Self maps of quaternionic projective spaces

by

Gustavo Granja

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Author ....
Department of Mathematics
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Certified by ....
Michael Hopkins
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Richard Melrose
Chairman, Departmental Committee on Graduate Students

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Abstract

The homology classification of self maps of quaternionic projective spaces is studied. A survey of the known results is given and some new self maps are shown to exist.

Thesis Supervisor: Michael Hopkins
Title: Professor of Mathematics

Chapter 1

The Feder-Gitler conjecture

Let $\mathbb{HP}^n$ denote n-dimensional quaternionic projective space with n a positive integer. Recall that $\mathrm{H}^*(\mathbb{HP}^n) = \mathbb{Z}[y]/y^{n+1}$ where y is a generator of $\mathrm{H}^4(\mathbb{HP}^n)$.

Definition 1.1 *The* degree *of a map* $f : \mathbb{HP}^n \longrightarrow \mathbb{HP}^n$ *is the integer* $\deg(f)$ *such that* $f^*(y) = \deg(f)y$

The homomorphism induced by a map on cohomology is determined by its degree. The determination of the possible degrees of self maps is the first step towards the homotopy classification of such maps[1]. It also amounts to the determination of the A_n self maps of $SU(2)$ [14, p. 300].

Consider the congruences

$$\prod_{i=0}^{m-1}(k - i^2) \equiv 0 \bmod \begin{cases} (2m)! & \text{if } m \text{ is even} \\ (2m)!/2 & \text{if } m \text{ is odd} \end{cases} \tag{1.1}$$

Feder and Gitler [4] have shown the following result:

Theorem 1.2 *If f is a self map of $\mathbb{HP}^n$ then $\deg(f)$ satisfies the congruences (1.1) for $m = 1, \ldots, n$.*

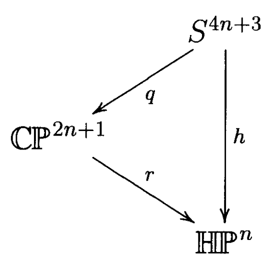
[1] The two problems are, however, not equivalent. One can see, for example, that there are at least 2 distinct homotopy classes of self maps of $\mathbb{HP}^3$ of each allowable degree (precisely 2 of degree 0). See also [7].

In the same paper, Feder and Gitler made the following conjecture:

Conjecture 1.3 *If k is an integer satisfying the congruences (1.1) for $m = 1, \dots, n$ then there is a self map of $\mathbb{HP}^n$ of degree k.*

There is some evidence in support of this conjecture. So far, it has been verified for $n = 1, 2, 3$ and ∞ and many of the conjectured maps have been shown to exist. In this chapter we prove theorem 1.2 and in the next chapter we present the available evidence in favor of conjecture 1.3.

We will now prove theorem 1.2. Embedding $\mathbb{C}$ in $\mathbb{H}$ in the usual way, $\mathbb{H}$ becomes a right vector space over $\mathbb{C}$ isomorphic to $\mathbb{C} \oplus \mathbb{C}$ and the left action of $Sp(1)$ on $\mathbb{H}$ is identified by this isomorphism with the usual action of $SU(2)$ on $\mathbb{C}^2$. Moreover if S^{4n+3} is the unit sphere in $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$, there is a commutative diagram



(1.2)

where h is the projection of the principal $Sp(1)$-bundle associated to the tautological quaternionic line bundle H, q is the projection of the principal S^1-bundle associated to the tautological complex line bundle L and r is the projection of a bundle with fiber $Sp(1)/S^1 = S^2$.

Lemma 1.4 $r^*(H) = L \oplus \overline{L}$

Proof. $S^1 \subset SU(2)$ acts on $\mathbb{C}^2$ by

$$\theta \longmapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

so we see that the canonical complex vector bundle map

$$\begin{array}{ccc} S^{4n+3} \times_{S^1} (\mathbb{C} \oplus \mathbb{C}) & \longrightarrow & S^{4n+3} \times_{Sp(1)} \mathbb{H} \\ \downarrow & & \downarrow \\ \mathbb{CP}^{2n+1} & \xrightarrow{\quad r \quad} & \mathbb{HP}^n \end{array}$$

defines an isomorphism between $L \oplus \overline{L}$ and $r^*(H)$. □

Lemma 1.5 *Let $z = [H] - 2$. Then* $\mathrm{K}(\mathbb{HP}^n) = \mathbb{Z}[z]/z^{n+1}$.

Proof. Let $x = c_1(L)$ and recall that $\mathrm{H}^*(\mathbb{CP}^{2n+1}) = \mathbb{Z}[x]/x^{2n+2}$. It follows from the Serre spectral sequence that the map r in (1.2) injects the cohomology of $\mathbb{HP}^n$ in the cohomology of $\mathbb{CP}^{2n+1}$ and if we define y by $r^*(y) = x^2$, then $\mathrm{H}^*(\mathbb{HP}^n) = \mathbb{Z}[y]/y^{n+1}$. Let $w = [L] - 1$, so that $\mathrm{Ch}(w) = x + x^2/2 + \ldots$. Since $c_1(\overline{L}) = -c_1(L)$, by naturality of the Chern character and lemma 1.4 we have

$$r^*(\mathrm{Ch}(z)) = \mathrm{Ch}(w + \overline{w}) = x + x^2/2 + \cdots - x + x^2/2 - \cdots$$

$$\mathrm{Ch}(z) = y + 2y^2/4! + \cdots$$

Note also that since $\mathrm{K}(\mathbb{HP}^n)$ and $\mathrm{K}(\mathbb{CP}^{2n+1})$ are torsion free the Chern character is a ring monomorphism [2, p. 19]. The lemma is clearly true for $n = 0$. Assume it has been shown for $n \leq k$. As $\mathbb{HP}^n$ admits an even cell decomposition, there is a short exact sequence

$$0 \longrightarrow \mathrm{K}(\mathbb{HP}^{k+1}, \mathbb{HP}^k) = \widetilde{\mathrm{K}}(\mathbb{HP}^{k+1}/\mathbb{HP}^k) \longrightarrow \mathrm{K}(\mathbb{HP}^{k+1}) \longrightarrow \mathrm{K}(\mathbb{HP}^k) \longrightarrow 0$$

Since $\mathbb{HP}^{k+1}/\mathbb{HP}^k = S^{4k+4}$, $\mathrm{Ch}(z^{k+1}) = y^{k+1}$ and the Chern character is integral on even dimensional spheres [6, p. 308] we see that z^{k+1} generates the summand $\mathrm{K}(\mathbb{HP}^{k+1}, \mathbb{HP}^k)$. Since $\mathrm{Ch}(z^{k+2}) = 0$, we have $\mathrm{K}(\mathbb{HP}^{k+1}) = \mathbb{Z}[z]/z^{k+2}$ as required. □

Recall that $\mathrm{K}(\mathbb{CP}^{2n+1}) = \mathbb{Z}[w]/w^{2n+2}$ where $w = [L] - 1$ (one can prove this exactly as the previous lemma). We have a commutative diagram

$$\begin{array}{ccc} \mathrm{K}(\mathbb{CP}^{2n+1}) & \xrightarrow{Ch} & \mathrm{H}^*(\mathbb{CP}^{2n+1};\mathbb{Q}) \\ {\scriptstyle r^*}\uparrow & & {\scriptstyle r^*}\uparrow \\ \mathrm{K}(\mathbb{HP}^n) & \xrightarrow{Ch} & \mathrm{H}^*(\mathbb{HP}^n;\mathbb{Q}) \end{array}$$

where all the maps except perhaps $r^* : \mathrm{K}(\mathbb{HP}^n) \longrightarrow \mathrm{K}(\mathbb{CP}^{2n+1})$ are injective. It follows that r^* is also injective in K-theory. This allows us to give an expression for the Adams operations. Since $L \otimes \overline{L} = \mathbb{C}$ and $\psi^k([L]) = [L]^k$, by lemma 1.4, we have $r^*(z) = w + \overline{w} = -w\overline{w}$ and so

$$r^*(\psi^k z) = (1+w)^k - 1 + (1+\overline{w})^k - 1 \tag{1.3}$$

To simplify notation, we will omit r^* and identify $\mathrm{K}(\mathbb{HP}^n)$ with its image in $\mathrm{K}(\mathbb{CP}^{2n+1})$. In particular, we have

$$\psi^2 z = 2(w+\overline{w}) + (w+\overline{w})^2 - 2w\overline{w} = 4z + z^2 \tag{1.4}$$

A self map f of $\mathbb{HP}^n$ determines a ring endomorphism $\varphi = f^*$ of $\mathrm{K}(\mathbb{HP}^n)$ that commutes with the Adams operations (which are themselves endomorphisms). Write

$$\varphi(z) = a_1 z + a_2 z^2 + \cdots$$

On $\mathrm{H}^4(\mathbb{HP}^n)$, the equation $\mathrm{Ch}(f^*(z)) = f^*(\mathrm{Ch}(z))$ yields $\deg(f) = a_1$. Given a ring endomorphism η of $\mathrm{K}(\mathbb{HP}^n)$, not necessarily arising from a map, we will also denote by $\deg(\eta)$ the coefficient of z in $\eta(z)$. Note that if η and λ are two ring endomorphisms we have $\deg(\eta\lambda) = \deg(\eta)\deg(\lambda)$.

Lemma 1.6 *Let a_1 be an integer. There is at most one endomorphism η of $\mathrm{K}(\mathbb{HP}^n)$ commuting with the Adams operations with $\deg(\eta) = a_1$.*

Proof. By (1.4) we have

$$\eta(\psi^2(z)) = \psi^2(\eta(z))$$

$$\eta(z^2 + 4z) = \eta(z)^2 + 4\eta(z)$$

Writing $\eta(z) = a_1 z + a_2 z^2 + \cdots$ the last equation determines the coefficients a_i in terms of a_1. For example, the first two coefficients are

$$\begin{aligned} a_2 &= \tfrac{1}{12} a_1(a_1 - 1) \\ a_3 &= \tfrac{1}{360} a_1(a_1 - 1)(a_1 - 4) \end{aligned} \tag{1.5}$$

□

Note that this lemma implies that η commutes with ψ^k for all k if and only if it commutes with ψ^2. Verily, if $\eta\psi^2 = \psi^2\eta$ then $\eta\psi^k$ and $\psi^k\eta$ also commute with ψ^2 and their degrees coincide so they are equal.

Lemma 1.7 *The image of* $\mathrm{KSp}(\mathbb{HP}^n)$ *in* $\mathrm{K}(\mathbb{HP}^n) = \mathbb{Z}[z]/z^{n+1}$ *is* $\{a_0 + a_1 z + \cdots | a_{2i} \in 2\mathbb{Z} \textit{ for } i > 0\}$

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathrm{K}}(S^{4n}) = \mathrm{K}(\mathbb{HP}^n, \mathbb{HP}^{n-1}) & \underset{\xleftarrow{a_n}}{\longrightarrow} & \mathrm{K}(\mathbb{HP}^n) & \longrightarrow & \mathrm{K}(\mathbb{HP}^{n-1}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \widetilde{\mathrm{KSp}}(S^{4n}) = \mathrm{KSp}(\mathbb{HP}^n, \mathbb{HP}^{n-1}) & \longrightarrow & \mathrm{KSp}(\mathbb{HP}^n) & \longrightarrow & \mathrm{KSp}(\mathbb{HP}^{n-1}) & \longrightarrow & 0 \end{array}$$

The splitting of the injection on the first row is given by a_n as was seen in the proof of lemma 1.5. The bottom row is exact by the long exact sequence of the pair for KO-theory. Indeed, $\mathrm{KSp} = \mathrm{KO}^4$, $\widetilde{\mathrm{KO}}^4(S^{4n}) = \mathbb{Z}$, $\widetilde{\mathrm{KO}}^5(S^{4n}) = 0$ and by the Atiyah-Hirzebruch spectral sequence $\mathrm{KO}^3(\mathbb{HP}^{n-1})$ is torsion. By induction we may assume that $\mathrm{KSp}(\mathbb{HP}^{n-1}) = \mathbb{Z}^n$ with a basis $\{v_0, \ldots, v_{n-1}\}$ such that the natural map sends v_i to $c_i z^i \in \mathrm{K}(\mathbb{HP}^{n-1})$ with $c_i = 1$ if i is 0 or odd and $c_i = 2$ if i is even and positive. This is certainly true for n=1. To conclude the proof we have only to check that the image of the leftmost map $\pi_{4n}(BSp) = \mathbb{Z} \longrightarrow \pi_{4n}(BU) = \mathbb{Z}$ has index 1 or 2 respectively when n is an odd or even positive integer. Indeed, in that case we can pick $v_n \in \mathrm{KSp}(\mathbb{HP}^n)$ mapping to $c_n z^n$ and then lifts of $v_0, \ldots, v_{n-1} \in \mathrm{KSp}(\mathbb{HP}^{n-1})$ to $\mathrm{KSp}(\mathbb{HP}^n)$ with the required properties. From the long exact homotopy sequence of the fibration $U/Sp \longrightarrow BSp \longrightarrow BU$ and the fact that $\pi_{4n}(U/Sp) = \pi_{4n-1}(BSp) = 0$, $\pi_{4n-1}(U/Sp) = \mathbb{Z}/2$ or 0 respectively when n is even or odd [10, p. 212] we obtain the required result, which completes the proof. □

We can now prove theorem 1.2:

Proof. Since a generator z of $\mathrm{K}(\mathbb{HP}^n)$ lies in the image of KSp it follows that $f^*(z) = a_1 z + a_2 z^2 + \cdots$ will also be in that image. The congruences (1.1) are just the conditions obtained when one requires that the coefficients a_i in (1.5) be in the image of KSp, described in lemma 1.7.

However, it does not seem easy to obtain formulas for the coefficients a_i in terms of the degree directly from the recursive relations we obtained in the proof of lemma 1.6. Instead, we will determine explicitly all the ring endomorphisms of $\mathrm{K}(\mathbb{HP}^n)$ commuting with the Adams operations.

ψ^k are such homomorphisms and the expression (1.3) suggests that we consider homomorphisms determined by

$$\psi^\mu(z) = (1+w)^\mu - 1 + (1+\overline{w})^\mu - 1 \tag{1.6}$$

where $z = w + \overline{w} = -w\overline{w}$. The right term of this equation is a symmetric power series in w,$\overline{w}$ with complex coefficients and so a power series in z. For certain complex values of μ the coefficients in the power series expansion will be integers and we will presently see that these values of μ give all the endomorphisms of $\mathrm{K}(\mathbb{HP}^n)$ commuting with the Adams operations.

The power series in (1.6) converge in a neighbourhood of 0 so we can regard ψ^μ as holomorphic functions. To compute the composite of two such functions we note that, in a neighbourhood of 0,

$$\psi^\mu(z) = (1+w)^\mu - 1 + (1+\overline{w})^\mu - 1 = -((1+w)^\mu - 1)((1+\overline{w})^\mu - 1)$$

Hence

$$\begin{aligned} \psi^\nu(\psi^\mu(z)) &= \psi^\nu((1+w)^\mu - 1 + (1+\overline{w})^\mu - 1) \\ &= (1+(1+w)^\mu - 1)^\nu - 1 + (1+(1+\overline{w})^\mu - 1)^\nu - 1) \\ &= \psi^{\mu\nu}(z) \end{aligned}$$

and we conclude that the endomorphisms of $\mathrm{K}(\mathbb{HP}^n) \otimes \mathbb{C}$ determined by $\psi^\mu(z)$ commute with each other.

We define the degree of an endomorphism of $\mathbb{C}[z]/z^n$ in the obvious way. Just as in lemma 1.6 we see that the degree of an endomorphism of $\mathrm{K}(\mathbb{HP}^n) \otimes \mathbb{C}$ commuting with ψ^2 actually determines the endomorphism. Since $\deg(\psi^\mu) = \mu^2$ it follows that the ψ^μ are all such endomorphisms.

It remains only to determine the coefficients of z^n in $\psi^\mu(z)$ as a function of μ. For this we write

$$w = \frac{z + \sqrt{z^2+4z}}{2}$$

$$\overline{w} = \frac{z - \sqrt{z^2+4z}}{2}$$

and we get

$$\begin{aligned} \psi^\mu(z) &= (1+w)^\mu - 1 + (1+\overline{w})^\mu - 1 \\ &= \left(1 + \frac{z+\sqrt{z^2+4z}}{2}\right)^\mu - 1 + \left(1 + \frac{z-\sqrt{z^2+4z}}{2}\right)^\mu - 1 \\ &= \sum_{k=1}^{\infty} \binom{\mu}{k} \left(\left(\frac{z+\sqrt{z^2+4z}}{2}\right)^k + \left(\frac{z-\sqrt{z^2+4z}}{2}\right)^k \right) \\ &= \sum_{k=1}^{\infty} \sum_{l=0}^{k} 2^{-k} \binom{\mu}{k} \binom{k}{l} \left((z^2+4z)^{l/2} z^{k-l} + (-1)^l (z^2+4z)^{l/2} z^{k-l} \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}\sum_{l=0}^{k} 2^{-k}\binom{\mu}{k}\binom{k}{l}(1+(-1)^l)z^{k-l/2}(z+4)^{l/2} \\
&= \sum_{k=1}^{\infty}\sum_{m=0}^{k/2} 2^{2m-k+1}\binom{\mu}{k}\binom{k}{2m}z^{k-m}(1+z/4)^m \\
&= \sum_{k=1}^{\infty}\sum_{m=0}^{k/2}\sum_{j=0}^{m} 2^{2m-k+1-2j}\binom{\mu}{k}\binom{k}{2m}\binom{m}{j}z^{k-m+j}
\end{aligned}
$$

The coefficient of z^n is

$$
\begin{aligned}
a_n &= \sum_{k=1}^{\infty}\sum_{m=0}^{k/2} 2^{1-k+2m-2(n-k+m)}\binom{\mu}{k}\binom{k}{2m}\binom{m}{m+n-k} \\
&= \sum_{k=n}^{2n}\sum_{m=0}^{k/2} 2^{1+k-2n}\binom{\mu}{k}\binom{k}{2m}\binom{m}{m+n-k}
\end{aligned}
$$

Now $\deg(\psi^\mu) = \mu^2$ so $\psi^\mu = \psi^{-\mu}$ (they both have the same degree). Consequently, the coefficients in the power series expansion of ψ^μ are even functions of μ and hence $a_n = a_n(\mu)$ is a polynomial of degree n in μ^2. Its roots include $0, 1, \ldots n-1$ so it can be written as

$$
A\prod_{i=0}^{n-1}(\mu^2 - i^2)
$$

for some $A \in \mathbb{C}$. As $a_n(n) = 1$ at we conclude that $A = 2/(2n)!$ and hence a necessary condition for the existence of a self map of $\mathbb{HP}^n$ of degree k is that

$$
\frac{2}{(2m)!}\prod_{i=0}^{m-1}(k^2 - i^2) \in \begin{cases} \mathbb{Z} & \text{if } m \text{ is odd} \\ 2\mathbb{Z} & \text{otherwise} \end{cases}
$$

for each $m = 1, \ldots, n$. $\square$

Chapter 2

Construction of self maps

We will now show the existence of some self maps of $\mathbb{HP}^n$. The method consists in analysing the obstruction to extension of a self map of $\mathbb{HP}^n$ to $\mathbb{HP}^{n+1}$.

First, recall that there is a cofiber sequence

$$S^{4n+3} \xrightarrow{h} \mathbb{HP}^n \xrightarrow{i} \mathbb{HP}^{n+1}$$

and a fiber sequence

$$S^3 \to S^{4n+3} \xrightarrow{h} \mathbb{HP}^n \tag{2.1}$$

where h denotes the Hopf map and i the inclusion.

Lemma 2.1 *There is an isomorphism*

$$\pi_{i-1}S^3 \oplus \pi_i S^{4n+3} \longrightarrow \pi_i \mathbb{HP}^n$$
$$(\alpha, \beta) \longmapsto j_*(\Sigma\alpha) + h_*\beta$$

where $j : S^4 \to \mathbb{HP}^n$ denotes the inclusion and Σ the suspension homomorphism.

Proof. The diagram

$$
\begin{array}{ccccccccc}
\Omega S^{4n+3} & \longrightarrow & \Omega\mathbb{HP}^n & \xrightarrow{\quad\delta\quad} & S^3 & \longrightarrow & S^{4n+3} & \longrightarrow & \mathbb{HP}^n \\
 & & {\scriptstyle \Omega j}\nwarrow & & \swarrow{\scriptstyle \Sigma} & & & & \\
 & & & \Omega S^4 & & & & &
\end{array}
$$

commutes up to sign, since all maps in the triangle induce isomorphisms of $\pi_3 = \mathbb{Z}$. Hence the long exact homotopy sequence of the fibration (2.1) breaks up into split short exact sequences

$$
0 \longrightarrow \pi_i S^{4n+3} \xrightarrow{h_*} \pi_i \mathbb{HP}^n \underset{\pm j_*\Sigma}{\overset{\delta}{\rightleftarrows}} \pi_{i-1} S^3 \longrightarrow 0
$$

and the proof is complete. $\square$

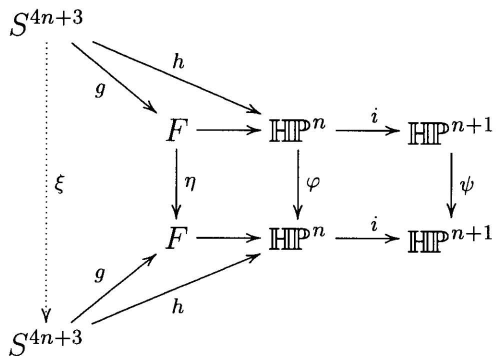
In what follows, we will often denote by l a map whose homotopy class is determined by the integer l (for instance a self map of a sphere).

Proposition 2.2 *There exists a self map of degree k of $\mathbb{HP}^{n+1}$ iff there exists a self map φ of $\mathbb{HP}^n$ of degree k such that the following diagram commutes up to homotopy.*

$$
\begin{array}{ccc}
S^{4n+3} & \xrightarrow{h} & \mathbb{HP}^n \\
\downarrow{\scriptstyle k^{n+1}} & & \downarrow{\scriptstyle \varphi} \\
S^{4n+3} & \xrightarrow{h} & \mathbb{HP}^n
\end{array}
\tag{2.2}
$$

Proof. If the diagram commutes up to homotopy, a choice of homotopy determines a map between the cofibres $\mathbb{HP}^{n+1}$ which extends φ and hence has degree k.

Conversely, suppose ψ is a map of degree k which we may assume to be cellular and let φ be its restriction to $\mathbb{HP}^n$. Letting F denote the homotopy fibre of the inclusion $\mathbb{HP}^n \xrightarrow{i} \mathbb{HP}^{n+1}$ we have a homotopy commutative diagram



where g is the map to the fibre determined by the canonical null homotopy of ih and η is the map induced on the fibre. The exact homotopy sequence of the fibration $F \to \mathbb{HP}^n \to \mathbb{HP}^{n+1}$ together with lemma 2.1 imply that the map g is a $(4n+5)$-equivalence and so there is a map ξ making the diagram commute up to homotopy.

Choosing a homotopy we can form the map of cofiber sequences

$$\begin{array}{ccccccc} S^{4n+3} & \xrightarrow{h} & \mathbb{HP}^n & \longrightarrow & \mathbb{HP}^{n+1} & \longrightarrow & S^{4n+4} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow{\scriptstyle \Sigma\xi} \\ S^{4n+3} & \xrightarrow{h} & \mathbb{HP}^n & \longrightarrow & \mathbb{HP}^{n+1} & \longrightarrow & S^{4n+4} \end{array}$$

and by looking at the diagram in cohomology we see that $\Sigma\xi$ has degree k^{n+1}. Hence ξ has degree k^{n+1}. □

The previous proposition tells us that the obstruction to extension of a self map φ is the element $\mathrm{o}(\varphi) \stackrel{\text{def}}{=} \varphi_*[h] - k^{n+1}[h] \in \pi_{4n+3}\mathbb{HP}^n$.

We will write $X_{(0)}$ for the rationalization [5] of the nilpotent space X.

Lemma 2.3 *There exists* $\alpha \in \pi_{4n+2}S^3$ *such that* $\mathrm{o}(\varphi) = j_*\Sigma\alpha$.

Proof. Let $k = \deg(\varphi)$. Since $\pi_{4n+2}S^3$ is a finite group, we have only to check that $\varphi_*([h]) = k^{4n+4}[h]$ in $\pi_{4n+3}\mathbb{HP}^n \otimes \mathbb{Q}$, or equivalently that the diagram (2.2) commutes for any map φ after rationalization. Rationally, since $\pi_i(\mathbb{HP}^\infty) = \pi_{i-1}(S^3)$ is torsion for $i > 4$,the fiber sequence $S^{4n+3} \to \mathbb{HP}^n \to \mathbb{HP}^\infty$ becomes $\mathrm{K}(\mathbb{Q}, 4n+3) \to \mathbb{HP}^n_{(0)} \to$

$K(\mathbb{Q},4)$ where the second map classifies the fundamental class in $H^4(\mathbb{HP}^n;\mathbb{Q})$. Then we have a homotopy commutative diagram

$$\begin{array}{ccccc} K(\mathbb{Q},4n+3) & \xrightarrow{h} & \mathbb{HP}^n_{(0)} & \longrightarrow & K(\mathbb{Q},4) \\ \downarrow l & & \downarrow \varphi & & \downarrow k \\ K(\mathbb{Q},4n+3) & \xrightarrow{h} & \mathbb{HP}^n_{(0)} & \longrightarrow & K(\mathbb{Q},4) \end{array}$$

where l is the map induced on the fibre. But since the cofibre of h is $\mathbb{HP}^{n+1}_{(0)}$ we see by a cohomology calculation as in the proof of proposition 2.2 that $l = k^{n+1}$ as required. □

So the obstruction to extension is actually an element in $\pi_{4n+2}S^3$. We will need a formula for the behaviour of the obstruction under the composition of maps:

Lemma 2.4 *Let φ, ψ be self maps of $\mathbb{HP}^n$, $k = \deg(\varphi)$, $l = \deg(\psi)$. Then* $o(\varphi\psi) = l^{n+1}\, o(\varphi) + k\, o(\psi)$.

Proof. We have

$$\begin{aligned} o(\varphi\psi) &= (\varphi\psi)_*[h] - (kl)^{n+1}[h] \\ &= \varphi_*(l^{n+1}[h] + o(\psi)) - (kl)^{n+1}[h] \\ &= l^{n+1}(k^{n+1}[h] + o(\varphi)) + \varphi_*\, o(\psi) - (kl)^{n+1}[h] \\ &= l^{n+1}\, o(\varphi) + k\, o(\psi) \end{aligned}$$

where the last equality holds because, by lemma 2.3, $o(\psi)$ is a suspended class in the image of the inclusion of the bottom cell of $\mathbb{HP}^n$ and φ has degree k on this cell [17, p. 479]. □

We can now give some evidence in support of conjecture 1.3. The following result is due to Arkowitz and Curjel [1, Proposition D] and verifies the conjecture for $n = 2$.

Proposition 2.5 *There exists a self map of $\mathbb{HP}^2$ of degree k iff* $\binom{k}{2} \equiv 0 \mod 12$.

Proof. Since $\mathbb{HP}^1 = S^4$ we can apply the distributivity formula [17, p. 494] and get

$$k_*[h] = k[h] + H_0([h])\binom{k}{2}[\iota_4, \iota_4]$$

where ι_4 denotes the class of the identity map on S^4, and H_0 is the Hopf invariant. Recall that $H_0([h]) = \pm 1$ [17, p. 506]. Let $\overline{h} = \pm h$, so that $H_0([\overline{h}]) = 1$. It follows from the EHP sequence [17, p. 550] that $[\iota_4, \iota_4]$ generates the kernel of the suspension map $\Sigma : \pi_7 S^4 \longrightarrow \pi_8 S^5$. By [16, p. 42 and p. 177] this is generated by $2[\overline{h}] - \Sigma\nu'$ where ν' generates $\pi_6 S^3 = \mathbb{Z}/12$, so $[\iota_4, \iota_4] = \pm(2[\overline{h}] - \Sigma\nu')$. But [17, p. 495] $H_0([\iota_4, \iota_4]) = 2$ so $[\iota_4, \iota_4] = 2[\overline{h}] - \Sigma\nu'$ and hence

$$\begin{aligned} k_*[\overline{h}] &= k^2[\overline{h}] - \binom{k}{2}\Sigma\nu' \\ \mathrm{o}([k]) &= \mp\binom{k}{2}\Sigma\nu' \end{aligned}$$

Since, by lemma 2.1 with $n = 1$, the map $\Sigma : \pi_* S^3 \longrightarrow \pi_{*+1} S^4$ is injective, we conclude that $\mathrm{o}([k]) = 0$ iff $\binom{k}{2} \equiv 0 \mod 12$ as required. □

It does not seem easy to get explicit formulas for the obstruction such as in the previous proposition for $n > 2$. Instead, we can try to analyse the obstruction one prime at a time.

Let $\mathcal{P} \subset \mathbb{Z}$ denote the set of primes. Given a subset $\mathcal{L} \subset \mathcal{P}$, write $X_{\mathcal{L}}$ for the localization of the nilpotent space X at $\mathcal{L}$ [5]. If $\mathcal{L} = \{p\}, \emptyset, \mathcal{P}$ we write respectively $X_{(p)}, X_{(0)}, X$ for the localizations.

Recall [5, p. 88] that for $\mathcal{L}, \mathcal{L}' \subset \mathcal{P}$ we have Sullivan's homotopy pull-back square

$$\begin{array}{ccc} X_{\mathcal{L}\cup\mathcal{L}'} & \longrightarrow & X_{\mathcal{L}} \\ \downarrow & & \downarrow \\ X_{\mathcal{L}'} & \longrightarrow & X_{\mathcal{L}\cap\mathcal{L}'} \end{array} \tag{2.3}$$

where all maps are localization maps. Self maps of $X_{\mathcal{L}}$, $X_{\mathcal{L}'}$ and $X_{\mathcal{L}\cap\mathcal{L}'}$ compatible up to homotopy together with a choice of homotopies give a self map of the pullback.

We are interested in the case when $X = \mathbb{HP}^n$ and $\mathcal{L} \cap \mathcal{L}' = \emptyset$. Then to give a self map of the diagram

$$\begin{array}{ccc} & & \mathbb{HP}^n_{\mathcal{L}} \\ & & \big\downarrow \\ \mathbb{HP}^n_{\mathcal{L}'} & \longrightarrow & \mathbb{HP}^n_{(0)} \end{array}$$

amounts to giving self maps of $\mathbb{HP}^n_{\mathcal{L}}$ and $\mathbb{HP}^n_{\mathcal{L}'}$ with the same degree. Verily, the obstructions to homotopy between maps $\mathbb{HP}^n_{\mathcal{M}} \longrightarrow \mathbb{HP}^n_{(0)}$ of the same degree lie in $\pi_{4i}\mathbb{HP}^n_{(0)} \otimes \mathbb{Z}_{\mathcal{M}} = \pi_{4i}\mathbb{HP}^n \otimes \mathbb{Q} = 0$, for $i > 1$, so $[\mathbb{HP}^n_{\mathcal{L}}, \mathbb{HP}^n_{(0)}] = \mathbb{Q}$ with the identification being given by the degree of the map.

We have seen in lemma 2.3 that the obstruction to extending a self map of $\mathbb{HP}^n$ one more stage lies in $\pi_{4n+2}S^3$. At an odd prime p, we have [16, p.177]

$$\pi_{4n+2}S^3 = \begin{cases} \mathbb{Z}/p & \text{if } n = i\frac{p-1}{2}, 1 \leq i \leq p \\ 0 & \text{otherwise for } n \leq p\frac{p-1}{2} \end{cases} \tag{2.4}$$

In particular, if $\mathcal{L} = \{p \in \mathcal{P} | p \geq 2n\}$ then there are no obstructions to extending a self map of $\mathbb{HP}^1_{\mathcal{L}}$ to $\mathbb{HP}^n_{\mathcal{L}}$ and we see that Sullivan's pullback square (2.3) implies the following result:

Lemma 2.6 *There exists a self map of $\mathbb{HP}^n$ of degree k iff there is a self map of $\mathbb{HP}^n_{(p)}$ of degree k for each prime $p < 2n$.*

We will need the following result of Rector [12, p. 103]:

Theorem 2.7 *If $k \in \mathbb{Z}_{(p)}$ is a p-adic square and a p-adic unit, there is a self map of $\mathbb{HP}^\infty_{(p)}$ of degree k.*

It follows that self maps $\mathbb{HP}^n_{(p)}$ with such degrees exist for all n. In fact, given a self map φ of $\mathbb{HP}^\infty_{(p)}$, we have

$$\begin{array}{ccccc} & & \mathbb{HP}^n_{(p)} & \hookrightarrow & \mathbb{HP}^\infty_{(p)} \\ & & \vdots\downarrow & & \big\downarrow \varphi \\ Y & \longrightarrow & \mathbb{HP}^n_{(p)} & \hookrightarrow & \mathbb{HP}^\infty_{(p)} \end{array}$$

where Y denotes the homotopy fibre of the inclusion. Since Y is $(4n+2)$-connected, φ factors through $\mathbb{HP}^n_{(p)}$ up to homotopy.

Recall [3, p. 184] that, if p is odd, $k \in \mathbb{Z}_p$ is a unit and a square iff k is a nonzero square mod p and if $p = 2$, this happens iff $k \equiv 1$ mod 8.

Lemma 2.8 *Let p be an odd prime and $k \in \mathbb{Z}$. If $p|k$ there exists a self map of $\mathbb{HP}^{(p+1)/2}_{(p)}$ of degree k.*

Proof. We begin by showing that $\mathbb{HP}^n$ and the James construction [17, p. 326] $\mathrm{J}^n(S^4)$ are equivalent at p for $n \leq (p-1)/2$. Consider the following diagram, where the rows are cofiber sequences:

$$\begin{array}{ccccccc} S^{4k+3}_{(p)} & \xrightarrow{g} & \mathrm{J}^k(S^4)_{(p)} & \longrightarrow & \mathrm{J}^{k+1}(S^4)_{(p)} & \longrightarrow & S^{4k+4}_{(p)} \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\lambda} \\ S^{4k+3}_{(p)} & \xrightarrow{h} & \mathbb{HP}^k_{(p)} & \longrightarrow & \mathbb{HP}^{k+1}_{(p)} & \longrightarrow & S^{4k+4}_{(p)} \end{array}$$

Here g is the attaching map for the standard cell decomposition of $\mathrm{J}(S^4)$ and ξ is a homotopy equivalence constructed by induction (we take the identity for $k = 1$). Assume $k < (p-1)/2$. By (2.4) and lemma 2.1, $\pi_{4k+3}\mathbb{HP}^k_{(p)} = \mathbb{Z}_{(p)}$ generated, as a $\mathbb{Z}_{(p)}$-module, by $[h]$ so there is $\lambda \in \mathbb{Z}_{(p)}$ such that the diagram commutes. Let χ be an induced map on the cofiber and v be a generator of $\mathrm{H}^4(\mathbb{HP}^{k+1}_{(p)}; \mathbb{Z}_{(p)})$. Then v^{k+1} generates $\mathrm{H}^{4k+4}(\mathbb{HP}^{k+1}_{(p)}; \mathbb{Z}_{(p)})$ and, as $\mathrm{H}^*(\mathrm{J}(S^4)) = \mathrm{H}^*(\Omega S^5)$ is a divided polynomial algebra [17, p. 326] , $\chi^*(v^{k+1})$ generates $\mathrm{H}^{4k+4}(\mathrm{J}^{k+1}(S^4)_{(p)}; \mathbb{Z}_{(p)})$. We conclude that λ is a unit in $\mathbb{Z}_{(p)}$ and hence χ is a homotopy equivalence.

We can now use this equivalence to prove that the obstruction to extending a self map of $\mathbb{HP}^{(p-1)/2}_{(p)}$ of degree p vanishes. Consider the diagram:

$$\begin{array}{ccccccc} S^{2p+1}_{(p)} & \xrightarrow{h} & \mathbb{HP}^{(p-1)/2}_{(p)} & \xrightarrow{e} & \mathrm{J}^{(p-1)/2}(S^4)_{(p)} & \hookrightarrow & \mathrm{J}(S^4)_{(p)} \\ & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle P} \\ & & \mathbb{HP}^{(p-1)/2}_{(p)} & \xrightarrow{e} & \mathrm{J}^{(p-1)/2}(S^4)_{(p)} & \hookrightarrow & \mathrm{J}(S^4)_{(p)} \end{array}$$

where p denotes a degree p map, e a homotopy equivalence and P the p^{th}-power map on the loop space $J(S^4) = \Omega S^5$. The diagram commutes up to homotopy because homotopy classes of maps from $\mathbb{HP}^{(p-1)/2}_{(p)}$ to $J(S^4)_{(p)}$ are classified by their degree (there are no obstructions to homotopy between maps of the same degree). We want to show that $o(p)$, the component of $p_*[h]$ in $\pi_{2p}(S^3_{(p)}) = \mathbb{Z}/p$, is 0. On this subgroup of $\pi_{2p+1}\mathbb{HP}^{(p-1)/2}_{(p)}$, the homomorphism induced by $\mathbb{HP}^{(p-1)/2}_{(p)} \longrightarrow J(S^4)_{(p)}$ is just the double suspension, which is an isomorphism [16, p. 177], so it is enough to check that applying this map to the obstruction yields 0. The result now follows from the fact that $\pi_{2p+1} J(S^4)_{(p)} = \pi_{2p+2}S^5_{(p)} = \mathbb{Z}/p$ and the power map P induces multiplication by p on the homotopy groups.

By (2.4), we have self maps of $\mathbb{HP}^{(p-1)/2}_{(p)}$ of any degree. Lemma 2.4 now implies that for any $l \in \mathbb{Z}_{(p)}$, the composite $pl : \mathbb{HP}^{(p-1)/2}_{(p)} \longrightarrow \mathbb{HP}^{(p-1)/2}_{(p)}$, extends to $\mathbb{HP}^{(p+1)/2}_{(p)}$. Indeed, $o(p) = 0$ and $o(l)$ is squashed like a bug by p. □

Lemma 2.9 *Let p be an odd prime. If $p^2|k$ there exists a self map of $\mathbb{HP}^p_{(p)}$ of degree k.*

Proof. By (2.4) there are no obstructions to extending self maps of $\mathbb{HP}^{(p-1)/2}_{(p)}$ to $\mathbb{HP}^{p-1}_{(p)}$. By lemma 2.8, for any $l \in \mathbb{Z}_{(p)}$ there are self maps φ, ψ of $\mathbb{HP}^{p-1}_{(p)}$ of degrees p and lp respectively. By lemma 2.4 and the fact (2.4) that p is an exponent for $\pi_{4p-2}(S^3_{(p)})$, $o(\varphi\psi) = 0$ so the composite of the two maps extends to give a map of the required degree $k = lp^2$. □

We can now give a different proof of the following result of McGibbon [8, Theorem 8, p. 483], which verifies the conjecture for $n = 3$:

Proposition 2.10 *If k satisfies the congruences (1.1) for $m = 1, 2, 3$ then there is a self map of $\mathbb{HP}^3$ of degree k.*

Proof. The only primes involved are 2,3, and 5. At the prime 2, the congruences (1.1) reduce to

$$k(k-1) \equiv 0 \mod 8$$

The proof of proposition 2.5 shows that, if k is a solution of this congruence, the obstruction to extending a map of degree k on $S^4_{(2)}$ to $\mathbb{HP}^2_{(2)}$ vanishes. Since [16, p. 186] $\pi_{10}S^3 = \mathbb{Z}/15$ has no 2-primary component, all of these maps extend to $\mathbb{HP}^3_{(2)}$.

At the prime 3, the congruences reduce to

$$k \equiv 0 \mod 9 \qquad \text{or} \qquad k \equiv 1 \mod 3$$

and the required maps are provided by lemma 2.9 and theorem 2.7 respectively.

At the prime 5, the congruences reduce to

$$k(k-1)(k-4) \equiv 0 \mod 5$$

and the required maps are provided by theorem 2.7 and lemma 2.8. □

Using the same argument, we can improve Proposition 2.4.(i) of [9]:

Proposition 2.11 *If $k \equiv 1$ mod 8 , k is a nonzero square mod p for $p < 2n/3 + 1$ odd and k satisfies the congruences (1.1) for $m = 1, \ldots, n$ then there is a self map of $\mathbb{HP}^n$ of degree k.*

Proof. We can assume that $n > 1$. By lemma 2.6, we have only to show that there is a self map of degree k of $\mathbb{HP}^n_{(p)}$ for $p < 2n$.

If $n < p < 2n$ then the set of congruences (1.1) reduces mod p to the single congruence

$$k(k-1)\ldots(k - \left(\frac{p-1}{2}\right)^2) \equiv 0 \mod p$$

k satisfies this equation iff k is a nonzero square mod p or $p|k$. In the first case, a self map of degree k is provided by theorem 2.7 and in the second case by lemma 2.8 and the observation (2.4) that there are no obstructions to the extension of a self map of $\mathbb{HP}^{(p-1)/2}_{(p)}$ to $\mathbb{HP}^n_{(p)}$.

If $2n/3+1 \leq p \leq n$ then the set of congruences 1.1 reduces mod p to

$$\begin{cases} k(k-1)\dots(k-\left(\frac{p-1}{2}\right)^2) \equiv 0 \mod p \\ k(k-1)\dots(k-(p-1)^2) \equiv 0 \mod p^2 \end{cases}$$

k satisfies these congruences iff k is a nonzero square mod p or $p^2|k$. In the first case, a self map of degree k exists by theorem 2.7 and in the second case by lemma 2.9 together with the fact (2.4) that there are no obstructions to the extension of a self map of $\mathbb{HP}^p_{(p)}$ to $\mathbb{HP}^n_{(p)}$.

Finally, if $p < 2n/3+1$ the maps are provided by theorem 2.7. □

The formula given in lemma 2.4 can be used to show [9, Proposition 2.4.(ii)] that if k is any even integer then there is a self map of $\mathbb{HP}^n$ of degree k^i for some $i \in \mathbb{N}$. This is a special case of the following

Proposition 2.12 *Let* $\{p_1 = 2, \dots, p_m\} = \{p \in \mathcal{P} | p < 2n\}$. *Let*

$$q_1 = \begin{cases} 8 & \textit{if } n \leq 3 \\ 2^{2^{n-3}3} & \textit{if } n > 3 \end{cases}$$

$$q_i = \begin{cases} p_i & \textit{if } p_i > n \\ p_i^{2^{n-p_i+1}} & \textit{if } p_i \leq n \end{cases}$$

If for each i, *either* $q_i|k$ *or* k *is a* p_i*-adic square and a* p_i*-adic unit then there is a self map of* $\mathbb{HP}^n$ *of degree* k.

Proof. We have only to check that if $q_i|k$ then there is a self map of degree k of $\mathbb{HP}^n_{(p_i)}$. This has already been seen in the cases when $i = 1, n \leq 3$ and $i > 1, n \leq p_i$.

The remaining cases follow by induction. If we have self maps of degree k of $\mathbb{HP}^n_{(p_i)}$ whenever $p_i^l|k$ for some $l > 2$ then, by lemma 2.4 and the facts [13] that 4 is an exponent for $\pi_i S^3_{(2)}$ and p odd is an exponent for $\pi_i S^3_{(p)}$, there are self maps of degree m of $\mathbb{HP}^{n+1}_{(p_i)}$ whenever $p_i^{2l}|m$. □

Finally we note that the set of integers satisfying (1.1) for all n are precisely the odd squares [4]. It was proved by Sullivan [15, p. 5.93] that there exist self maps of $\mathbb{HP}^\infty$ of these degrees (this also follows from theorem 2.7) thus the conjecture is true for $n = \infty$. Moreover, in this case, the self maps are classified up to homotopy by their degrees [11]. One should also note that McGibbon [9] has verified that the conjecture holds stably, in the sense that for each k satisfying (1.1) for $m = 1, \dots, n$ there is a stable self map of $\mathbb{HP}^n$ inducing the same homomorphism on homology as that which an unstable map of degree k would induce.

Bibliography

[1] M. Arkowitz and C. Curjel. On maps of H-spaces. *Topology*, 6:137–148, 1967.

[2] M. Atiyah and F. Hirzebruch. *Vector bundles and homogeneous spaces*, volume 3 of *Proceedings of Symposia in Pure Mathematics*, pages 7–38. AMS, 1961.

[3] D. Eisenbud. *Commutative algebra with a view towards algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, 1995.

[4] S. Feder and S. Gitler. Mappings of quaternionic projective spaces. *Bol. Soc. Mat. Mexicana*, 18:33–37, 1973.

[5] P. Hilton, G. Mislin, and J. Roitberg. *Localization of nilpotent groups and spaces*, volume 15 of *North-Holland Mathematics Studies*. North-Holland, 1975.

[6] D. Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer, 1994.

[7] H. Marcum and D. Randall. A note on self-mappings of quaternionic projective spaces. *An. Acad. Brasil. Ciênc.*, 48:7–9, 1976.

[8] C. McGibbon. Multiplicative properties of power maps. II. *Trans. Amer. Math. Soc.*, 274:479–508, 1982.

[9] C. McGibbon. Self maps of projective spaces. *Trans. Amer. Math. Soc.*, 271:325–346, 1982.

[10] M. Mimura and H. Toda. *Topology of Lie Groups*, volume 91 of *Translations of mathematical monographs*. AMS, 1991.

[11] G. Mislin. The homotopy classification of self-maps of infinite quaternionic projective space. *Quart. J. Math. Oxford (2)*, 38:245–257, 1987.

[12] D. Rector. *Loop structures on the homotopy type of S^3*, volume 249 of *Lecture Notes in Mathematics*, pages 99–105. Springer, 1971.

[13] P. Selick. Odd primary torsion in $\pi_k S^3$. *Topology*, 17:407–412, 1978.

[14] J. Stasheff. Homotopy associativity of H-spaces. II. *Trans. Amer. Math. Soc.*, 108:293–312, 1963.

[15] D. Sullivan. Geometric topology. I: Localization, periodicity and Galois symmetry. *MIT notes*, 1970.

[16] H. Toda. *Composition methods in homotopy groups of spheres*, volume 49 of *Annals of Mathematics Studies.* Princeton University Press, 1962.

[17] G. Whitehead. *Elements of Homotopy Theory*, volume 61 of *Graduate Texts in Mathematics.* Springer, 1978.