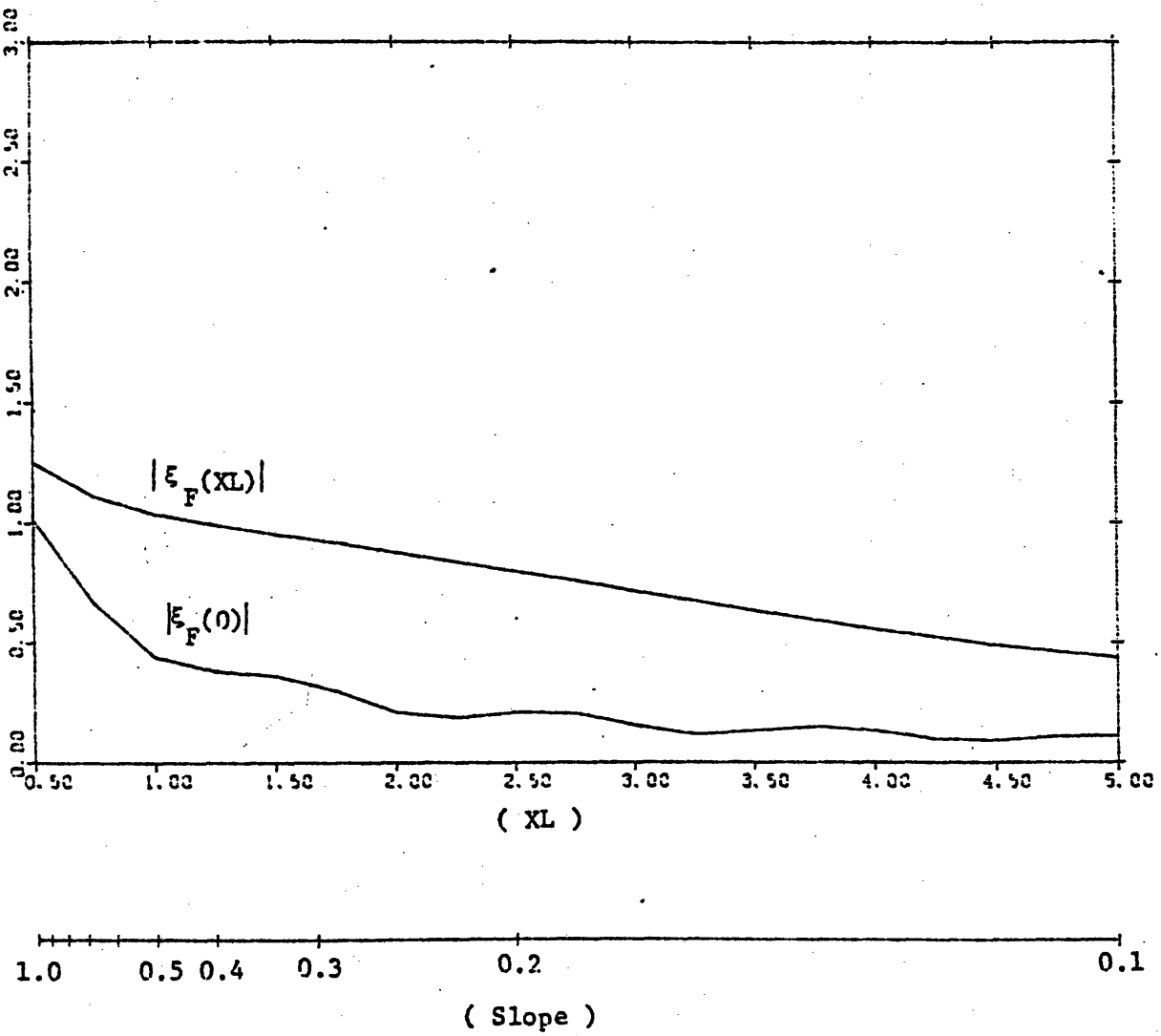
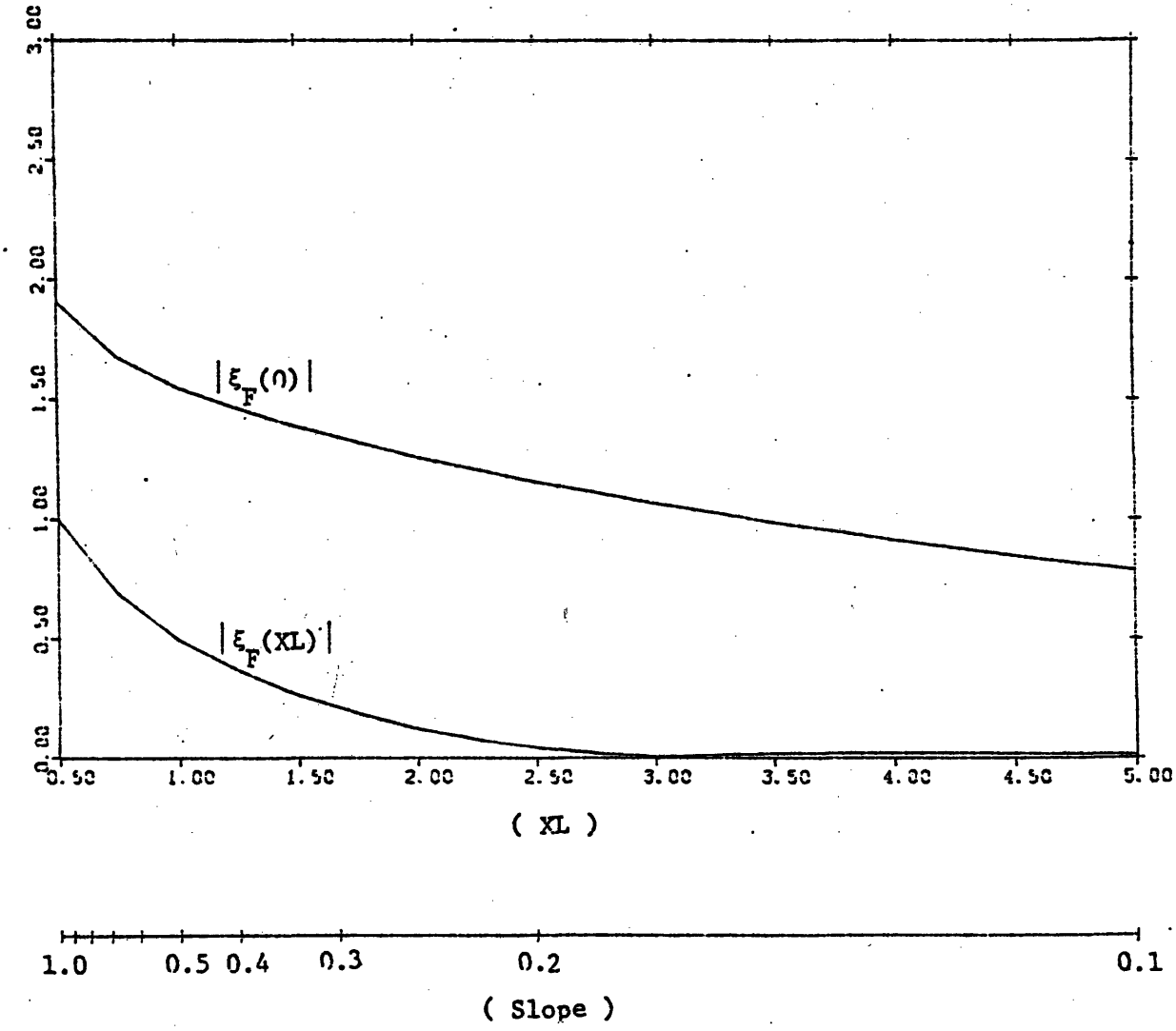


Figure 5.8 (Cont.)

(b) Oblique Incidence $\alpha_0 = \frac{\pi}{4}$



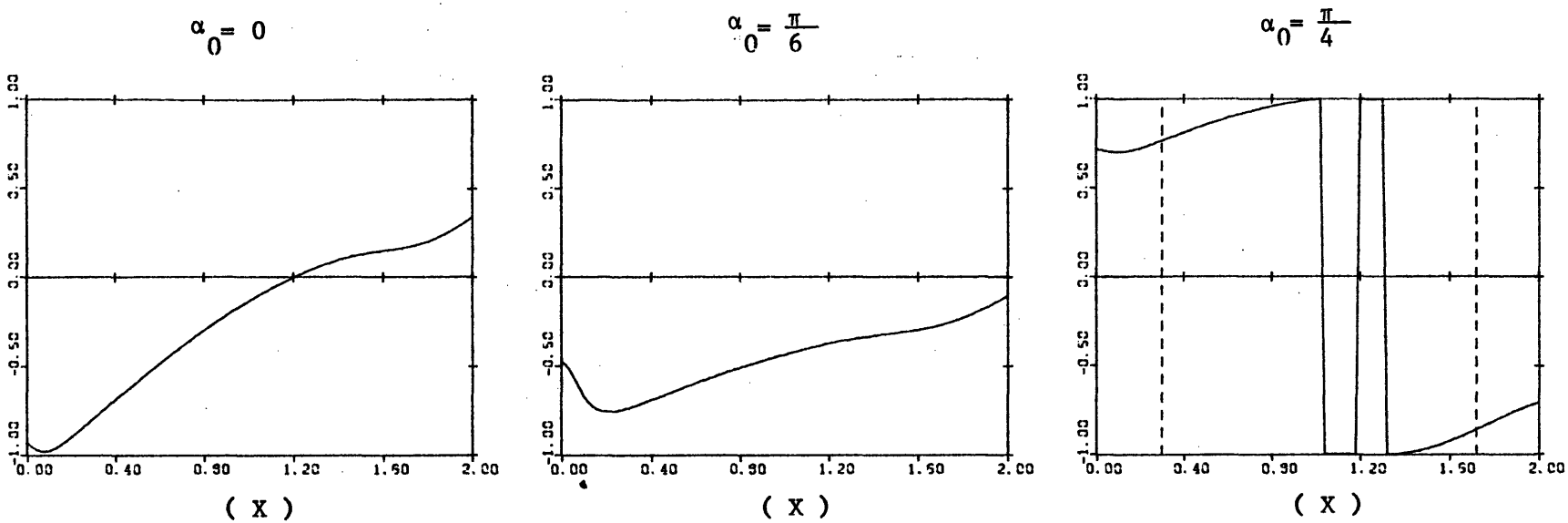
5.h. Canyons and Ridges

In this subsection we consider the depth variation to be a cosine profile:

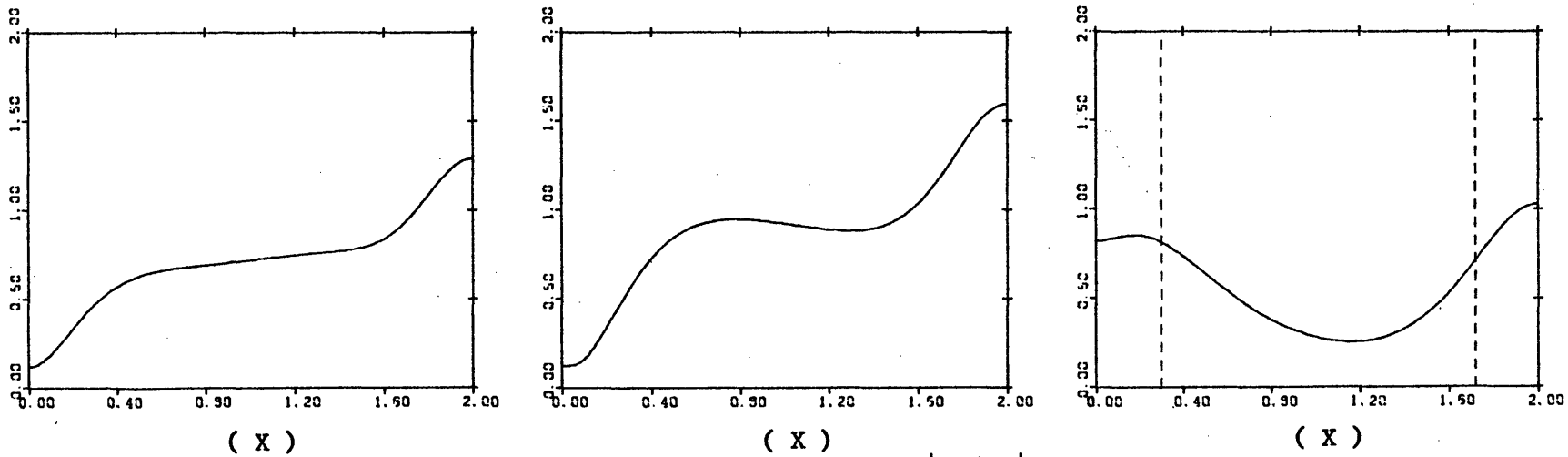
$$h = h_o + \frac{dh}{2}\left(1 - \cos\frac{2\pi X}{XL}\right)$$

The bottom profile corresponds to a canyon if $dh > 0$ and to a ridge if $dh < 0$. Figs. (5.9) and (5.10) show the forced wave amplitude and phase in the case of a canyon: $h_o = 0.5$, $dh = 0.5$. The length of the transition zone is $XL = 2$ in Fig. (5.9) and $XL = 10$ in Fig. (5.10). For such depth variations progressive outgoing forced waves are present on both sides for normal incidence $\alpha_o = 0$, and oblique incidence $\alpha_o = \pi/6$ and $\pi/4$. However for $\alpha_o = 0$ and $\pi/6$, λ^2 is positive everywhere. The wave amplitude is larger on the right hand side than on the left hand side. The phase distribution also shows that in the zone of varying depth the forced waves are propagating towards the right except in a small region in the neighborhood of $X = 0$. For $\alpha_o = \pi/4$, λ^2 is negative for $h > 0.6$. The location of the turning points is marked on the plots by the dashed line. The right and left going waves have amplitudes of the same order of magnitude. In the case of normal incidence, the variations of the right going wave amplitude $|\xi_F|_R$ and left going wave amplitude $|\xi_F|_L$ with the length of the transition zone XL are displayed in Fig. (5.11). $|\xi_F|_L$ decreases rapidly towards zero with small oscillations while $|\xi_F|_R$ is much bigger and undergoes large oscillations. Depending on the length

Figure 5.9 Forced Waves over a Canyon, h_0= 0.5, dh= 0.5, XL= 2

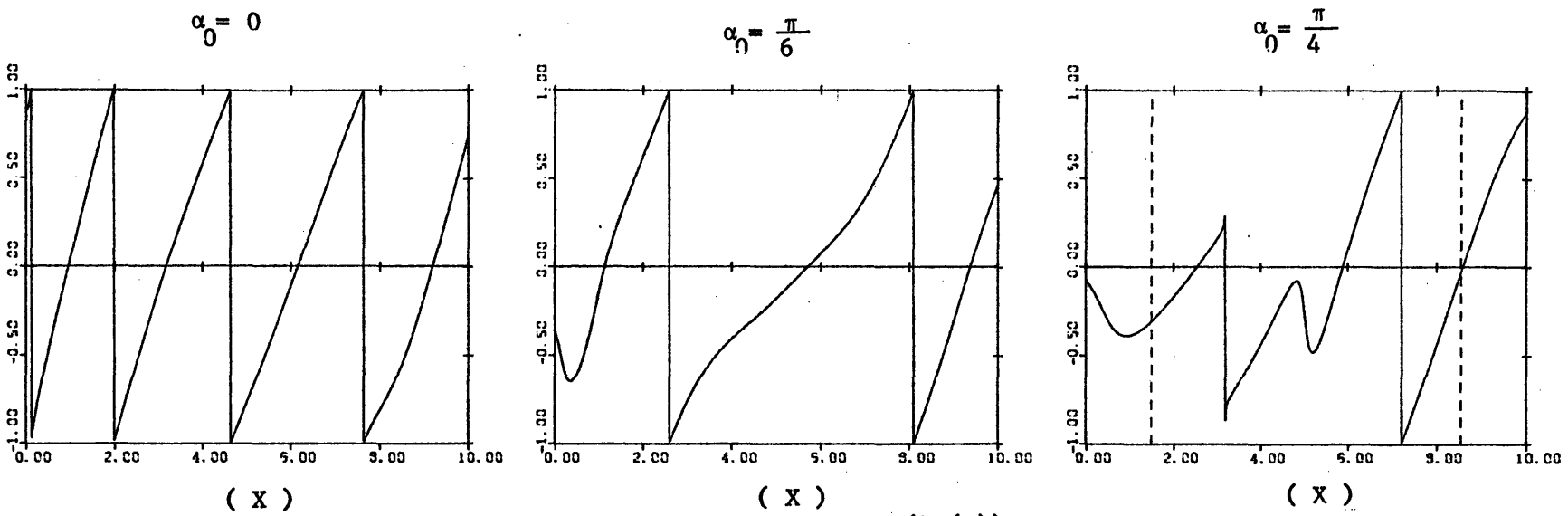


(a) Forced Wave Phase $\frac{\mathrm{Arg}(\xi_F(X))}{\pi}$

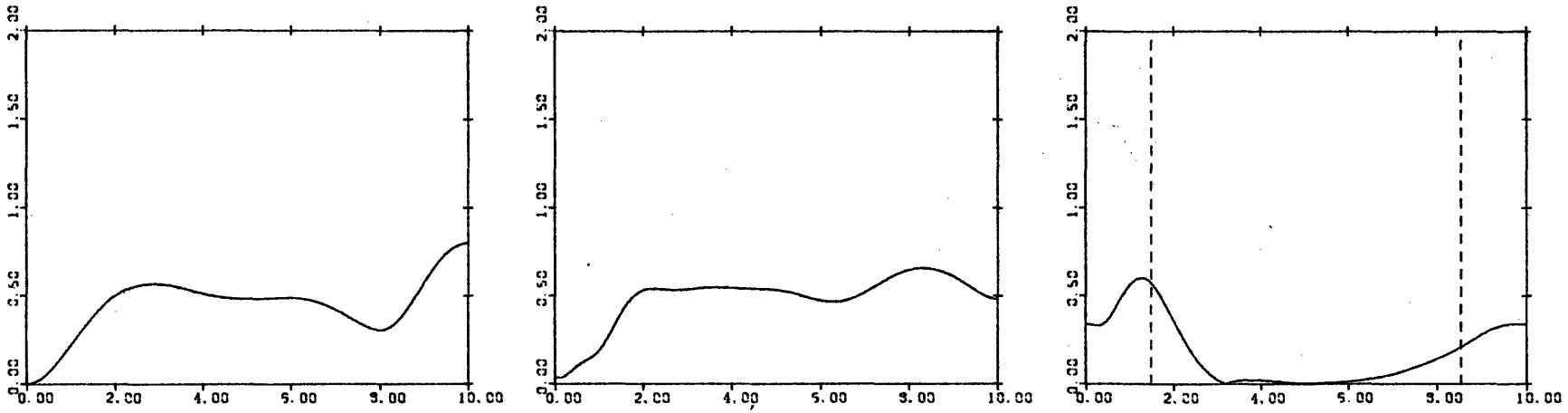


(b) Forced Wave Amplitude $|\xi_F(X)|$

Figure 5.10 Forced Waves over a Canyon, h_0= 0.5, dh= 0.5, XL= 10

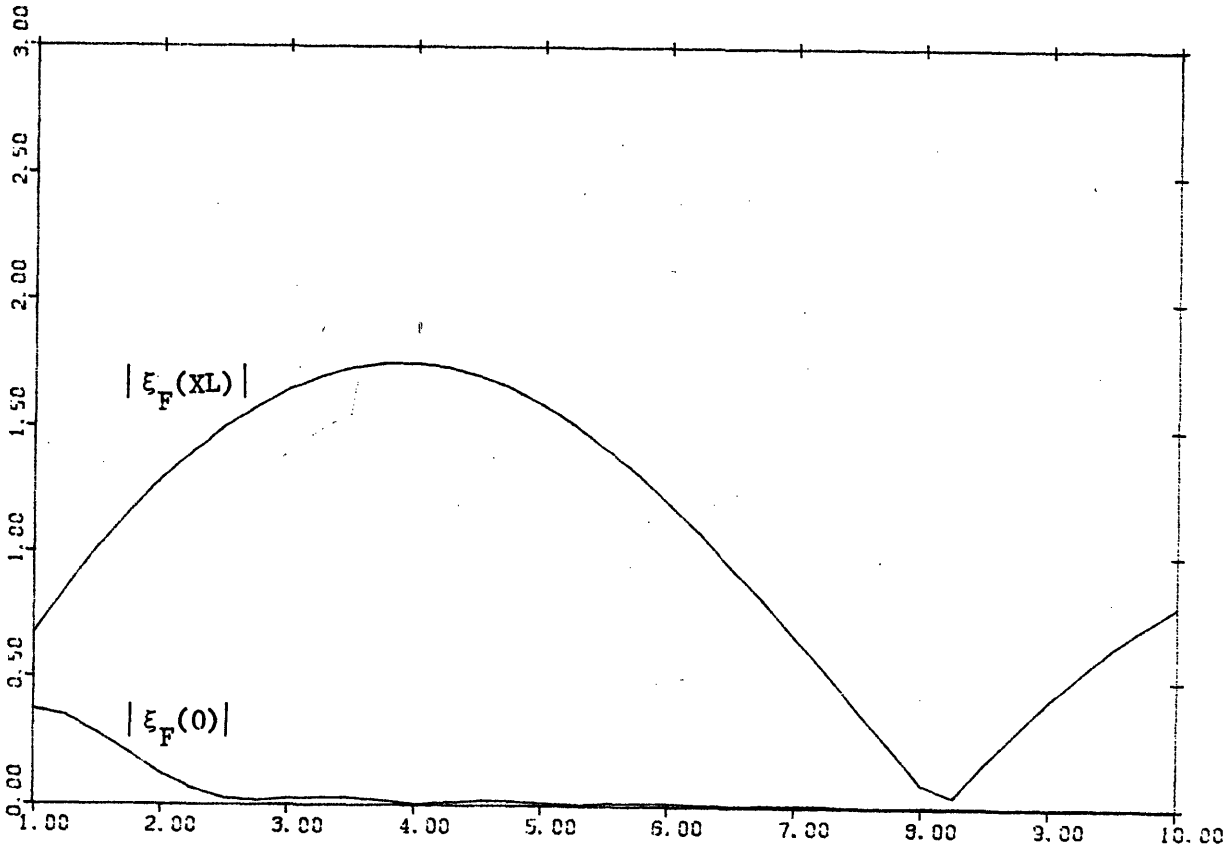


(a) Forced Wave Phase $\frac{\text{Arg}(\xi_F(X))}{\pi}$



(b) Forced Wave Amplitude $|\xi_F(X)|$

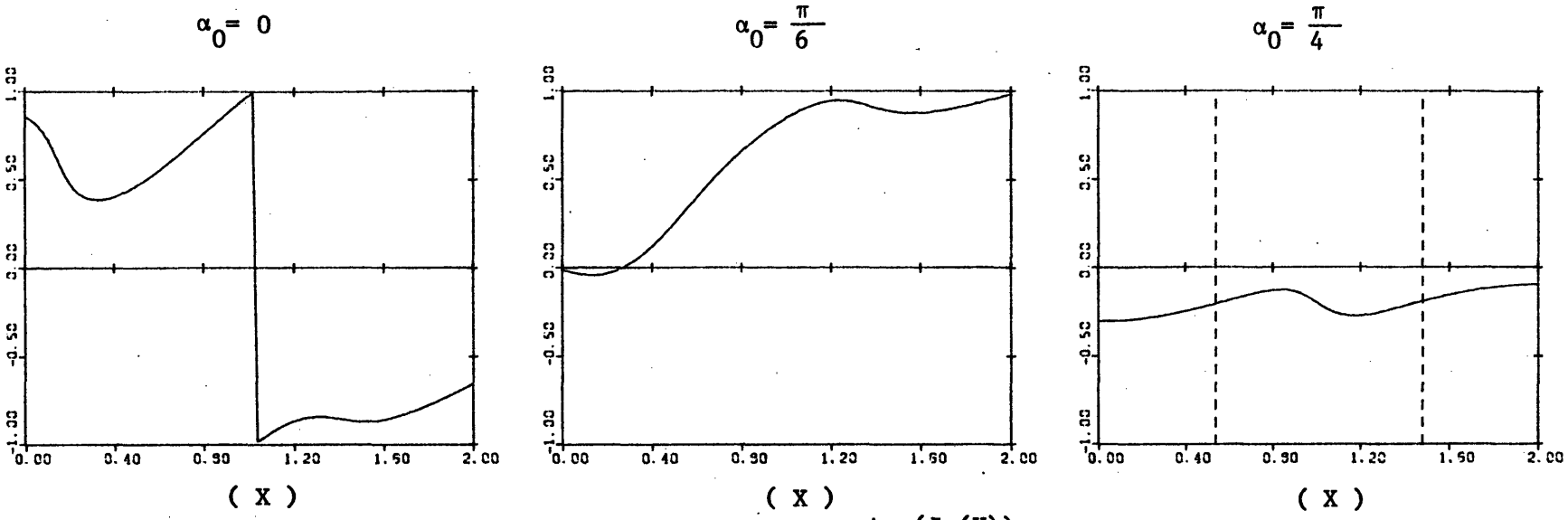
Figure 5.11 Forced Wave Amplitudes on the Left and Right Hand Sides of a canyon, Normal Incidence, h_0= 0.5, dh= 0.5



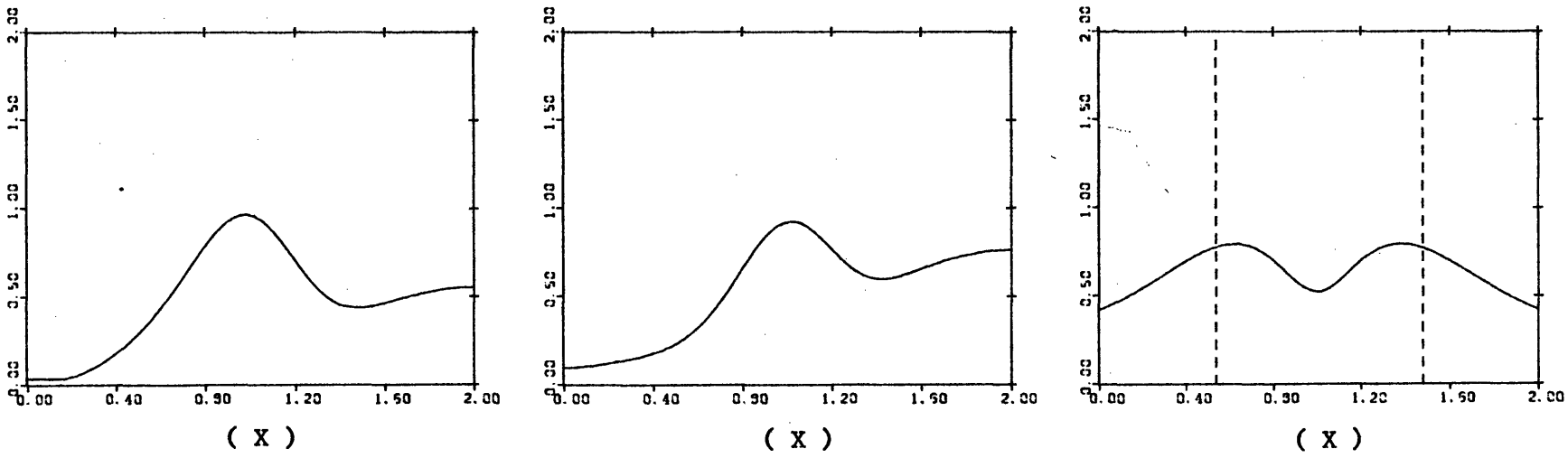
of the transition zone the right going wave can either have a very large amplitude or be absent. For XL = 8 there is hardly any ξ_F in both directions.

Fig. (5.12) and (5.13) correspond to the case of a ridge $h_o = 1$, dh = - 0.5, $\alpha_o = 0, \pi/6, \pi/4$. The length of the transition zone is XL = 2 in Fig. (5.12) and XL = 10 in Fig. (5.13). For $\alpha_o = 0$ and $\pi/6$ progressive outgoing waves are present in both sides of the region of variable depth. The left going wave amplitude is much smaller than the right going wave amplitude. For $\alpha_o = \pi/4$ forced waves are trapped on both sides, their amplitudes at X = 0 and X = XL are of the same order of magnitude. The ridge acts as a wave guide for the forced waves. In the case where progressive outgoing waves are present (i.e. $\alpha_o = 0$, $\alpha_o = \pi/6$), from the comparison of Fig. (5.12) and (5.13) we note that $|\xi_F|_R$ is larger for XL = 10 than for XL = 2. This seems to contradict our earlier experience that $|\xi_F|$ decreases with increasing XL. In Fig. (5.14) we have plotted the variations of $|\xi_F|_R$ and $|\xi_F|_L$ with XL for the case of normal incidence. It shows that $|\xi_F|_R$ is oscillating with a long period. The problem has now two different length scales: a length scale characterizing the oscillations and a much larger length scale over which the effect of the slope becomes important. On this long scale the maximum amplitude of the right going forced wave decreases.

Figure 5.12 Forced Waves over a Ridge, h_0= 1, dh=-0.5, XL= 2

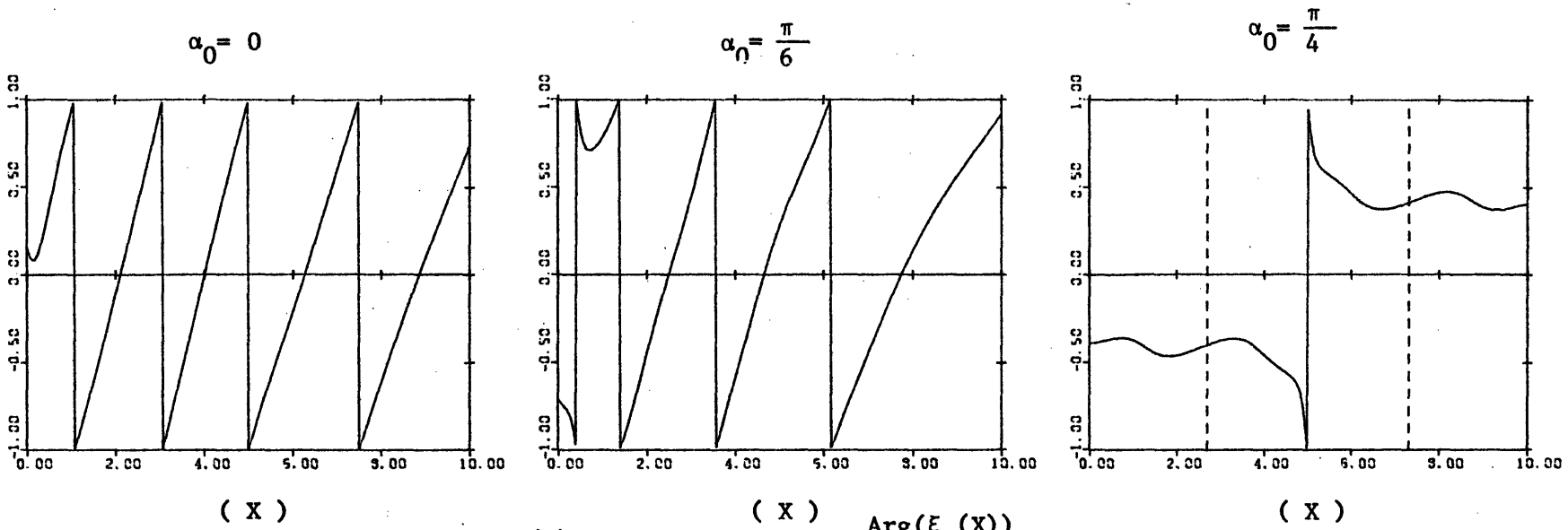


(a) Forced Wave Phase $\frac{\text{Arg}(\xi_F(X))}{\pi}$

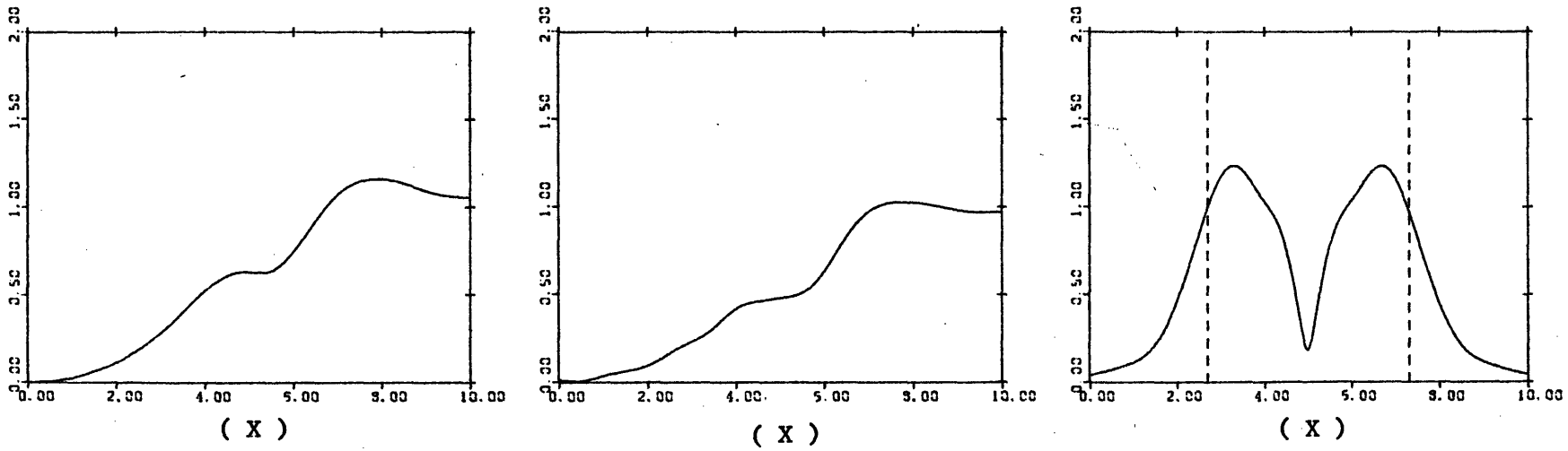


(b) Forced Wave Amplitude $|\xi_F(X)|$

Figure 5.13 Forced Waves over a Ridge, $h_0 = 1$, dh=-0.5, XL= 10

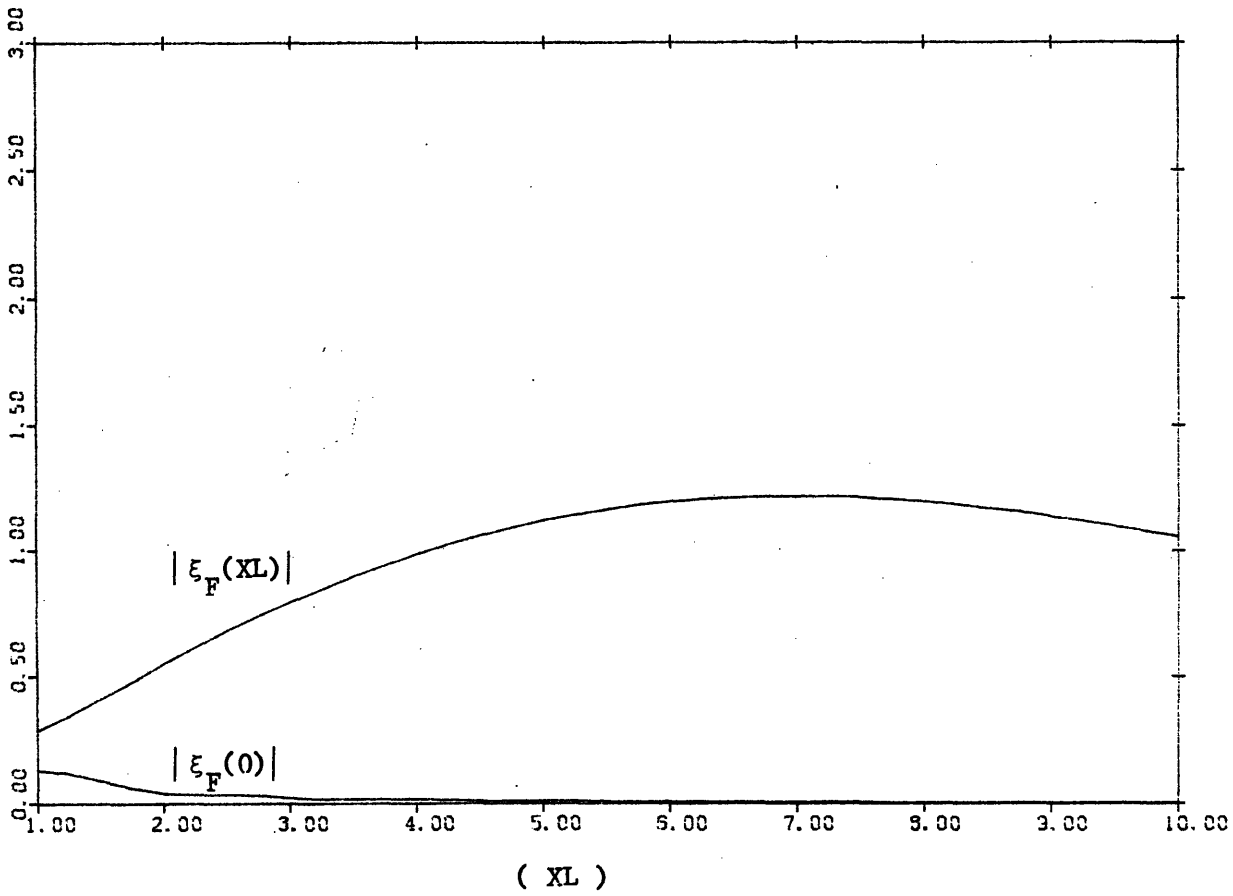


(a) Forced Wave Phase $\frac{\mathrm{Arg}(\xi_F(X))}{\pi}$



(b) Forced Wave Amplitude $|\xi_F(X)|$

Figure 5.14 Forced Wave Amplitudes on the Left and Right Hand Sides of a Ridege, Normal Incidence, h_0= 1, dh= -0.5



6. Conclusion

A slowly modulated wave train propagating on varying depth generates second order long waves which can be decomposed into 'locked' and 'forced' waves. The locked waves propagate in the same direction and with the same velocity as the wave envelope. They are bound to it. The forced waves exist in the region of depth variation. In the case of normal incidence ($\lambda^2 > 0$ everywhere), they are radiated in the form of outgoing waves with constant amplitude. Their phase velocity and direction of propagation are different from those of the wave envelope. In the case of oblique incidence, the forced waves could be trapped along the zone of variable depth if the constant depth region is deep enough ($\lambda^2 < 0$). Their amplitude then decays exponentially as we move away from the transition zone.

In the case where λ^2 is everywhere positive, the forced waves are right going on most part of the region of depth variation. Their amplitude on the right hand side is in general larger than on the left hand side. In the case where λ^2 is somewhere negative, for linear depth variation the forced wave amplitude is smaller on the side where it is trapped, for a canyon or a ridge the forced wave amplitudes are of the same order on both sides whether they are trapped or not.

The effect of increasing depth is a rapid decrease of the forced wave amplitude. The effect of increasing the length of the transition

zone is a decrease of the right going and left going wave amplitude accompanied with some oscillations. These oscillations are small for a linear depth profile but have large amplitude and large period for a canyon or a ridge.

It should be noted that these results are valid everywhere but near a caustic. In the neighborhood of a caustic a more refined theory is needed to take into account the finiteness of the short wave amplitude. The numerical part of this study could also be extended by considering a two dimensional topography.

Figure Captions: Part II

Figure 3.1: Wavenumbers in the X-direction of the short waves (k_1) and their envelope (K) for different angles of incidence (a) $\alpha_o = 0$, (b) $\alpha_o = \pi/6$, (c) $\alpha_o = \pi/4$, (d) $\alpha_o = \pi/3$. The reference depth is $h_o = 0.5$.

Figure 3.2: Wavenumbers in the X-direction of the short waves (k_1) and their envelope (K) for different angles of incidence (a) $\alpha_o = 0$, (b) $\alpha_o = \pi/6$, (c) $\alpha_o = \pi/4$, (d) $\alpha_o = \pi/3$. The reference depth is $h_o = 1$.

Figure 3.3: Mean set-down for different angles of incidence $\alpha_o = 0$, $\pi/6$, $\pi/4$, $\pi/3$ and reference depths (a) $h_o = 0.5$, (b) $h_o = 1$.

Figure 3.4: Set-down associated with the locked waves for different angles of incidence $\alpha_o = 0$, $\pi/6$, $\pi/4$, $\pi/3$ and reference depths (a) $h_o = 0.5$, (b) $h_o = 1$.

Figure 5.1: Forced wave (a) amplitude $|\xi_F(X)|$, and (b) phase $\frac{\text{Arg}(\xi_F(X))}{\pi}$ over linearly increasing depth. $h_o = 0.5$, dh = 0.5, XL = 1.

Figure 5.2: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{\text{Arg}(\xi_F(X))}{\pi}$ over linearly decreasing depth. $h_o = 1$, dh = - 0.5, XL = 1.

Figure 5.3: Forced wave (a) amplitude $|\xi_F(X)|$, and (b) phase $\frac{\text{Arg}(\xi_F(X))}{\pi}$ over linearly increasing depth. $h_o = 1$, dh = 0.5, XL = 1.

Figure 5.4: Forced wave amplitudes on the left $|\xi_F(0)|$ and right $|\xi_F(XL)|$ hand sides of a linearly increasing depth region, for variable h_o. Normal incidence, dh = 0.5, XL = 1.

Figure 5.5: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{Arg(\xi_F(X))}{\pi}$ for linearly increasing depth. $h_o = 0.5$, dh = 0.5, XL = 5.

Figure 5.6: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{Arg(\xi_F(X))}{\pi}$ for linearly decreasing depth. $h_o = 1$, dh=- 0.5, XL = 5.

Figure 5.7: Forced wave amplitudes on the left $|\xi_F(0)|$ and right $|\xi_F(XL)|$ hand sides of a linearly decreasing depth region, for variable XL. (a) Normal incidence, (b) oblique incidence $\alpha_o = \pi/4$. $h_o = 1$, dh = - 0.5.

Figure 5.8: Forced wave amplitudes on the left $|\xi_F(0)|$ and right $|\xi_F(XL)|$ hand sides of a linearly increasing depth region, for variable XL. (a) Normal incidence, (b) oblique incidence $\alpha_o = \pi/4$. $h_o = 0.5$, dh = 0.5.

Figure 5.9: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{Arg(\xi_F(X))}{\pi}$ over a canyon. $h_o = 0.5$, dh = 0.5, XL = 2.

Figure 5.10: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{Arg(\xi_F(X))}{\pi}$ over a canyon. $h_o = 0.5$, dh = 0.5, XL = 10.

Figure 5.11: Forced wave amplitudes on the left $|\xi_F(0)|$ and right $|\xi_F(XL)|$ hand sides of a canyon, for variable XL. Normal incidence, $h_o = 0.5$, dh = 0.5.

Figure 5.12: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{Arg(\xi_F(X))}{\pi}$ over a ridge. $h_o = 1$, dh = - 0.5, XL = 2.

Figure 5.13: Forced wave (a) amplitude $|\xi_F(X)|$ and (b) phase $\frac{Arg(\xi_F(X))}{\pi}$ over a ridge. $h_o = 1$, dh = - 0.5, XL = 10.

Figure 5.14: Forced wave amplitudes on the left $|\xi_F(0)|$ and right $|\xi_F(XL)|$ hand sides of a ridge, for variable XL. Normal incidence, $h_o = 1$, dh = - 0.5.

ANNEXES

APPENDIX A1

The lower order terms which appear in Eqs. (3.10) are listed below:

$$I_{10} = I_{11} = 0$$

$$I_{20} = -2\rho(2k^2 u_{11}u_{1-1} - ik\frac{\partial u_{11}}{\partial z}W_{1-1} + ik\frac{\partial u_{1-1}}{\partial z}W_{11})$$

$$I_{21} = -\frac{2ik}{C_g}\frac{\partial p_{11}}{\partial \tau} - 2ik\rho\frac{\partial u_{10}}{\partial z}W_{11}$$

$$I_{22} = -2\rho(-k^2 u_{11}u_{11} + ik\frac{\partial u_{11}}{\partial z}W_{11})$$

$$I_{31} = -\frac{2ik}{C_g}\frac{\partial p_{21}}{\partial \tau} - 2ik\frac{\partial p_{11}}{\partial X} - \frac{i\partial k}{\partial X}p_{11} - \frac{1}{C_g^2}\frac{\partial^2 p_{11}}{\partial \tau^2} - 2\rho(2ik\frac{\partial U}{\partial X}u_{11}$$
$$+ \frac{2ik}{C_g}\frac{\partial u_{10}}{\partial \tau}u_{11} + 4k^2 u_{1-1}u_{22} + ik\frac{\partial u_{20}}{\partial z}W_{11} - ik\frac{\partial u_{22}}{\partial z}W_{1-1}$$
$$+ 2ik\frac{\partial u_{1-1}}{\partial z}W_{22})$$

$$J_{10} = J_{11} = J_{20} = J_{21} = J_{22} = 0$$

$$J_{31} = -ik\frac{\partial h}{\partial X}p_{11}$$

$$K_{10} = K_{11} = 0$$

$$K_{20} = -\frac{\partial u_{10}}{\partial \tau} - \frac{U}{C_g}\frac{\partial u_{10}}{\partial \tau} - W_{10}\frac{\partial u_{10}}{\partial z} - W_{11}\frac{\partial u_{1-1}}{\partial z} - W_{1-1}\frac{\partial u_{11}}{\partial z} - \frac{1}{\rho C_g}\frac{\partial p_{10}}{\partial \tau}$$

$$K_{21} = \frac{\partial u_{11}}{\partial \tau} - \frac{U}{C_g}\frac{\partial u_{11}}{\partial \tau} - ik\,u_{11}u_{10} - W_{10}\frac{\partial u_{11}}{\partial z} - W_{11}\frac{\partial u_{10}}{\partial z} - \frac{1}{\rho C_g}\frac{\partial p_{11}}{\partial \tau}$$

$$K_{22} = -ik\,u_{11}u_{1-1} - W_{11}\frac{\partial u_{11}}{\partial z}$$

$$K_{30} = \frac{\partial u_{20}}{\partial \tau} - \frac{\partial u_{10}}{\partial T} - \frac{\partial U}{\partial X} u_{10} - \frac{U}{C_g} \frac{\partial u_{20}}{\partial \tau} - U \frac{\partial u_{10}}{\partial X} + (z+h) \frac{\partial U}{\partial X} \frac{\partial u_{10}}{\partial z}$$

$$+ \frac{\partial h}{\partial X} U \frac{\partial u_{10}}{\partial z} - \frac{1}{C_g} u_{10} \frac{\partial u_{10}}{\partial \tau} + ik\, u_{11} u_{2-1} - ik\, u_{1-1}\, u_{21}$$

$$- \frac{1}{C_g} u_{11} \frac{\partial u_{1-1}}{\partial \tau} - \frac{1}{C_g} u_{1-1} \frac{\partial u_{11}}{\partial \tau} + ik\, u_{1-1} u_{21} - ik\, u_{11} u_{2-1}$$

$$- W_{10} \frac{\partial u_{20}}{\partial z} - W_{1-1} \frac{\partial u_{21}}{\partial z} - W_{11} \frac{\partial u_{2-1}}{\partial z} - W_{2-1} \frac{\partial u_{11}}{\partial z} - \frac{1}{\rho C_g} \frac{\partial p_{20}}{\partial \tau}$$

$$- \frac{1}{\rho} \frac{\partial p_{10}}{\partial X}$$

$$L_{10} = L_{11} = 0$$

$$L_{20} = - \frac{1}{C_g} \frac{\partial u_{10}}{\partial \tau}$$

$$L_{21} = - \frac{1}{C_g} \frac{\partial u_{11}}{\partial \tau}$$

$$L_{22} = 0$$

$$M_{10} = M_{20} = 0$$

$$M_{20} = \frac{\partial \eta_{10}}{\partial \tau} - \frac{U}{C_g} \frac{\partial \eta_{10}}{\partial \tau} + \eta_{10} \frac{\partial W_{10}}{\partial z} + ik\, u_{11} \eta_{1-1} - ik\, u_{1-1} \eta_{11}$$

$$+ \eta_{1-1} \frac{\partial W_{11}}{\partial z} + \eta_{11} \frac{\partial W_{1-1}}{\partial z}$$

$$M_{21} = \frac{\partial \eta_{11}}{\partial \tau} - \frac{U}{C_g} \frac{\partial \eta_{11}}{\partial \tau} + \eta_{11} \frac{\partial W_{10}}{\partial z} + \eta_{10} \frac{\partial W_{11}}{\partial z} - ik\, u_{10} \eta_{11}$$

$$M_{22} = \eta_{11} \frac{\partial W_{11}}{\partial z} - ik\, u_{11} \eta_{11}$$

$$N_{10} = N_{11} = 0$$

$$N_{20} = \frac{\partial p_{10}}{\partial \tau} - \frac{U}{C_g}\frac{\partial p_{10}}{\partial \tau} - i\sigma\eta_{11}\frac{\partial p_{1-1}}{\partial z} + i\sigma\eta_{1-1}\frac{\partial p_{11}}{\partial z} + ik\, u_{11}p_{1-1}$$

$$-iku_{1-1}p_{11} - W_{10}\frac{\partial p_{10}}{\partial z} - W_{11}\frac{\partial p_{1-1}}{\partial z} - W_{1-1}\frac{\partial p_{11}}{\partial z} + \rho g\,\eta_{10}\frac{\partial W_{10}}{\partial z}$$

$$+ \rho g\eta_{11}\frac{\partial W_{1-1}}{\partial z} + \rho g\eta_{1-1}\frac{\partial W_{11}}{\partial z}$$

$$N_{21} = \frac{\partial p_{11}}{\partial \tau} - \frac{U}{C_g}\frac{\partial p_{11}}{\partial \tau} + i\sigma\eta_{10}\frac{\partial p_{11}}{\partial z} - ik\, u_{10}p_{11} - W_{10}\frac{\partial p_{11}}{\partial z} - W_{11}\frac{\partial p_{10}}{\partial z}$$

$$+ \rho g\,\eta_{10}\frac{\partial W_{11}}{\partial z} + \rho g\eta_{11}\frac{\partial W_{10}}{\partial z}$$

$$N_{22} = i\sigma\eta_{11}\frac{\partial p_{11}}{\partial z} - ik\, u_{11}p_{11} - W_{11}\frac{\partial p_{11}}{\partial z} + \rho g\eta_{11}\frac{\partial W_{11}}{\partial z}$$

To simplify the expressions for N_{30} and N_{31} we use the result (3.24): $\eta_{10} = u_{10} = W_{10} = p_{10} = 0$

$$N_{30} = \frac{\partial p_{20}}{\partial \tau} - \frac{U}{C_g}\frac{\partial p_{20}}{\partial \tau} - i\sigma\eta_{21}\frac{\partial p_{1-1}}{\partial z} + i\sigma\eta_{2-1}\frac{\partial p_{11}}{\partial z} - i\sigma\eta_{11}\frac{\partial p_{2-1}}{\partial z}$$

$$+ i\sigma\eta_{1-1}\frac{\partial p_{21}}{\partial z} + \eta_{11}\frac{\partial^2 p_{1-1}}{\partial \tau \partial z} - \frac{U}{C_g}\eta_{11}\frac{\partial^2 p_{1-1}}{\partial \tau \partial z} + \eta_{1-1}\frac{\partial^2 p_{11}}{\partial \tau \partial z}$$

$$- \frac{U}{C_g}\eta_{1-1}\frac{\partial^2 p_{11}}{\partial \tau \partial z} - ik\, u_{2-1}p_{11} + ik\, u_{21}p_{1-1} - ik\, u_{1-1}p_{21}$$

$$+ ik\, u_{11}p_{2-1} - \frac{1}{C_g}u_{11}\frac{\partial p_{1-1}}{\partial \tau} - \frac{1}{C_g}u_{1-1}\frac{\partial p_{11}}{\partial \tau} - W_{21}\frac{\partial p_{1-1}}{\partial z} - W_{2-1}\frac{\partial p_{11}}{\partial z}$$

$$- W_{11}\frac{\partial p_{2-1}}{\partial z} - W_{1-1}\frac{\partial p_{21}}{\partial z} + \rho g\eta_{11}\frac{\partial W_{2-1}}{\partial z} + \rho g\eta_{1-1}\frac{\partial W_{21}}{\partial z} + \rho g\eta_{21}\frac{\partial W_{1-1}}{\partial z}$$

$$+ \rho g\eta_{2-1}\frac{\partial W_{11}}{\partial z}$$

$$N_{31} = \frac{\partial p_{21}}{\partial \tau} - \frac{U}{C_g}\frac{\partial p_{21}}{\partial \tau} - \frac{\partial p_{11}}{\partial T} - \frac{U\partial p_{11}}{\partial X} + i\sigma\eta_{20}\frac{\partial p_{11}}{\partial z} - i\sigma\eta_{22}\frac{\partial p_{1-1}}{\partial z}$$

$$- 2i\sigma\eta_{1-1}\frac{\partial p_{22}}{\partial z} + \frac{i\sigma}{2}\left(\eta_{11}\,\eta_{11}\frac{\partial^2 p_{1-1}}{\partial z^2} + 2\eta_{11}\eta_{1-1}\frac{\partial^2 p_{11}}{\partial z^2}\right) - ik\,u_{20}p_{11}$$

$$+ ik\,u_{22}p_{1-1} - 2ik\,u_{1-1}p_{22} - ik\left(\eta_{11}p_{11}\frac{\partial u_{1-1}}{\partial z} - \eta_{11}\,p_{1-1}\right.$$

$$\left.\frac{\partial u_{11}}{\partial z} + \eta_{1-1}p_{11}\frac{\partial u_{11}}{\partial z}\right) - ik\left(-\eta_{11}u_{11}\frac{\partial p_{1-1}}{\partial z} + u_{11}\eta_{1-1}\frac{\partial p_{11}}{\partial z}\right.$$

$$\left.+ u_{1-1}\,\eta_{11}\frac{\partial p_{11}}{\partial z}\right) - W_{22}\frac{\partial p_{1-1}}{\partial z} - W_{1-1}\frac{\partial p_{22}}{\partial z} - W_{11}\frac{\partial p_{20}}{\partial z} - \left(\eta_{11}\frac{\partial p_{11}}{\partial z}\right.$$

$$\left.\frac{\partial W_{1-1}}{\partial z} + \eta_{11}\frac{\partial p_{1-1}}{\partial z}\frac{\partial W_{11}}{\partial z} + \eta_{1-1}\frac{\partial p_{11}}{\partial z}\frac{\partial W_{11}}{\partial z}\right) - \left(\eta_{11}W_{11}\frac{\partial^2 p_{1-1}}{\partial z^2}\right.$$

$$\left.+ \eta_{11}W_{1-1}\frac{\partial^2 p_{11}}{\partial z^2} + \eta_{1-1}W_{11}\frac{\partial^2 p_{11}}{\partial z^2}\right) - \rho g\frac{\partial \xi}{\partial X}u_{11} + (\xi+h)\frac{\partial U}{\partial X}\frac{\partial p_{11}}{\partial z}$$

$$+ U\frac{\partial h}{\partial X}\frac{\partial p_{11}}{\partial z} - \rho g\frac{\partial U}{\partial X}\eta_{11} + \rho g\eta_{1-1}\frac{\partial W_{22}}{\partial z} + \rho g\eta_{22}\frac{\partial W_{1-1}}{\partial z} + \rho g\eta_{20}$$

$$\frac{\partial W_{11}}{\partial z} + \frac{\rho g}{2}\left(\eta_{11}^2\frac{\partial^2 W_{1-1}}{\partial z^2} + 2\eta_{11}\eta_{1-1}\frac{\partial^2 W_{11}}{\partial z^2}\right)$$

$$Q_{31} = -\underline{\frac{\partial W_{11}}{\partial T}} + \left(1 - \frac{U}{C_g}\right)\frac{\partial W_{21}}{\partial \tau} - \underline{\frac{U\partial W_{11}}{\partial X}} + W_{11}\frac{\partial U}{\partial X} + (z+h)\frac{\partial U}{\partial X}\frac{\partial W_{11}}{\partial z}$$

$$+ \frac{\partial h}{\partial X}U\frac{\partial W_{11}}{\partial z} - 2iku_{1-1}W_{22} + \underline{ik\,W_{1-1}u_{22}} - ik\,W_{11}u_{20} - W_{11}\frac{\partial W_{20}}{\partial z}$$

$$- W_{1-1}\frac{\partial W_{22}}{\partial z} - W_{20}\frac{\partial W_{11}}{\partial z} - W_{22}\frac{\partial W_{1-1}}{\partial z}$$

The underlined terms are those which differ from the similar expression found by Turpin (1981)

Appendix A 2

FILE: WAVE FORTRAN A

```
*PROCESS SC(AXIS)                                                    WAV00010
C     ******************************************************         WAV00020
C     *                                                    *         WAV00030
C     *    COMPUTE THE EVOLUTION OF A SECH PROFILE         *         WAV00040
C     *                                                    *         WAV00050
C     *         OVER VARIABLE DEPTH AND CURRENT            *         WAV00060
C     *                                                    *         WAV00070
C     ******************************************************         WAV00080
C     A=AMPL.;H=DEPTH;XHI=TOTAL DEPTH(WITH CURRENT);                 WAV00090
C     T=ADIM.PERIOD;KA=WAVE NUMBER;Y1,Y2,Y3=COEF.OF EQUA.GOVER.A;     WAV00100
C     CGG=GROUP VELOC.IN FIXED FRAME;UINF=CURRENT VEL. AT X=0        WAV00110
C     ERR=CONSERVED QUANTITY( EVOLU.LAW)                             WAV00120
C     NA,2*JA1 ARE DIMEN. OF MATRIX A(AMPL.)                         WAV00130
C     NAV,DH,L=PARAM. OF DEPTH PROFILE                               WAV00140
C     DX,DTO=DISCR. LENGTH                                           WAV00150
      COMPLEX A(302,151)                                             WAV00160
      DIMENSION H(76),XHI(76),KA(76),Y1(76),Y2(76),Y3(76)            WAV00170
     S,CGG(76),ERR(500)                                              WAV00180
      REAL KA,KAA,L                                                  WAV00190
      WRITE(10,100)                                                  WAV00200
      READ(5,*)UINF,T,DH1,NAV,NA,JA1                                 WAV00210
  100 FORMAT(5X,'UINF,T,DH1,NAV,NA,JA1')                             WAV00220
      DTO=0.1                                                        WAV00230
      JA=2*JA1                                                       WAV00240
      L=1.                                                           WAV00250
      DX=L/FLOAT(NAV-1)                                              WAV00260
      OME=2.*3.14159/T                                               WAV00270
      CALL DEPTH(NAV,DH1,H)                                          WAV00280
C     GIVES THE DEPTH VARIATION                                      WAV00290
      DO 1 I=1,NAV                                                   WAV00300
      HH=H(I)                                                        WAV00310
      IF(ABS(UINF).LE.0.0001) GO TO 10                               WAV00320
      CALL MEAFS(HH,UINF,XHII)                                       WAV00330
C     GIVES THE SET UP DUE TO THE CURRENT                            WAV00340
      GO TO 11                                                       WAV00350
   10 XHII=H(I)                                                      WAV00360
   11 XHI(I)=XHII                                                    WAV00370
      CALL WAVENU(OME,UINF,XHII,KAA)                                 WAV00380
C     GIVES THE WAVENUMBER                                           WAV00390
C     IF WAVENUMBER NEGATIVE WAVES ARE STOPPED                       WAV00400
      KA(I)=KAA                                                      WAV00410
      IF(KA(I).GT.0.)GO TO 1                                         WAV00420
      WRITE(10,200)                                                  WAV00430
      WRITE(10,300)UINF,T,H(I),XHII,KA(I)                            WAV00440
  200 FORMAT(5X,'WAVES CANNOT PROP. AGAINST THIS CURRENT')           WAV00450
  300 FORMAT(5E12.4)                                                 WAV00460
      GO TO 20                                                       WAV00470
    1 CONTINUE                                                       WAV00480
      KA(NAV+1)=KA(NAV)                                              WAV00490
      XHI(NAV+1)=XHI(NAV)                                            WAV00500
      CALL COEFF(NAV,XHI,KA,H,UINF,OME,DX,Y1,Y2,Y3,CGG)              WAV00510
C     GIVES THE COEFF. OF GOV. EQ.                                   WAV00520
      Y21=Y2(1)                                                      WAV00530
      XX=ABS(Y3(1))                                                  WAV00540
      CALL INIT(JA,DTO,Y21,XX,A,IS,ERR)                              WAV00550
```

```
C     GIVES INITIAL PROFILE AT X2=0.                                    WAV00560
      CALL SOLUT(NA,NAV,JA,DX,DTO,Y1,Y2,Y3,IS,A,ERR,ADR,AN)             WAV00570
C     SOLVES THE LINEAR SYSTEM OF EQS.                                  WAV00580
      CALL IMPRES(NA,NAV,KA,XHI,Y2,Y3,ERR,ADR,AN)                       WAV00590
      CALL WPLOT(NAV,NA,JA,DTO,DX,A)                                    WAV00600
 20   CONTINUE                                                          WAV00610
      STOP                                                              WAV00620
      END                                                               WAV00630
      SUBROUTINE DEPTH(NAV,DH1,H)                                       WAV00640
      DIMENSION H(1)                                                    WAV00650
      DO 1 N=1,NAV                                                      WAV00660
      R=FLOAT(N-1)*3.14159/FLOAT(NAV-1)                                 WAV00670
      H(N)=1.+DH1*(COS(R)-1.)/2.                                        WAV00680
 1    CONTINUE                                                          WAV00690
      H(NAV+1)=H(NAV)                                                   WAV00700
      RETURN                                                            WAV00710
      END                                                               WAV00720
      SUBROUTINE MEAFS(HH,UINF,XHII)                                    WAV00730
      EPSI=0.001                                                        WAV00740
      A2=(UINF**2)/2.                                                   WAV00750
      X2=(A2**2)+4.*A2                                                  WAV00760
      X1=(A2-SQRT(X2))/2.                                               WAV00770
      X2=(A2+SQRT(X2))/2.                                               WAV00780
      DELT=0.25*ABS(1.-X2)                                              WAV00790
      IF(DELT.LE.0.05) DELT=0.05                                        WAV00800
      XHII=1.                                                           WAV00810
      N=0                                                               WAV00820
      Z=HH-1.                                                           WAV00830
      ER=ABS(Z)                                                         WAV00840
      IF(ER.LE.EPSI) GO TO 20                                           WAV00850
      PTE=2.*(HH-1)-(1.-X1)*(1.-X2)                                     WAV00860
      IF(HH.LT.1.) GO TO 1                                              WAV00870
      IF(UINF.GT.1.) GO TO 10                                           WAV00880
      GO TO 11                                                          WAV00890
 12   IF(UINF.LT.1..AND.Z1.LE.0.) DELT=DELT/2.                          WAV00900
      IF(UINF.GE.1..AND.Z1.GT.0.) DELT=DELT/2.                          WAV00910
      N=N+1                                                             WAV00920
      IF(N.GT.500) GO TO 20                                             WAV00930
 11   XHII=XHII+DELT                                                    WAV00940
      GO TO 14                                                          WAV00950
 13   IF(UINF.LT.1..AND.Z1.GT.0.) DELT=DELT/2.                          WAV00960
      IF(UINF.GE.1..AND.Z1.LE.0.) DELT=DELT/2.                          WAV00970
      N=N+1                                                             WAV00980
      IF(N.GT.500) GO TO 20                                             WAV00990
 10   XHII=XHII-DELT                                                    WAV01000
 14   Z1=Z                                                              WAV01010
      Z=(XHII-1.)*(XHII-X1)*(XHII-X2)                                   WAV01020
      Z=(HH-1.)*(XHII**2)-Z                                             WAV01030
      ER=ABS(Z)                                                         WAV01040
      IF(ER.LE.EPSI) GO TO 20                                           WAV01050
      IF(Z.GE.0..AND.UINF.LT.1.) GO TO 12                               WAV01060
      IF(Z.LE.0..AND.UINF.LT.1.) GO TO 13                               WAV01070
      IF(Z.LE.0..AND.UINF.GE.1.) GO TO 12                               WAV01080
      IF(Z.GE.0..AND.UINF.GE.1.) GO TO 13                               WAV01090
      GO TO 20                                                          WAV01100
 1    IF(UINF.GT.1.) GO TO 15                                           WAV01110
      GO TO 16                                                          WAV01120
 17   IF(UINF.GE.1.AND.(Z1.GT.0..OR.PT1.LE.0.)) DELT=DELT/2.            WAV01130
      IF(UINF.LT.1.AND.Z1.LE.0.AND.PT1.LE.0.) DELT=DELT/2.              WAV01140
      N=N+1                                                             WAV01150
      IF(N.GT.500) GO TO 20                                             WAV01160
 15   XHII=XHII+DELT                                                    WAV01170
      GO TO 18                                                          WAV01180
 19   IF(UINF.GE.1.AND.Z1.LE.0.AND.PT1.GT.0.) DELT=DELT/2.              WAV01190
      IF(UINF.LT.1.AND.(Z1.GT.0..OR.PT1.GT.0.)) DELT=DELT/2.            WAV01200
      N=N+1                                                             WAV01210
      IF(N.GT.500) GO TO 20                                             WAV01220
```

```
 16   XHII=XHII-DELT                                                    WAVO1230
 18   Z1=Z                                                              WAVO1240
      PT1=PTE                                                           WAVO1250
      Z=(XHII-1.)*(XHII-X1)*(XHII-X2)                                   WAVO1260
      Z=(HH-1.)*(XHII**2)-Z                                             WAVO1270
      ER=ABS(Z)                                                         WAVO1280
      IF(ER.LE.EPSI) GO TO 20                                           WAVO1290
      PTE=2.*(HH-1.)*XHII-(XHII-X1)*(XHII-X2)-(XHII-1.)*(2.*XHI         WAVO1300
     SI-X1-X2)                                                          WAVO1310
      IF((Z.GT.O..OR.PTE.LE.O.).AND.UINF.GE.1.) GO TO 19                WAVO1320
      IF(Z.LE.O..AND.PTE.GT.O..AND.UINF.GE.1.) GO TO 17                 WAVO1330
      IF(Z.LE.O..AND.PTE.LE.O..AND.UINF.LT.1.) GO TO 19                 WAVO1340
      IF((Z.GT.O..OR.PTE.GT.O.).AND.UINF.LT.1.) GO TO 17                WAVO1350
 20   RETURN                                                            WAVO1360
      END                                                               WAVO1370
      SUBROUTINE WAVENU(OME,UINF,XHII,KAA)                              WAVO1380
      REAL K1,K2,KAA,K1INF,K2INF                                        WAVO1390
      EPSI=0.0001                                                       WAVO1400
      X1=1.                                                             WAVO1410
      X2=1.                                                             WAVO1420
      X3=0.                                                             WAVO1430
C     IF X1=1.(RES.-1.),X2=1.(RES.O.),X3=O.(RES.1.),WE STUD.THE         WAVO1440
C     SMALLEST(RES.LARGEST)ROOT                                         WAVO1450
      UN=UINF/XHII                                                      WAVO1460
      N=O                                                               WAVO1470
C     APPROXI.PLAC.OF ROOTS                                             WAVO1480
      K1=14.                                                            WAVO1490
  1   K1=K1/2.                                                          WAVO1500
      FMG=(OME**2)/K1-2.*UN*OME+(UN**2)*K1-TANH(K1*XHII)                WAVO1510
      PTE=UN**2-(OME/K1)**2-XHII/((COSH(K1*XHII))**2)                   WAVO1520
      IF(FMG.LE.O..OR.PTE.GT.O.) GO TO 1                                WAVO1530
      DELT=K1                                                           WAVO1540
      K2=K1                                                             WAVO1550
  2   FMG=(OME**2)/K2-2.*UN*OME+(UN**2)*K2-TANH(K2*XHII)                WAVO1560
      IF(FMG.LE.O.) GO TO 10                                            WAVO1570
      PTE=UN**2-(OME/K2)**2-XHII/((COSH(K2*XHII))**2)                   WAVO1580
      IF(PTE.GT.O.) GO TO 11                                            WAVO1590
      K2=K2+DELT                                                        WAVO1600
      GO TO 2                                                           WAVO1610
 11   DELT=DELT/2.                                                      WAVO1620
      K2=K2-DELT                                                        WAVO1630
      IF((DELT.LT.O.0005).AND.(UINF.LT.O.))GO TO 900                    WAVO1640
C     K DOESNOT EXIST                                                   WAVO1650
      GO TO 2                                                           WAVO1660
 10   K1INF=K2-DELT                                                     WAVO1670
      FMG=1.                                                            WAVO1680
      IF(X1.GE.O.) GO TO 131                                            WAVO1690
 12   K2=K2+DELT                                                        WAVO1700
      FMG=OME**2/K2-2.*UN*OME+(UN**2)*K2-TANH(K2*XHII)                  WAVO1710
      IF(FMG.LE.O.) GO TO 12                                            WAVO1720
      K2INF=K2                                                          WAVO1730
C     AT THIS PT.THE 2 ROOTS ARE K1(K2)BETW. K1INF AND K1INF+DE         WAVO1740
C     LT.(K2INF)                                                        WAVO1750
 131  K2=X2*K1INF+X3*K2INF                                              WAVO1760
      GO TO 130                                                         WAVO1770
 13   IF(FM1.LE.O.) DELT=DELT/2.                                        WAVO1780
 130  K2=K2+X1*DELT                                                     WAVO1790
      N=N+1                                                             WAVO1800
      IF(N.GT.50) GO TO 1000                                            WAVO1810
      GO TO 15                                                          WAVO1820
 14   IF(FM1.GT.O.) DELT=DELT/2.                                        WAVO1830
      K2=K2-X1*DELT                                                     WAVO1840
      N=N+1                                                             WAVO1850
      IF(N.GT.50) GO TO 1000                                            WAVO1860
 15   FM1=FMG                                                           WAVO1870
      FMG=(OME**2)/K2-2.*UN*OME+(UN**2)*K2-TANH(K2*XHII)                WAVO1880
      ER=ABS(FMG)                                                       WAVO1890
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      IF(FMG.GE.O..AND.ER.GT.EPSI) GO TO 13                              WAV01900
      IF(FMG.LT.O..AND.ER.GT.EPSI) GO TO 14                              WAV01910
      KAA=K2                                                             WAV01920
      GO TO 1000                                                         WAV01930
  900 KAA=-K2-DELT                                                       WAV01940
C     THE WAVE NUMBER IS SET TO BE NEGATIVE FOR STOPPED WAVES            WAV01950
 1000 RETURN                                                             WAV01960
      END                                                                WAV01970
      SUBROUTINE COEFF(NAV,XHI,KA,H,UINF,OME,DX,Y1,Y2,Y3,CGG)            WAV01980
      DIMENSION XHI(1),KA(1),Y1(1),Y2(1),Y3(1),H(1),CGG(1)               WAV01990
      REAL KA,KX1,KX,KH                                                  WAV02000
      DO 1 N=1,NAV                                                       WAV02010
      UN=UINF/XHI(N)                                                     WAV02020
      C=-UN+(OME/KA(N))                                                  WAV02030
      KX=KA(N)*XHI(N)                                                    WAV02040
      BET=TANH(KX)                                                       WAV02050
      CO=COSH(KX)                                                        WAV02060
      SI=SINH(KX)                                                        WAV02070
      CGM=C*O.5*(1.+KX/(SI*CO))                                          WAV02080
      CG=CGM+UN                                                          WAV02090
      CGG(N)=CG                                                          WAV02100
      Y2(N)=1.-XHI(N)*(1.-BET*BET)*(1.-BET*KX)/(CGM**2)                  WAV02110
      Y2(N)=Y2(N)*(CGM**2)*KA(N)/(2.*C*(CG**3))                          WAV02120
      Y3(N)=4.*((C/CGM)**2)+4.*C/(CGM*CO*CO)+XHI(N)/((CGM*CO*CO)**2)     WAV02130
      Y3(N)=-2.*Y3(N)*((BET*CGM)**2)/(XHI(N)-(CGM**2))+9.-10.*(          WAV02140
     SBET**2)+9.*(BET**4)                                                WAV02150
      Y3(N)=KA(N)*Y3(N)/(4.*CG*C*((C*BET)**2))                           WAV02160
      KX1=KA(N+1)*XHI(N+1)                                               WAV02170
      CG1OS=O.5*(1.+KX1/(SINH(KX1)*COSH(KX1)))/KA(N+1)                   WAV02180
      CG1OS=CG1OS+UINF/(XHI(N+1)*(OME-UINF*KA(N+1)/XHI(N+1)))            WAV02190
      Y1(N)=(CG1OS-CG/(C*KA(N)))/DX                                      WAV02200
      Y1(N)=Y1(N)/(CG1OS+CG/C*KA(N))                                     WAV02210
    1 CONTINUE                                                           WAV02220
      RETURN                                                             WAV02230
      END                                                                WAV02240
      SUBROUTINE INIT(JA,DTO,Y21,XX,A,IS,ERR)                            WAV02250
      COMPLEX A(302,1)                                                   WAV02260
      DIMENSION ERR(1)                                                   WAV02270
      IS=1                                                               WAV02280
C     IF IS=1 INIT. PROF.SYM.IN TO                                       WAV02290
C     IF IS=1 GIVE A AT X2=O FOR TO=-DTO TO JA*DTO                       WAV02300
C     IF IS=O INIT.PROF.NOT SYM.IN TO                                    WAV02310
      QO=1.                                                              WAV02320
      W=QO*SQRT(ABS(XX/(2.*Y21)))/O.75                                   WAV02330
      JA2=JA+1                                                           WAV02340
      E=O.                                                               WAV02350
      DO1 I=1,JA2                                                        WAV02360
      R=FLOAT(I-2)                                                       WAV02370
      A(I,1)=CMPLX(QO/COSH(W*R*DTO),O.)                                  WAV02380
      IF(I.LE.2) GO TO 1                                                 WAV02390
      E=E+CABS(A(I,1))**2+CABS(A(I-1,1))**2                              WAV02400
    1 CONTINUE                                                           WAV02410
      ERR(1)=E*DTO/2.                                                    WAV02420
      RETURN                                                             WAV02430
      END                                                                WAV02440
      SUBROUTINE SOLUT(NA,NAV,JA,DX,DTO,Y1,Y2,Y3,IS,A,ERR,ADR,AN)        WAV02450
      COMPLEX AL(500),BE(500),GA(500),W(500),X(500),Y(500),              WAV02460
     SA(302,1)                                                           WAV02470
      DIMENSION Y1(1),Y2(1),Y3(1),ERR(1)                                 WAV02480
C     IF IS=O A IS NOT SYMET.IN TO                                       WAV02490
C     IF IS=1 A IS SYMET.IN TO                                           WAV02500
      JA1=JA-1                                                           WAV02510
      YY=O.                                                              WAV02520
      KK=3-IS                                                            WAV02530
      DO 1 I=2,NA                                                        WAV02540
      E=O.                                                               WAV02550
      A(JA+1,I)=CMPLX(O.,O.)                                             WAV02560
```

```
      IF(IS.EQ.0) A(1,I)=CMPLX(0.,0.)                                   WAV02570
      IF(I.GT.NAV) GO TO 100                                            WAV02580
      Y111=Y1(I-1)                                                      WAV02590
      Y222=Y2(I-1)                                                      WAV02600
      Y333=Y3(I-1)                                                      WAV02610
      Y11=(Y1(I-1)+Y1(I))/2.                                            WAV02620
      Y22=(Y2(I-1)+Y2(I))/2.                                            WAV02630
      Y33=(Y3(I-1)+Y3(I))/2.                                            WAV02640
      GO TO 101                                                         WAV02650
  100 IF(I.GT.(NAV+1)) GO TO 101                                        WAV02660
      Y111=0.                                                           WAV02670
      Y222=Y2(NAV)                                                      WAV02680
      Y333=Y3(NAV)                                                      WAV02690
      Y11=0.                                                            WAV02700
      Y22=Y222                                                          WAV02710
      Y33=Y333                                                          WAV02720
  101 DO 2 J=1,JA1                                                      WAV02730
      S=CABS(A(J+1,I-1))                                                WAV02740
      AL(J)=CMPLX(0.,Y22*DX/(2.*(DTO**2)))                              WAV02750
      GA(J)=AL(J)                                                       WAV02760
      W(J)=A(J+1,I-1)*CMPLX(1.-DX*Y11/2.,DX*Y22/(DTO**2)-Y33*S*         WAV02770
     SS*DX/2.)                                                          WAV02780
      W(J)=W(J)-(A(J+2,I-1)+A(J,I-1))*AL(J)                             WAV02790
      BE(J)=A(J+1,I-1)*CMPLX(1.-DX*Y111,DX*2.*Y222/(DTO**2)-Y33         WAV02800
     S3*S*S*DX)                                                         WAV02810
      BE(J)=BE(J)-(A(J+2,I-1)+A(J,I-1))*CMPLX(0.,Y222*DX/(DTO**         WAV02820
     S2))                                                               WAV02830
      S=CABS(BE(J))                                                     WAV02840
      BE(J)=CMPLX(1.+DX*Y11/2.,-DX*Y22/(DTO**2)+DX*Y33*S*S/2.)          WAV02850
    2 CONTINUE                                                          WAV02860
      X(JA-1)=-AL(JA-1)/BE(JA-1)                                        WAV02870
      Y(JA-1)=W(JA-1)/BE(JA-1)                                          WAV02880
      DO 3 J=3,JA                                                       WAV02890
      K=JA+1-J                                                          WAV02900
      X(K)=-AL(K)/(GA(K)*X(K+1)+BE(K))                                  WAV02910
      Y(K)=(W(K)-GA(K)*Y(K+1))/(GA(K)*X(K+1)+BE(K))                     WAV02920
    3 CONTINUE                                                          WAV02930
      IF(IS.EQ.0) A(2,I)=(W(1)-GA(1)*Y(2))/(GA(1)*X(2)+BE(1))           WAV02940
      IF(IS.EQ.1) A(1,I)=(X(2)*Y(1)+Y(2))/(1.-X(2)*X(1))                WAV02950
      DO 4 J=KK,JA                                                      WAV02960
      A(J,I)=A(J-1,I)*X(J-1)+Y(J-1)                                     WAV02970
      IF(J.LE.2) GO TO 4                                                WAV02980
      E=E+CABS(A(J,I))**2+CABS(A(J-1,I))**2                             WAV02990
    4 CONTINUE                                                          WAV03000
      IF(I.GT.NAV) GO TO 103                                            WAV03010
      YY=YY+Y1(I)+Y1(I-1)                                               WAV03020
  103 ERR(I)=0.5*DTO*E*EXP(DX*YY)                                       WAV03030
    1 CONTINUE                                                          WAV03040
C     AN,ADR ARE THE NUM. VALUE & THE D-R THEORY VALUE OF THE AREA      WAV03050
C     UNDER THE WAVE PROFILE AT THE END OF THE DEPTH VARIATION          WAV03060
      AN=0.                                                             WAV03070
      ADR=3.14159*SQRT(2.*Y2(1)/(ABS(Y3(1))*EXP(DX*YY)))                WAV03080
      JA1=JA+1                                                          WAV03090
      DO5 J=3,JA1                                                       WAV03100
      SV=CABS(A(J,NAV))+CABS(A(J-1,NAV))                                WAV03110
      AN=AN+SV                                                          WAV03120
    5 CONTINUE                                                          WAV03130
      AN=AN*DTO                                                         WAV03140
      RETURN                                                            WAV03150
      END                                                               WAV03160
      SUBROUTINE IMPRES(NA,NAV,KA,XHI,Y2,Y3,ERR,ADR,AN)                 WAV03170
      DIMENSION Y2(1),Y3(1),ERR(1),XHI(1)                               WAV03180
      REAL KA(1),KH1,K1,KH2,K2                                          WAV03190
      K1=Y3(1)/Y2(1)                                                    WAV03200
      KH1=XHI(1)*KA(1)                                                  WAV03210
      K2=Y3(NAV)/Y2(NAV)                                                WAV03220
      K2=K2*((ADR/3.14159)**2)*ABS(K1)/2.                               WAV03230
```

```
C       PLOT THE WAVE AMPL.                                              WPL00460
        DO1 I=2,JA1,IDELT                                                WPL00470
        I1=I-2                                                           WPL00480
        XO=-FLOAT(I1)*SI*COS(AL)                                         WPL00490
        YO=-FLOAT(I1)*SI*SIN(AL)+CABS(A(I,1))*1.4                        WPL00500
        CALL PLOT(XO,YO,3)                                               WPL00510
        DO2 J=1,NA                                                       WPL00520
        J1=J-1                                                           WPL00530
        XP=XO+FLOAT(J1)*SJ*COS(AL)                                       WPL00540
        YP=-FLOAT(I1)*SI*SIN(AL)-FLOAT(J1)*SJ*SIN(AL)                    WPL00550
        YP=YP+CABS(A(I,J))*1.4                                           WPL00560
        CALL PLOT(XP,YP,2)                                               WPL00570
 2      CONTINUE                                                         WPL00580
 1      CONTINUE                                                         WPL00590
        DO3 J=1,NA,JDELT                                                 WPL00600
        J1=J-1                                                           WPL00610
        XO=FLOAT(J1)*SJ*COS(AL)                                          WPL00620
        YO=-FLOAT(J1)*SJ*SIN(AL)+CABS(A(2,J))*1.4                        WPL00630
        CALL PLOT(XO,YO,3)                                               WPL00640
        DO4 I=2,JA1                                                      WPL00650
        I1=I-2                                                           WPL00660
        XP=XO-FLOAT(I1)*SI*COS(AL)                                       WPL00670
        YP=-FLOAT(J1)*SJ*SIN(AL)-FLOAT(I1)*SI*SIN(AL)                    WPL00680
        YP=YP+CABS(A(I,J))*1.4                                           WPL00690
        CALL PLOT(XP,YP,2)                                               WPL00700
 4      CONTINUE                                                         WPL00710
 3      CONTINUE                                                         WPL00720
C       PLOT THE WAVE AMPL. AND ARG. AT XO=(IN(I)-1)*DX                  WPL00730
C       NU:NUMBER OF CURVES                                              WPL00740
C       THE CURVES ARE PLOTTED FROM T=0 TO T=ISTOP                       WPL00750
C       D1:SUBDIVISION OF THE T AXIS                                     WPL00760
C       WX:LENGTH OF THE T AXIS                                          WPL00770
        NU=3                                                             WPL00780
        IN(1)=1                                                          WPL00790
        IN(2)=NAV                                                        WPL00800
        WRITE(10,300)                                                    WPL00810
 300    FORMAT(5X,'ISTOP,IN(3),D1')                                      WPL00820
        READ(5,*)ISTOP,IN(3),D1                                          WPL00830
        WX=(ISTOP-2)*DT/D1                                               WPL00840
        CALL PLOT(-9.5,-9.,-3)                                           WPL00850
C       PLOT THE WAVE ARG. AT XO                                         WPL00860
C       SUB. LIN PLOTS HOR. AND VERT. AXIS                               WPL00870
        XI=0.5+WX                                                        WPL00880
        CALL AXIS(0.5,1.5,'  ',2,4.,90.,-1.,0.5)                         WPL00890
        CALL AXIS(0.5,1.5,'  ',-2,WX,0.,0.,D1)                           WPL00900
        CALL LIN(0.5,5.5,WX,0.)                                          WPL00910
        CALL LIN(0.5,3.5,WX,0.)                                          WPL00920
        CALL LIN(XI,1.5,4.,90.)                                          WPL00930
        DO5 J=1,NU                                                       WPL00940
        XP=0.5                                                           WPL00950
        W=AIMAG(A(2,IN(J)))                                              WPL00960
        S=CABS(A(2,IN(J)))                                               WPL00970
        YP=ARCOS(REAL(A(2,IN(J))/S))*SIGN(1.,W)                          WPL00980
        YP=(2*YP)/3.14159+3.5                                            WPL00990
        CALL PLOT(XP,YP,3)                                               WPL01000
        DO6 I=2,ISTOP                                                    WPL01010
        XP=0.5+(I-2)*WX/(ISTOP-2)                                        WPL01020
        W=AIMAG(A(I,IN(J)))                                              WPL01030
        S=CABS(A(I,IN(J)))                                               WPL01040
        IF(S.LT.0.001)GO TO 5                                            WPL01050
        YP=ARCOS(REAL(A(I,IN(J))/S))*SIGN(1.,W)                          WPL01060
        YP=(2*YP)/3.14159+3.5                                            WPL01070
        CALL PLOT(XP,YP,2)                                               WPL01080
 6      CONTINUE                                                         WPL01090
 5      CONTINUE                                                         WPL01100
```

```
      KH2=XHI(NAV)*KA(NAV)                                              WAV03240
      WRITE(10,100)K1,KH1,K2,KH2                                        WAV03250
100   FORMAT(5X,'K1=',E12.4,2X,'KH1=',E12.4,/,5X,'K2=',                 WAV03260
     &E12.4,2X,'KH2=',E12.4)                                            WAV03270
      WRITE(10,200)ADR,AN                                               WAV03280
      DO3 J=1,NA,20                                                     WAV03290
      WRITE(10,300)J,ERR(J)                                             WAV03300
3     CONTINUE                                                          WAV03310
200   FORMAT(2X,'D-R=',E12.4,2X,'NUM=',E12.4)                           WAV03320
300   FORMAT(5X,'QUANTITY CONSER.AT(',I3,')=',E12.4)                    WAV03330
1000  RETURN                                                            WAV03340
      END                                                               WAV03350
```

FILE: WPLOT FORTRAN A

```
@PROCESS SC(AXIS)                                                       WPL00010
      SUBROUTINE WPLOT(NAV,NA,JA,DT,DX,A)                               WPL00020
C     3-DIMENSIONAL PLOT OF AMPL. A(T,X)                                WPL00030
      DIMENSION IN(4)                                                   WPL00040
      COMPLEX A(302,1)                                                  WPL00050
      WRITE(10,100)                                                     WPL00060
 100  FORMAT(5X,'DELX,DELT,JDELT,IDELT')                                WPL00070
      READ(5,*)DELX,DELT,JDELT,IDELT                                    WPL00080
C     DELX,DELT:SUBDIVISION OF X AND T AXIS                             WPL00090
C     JDELT,IDELT:X AND T PLOTTING STEPS                                WPL00100
C     WX,WY:LENGTH OF T AND X AXIS                                      WPL00110
C     AL:PROJECTION ANGLE                                               WPL00120
      JA1=JA+1                                                          WPL00130
      WX=JA*DT/DELT                                                     WPL00140
      WY=(NA-1)*DX/DELX                                                 WPL00150
      JA2=JA-1                                                          WPL00160
      NA2=NA-1                                                          WPL00170
      SI=WX/FLOAT(JA2)                                                  WPL00180
      SJ=WY/FLOAT(NA2)                                                  WPL00190
      AL=35.*3.1416/180.                                                WPL00200
      XO=9.5                                                            WPL00210
      YO=9.                                                             WPL00220
      CALL PLOTS(120,IDUM2,08)                                          WPL00230
      CALL FACTOR(0.6)                                                  WPL00240
      CALL PLOT(XO,YO,-3)                                               WPL00250
C     PLOT THE AXIS                                                     WPL00260
      XO=-JA2*SI*COS(AL)                                                WPL00270
      YO=-JA2*SI*SIN(AL)                                                WPL00280
      CALL PLOT(XO,YO,2)                                                WPL00290
      CALL PLOT(0.,0.,3)                                                WPL00300
      XP=NA2*SJ*COS(AL)                                                 WPL00310
      YP=-NA2*SJ*SIN(AL)                                                WPL00320
      CALL PLOT(XP,YP,2)                                                WPL00330
      XP=XO+WY*COS(AL)                                                  WPL00340
      YP=YO-WY*SIN(AL)                                                  WPL00350
      CALL AXIS(XO,YO,'  ',-2,WY,-35.,0.,DELX)                          WPL00360
      C=WX*DELT                                                         WPL00370
      B=-DELT                                                           WPL00380
      CALL AXIS(XP,YP,'  ',-2,WX,35.,C,B)                               WPL00390
      CALL PLOT(0.,0.,3)                                                WPL00400
      CALL PLOT(0.,2.8,2)                                               WPL00410
      CALL PLOT(0.,1.4,3)                                               WPL00420
      CALL PLOT(0.1,1.4,2)                                              WPL00430
      CALL PLOT(0.,2.8,3)                                               WPL00440
      CALL PLOT(0.1,2.8,2)                                              WPL00450
```

```
C       PLOT THE WAVE AMPL. AT XO                                       WPLO1110
        CALL AXIS(0.5,7.2,'  ',-2,WX,0.,0.,D1)                          WPLO1120
        CALL AXIS(0.5,7.2,'  ',2,4.,90.,0.,0.5)                         WPLO1130
        CALL LIN(0.5,11.2,WX,0.)                                        WPLO1140
        CALL LIN(XI,7.2,4.,90.)                                         WPLO1150
        DO7 J=1,NU                                                      WPLO1160
        XP=0.5                                                          WPLO1170
        YP=CABS(A(2,IN(J)))/0.5+7.2                                     WPLO1180
        CALL PLOT(XP,YP,3)                                              WPLO1190
        DO8 I=2,ISTOP                                                   WPLO1200
        XP=0.5+(I-2)*WX/(ISTOP-2)                                       WPLO1210
        YP=CABS(A(I,IN(J)))/0.5+7.2                                     WPLO1220
        CALL PLOT(XP,YP,2)                                              WPLO1230
 8      CONTINUE                                                        WPLO1240
 7      CONTINUE                                                        WPLO1250
        CALL ENDPLT(15.,0.,999)                                         WPLO1260
        STOP                                                            WPLO1270
        END                                                             WPLO1280
```

FILE: LIN FORTRAN A

```
        SUBROUTINE LIN(XO,YO,XL,XA)                                     LIN00010
C       PLOTS HOR. AND VERT. AXIS                                       LIN00020
C       XO,YO=COORD. OF AXIS                                            LIN00030
C       XL=LENGTH OF AXIS                                               LIN00040
C       XA=ANGLE OF AXIS (XA=0. OR 90.)                                 LIN00050
        CALL PLOT(XO,YO,3)                                              LIN00060
        N=INT(XL)+1                                                     LIN00070
        IF(ABS(XA-90.).LT.0.1)GO TO 2                                   LIN00080
        X1=XO+XL                                                        LIN00090
        CALL PLOT(X1,YO,2)                                              LIN00100
        Y2=YO-0.05                                                      LIN00110
        Y1=YO+0.05                                                      LIN00120
        X1=XO                                                           LIN00130
        DO1 I=1,N                                                       LIN00140
        CALL PLOT(X1,Y2,3)                                              LIN00150
        CALL PLOT(X1,Y1,2)                                              LIN00160
        X1=XO+I                                                         LIN00170
 1      CONTINUE                                                        LIN00180
        GO TO 3                                                         LIN00190
 2      Y1=YO+XL                                                        LIN00200
        CALL PLOT(XO,Y1,2)                                              LIN00210
        X2=XO-0.05                                                      LIN00220
        X1=XO+0.05                                                      LIN00230
        Y1=YO                                                           LIN00240
        DO3 I=1,N                                                       LIN00250
        CALL PLOT(X2,Y1,3)                                              LIN00260
        CALL PLOT(X1,Y1,2)                                              LIN00270
        Y1=YO+I                                                         LIN00280
 3      CONTINUE                                                        LIN00290
        RETURN                                                          LIN00300
        END                                                             LIN00310
```

Appendix B1

The lower order terms appearing in eqs. (2.12) are listed below

$$R_{10} = F_{10} = G_{10} = H_{10} = 0$$

$$R_{20} = F_{20} = G_{20} = 0$$

$$H_{20} = \frac{\omega^2 a^2}{4 \sinh^2 kh} + \phi_{10_T}$$

$$R_{21} = -2i \vec{k}.\nabla\phi_{11} - i(\nabla.\vec{k})\phi_{11}$$

$$F_{21} = -\nabla h.(i\vec{k}\,\phi_{11})_{z=-h}$$

$$G_{21} = 2i\omega\phi_{11_T}\big|_{z=0}$$

$$H_{21} = \phi_{11_T}\big|_{z=0}$$

$$R_{30} = -\nabla^2 \phi_{10}$$

$$F_{30} = -\nabla h.\nabla\phi_{10}\big|_{z=-h}$$

$$G_{30} = g\,\eta_{20_T} - 2\nabla.(\vec{k}\,\omega\phi_{11}^2)\big|_{z=0}$$

Appendix B2

FILE: FORCED FORTRAN A

```
@PROCESS SC(AXIS)                                                    FOR00010
C     ***********************************************************    FOR00020
C     *      COMPUTE FORCED WAVES ON VARIABLE DEPTH             *    FOR00030
C     *                    USING                                *    FOR00040
C     *          HYBRID FINITE-ELEMENT METHOD                 *    FOR00050
C     ***********************************************************    FOR00060
C                                                                    FOR00070
C     IP=1 THE PRG. COMPUTES & PLOTS THE FORCED WAVE AMPL.           FOR00080
C     FOR N2(<4) DIFF. ANGLES OF INCIDENCE ALO;SAME LENGTH OF        FOR00090
C     TRANSITION ZONE=XLO(DXL=O)                                     FOR00100
C     IP=O THE PRG. COMPUTES THE AMPL. OF THE WAVES AT THE ENDS OF   FOR00110
C     THE REGION OF VAR. DEPTH;THE LENGTH OF THIS ZONE INCREASES     FOR00120
C     FROM XLO WITH INCR. DXL;ANGLE OF INCID. CONST.=ALO(IAL)        FOR00130
C     H=DEPTH,KA=WAVENUMBER,CG=GROUP VEL.,                           FOR00140
C     AL=DIR. OF PROP. OF SHORT WAVES                                FOR00150
C     ALO=DIR. OF PROP. AT REF. DEPTH                                FOR00160
C     KL=ENVELOPE WAVENUMBER IN X-DIR.                               FOR00170
C     A,ET= FORCED WAVE ,AML,AMR= AMPL. ON LEFT & RIGHT HAND SIDE    FOR00180
C     OF VARYING DEPTH REGION                                        FOR00190
C     Z=DIFF. MEAN SET-DOWN AND LOCKED WAVE AMPL.                    FOR00200
C     LA,LAM=COEF. OF GOV. EQ.                                       FOR00210
C     ID=O :LINEAR DEPTH VAR., ID=1 : COSINE DEPTH VAR.              FOR00220
C     XL=LENGTH OF TRANSITION REGION                                 FOR00230
C     XHO=REF. DEPTH, DH= AMPL. OF DEPTH VAR.                        FOR00240
C     DX=DISCR. LENGTH, NA=NUMB. OF INT.                             FOR00250
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                        FOR00260
      REAL*8 H(1005),KA(1005),CG(1005),AL(1005),KL(1005)             FOR00270
      REAL*8 LA(1005),LAM(4,1005),AML(1005),AMR(1005),ALO(4),KAA     FOR00280
      COMPLEX*16  F(1005),A(1005),ET(4,1005)                         FOR00290
      WRITE(10,200)                                                  FOR00300
      READ(5,*)IP,ID,IAL,XLO,DXL,XHO,DH,N2                           FOR00310
  200 FORMAT(5X,'IP(O NO PLOT,1 PLOT),ID(O LIN.DEPTH,1 COS.)',/,     FOR00320
     &5X,'IAL(ANGLE),XLO,DXL,XHO,DH,N2')                             FOR00330
      DX=0.02                                                        FOR00340
      ALO(1)=O.                                                      FOR00350
      ALO(2)=3.14159DO/6.                                            FOR00360
      ALO(3)=3.14159DO/4.                                            FOR00370
      ALO(4)=3.14159DO/3.                                            FOR00380
      H(1)=XHO                                                       FOR00390
      K=1                                                            FOR00400
      IF(IP.EQ.O)AL(1)=ALO(IAL)                                      FOR00410
      IF(IP.EQ.1)AL(1)=ALO(1)                                        FOR00420
   7  CONTINUE                                                       FOR00430
      XL=XLO+(K-1)*DXL                                               FOR00440
      NA=3+IDINT(XL/DX)                                              FOR00450
      CALL DEPTH(ID,H,NA,DH)                                         FOR00460
C     GIVES THE DEPTH VAR.                                           FOR00470
      DO1 I=1,NA                                                     FOR00480
      HH=H(I)                                                        FOR00490
      CALL WAVENU(HH,KAA)                                            FOR00500
C     GIVES THE WAVENUMBER                                           FOR00510
      KA(I)=KAA                                                      FOR00520
      CG(I)=(1.+2.*KA(I)*HH/DSINH(2.*KA(I)*HH))/(2.*KA(I))           FOR00530
   1  CONTINUE                                                       FOR00540
C     IF=O NO CAUSTIC, IF>O THERE IS A CAUSTIC                       FOR00550
```

```
      IF=0                                                              FOR00560
 8    CONTINUE                                                          FOR00570
      CALL FORC(IF,H,KA,AL,LA,DX,NA,F,CG)                               FOR00580
C     COMPUTES THE FORCING FCT.                                         FOR00590
      IF(IF.GT.1)GO TO 10                                               FOR00600
      CALL SOLUT(H,LA,DX,NA,F,A)                                        FOR00610
C     SOLVES THE LINEAR SET OF EQS.                                     FOR00620
      IF(IP.EQ.0)GO TO 6                                                FOR00630
      DO3 I=1,NA                                                        FOR00640
      ET(K,I)=A(I)                                                      FOR00650
      LAM(K,I)=LA(I)                                                    FOR00660
 3    CONTINUE                                                          FOR00670
      K=K+1                                                             FOR00680
      IF(K.LE.N2)AL(1)=ALO(K)                                           FOR00690
      IF(K.LE.N2)GO TO 8                                                FOR00700
      CALL DRAW(XL,N2,NA,DX,ET,LAM)                                     FOR00710
C     PLOTS THE WAVE AMPL. AND PHASE                                    FOR00720
      GO TO 1000                                                        FOR00730
 6    AML(K)=CDABS(A(1))                                                FOR00740
      AMR(K)=CDABS(A(NA))                                               FOR00750
      K=K+1                                                             FOR00760
      IF(K.LE.N2)GO TO 7                                                FOR00770
      WRITE(7,100)(AML(K),K=1,N2),(AMR(K),K=1,N2)                       FOR00780
      GO TO 1000                                                        FOR00790
 10   WRITE(10,400)                                                     FOR00800
      WRITE(10,401)IF,H(IF),AL(1)                                       FOR00810
 400  FORMAT(6X,'CAUSTIC')                                              FOR00820
 401  FORMAT(2X,'I=',I4,3X,'H(I)=',E10.4,3X,'AL(1)=',E10.4)             FOR00830
 1000 CONTINUE                                                          FOR00840
 100  FORMAT(4(2X,E10.4))                                               FOR00850
      STOP                                                              FOR00860
      END                                                               FOR00870
      SUBROUTINE DEPTH(ID,H,NA,DH)                                      FOR00880
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                           FOR00890
      REAL*8 H(1005)                                                    FOR00900
      N=NA-1                                                            FOR00910
      IF(ID.EQ.0)GO TO 3                                                FOR00920
      DO1 I=2,N                                                         FOR00930
      H(I)=H(1)+DH*(1.-DCOS(2.*3.14158DO*(I-2)/(N-2)))/2.               FOR00940
 1    CONTINUE                                                          FOR00950
      H(NA)=H(N)                                                        FOR00960
      GO TO 1000                                                        FOR00970
 3    DO2 I=2,N                                                         FOR00980
      H(I)=H(1)+DH*(I-2)/(N-2)                                          FOR00990
 2    CONTINUE                                                          FOR01000
      H(NA)=H(N)                                                        FOR01010
 1000 CONTINUE                                                          FOR01020
      RETURN                                                            FOR01030
      END                                                               FOR01040
      SUBROUTINE WAVENU(HH,KAA)                                         FOR01050
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                           FOR01060
      REAL*8 K1,K2,KAA,K1INF,K2INF                                      FOR01070
      OME=1.                                                            FOR01080
      EPSI=1.E-8                                                        FOR01090
      X1=1.                                                             FOR01100
      X2=1.                                                             FOR01110
      X3=0.                                                             FOR01120
C     IF X1=1.(RES.-1.),X2=1.(RES.0.),X3=0.(RES.1.),WE STUD.THE         FOR01130
C     SMALLEST(RES.LARGEST)ROOT                                         FOR01140
      N=0                                                               FOR01150
C     APPROXI.PLAC.OF ROOTS                                             FOR01160
      K1=28.                                                            FOR01170
 1    K1=K1/2.                                                          FOR01180
      FMG=(OME**2)/K1-DTANH(K1*HH)                                      FOR01190
      PTE=-(OME/K1)**2-HH/((DCOSH(K1*HH))**2)                           FOR01200
      IF(FMG.LE.0..OR.PTE.GT.0.) GO TO 1                                FOR01210
      DELT=K1                                                           FOR01220
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      K2=K1                                                              FORO1230
 2    FMG=(OME**2)/K2-DTANH(K2*HH)                                       FORO1240
      IF(FMG.LE.O.) GO TO 10                                             FORO1250
      PTE=-(OME/K2)**2-HH/((DCOSH(K2*HH))**2)                            FORO1260
      IF(PTE.GT.O.) GO TO 11                                             FORO1270
      K2=K2+DELT                                                         FORO1280
      GO TO 2                                                            FORO1290
 11   DELT=DELT/2.                                                       FORO1300
      K2=K2-DELT                                                         FORO1310
      GO TO 2                                                            FORO1320
 10   K1INF=K2-DELT                                                      FORO1330
      FMG=1.                                                             FORO1340
      IF(X1.GE.O.) GO TO 131                                             FORO1350
 12   K2=K2+DELT                                                         FORO1360
      FMG=OME**2/K2-DTANH(K2*HH)                                         FORO1370
      IF(FMG.LE.O.) GO TO 12                                             FORO1380
      K2INF=K2                                                           FORO1390
C     AT THIS PT.THE 2 ROOTS ARE K1(K2)BETW. K1INF AND K1INF+DE          FORO1400
C     LT.(K2INF)                                                         FORO1410
 131  K2=X2*K1INF+X3*K2INF                                               FORO1420
      GO TO 130                                                          FORO1430
 13   IF(FM1.LE.O.) DELT=DELT/2.                                         FORO1440
 130  K2=K2+X1*DELT                                                      FORO1450
      N=N+1                                                              FORO1460
      IF(N.GT.150) GO TO 1000                                            FORO1470
      GO TO 15                                                           FORO1480
 14   IF(FM1.GT.O.) DELT=DELT/2.                                         FORO1490
      K2=K2-X1*DELT                                                      FORO1500
      N=N+1                                                              FORO1510
      IF(N.GT.150) GO TO 1000                                            FORO1520
 15   FM1=FMG                                                            FORO1530
      FMG=(OME**2)/K2-DTANH(K2*HH)                                       FORO1540
      ER=DABS(FMG)                                                       FORO1550
      IF(FMG.GE.O..AND.ER.GT.EPSI) GO TO 13                              FORO1560
      IF(FMG.LT.O..AND.ER.GT.EPSI) GO TO 14                              FORO1570
      KAA=K2                                                             FORO1580
 1000 RETURN                                                             FORO1590
      END                                                                FORO1600
      SUBROUTINE FORC(IF,H,KA,AL,LA,DX,NA,F,CG)                          FORO1610
C     KY=WAVENUMBER IN Y-DIR.                                            FORO1620
C     Z=DIFF. MEAN SET-DOWN AND LOCKED WAVE AMPL.                        FORO1630
C     XNU=DEF. RAIO OF FREQ. OF WAVE ENVEL. & SHORT WAVES                FORO1640
C     F=FORCING FCT. TIMES DX                                            FORO1650
C     THE MATCHING COND. ARE APPLIED AT X=DX & X=(NA-1)*DX               FORO1660
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER(I-N)                             FORO1670
      REAL*8 KA(1005),H(1005),CG(1005),Z(1005)                           FORO1680
      REAL*8 KL(1005),AL(1005),LA(1005)                                  FORO1690
      REAL*8 KY                                                          FORO1700
      COMPLEX*16 F(1005)                                                 FORO1710
      DO1 I=1,NA                                                         FORO1720
      KY=KA(1)*DSIN(AL(1))                                               FORO1730
      W=KY/KA(I)                                                         FORO1740
      IF(W.GE.1)GO TO 6                                                  FORO1750
      AL(I)=DARSIN(W)                                                    FORO1760
      KL(I)=CG(1)*KA(1)/(CG(I)*DCOS(AL(I)))                              FORO1770
      KL(I)=KL(I)-KY*DTAN(AL(I))                                         FORO1780
      Z(I)=2.*KA(I)*H(I)*((KL(I)**2)+(KY**2))/DSINH(KA(I)*H(I)*2.)       FORO1790
      Z(I)=Z(I)+2.*KA(I)*((CG(1)*KA(1))**2)/CG(I)                        FORO1800
      Z(I)=-Z(I)/(((CG(1)*KA(1))**2)-H(I)*((KL(I)**2)+(KY**2)))          FORO1810
      Z(I)=Z(I)-1./(DSINH(KA(I)*H(I))**2)                                FORO1820
      Z(I)=Z(I)*CG(1)*DCOS(AL(1))/(8.*CG(I)*DCOS(AL(I)))                 FORO1830
 1    CONTINUE                                                           FORO1840
      XNU=1./(CG(1)*KA(1))                                               FORO1850
      DO8 I=1,NA                                                         FORO1860
      LA(I)=(CG(1)**2-H(I)*(DSIN(AL(1))**2))*(KA(1)**2)                  FORO1870
      LA(I)=LA(I)*(XNU**2)                                               FORO1880
 8    CONTINUE                                                           FORO1890
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      TETA=0.                                                          FORO1900
      N=NA-1                                                           FORO1910
      DO2 I=2,N                                                        FORO1920
      IF(I.EQ.2)GO TO 3                                                FORO1930
      IF(I.EQ.N)GO TO 4                                                FORO1940
      DZ=(Z(I+1)-Z(I-1))/(2.*DX)                                       FORO1950
      DDZ=(Z(I+1)-2.*Z(I)+Z(I-1))/((DX**2))                            FORO1960
      DH=(H(I+1)-H(I-1))/(2.*DX)                                       FORO1970
      DA=KL(I)*DH+H(I)*(KL(I+1)-KL(I-1))/(2.*DX)                       FORO1980
      DB=(KA(I+1)-KA(I-1))-KA(I)*(CG(I+1)-CG(I-1))/CG(I)               FORO1990
      DB=DB/(2.*DX*CG(I))                                              FORO2000
      GO TO 5                                                          FORO2010
3     DZ=(Z(I+1)-Z(I))/DX                                              FORO2020
      DDZ=(Z(I+2)-2.*Z(I+1)+Z(I))/(DX**2)                              FORO2030
      DH=(H(I+1)-H(I))/DX                                              FORO2040
      DA=DH*KL(I)+H(I)*(KL(I+1)-KL(I))/DX                              FORO2050
      DB=(KA(I+1)-KA(I))-KA(I)*(CG(I+1)-CG(I))/CG(I)                   FORO2060
      DB=DB/(DX*CG(I))                                                 FORO2070
      GO TO 5                                                          FORO2080
4     DZ=(Z(I)-Z(I-1))/DX                                              FORO2090
      DDZ=(Z(I)-2.*Z(I-1)+Z(I-2))/(DX**2)                              FORO2100
      DH=(H(I)-H(I-1))/DX                                              FORO2110
      DA=DH*KL(I)+H(I)*(KL(I)-KL(I-1))/DX                              FORO2120
      DB=(KA(I)-KA(I-1))-KA(I)*(CG(I)-CG(I-1))/CG(I)                   FORO2130
      DB=DB/(DX*CG(I))                                                 FORO2140
5     X1=DDZ*H(I)+DZ*DH                                                FORO2150
      X2=DA*Z(I)+2.*H(I)*KL(I)*DZ-DB*(CG(1)**2)*KA(1)*DCOS(AL(1))/4.   FORO2160
      X2=2.*XNU*X2                                                     FORO2170
      F(I)=DCMPLX(X1,X2)*DCMPLX(DCOS(TETA),DSIN(TETA))*DX              FORO2180
      IF((I.EQ.2).OR.(I.EQ.N))F(I)=F(I)/2.                             FORO2190
      TETA=TETA+(KL(I)+KL(I+1))*XNU*DX                                 FORO2200
2     CONTINUE                                                         FORO2210
      F(1)=0.                                                          FORO2220
      F(NA)=0.                                                         FORO2230
      GO TO 1000                                                       FORO2240
6     IF=I                                                             FORO2250
1000  CONTINUE                                                         FORO2260
      RETURN                                                           FORO2270
      END                                                              FORO2280
      SUBROUTINE SOLUT(H,LA,DX,NA,F,A)                                 FORO2290
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                          FORO2300
      REAL*8 H(1005),LA(1005)                                          FORO2310
      COMPLEX*16 F(1005),A(1005)                                       FORO2320
      COMPLEX*16 W(1005),X(1005),Y(1005)                               FORO2330
      COMPLEX*16 GA(1005),BE(1005),AL(1005)                            FORO2340
      COMPLEX*16 S1,SN                                                 FORO2350
      N=NA-1                                                           FORO2360
      IF(LA(1).GE.0.)S1=DCMPLX(0.0D0,1.0D0)                            FORO2370
      IF(LA(1).LT.0.)S1=DCMPLX(-1.0D0,0.0D0)                           FORO2380
      IF(LA(NA).GE.0.)SN=DCMPLX(0.0D0,1.0D0)                           FORO2390
      IF(LA(NA).LT.0.)SN=DCMPLX(-1.0D0,0.0D0)                          FORO2400
      GA(1)=(2.*DX*LA(1)/3.+H(1)/DX)                                   FORO2410
      BE(1)=(4.*DX*LA(1)/3.-H(1)/DX)                                   FORO2420
      BE(1)=BE(1)+2.*S1*DSQRT(H(1)*DABS(LA(1)))                        FORO2430
      BE(NA)=(4.*DX*LA(NA)/3.-H(NA)/DX)                                FORO2440
      BE(NA)=BE(NA)+2.*SN*DSQRT(H(NA)*DABS(LA(NA)))                    FORO2450
      AL(NA)=(2.*DX*LA(NA)/3.+H(NA)/DX)                                FORO2460
      W(1)=0.                                                          FORO2470
      W(NA)=0.                                                         FORO2480
      DO1 I=2,N                                                        FORO2490
      GA(I)=(LA(I+1)+LA(I))*DX/3.+(H(I+1)+H(I))/(2.*DX)                FORO2500
      BE(I)=2.*(LA(I+1)+2.*LA(I)+LA(I-1))*DX/3.                        FORO2510
      BE(I)=BE(I)-(H(I+1)+2.*H(I)+H(I-1))/(2.*DX)                      FORO2520
      AL(I)=(LA(I)+LA(I-1))*DX/3.+(H(I)+H(I-1))/(2.*DX)                FORO2530
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      W(I)=F(I)                                                          FORO2540
 1    CONTINUE                                                           FORO2550
      X(N)=-AL(NA)/BE(NA)                                                FORO2560
      Y(N)=W(NA)/BE(NA)                                                  FORO2570
      NB=N-1                                                             FORO2580
      DO2 I=1,NB                                                         FORO2590
      I1=N-I                                                             FORO2600
      I2=I1+1                                                            FORO2610
      X(I1)=-AL(I2)/(GA(I2)*X(I2)+BE(I2))                                FORO2620
      Y(I1)=(-GA(I2)*Y(I2)+W(I2))/(GA(I2)*X(I2)+BE(I2))                  FORO2630
 2    CONTINUE                                                           FORO2640
      A(1)=(W(1)-GA(1)*Y(1))/(GA(1)*X(1)+BE(1))                          FORO2650
      DO3 I=1,N                                                          FORO2660
      A(I+1)=A(I)*X(I)+Y(I)                                              FORO2670
 3    CONTINUE                                                           FORO2680
      RETURN                                                             FORO2690
      END                                                                FORO2700
@PROCESS SC(AXIS)                                                        FORO2710
      SUBROUTINE DRAW(XL,N2,NA,DX,ET,LAM)                                FORO2720
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER(I-N)                             FORO2730
      COMPLEX*16 ET(4,1005)                                              FORO2740
      REAL*8 LAM(4,1005)                                                 FORO2750
      CALL PLOTS(120,IDUM2,08)                                           FORO2760
      CALL FACTOR(0.5)                                                   FORO2770
      XO=1.                                                              FORO2780
      YO=1.5                                                             FORO2790
      N=NA-1                                                             FORO2800
      DEL=N*DX/5.                                                        FORO2810
      DEL=IDINT(10.*DEL)/10.                                             FORO2820
      DO6 K=1,N2                                                         FORO2830
      X1=XO+5.                                                           FORO2840
      Y1=YO+4.                                                           FORO2850
      CALL AXIS(XO,YO,'  ',-2,5.,0.,0.,DEL)                              FORO2860
      CALL AXIS(XO,YO,'  ',2,4.,90.,0.,0.5)                              FORO2870
      CALL LIN(XO,Y1,5.,0.)                                              FORO2880
      CALL LIN(X1,YO,4.,90.)                                             FORO2890
      DO2 I=2,N                                                          FORO2900
      XP=XO+(I-2)*DX/DEL                                                 FORO2910
      YP=YO+CDABS(ET(K,I))/0.5                                           FORO2920
      IF(I.EQ.2)CALL PLOT(XP,YP,3)                                       FORO2930
      CALL PLOT(XP,YP,2)                                                 FORO2940
 2    CONTINUE                                                           FORO2950
      DO3 I=2,N                                                          FORO2960
      S=LAM(K,I)*LAM(K,I-1)                                              FORO2970
      IF(S.GT.0.)GO TO 3                                                 FORO2980
      XP=XO+(I-2)*DX/DEL                                                 FORO2990
      YP=YO                                                              FORO3000
      CALL PLOT(XP,YP,3)                                                 FORO3010
 5    YP=YP+0.1                                                          FORO3020
      IF(YP.GT.Y1)GO TO 3                                                FORO3030
      CALL PLOT(XP,YP,2)                                                 FORO3040
      YP=YP+0.1                                                          FORO3050
      CALL PLOT(XP,YP,3)                                                 FORO3060
      GO TO 5                                                            FORO3070
 3    CONTINUE                                                           FORO3080
      YO=YO+6.                                                           FORO3090
      Y1=YO+2.                                                           FORO3100
      Y2=YO+4.                                                           FORO3110
      CALL AXIS(XO,YO,'  ',-2,5.,0.,0.,DEL)                              FORO3120
      CALL AXIS(XO,YO,'  ',2,4.,90.,-1.,0.5)                             FORO3130
      CALL LIN(XO,Y1,5.,0.)                                              FORO3140
      CALL LIN(XO,Y2,5.,0.)                                              FORO3150
      CALL LIN(X1,YO,4.,90.)                                             FORO3160
      DO4 I=2,N                                                          FORO3170
      ET(K,I)=ET(K,I)/CDABS(ET(K,I))                                     FORO3180
      WP=DARSIN(DIMAG(ET(K,I)))                                          FORO3190
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      WS=SIGN(DARCOS(DREAL(ET(K,I))),WP)                                FOR03200
      XP=XO+(I-2)*DX/DEL                                                FOR03210
      YP=Y1+WS/(0.5*3.14159)                                            FOR03220
      IF(I.EQ.2)CALL PLOT(XP,YP,3)                                      FOR03230
      CALL PLOT(XP,YP,2)                                                FOR03240
 4    CONTINUE                                                          FOR03250
      DO7 I=2,N                                                         FOR03260
      S=LAM(K,I)*LAM(K,I-1)                                             FOR03270
      IF(S.GT.0.)GO TO 7                                                FOR03280
      XP=XO+(I-2)*DX/DEL                                                FOR03290
      YP=YO                                                             FOR03300
      CALL PLOT(XP,YP,3)                                                FOR03310
 8    YP=YP+0.1                                                         FOR03320
      IF(YP.GT.Y2)GO TO 7                                               FOR03330
      CALL PLOT(XP,YP,2)                                                FOR03340
      YP=YP+0.1                                                         FOR03350
      CALL PLOT(XP,YP,3)                                                FOR03360
      GO TO 8                                                           FOR03370
 7    CONTINUE                                                          FOR03380
      XO=XO+6.                                                          FOR03390
      YO=1.5                                                            FOR03400
 6    CONTINUE                                                          FOR03410
      CALL ENDPLT(20.,0.,999)                                           FOR03420
      RETURN                                                            FOR03430
      END                                                               FOR03440
```

FILE: LIN FORTRAN A

```
      SUBROUTINE LIN(XO,YO,XL,XA)                                       LIN00010
C     PLOTS HOR. AND VERT. AXIS                                         LIN00020
C     XO,YO=COORD. OF AXIS                                              LIN00030
C     XL=LENGTH OF AXIS                                                 LIN00040
C     XA=ANGLE OF AXIS (XA=0. OR 90.)                                   LIN00050
      CALL PLOT(XO,YO,3)                                                LIN00060
      N=INT(XL)+1                                                       LIN00070
      IF(ABS(XA-90.).LT.0.1)GO TO 2                                     LIN00080
      X1=XO+XL                                                          LIN00090
      CALL PLOT(X1,YO,2)                                                LIN00100
      Y2=YO-0.05                                                        LIN00110
      Y1=YO+0.05                                                        LIN00120
      X1=XO                                                             LIN00130
      DO1 I=1,N                                                         LIN00140
      CALL PLOT(X1,Y2,3)                                                LIN00150
      CALL PLOT(X1,Y1,2)                                                LIN00160
      X1=XO+I                                                           LIN00170
 1    CONTINUE                                                          LIN00180
      GO TO 3                                                           LIN00190
 2    Y1=YO+XL                                                          LIN00200
      CALL PLOT(XO,Y1,2)                                                LIN00210
      X2=XO-0.05                                                        LIN00220
      X1=XO+0.05                                                        LIN00230
      Y1=YO                                                             LIN00240
      DO3 I=1,N                                                         LIN00250
      CALL PLOT(X2,Y1,3)                                                LIN00260
      CALL PLOT(X1,Y1,2)                                                LIN00270
      Y1=YO+I                                                           LIN00280
 3    CONTINUE                                                          LIN00290
      RETURN                                                            LIN00300
      END                                                               LIN00310
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FILE: TEST2 FORTRAN A

```
C     ************************************************                TES00010
C     *        FORCED WAVES ON VARIABLE DEPTH          *             TES00020
C     *  ANALYTICAL SOL. FOR LINEARLY INCREASING DEPTH *             TES00030
C     ************************************************              TES00040
C                                                                     TES00050
C     H:DEPTH ; AL:SLOPE ; XI,XF:END POINTS                         TES00060
C     A:FORCED WAVES AMPLITUDE ; F:FORCING TERM                     TES00070
C     Z:REL. DISPLACEMENT (ETO-ETL)                                 TES00080
C     KA:WAVELENGTH ; CG:GROUP VELOCITY                             TES00090
C     DX:LENGTH INT. :NA-1:NUM. OF INT                              TES00100
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                       TES00110
      REAL*8 KAA                                                    TES00120
      REAL*8 H(1000),KA(1000),AM(1000)                              TES00130
      COMPLEX*16  F(1000),A(1000)                                   TES00140
      WRITE(10,200)                                                 TES00150
      READ(5,*)XI,XF,AL                                             TES00160
200   FORMAT(5X,'XI,XF,AL')                                         TES00170
      DX=0.02                                                       TES00180
      NA=1+IDINT((XF-XI)/DX)                                        TES00190
      DO2 I=1,NA                                                    TES00200
      H(I)=AL*(XI+(XF-XI)*(I-1)/(NA-1))                             TES00210
2     CONTINUE                                                      TES00220
C     FIND WAVELENGTH                                               TES00230
      DO1 I=1,NA                                                    TES00240
      HH=H(I)                                                       TES00250
      IF(DABS(HH).LT.0.01)GO TO 1000                                TES00260
      CALL WAVENU(HH,KAA)                                           TES00270
      KA(I)=KAA                                                     TES00280
1     CONTINUE                                                      TES00290
C     FINDS FORCING TERM                                            TES00300
      CALL FORC(H,KA,DX,NA,F)                                       TES00310
C     GIVES FORCED WAVE AMPLITUDE                                   TES00320
      CALL SOLUT(XI,XF,AL,DX,NA,F,A)                                TES00330
      DO3 I=1,NA                                                    TES00340
      AM(I)=CDABS(A(I))                                             TES00350
3     CONTINUE                                                      TES00360
      WRITE(7,100)(AM(I),I=1,NA)                                    TES00370
100   FORMAT(2X,4(2X,E10.4))                                        TES00380
1000  CONTINUE                                                      TES00390
      STOP                                                          TES00400
      END                                                           TES00410
      SUBROUTINE WAVENU(HH,KAA)                                     TES00420
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                       TES00430
      REAL*8 K1,K2,KAA,K1INF,K2INF                                  TES00440
      OME=1.                                                        TES00450
      EPSI=1.E-8                                                    TES00460
      X1=1.                                                         TES00470
      X2=1.                                                         TES00480
      X3=0.                                                         TES00490
C     IF X1=1.(RES.-1.),X2=1.(RES.0.),X3=0.(RES.1.),WE STUD.THE     TES00500
C     SMALLEST(RES.LARGEST)ROOT                                     TES00510
      N=0                                                           TES00520
C     APPROXI.PLAC.OF ROOTS                                         TES00530
      K1=28.                                                        TES00540
 1    K1=K1/2.                                                      TES00550
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      FMG=(OME**2)/K1-DTANH(K1*HH)                                      TES00560
      PTE=-(OME/K1)**2-HH/((DCOSH(K1*HH))**2)                           TES00570
      IF(FMG.LE.O..OR.PTE.GT.O.) GO TO 1                                TES00580
      DELT=K1                                                           TES00590
      K2=K1                                                             TES00600
    2 FMG=(OME**2)/K2-DTANH(K2*HH)                                      TES00610
      IF(FMG.LE.O.) GO TO 10                                            TES00620
      PTE=-(OME/K2)**2-HH/((DCOSH(K2*HH))**2)                           TES00630
      IF(PTE.GT.O.) GO TO 11                                            TES00640
      K2=K2+DELT                                                        TES00650
      GO TO 2                                                           TES00660
   11 DELT=DELT/2.                                                      TES00670
      K2=K2-DELT                                                        TES00680
      GO TO 2                                                           TES00690
   10 K1INF=K2-DELT                                                     TES00700
      FMG=1.                                                            TES00710
      IF(X1.GE.O.) GO TO 131                                            TES00720
   12 K2=K2+DELT                                                        TES00730
      FMG=OME**2/K2-DTANH(K2*HH)                                        TES00740
      IF(FMG.LE.O.) GO TO 12                                            TES00750
      K2INF=K2                                                          TES00760
C     AT THIS PT.THE 2 ROOTS ARE K1(K2)BETW. K1INF AND K1INF+DE         TES00770
C     LT.(K2INF)                                                        TES00780
  131 K2=X2*K1INF+X3*K2INF                                              TES00790
      GO TO 130                                                         TES00800
   13 IF(FM1.LE.O.) DELT=DELT/2.                                        TES00810
  130 K2=K2+X1*DELT                                                     TES00820
      N=N+1                                                             TES00830
      IF(N.GT.150) GO TO 1000                                           TES00840
      GO TO 15                                                          TES00850
   14 IF(FM1.GT.O.) DELT=DELT/2.                                        TES00860
      K2=K2-X1*DELT                                                     TES00870
      N=N+1                                                             TES00880
      IF(N.GT.50) GO TO 1000                                            TES00890
   15 FM1=FMG                                                           TES00900
      FMG=(OME**2)/K2-DTANH(K2*HH)                                      TES00910
      ER=DABS(FMG)                                                      TES00920
      IF(FMG.GE.O..AND.ER.GT.EPSI) GO TO 13                             TES00930
      IF(FMG.LT.O..AND.ER.GT.EPSI) GO TO 14                             TES00940
      KAA=K2                                                            TES00950
 1000 RETURN                                                            TES00960
      END                                                               TES00970
      SUBROUTINE FORC(H,KA,DX,NA,F)                                     TES00980
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER(I-N)                            TES00990
      REAL*8 KA(1000),H(1000),CG(1000),Z(1000)                          TES01000
      COMPLEX*16 F(1000)                                                TES01010
      DO1 I=1,NA                                                        TES01020
      CG(I)=(1.+2.*KA(I)*H(I)/DSINH(2.*KA(I)*H(I)))/(2.*KA(I))          TES01030
      Z(I)=(4.*CG(I)*KA(I)-1.)/((CG(I)**2)-H(I))                        TES01040
      Z(I)=Z(I)+1./(DSINH(KA(I)*H(I))**2)                               TES01050
      Z(I)=-Z(I)*CG(1)/(8.*CG(I))                                       TES01060
    1 CONTINUE                                                          TES01070
      TETA=O.                                                           TES01080
      DO2 I=1,NA                                                        TES01090
      IF(I.EQ.1)GO TO 3                                                 TES01100
      IF(I.EQ.NA)GO TO 4                                                TES01110
      DZ=(Z(I+1)-Z(I-1))/(2.*DX)                                        TES01120
      DDZ=(Z(I+1)-2.*Z(I)+Z(I-1))/(DX**2)                               TES01130
      DH=(H(I+1)-H(I-1))/(2.*DX)                                        TES01140
      DA=DH/CG(I)-H(I)*(CG(I+1)-CG(I-1))/(2.*DX*(CG(I)**2))             TES01150
      DB=(KA(I+1)-KA(I-1))-KA(I)*(CG(I+1)-CG(I-1))/CG(I)                TES01160
      DB=DB/(2.*DX*CG(I))                                               TES01170
      GO TO 5                                                           TES01180
    3 DZ=(Z(I+1)-Z(I))/DX                                               TES01190
      DH=(H(I+1)-H(I))/DX                                               TES01200
      DDZ=(Z(I+2)-2.*Z(I+1)+Z(I))/(DX**2)                               TES01210
      DA=DH/CG(I)-H(I)*(CG(I+1)-CG(I))/(DX*(CG(I)**2))                  TES01220
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      DB=(KA(I+1)-KA(I))-KA(I)*(CG(I+1)-CG(I))/CG(I)                    TES01230
      DB=DB/(DX*CG(I))                                                  TES01240
      GO TO 5                                                           TES01250
4     DZ=(Z(I)-Z(I-1))/DX                                               TES01260
      DH=(H(I)-H(I-1))/DX                                               TES01270
      DDZ=(Z(I)-2.*Z(I-1)+Z(I-2))/(DX**2)                               TES01280
      DA=DH/CG(I)-H(I)*(CG(I)-CG(I-1))/(DX*(CG(I)**2))                  TES01290
      DB=(KA(I)-KA(I-1))-KA(I)*(CG(I)-CG(I-1))/CG(I)                    TES01300
      DB=DB/(DX*CG(I))                                                  TES01310
5     X1=DDZ*H(I)+DH*DZ                                                 TES01320
      X2=(2.*DA*Z(I)+4.*H(I)*DZ/CG(I)-DB*CG(1)/2.)                      TES01330
      F(I)=DCMPLX(DCOS(TETA),DSIN(TETA))*X1                             TES01340
      F(I)=F(I)+DCMPLX(-DSIN(TETA),DCOS(TETA))*X2                       TES01350
      TETA=TETA+(1./CG(I)+1./CG(I+1))*DX                                TES01360
2     CONTINUE                                                          TES01370
      RETURN                                                            TES01380
      END                                                               TES01390
      SUBROUTINE SOLUT(XI,XF,AL,DX,NA,F,A)                              TES01400
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER (I-N)                           TES01410
      COMPLEX*16 F(1000),A(1000)                                        TES01420
      REAL*8 X(1000),Y(1000),ET(1000)                                   TES01430
      COMPLEX*16 C1,C2,C3,C4,B1,B2                                      TES01440
      COMPLEX*16 GA(1000),BE(1000)                                      TES01450
      DO1 I=1,NA                                                        TES01460
      XX=XI+(XF-XI)*(I-1)/(NA-1)                                        TES01470
      XX=DSQRT(XX/AL)*4.                                                TES01480
      CALL BESSEL(0,XX,XJ,XY)                                           TES01490
      X(I)=XJ                                                           TES01500
      Y(I)=XY                                                           TES01510
1     CONTINUE                                                          TES01520
      GA(1)=0.                                                          TES01530
      BE(1)=0.                                                          TES01540
      DO2 I=2,NA                                                        TES01550
      GA(I)=GA(I-1)+(F(I)*Y(I)+F(I-1)*Y(I-1))*DX/2.                     TES01560
      BE(I)=BE(I-1)+(F(I)*X(I)+F(I-1)*X(I-1))*DX/2.                     TES01570
2     CONTINUE                                                          TES01580
      XX=4.*DSQRT(XI/AL)                                                TES01590
      CALL BESSEL(1,XX,XJ,XY)                                           TES01600
      C1=DCMPLX(-XJ,X(1))                                               TES01610
      C2=DCMPLX(-XY,Y(1))                                               TES01620
      XX=4.*DSQRT(XF/AL)                                                TES01630
      CALL BESSEL(1,XX,XJ,XY)                                           TES01640
      C3=DCMPLX(-XJ,-X(NA))                                             TES01650
      C4=DCMPLX(-XY,-Y(NA))                                             TES01660
      B1=C2*(C3*GA(NA)-C4*BE(NA))/(C2*C3-C1*C4)                         TES01670
      B2=-C1*B1/C2                                                      TES01680
      DO3 I=1,NA                                                        TES01690
      A(I)=(-GA(I)+B1)*X(I)+(BE(I)+B2)*Y(I)                             TES01700
      A(I)=A(I)*3.14159/AL                                              TES01710
3     CONTINUE                                                          TES01720
      RETURN                                                            TES01730
      END                                                               TES01740
      SUBROUTINE BESSEL(IO,X,XJ,XY)                                     TES01750
C     GIVES BESSEL FUNCTIONS J AND Y ACCORDING MATH. TABLE              TES01760
C     IO:ORDER OF BESSEL FUNCTION ,IO=0 OR 1                            TES01770
      IMPLICIT REAL*8 (A-H,O-Z),INTEGER(I-N)                            TES01780
      IF(IO.EQ.1)GO TO 3                                                TES01790
      IF(DABS(X).GT.3.)GO TO 2                                          TES01800
      X1=(X/3.)**2                                                      TES01810
      XJ=1.+X1*(-2.2499997+X1*(1.2656208+X1*(-0.3163866+X1*(            TES01820
     &0.0444479+X1*(-0.0039444+X1*0.00021)))))                          TES01830
      XY=.36746691+X1*(.60559366+X1*(-.74350384+X1*(.25300117+          TES01840
     &X1*(-0.04261214+X1*(0.00427916+X1*(-0.00024846))))))              TES01850
      XY=XY+2.*DLOG(X/2.)*XJ/3.14159                                    TES01860
      GO TO 11                                                          TES01870
2     X1=(3./X)                                                         TES01880
      XF=.79788456+X1*(-.00000077+X1*(-.0055274+X1*(-.00009512          TES01890
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     &+X1*(.00137237+X1*(-.00072805+X1*.00014476)))))                    TES01900
      XT=X-.78539816+X1*(-.04166397+X1*(-.00003954+X1*(                  TES01910
     &.00262573+X1*(-.00054125+X1*(-.00029333+X1*.00013558)))))         TES01920
      XJ=XF*DCOS(XT)/DSQRT(X)                                           TES01930
      XY=XF*DSIN(XT)/DSQRT(X)                                           TES01940
      GO TO 11                                                          TES01950
3     CONTINUE                                                          TES01960
      IF(DABS(X).GT.3.)GO TO 4                                          TES01970
      X1=(X/3.)**2                                                      TES01980
      XJ=(.5+X1*(-.56249985+X1*(.21093573+X1*(-.03954289+X1*(           TES01990
     &.00443319+X1*(-.00031761+X1*.00001109))))))*X                     TES02000
      XY=-.6366198+X1*(.2212091+X1*(2.1682709+X1*(-1.3164827+X1*(       TES02010
     &.3123951+X1*(-.0400976+.0027873*X1)))))                           TES02020
      XY=XY/X+2.*DLOG(X/2.)*XJ/3.14159                                  TES02030
      GO TO 11                                                          TES02040
4     X1=3./X                                                           TES02050
      XF=.79788456+X1*(.00000156+X1*(.01659667+X1*(.00017105+X1*(       TES02060
     &-.00249511+X1*(.00113653-X1*.00020033)))))                        TES02070
      XT=X-2.35619449+X1*(.12499612+X1*(.0000565+X1*(-.00637879+        TES02080
     &X1*(.00074348+X1*(.00079824-X1*.00029166)))))                     TES02090
      XJ=XF*DCOS(XT)/DSQRT(X)                                           TES02100
      XY=XF*DSIN(XT)/DSQRT(X)                                           TES02110
11    CONTINUE                                                          TES02120
      RETURN                                                            TES02130
      END                                                               TES02140
```

Amplitude of Forced Waves over Linearly Increasing Depth,

Normal Incidence, XL= 1, h_0= 0.5, h_1= 1

Numerical Results

0.4390D+00	0.4422D+00	0.4511D+00	0.4642D+00
0.4805D+00	0.4990D+00	0.5190D+00	0.5399D+00
0.5613D+00	0.5830D+00	0.6046D+00	0.6262D+00
0.6474D+00	0.6683D+00	0.6888D+00	0.7089D+00
0.7284D+00	0.7474D+00	0.7659D+00	0.7838D+00
0.8011D+00	0.8178D+00	0.8339D+00	0.8494D+00
0.8642D+00	0.8784D+00	0.8920D+00	0.9049D+00
0.9172D+00	0.9288D+00	0.9398D+00	0.9501D+00
0.9598D+00	0.9689D+00	0.9773D+00	0.9850D+00
0.9922D+00	0.9988D+00	0.1005D+01	0.1010D+01
0.1015D+01	0.1019D+01	0.1023D+01	0.1026D+01
0.1029D+01	0.1031D+01	0.1033D+01	0.1034D+01
0.1035D+01	0.1036D+01	0.1036D+01	

Analytical Results

0.4389D+00	0.4422D+00	0.4510D+00	0.4642D+00
0.4805D+00	0.4989D+00	0.5189D+00	0.5398D+00
0.5612D+00	0.5829D+00	0.6046D+00	0.6261D+00
0.6473D+00	0.6682D+00	0.6887D+00	0.7088D+00
0.7283D+00	0.7473D+00	0.7658D+00	0.7837D+00
0.8010D+00	0.8177D+00	0.8338D+00	0.8493D+00
0.8641D+00	0.8784D+00	0.8919D+00	0.9049D+00
0.9171D+00	0.9288D+00	0.9397D+00	0.9501D+00
0.9598D+00	0.9688D+00	0.9772D+00	0.9850D+00
0.9922D+00	0.9987D+00	0.1005D+01	0.1010D+01
0.1015D+01	0.1019D+01	0.1023D+01	0.1026D+01
0.1029D+01	0.1031D+01	0.1033D+01	0.1034D+01
0.1035D+01	0.1036D+01	0.1036D+01	

The relative difference is less than 0.1% .

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