Noncommutative Ruled Surfaces

by

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Carnegie Mellon University, 1992

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1997

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Abstract

We construct noncommutative ruled surfaces, over a smooth curve X, as noncommutative projectivizations of rank 2 locally free $\mathcal{O}_X$-bimodules. The noncommutative analog of the projectivized vector bundle $\mathbf{P}_X(\mathcal{E})$ is discussed, and conditions are described for this construction to extend to the noncommutative situation. Locally free bimodules over the generic point of X are described, and many cases of noncommutative ruled surfaces are classified up to birational equivalence. Global bimodules are also described in detail, and geometric conditions are given for a bimodule to give a noncommutative surface. Finally, three examples over $X = \mathbf{P}^1$ are presented: sheaves of differential algebras, noncommutative deformations of commutative quadric surfaces, and quantum quadrics arising from Sklyanin algebras.

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Acknowledgments

First, I must thank my advisor, Michael Artin, for introducing me to the subject of noncommutative algebraic geometry, and for many, many helpful discussions on my research. This thesis would not have gotten off the ground without his insight, experience, and suggestions.

I would also like to thank the following people for many helpful ideas or conversations at one point or another: Daniel Chan, Mathieu Gagné, Colin Ingalls, Karen Smith, Toby Stafford, and Michel Van den Bergh.

Thanks to Phyllis Ruby and Linda Okun for administrative help.

Finally, thanks to my parents, Robert and Christine, for love and encouragement throughout.

This material is based upon work supported under a National Science Foundation Graduate Student Fellowship.

Contents

Chapter 1

Introduction

Much work has been done by Artin, Schelter, Stafford, Tate, Van den Bergh, and others attempting to classify graded rings of Gelfand-Kirillov-dimension 3 which are "nearly commutative" in some sense. For example, [ATV] describes those graded rings of GK-dimension 3 which are of finite global dimension and Gorenstein: these rings can be described as homogeneous coordinate rings of quantum projective planes.

The goal of this thesis is to develop a method for constructing GK-dimension 3 graded rings as homogeneous coordinate rings of a quantum ruled surface, which is a noncommutative analog of a ruled surface. Since dimension 3 commutative rings arise naturally as homogeneous coordinate rings of algebraic surfaces, it is reasonable to expect that noncommutative GK-dimension 3 graded rings may be constructed as homogeneous coordinate rings of quantized versions of algebraic surfaces. Many examples of GK-dimension 3 rings can be formed in this fashion, including homogeneous coordinate rings of quantum quadric surfaces, quotients of 4-dimensional Sklyanin algebras, and quotients of quantum homogeneous enveloping algebras of the Lie algebra $\mathfrak{sl}_2$.

We start with the usual construction of an algebraically ruled surface S over a smooth projective curve X, namely $S = \mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle over X. This construction describes S as a $\mathbf{P}^1$-bundle over X. Specifically, $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}\, S(\mathcal{E})$ where $S(\mathcal{E})$ is the sheaf of symmetric algebras of $\mathcal{E}$.

Van den Bergh has proposed the idea of constructing a noncommutative ruled surface by taking $\mathcal{E}$ to be, instead of a vector bundle, a rank 2 *bimodule* over X; i.e. $\mathcal{E}$ is an $\mathcal{O}_X$-bimodule which is locally free of rank 2 as both a left module and as a right module.

In chapter 2 we define bimodules, following the definitions of Artin and Van den Bergh in [AV] and [V]: $\mathcal{E}$ is defined as a quasi-coherent sheaf on the product space $X \times X$, such that the support of $\mathcal{E}$ is relatively locally finite over each factor of X (i.e. each coherent subsheaf has support which is finite over each factor of X). We think of $\mathcal{E}$ as a left module via its image under the projection to the first factor of X, and as a right module via its image under the projection to the second factor of X.

Next, we make the noncommutative definition of $\mathbf{P}(\mathcal{E})$. The resulting object that we get is a noncommutative scheme. If this scheme has a polarization, which is a noncommutative analog of a projective embedding, then we may construct a homogeneous coordinate ring from the scheme. However, not all rank 2 bimodules yield noncommutative schemes, as there are additional conditions on $\mathcal{E}$ in order to form the noncommutative analog of $\mathbf{P}(\mathcal{E})$. In order for $\mathcal{E}$ to generate a quantum ruled surface, it is necessary that $\mathcal{E} \otimes \mathcal{E}$ contains a rank 1 subbimodule. This is necessary for the existence of the noncommutative analog of $S(\mathcal{E})$. Recall that in the commutative case $S(\mathcal{E}) = T(\mathcal{E})/\mathcal{R}$, where $T(\mathcal{E})$ is the sheaf of tensor algebras of $\mathcal{E}$ and $\mathcal{R}$ is the subsheaf of $T(\mathcal{E})$ generated by sections of the form $(x \otimes y - y \otimes x)$. The noncommutative analog of $\mathcal{R}$ is generated by a rank 1 subbimodule $\mathcal{Q}$ of $\mathcal{E} \otimes \mathcal{E}$, which is not decomposable into a tensor product $\mathcal{Q} \cong \mathcal{L} \otimes \mathcal{M}$ of rank 1 subbimodules of $\mathcal{E}$. If such a subbimodule $\mathcal{Q}$ exists, we call $\mathcal{E}$ *admissible*. In this case, set $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$, where $(\mathcal{Q})$ is the subbimodule of $T(\mathcal{E})$ generated by $\mathcal{Q}$. The quantum ruled surface is then $\underline{\text{Proj}}\, \mathcal{B}$.

By analyzing the behavior of the bimodule $\mathcal{E}$ at the generic point η of X, we may develop a theory of birational equivalence of quantum ruled surfaces. In the commutative case, we say that two surfaces S_1 and S_2 are birationally equivalent if the functions fields $K(S_1), K(S_2)$ are isomorphic. Also recall that the function field of Proj R, where R is a commutative graded domain, is given by $(\text{Frac}\, R)_0$, the degree-0 component of the graded field of fractions of R. In chapter 3, we discuss this concept for the noncommutative ruled surfaces described above.

We first classify the possible rank 2 bimodules at the generic point. This classification, given in Theorem 3.2, categorizes bimodules into three cases, depending on how the right bimodule action is expressed in terms of the left bimodule action.

More specifically, let E be a locally free rank 2 bimodule over $K = K(X)$, with generators denoted y and z. However, the action of K on the left differs from the action of K on the right. If we let the row vector $\begin{pmatrix} p & q \end{pmatrix}$ denote the element $py + qz$ for any $p, q \in K$, then the right action can be described, for any $\lambda \in K$, as

$$\begin{pmatrix} p & q \end{pmatrix} \cdot \lambda = \begin{pmatrix} p & q \end{pmatrix} M(\lambda),$$

where M is a homomorphism from K to $M_2(K)$, the algebra of 2×2 matrices with elements in K. The possibilites for M, and hence the possible K-bimodules E of rank 2, are explicitly described.

It follows that the structure of the bimodule algebra $\mathcal{B}$ at η can be described in terms of M and the subbimodule $\mathcal{Q}$, which corresponds to an eigenvector of $M \circ M$, the composition of M with M (which will be defined precisely in chapter 3).

Let $\mathcal{E}_1, \mathcal{E}_2$ be two rank 2 locally free admissible bimodules, with corresponding bimodule algebras $\mathcal{B}_1, \mathcal{B}_2$; here $\mathcal{B}_i = T(\mathcal{E}_i)/(\mathcal{Q}_i)$ for suitable $\mathcal{Q}_i \subseteq \mathcal{E}_i \otimes \mathcal{E}_i$, for $i = 1, 2$. We call the two quantum ruled surfaces given by the bimodule algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ *birationally equivalent* if the degree-0 components of the graded skew fields of fractions $\operatorname{Frac}(\mathcal{B}_1), \operatorname{Frac}(\mathcal{B}_2)$ at η are isomorphic as skew algebras over $K(X)$. Birational equivalence classes for the bimodule algebras resulting from the three cases of rank 2 locally free K-bimodules of Theorem 3.2 are determined.

We begin chapter 4 by describing the structure of locally free bimodules of low rank. Rank 1 locally free bimodules are easily described in Theorem 2.8 as being supported on the graph of an automorphism of X. We classify rank 2 bimodules $\mathcal{E}$ by Theorem 4.5 into four different classes, based on the geometry of the support of $\mathcal{E}$ inside of $X \times X$. If $Y = \operatorname{Supp} \mathcal{E}$, then the four classes are

(1) $\mathcal{E}$ is defined over a nonreduced bidegree $(1, 1)$ divisor,

(2) Y is a reduced bidegree $(1,1)$ divisor,

(3) Y is a reducible bidegree $(2,2)$ divisor,

(4) Y is an irreducible bidegree $(2,2)$ divisor.

We then describe precise necessary and sufficient conditions for $\mathcal{E}$ to be admissible in the four cases.

In case (1), every $\mathcal{E}$ is admissible. Specific admissibility conditions for case (2) are unknown. In case (3), the geometry of $\mathcal{E}$ is determined by two automorphisms of X, and we show that $\mathcal{E}$ is admissible only if the two automorphisms commute or have equal squares. In case (4), we have the following result:

Theorem 4.12 *Let $\mathcal{E}$ be a rank 2 locally free bimodule supported on an irreducible curve Y in $X \times X$, such that the projections π_1, π_2 from Y onto each factor of X are finite of degree 2. Then $\mathcal{E}$ is admissible if and only if there exists a birational automorphism ϕ of Y and an automorphism σ of X such that $\pi_2 = \pi_1\phi$ and $\pi_1\phi^2 = \sigma\pi_1$.*

In chapter 5, we conclude with examples of homogeneous coordinate rings of quantum ruled surfaces over $X = \mathbf{P}^1$ using the techniques developed. The first class of examples is quantum ruled surfaces arising from homogenized differential algebras, which is locally given by the algebra $k[u]\langle y, z\rangle$, where u is a local coordinate on $\mathbf{P}^1$, with relations

$$yu - uy - z, yz - zy, uz - zu.$$

The degree 1 component of this algebra is a rank 2 locally free bimodule which is in case (1). The surface constructed has as its homogeneous coordinate ring a central quotient of a homogeneous quantum enveloping algebra of $\mathfrak{sl}_2$. This algebra has been studied in [LbS].

Another class of examples is deformations of quadric surfaces. These belong to cases (2) and (3) over the curve $\mathbf{P}^1$.

The last example, constructed by Van den Bergh in [V] though using different techniques, is a bimodule of case (4) supported on an elliptic curve in $\mathbf{P}^1 \times \mathbf{P}^1$ which gives a quantum quadric surface over $\mathbf{P}^1$ whose homogeneous coordinate ring is a quotient of a 4-dimensional Sklyanin algebra by a central quadratic element. This algebra has been studied extensively in [SS] and [LvS].

Chapter 2

Bimodules

2.1 Definition

Recall that in the affine setting, if R is a commutative ring and M is an (R, R)-bimodule, we may think of M as a left $(R \otimes R)$-module, where the module action is defined as follows (for $r \in R, m \in M$):

$$\begin{aligned} rm &= (r \otimes 1)m \\ mr &= (1 \otimes r)m \end{aligned}$$

More generally, if A is an arbitrary (non-commutative) ring, then (A, A)-bimodules are equivalent to left $(A \otimes A^{op})$-modules, where A^{op} is the opposite ring to A, in which multiplication order is reversed.

We wish to extend this idea of representing bimodules to sheaves of bimodules over a projective scheme X. The definitions presented here follow the conventions set forth in [AV] and [V].

Fix throughout an algebraically closed field k of characteristic 0. Let X be a projective scheme over k, and $\mathcal{O}_X$ its structure sheaf. Let $\mathrm{pr}_1, \mathrm{pr}_2$ denote the canonical projections from $X \times X$ to X.

Definition 2.1 *Let $f : Y \to X$ be a morphism of finite type between noetherian schemes, and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_Y$-module. We say that $\mathcal{M}$ is* relatively locally finite (rlf) *for* f *if, for all coherent* $\mathcal{M}' \subseteq \mathcal{M}$ *with support* $Z \subseteq Y$*, the restriction* $f|_Z : Z \to X$ *is finite.*

Definition 2.2 *An* $\mathcal{O}_X$-bimodule *is a quasi-coherent sheaf on* $X \times X$ *which is rlf for the two projections* $\mathrm{pr}_1, \mathrm{pr}_2$ *from* $X \times X$ *to* X.

This definition has the effect of defining an $\mathcal{O}_X$-bimodule to be an $\mathcal{O}_{X\times X}$-module. This definition is consistent with the affine case described above.

If $\mathcal{M}$ is an $\mathcal{O}_X$-bimodule, then we think of $\mathcal{M}$ as a left $\mathcal{O}_X$-module via the projection to the first factor, and $\mathcal{M}$ as a right $\mathcal{O}_X$-module via the projection to the second factor. In situations where there is no ambiguity, we will use the notation $\mathcal{M}$ to denote both the bimodule on X and the sheaf on $X \times X$.

We introduce the convention that for a bimodule $\mathcal{M}$, the notation $\mathcal{M}(U)$ for an open set $U \subseteq X$ is taken to mean the sections of the left module structure of $\mathcal{M}$; i.e.

$$\mathcal{M}(U) = (\mathrm{pr}_{1*}\mathcal{M})(U) = \mathcal{M}(U \times X),$$

where $\mathcal{M}(U \times X)$ are the sections of $\mathcal{M}$ over the open set $U \times X$ in $X \times X$.

Under this convention, $\mathcal{M}(U)$ is a left $\mathcal{O}_X(U)$-module. However, $\mathcal{M}(U)$ does not in general have a bimodule structure, as $\mathcal{M}(U)$ is not a right $\mathcal{O}_X(U)$-module. The sections which have a right $\mathcal{O}_X(U)$-module structure are precisely $\mathcal{M}(X \times U)$, which are not the same sections as $\mathcal{M}(U \times X) = \mathcal{M}(U)$.

Some additional notation: If $\mathcal{F}$ is a coherent sheaf on X, then we define $\mathcal{F}^\sigma := \sigma^*\mathcal{F}$, so that $\mathcal{F}^\sigma(U) = \mathcal{F}(\sigma U)$. Also denote by f^σ the image of f under the map $\sigma^* : \mathcal{F} \to \mathcal{F}$. So if $f \in \mathcal{F}(U)$, then $f^\sigma \in \mathcal{F}(\sigma U)$.

If σ is an automorphism of X and $\mathcal{L}$ is a coherent sheaf over X, then we define the bimodule $\mathcal{L}_\sigma$ by

$$\mathcal{L}_\sigma = (\mathrm{pr}_1^*\mathcal{L}) \otimes \mathcal{O}_\Gamma = \pi_1^*\mathcal{L},$$

where Γ is the graph of σ in $X \times X$, $\mathrm{pr}_1 : X \times X \to X$ is the projection in the first factor, and $\pi_1 : \Gamma \to X$ is the restriction of pr_1 to Γ.

$\mathcal{L}_\sigma$, thought of as a left module, is the original sheaf $\mathcal{L}$, and the right module structure of $\mathcal{L}_\sigma$ is given by

$$\mathrm{pr}_{2*}\mathcal{L}_\sigma = \sigma_*\mathcal{L} = \mathcal{L}^{\sigma^{-1}}.$$

Locally, we can view $\mathcal{L}_\sigma(U)$ as an $(\mathcal{O}_X(U), \mathcal{O}_X(\sigma U))$-bimodule, and in particular for $x \in \mathcal{L}_\sigma(U), a \in \mathcal{O}_X(U)$,

$$ax = xa^\sigma.$$

We also have a notion of the tensor product of two bimodules. This is done by lifting both bimodules up to $X^3 = X \times X \times X$. Let $\mathrm{pr}_{12}, \mathrm{pr}_{13}, \mathrm{pr}_{23}$ denote the three projections from X^3 to $X \times X$ (so, for instance, pr_{12} is the projection in the first two factors). If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{O}_X$-bimodules, then we define the $\mathcal{O}_X$-bimodule $\mathcal{M} \otimes \mathcal{N}$ by

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = \mathrm{pr}_{13*}\left((\mathrm{pr}_{12}^*\mathcal{M}) \otimes_{\mathcal{O}_{X^3}} (\mathrm{pr}_{23}^*\mathcal{N})\right).$$

In other words, we pull $\mathcal{M}$ up from $X \times X$ to X^3 via the projection to the first two factors, we pull $\mathcal{N}$ up via the projection to the last two factors, tensor them together (as $\mathcal{O}_{X^3}$ modules), and then push the resulting module back down to $X \times X$ by projection to the first and last factors. This is analogous to the affine case, in that the right action of $\mathcal{M}$ "commutes through the tensor" with the left action of $\mathcal{N}$.

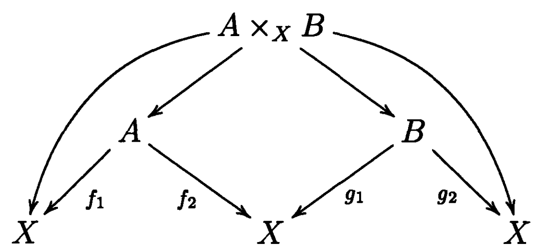
The geometric properties of the bimodule tensor operation can be described by the following lemma.

Lemma 2.3 *Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{O}_X$-bimodules, with supports A, B (respectively) in $X \times X$. Denote the projection maps from A to the two copies of X by f_1, f_2 and the projection maps from B to the two copies of X by g_1, g_2. Let $A \times_X B$ denote the fibre product of A and B given by the following diagram:*

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle g_1} \\ A & \xrightarrow{f_2} & X \end{array}$$

Then the support of $\mathcal{M} \otimes \mathcal{N}$ is the image of $A \times_X B$ in $X \times X$, via the projections

f_1 from A to X and g_2 from B to X, as in the following diagram:



Proof: Let $\mathcal{P}$ be the $\mathcal{O}_{X^3}$-module $(\mathrm{pr}_{12}^*\mathcal{M} \otimes_{\mathcal{O}_{X^3}} \mathrm{pr}_{23}^*\mathcal{N})$. Then for an open set $U \times V \times W \subseteq X^3$,

$$\mathcal{P}(U \times V \times W) = \mathcal{M}(U \times V) \otimes_{\mathcal{O}_{X^3}(U\times V\times W)} \mathcal{N}(V \times W)$$

where $\mathcal{M}, \mathcal{N}$ are thought of as modules over $\mathcal{O}_{X^3}$ via $\mathrm{pr}_{12}^*, \mathrm{pr}_{23}^*$ respectively. Hence for any point $p \in X^3$,

$$\begin{aligned} p \in \operatorname{Supp}\mathcal{P} &\Leftrightarrow (\mathrm{pr}_{12}(p) \in \operatorname{Supp}\mathcal{M}) \text{ and } (\mathrm{pr}_{23}(p) \in \operatorname{Supp}\mathcal{N}) \\ &\Leftrightarrow (\mathrm{pr}_{12}(p) \in A) \text{ and } (\mathrm{pr}_{23}(p) \in B). \end{aligned}$$

But

$$\begin{aligned} A \times_X B &= \{(r,s) \in A \times B \mid f_2(r) = g_1(s)\} \\ &= \{(r_1, r_2, s_1, s_2) \in X^4 \mid (r_1, r_2) \in A, (s_1, s_2) \in B, r_2 = s_1\} \\ &= \{(t_1, t_2, t_3) \in X^3 \mid (t_1, t_2) \in A, (t_2, t_3) \in B\} \\ &= \operatorname{Supp}\mathcal{P}. \end{aligned}$$

Finally, the support of $\mathcal{M} \otimes \mathcal{N}$ on $X \times X$ is given by $\operatorname{Supp}\mathrm{pr}_{13*}\mathcal{P} = \mathrm{pr}_{13}(\operatorname{Supp}\mathcal{P})$. But this is the set $\mathrm{pr}_{13}(A \times_X B)$, mapped to $X \times X$ via the outer curved arrows in the diagram above. $\square$

For bimodules of the form $\mathcal{L}_\sigma$, described above, we have the following result (from [AV]). The proof is a matter of chasing the definitions.

Lemma 2.4 ([AV, 2.14]) $\mathcal{L}_\sigma \otimes_{\mathcal{O}_X} \mathcal{M}_\tau \cong (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}^\sigma)_{\tau\sigma}$.

Let us interpret the tensor product in Lemma 2.4 locally. Recall that $\mathcal{L}(U)$ is an $(\mathcal{O}_X(U), \mathcal{O}_X(\sigma U))$-bimodule, and that $\mathcal{M}^\sigma(U) = \mathcal{M}(\sigma U)$ is an $(\mathcal{O}_X(\sigma U), \mathcal{O}_X(\tau\sigma U))$-bimodule. Therefore, $(\mathcal{L} \otimes \mathcal{M}^\sigma)(U) = \mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{M}(\sigma(U))$ has the structure of an $(\mathcal{O}_X(U), \mathcal{O}_X(\tau\sigma U))$-bimodule.

Definition 2.5 *A bimodule $\mathcal{L}$ is called* invertible *if there exists a bimodule $\mathcal{M}$ such that $\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O} \cong \mathcal{M} \otimes \mathcal{L}$. $\mathcal{M}$ is called the* inverse *of $\mathcal{L}$, and is denoted $\mathcal{L}^{-1}$.*

Proposition 2.6 ([AV, 2.15]) *An $\mathcal{O}_X$-bimodule $\mathcal{M}$ is invertible if and only if it is isomorphic to a bimodule of the form $\mathcal{L}_\sigma$, where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module and σ is an automorphism of X.*

We will be especially concerned with bimodules which are locally free when considered as left or right modules, as in the following definition:

Definition 2.7 *Let $\mathcal{M}$ be an $\mathcal{O}_X$-bimodule, thought of as a coherent sheaf on $X \times X$. We say that $\mathcal{M}$ is* locally free *if $\mathrm{pr}_{1*}(\mathcal{M})$ and $\mathrm{pr}_{2*}(\mathcal{M})$ are each locally free on X. If both of these modules are locally free of the same rank r, then we say that $\mathcal{M}$ is* locally free of rank r *as well.*

This definition states that $\mathcal{M}$ is a locally free bimodule if $\mathcal{M}$ is locally free as a left module and as a right module, and has rank r if the left and right module ranks are both r.

It is easy to characterize the rank 1 locally free bimodules.

Theorem 2.8 *Let X be a smooth curve, and suppose $\mathcal{M}$ is a rank 1 locally free $\mathcal{O}_X$-bimodule. Then $\mathcal{M} \cong \mathcal{L}_\sigma$ for some rank 1 locally free module $\mathcal{L}$ over X and some automorphism σ of X.*

Proof: Let us consider the affine case first. A commutative ring R is given, and we wish to determine the possible commutative rings S with embeddings $R \hookrightarrow S$, and the possible S-modules M, with $\operatorname{Ann} M = 0$ (i.e. M is a faithful S-module), such that M, when considered as an R-module by restriction of scalars, is isomorphic to the rank 1 free module R.

Let e be a generator of M. We claim that $S \cong R$. Suppose not, and let $s \in S \backslash R$ be given. Then $se = ae$ for some $a \in R$. But then for any $r \in R$, $s(re) = a(re)$. Thus s acts on M the same way that a does, and hence $s - a \neq 0$ is in the annihilator of M. But this contradicts the fact that M is faithful. Hence $S \cong R$.

Geometrically, this says that over any affine open subset $U \subseteq X$, the support of $\mathcal{M}$ is isomorphic to U, and $\mathcal{M}$ is locally free of rank 1 on that support. Thus, when considered globally, $\mathcal{M}$ must be supported on a bidegree (1,1) divisor isomorphic to each factor of X, and be locally free of rank 1 over that divisor. But every such bidegree (1,1) divisor of $X \times X$ is the graph of some automorphism σ of X, and the result follows. □

2.2 Bimodule Algebras

We have the following definition of a bimodule algebra from Van den Bergh:

Definition 2.9 ([V]) *An $\mathcal{O}_X$-bimodule $\mathcal{B}$ is called a* bimodule algebra *if it is endowded with $\mathcal{O}_X$-linear maps $u : \mathcal{O}_X \to \mathcal{B}$ and $\mu : \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{B} \to \mathcal{B}$ which satisfy the usual algebra axioms; i.e. the following diagrams commute, where id denotes the appropriate identity map and ${}_{\mathcal{O}}\mu, \mu_{\mathcal{O}}$, denote respectively left and right scalar multiplication:*

Associativity:

$$\begin{array}{ccc} \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & \xrightarrow{id \otimes \mu} & \mathcal{B} \otimes \mathcal{B} \\ {\scriptstyle \mu \otimes id}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{B} \otimes \mathcal{B} & \xrightarrow[\mu]{} & \mathcal{B} \end{array}$$

Left identity:

$$\begin{array}{ccc} \mathcal{O} \otimes \mathcal{B} & \xrightarrow{u \otimes id} & \mathcal{B} \otimes \mathcal{B} \\ & {\scriptstyle {}_{\mathcal{O}}\mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & \mathcal{B} & \end{array}$$

Right identity:

$$\begin{array}{ccc} \mathcal{B}\otimes\mathcal{O} & \xrightarrow{id\otimes u} & \mathcal{B}\otimes\mathcal{B} \\ & {\scriptstyle \mu_{\mathcal{O}}} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & \mathcal{B} & \end{array}$$

This is not a local definition. In general, $\mathcal{B}$ does not come equipped with local multiplication maps

$$\mathcal{B}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{B}(U) \to \mathcal{B}(U).$$

More specifically, multiplication provides local maps

$$(\mathcal{B} \otimes \mathcal{B})(U) \to \mathcal{B}(U),$$

and recall that

$$\begin{aligned} (\mathcal{B}\otimes\mathcal{B})(U) &= (\mathcal{B}\otimes\mathcal{B})(U\times X) \\ &= (\mathrm{pr}_{13*}(\mathrm{pr}_{12}^*\mathcal{B}\otimes\mathrm{pr}_{23}^*\mathcal{B}))(U\times X) \\ &= (\mathrm{pr}_{12}^*\mathcal{B}\otimes\mathrm{pr}_{23}^*\mathcal{B})(U\times X\times X) \\ &= (\mathrm{pr}_{12}^*\mathcal{B})(U\times X\times X)\otimes(\mathrm{pr}_{23}^*\mathcal{B})(U\times X\times X). \end{aligned}$$

In particular, let $b_1 \in \mathcal{B}(U), b_2 \in \mathcal{B}(V)$ be sections with $Y_1 = \mathrm{Supp}(b_1), Y_2 = \mathrm{Supp}(b_2)$, with $Y_1, Y_2 \subseteq X \times X$. Let the projections be denoted $\pi_i^j : Y_j \to X_i$; hence $\pi_1^1(Y_1) = U$ and $\pi_1^2(Y_2) = V$. Then $b_1 \otimes b_2$ is defined in $\mathcal{B}(U)$ if and only if $\pi_2^1(Y_1) = \pi_1^2(Y_2)$. In other words, we require that the open set upon which the right action on b_1 is defined is equal to the open set upon which the left action on b_2 is defined.

The definition of bimodule algebra can easily be extended to give a definition for graded bimodule algebras. This again is presented by Van den Bergh.

Definition 2.10 ([V]) *An $\mathcal{O}_X$-bimodule $\mathcal{B}$ which is a direct sum $\mathcal{B} = \oplus\mathcal{B}_n$ of bimodules is called a* graded bimodule algebra *if it is equipped with $\mathcal{O}_X$-linear maps $u : \mathcal{O}_X \to \mathcal{B}_0$ and $\mu_{m,n} : \mathcal{B}_m \otimes_{\mathcal{O}_X} \mathcal{B}_n \to \mathcal{B}_{m+n}$ which satisfy the usual graded algebra axioms.*

As above, a graded bimodule algebra in general does not have local graded algebra structure.

Given a bimodule $\mathcal{E}$, we wish to construct a bimodule algebra which is the noncommutative analog of the symmetric algebra of $\mathcal{E}$. Recall the definition from commutative algebraic geometry: If $\mathcal{E}$ is a (commutative) sheaf over X, then the symmetric algebra $S(\mathcal{E})$ is defined as

$$S(\mathcal{E}) = \frac{T(\mathcal{E})}{(x \otimes y - y \otimes x)}$$

where $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ and x and y range over all sections of $\mathcal{E}$. This can also be defined locally as

$$S(\mathcal{E})(U) = S(\mathcal{E}(U))$$

for any open set $U \subseteq X$, where $S(\mathcal{E}(U))$ is the usual symmetric algebra of $\mathcal{E}(U)$. However, as bimodules do not have such a nice local structure, this definition is not directly extendible to the noncommutative situation, and must be suitably modified.

We begin our definition by explicitly stating the definition of the twisted tensor algebra of a bimodule $\mathcal{E}$.

Definition 2.11 *Let $\mathcal{E}$ be a bimodule over X. The* twisted tensor algebra *of $\mathcal{E}$, denoted $T(\mathcal{E})$, is defined as*

$$T(\mathcal{E}) = \bigoplus_{i=0}^{\infty} \mathcal{E}_n$$

where $\mathcal{E}_0 = \mathcal{O}_X$ (the trivial bimodule), and

$$\mathcal{E}_n = \mathcal{E}^{\otimes n} = \underbrace{\mathcal{E} \otimes \mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{n \text{ times}}.$$

The multiplication $\mathcal{E}_i \otimes \mathcal{E}_j \to \mathcal{E}_{i+j}$ is the canonical isomorphism given by the tensor product of the bimodules $\mathcal{E}_i$ and $\mathcal{E}_j$, and the unit map is the canonical isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathcal{E}_0$.

$T(\mathcal{E})$ is itself a bimodule, since any coherent submodule of $T(\mathcal{E})$ as an $\mathcal{O}_{X\times X}$-module must be contained in the sum of finitely many of the $\mathcal{E}_i$, and hence has finite support. However, in general $T(\mathcal{E})$ is not a sheaf of algebras in the usual commutative sense, as it does not have an algebra structure locally.

However, for all $\mathcal{E}$, there are two natural ways in which $T(\mathcal{E})$ yields an algebra structure. First, let $\eta \in X$ be the generic point of X. Then $\mathcal{E}_\eta$ has the same sections as a left module and as a right module (see [V, 2.9]), so we can think of $\mathcal{E}_\eta$ as a $K(X)$-bimodule, where $K(X) = \mathcal{O}_{X,\eta}$, the function field of X. Also, by ([V, 2.10]),

$$(\mathcal{E} \otimes \mathcal{E})_\eta = \mathcal{E}_\eta \otimes_{K(X)} \mathcal{E}_\eta.$$

Thus, $T(\mathcal{E})_\eta$ is an skew K-algebra. We will examine these skew algebras in more detail in Chapter 3.

Second, we may think of global sections of $\mathcal{E}$ as a k-bimodule, since

$$\mathrm{pr}_{1*}\mathcal{E}(X) = \mathrm{pr}_{2*}\mathcal{E}(X) = \mathcal{E}(X \times X)$$

(i.e. $\mathcal{E}$ has the same global sections when thought of as a left module and as a right module). Also, tensor product gives us a map

$$\mathcal{E}(X) \otimes \mathcal{E}(X) \to (\mathcal{E} \otimes \mathcal{E})(X),$$

and hence $H^0(X, T(\mathcal{E}))$ has the structure of a k-algebra. This algebra will be considered in Section 2.4.

Recall that we wish to make the noncommutative analog of the commutative construction

$$S(\mathcal{E}) = \frac{T(\mathcal{E})}{(x \otimes y - y \otimes x)}.$$

We may write this as

$$S(\mathcal{E}) = \frac{T(\mathcal{E})}{(\mathcal{Q})},$$

where $\mathcal{Q}$ is the locally free submodule of $\mathcal{E} \otimes \mathcal{E}$ generated by all sections of the form $(x \otimes y - y \otimes x)$.

In the noncommutative case, let $\mathcal{Q}$ be a locally free subbimodule of the bimodule $\mathcal{E} \otimes \mathcal{E}$. Here $\mathcal{Q}$ is the noncommutative analog of the submodule $(x \otimes y - y \otimes x)$ in degree 2. We then form the quotient

$$\mathcal{B} = \frac{T(\mathcal{E})}{(\mathcal{Q})}$$

where $(\mathcal{Q})$ is the subbimodule ideal of $T(\mathcal{E})$ generated by $\mathcal{Q}$ (e.g. the degree 3 component of $(\mathcal{Q})$ is $(\mathcal{Q} \otimes \mathcal{E}) + (\mathcal{E} \otimes \mathcal{Q}))$. This quotient is well-defined as a bimodule, since we may consider it as a quotient of $\mathcal{O}_{X \times X}$-modules.

In general, in the noncommutative situation, we will not simply be able to take $\mathcal{Q}$ to be the subbimodule of $\mathcal{E} \otimes \mathcal{E}$ generated by sections of the form $x \otimes y - y \otimes x$, as these sections are generally not defined. For example, suppose sections $x \in \mathcal{E}(U)$, $y \in \mathcal{E}(V)$ for open sets $U, V \subseteq X$ are given. Then the section $x \otimes y \in (\mathcal{E} \otimes \mathcal{E})(U)$ is defined if and only if $\pi_2^{-1}(U) = \pi_1^{-1}(V)$. Similarly, the section $y \otimes x \in (\mathcal{E} \otimes \mathcal{E})(V)$ is defined if and only if $\pi_1^{-1}(U) = \pi_2^{-1}(V)$. In general these conditions are not simultaneously satisfied, so the section $x \otimes y - y \otimes x$ is undefined. In fact, $x \otimes y$ and $y \otimes x$, if defined, are defined as sections over different open sets unless $U = V$, and even in this case we require that $\pi_1^{-1}(U) = \pi_2^{-1}(U)$.

If $\mathcal{E}$ is a locally free rank 2 bimodule, then we will want $\mathcal{Q}$ to be a locally free rank 1 subbimodule of $\mathcal{E} \otimes \mathcal{E}$. Later, we will impose additional conditions on $\mathcal{Q}$, but for now we suffice to find *any* rank 1 locally free subbimodule of $\mathcal{E} \otimes \mathcal{E}$.

Definition 2.12 *We say that a rank 2 locally free bimodule $\mathcal{E}$* admits a quadratic relation *if there exists a rank 1 locally free subbimodule $\mathcal{Q}$ of $\mathcal{E} \otimes \mathcal{E}$.*

Proposition 2.13 *If $\mathcal{E}$ admits a quadratic reltaion, then the support of $\mathcal{E} \otimes \mathcal{E}$ in $X \times X$ has a component which is a divisor of bidegree (1,1).*

Proof: Let W be the support of $\mathcal{E} \otimes \mathcal{E}$ in $X \times X$. If $\mathcal{Q} \subseteq \mathcal{E} \otimes \mathcal{E}$ is a subbimodule of rank 1, let C be the support of $\mathcal{Q}$, which is necessarily a component of W. Then Theorem 2.8 states that C is a divisor of bidegree (1,1). □

We wish to determine the conditions under which the bimodule algebra $\mathcal{B}$ has suitably nice properties, given that $\mathcal{B}$ should be a noncommutative analog of a symmetric algebra sheaf. One such property should be that the Hilbert series of $\mathcal{B}$ should coincide with the Hilbert series of a symmetric algebra. Recall that the Hilbert series of $\operatorname{Sym}\mathcal{F}$ for $\mathcal{F}$ a commutative locally free rank 2 module is

$$\operatorname{rank}(\operatorname{Sym}\mathcal{F})_n = n+1.$$

Hence, we want that $\operatorname{rank}\mathcal{B}_n = n+1$ as well.

We will make use of the following lemma, due to Van den Bergh:

Lemma 2.14 ([V, 3.4]) *Let $\mathcal{B}$ be a bimodule algebra, $\mathcal{M}$ an $\mathcal{O}_X$-module, and $\mathcal{N}$ a $\mathcal{B}$-module. Then there is an isomorphism*

$$\operatorname{Hom}_{\mathcal{B}}(\mathcal{B}\otimes\mathcal{M},\mathcal{N}) \cong \operatorname{Hom}_{\mathcal{O}}(\mathcal{M},\mathcal{N}).$$

Since we have the following $\mathcal{O}_X$-module maps:

$$\begin{aligned} \mathcal{O} &\to \mathcal{O} \\ \mathcal{E} &\to \mathcal{B} \\ \mathcal{Q} &\to \mathcal{B}\otimes\mathcal{E}, \end{aligned}$$

we may use Lemma 2.14 to extend these maps to $\mathcal{B}$-module maps:

$$\begin{aligned} \mathcal{B}\otimes\mathcal{O} &\to \mathcal{O} \\ \mathcal{B}\otimes\mathcal{E} &\to \mathcal{B} \\ \mathcal{B}\otimes\mathcal{Q} &\to \mathcal{B}\otimes\mathcal{E}, \end{aligned}$$

and get a sequence of $\mathcal{B}$-module maps

$$(\mathcal{B}\otimes\mathcal{Q}) \longrightarrow (\mathcal{B}\otimes\mathcal{E}) \longrightarrow \mathcal{B} \longrightarrow \mathcal{O}.$$

These are in fact graded $\mathcal{B}$-module maps (a $\mathcal{B}$-module map $\mathcal{M} \to \mathcal{N}$ of graded $\mathcal{B}$-modules is a *graded* $\mathcal{B}$-module map if $\mathcal{M}_n$ maps to $\mathcal{N}_n$ for all n), with appropriate degree shifts:

$$(\mathcal{B} \otimes \mathcal{Q})(-2) \longrightarrow (\mathcal{B} \otimes \mathcal{E})(-1) \longrightarrow \mathcal{B} \longrightarrow \mathcal{O},$$

where the notations (-1) and (-2) indicate graded shifts, so that for instance

$$(\mathcal{B} \otimes \mathcal{E})(-1)_1 = (\mathcal{B} \otimes \mathcal{E})_0 = \mathcal{E}.$$

Definition 2.15 *Let $\mathcal{B}$ be a bimodule algebra constructed by $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$ as above. We say that $\mathcal{B}$ has an* admissible resolution *if the sequence*

$$0 \longrightarrow (\mathcal{B} \otimes \mathcal{Q})(-2) \longrightarrow (\mathcal{B} \otimes \mathcal{E})(-1) \longrightarrow \mathcal{B} \longrightarrow \mathcal{O} \longrightarrow 0$$

is exact.

This gives us the desired Hilbert series for $\mathcal{B}$, although we need an extra condition on $\mathcal{Q}$.

Definition 2.16 *Let $\mathcal{E}$ be a rank 2 locally free bimodule which admits a quadratic relation, and let $\mathcal{Q}$ be a rank 1 subbimodule of $\mathcal{E} \otimes \mathcal{E}$. Set $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$. We say that $\mathcal{Q}$ is* fully saturated *if $\mathcal{B}_n$ is locally free for all $n \geq 0$.*

Theorem 2.17 *Suppose $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$ is a bimodule algebra with an admissible resolution, where $\mathcal{Q}$ is fully saturated. Then*

$$\operatorname{rank} \mathcal{B}_n = n + 1.$$

Proof: Let $h_n = \operatorname{rank} \mathcal{B}_n$; this is well-defined since $\mathcal{Q}$ is fully saturated. Then $h_0 = 1$, $h_1 = 2$, and

$$\operatorname{rank}(\mathcal{B} \otimes \mathcal{Q})(-2)_n = h_{n-2}$$

and

$$\operatorname{rank}(\mathcal{B} \otimes \mathcal{E})(-1)_n = 2h_{n-1},$$

so the exact sequence gives for all $n > 1$:

$$h_n = 2h_{n-1} - h_{n-2}.$$

This solves to give $h_n = n + 1$. $\square$

Suppose $\mathcal{E}$ has an admissible resolution. If we examine the exact sequence of Definition 2.15 in degree 3, we conclude that the sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{Q} \longrightarrow \frac{\mathcal{E} \otimes \mathcal{E}}{\mathcal{Q}} \otimes \mathcal{E} \longrightarrow \frac{\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E}}{\mathcal{E} \otimes \mathcal{Q} + \mathcal{Q} \otimes \mathcal{E}} \longrightarrow 0$$

is exact. But this sequence is exact if and only if the sum $\mathcal{E} \otimes \mathcal{Q} + \mathcal{Q} \otimes \mathcal{E}$ is direct; i.e. if

$$(\mathcal{E} \otimes \mathcal{Q}) \cap (\mathcal{Q} \otimes \mathcal{E}) = 0.$$

This condition is therefore necessary for $\mathcal{E}$ to have an admissible resolution.

Note that if this condition holds, then the sequence of Definition 2.15 is precisely the Koszul complex corresponding to $\mathcal{B}$, where the definition of Koszul algebra (as in [Ma]) is extended to bimodule sheaves. So the condition that $\mathcal{E}$ has an admissible complex may be equivalent to $\mathcal{B}$ being a "Koszul bimodule algebra."

We can express an equivalent algebraic condition for $\mathcal{E}$ to have an admissible resolution as follows:

Let $\mathcal{R}$ denote the subbimodule of $\mathcal{B}$ generated by $\mathcal{Q}$, so that $\mathcal{B} = T(\mathcal{E})/\mathcal{R}$. Then the canonical sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow T(\mathcal{E}) \longrightarrow \mathcal{B} \longrightarrow 0$$

is exact. Furthermore, $\mathcal{E}$ is a flat $\mathcal{O}_X$-bimodule, since it is locally free; flatness of locally free bimodules follows directly from flatness of locally free $\mathcal{O}_X$-modules. The operation $(\cdot \otimes \mathcal{E})$ locally is tensoring by a free module, although it rearranges the open sets. Thus, the sequence

$$0 \longrightarrow \mathcal{R} \otimes \mathcal{E} \longrightarrow T(\mathcal{E}) \otimes \mathcal{E} \longrightarrow \mathcal{B} \otimes \mathcal{E} \longrightarrow 0$$

is also exact.

We then get the following commutative diagram, where the two rows are exact:

$$\begin{array}{ccccccccc}
 & & & & 0 & & \ker & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}\otimes\mathcal{E} & \longrightarrow & T(\mathcal{E})\otimes\mathcal{E} & \longrightarrow & \mathcal{B}\otimes\mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & T(\mathcal{E}) & \longrightarrow & \mathcal{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{R}/(\mathcal{R}\otimes\mathcal{E}) & & \mathcal{O} & = & \mathcal{O} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

There is a natural surjection $(\mathcal{B}\otimes\mathcal{Q}) \to \ker$, hence by the Snake Lemma there is a natural map

$$\mathcal{B}\otimes\mathcal{Q} \to \frac{\mathcal{R}}{\mathcal{R}\otimes\mathcal{E}},$$

and $\mathcal{B}$ has an admissible resolution if and only if this map is an isomorphism. In degree n, this map is

$$\mathcal{B}_{n-2}\otimes\mathcal{Q} \to \frac{\mathcal{R}_n}{\mathcal{R}_{n-1}\otimes\mathcal{E}}.$$

For $n = 0, 1$, this is trivial (both sides are 0). For $n = 2$, it is also trivial (both sides are $\mathcal{Q}$). For $n = 3$, $\mathcal{R}_3 = (\mathcal{E}\otimes\mathcal{Q}) + (\mathcal{Q}\otimes\mathcal{E})$ by definition, and the condition is

$$\mathcal{E}\otimes\mathcal{Q} \cong \frac{(\mathcal{E}\otimes\mathcal{Q}) + (\mathcal{Q}\otimes\mathcal{E})}{\mathcal{Q}\otimes\mathcal{E}},$$

and this holds if and only if the sum $(\mathcal{E}\otimes\mathcal{Q}) + (\mathcal{Q}\otimes\mathcal{E})$ is direct, as discussed above.

In degree n,

$$(\mathcal{B}\otimes\mathcal{Q})_n = \frac{\mathcal{E}^{n-2}\otimes\mathcal{Q}}{\mathcal{R}_{n-2}\otimes\mathcal{Q}},$$

hence the condition is that the kernel of the natural map

$$\mathcal{E}^{n-2}\otimes\mathcal{Q} \to \frac{\mathcal{R}_n}{\mathcal{R}_{n-1}\otimes\mathcal{E}}$$

is precisely $\mathcal{R}_{n-2} \otimes \mathcal{Q}$. But this map is simply the composition of the inclusion of $\mathcal{E}^{n-2} \otimes \mathcal{Q}$ into $\mathcal{R}_n$, with the quotient by $\mathcal{R}_{n-1} \otimes \mathcal{E}$. Hence $\mathcal{E}$ has an admissible resolution if and only if

$$(\mathcal{E}^{n-2} \otimes \mathcal{Q}) \cap (\mathcal{R}_{n-1} \otimes \mathcal{E}) = (\mathcal{R}_{n-2} \otimes \mathcal{Q})$$

for all $n \geq 2$.

2.3 Admissibility

Unfortunately, admitting a quadratic relation is not a strong enough condition for $\mathcal{E}$, even if we include the condition that $\mathcal{Q} \otimes \mathcal{E}$ and $\mathcal{E} \otimes \mathcal{Q}$ do not meet. For example, consider the situation of algebras of global dimension 2. In some sense, the algebra

$$k\langle x, y\rangle/(xy - yx)$$

is a "good" algebra, whereas the algebra

$$k\langle x, y\rangle/(xy)$$

is a "bad" algebra (in particular, it is not a domain), even though both $(xy - yx)$ and (xy) have no overlaps in degree 3:

$$(kx \oplus ky)(xy - yx) \cap (xy - yx)(kx \oplus ky) = 0,$$

and

$$(kx \oplus ky)(xy) \cap (xy)(kx \oplus ky) = 0.$$

The precise condition we wish to impose here is called *regularity*, in the sense of [AS]:

Definition 2.18 ([AS]) *A graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is called* regular *if:*

(i) A has finite global dimension d,

(ii) A has finite gk-dimension,

(iii) A is Gorenstein; i.e.

$$\mathrm{Ext}^q_A(k, A) \cong \begin{cases} 0 & \text{if } q \neq d \\ k & \text{if } q = d \end{cases}$$

We would like to extend this definition of regularity to bimodule algebras. Before we can state such a definition, we need to discuss the definition of $\underline{\mathrm{Hom}}$ for bimodules.

First, let us recall the definition of global Hom. Let $\mathcal{B}$ be an arbitrary bimodule algebra, and let $\mathcal{M}, \mathcal{N}$ be $\mathcal{B}$-modules. An element $f \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{M}, \mathcal{N})$ is an element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ which satisfies the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{N} \\ {\scriptstyle \mu_M}\uparrow & & \uparrow{\scriptstyle \mu_N} \\ \mathcal{B} \otimes \mathcal{M} & \xrightarrow{1 \otimes f} & \mathcal{B} \otimes \mathcal{N} \end{array}$$

where μ_M, μ_N are the $\mathcal{B}$-module multiplication maps.

To define $\underline{\mathrm{Hom}}$, we need to require the above diagram for all open sets; i.e. for an open set $U \subseteq X$, we define the sections $\underline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{M}, \mathcal{N})$ to be those sections $f \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(U) & \xrightarrow{f} & \mathcal{N}(U) \\ {\scriptstyle \mu_M}\uparrow & & \uparrow{\scriptstyle \mu_N} \\ (\mathcal{B} \otimes \mathcal{M})(U) & \xrightarrow{1 \otimes f} & (\mathcal{B} \otimes \mathcal{N})(U) \end{array}$$

However, the added difficulty here is that in general, $(\mathcal{B} \otimes \mathcal{M})(U) \neq \mathcal{B}(U) \otimes \mathcal{M}(U)$. In particular,

$$\begin{aligned} (\mathcal{B} \otimes M)(U) &= (\mathrm{pr}_{1*}(\mathcal{B} \otimes \mathrm{pr}_2^* \mathcal{M}))(U) \\ &= (\mathcal{B} \otimes \mathrm{pr}_2^* \mathcal{M})(U \times X) \\ &= \mathcal{B}(U \times X) \otimes (\mathrm{pr}_2^* \mathcal{M})(U \times X). \end{aligned}$$

In other words, if $b \in \mathcal{B}(U), m \in \mathcal{M}(V)$ such that $b \otimes m \in (\mathcal{B} \otimes \mathcal{M})(U)$, then $\pi_2^{-1}(U) = \pi_1^{-1}(V)$, where π_1, π_2 are the projection maps from the support of b to each factor of X.

Define $\underline{\text{Ext}}$ to be the derived functor of $\underline{\text{Hom}}$ in the usual fashion. We may now extend the definition of regular to bimodule algebras:

Definition 2.19 *Suppose $\mathcal{E}$ is a rank 2 locally free bimodule which admits a quadratic relation, with $\mathcal{Q} \subseteq \mathcal{E} \otimes \mathcal{E}$ a rank 1 locally free subbimodule. Let $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$. We say that $\mathcal{B}$ is* regular *if*

$$\underline{\text{Ext}}^i_{\mathcal{B}}(\mathcal{O}, \mathcal{B}) = \begin{cases} 0 & \text{if } i \neq 2 \\ \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{Q}, \mathcal{O}) & \text{if } i = 2 \end{cases}$$

To see the motivation of this definition, suppose $\mathcal{B}$ has an admissible resolution, and take the exact sequence from Definition 2.15:

$$0 \longrightarrow (\mathcal{B} \otimes \mathcal{Q})(-2) \longrightarrow (\mathcal{B} \otimes \mathcal{E})(-1) \longrightarrow \mathcal{B} \longrightarrow \mathcal{O} \longrightarrow 0$$

and apply the functor $\underline{\text{Hom}}_{\mathcal{B}}(-, \mathcal{B})$. We use the facts that

$$\underline{\text{Hom}}_{\mathcal{B}}(\mathcal{B} \otimes \mathcal{M}, \mathcal{N}) = \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$$

(this is an extension of Lemma 2.14 to $\underline{\text{Hom}}$) and that if $\mathcal{M}$ is locally free, then

$$\underline{\text{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{B}) = \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{O}) \otimes \mathcal{B}.$$

This gives us a complex (where $\mathcal{M}^\vee = \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{O})$):

$$0 \longleftarrow (\mathcal{Q}^\vee \otimes \mathcal{B})(2) \longleftarrow (\mathcal{E}^\vee \otimes \mathcal{B})(1) \longleftarrow \mathcal{B} \longleftarrow 0.$$

The Gorenstein condition says that this resolution should be exact in the first two positions, and that the last map should have cokernel $\mathcal{Q}^\vee$ in degree -2.

Unfortunately, regularity is technically difficult to work with. So for now we will work with a simpler condition, motivated by the following discussion of regular algebras of dimension 2.

Consider an algebra of the form

$$A = k\langle x, y\rangle/(f)$$

where f is a quadratic polynomial. Then A is regular if and only if f is not decomposable into a product of linear polynomials (see [AS] and [I]). We wish to mimic this definition for bimodule algebras:

Definition 2.20 *Let $\mathcal{E}$ be an rank 2 locally free bimodule which admits a quadratic relation. We say that $\mathcal{E}$ is* admissible *if there exists a rank 1 locally free subbimodule $\mathcal{Q} \subseteq \mathcal{E} \otimes \mathcal{E}$ which cannot be decomposed as $\mathcal{Q} = \mathcal{L} \otimes \mathcal{M}$, for rank 1 subbimodules $\mathcal{L}, \mathcal{M} \subseteq \mathcal{E}$.*

Note: If $\mathcal{E}$ is admissible as above, we will often say that $\mathcal{Q}$ is an *admissible* rank 1 subbimodule.

This is clearly a weaker definition than the AS-regular condition for algebras given above, as this definition in some sense only addresses the "local" properties of $\mathcal{E}$. For a more precise definition, we need to use the Gorenstein condition for bimodule algebras. However, as this definition is not well understood, we proceed using the more basic admissibility condition.

2.4 Noncommutative Projective Schemes

Recall the definition of a noncommutative projective scheme from [AZ]:

Definition 2.21 ([AZ]) *Let B be a noncommutative graded ring. We define the category*

$$(B - \mathrm{qgr}) = (B - \mathrm{gr})/(\mathrm{tors})$$

where $(B - \mathrm{gr})$ *is the category of all finitely generated* B*-graded left modules, and* (tors) *is the subcategory of all torsion* B*-modules, that is those modules* M *where* $B_n M = 0$ *for all* $n \gg 0$.

In this definition, we are not specifying a particular polarization for the scheme.

The goal now is to form a noncommutative projective scheme in the sense of [AZ] for a bimodule algebra $\mathcal{B}$. Define

$$(\mathcal{B} - \mathrm{qgr}) = (\mathcal{B} - \mathrm{gr})/(\mathrm{tors})$$

where $(\mathcal{B} - \mathrm{gr})$ is the category of finitely generated graded $\mathcal{B}$-modules, and (tors) is the subcategory of torsion $\mathcal{B}$-modules: $\mathcal{M}$ is *torsion* if $\mathcal{M}_n = 0$ for all $n \gg 0$.

We may then use a result of Van den Bergh to describe $(\mathcal{B} - \mathrm{qgr})$; first, we need a notion of ampleness.

Definition 2.22 ([V, 5.1]) *We say that* $\mathcal{B}$ *is* ample *if for any coherent module* $\mathcal{M}$ *over* X*, and for* $n \gg 0$, $\mathcal{B}_n \otimes \mathcal{M}$ *is generated by global sections and* $H^q(X, \mathcal{B}_n \otimes \mathcal{M}) = 0$ *for all* $q > 0$.

Theorem 2.23 ([V, 5.2]) *Let* $\mathcal{B}$ *be a left-noetherian bimodule algebra over* X *and assume* $\mathcal{B}$ *is ample. Then* $B = \Gamma(\mathcal{B})$ *is noetherian and there are inverse equivalences*

$$(B - \mathrm{qgr}) \longleftrightarrow (\mathcal{B} - \mathrm{qgr}).$$

Therefore, via this theorem, to understand $\underline{\mathrm{Proj}}\, \mathcal{B}$, we need only compute B, the algebra of global sections of $\mathcal{B}$. Then $\underline{\mathrm{Proj}}\, \mathcal{B} = \mathrm{Proj}\ B$.

We now ask whether two different bimodule algebras give the same noncommutative projective scheme. Recall that for commutative ruled surfaces, we have the following result (see, for instance, [B, III.7]):

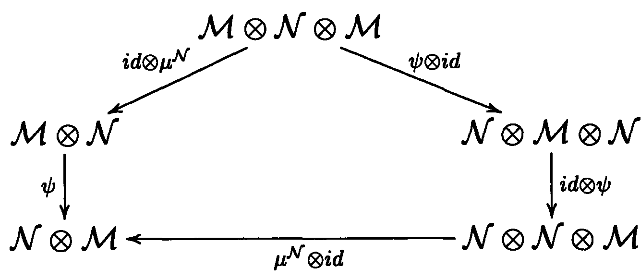
Theorem 2.24 *Let* X *be a curve,* $\mathcal{E}$ *a vector bundle over* X*, and* $\mathcal{L}$ *a line bundle over* X*. Then* $\mathbf{P}_X(\mathcal{E}) = \mathbf{P}_X(\mathcal{E} \otimes \mathcal{L})$.

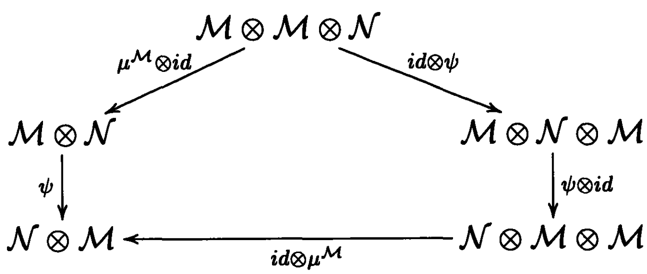
In other words, we can twist a locally free vector bundle by any rank 1 line bundle without changing the resulting ruled surface. It is reasonable to inquire if a similar result holds for noncommutative ruled surfaces. However, it is not so simple to tensor a bimodule by a line bundle. Given a line bundle with suitable nice bimodule properties, however, we may state a result similar to the commutative result above.

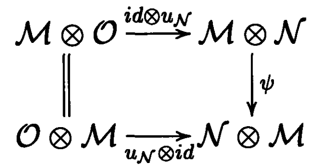
Definition 2.25 *Let $\mathcal{M}$,$\mathcal{N}$ be bimodule algebras, with multiplication maps $\mu^{\mathcal{M}}, \mu^{\mathcal{N}}$ and unit maps $u_{\mathcal{M}}, u_{\mathcal{N}}$. We say that $\mathcal{M}$,$\mathcal{N}$ are* compatible (through ψ) *if there exists an $\mathcal{O}$-linear isomorphism*

$$\psi : \mathcal{M} \otimes \mathcal{N} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{M}$$

which is compatible with multiplication and the unit map; i.e. the following diagrams commute (where id denotes the appropriate identity map)







(three other unit compatibility diagrams omitted)

Proposition 2.26 *Suppose $\mathcal{M}$,$\mathcal{N}$ are graded bimodule algebras which are compatible through ψ, such that the compatibility maps preserve the grading. Let $\mu^{\mathcal{M}}_{m,n}, \mu^{\mathcal{N}}_{m,n}$*

denote the multiplication maps and $u_{\mathcal{M}}, u_{\mathcal{N}}$ denote the unit maps on $\mathcal{M}, \mathcal{N}$ respectively. Define $\mathcal{N}'_n = \mathcal{N}_n \otimes \mathcal{M}_n$. Then there exist canonical multiplication maps $\mu^{\mathcal{N}'}_{m,n} : \mathcal{N}'_m \otimes \mathcal{N}'_n \to \mathcal{N}'_{m+n}$, and a unit map $u^{\mathcal{N}'} : \mathcal{O} \to \mathcal{N}'_0$ such that $\mathcal{N}'$ is a bimodule algebra.

Proof: Define $\mu^{\mathcal{N}'}_{m,n}$ to be the composition

$$\begin{aligned}
\mathcal{N}'_m \otimes \mathcal{N}'_n &= (\mathcal{N}_m \otimes \mathcal{M}_m) \otimes (\mathcal{N}_n \otimes \mathcal{M}_n) \\
&= \mathcal{N}_m \otimes (\mathcal{M}_m \otimes \mathcal{N}_n) \otimes \mathcal{M}_n \\
&\xrightarrow{id \otimes \psi \otimes id} \mathcal{N}_m \otimes (\mathcal{N}_n \otimes \mathcal{M}_m) \otimes \mathcal{M}_n \\
&= (\mathcal{N}_m \otimes \mathcal{N}_n) \otimes (\mathcal{M}_m \otimes \mathcal{M}_n) \\
&\xrightarrow{\mu^{\mathcal{N}} \otimes \mu^{\mathcal{M}}} \mathcal{N}_{m+n} \otimes \mathcal{M}_{m+n} \\
&= \mathcal{N}'_{m+n}.
\end{aligned}$$

Define $u_{\mathcal{N}'}$ to be the composition

$$\mathcal{O} = \mathcal{O} \otimes \mathcal{O} \xrightarrow{u_{\mathcal{M}} \otimes u_{\mathcal{N}}} \mathcal{M}_0 \otimes \mathcal{N}_0 = \mathcal{M}'_0$$

Associativity of $\mu^{\mathcal{N}'}$ is given by the diagram in Figure 2-1. The upper two triangles are commutative trivially. The middle two triangles are commutative by the compatibility conditions. The lower rectangle is commutative by the associativity of multiplication on $\mathcal{M}$ and $\mathcal{N}$.

The right unit axiom is verified by the commutative diagram in Figure 2-2. The upper and lower triangles are commutative by the right unit axioms on $\mathcal{M}$ and $\mathcal{N}$ respectively. The middle rectangle is commutative by the compatibility axioms. A similar diagram exists for left units. □

Note: Some general results regarding algebra objects in tensor categories can be found in [MacL]. Specifically, the category of $\mathcal{B}$-modules is a tensor category. Then the condition for algebra objects $\mathcal{M}, \mathcal{N}$ to give an algebra object $\mathcal{M} \otimes \mathcal{N}$ is precisely the compatibility condition of Definition 2.25.

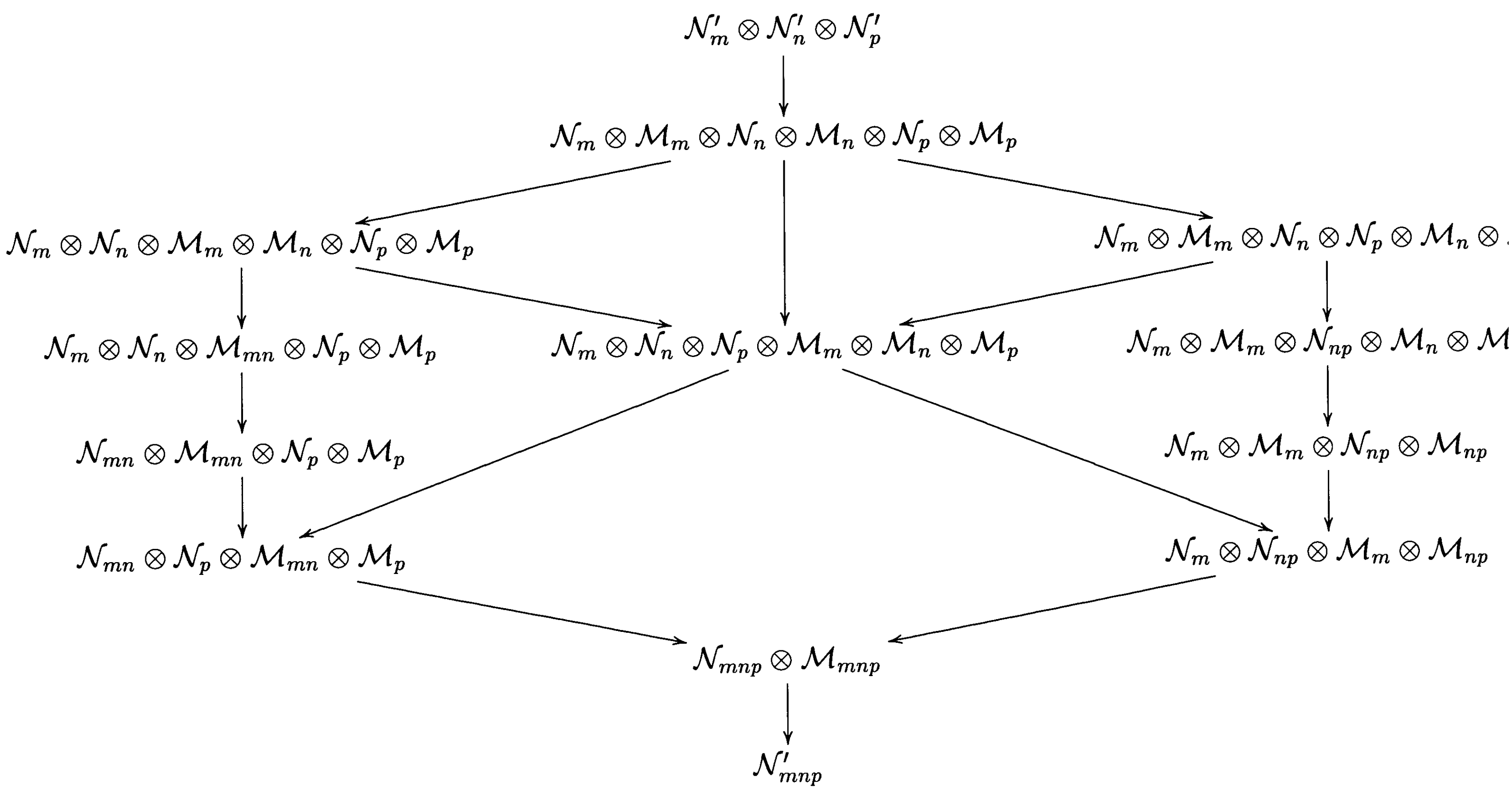


Figure 2-1: Associative law

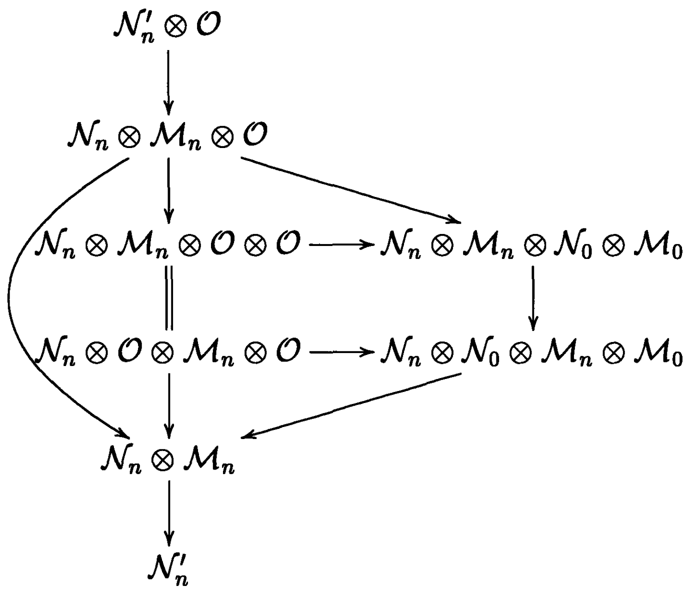


Figure 2-2: Unit axiom

Theorem 2.27 *Let $\mathcal{B}$ be a graded bimodule algebras, and $\mathcal{L}$ a* **Z***-graded bimodule algebra; i.e.*

$$\mathcal{L} = \bigoplus_{n=-\infty}^{\infty} \mathcal{L}_n$$

such that $\mathcal{L}_n$ is an invertible bimodule for all $n \geq 0$, and $\mathcal{L}_n^{-1} = \mathcal{L}_{-n}$. Let $\mathcal{L}_+$ denote the subbimodule

$$\mathcal{L}_+ = \bigoplus_{n=0}^{\infty} \mathcal{L}_n.$$

Suppose that $\mathcal{B}$ and $\mathcal{L}$ are graded compatible, and let $\mathcal{B}' = \mathcal{B} \otimes \mathcal{L}_+$ be the bimodule algebra described in Proposition 2.26. Then there is an equivalence of categories

$$(\mathcal{B} - \mathrm{qgr}) = (\mathcal{B}' - \mathrm{qgr}).$$

Proof: To show the equivalence of categories, we construct an invertible functor

$$F : (\mathcal{B} - \mathrm{gr}) \to (\mathcal{B}' - \mathrm{gr})$$

so that, for any $\mathcal{B}$-module $\mathcal{M}$,

$$(F(\mathcal{M}))_n = \mathcal{L}_n \otimes \mathcal{M}_n.$$

Let $\mu^{\mathcal{M}}$ denote the multiplication map $\mathcal{B} \otimes \mathcal{M} \to \mathcal{M}$. We define a multiplication map $\mathcal{B}' \otimes \mathcal{M}' \to \mathcal{M}'$ by the composition

$$\begin{array}{rcl} \mathcal{B}'_m \otimes \mathcal{M}'_n & = & \mathcal{B}_m \otimes \mathcal{L}_m \otimes \mathcal{L}_n \otimes \mathcal{M}_n \\ & \xrightarrow{id \otimes \mu^L_{m,n} \otimes id} & \mathcal{B}_m \otimes \mathcal{L}_{m+n} \otimes \mathcal{M}_n \\ & \xrightarrow{\psi_{m,m+n} \otimes id} & \mathcal{L}_{m+n} \otimes \mathcal{B}_m \otimes \mathcal{M}_n \\ & \xrightarrow{id \otimes \mu^{\mathcal{M}}_{m,n}} & \mathcal{L}_{m+n} \otimes \mathcal{M}_{m+n} \\ & = & \mathcal{M}'_{m+n}. \end{array}$$

It is easy (but tedious) to check that these maps satisfy the module multiplication and unit axioms.

The inverse functor is

$$(F^{-1}(\mathcal{M}'))_n = (\mathcal{L}_n)^{-1} \otimes \mathcal{M}_n.$$

Then

$$\begin{array}{rcl} (F^{-1}(F(\mathcal{M})))_n & = & (\mathcal{L}_n)^{-1} \otimes F(\mathcal{M})_n \\ & = & (\mathcal{L}_n)^{-1} \otimes \mathcal{L}_n \otimes \mathcal{M}_n \\ & = & \mathcal{M}_n. \end{array}$$

The multiplication on $F^{-1}(F(\mathcal{M}))$ is the same as the multiplication on $\mathcal{M}$, due to the compatibility axioms.

Finally, F sends torsion $\mathcal{B}$-modules to torsion $\mathcal{B}'$-modules. □

A special case of Theorem 2.27 is the case where $\mathcal{B}$ is the algebra

$$\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$$

where $\mathcal{E}$ is an admissible rank 2 locally free bimodule, and $\mathcal{F}$ is an invertible bimodule such that

$$\mathcal{L} = \bigoplus_{n=-\infty}^{\infty} \mathcal{F}^{\otimes n}$$

where $\mathcal{F}^{\otimes -n} = (\mathcal{F}^{-1})^{\otimes n} = (\mathcal{F}^{\otimes n})^{-1}$. If the supports of $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{F} \otimes \mathcal{E}$ are equal, then it is reasonable to expect that $\mathcal{B}$ and $\mathcal{L}$ will be compatible. This is the closest noncommutative analog of the commutative algebraic geometry equivalence $\mathbf{P}_X(\mathcal{E} \otimes \mathcal{F}) = \mathbf{P}_X(\mathcal{E})$

There is another case of equivalence of categories, due to Van den Bergh, which does not have a commutative analog.

Proposition 2.28 **([V])** *Let $\mathcal{B}$ be a bimodule algebra, and $\mathcal{L}$ an invertible bimodule. Define the bimodule $\mathcal{B}'$ by*

$$\mathcal{B}' = \mathcal{L}^{-1} \otimes \mathcal{B} \otimes \mathcal{L}.$$

Then

$$\underline{\mathrm{Proj}}\, \mathcal{B} = \underline{\mathrm{Proj}}\, \mathcal{B}'.$$

Proof: $\mathcal{B}'$ is a bimodule algebra, since we can define multiplication as follows:

$$\begin{aligned} \mathcal{B}'_m \otimes \mathcal{B}'_n &= (\mathcal{L}^{-1} \otimes \mathcal{B}_m \otimes \mathcal{L}) \otimes (\mathcal{L}^{-1} \otimes \mathcal{B}_n \otimes \mathcal{L}) \\ &= \mathcal{L}^{-1} \otimes \mathcal{B}_m \otimes \mathcal{B}_n \otimes \mathcal{L} \\ &\xrightarrow{id \otimes \mu \otimes id} \mathcal{L}^{-1} \otimes \mathcal{B}_{m+n} \otimes \mathcal{L} \\ &= \mathcal{B}'_{m+n}. \end{aligned}$$

Then we may define an equivalence of categories

$$(\mathcal{B} - \mathrm{gr}) \to (\mathcal{B}' - \mathrm{gr})$$

by sending a $\mathcal{B}$-module $\mathcal{M}$ to the $\mathcal{B}'$-module $(\mathcal{L}^{-1} \otimes \mathcal{M})$. The inverse equivalence is given by tensoring on the left by $\mathcal{L}$. □

Chapter 3

Birational Equivalence

3.1 Algebras at the Generic Point

Let X be a reduced, irreducible curve; for most examples X will be smooth. Let $K = K(X)$ be the function field of X. Let $\mathcal{E}$ be a rank 2 locally free $\mathcal{O}_X$-bimodule. Recall that at the generic point $\eta \in X$, the tensor algebra $T(\mathcal{E})_\eta$ has the structure of a skew K-algebra, since

$$(\mathcal{E} \otimes \mathcal{E})_\eta = \mathcal{E}_\eta \otimes_K \mathcal{E}_\eta.$$

At the generic point, $\mathcal{E}_\eta$ is a rank 2 locally free K-bimodule; that is, $\mathcal{E}_\eta$ is a rank 2 K-vector space when thought of as a left module and when thought of as a right module. Choose a basis y, z for the left module ${}_K(\mathcal{E}_\eta)$, such that y, z is also a basis for the right module $(\mathcal{E}_\eta)_K$. (A generic left-basis y, z for $\mathcal{E}_\eta$ will satisfy this condition.) We wish to analyze how the left and right structures of $\mathcal{E}_\eta$ differ.

For any $\lambda \in K$, we may express right multiplication by λ in terms of left multiplication by elements of K. In particular:

$$\begin{aligned} y\lambda &= a(\lambda)y + b(\lambda)z \\ z\lambda &= c(\lambda)y + d(\lambda)z \end{aligned}$$

where a, b, c, d are functions from K to K. More generally, we may express any element of $\mathcal{E}_\eta$ as a rank 2 row vector, where the vector

$$\begin{pmatrix} p & q \end{pmatrix}$$

corresponds to the element $py+qz$, for any $p, q \in K$. Then we may express the above equations in matrix form:

$$\begin{pmatrix} p & q \end{pmatrix} \cdot \lambda = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}.$$

On the left side of the above equation, the notation

$$\begin{pmatrix} p & q \end{pmatrix} \cdot \lambda$$

does not denote scalar multiplication of p and q by λ; it instead denotes the right action of λ on the vector $py+qz$ in the right module structure of $\mathcal{E}_\eta$. However, the matrix multiplication on the right side of the equation is standard matrix multiplication over K.

If we let M denote the map from K to the ring of 2-by-2 matrices with elements in K (denoted $K^{2\times 2}$), which sends λ to the matrix

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix},$$

then

$$\begin{pmatrix} p & q \end{pmatrix} \cdot \lambda = \begin{pmatrix} p & q \end{pmatrix} M(\lambda).$$

Proposition 3.1 *$M : K \to K^{2\times 2}$ is a k-algebra homomorphism.*

Proof: Clearly $M(\lambda+\mu) = M(\lambda) + M(\mu)$ and for $s \in k$,

$$M(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

since the action of k is central; i.e. it acts the same on the left and on the right. Furthermore,

$$\begin{aligned}\begin{pmatrix} p & q \end{pmatrix} \cdot (\lambda\mu) &= (\begin{pmatrix} p & q \end{pmatrix} M(\lambda)) \cdot \mu \\ &= \begin{pmatrix} p & q \end{pmatrix} M(\lambda)M(\mu)\end{aligned}$$

so $M(\lambda\mu) = M(\lambda)M(\mu)$. □

We now classify the possible K-bimodule structures, based on the structure of the map M.

Theorem 3.2 *Let E be a free rank 2 K-bimodule. Then there exists a basis y, z of E as a left module, with right action given by*

$$\begin{pmatrix} p & q \end{pmatrix} \cdot \lambda = \begin{pmatrix} p & q \end{pmatrix} M(\lambda),$$

such that $M(\lambda)$ is one of the following:

(i)

$$M(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & d(\lambda) \end{pmatrix}$$

with $a, d \in \mathrm{Aut}_k(K)$, $a \neq d$.

(ii)

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ 0 & a(\lambda) \end{pmatrix}$$

with $a \in \mathrm{Aut}_k(K)$, and b an a-derivation; i.e. for all $\lambda, \mu \in K$,

$$b(\lambda\mu) = a(\lambda)b(\mu) + b(\lambda)a(\mu).$$

(iii)

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ mb(\lambda) & a(\lambda) \end{pmatrix}$$

such that

(a) a is a k-linear map,

(b) b is a non-zero a-derivation: i.e. $b(c) = 0$ for $c \in k$ and for all $\lambda, \mu \in K$,

$$b(\lambda)b(\mu) = a(\lambda)b(\mu) + b(\lambda)a(\mu),$$

and

(c) $m \in K$ is not a perfect square, such that for all $\lambda, \mu \in K$,

$$a(\lambda\mu) = a(\lambda)a(\mu) + mb(\lambda)b(\mu).$$

Proof: Choose a basis y, z of E, and denote

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$$

for some functions $a, b, c, d : K \to K$.

We divide the argument into three cases, which correspond to the three cases in the result of the theorem.

Case (i): Suppose there exists some $\bar{\lambda} \in K$ such that $M(\bar{\lambda})$ has two distinct eigenvalues in K. Choose a basis y, z such that $M(\bar{\lambda})$ is diagonal; then

$$M(\bar{\lambda}) = \begin{pmatrix} a(\bar{\lambda}) & 0 \\ 0 & d(\bar{\lambda}) \end{pmatrix}.$$

We now use the fact that in $K^{2\times 2}$, diagonal matrices with distinct nonzero elements only commute with other diagonal matrices; to be precise, the centralizer of a matrix

$$\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$$

with $m \neq n$ both nonzero is

$$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}.$$

Hence, since the image of M is commuative in $K^{2\times 2}$, every matrix $M(\lambda)$ is diagonal for all $\lambda \in K$. Thus

$$M(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & d(\lambda) \end{pmatrix},$$

where $a \neq d$. Comparing the upper-left elements in the identity

$$M(\lambda\mu) = M(\lambda)M(\mu)$$

shows that a is an automorphism, and comparing the lower-right elements in the same identity shows that d is an automorphism. This is case (i) of the theorem.

Case (ii): Suppose that every matrix $M(\lambda)$ has a unique eigenvalue. Since the characteristic of K is 0, this eigenvalue must be in K. Choose a basis y, z of E such that y is an eigenvector of $M(\lambda)$ for all λ (since all the $M(\lambda)$ commute, they must have a common eigenvector, if one exists). Hence

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ 0 & a(\lambda) \end{pmatrix}.$$

Comparing the upper-left entries of the identity

$$M(\lambda\mu) = M(\lambda)M(\mu)$$

shows that a is an automorphism, and comparing the upper-right entries of the same identity establishes that b is an a-derivation. This is case (ii) of the theorem.

Case (iii): If neither case (i) or case (ii) above, then some matrix $M(\bar{\lambda})$ for some $\bar{\lambda} \in K$ has eigenvalues not in K. Write

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$$

with a, b, c, d all nonzero. Suppose $a \neq d$. We will show that there is a change of basis which makes $a = d$.

For any $\lambda, \mu \in K$, $b(\lambda\mu)$ is the upper right component of $M(\lambda\mu) = M(\lambda)M(\mu)$. Hence

$$b(\lambda\mu) = a(\lambda)b(\mu) + b(\lambda)d(\mu).$$

But also $b(\lambda\mu) = b(\mu\lambda)$ is the upper right component of $M(\mu\lambda) = M(\mu)M(\lambda)$, hence

$$b(\lambda\mu) = a(\mu)b(\lambda) + b(\mu)d(\lambda).$$

Hence

$$a(\lambda)b(\mu) + b(\lambda)d(\mu) = a(\mu)b(\lambda) + b(\mu)d(\lambda).$$

Here $b(\bar{\lambda}) \neq 0$, since $M(\bar{\lambda})$ has no eigenvector. Set $\mu = \bar{\lambda}$, and let $B = b(\bar{\lambda}), A = a(\bar{\lambda}), D = d(\bar{\lambda})$. Then

$$a(\lambda)B + b(\lambda)D = Ab(\lambda) + Bd(\lambda).$$

Since $A \neq D$, solve for $b(\lambda)$ to get

$$b(\lambda) = \frac{B(d(\lambda) - a(\lambda))}{D - A}.$$

Let $m_1 = \frac{B}{D-A}$; then $m_1 \neq 0$ and

$$M(\lambda) = \begin{pmatrix} a(\lambda) & m_1(d(\lambda) - a(\lambda)) \\ c(\lambda) & d(\lambda) \end{pmatrix}.$$

Since K is commutative,

$$M(\lambda)M(\mu) = M(\lambda\mu) = M(\mu\lambda) = M(\mu)M(\lambda)$$

for any $\lambda, \mu \in K$. Comparing the upper-left term of $M(\lambda)M(\mu)$ and $M(\mu)M(\lambda)$, we get that

$$a(\lambda)a(\mu) + b(\lambda)c(\mu) = a(\mu)a(\lambda) + b(\mu)c(\lambda)$$

hence

$$b(\lambda)c(\mu) = b(\mu)c(\lambda).$$

Since neither b nor c is zero, there must exist $m_0 \in K$ such that $c = m_0 b = m_2(d(\lambda) - a(\lambda))$ where $m_2 = m_0 m_1$. Thus

$$M(\lambda) = \begin{pmatrix} a(\lambda) & m_1(d(\lambda) - a(\lambda)) \\ m_2(d(\lambda) - a(\lambda)) & d(\lambda) \end{pmatrix}.$$

Now make the change of basis

$$\begin{aligned} y &\leftarrow y/m_1 \\ z &\leftarrow z \end{aligned}$$

to give

$$M(\lambda) = \begin{pmatrix} a(\lambda) & d(\lambda) - a(\lambda) \\ m(d(\lambda) - a(\lambda)) & d(\lambda) \end{pmatrix}$$

where $m = m_2 m_1$. Then the change of basis

$$\begin{aligned} y &\leftarrow y + z/2 \\ z &\leftarrow z \end{aligned}$$

gives

$$M(\lambda) = \begin{pmatrix} (d(\lambda) + a(\lambda))/2 & d(\lambda) - a(\lambda) \\ f(\lambda) & (d(\lambda) + a(\lambda))/2 \end{pmatrix}$$

for some function $f(\lambda)$. Hence the upper left and lower right elements of $M(\lambda)$ are equal for all λ.

So now assume that we have a basis y, z such that

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & a(\lambda) \end{pmatrix}.$$

By the same argument as above, we must have $c = mb$ for some non-zero constant $m \in K$. Since this matrix is not diagonalizable, we conclude that m does not have a square root in K; otherwise, the transformation

$$\begin{aligned} y &\leftarrow \sqrt{m}y + z \\ z &\leftarrow z \end{aligned}$$

would diagonalize M. This is case (iii) of the theorem, and the functional equations for a and b are easily checked by comparing elements of the identity

$$M(\lambda\mu) = M(\lambda)M(\mu).$$

□

3.2 Admissibility of Rank 2 Bimodules Over K

We wish to examine admissibility, at the generic point, of a rank 2 locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$. Globally, admissibility is the existence of a rank 1 subbimodule $\mathcal{Q} \subseteq (\mathcal{E} \otimes \mathcal{E})$ such that $\mathcal{Q} \not\cong \mathcal{L} \otimes \mathcal{M}$ for $\mathcal{L}, \mathcal{M} \subseteq \mathcal{E}$. We simply restrict this definition to the generic point η:

Definition 3.3 *Let E be a rank 2 free bimodule over K. We say that E is* admissible *if there exists a rank 1 free subbimodule $Q \subseteq (E \otimes_K E)$ such that $Q \not\cong L \otimes M$ where L, M are rank 1 free subbimodules of E.*

To determine admissibility at the generic point, we need to examine how the left and right actions differ on $E \otimes_K E$. Let y, z be a fixed left and right basis of E. We compute $(y \otimes y)\lambda$ for an element $\lambda \in K$. (Note: we will omit the tensor sign in our notation, so yy is shorthand for $y \otimes y$.)

$$\begin{aligned}
(yy)\lambda &= y(a(\lambda)y + b(\lambda)z) \\
&= \Big(a(a(\lambda))y + b(a(\lambda))z\Big)y + \Big(a(b(\lambda))y + b(b(\lambda))z\Big)z \\
&= a(a(\lambda))yy + a(b(\lambda))yz + b(a(\lambda))zy + b(b(\lambda))zz
\end{aligned}$$

Similar calculations can be done for the other three generators of $E \otimes E$. If we represent the element $pyy + qyz + rzy + szz$ by the rank 4 row vector

$$\begin{pmatrix} p, & q, & r, & s \end{pmatrix}$$

this gives us the matrix form

$$\begin{pmatrix} p, & q, & r, & s \end{pmatrix} \cdot \lambda = \begin{pmatrix} p, & q, & r, & s \end{pmatrix} ((M \circ M)(\lambda)),$$

where $M \circ M$ is the "composition" of M and M, defined as

$$\begin{aligned}
(M \circ M)(\lambda) &= \begin{pmatrix} a(M(\lambda)) & b(M(\lambda)) \\ c(M(\lambda)) & d(M(\lambda)) \end{pmatrix} \\
&= \begin{pmatrix} a(a(\lambda)) & a(b(\lambda)) & b(a(\lambda)) & b(b(\lambda)) \\ a(c(\lambda)) & a(d(\lambda)) & b(c(\lambda)) & b(d(\lambda)) \\ c(a(\lambda)) & c(b(\lambda)) & d(a(\lambda)) & d(b(\lambda)) \\ c(c(\lambda)) & c(d(\lambda)) & d(c(\lambda)) & d(d(\lambda)) \end{pmatrix}
\end{aligned}$$

Theorem 3.4 *Let E be a rank 2 free bimodule over K. Then E is admissible if and only if there exists an common eigenvector v of $(M \circ M)(\lambda)$ for all $\lambda \in K$, such that $v \neq x_1 \otimes x_2$ for $x_1, x_2 \in E$.*

Proof: E is admissible if and only if there exists an element $x \in E \otimes E$ such that $xK = Kx$. But this means that for any $\lambda \in K$,

$$x \cdot \lambda = \mu(\lambda)x$$

for some automorphism μ of K. However we know that

$$x \cdot \lambda = x((M \circ M)(\lambda)),$$

so we need that

$$x((M \circ M)(\lambda)) = (\mu(\lambda))x,$$

in other words, x is an eigenvector of $(M \circ M)(\lambda)$. And, since this must hold for all $\lambda \in K$, we need that x is an eigenvector for all $(M \circ M)(\lambda)$ where λ ranges over all of K.

Finally, since the definition of admissibility states that $Q \ncong L \otimes M$, we require that $x \neq x_1 \otimes x_2$ with $x_1, x_2 \in E$. $\square$

We now determine admissibility conditions for bimodules of each of the three cases described in Theorem 3.2.

Theorem 3.5 *Let E be a rank 2 free K-bimodule with basis y, z such that $M(\lambda)$ is as in Case (i) of Theorem 3.2. Then E is admissible if and only if $ad = da$ or $a^2 = d^2$.*

Proof: Write

$$M(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & d(\lambda) \end{pmatrix},$$

then

$$(M \circ M)(\lambda) = \begin{pmatrix} a^2(\lambda) & 0 & 0 & 0 \\ 0 & ad(\lambda) & 0 & 0 \\ 0 & 0 & da(\lambda) & 0 \\ 0 & 0 & 0 & d^2(\lambda) \end{pmatrix}.$$

There are four obvious eigenvectors, but each of these are nonadmissible; for example, the eigenvector

$$\Big(1,\ 0,\ 0,\ 0\Big)$$

is not admissible since it represents the element $yy \in E \otimes E$, which is the product of two elements of E. To get an admissible eigenvector, we need that two of the eigenvalues

$$a^2(\lambda), ad(\lambda), da(\lambda), d^2(\lambda)$$

coincide for all $\lambda \in K$, which means that, since $a \neq d$, we have either $ad = da$ or $a^2 = d^2$. $\square$

For example, if $ad = da$, then

$$\Big(0,\ 1,\ q,\ 0\Big)$$

is an eigenvector for any $q \in K$. Let Q be the subbimodule of $E \otimes E$ given by this eigenvector; i.e. Q is generated by $yz - qzy$. Then the skew algebra $B = T(E)/(Q)$ is

$$B = \frac{K\langle y, z\rangle}{(y\lambda - a(\lambda)y, z\lambda - d(\lambda)z, yz - qzy)}$$

where λ ranges over all elements of K.

Theorem 3.6 *Let E be a rank 2 free K-bimodule with basis y, z such that $M(\lambda)$ is as in Case (ii) of Theorem 3.2. Then E is admissible if and only if $ab = \mu ba$ for some $\mu \in K$.*

Proof: Write

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ 0 & a(\lambda) \end{pmatrix}$$

where b is a non-zero a-derivation. Then

$$(M \circ M)(\lambda) = \begin{pmatrix} a^2(\lambda) & ab(\lambda) & ba(\lambda) & b^2(\lambda) \\ 0 & a^2(\lambda) & 0 & ba(\lambda) \\ 0 & 0 & a^2(\lambda) & ab(\lambda) \\ 0 & 0 & 0 & a^2(\lambda) \end{pmatrix}$$

This has an obvious eigenvector

$$\Big(0,\ 0,\ 0,\ 1\Big)$$

but this eigenvector is not admissible, since it represents the element $z \otimes z$ in $E \otimes E$. We look for an eigenvector of the form

$$\Big(p,\ q,\ r,\ 0\Big)$$

for $p, q, r \in K$. Then

$$\begin{aligned} &\Big(p,\ q,\ r,\ 0\Big)\,(M \circ M)(\lambda) \\ &\qquad = \Big(a^2(\lambda)p,\ ab(\lambda)p + a^2(\lambda)q,\ ba(\lambda)p + a^2(\lambda)r,\ b^2(\lambda)p + ba(\lambda)q + ab(\lambda)r\Big) \end{aligned}$$

We wish this to be an eigenvector with eigenvalue $a^2(\lambda)$, that is

$$\begin{aligned} &\Big(a^2(\lambda)p,\ ab(\lambda)p + a^2(\lambda)q,\ ba(\lambda)p + a^2(\lambda)r,\ b^2(\lambda)p + ba(\lambda)q + ab(\lambda)r\Big) \\ &\qquad\qquad = \Big(a^2(\lambda)p,\ a^2(\lambda)q,\ a^2(\lambda)r,\ 0\Big). \end{aligned}$$

Comparing the second and third elements of the vectors above, we see that

$$ab(\lambda)p = ba(\lambda)p = 0,$$

hence $p = 0$. Then by comparing the fourth element, we get

$$ab(\lambda)q + ba(\lambda)r = 0$$

for all $\lambda \in K$, which means that

$$\frac{ab}{ba}$$

should be constant. In particular, if $\mu ab = ba$, then

$$\Big(0,\ 1,\ -\mu,\ 0\Big)$$

is an eigenvector. □

Suppose that $qab = ba$, and let Q be the rank 1 subbimodule of $E \otimes E$ generated by $yz - qzy - rz^2$. Then the skew algebra $B = T(E)/(Q)$ is given by

$$B = \frac{K\langle y, z\rangle}{(y\lambda - a(\lambda)y - b(\lambda)z, z\lambda - a(\lambda)z, yz - qzy - rz^2)},$$

where λ ranges over all elements of K.

In the special case where a is the identity on K, the ratio

$$\frac{ab}{ba}$$

is equal to the constant 1 for any derivation b, and we get an eigenvector

$$\Big(0,\ -1,\ 1,\ 0\Big).$$

Theorem 3.7 *Let E be a rank 2 free K-bimodule with basis y, z such that $M(\lambda)$ is as in Case (iii) of Theorem 3.2. Let $t = a(m)$ and $u = b(m)$. Then E is admissible* *if and only if the matrix*

$$\begin{pmatrix} a^2(\lambda) & ab(\lambda) & ba(\lambda) & b^2(\lambda) \\ (tab+mub^2)(\lambda) & a^2(\lambda) & (uab+tb^2)(\lambda) & ba(\lambda) \\ mba(\lambda) & mb^2(\lambda) & a^2(\lambda) & ab(\lambda) \\ m(uab+tb^2)(\lambda) & ba(\lambda) & (tab+mub^2)(\lambda) & a^2(\lambda) \end{pmatrix}.$$

has a common eigenvector in K for all $\lambda \in K$.

Proof: Write

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ mb(\lambda) & a(\lambda) \end{pmatrix}$$

as in case (iii) of Theorem 3.2. Then

$$(M \circ M)(\lambda) = \begin{pmatrix} a^2(\lambda) & ab(\lambda) & ba(\lambda) & b^2(\lambda) \\ (tab+mub^2)(\lambda) & a^2(\lambda) & (uab+tb^2)(\lambda) & ba(\lambda) \\ mba(\lambda) & mb^2(\lambda) & a^2(\lambda) & ab(\lambda) \\ m(uab+tb^2)(\lambda) & ba(\lambda) & (tab+mub^2)(\lambda) & a^2(\lambda) \end{pmatrix}.$$

□

3.3 Birational Equivalence Classes

In commutative algebraic geometry, we say that two schemes X and Y are birationally equivalent if the 0-degree components of the function fields $K(X), K(Y)$ are isomorphic. We make a similar definition for noncommutative schemes over the generic point:

Definition 3.8 *Let E_1, E_2 be admissible free rank 2 K-bimodules, with $Q_i \subseteq E_i \otimes E_i$ rank 1 subbimodules, for $i = 1, 2$. Let $B_i = T(E_i)/(Q_i)$, for $i = 1, 2$. We say that B_1, B_2 are* birationally equivalent *if the graded (left) fields of fractions $D_i = \mathrm{Frac}(B_i)$ exists and if the degree 0 components $(D_i)_0$ are isomorphic as skew K-algebras.*

We can use our classification of admissible rank 2 free K-bimodules above to describe the birational equivalence classes. Some notation: if A is an algebra, α an automorphism of A, and δ an α-derivation of A, then the algebra

$$A[x;\alpha,\delta] = \frac{A\langle x\rangle}{xa - \alpha(a)x - \delta(a)},$$

where a ranges over all elements of A, is called a *(left) Ore extension* of A. If $\delta = 0$ we will omit it from the notation and simply write $A[x;\alpha]$.

We will also use the following standard results from noncommutative algebra:

Proposition 3.9 ([C, 9.3.3],[MR, 2.1.15]) *(i) If A is a left noetherian domain, then $A[x;\alpha,\delta]$ is also a left noetherian domain.*
(ii) If A is a left noetherian domain, then A has a left field of fractions.

So in particular, left Ore extensions of noetherian domains have left fields of fractions.

Case (i) of Theorem 3.2

Here E is given by

$$M(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & d(\lambda) \end{pmatrix}.$$

E is admissible if either $ad = da$ or $a^2 = d^2$. Suppose $ad = da$, and Q is the rank 1 subbimodule of $E \otimes E$ generated by

$$yz - qzy$$

for some nonzero $q \in K$.

Then $B = T(E)/(Q)$ is given by

$$B = \frac{K\langle y, z\rangle}{(yk - a(k)y, zk - d(k)z, yz - qzy)}.$$

Let $C = K[z;a]$, and extend a to an automorphism of C by $a(z) = qz$, hence

$$a(z^n) = qa(q)a^2(q)\cdots a^{n-1}(q)z^n.$$

(This shows that C extended as above is indeed an automorphism; the inverse is

$$a^{-1}(z^n) = (qa(q)a^2(q)\cdots a^{n-1}(q))^{-1}z^n,$$

since the coefficient is nonzero.) Then $B = C[y; a]$. In particular, B has a left field of fractions D, by Proposition 3.9.

Theorem 3.10 *Let $B = K[z; a][y; d]$ as above, and let D be the field of fractions of K. Then*

$$D_0 = \frac{K(t)}{t\lambda - d^{-1}a(\lambda)t}.$$

Proof: Every element of D_0 can be written as

$$f(y, z)^{-1}g(y, z)$$

where f, g are homogeneous polynomials with $\deg f = \deg g = n$. Write

$$f(y, z) = \sum_{i=0}^{n} a_i z^{n-i} y^i$$

and

$$g(y, z) = \sum_{i=0}^{n} b_i z^{n-i} y^i.$$

We can always write our polynomials this way since the relation given by Q allows us to skew-commute z's past y's. Let us rewrite f and g as

$$f(y, z) = \sum_{i=0}^{n} a_i z^n z^{-i} y^i$$

and

$$g(y, z) = \sum_{i=0}^{n} b_i z^n z^{-i} y^i.$$

We now need to commute the z^n term past the constant terms, using the rule $z\lambda = d(\lambda)z$. This gives us

$$f(y, z) = \sum_{i=0}^{n} z^n a_i' z^{-i} y^i$$

and

$$g(y,z) = \sum_{i=0}^{n} z^n b'_i z^{-i} y^i,$$

where $a'_i = d^{-n}(a_i)$ and $b'_i = d^{-n}(b_i)$. Also, define a new variable t by

$$t = z^{-1}y.$$

Since the y's and z's skew commute, we can write

$$z^{-i}y^i = c_i t^i$$

for some constants $c_i \in K$.

Proposition 3.11 *$z^{-i}y^i = c_i t^i$, where*

$$c_i = \prod_{j=0}^{i-2} \prod_{k=j+2}^{i} a^j d^{-k}(q).$$

Proof: Consider the relation

$$yz = qzy.$$

Thus

$$yz = zd^{-1}(q)y$$

and hence

$$z^{-1}y = d^{-1}(q)yz^{-1}.$$

Next, consider

$$zd^{-1}(\lambda) = \lambda z,$$

and multiply on the left and on the right by z^{-1} to get

$$z^{-1}\lambda = d^{-1}(\lambda)z^{-1}.$$

We now show by induction that

$$z^{-i}y = (\prod_{j=1}^{i} d^{-j}(q))yz^{-i}.$$

Let $d_i \in K$ satisfy

$$z^{-i}y = d_i y z^{-i}.$$

Recall that $d_1 = d^{-1}(q)$ by the previous step, and that we can write d_i in terms of d_{i-1} for $i > 1$:

$$\begin{aligned} z^{-i}y &= z^{-1}z^{-(i-1)}y \\ &= z^{-1}d_{i-1}yz^{-(i-1)} \\ &= d^{-1}(d_{i-1})z^{-1}yz^{-(i-1)} \\ &= d^{-1}(d_{i-1})d^{-1}(q)yz^{-i}. \end{aligned}$$

So

$$d_i = d^{-1}(qd_{i-1}).$$

The formula can be easily checked to satisfy this recurrence.

To complete the proof, observe that $c_1 = 1$, and we may express c_i in terms of c_{i-1} for $i > 1$ as follows:

$$\begin{aligned} z^{-i}y^i &= z^{-1}z^{-(i-1)}yy^{i-1} \\ &= z^{-1}d_{i-1}yz^{-(i-1)}y^{i-1} \\ &= z^{-1}d_{i-1}yc_{i-1}t^{i-1} \\ &= d^{-1}(d_{i-1})d^{-1}a(c_{i-1})z^{-1}yt^{i-1} \\ &= d^{-1}(d_{i-1})d^{-1}a(c_{i-1})t^i \end{aligned}$$

So thus

$$c_i = d^{-1}(d_{i-1})d^{-1}a(c_{i-1}).$$

The formula can be easily checked to satisfy this recurrence. (Recall that a and d commute.) □

Now we may write

$$f(y,z) = \sum_{i=0}^{n} z^n a_i'' t^i$$

and

$$g(y,z) = \sum_{i=0}^{n} z^n b_i'' t^i$$

where $a_i'' = c_i a_i'$ and $b_i'' = c_i b_i'$. Then

$$\begin{aligned}
(f(y,z))^{-1} g(y,z) &= (\sum_{i=0}^{n} z^n a_i'' t^i)^{-1} \sum_{i=0}^{n} z^n b_i'' t^i \\
&= (\sum_{i=0}^{n} a_i'' t^i)^{-1} z^{-n} z^n (\sum_{i=0}^{n} b_i'' t^i) \\
&= (\sum_{i=0}^{n} a_i'' t^i)^{-1} (\sum_{i=0}^{n} b_i'' t^i)
\end{aligned}$$

Hence, D_0 is simply $K(t)$; however, the action of K on t differs on the left and on the right. Specifically, we can write for any $\lambda \in K$,

$$\begin{aligned}
t\lambda &= z^{-1} y \lambda \\
&= z^{-1} a(\lambda) y \\
&= d^{-1} a(\lambda) z^{-1} y \\
&= d^{-1} a(\lambda) t.
\end{aligned}$$

□

Note that the structure of D_0 does not depend on the particular choice of Q in $E \otimes E$, since D_0 is independent of q.

We then get the following corollary describing birational equivalence classes of bimodules in case (i).

Corollary 3.12 *Let E_1 and E_2 be two rank 2 free K-bimodules in Case (i) of Theorem 3.2, with basis y_i, z_i and multiplication on E_i given by*

$$M_i(\lambda) = \begin{pmatrix} a_i(\lambda) & 0 \\ 0 & d_i(\lambda) \end{pmatrix}$$

for $i = 1, 2$, with $a_i \neq d_i$ and $a_i d_i = d_i a_i$. Let Q_i be the free rank 1 subbimodule of $E_i \otimes E_i$ generated by

$$y_i z_i - q_i z_i y_i,$$

for some nonzero $q_i \in K$, and let $B_i = T(E_i)/(Q_i)$ Then B_1 and B_2 are birationally equivalent if and only if $d_1^{-1} a_1 = d_2^{-1} a_2$.

Proof: By Theorem 3.10,

$$(\operatorname{Frac} B_1)_0 = \frac{K(t)}{t\lambda - d_1^{-1} a_1(\lambda) t}$$

and

$$(\operatorname{Frac} B_2)_0 = \frac{K(t)}{t\lambda - d_2^{-1} a_2(\lambda) t}.$$

Hence they are isomorphic as skew K-algebras if and only if $d_1^{-1} a_1 = d_2^{-1} a_2$. □

Case (ii) of Theorem 3.2

Here E is given by

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ 0 & a(\lambda) \end{pmatrix}.$$

For admissibility, we require that $qab = ba$. Then Q is the rank 1 subbimodule of $E \otimes E$ generated by

$$yz - qzy - rz^2.$$

Then $B = T(E)/(Q)$ is given by

$$B = \frac{K\langle y, z \rangle}{(y\lambda - a(\lambda) y + b(\lambda) z, z\lambda - a(\lambda) z, yz - qzy - rz^2)}.$$

Let $C = K[z;a]$, and extend a to an automorphism of C by $a(z) = qz$ (as in case (i) above). Also, let δ denote the a-derivation of C defined by $\delta(k) = b(k)z$ for $k \in K$ and $\delta(z) = rz^2$. Then $B = C[y;a,\delta]$. In particular, B has a left field of fractions D.

Theorem 3.13 *Let $B = K[z;a][y;a,\delta]$ as above, and let D be the field of fractions of K. Then*

$$D_0 = \frac{K(t)}{t\lambda - \lambda t - a^{-1}b(\lambda)}$$

Proof: Again, every element of D_0 can be written as

$$f(y,z)^{-1}g(y,z)$$

where f, g are homogeneous polynomials with $\deg f = \deg g = n$. Write f and g as

$$f(y,z) = \sum_{i=0}^{n} a_i z^{n-i} y^i$$

and

$$g(y,z) = \sum_{i=0}^{n} b_i z^{n-i} y^i.$$

We can always write f and g in this form since the relation given by Q allows us to skew commute y's past z's. Rewrite f and g as

$$f(y,z) = \sum_{i=0}^{n} a_i z^n z^{-i} y^i$$

and

$$g(y,z) = \sum_{i=0}^{n} b_i z^n z^{-i} y^i.$$

We can commute the z^n past the constants, using the rule $z\lambda = a(\lambda)z$. This gives

$$f(y,z) = \sum_{i=0}^{n} z^n a_i' z^{-i} y^i$$

and

$$g(y,z) = \sum_{i=0}^{n} z^n b_i' z^{-i} y^i.$$

Define a new variable t by

$$t = z^{-1}y.$$

Proposition 3.14

$$z^{-i}y^i = \sum_{j=1}^{i} c_{i,j}t^j$$

for some constants $c_{i,j} \in K$*, satisfying the recurrence*

$$c_{i,j} = a^{-1}(d_{i-1})c_{i-1,j-1} + a^{-1}(d_{i-1})a^{-1}b(c_{i-1,j}) + a^{-1}(e_{i-1})c_{i-1,j}.$$

with $c_{1,1} = 1$. *In particular* $c_{i,i} \neq 0$.

Proof: We know that

$$yz = qzy + rz^2.$$

Thus

$$yz = za^{-1}(q)y + za^{-1}(r)z,$$

and multiplying by z^{-1} on the left and on the right gives

$$z^{-1}y = a^{-1}(q)yz^{-1} + a^{-1}(r).$$

Next, we show inductively that $z^{-i}y = d_i yz^{-i} + e_i z^{-(i-1)}$, where

$$d_i = \prod_{j=1}^{i} a^{-j}(q)$$

and

$$e_i = \sum_{j=1}^{i} \left(a^{-j}(r) \prod_{k=j+1}^{i} a^{-k}(q) \right).$$

Recall that $d_1 = a^{-1}(q)$ and $e_1 = a^{-1}(r)$ by the above step. Then for $d > 1$

$$\begin{aligned}
z^{-i}y &= z^{-1}z^{-(i-1)}y \\
&= z^{-1}(d_{i-1}yz^{-(i-1)} + e_{i-1}z^{-(i-2)}) \\
&= a^{-1}(d_{i-1})z^{-1}yz^{-(i-1)} + a^{-1}e_{i-1}z^{-(i-1)} \\
&= a^{-1}(d_{i-1})(a^{-1}(q)yz^{-1} + a^{-1}(r))z^{-(i-1)} + a^{-1}(e_{i-1})z^{-(i-1)} \\
&= a^{-1}(d_{i-1})yz^{-i} + (a^{-1}(d_{i-1})a^{-1}(r) + a^{-1}(e_{i-1}))z^{-(i-1)}.
\end{aligned}$$

So

$$d_i = a^{-1}(d_{i-1})a^{-1}(q)$$

and

$$e_i = a^{-1}(d_{i-1})a^{-1}(r) + a^{-1}(e_{i-1}),$$

which solve to give the expressions given.

Finally, observe that $c_{1,1} = 1$. Then for $i > 1$,

$$\begin{aligned}
z^{-i}y^i &= z^{-1}z^{-(i-1)}yy^{i-1} \\
&= z^{-1}(d_{i-1}yz^{-(i-1)} + e_{i-1}z^{-(i-2)})y^{i-1} \\
&= z^{-1}d_{i-1}yz^{-(i-1)}y^{i-1} + z^{-1}e_{i-1}z^{-(i-2)}y^{i-1} \\
&= a^{-1}(d_{i-1})z^{-1}yz^{-(i-1)}y^{i-1} + a^{-1}(e_{i-1})z^{-(i-1)}y^{i-1} \\
&= a^{-1}(d_{i-1})z^{-1}y\sum_{j=1}^{i-1} c_{i-1,j}t^j + a^{-1}(e_{i-1})\sum_{j=1}^{i-1} c_{i-1,j}t^j \\
&= a^{-1}(d_{i-1})\sum_{j=1}^{i-1} c_{i-1,j}t^{j+1} + a^{-1}(d_{i-1})\sum_{j=1}^{i-1} a^{-1}b(c_{i-1,j})t^j + a^{-1}(e_{i-1})\sum_{j=1}^{i-1} c_{i-1,j}t^j.
\end{aligned}$$

So by collecting the t^j terms, we get the given recurrence.

In particular,

$$c_{i,i} = a^{-1}(d_{i-1})c_{i-1,i-1}$$

with $c_{1,1} = 1$, so

$$c_{i,i} = \prod_{j=2}^{i}\prod_{k=2}^{j} a^{-k}(q).$$

□

So we may write

$$f(y,z) = \sum_{i=0}^{n} z^n a_i'' t^i$$

and

$$g(y,z) = \sum_{i=0}^{n} z^n b_i'' t^i,$$

where a'' and b'' depend on a' and b' and the $c_{i,j}$ constants. This provides a bijection between D and $K(t)$, as before. And for any $\lambda \in K$,

$$\begin{aligned} t\lambda &= z^{-1}y\lambda \\ &= z^{-1}(a(\lambda)y + b(\lambda)z) \\ &= \lambda z^{-1}y + a^{-1}b(\lambda) \\ &= \lambda t + a^{-1}b(\lambda). \end{aligned}$$

Then $a^{-1}b$ is a K-derivation:

$$\begin{aligned} (a^{-1}b)(\lambda\mu) &= a^{-1}(a(\lambda)b(\mu) + a(\mu)b(\lambda)) \\ &= \lambda a^{-1}b(\mu) + \mu a^{-1}b(\lambda). \end{aligned}$$

This does not depend on q or r. □

This gives the following corollary regarding the birational equivalence classes:

Corollary 3.15 *Let E_1, E_2 be two admissible rank 2 free K-bimodules in Case (ii) of Theorem 3.2, with multiplication on E_i given by*

$$M_i(\lambda) = \begin{pmatrix} a_i(\lambda) & b_i(\lambda) \\ 0 & a_i(\lambda) \end{pmatrix}$$

for $i = 1, 2$. Let Q_i be any admissible rank 1 subbimodule of $E_i \otimes E_i$, and set $B_i = T(E_i)/(Q_i)$. Then B_1 and B_2 are birationally equivalent if and only if $a_1^{-1}b_1 = a_2^{-1}b_2$.

Proof: By Theorem 3.13,

$$(\operatorname{Frac} B_1)_0 = \frac{K(t)}{t\lambda - \lambda t - a_1^{-1}b_1(\lambda)}$$

and

$$(\operatorname{Frac} B_2)_0 = \frac{K(t)}{t\lambda - \lambda t - a_2^{-1}b_2(\lambda)}$$

Hence they are isomorphic as skew K-algebras if and only if $a_1^{-1}b_1 = a_2^{-1}b_2$. □

If we only care about birational equivalence, we may use this corollary to set a equal to the identity automorphism of K; then a birational equivalence class is given by a choice of $b \in \operatorname{Der} K$.

Chapter 4

Global Bimodules and Noncommutative Ruled Surfaces

4.1 Geometry of Rank 2 Locally Free Bimodules

We may now characterize locally free rank 2 bimodules. As in the proof of Theorem 2.8, we will first consider the affine case. It is convenient to first consider the situation over a field.

Lemma 4.1 *Let K be a field. Suppose that S is a commutative ring with an injection $K \hookrightarrow S$, and that M is a faithful left S-module, such that as a K-module, $M_K \cong K \oplus K$; i.e. M_K is a vector space of rank 2 over K. Then S is isomorphic to one of the following:*

1. K
2. $\dfrac{K[y]}{y^2}$
3. $\dfrac{K[y]}{y^2 - 1}$
4. $\dfrac{K[y]}{y^2 - m}$ *where m does not have a square root in K.*

Proof: Denote elements of M by ordered pairs $(k_1, k_2) \in K \oplus K$ such that for any $c \in K$,

$$c(k_1, k_2) = (ck_1, ck_2).$$

Let any element $s \in S$ be given, and suppose that

$$\begin{aligned} s(1,0) &= (a,c) \\ s(0,1) &= (b,d) \end{aligned}$$

Then for an arbitrary element $(k_1, k_2) \in M$,

$$\begin{aligned} s(k_1,k_2) &= s(k_1(1,0) + k_2(0,1)) \\ &= k_1 s(1,0) + k_2 s(0,1) \\ &= k_1(a,c) + k_2(b,d) \\ &= (k_1 a + k_2 b, k_1 c + k_2 d) \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

Thus the action of s on M is determined by its action on $(1,0)$ and $(0,1)$; furthermore, two elements s_1, s_2 cannot have the same action, since in that case $(s_1 - s_2)M = 0$, contradicting the assumption that M is a faithful S-module. Thus we may think of S sitting inside of $K^{2\times 2}$, with the action on $M = K^2$ given by standard matrix multiplication.

So it remains to classify those commutative rings S with

$$K \hookrightarrow S \hookrightarrow K^{2\times 2}.$$

If $S = K$ then we have case 1, so assume that there exists an element $y \in S - K$. We claim that S is the subring of $K^{2\times 2}$ generated by K and y. Let $z \in S$ be any other element such that $z \notin K$. Then

$$y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$z = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Let $y_1 = y - d$ and $z_1 = z - d'$. Then $y_1, z_1 \in S$ with

$$y_1 = \begin{pmatrix} r & b \\ c & 0 \end{pmatrix}$$

and

$$z_1 = \begin{pmatrix} r' & b' \\ c' & 0 \end{pmatrix}$$

with $r = a - d, r' = a' - d'$. Observe that $y_1, z_1 \neq 0$ since we assume $y, z \notin K$. But S is commutative, so $y_1 z_1 = z_1 y_1$. Comparing matrix entries, we conclude that

$$\begin{aligned} rb' &= r'b \\ rc' &= r'c \\ bc' &= b'c \end{aligned}$$

These equations imply that $z_1 \in Ky_1$, hence $z \in (K + Ky)$.

Since $y^2 \in (K + Ky)$, we conclude that $y^2 - py - q = 0$ for some $p, q \in K$. Replacing y by $y - (p/2)$ gives

$$y^2 - m = 0$$

for some $m \in K$. If $m = 0$ we have case 2. If m has no square root then we have case 4. Otherwise, if $m = m_0^2$ for some $m_0 \in K$, replacing y by y/m_0 gives case 3. □

Lemma 4.2 *Let R be a commutative integrally closed domain, with field of fractions K. Suppose that S is a commutative ring with an injection $R \hookrightarrow S$, and that M is a faithful left S-module, such that as an R-module, $M_R \cong R \oplus R$; i.e. M_R is a free* *module of rank 2 over R. Then S is isomorphic to one of the following:*

1. R
2. $\dfrac{R+Iy}{y^2}$ *for some nonzero ideal I of R*
3. $\dfrac{R+Iy}{y^2-1}$ *for some nonzero ideal I of R*
4. $\dfrac{R+Iy}{y^2-m}$ *for some nonzero ideal I of R, and some $m \in K$ such that m has no square root in R and $I^2 m \subseteq R$.*

Proof: By the same argument as in Lemma 4.1, we may consider S as a commutative subring of $R^{2\times 2}$, containing R.

Let K be the field of fractions of R, and let $\bar{S} = S \otimes_R K$. Then by Lemma 4.1, $\bar{S}$ is isomorphic to one of the four cases shown. In case 1, if $\bar{S}$ is K, then we must have $S = R$, since in all cases $R \subseteq S$.

In the other three cases,

$$\bar{S} = \frac{K + Ky}{y^2 - m}$$

where $m = 0, 1$ or m has no square root in K. In the last case, take a sufficient multiple of y so that $m \in R$. Think of $\bar{S}$ as subring of $K^{2\times 2}$, and write

$$y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

for some $y_1, y_2, y_3, y_4 \in K$.

Let

$$J_i = \{c \in K \mid c y_i \in R\}$$

for $1 \le i \le 4$. Then J_i is a fractional ideal of R, and if $y_i \neq 0$, then $J_i \cong R$ as an R-module, since J_i is the preimage of R under the R-module isomorphism of K which sends $x \in K$ to $x y_i$.

Let $J_0 = J_1 \cap J_2 \cap J_3 \cap J_4$, and choose i such that $y_i \neq 0$. Then J_0 is isomorphic as an R-module to some ideal I_0 of R, since we may think of J_0 as

$$J_0 = J_i \cap \bigcap_{j \neq i} J_j$$

which is an R-submodule of the R-module $J_i \cong R$. Furthermore, the map from J_0 to I_0 send an element $x \in J_0$ to the element $y_i^{-1}x \in I_0$.

By construction, $J_0 y = Ky \cap R^{2\times 2}$. Hence,

$$\bar{S} \cap R^{2\times 2} \cong \frac{R + I_0 y}{y^2 - m}$$

as R-modules. But S is an R-submodule of $\bar{S} \cap R^{2\times 2}$, hence

$$S = \frac{R + Iy}{y^2 - m}$$

for some ideal $I \subseteq I_0$.

However, as an algebra,

$$S = \frac{R + Jy}{y^2 - m}$$

for some fractional ideal $J \subseteq J_0$, with $mJ^2 \subseteq R$, and where $J \cong I$ as R-modules by the R-module isomorphism $x \mapsto y_i^{-1}x$. If $m = 1$, then $J^2 \subseteq R$, hence $J \subseteq R$ since R is integrally closed. This gives case 3.

Otherwise, as an R-algebra,

$$S = \frac{R + Iy}{y^2 - y_i^{-2}m}.$$

If $m = 0$ this gives case 2. If m does not have a square root, then neither does $y_i^{-2}m$, and this gives case 4 (since $(Iy)^2 \subseteq R$ in the R-algebra S). □

The following technical lemma is useful.

Lemma 4.3 *Let R be a commutative domain with elements $f, g \in R$. Then, as R-algebras,*

$$\frac{R+(f)y}{y^2-g} \cong \frac{R[y]}{y^2-f^2g}.$$

Proof: The isomorphism from the algebra on the right to the algebra on the left is given by the algebra morphism which sends the element $a+by$ to the element $a+fby$. This map is an isomorphism since R is a domain (so $fb=0$ only if $b=0$). $\square$

Also recall the following standard fact from commutative algebra: If I is an ideal of R and $\mathfrak{q}$ is a prime ideal of R, then

$$(I \otimes_R R_{\mathfrak{q}} = R_{\mathfrak{q}}) \Leftrightarrow (I \not\subseteq \mathfrak{q}).$$

We now consider the geometry of the four cases described above.

Lemma 4.4 *Let R, S be commutative rings. Suppose $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ are curves, X smooth and irreducible, with a projection $f : Y \to X$, and $\mathcal{M}$ is a coherent $\mathcal{O}_Y$-module, with $\operatorname{Ann} \mathcal{M} = 0$, such that $f_*(\mathcal{M}) \cong \mathcal{O}_X \oplus \mathcal{O}_X$. Then one of the following cases holds:*

1. *$Y \cong X$, and f is an isomorphism*

2. *Y is a nonreduced curve, generically degree 2, and $f_{red} : Y_{red} \to X$ is an isomorphism*

3. *f is finite of degree 2, and Y consists of two irreducible components Y_1, Y_2, intersecting in a finite number of points, such that $f_i = f|_{Y_i} : Y_i \to X$ is an isomorphism.*

4. *Y is irreducible and f is a degree 2 finite morphism.*

Proof: Observe that the geometric conditions imply that R is a commutative integrally closed domain and that R embeds into the commutative ring S. Let $M = H^0(\mathcal{M}, S)$. Then M is a faithful S-module, and $M_R \cong R \oplus R$. Hence, the four geometric cases above correspond to the four algebraic cases of Lemma 4.2.

Case 1: $S = R$. This is immediate.

Case 2:

$$S = \frac{R + Iy}{y^2}.$$

Let $\mathfrak{q}$ be a point of X, such that $I \not\subseteq \mathfrak{q}$. Then

$$S_{\mathfrak{q}} = \frac{R_{\mathfrak{q}}[y]}{y^2}.$$

Since R is noetherian of dimension 1, this happens at all but a finite number of points of X. Thus, except at a finite number of points, $f_{red} : Y_{red} \to X$ is an isomorphism; hence it must be an isomorphism over all of Y.

Case 3 and 4:

$$S = \frac{R + Iy}{y^2 - c},$$

where either $c = 1$ or c has no square root. At any prime $\mathfrak{q}$ such that $I \not\subseteq \mathfrak{q}$,

$$S_{\mathfrak{q}} = \frac{R_{\mathfrak{q}}[y]}{y^2 - c}.$$

Thus, except over a finite number of points, Y is a 2-to-1 cover of X, and Y is reduced (since S has no nilpotent elements). Then the two cases are distinguished by whether S is a domain. If it is a domain, as in case 4, then Y is irreducible; otherwise, as in case 3, Y has two reducible components, each of which must then necessarily be isomorphic to X. □

We can now determine the possible structures of a rank 2 locally free bimodule over a curve X.

Theorem 4.5 *Let $\mathcal{E}$ be a bimodule over X which is locally free of rank 2. Let Y be the support of $\mathcal{E}$ in $X \times X$. Then Y is finite over each factor of X, and one of the following cases holds:*

1. *Y is irreducible of bidegree (1,1), and $\mathcal{E}$ is locally free of rank 2 on Y.*

2. *Y is irreducible of bidegree (1,1), and $\mathcal{E}$ is generically locally free of rank 1 over a nonreduced scheme Z, with $Z_{red} = Y$, such that Z is degree 2 over Y.*

3. Y is of bidegree (2,2), reducible with irreducible components Y_1, Y_2, such that Y_i is of bidegree (1,1).

4. Y is irreducible of bidegree (2,2).

Proof: Affinely, the situation is as in Lemma 4.4: if $U = \operatorname{Spec} R$ is an affine open set of X, with $V = \operatorname{Spec} S$ its cover in Y and $M = \mathcal{E}(U)$, then M is free of rank 2 over R, and hence V must lie over U as in the Lemma. Extending the local situations of Lemma 4.4 to the projective smooth curve X, and considering the geometry over both projections to X, gives the four cases listed in the statement above.

Since Y is finite over either factor of X over any affine open set $U \subseteq X$, this is also true globally.

In case 1, since $\mathrm{pr}_1, \mathrm{pr}_2$ are isomorphisms, we conclude that $\mathcal{E}$ is locally free of rank 2 over Y. In case 2, locally, we get $\mathcal{E} \cong \mathcal{O}_Z$; this gives the statement about $\mathcal{E}$ in case 2. □

4.2 Admissibility of Rank 2 Locally Free Bimodules

Before considering each of the cases, we need the following lemma.

Lemma 4.6 *Let X be a smooth curve and $\mathcal{F}$ a locally free sheaf of rank 2 on X. Then there exists a subbundle $\mathcal{Q} \subseteq \mathcal{F} \otimes \mathcal{F}$ such that $\mathcal{Q}$ cannot be written as $\mathcal{Q} = \mathcal{L} \otimes \mathcal{M}$ for some subbundles $\mathcal{L}, \mathcal{M}$ of $\mathcal{F}$ which are locally free of rank 1.*

Proof: Take $\mathcal{Q} = \bigwedge^2 \mathcal{F} \subseteq \mathcal{F} \otimes \mathcal{F}$; i.e. $\mathcal{Q}$ is defined locally as

$$\mathcal{Q}(U) = \{s \in \mathcal{F}(U) \otimes \mathcal{F}(U) \mid \sigma s = -s\}$$

where σ is the map which sends $x \otimes y$ to $y \otimes x$. Then $\mathcal{Q}$ is locally free of rank 1, but cannot be written as a tensor product of two subbundles of $\mathcal{F}$, since this cannot happen locally: for any open set U, we cannot have $\mathcal{Q}(U) = \mathcal{L}(U) \otimes \mathcal{M}(U)$, because sections of $\mathcal{Q}(U)$ of the form $x \otimes y - y \otimes x$ with x and y linearly independent cannot be rewritten as a single tensor product $l \otimes m$. □

We now consider admissibility conditions for each of the four cases described in Theorem 4.5.

4.2.1 Case 1: $\mathcal{E}$ Defined on a Reduced (1,1) Divisor

Suppose $\mathcal{E}$ is a rank 2 locally free bimodule, supported on an irreducible bidegree (1,1) divisor Y in $X \times X$, such that the projection maps $\pi_1, \pi_2 : Y \to X$ are isomorphisms. Then $Y = \Gamma_\sigma$ for some $\sigma \in \operatorname{Aut} X$.

Theorem 4.7 *Let $\mathcal{E}$ be a locally free rank 2 bimodule supported on an irreducible divisor of bidegree (1,1) which is finite over each factor. Then $\mathcal{E}$ is admissible.*

Proof: Let W be the support of $\mathcal{E} \otimes \mathcal{E}$. Then $W = \Gamma_{\sigma^2}$, which is a divisor of bidegree (1,1), so $\mathcal{E}$ is admissible. Also, $\mathcal{E}$, considered as a sheaf over W, is locally free of rank 4. Then by Lemma 4.6, we may choose $\mathcal{Q} \subseteq \mathcal{E} \otimes \mathcal{E}$ which is admissible. □

4.2.2 Case 2: $\mathcal{E}$ Supported on a Nonreduced (1,1) Divisor

Conditions for the admissibility of $\mathcal{E}$ are unknown in this case.

4.2.3 Case 3: $\mathcal{E}$ Supported on Reducible (2,2) Divisor

Suppose that $\mathcal{E}$ is supported on a reducible curve Y of bidegree (1,1) in $X \times X$, where each component is finite over each factor of X. The components are then necessarily the graphs of automorphisms σ, τ of X, with $\sigma \neq \tau$. Denote $S = \Gamma_\sigma$, $T = \Gamma_\tau$; then $Y = S \cup T$.

Let π_1, π_2 denote the projections from Y to the two copies of X. Since $\pi_{1*}\mathcal{E}$ and $\pi_{2*}\mathcal{E}$ are each locally free of rank 2, we conclude that $\mathcal{E} \otimes \mathcal{O}_S$ and $\mathcal{E} \otimes \mathcal{O}_T$ are each generically locally free of rank 1 over S and T respectively. This is also clear by examining the local algebraic situation in Case 3 of Lemma 4.4: the coordinate ring of each component annihilates a saturated rank 1 submodule of the free module of rank 2 over $R_{\mathfrak{q}}$.

Let $\mathcal{E}_S = \mathcal{E} \otimes \mathcal{O}_S$ and $\mathcal{E}_T = \mathcal{E} \otimes \mathcal{O}_T$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_S \oplus \mathcal{E}_T \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is a sheaf supported on $S \cap T$.

$\mathcal{E} \otimes \mathcal{E}$ is supported on

$$W = \Gamma_{\sigma^2} \cup \Gamma_{\sigma\tau} \cup \Gamma_{\tau\sigma} \cup \Gamma_{\tau^2}.$$

If these four graphs are distinct components of W, then $\mathcal{E} \otimes \mathcal{E}$ is generically locally free of rank 1 on each of these components (specifically, $\mathcal{E} \otimes \mathcal{E}$ is locally free of rank 1 except possibly at intersection points of two or more components). However, choosing $\mathcal{Q}$ to be the rank 1 locally free bimodule supported on one of the components is not admissible, since for example

$$(\mathcal{E} \otimes \mathcal{E}) \otimes \mathcal{O}_{\Gamma_{\sigma^2}} = \mathcal{E}_S \otimes \mathcal{E}_S.$$

Hence, two of the components must coincide:

Proposition 4.8 *Let $\mathcal{E}$ be a rank 2 locally free bimodule supported on a reducible curve Y whose with two irreducible components, each finite over each copy of X, and of bidegree (1,1). Let σ, τ be automorphisms of X such that $Y = \Gamma_\sigma \cup \Gamma_\tau$. Then $\mathcal{E}$ is admissible only if $\sigma\tau = \tau\sigma$ or $\sigma^2 = \tau^2$.*

If $\mathcal{E} = \mathcal{E}_S \oplus \mathcal{E}_T$, then it is clear that the converse of Proposition 4.8 is also true, because in that case

$$\begin{aligned} \mathcal{E} \otimes \mathcal{E} &= (\mathcal{E}_S \otimes \mathcal{E}_S^{\sigma})_{\sigma^2} \\ &\oplus (\mathcal{E}_S \otimes \mathcal{E}_T^{\sigma})_{\sigma\tau} \\ &\oplus (\mathcal{E}_T \otimes \mathcal{E}_S^{\tau})_{\tau\sigma} \\ &\oplus (\mathcal{E}_T \otimes \mathcal{E}_T^{\tau})_{\tau^2}. \end{aligned}$$

Suppose $\sigma\tau = \tau\sigma$. Then $\mathcal{E} \otimes \mathcal{E}$ restricted to $\Gamma_{\sigma\tau} = \Gamma_{\tau\sigma}$ is locally free of rank 2, and a general rank 1 subbundle will be admissible. A similar argument holds for $\sigma^2 = \tau^2$.

Finally, let $E = \mathcal{E}_\eta$ be the bimodule at the generic point. Then using the notation of Chapter 3, the right $K(X)$-action on E is given by the matrix

$$M(\lambda) = \begin{pmatrix} \sigma(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}$$

and by Theorem 3.5 E is admissible (in the generic sense) if and only if $\sigma\tau = \tau\sigma$ or $\sigma^2 = \tau^2$

4.2.4 Case 4: $\mathcal{E}$ Supported on an Irreducible (2,2) Divisor

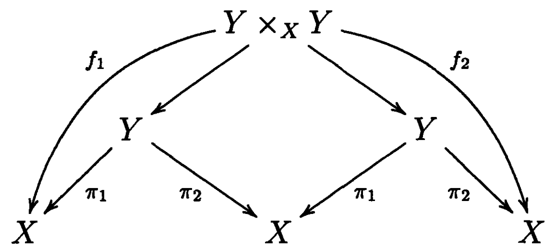
Let us suppose that $\mathcal{E}$ is supported on a curve Y in $X \times X$, which is irreducible, and the projections π_1, π_2 from Y to X are each finite of degree 2. We can describe the support of $\mathcal{E} \otimes \mathcal{E}$ geometrically. Recall that

$$\mathcal{E} \otimes \mathcal{E} = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*\mathcal{E} \otimes_{\mathcal{O}_{X^3}} \mathrm{pr}_{23}^*\mathcal{E}).$$

Thus, by Lemma 2.3, the support of $\mathcal{E} \otimes \mathcal{E}$ is the image in $X \times X$ of the fibre product $Y \times_X Y$ given by the diagram

$$\begin{array}{ccc} Y \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow \pi_1 \\ Y & \xrightarrow{\pi_2} & X \end{array}$$

This set is mapped into $X \times X$ via the maps f_1 and f_2, as given in the following diagram:



Note f_1 and f_2 are degree 4 maps. Let $Z = Y \times_X Y$ (where this is the fibre product of Y over X via π_2 with Y over X via π_1 as described above), and let W be the image of Z in $X \times X$ via f_1 and f_2, as in the diagram above.

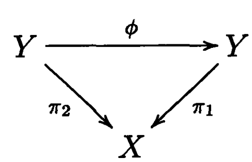
Proposition 4.9 *$\mathcal{E}$ is admissible if W has a component isomorphic to X, such that the projections from W onto each factor of X are isomorphisms.*

Proof: By Theorem 2.8, a bimodule of rank 1 must be supported on a (1,1) divisor in $X \times X$, which is the graph of an automorphism of X, which necessarily is isomorphic to X. □

So the question becomes: when does W has a component isomorphic to X? In general W will be birational to Z. If Z itself has a component isomorphic to X, then this component would be just a point in W, since the maps from Z to X go through Y, and any component of Z which is isomorphic to X would map to a point in Y.

The next case to consider is the case where Z has a component birational to Y.

Proposition 4.10 *Z has a component birational to Y which maps birationally onto each copy of Y if and only if the two projections π_1, π_2 from Y to X differ by a birational map $\phi : Y \to Y$; i.e. the following diagram commutes:*

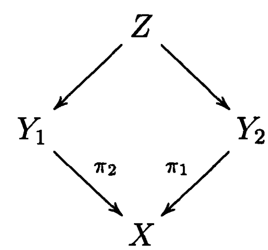


In the above case, Z consists of two components, each of which is birational to Y and which map birationally onto each factor of Y. Otherwise, Z is irreducible, and maps 2-to-1 onto each factor of Y.

Proof: Set-theoretically, Z can be written as

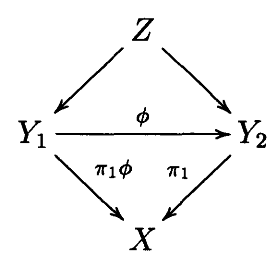
$$Z = \{(y_1, y_2) \in Y \times Y \mid \pi_2 y_1 = \pi_1 y_2\}.$$

Consider the diagram



(The copies of Y are labeled separately to avoid confusion.)

To prove the "if" direction: Suppose that $\pi_2 = \phi\pi_1$. Then there is the following commutative diagram:



Then for a sufficiently general point $p \in Y_1$, the points of Z lying over p are $(p, \phi(p))$ and $(p, \tau\phi(p))$, where τ is the automorphism of Y which interchanges the fibres of π_1. Then the subset of Z given by

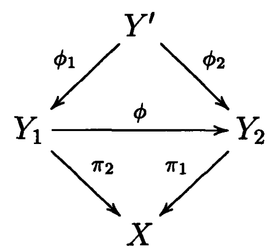
$$\{(p, \phi(p)) \mid p \in Y_1,\ \phi(p) \text{ defined}\}$$

is a dense open subset of a component of Z birational to Y. In this case,

$$\{(p, \tau\phi(p)) \mid p \in Y_1,\ \phi(p) \text{ defined}\}$$

is also a dense open subset of a component of Z birational to Y, and Z is composed entirely of these two components.

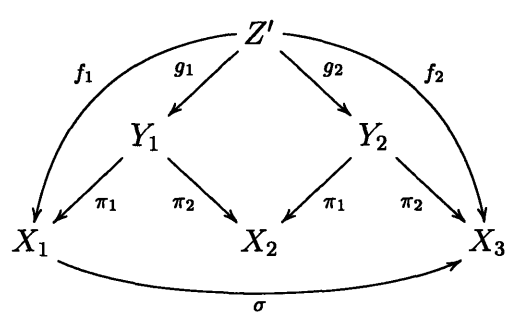
Conversely: suppose that Z has a component Y' with birational maps ϕ_1, ϕ_2 onto Y_1, Y_2 respectively. Define $\phi = \phi_2 {\phi_1}^{-1}$. Then ϕ has the required property; i.e. the following diagram commutes:



If Z does not have a component birational to Y, then any component of Z must map 2-to-1 onto each factor of Y, since Y is irreducible. But the maps from Z to each copy of Y are already 2-to-1, hence Z has just a single component. □

Proposition 4.11 *Suppose Z has a component Z' whose image W' in $X \times X$ maps birationally to each factor of X. Then Z' is birational to Y, and hence the two maps from Y to X differ by a birational automorphism ϕ of Y.*

Proof: Suppose Z' exists as in the statement. Then W' is the graph of some automorphism σ of X, and the following diagram is commutative:



(The X and Y are labeled separately to avoid confusion.) Suppose Z' is not birational to Y; then by Proposition 4.10 the maps g_1, g_2 from Z' to Y_1 and Y_2 are generically 2-to-1. We will show this leads to a contradiction.

Consider a point $p \in X_1$. For a suitable general choice of p, the preimage of p in Z under f_1 is four points $z_1, \ldots, z_4$. Moreover, $f_2(z_i) = \sigma(p)$ for all $1 \le i \le 4$. Hence, the image $g_2(\{z_1, \ldots, z_4\})$ consists of two points in Y_2, since only two points of Y_2 map to $\sigma(p)$, namely the two points in the fibre $\pi_2^{-1}(\sigma(p))$.

Let $\{q_1, q_2\} = g_1(\{z_1, \ldots, z_4\})$ and $\{r_1, r_2\} = g_2(\{z_1, \ldots, z_4\})$. Recall that we can express the points of Z as ordered pairs of points of Y, so that

$$\{z_1, z_2, z_3, z_4\} = \{(q_1, r_1), (q_1, r_2), (q_2, r_1), (q_2, r_2)\}.$$

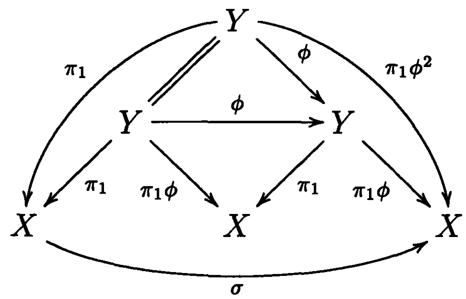
But this means that $\pi_2(q_1) = \pi_2(q_2) = \pi_1(r_1) = \pi_1(r_2) = s$ for some point $s \in X_2$. In particular, the fibres of π_1 and the fibres of π_2 coincide. But this means that the maps π_1 and π_2 differ only by an automorphism ρ of X, which means that Y sits as the graph of ρ in $X \times X$, contradicting the fact that Y is a bidegree (2,2) divisor. □

Theorem 4.12 *Let $\mathcal{E}$ be a rank 2 locally free bimodule supported on an irreducible curve Y in $X \times X$, such that the projections π_1, π_2 from Y onto each factor of X are finite of degree 2. Then $\mathcal{E}$ is admissible if and only if there exists a birational automorphism ϕ of Y and an automorphism σ of X such that (i) $\pi_2 = \pi_1\phi$ and (ii) the following diagram commutes:*

$$\begin{array}{ccc} Y & \xrightarrow{\phi^2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ X & \xrightarrow{\sigma} & X \end{array}$$

Proof: Suppose $\mathcal{E}$ is admissible. By Theorem 2.8, W must contain a component which maps isomorphically to X by each projection. Then by Proposition 4.11, $Y \times_X Y$ has a component birational to Y, and there is a birational automorphism ϕ

of Y and an automorphism σ of X which give us the following commutative diagram:



Following the three outside curved arrows gives the requisite commutative diagram.

Conversely, if the two projections π_1, π_2 differ by an automorphism ϕ, then $Y \times_X Y$ has a component birational to Y, and the existence of σ and the commutative diagram ensures that this component will project to the graph of σ in $X \times X$. We can then take $\mathcal{Q}$ to be any rank 1 locally free bimodule supported on the image of this component in W. Furthermore, since Y itself does not have any component isomorphic to X, it does not have any rank 1 locally free subbimodules, and hence any rank 1 locally free subbimodule of $\mathcal{E} \otimes \mathcal{E}$ will automatically be admissible. $\square$

Corollary 4.13 *Suppose X is a curve, with the genus of X not equal to 1, and let $\mathcal{E}$ be a rank 2 locally free admissible bimodule as above, supported on a smooth curve Y. Then the automorphisms ϕ, σ of Theorem 4.12 are of finite order.*

Proof: Let $g(X), g(Y)$ denote (respectively) the genus of X and the genus of Y. If $g(X) > 1$, then $g(Y) > 1$ as well, and all automorphisms of X and Y are finite. So we are left with the case where X is rational. Since Y is an irreducible divisor of $X \times X$ of bidegree (2,2), the genus formula gives that:

$$\begin{aligned} g(y) &= \frac{1}{2}Y.(Y+K)+1 \\ &= \frac{1}{2}(2,2).(0,0)+1 \\ &= 1. \end{aligned}$$

Hence, the Riemann-Hurwitz formula states that there are four points $p_1, \ldots, p_4$ of X over which the map π_1 is ramified, with preimages $q_1, \ldots, q_4$. But since ϕ^2 maps fibres of π_1 to fibres of π_1, it must preserve the q_i set-wise. Thus ϕ^{24} thus fixes the points $q_1, \ldots, q_4$ pointwise (since any element of the symmetric group Σ_4 has order dividing 12). But any automorphism which fixes four points of a curve must be finite order, so ϕ^{24} is of finite order, and hence so is ϕ. But by Theorem 4.12, $\pi_1\phi^{2n} = \sigma^n\pi_1$. Hence, for N such that $\phi^{2N} = id_Y$, we conclude $\pi_1 = \sigma^N\pi_1$, hence $\sigma^N = id_X$. $\square$

This argument does not hold for $g(X) = 1$, for in that case we could also have $g(Y) = 1$, with π_1 an étale cover of X.

Chapter 5

Examples

In this chapter we will describe some examples of bimodules and their homogeneous coordinate rings over the curve $X = \mathbf{P}^1$. Let u, v be projective coordinates for X.

We will use the following coordinate system: Consider the projection maps $\mathrm{pr}_1, \mathrm{pr}_2$ from $X \times X$ to X. Often, we will denote the two factors of X as X_1, X_2 to avoid confusion, where X_i is the image of pr_i for $i = 1, 2$. Let u_i, v_i be projective coordinates on X_i for $i = 1, 2$. Then we can write the homogeneous coordinate ring of $X_1 \times X_2$ as the subring of $k[u_1, u_2, v_1, v, 2]$ generated by the elements $u_1u_2, u_1v_2, v_1u_2, v_2u_2$.

5.1 Bimodule of Differential Operators

Let $\mathcal{D}$ be the sheaf of differential operators, given locally on $\mathbf{P}^1$ by

$$\mathcal{D} = \mathcal{O}\langle x \rangle / (xu - ux - 1),$$

where u is a local coordinate and y is to be thought of as the differential operator $\partial / \partial u$. It is more convenient to work with the homogenized version of this algebra, given by

$$\mathcal{A} = \mathcal{O}\langle x, z \rangle / (xu - ux - z,\ xz - zx,\ uz - zu).$$

We will construct this algebra as a bimodule algebra as follows.

Let Δ denote the diagonal of $X_1 \times X_2$; i.e.

$$\Delta = V(u_1 v_2 - v_1 u_2)$$

and let 2Δ denote the nonreduced curve of degree 2 over the diagonal; i.e. 2Δ is the closed subscheme corresponding to the ideal $(u_1 v_2 - v_1 u_2)^2$. Let $\mathcal{E} = \mathcal{O}_{2\Delta}$. Then $\mathcal{E}$ is a bimodule of Case 1.

Consider the affine subset $U_1 \times U_2$ of $X_1 \times X_2$ determined by $v_1 \neq 0$ and $v_2 \neq 0$, which has coordinate ring

$$R = k[u_1, u_2].$$

Then $\mathcal{E}|_U$ is the (affine) module corresponding the quotient

$$\frac{R}{(u_1 - u_2)^2}.$$

As a bimodule, recall that the convention is that the left action of u on X corresponds to action of u_1 on $X \times X$, and the right action of u on X corresponds to the action of u_2 on $X \times X$. Let x denote the identity element of $\mathcal{E}|_U$, and let $z = (u_1 - u_2)x$. Then, as a bimodule, the relations on $\mathcal{E}$ are

$$ux - xu = (u_1 - u_2)x = z$$

and

$$uz - zu = (u_1 - u_2)z = (u_1 - u_2)^2 x = 0.$$

This is simply the degree 1 component of the sheaf of differential operators.

Moreover, let $\mathcal{Q}|_U$ be the rank 1 subbimodule of $(\mathcal{E} \otimes \mathcal{E})(U)$ generated by the element $xz - zx$. We must verify that this indeed generates a rank 1 bimodule, by verifying that the left and right modules generated by this element coincide. But

$$\begin{aligned} u(xz - zx) &= uxz - uzx \\ &= xuz + z^2 - zux \\ &= xzu + z^2 - z^2 - zxu \\ &= (xz - zx)u. \end{aligned}$$

Then set $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$. Over each affine cover, the bimodule algebra $\mathcal{B}$ is given by

$$\mathcal{B}(U) = k[u]\langle x, z\rangle/(xu - ux - z,\ zu - uz,\ zx - xz),$$

which is precisely $\mathcal{A}(U)$ as described above.

The same calculations can be done on the other standard affine subset of $X_1 \times X_2$, namely the set where $u_1 \neq 0$ and $u_2 \neq 0$, and we conclude that $\mathcal{B} = \mathcal{A}$. Hence, to understand $\underline{\mathrm{Proj}}\,\mathcal{B}$, we need to compute the global sections of $\mathcal{A}$.

Let U, V be the standard affine cover of $\mathbf{P}^1$, so that

$$\mathcal{A}(U) = k[u]\langle x, z\rangle/(xu - ux - z, z\,\text{central}),$$

and

$$\mathcal{A}(V) = k[v]\langle y, z\rangle/(yv - vy - z,\ z\,\text{central}).$$

Alternatively, $\mathcal{A}(U)$ is the Ore extension $k[u,z][x; id, \delta]$ where δ is the derivation defined by $\delta(u) = z, \delta(z) = 0$, and $\mathcal{A}(V)$ is an Ore extension in a similar fashion. These are graded algebras, with $\deg u, v = 0$ and $\deg x, y, z = 1$. The transition functions from U to V are given by

$$\begin{aligned} u &\longmapsto v^{-1} \\ x &\longmapsto -vyv \\ z &\longmapsto z \end{aligned}$$

It is easy to check that the global sections of $\mathcal{A}$ are given by the following basis set of elements of $\mathcal{A}(U)$: $\{z, x, ux, u^2x + uz\}$. Let us in fact choose the following basis for the global sections:

$$\begin{aligned} e &= x \\ f &= u^2x + uz \\ h &= 2ux + z \\ z &= z \end{aligned}$$

The reason for these choices (and the names that we have given them) will be apparent when we compute the relations in the algebra $A = \Gamma(\mathcal{A})$.

$$\begin{aligned} [e, h] &= eh - he \\ &= x(2ux + z) - (2ux + z)x \\ &= 2(ux + z)x - 2uxx \\ &= 2zx \\ &= 2ez \end{aligned}$$

$$\begin{aligned} [f, h] &= fh - hf \\ &= (u^2x + uz)(2ux + z) - (2ux + z)(u^2x + uz) \\ &= 2u^2xux + 2uzux - 2uxu^2x - 2uxuz \\ &= 2u^2xux + 2uzux - 2u(ux + z)ux - 2(ux + z)z \\ &= -2uzux - 2z^2 \\ &= -2fz \end{aligned}$$

$$\begin{aligned}
[e,f] &= ef - fe \\
&= x(u^2x + uz) - (u^2x + uz)x \\
&= xu^2x + xuz - u^2x^2 - uzx \\
&= (ux + z)ux + (ux + z)z - u^2x^2 - uzx \\
&= u(ux + z)x + uzx + uxz + z^2 - u^2x^2 - uzx \\
&= 2uxz + z^2 \\
&= hz
\end{aligned}$$

We also get the relations $[e, z] = [f, z] = [h, z] = 0$, and also the relation $h^2 = 4ef - 2hz - z^2$.

These relations are exactly the relations of the homogenized enveloping algebra of $\mathfrak{sl}_2$ modulo a central quadratic element. Specifically,

$$A = \frac{H(\mathfrak{sl}_2)}{(h^2 - 4ef - 2hz - z^2)}$$

where $H(\mathfrak{sl}_2)$ is the homogenized $U(\mathfrak{sl}_2)$, given by the relations:

$$\frac{k\langle e, f, h, z\rangle}{[e,h] - 2ez,\ [f,h] + 2fz,\ [e,f] - hz,\ [e,z],\ [f,z],\ [h,z]}.$$

There is a more general theory relating quotients of homogenized $U(\mathfrak{sl}_2)$ and differential algebras over $\mathbf{P}^1$; see [LbS] and [V] for details.

5.2 Quantum $\mathbf{P}^1 \times \mathbf{P}^1$

Let σ, τ be automorphisms of $\mathbf{P}^1$, with graphs $S = \Gamma_\sigma$ and $T = \Gamma_\tau$ in $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\mathcal{E} = \mathcal{O}_{\Gamma_\sigma}(1) \oplus \mathcal{O}_{\Gamma_\tau}(1)$. This is a bimodule of case 2 or 3 (depending on whether $\sigma = \tau$ or $\sigma \neq \tau$).

$\mathcal{E}$ is a bimodule which is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ as either a left module or a right module, and hence the resulting noncommutative scheme may be thought of as a quantum version of $\mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathbf{P}^1 \times \mathbf{P}^1$.

Since we wish to discuss the left and right multiplication operations on this bimodule, it will be useful to write σ and τ in some "normalized" form. In other words, we will want to choose convenient projective coordinates $[u, v]$ for $\mathbf{P}^1$ in such a way that σ and τ, when written as elements of $\operatorname{Aut} \mathbf{P}^1 = PGL_2$, will have nice representations.

Recall that an automorphism of $\mathbf{P}^1$ has exactly one or two fixed points. We have four cases:

(1): σ,τ do not have a common fixed point. Choose projective coordinates u, v so that $[0, 1]$ is a fixed point for σ and $[1, 0]$ is a fixed point for τ.

(2): σ has two fixed points, with exactly one fixed point in common with τ. Choose projective coordinates u, v so that $[0, 1]$ is the common fixed point and $[1, 0]$ is the other fixed point of σ. If σ, τ have a fixed point in common with σ having only one fixed point and τ having two fixed points, interchange σ and τ.

(3): σ,τ each have two common fixed points. Choose projective coordinates x_0, x_1 so that $[0, 1]$ and $[1, 0]$ are the fixed points. This includes the case where $\sigma = \tau$.

(4): σ, τ have only one (common) fixed point. Choose projective coordinates u, v such that $[0, 1]$ is the fixed point, and so that $\sigma([1, 0]) = [1, 1]$.

Then we may represent σ, τ as elements of PGL_2 as follows:

Case	σ	τ
(1)	$\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix}$
(2)	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}$
(3)	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$
(4)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$

where, in each representation, a, b, c, d must be nonzero. These representations act on the point $[u, v]$ on the left; e.g. in Case (1),

$$\sigma([p, q]) = [ap + q, q]$$

since

$$\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + q \\ q \end{pmatrix}.$$

Let us now examine how the representations of σ and τ determine the left and right actions of $\mathcal{O}_{\mathbf{P}^1}$ on $\mathcal{E}$. Let s, t be non-zero global sections of $\mathcal{O}_S, \mathcal{O}_T$ respectively. As an $\mathcal{O}_{X_1 \times X_2}$-module,

$$\begin{aligned} (\sigma(u_1) - u_2)s &= 0 \\ (\tau(u_1) - u_2)t &= 0. \end{aligned}$$

Hence, as an $\mathcal{O}_X$-bimodule, for an open set $U \subset \mathbf{P}^1$ and a section $a \in \mathcal{O}(U)$,

$$\begin{aligned} sa &= \sigma(a)s \\ ta &= \tau(a)t. \end{aligned}$$

This allows us to write the bimodule relations for the bimodules in each of the four cases above (keeping in mind that u and v do not act directly on s and t, but instead rational functions of u and v act on s and t):

$$\begin{aligned} \text{Case (1)}: \quad su &= aus + vs \\ sv &= vs \\ tu &= ut \\ tv &= dut + cvt \end{aligned}$$

$$\begin{aligned} \text{Case (2)}: \quad su &= aus \\ sv &= vs \\ tu &= cut + vt \\ tv &= vt \end{aligned}$$

$$\begin{aligned} \text{Case (3)}: \quad su &= aus \\ sv &= vs \\ tu &= cut \\ tv &= vt \end{aligned}$$

$$\begin{aligned} \text{Case (4)}: \quad su &= us + vs \\ sv &= vs \\ tu &= cut + vt \\ tv &= vt \end{aligned}$$

These relations give us the full information on how the bimodule multiplication laws work. In particular, if x is any section of $\mathcal{E}$ over an open subset U, then x can be written as $x = x_S + x_T$, for suitable $x_S \in \mathcal{O}_S(U)$, $x_T \in \mathcal{O}_T(U)$, and for any $a \in \mathcal{O}(U)$, we can write $a = f(u, v)$ for a suitable rational function f. We then have

$$xa = (x_S + x_T)f(u, v) = f(\sigma(u), \sigma(v))(x_S) + f(\tau(u), \tau(v))(x_T).$$

Note that this bimodule action is a bit unusual, as $f(\sigma(u), \sigma(v)) \in \mathcal{O}(\sigma U)$ whereas $f(\tau(u), \tau(v)) \in \mathcal{O}(\tau U)$, which are in general different open sets of $\mathbf{P}^1$. So our bimodule $\mathcal{E}$ is not a bimodule locally on some open set U unless $\sigma U = \tau U$.

$$\begin{array}{llcl}
\text{Case (1)}: & (ss)u & = & (a^2u+(a+1)v)(ss) \\
& (ss)v & = & v(ss) \\
& (st)u & = & (au+v)(st) \\
& (st)v & = & (adu+(c+d)v)(st) \\
& (ts)u & = & ((a+d)u+cv)(ts) \\
& (ts)v & = & (du+cv)(ts) \\
& (tt)u & = & u(tt) \\
& (tt)v & = & ((cd+d)u+c^2v)(tt) \\
\\
\text{Case (2)}: & (ss)u & = & (a^2u)(ss) \\
& (st)u & = & (acu+v)(st) \\
& (ts)u & = & (acu+av)(ts) \\
& (tt)u & = & (c^2u+(c+1)v)(tt) \\
& v \text{ commutes} & & \\
\\
\text{Case (3)}: & (ss)u & = & (a^2u)(ss) \\
& (st)u & = & (acu)(st) \\
& (ts)u & = & (acu)(ts) \\
& (tt)u & = & (c^2u)(tt) \\
& v \text{ commutes} & & \\
\\
\text{Case (4)}: & (ss)u & = & (u+2v)(ss) \\
& (st)u & = & (cu+(c+1)v)(st) \\
& (ts)u & = & (cu+2v)(ts) \\
& (tt)u & = & (c^2u+(c+1)v)(tt) \\
& v \text{ commutes} & &
\end{array}$$

Figure 5-1: Bimodule action on $\mathcal{E} \otimes \mathcal{E}$

We can also write down the bimodule actions for the tensor product $\mathcal{E} \otimes \mathcal{E}$, as shown in Figure 5-1.

In order to find a rank 1 subbimodule $\mathcal{Q} \subseteq \mathcal{E} \otimes \mathcal{E}$, we must find a section y, given by

$$y = c_1 ss + c_2 st + c_3 ts + c_4 tt$$

such that $\mathcal{O}y = y\mathcal{O}$. Furthermore, for this subbimodule to be admissible, we need at least two of the coefficients c_i to be nonzero. This gives the following table:

Case	Non − zero coefficients	Conditions
(1)	c_1, c_4	$a = c = -1$
(2)	c_1, c_4	$c = -1, a = \pm 1$
	c_2, c_3	$a = 1$
(3)	c_2, c_3	any
	c_1, c_4	$a = \pm c$
	any	$a = c$
(4)	any	$c = 1$

Let us consider a specific example from case (3); i.e. σ, τ have two common fixed points. Let U be the open set defined by $v \neq 0$. Then $\mathcal{O}(U) = k[w]$, where $w = u/v$, and the bimodule relations are given by

$$\begin{aligned} sw &= aws \\ tw &= cwt \end{aligned}$$

Let us choose $\mathcal{Q}$ to be the sub-bimodule of $\mathcal{E}_2$ generated by $st - qts$ for some $q \in k$.

We now wish to compute global sections of $\mathcal{E}$. Since $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ as a right module, the global sections have as a basis $\{s, ws, t, wt\}$. Let these sections be denoted by $\{x_1, x_2, x_3, x_4\}$ respectively. Then the multiplication relations in the algebra are given by the following (note s and t commute because of the relation given by $\mathcal{Q}$):

$$\begin{aligned} x_2x_1 &= wss &= a^{-1}sws &= a^{-1}x_1x_2 \\ x_3x_1 &= ts &= q^{-1}st &= q^{-1}x_1x_3 \\ x_4x_1 &= wts &= q^{-1}a^{-1}swt &= q^{-1}a^{-1}x_1x_4 \\ x_3x_2 &= tws &= q^{-1}cstw &= q^{-1}cx_2x_3 \\ x_4x_2 &= wtws &= q^{-1}a^{-1}cwswt &= q^{-1}a^{-1}cx_2x_4 \\ x_4x_3 &= wtt &= c^{-1}twt &= c^{-1}x_3x_4 \\ x_1x_4 &= swt &= c^{-1}stw &= c^{-1}x_2x_3 \end{aligned}$$

So the algebra is the quotient of $k\langle x_1, x_2, x_3, x_4\rangle$ by the relations:

$$x_2x_1 - a^{-1}x_1x_2$$
$$x_3x_2 - q^{-1}cx_2x_3$$
$$x_3x_1 - q^{-1}x_1x_3$$
$$x_4x_2 - q^{-1}a^{-1}cx_2x_4$$
$$x_4x_1 - q^{-1}a^{-1}x_1x_4$$
$$x_4x_3 - c^{-1}x_3x_4$$
$$x_1x_4 - c^{-1}x_2x_3$$

and it is easy to see that this algebra has the correct Hilbert series to be a quantum quadric surface. In particular, if $a = c = q = 1$, then this is simply the homogeneous coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^1$.

5.3 Quantum Quadrics from Sklyanin Algebras

Let Y be an elliptic curve, σ be an automorphism of Y given by translation by a point of E, and $\mathcal{L}$ be an invertible sheaf on Y. Then one can define the *4-dimensional Sklyanin algebra* $A = A(Y, \sigma, \mathcal{L})$. The definition is rather technical; see [LvS] or [SS] for details.

The 4-dimensional Sklyanin algebra has very nice homological properties. In particular, if Ω is a degree 2 central element of A, then the quotient algebra $A/A\Omega$ is a "quantum quadric surface": it has the same Hilbert series as the homogeneous coordinate ring of the commutative quadric surface, namely

$$\frac{k[a, b, c, d]}{(ad - bc)},$$

Also, $A/A\Omega$ has two families of line modules, which correspond naturally to the two sets of ruling lines on a commutative quadric surface. (See [LvS] or [SS] for the specific constructions.)

Van den Bergh has shown (in [V]) that the 4-dimensional Sklyanin algebra may be constructed as the algebra of global sections of a bimodule

$$\mathcal{B} = T(\mathcal{E})/\mathcal{R}$$

where $\mathcal{E}$ is an $\mathcal{O}_{\mathbf{P}^1}$-bimodule and $\mathcal{R}$ is a subbimodule of $T(\mathcal{E})$ generated by a rank 1 subbimodule of $\mathcal{E} \otimes \mathcal{E}$. Moreover, $\mathcal{E}$ is supported on an elliptic curve in $\mathbf{P}^1 \times \mathbf{P}^1$, isomorphic to Y, which is finite of degree 2 over each factor of $\mathbf{P}^1$.

Bibliography

[AS] M. Artin and W. F. Schelter, "Graded algebras of global dimension 3," *Adv. Math.* **66** (1987), 171–216.

[ATV] M. Artin, J. Tate, and M. Van den Bergh, "Modules over regular algebras of dimension 3," *Invent. Math.* **106** (1991), no. 2, 335-388.

[AV] M. Artin and M. Van den Bergh, "Twisted homogeneous coordinate rings," *J. Algebra* **133** (1990), 249–271.

[AZ] M. Artin and J. J. Zhang, "Noncommutative projective schemes," *Adv. Math.* **109** (1994), 228–287.

[B] A. Beauville, *Complex Algebraic Surfaces*, Cambridge University Press, 1983.

[C] P. M. Cohn, *Algebra, Volume 3*, John Wiley & Sons, 1991.

[H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[I] R. S. Irving, "Prime ideals of Ore extensions over commutative rings, II," *J. Algebra* **58** (1979), 399–423.

[LbS] L. Le Bruyn and S. P. Smith, "Homogenized $\mathfrak{sl}(2)$," *Proc. Amer. Math. Soc.* **118** (1993), 725–730.

[LvS] T. Levasseur and S. P. Smith, "Modules over the 4-dimensional Sklyanin algebra," *Bull. Soc. Math. France* **121** (1993), 35–90.

[MacL] S. Mac Lane, *Categorical Algebra*, American Mathematical Society Colloquium Lectures, 1963.

[Ma] Y. I. Manin, *Quantum Groups and Non-Commutative Geometry*, Centre de Recherches Mathématiques, 1988.

[MR] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, 1987.

[SS] S. P. Smith and J. T. Stafford, "Regularity of the four dimensional Sklyanin algebra," *Comp. Math.* **83** (1992), 259–289.

[V] M. Van den Bergh, "A translation principle for the four-dimensional Sklyanin algebras," *J. Algebra* **184** (1996), 435–490.