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Complete Integrability and Geometrically Induced Representations

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Abstract

In the completely integrable case one can ask, when do geometrically induced representations split into one dimensional weight spaces. In this thesis this problem has been solved in an interesting case, that of the Bott tower. This is then applied to yield a geometric interpretation of the multiplicities of finite dimensional representations of compact Lie groups.

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Dedication.

To my father and mother,
Jack Grossberg and Anita Zang Grossberg,
whose love and support made this possible.

सम्पूर्ण क्रांति अब नारा है
भावी इतिहास हमारा है।

[Total Revolution is our slogan;
Future history belongs to us]
from Prison Diary
by Jayaprakash Narayan

" 'You are still young,' said Athos,
'and there is still time for your
bitter memories to turn to sweet ones.' "
the last line of
The Three Muskateers
by Alexander Dumas

" Next year in Jerusalem"
the Hagadah

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Introduction

Introduction

In the late 1950's R. Bott and H. Samelson intoduced certain manifolds which they used to study the cohomology of the K/T, the flag variety of a compact Lie group K, and the cohomology of $\Omega(K)$ the loop space of K. These manifolds, which throughout this paper are referred to as the Bott-Samelson manifolds, have periodically come up in various guises. Recently, in the summer of '89, Bott noticed that these manifolds supported an action of a real torus of half the dimension of the space [B1]. This is a special case of the phenomenon of complete integrability. One expects that complete integrability should be reflected in the simplification of some geometric structure. In his letter to M. Atiyah he conjectured that the Bott-Samelson manifolds should be realizable as some kind of tower of projecivized vector bundles. He conjectured that there should be some form of induced representation and that under the action of this large torus this representation should decompose into irreducible distict complex lines. Lastly he conjectured that given that the flag manifold is so closely linked with associated Bott-Samelson manifolds, such a complete splitting into lines could lead to some kind of canonical basis of irreducible finite dimensional representations of the compact Lie groups. In fact this very problem was solved my G. Lusztig at approximately the same time using the machinery of quantum groups. It seems likely that there may be a strong connection between quantum groups and this approach although this has not, as yet, revealed itself.

As to Bott's idea that the Bott-Samelson manifolds are projectivized towers, this is clarified in the the section on Bott towers. Bott towers are introduced, they are natural generalizations of Hirzebruch surfaces and they promise to provide interesting examples of various phonomena. In the first chapter geometrically induced representations are reviewed; the notation closely follows [A-B1.] The model for complete integrability and geometrically induced representations, is complex projective space. This example is discussed. A general fact about complete integrability and complex manifolds is spelled out as was communicated in [B1.] Some general conjectures are made about the almost complex case. Bott towers have nice almost complex structures which are preserved by the action of the large torus. This permits the construction of a equivariant elliptic differential operators $D_{\mathbf{L}}$. The kernel and cokernel provide the desired representations and complete integrability does yield a virtual splitting of the index. We call this the simple multiplicity result for Bott towers. There is a section on functoriality of the index which appears necessary for this proof although it is hoped that a proof can be found which avoids the excursion into topology. The second chapter explores the connections with representation theory.

If G is a group one way to seek representations of this group is via geometric induction. Let G act on a manifold M. For the rest of the paper we will assume M is compact without boundry. Let $(A^\bullet, d)$ be a G-equivariant elliptic complex. This then induces a representation of on each of the cohomology spaces, $H^i(A^\bullet, d)$. While it is often difficult to undersand the cohomology itself it is often possible to study the index. If we let the index of the elliptic complex to be $\sum(-1)^i H^i(A^\bullet, d)$, then G induces a virtual representation on the index of the complex. Denote the representation ring of G by $R(G)$. If $\hat{G}$ is the set of inequivalent irreducible representations of G then $R(G)$ is the free abelian group generated by $\hat{G}$. Hence we may write

$$\sum(-1)^i H^i(A^\bullet, d) = \sum_{V \in \hat{G}} M_V \cdot V \tag{0.1}$$

.

In particular if $G = T^n = \overbrace{U(1) \times U(1) \times \cdots \times U(1)}^{n \text{ times}}$ (an $n-$torus), then $\hat{T} = \mathrm{Hom}(T^n, U(1))$, is the set of characters of T. An element of $\hat{T}$ is a map

$$e^\lambda : T^n \to U(1) \tag{0.2}$$

where $\lambda = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n$ and $e^\lambda(e^{i2\pi\theta_1}, e^{i2\pi\theta_2}, \ldots, e^{i2\pi\theta_n}) = e^{i2\pi\sum_{i=1}^n d_i\theta_i}$. So in this case

$$\sum(-1)^i H^i(A^\bullet, d) = \sum_\lambda m_\lambda \cdot e^\lambda. \tag{0.3}$$

Here m_λ is called the *multiplicity* of the weight λ and if $m_\lambda = 1$ whenever $m_\lambda \neq 0$ the representation is called *multiplicity free*. A weaker condition which will be of more interest here is requiring $m_\lambda = \pm 1$ when $m_\lambda \neq 0$ which will be called *simple multiplicity*. Note that the presence of multiplicities -1 indicate a *virtual* representation. The case of the torus is of course important because when G is complex semi-simple or compact its finite dimensional complex representations are completely determined by the character of a maximal torus. The case of multiplicity free is important (respectively simple multiplicity) because the decomposition into one dimensional subspaces gives us a canonical basis of the representation (respectively virtual representation.)

R. Bott points out in a letter to M. F. Atiyah [B1] that complete integrability of a manifold can imply the existence of a canonical basis for geometrically induced representations. Complete integrability is taken to mean the existence of a torus T^n (as above) acting smoothly on a real $2n-$dimensional manifold with isolated fixed points. Bott shows in the case of T^n acting holomorphicly on a complex $n-$dimensional manifold M with equivariant holomorphic line bundle $\mathbf{L}$ the induced representation on $\Gamma(M, \mathbf{L})$ splits into complex lines. This is accomplished by choosing a fixed point and filtering the sections by degrees of vanishing at that point. It was hoped that there was such an action on a manifold closely related to the the flag manifold G/B of a complex semi-simple Lie group G with Borel subgroup B. This in turn would lead to a canonical basis for finite dimensional representations of G via Borel-Weil-Bott theory. This turns out not to be successful in the original context however it will be useful to review this theory briefly [B2]. As an aside, the problem of finding a canonical basis was subsequently solved by G. Lusztig using quantum group methods [L].

Let Δ_+ be a positive root system for the Lie group G. We recall that there is a 1-1 correspondence between irreducible representations and the set of dominant weights with respect to Δ_+. Let λ be an integral weight and B be the Borel subgroup with Lie algebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}^{-\alpha}$, the negative root spaces. If r is the rank of G then e^λ is an irreducible representation $\mathbf{C}_\lambda$ of T^r on $\mathbf{C}$ which extends naturally to $B \supset T^r$. Thus the line bundle $\mathbf{L}_\lambda := G \times_B \mathbf{C}_\lambda \to G/B$ has an action of G by left multiplication. This induces a representation of G on $H^0(G/B, \mathbf{L}_\lambda) \equiv \Gamma(G/B, \mathbf{L}_\lambda)$ (Dolbeaut cohomology with coefficients in $\mathbf{L}_\lambda$). The Borel-Weil theorem tells us that if λ is dominant, this is simply the unique irreducible representation, V_λ, with highest weight λ and further (Bott,) that $H^q(G/B, \mathbf{L}_\lambda) = 0$ for $q > 0$. So in fact the index of the rolled up Dolbeaut operator twisted by the line bundle, $\bar{\partial} + \bar{\partial}^* \otimes \nabla_{\mathbf{L}}$, which is $\sum(-1)^i H^i(G/B, \mathbf{L}_\lambda)$ is simply V_λ. We can restrict to the action of T^r on G/B which then acts on the index of $\bar{\partial} + \bar{\partial}^* \otimes \mathbf{L}_\lambda$. Hence the character index is just the character of V_λ. Of course as mentioned above

$$V_\lambda \cong \sum(-1)^i H^i(G/B, \mathbf{L}_\lambda) = \sum_\mu m_\mu \cdot e^\mu \tag{0.4}$$

with m_μ is the multiplicity of the weight μ. When $\mu = \lambda$, $m_\mu = 1$ but in general $m_\mu > 1$ when $m_\mu \neq 0$.

In order to try to further split these weight spaces into one dimensional pieces one first passes to a Bott-Samelson manifold M defined in [B-S]. If w_0 is the longest element of the Weyl group of G then M is built out of M and a reduced expression for w_0 in terms of simple reflections. M is a sequence of blows up of G/B which maintain the action of T^r. Demazure showed that a map $\Psi : M \to G/B$ induces an isomorphism $H^q(G/B, \mathbf{L}_\lambda) \cong H^q(M, \Psi^*\mathbf{L}_\lambda)$ as T^r modules [D]. Composing this isomorphism with that of 0.4 we obtain $\sum(-1)^i H^i(M, \Psi^*\mathbf{L}_\lambda) \cong V_\lambda$.The surprising fact is that M admits an action of a $n-$torus, $\tilde{T}$ with n the number of positive roots [B1]. Hence we are in the completely integrable case. This action lifts to an action on $\Psi^*\mathbf{L}_\lambda$ and so by Bott's observation above, this should split V_λ into one dimensional pieces. The character can be computed via the Atiyah-Bott fixed point formula. Some of the multiplicities were discovered to be negative integers implying, that $\tilde{T}$ does not act holomorphicly.

The author's solution is to relax the category to that of an almost complex structure J and to study the G-equivariant index of a a twisted Dolbeault or equivalently Spin^c twisted Dirac operator asscociated to the almost complex structure $D_\mathbf{F}$ with $\mathbf{F}$ a line bundle [L-M]. This is in done in the more general setting of a manifold that is the successive projectivization of rank-2 complex plane bundles. We assume that at each stage of the projectivization, there is a section. Again one can find a torus $\tilde{T}$ of half the dimension of the manifold acting on it. One can also find an equivariant almost complex structure and lift the action to an auxiliary line bundle.

Theorem 0.1 *Let M be a successive projectivization of complex two-plane bundles, with complex dimension n. Suppose further that at each step in the tower of two-sphere fiber bundles, there is a section. Let $\mathbf{L} \to M$ be a line bundle. For any effective action of an n-torus on the tower preserving the sections and an equivariant operator $(\bar\partial + \bar\partial^*) \otimes \mathbf{L}$, the virtual character index has simple multiplicity.*

This is proven by induction. The families index theorem [A-S1] is used in stages.

When applied to a Bott-Samelson manifold M, which is a successive projectivization, one obtains a virtual representation of a torus $\tilde{T}$ that naturally contains the maximal torus T of the group as mentioned above. Restricting the character to the maximal torus $T \subset \tilde{T}$, gives the ordinary character of the representation V_λ. This is proven by finding an almost complex structure, $\tilde{J}$ which is invariant under $\tilde{T}$. Demazure showed that the manifolds Bott and Samelson discovered were algebraic varieties and hence carry an honest complex structure J. The maximal torus $T \subset K$ acts on M preserving both J and $\tilde{J}$.

Theorem 0.2 *There is a family of T-equivariant almost complex structures J_ϵ, $0 \leq \epsilon \leq 1$, with $J_0 = J$ and $J_1 = \tilde{J}$.*

The homotopy invariance of the index shows that the $\tilde{T}$ character extends the ordinary T character. Studying the form of our extended character reveals that the weights in the dual Lie algebra of the larger torus, Lie$(\tilde{T}) = \tilde{\mathfrak{t}}^*$, live in a non-convex shape the author calls a twisted cube. Counting the signed lattice points in the twisted cube yields the dimension of the representation while first intersecting with subspaces (inverse images of points under the projection $\tilde{\mathfrak{t}}^* \to \mathfrak{t}^*$) yields multiplicities of weights in the original representation.

Chapter 1

Simple multiplicity in projectivized towers

1.1 Geometrically induced representations

In this section we will briefly review the machinery of geometric induction. This is not meant to be an exposition of the material but rather an opportunity to set notation. The general idea is to start with a C^∞-manifold M with some kind of geometric structure, and an action of a group G on M that preserves that geometric structure. Out of this data one constructs a representation of the group G. Probably the most familiar example is the case of a complex semisimple Lie group G acting on $M = G/B$ with B a Borel subgroup. This induces a representation on the Dolbeault cohomology with coefficients in a line bundle $H^i(G/B,\ \mathbf{L})$. In the case that $\mathbf{L}$ comes from a dominant weight of G, the geometrically *induced* representation on $H^0(G/B,\ \mathbf{L})$, coincides with the ordinary notion of an induced representation.

1.1.1 Elliptic Complexes

Let M be a C^∞-manifold. Suppose we have vector bundles $\{\mathbf{E}_i\}$ over M. Denote by $\Gamma(\mathbf{E}_i)$, C^∞-sections of $\mathbf{E}_i$. A sequence of kth order differential operators

$$\cdots \to \Gamma(\mathbf{E}_{i-1}) \overset{d_{i-1}}{\to} \Gamma(\mathbf{E}_i) \overset{d_i}{\to} \Gamma(\mathbf{E}_{i+1}) \cdots \tag{1.1}$$

is called a *differential complex* if $d_i d_{i-1} = 0$ (sometimes abbreviated $d^2 = 0$.) Let $\Gamma(E)$ denote this complex. The vector spaces

$$H^i(\Gamma(E)) \overset{\text{def}}{=} \ker d_i / \mathrm{im}\ d_{i-1} \tag{1.2}$$

are the *cohomology groups* (thought of as additive groups) of the complex $\Gamma(E)$. Locally we can write a kth order differential operator

$$d_i = \sum_{|\alpha| \leq k} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{1.3}$$

here α here is a multi-index. Locally then the *principal symbol* of this operator is defined to be

$$\sigma(d_i) = \sum_{|\alpha| = k} A_\alpha(x) \xi^\alpha. \tag{1.4}$$

The variables ξ are coordinates in the fibers of the cotangent bundle, T^*M. It turns out that the principal symbol is a globally well defined object. Let $\pi^* : T^*M \rightarrow M$. The symbols of the operators of the complex $\Gamma(E)$ induce a sequence of bundles called the *symbol sequence*

$$\cdots \rightarrow \pi^*\mathbf{E}_{i-1} \stackrel{\sigma(d_{i-1})}{\rightarrow} \pi^*\mathbf{E}_i \stackrel{\sigma(d_i)}{\rightarrow} \pi^*\mathbf{E}_{i+1} \rightarrow \cdots \tag{1.5}$$

over T^*M A differential complex is said to be *elliptic* if its symbol sequence is exact off of the zero section. An important property of a differential complex that is elliptic is that if the manifold M is also compact, then the cohomology of the complex is finite dimensional.

1.1.2 Lifting actions

Suppose $f : M \rightarrow M$ is a smooth map. Suppose as above we have an elliptic differential complex $\Gamma(E)$. Consider maps

$$\varphi^i : f^*\mathbf{E}_i \rightarrow \mathbf{E}_i \tag{1.6}$$

for each i, which covers f. Composing φ^i with f^* we get a map

$$T^i : \Gamma(\mathbf{E}_i) \stackrel{f^*}{\rightarrow} \Gamma(f^*\mathbf{E}_i) \stackrel{\varphi}{\rightarrow} \Gamma(\mathbf{E}_i). \tag{1.7}$$

If T^i commute with the differential operators, i.e.. $d_i T^i = T^{i+1} d_i$, then we call the collection of maps $\{\varphi^i\}$ a *lifting* of f to the complex $\Gamma(E)$. A lifting of f, $\{\varphi^i\}$, to $\Gamma(E)$ induces maps

$$T^i : H^i(\Gamma(\mathbf{E})) \rightarrow H^i(\Gamma(\mathbf{E})) \tag{1.8}$$

which by abuse of notation we also denote T^i. In particular suppose we have a left action of a group G on a manifold M. Let $f_{g^{-1}}$ be the map

$$\begin{array}{rcl} f_{g^{-1}} : M & \rightarrow & M \\ m & \mapsto & g^{-1} \cdot m \end{array} \tag{1.9}$$

and assume there is a natural lifting of $f_{g^{-1}}$. Call the map induced from the lifting of $f_{g^{-1}}$, T^i_g (later we shall see why this convention makes sense.) Suppose further that if $g, h \in G$ then $T_g^i T_h^i = T_{gh}^i$. Then we call the resulting representation on $H^i(\Gamma(E))$ a *geometrically induced representation.*

1.1.3 Almost Complex Structures and The Dolbeault Operator

Suppose M is an even dimensional manifold with an almost complex structure J. If $m \in M$ then $J_m : T_mM \rightarrow T_mM$ is a linear map such that $(J_m)^2 = -\text{Identity}$. Thus the complexification splits as follows

$$TM \otimes \mathbf{C} \cong (TM)^{1,0} \oplus (TM)^{0,1} \tag{1.10}$$

with $(TM)^{1,0}$ the $+i$ eigenspace and $(TM)^{0,1}$ the -i eigenspace of J acting on $TM \otimes \mathbf{C}$. There is a similar splitting for $T^*M \otimes \mathbf{C}$. With this splitting one obtains a bigrading on $\wedge^*(T^*M \otimes \mathbf{C})$ (here $\wedge$ taken to be over $\mathbf{C}$) such that

$$\wedge^r(T^*M \otimes \mathbf{C}) = \bigoplus_{r=p+q} \wedge^{p,q} T^*M. \tag{1.11}$$

Were we define $\wedge^{p,q}T^*M = \wedge^p(T^*M)^{1,0} \otimes \wedge^q(T^*M)^{0,1}$. Let

$$\pi_{p,q} : \wedge^r T^*M \otimes \mathbf{C} \rightarrow \wedge^{p,q} T^*M \tag{1.12}$$

be the projection. The ordinary exterior derivative is an operator

$$d : \Gamma(\wedge^r T^*M \otimes \mathbf{C}) \rightarrow \Gamma(\wedge^{r+1} T^*M \otimes \mathbf{C}). \tag{1.13}$$

Define the operator

$$\bar{\partial} : \Gamma(\wedge^{p,q} T^*M) \rightarrow \Gamma(\wedge^{p,q+1} T^*M) \tag{1.14}$$

where $\bar{\partial} = \pi_{p,q+1} d$. If $\bar{\partial}^2 = 0$, then we can form a complex

$$\cdots \xrightarrow{\bar{\partial}} \Gamma(\wedge^{0,q-1}(T^*M)) \xrightarrow{\bar{\partial}} \Gamma(\wedge^{0,q}(T^*M)) \xrightarrow{\bar{\partial}} \Gamma(\wedge^{0,q+1}(T^*M)) \xrightarrow{\bar{\partial}} \cdots. \tag{1.15}$$

The symbol of $\bar{\partial}$ is $\sigma(\bar{\partial})(\eta, \varphi) = \eta^{0,1} \wedge \varphi$, with $\eta \in T^*M$, $\eta = \eta^{1,0} + \eta^{0,1}$, and $\varphi \in \wedge^{0,q}$. The symbol sequence is exact, so the complex is elliptic. The condition that $\bar{\partial}^2 = 0$, is equivalent to the almost complex structure being integrable, which by [N-N] is equivalent to the almost complex structure coming from an honest complex structure. Suppose, in this case, that $\mathbf{L} \rightarrow M$ is a holomorphic line bundle. If we choose a hermitian metric h on $\mathbf{L}$, this determines a unique connection $\nabla_{\mathbf{L}}$ on $\mathbf{L}$ as follows. If s is a local holomorphic trivializing section, and $\nabla_{\mathbf{L}} s = \theta \otimes s$ is determined by the connection 1 form θ, which is defined by the equation $\theta = h^{-1}\partial h$. The connection decomposes as $\nabla_{\mathbf{L}} = \nabla'_{\mathbf{L}} + \nabla''_{\mathbf{L}}$, where $\nabla'_{\mathbf{L}} = \partial + \theta$ and $\nabla''_{\mathbf{L}} = \bar{\partial}$. Suppose $\varphi \otimes s \in \Gamma(\Lambda^{0,q}(T^*M) \otimes \mathbf{L})$. Then define the operator $\bar{\partial}_{\mathbf{L}}$

$$\bar{\partial}_{\mathbf{L}}(\varphi \otimes s) = (\bar{\partial} \otimes \nabla''_{\mathbf{L}})(\varphi \otimes s) = (\bar{\partial}\varphi) \otimes s + (-1)^q \varphi \otimes \nabla''_{\mathbf{L}} s. \tag{1.16}$$

If s is a holomorphic section then this operator simplifies to $\bar{\partial} \otimes \mathbf{1}$. From this we build the twisted complex

$$\cdots \xrightarrow{\bar{\partial}\otimes\mathbf{1}} \Gamma(\Lambda^{0,q-1}(T^*M) \otimes \mathbf{L}) \xrightarrow{\bar{\partial}\otimes\mathbf{1}} \Gamma(\Lambda^{0,q}(T^*M) \otimes \mathbf{L}) \xrightarrow{\bar{\partial}\otimes\mathbf{1}} \Gamma(\Lambda^{0,q+1}(T^*M) \otimes \mathbf{L}) \xrightarrow{\bar{\partial}\otimes\mathbf{1}} \cdots. \tag{1.17}$$

which is again elliptic. We write the cohomology of this complex as $H^j(M, \mathbf{L})$. If we suppose that we have a hermitian metric on T^*M then we can form the adjoint $\bar{\partial}^*$ to the operator $\bar{\partial}$. Putting this together with the connection we can "roll up" the complex (1.17) into one operator

$$D_{\mathbf{L}} = (\bar{\partial} + \bar{\partial}^*) \otimes \nabla''_{\mathbf{L}} : \Gamma(\Lambda^{0,\mathrm{even}}(T^*M) \otimes \mathbf{L}) \rightarrow \Gamma(\Lambda^{0,\mathrm{odd}}(T^*M) \otimes \mathbf{L}). \tag{1.18}$$

This is an elliptic operator. The even cohomology of (1.17) is captured in the kernel, ker $D_{\mathbf{L}} = H^{\mathrm{even}}(M, \mathbf{L})$ while the odd cohomology is captured in the cokernel, coker $D_{\mathbf{L}} = H^{\mathrm{odd}}(M, \mathbf{L})$. We call this operator the (twisted) Dolbeault operator. The operator $D_{\mathbf{L}}$ is well defined and elliptic even if $\bar{\partial}^2 \neq 0$. In that case $H^j(M, \mathbf{L})$ no longer makes sense, and we do not have a canonical choice of $\nabla_{\mathbf{L}}$. Nevertheless we may choose a $\nabla_{\mathbf{L}}$, and thus $\nabla''_{\mathbf{L}}$ and define $D_{\mathbf{L}}$. Though we do not have cohomology we still have the kernel and cokernel of $D_{\mathbf{L}}$. It is precisely in the case in which the almost complex structure is not integrable that the operator $D_{\mathbf{L}}$ is necessary to provide "cohomology." In fact such an operator may be defined more generally. Breifly, if our even dimentional manifold, M, has a Spin^c structure, we can define a Spin^c

operator analogous to the Dirac operator. If the Spin^c comes from an almost complex structure, then is operator is simply $\bar{\partial} + \bar{\partial}^*$. As before we can twist the operator by choosing a connection on a line bundle over M.

1.2 Complete Itegrability and the Twisted Dolbeault Operator

In this section we define the G -index of the iwisted Dolbeault operator. We study the case of the torus action on complex projective space. We define what we mean by complete integrablity and using complex projective space as a model generalize to complex manifolds. From this we form a conjecture about almost complex manifolds.

1.2.1 The G -Index of the Twisted Dolbeault Operator

Suppose G is a compact Lie group acting on an almost complex manifold M (on the left)with almost complex structure J. We assume that the group action preserves this structure J. Let $g \in G$, and let $l_g : M \to M$ be the diffeomorphism obtained by left translation by g. As mentioned before it will actually make more sense to work with $l_{g^{-1}}$. The linearization of left translation at a point is

$$T(l_{g^{-1}}) : T_{g \cdot m}M \rightarrow T_m M. \tag{2.19}$$

If we complexify the tangent bundle and the map and take the transpose, we have a map

$$T(l_{g^{-1}})^* : T_m^* M \otimes \mathbf{C} \rightarrow T_{g \cdot m}^* M \otimes \mathbf{C}. \tag{2.20}$$

Since the action preserves the almost complex structure J, this map preserves the -i eigenspace of J,

$$T(l_{g^{-1}})^* : (T_m^* M)^{0,1} \rightarrow (T_{g \cdot m}^* M)^{0,1}. \tag{2.21}$$

Now the g^{-1} convention becomes clear. Since $T(l_{g^{-1}})^* T(l_{h^{-1}})^* = T(l_{(gh)^{-1}})^*$, a left action is induced on $\Gamma(\wedge^{p,q} T^* M)$. In particular if $\bar{\partial}^2 = 0$ and $\mathbf{L} \to M$ is a holomorphic equivariant line bundle, then this induces representations of G on $H^q(M, \mathbf{L})$. Even if $\bar{\partial}^2 \neq 0$, if we suppose $\mathbf{L} \rightarrow M$ is an equivariant line bundle, with equivariant connection $\nabla_{\mathbf{L}}$, and an equivariant metric, then we can define $D_{\mathbf{L}}$ as in (1.18). This will be an equivariant operator and so we will induce representations of G on ker $D_{\mathbf{L}}$ and coker $D_{\mathbf{L}}$.

Recall that in Section 1.1.2, if $g \in G$, we let T_g^i denoted the associated endomorphism the cohomology $H^i(\Gamma(E))$ of the elliptic complex $\Gamma(E)$. The G-index or character index of $\Gamma(E)$ is a character of the group G, $ind_G(\Gamma(E))$ defined by

$$ind_G(\Gamma(E))(g) = \sum(-1)^i \text{trace } (T_g^i) \tag{2.22}$$

with $g \in G$ and the trace taken by considering T_g^i as an endomorphism of the finite dimensional vector space $H^i(\Gamma(E))$. Applying this definition to the above $2n$ dimensional almost complex manifold M, with equivariant line bundle $\mathbf{L}$ and equivariant connection $\nabla_{\mathbf{L}}$, we have

$$ind_G\left(\Gamma(\Lambda^{0,*}T^*M \otimes \mathbf{L}), par\bar{t}ial\right) = \sum_{i=1}^{n}(-1)^i ch(H^i(M, \mathbf{L})) \tag{2.23}$$

with ch the character of the representation of G on $H^i(M, \mathbf{L})$, in the case where $\bar{\partial}^2 = 0$ and $\mathbf{L}$ is holomorphic. Even in the non-integrable case, where $\bar{\partial}^2 \neq 0$ we have

$$ind_G(D_{\mathbf{L}}) = ch(kerD_{\mathbf{L}}) - ch(cokerD_{\mathbf{L}}) \tag{2.24}$$

While it is often very difficult to study the geometrically induced representations specifically, it is often possible to obtain information about the index.

1.2.2 Complex Projective Space

Perhaps the most beautiful example of geometrically induced representations is on $\mathbf{CP}^n$. This example is very much a model situation in that we can induce interesting representations on sections of holomorphic line bundles, and we hace an action of an n torus which is what we call complete integrability. More specifically have the principal bundle

$$\begin{array}{ccc} H := \mathbf{C}^\times & \rightarrow & \mathbf{C}^{n+1} \setminus \{0\} =: E \\ & & \downarrow \\ & & \mathbf{CP}^n \end{array}$$

where $\mathbf{C}^\times := \mathbf{C} \setminus \{0\}$, and the right bundle action is given by $(z_1, \dots z_n) \cdot t = (z_1 t, \dots z_n t)$ Now let $k \in \mathbf{Z}$ we can form the associated line bundle

$$\begin{array}{ccc} E \times_{\mathbf{C}^\times} \mathbf{C}_k & = & \mathbf{L} \\ \downarrow & & \\ \mathbf{CP}^n & & \end{array}$$

with $\mathbf{C}_k$ is the $\mathbf{C}^\times$ module obtained by left multiplication by t^k, $t \in \mathbf{C}$. Holomorphic sections of this bundle come from maps $f : E \rightarrow \mathbf{C}_k$ such that $f(z_0 t, \dots, z_n t) = t^{-k} f(z_0, \dots, z_n)$. In other words, f is homogeneous of degree $-k$. If $k \leq 0$ then $\mathbf{L}$ has sections which are given by the homogeneous polynomials of degree $-k$.

We can now induce interesting representaions geometricall. The group SU(n+1) acts linearly on $\mathbf{C}^{n+1} \setminus \{0\}$ by the restriction of its defining representation. This action commutes with the right bundle action on E. Thus this induces an action on $\mathbf{CP}^n$, on $\mathbf{L}$ and on the space of holomorphic sections $\Gamma(\mathbf{CP}^n, \mathbf{L})$. This is in fact an irreducible representation of SU(n+1) and the module, as we have seen above, is the space of homogeneous polynomials on $\mathbf{C}^{n+1}$ of degree $-k$. A canonical basis for this space is the set of monomials

$$z_0^{p_0} \cdots z_n^{p_n},$$
$$\sum_{i=1}^n p_i = -k, \text{ and } p_i \geq 0.$$

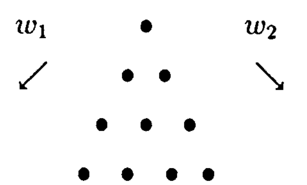
We would like to see this basis geometrically. Suppose we restrict to a maximal torus T in SU(n+1) which could be represented as the diagonal determinant one matrices. The torus T has dimension n which is the complex dimension of $\mathbf{CP}^n$. This case of a torus acting on a manifold of half the dimension (real) of the manifold we will call *complete integrability.* The T thus acts on $\Gamma(\mathbf{CP}^n, \mathbf{L})$ and splits it into weight spaces. In particular the basis above is preserved by this action and these monomials span the weight spaces. We can take generators $w_j = e^{2i\pi\theta_j}$. Then an element of T looks like

$$\begin{pmatrix} w_1 & & & & \\ & w_1^{-1} w_2 & & & \\ & & \ddots & & \\ & & & w_{n-1}^{-1} w_n & \\ & & & & w_n^{-1} \end{pmatrix}$$

In this notation the weight associated to the monomial $z_0^{p_0} \cdots z_n^{p_n}$ is $w_1^{p_0-p_1} \cdots w_n^{p_{n-1}-p_n}$ It is important to note that each basis element has a different weight. We call this situation *multiplicity free.* Suppose $n = 2$ and $k = 3$ then the weights are

$$\begin{array}{c} 1 \\ w_1\ w_2 \\ w_1^2\ w_1w_2\ w_2^2 \\ w_1^3\ w_1^2w_1\ w_1w_2^2w_2^3 \end{array} \tag{2.25}$$

which can be graphically represented as the weight diagram:



(2.26)

1.2.3 Complete Integrability and the Complex Case

One would hope that, just as in the case of complex projective space, complete integability would split geometrically induced representaions into distinct one dimentional weight spaces. Bott explains that this is the case for sections of a holomorphic line bundle over complex manifold [B]. Suppose we have an n dimensional complex manifold M with a holomorphic effective action of an n torus $T^{(n)}$ with isolated fixed points.

Theorem 1.1 *Suppose there is also a holomorphic equivariant line bundle* $\mathbf{L}$ *over* M*. Then if* $\Gamma_{hol}(M, \mathbf{L})$*, the space of holomorphic sections, doesn't vanish it is multiplicity free.*

Proof. Let $m \in M$, be a fixed point. We can filter $\Gamma_{\text{hol}}(M, \mathbf{L})$ by the degree of a section vanishing at the point m.

$$\Gamma_{\text{hol}}(M, \mathbf{L}) = F_0 \supset F_1 \cdots \supset F_p \tag{2.27}$$

Since we are taking holomorphic sections this descending filtration terminates. The graded pieces F_i/F_{i-1}, by the Taylor expansion in n variables is just $S^i(T_mM) \otimes \mathbf{L}_m$, where S^i is the ith symmetric power. This yields an embedding

$$\Gamma_{\text{hol}}(M, \mathbf{L}) \hookrightarrow S^*(T_m M) \otimes \mathbf{L}_m. \tag{2.28}$$

Since the fixed points are isolated an we have an effective torus action, the weights (or normal exponents) are linearly independent at $T_m M$. Hence $S^*(T_m M)$ splits into one dimensional pieces which are multiplicity free and so does $S^*(T_m M) \otimes \mathbf{L}_m$. This then induces a multiplicity free splitting of $\Gamma_{\text{hol}}(M, \mathbf{L})$. ■

The proof above would seem to indicate that the representation is more or less determined by the isotropy representations at the fixed points. For sufficiently positive line bundles this is certainly true. This might tempt one to generalize this for almost complex structures which are the pointwise analog of complex structures. Here things become more complicated but potentially workable. Let M be an almost complex manifold with almost complex structure J. Suppose T is an n torus acting on the $2n$ dimensional manifold M. Further suppose that T acts by isometries preserving the almost complex structure J, and that there is a line bundle $\mathbf{L}$, and a connection $\nabla_{\mathbf{L}}$ which are equivariant. Then there are induced representations on $\ker D_{\mathbf{L}}$ and $\operatorname{coker} D_{\mathbf{L}}$. We define the T index of $D_{\mathbf{L}}$ as the difference of the characters of these representations

$$ind_T D_{\mathbf{L}} = ch_T(\ker D_{\mathbf{L}}) - ch_T(\operatorname{coker} D_{\mathbf{L}}). \tag{2.29}$$

If the almost complex structure is integrable, i.e. $\bar{\partial}^2 = 0$, then we can express the index in terms of the representations on the cohomology

$$ind_T D_{\mathbf{L}} = ch_T\left(H^{even}(M, \mathbf{L})\right) - ch_T\left(H^{odd}(M, \mathbf{L})\right). \tag{2.30}$$

More particularly if the higher cohomology vanishes then

$$ind_T D_{\mathbf{L}} = ch_T\left(\Gamma_{\text{hol}}(M, \mathbf{L})\right). \tag{2.31}$$

As shown above T splits $\Gamma_{\text{hol}}(M, \mathbf{L})$ into multiplicity one dimensional pieces and so $ind_T D_{\mathbf{L}}$ is a Laurent polynomial with all $+1$ coefficients. A less stringent condition would be that the coefficients would be ± 1, which we will call simple multiplicity. In general there is the following wild conjecture:

Conjecture 1.1 *Let M be a $2n$ dimensional almost complex manifold with all the structure above including a torus T of dimension n acting effectively with isolated fixed points, then $ind_T D_{\mathbf{L}}$ has simple multiplicity.*

Or even a bit wilder:

Conjecture 1.2 *This is still true when $D_{\mathbf{L}}$ is manufactured from a $Spin^c$ structure.*

On possible approach is to first look at the symplectic version of this problem (which there will unfortunately not be time for here) which is presumably easier to understand.

1.3 Bott Towers and the Simple Multiplicity Theorem

In this section Bott-towers are defined. We show first that for Bott-towers with an almost complex structure, an action of an n-torus $T^{(n)}$, an equivariant complex line bundle and an equivariant connection, we can construct an operator $D_{\mathbf{L}}$ whose $T^{(n)}$-index is a character of $T^{(n)}$ with simple multiplicity. This will be done by induction on the stages of the tower (dimension). The result is established on $\mathbf{CP}^1$ first. Then an argument is given exactly analogous to a spectral sequence argument. By computing the index "along the fibers" and assuming the index on the base has a particular form on the base, the result can be established on the bundle. A crucial piece of the proof uses the index theorem and the multiplicitivity axiom to prove functioriality. This will be explained in the third and fourth subsections. The this part is not entirely satisfactory since there should be a direct proof that does not involve the brief sidetrack into topology. Throughout this section it will be assumed that an almost complex structure comes with a compatible riemannian metric which is guaranteed since $T^{(n)}$ is compact.In the following section we establish that the extra structure that was assumed on the Bott-tower always exist.

1.3.1 The Simple Multiplicity Theorem

Suppose we start with a line bundle $\mathbf{L}_1$ over, $\mathbf{CP}^1$:

$$\begin{array}{c} \mathbf{L}_1 \\ \downarrow \\ \mathbf{CP}^1 \stackrel{\text{def}}{=} M_1 \end{array} \tag{3.32}$$

Now take the direct sum of $\mathbf{L}_1$ and a trivial bundle and projectivize the fibers:

$$\begin{array}{c} \mathbf{P}(1 \oplus \mathbf{L}_1) \stackrel{\text{def}}{=} M_2 \\ \downarrow \quad \pi_2 \\ M_1 \end{array} \tag{3.33}$$

If we assume that the line bundle $\mathbf{L}_1$ is holomorphic then M_1 is a Hirzebruch surface. We can generalize this as follows. Again suppose we take a line bundle $\mathbf{L}_2$

over M_2. We can repeat the process n-times.

$$
\begin{array}{cccccc}
 & & & & \mathsf{P}(1 \oplus \mathbf{L}_n) & = \; M_n \\
 & & & & \downarrow \pi_n & \\
 & & & & M_{n-1} & \\
 & & & \cdots & & \\
 & & \mathsf{P}(1 \oplus \mathbf{L}_3) & & & \\
 & & \downarrow \pi_3 & & & \\
 & \mathsf{P}(1 \oplus \mathbf{L}_2) \; = & M_2 & & & \\
 & \downarrow \pi_2 & & & & \\
\mathsf{CP}^1 \; = & M_1 & & & &
\end{array}
\tag{3.34}
$$

Such a collection of spaces and maps, $\{M_i, \pi_i\}$ will be called a *Bott tower.* An ***almost complex structure on a Bott tower*** is then a collection of almost complex structures $\{J^i\}$ such that J^i is an almost complex structure on M_i and

a) $T(\pi_i)J^i = J^{i-1}T(\pi_i)$

b) The almost complex structure restricted to fiber $J^i|_{\mathsf{CP}^1}$ is the ordinary complex structure on CP^1 .

A *complete almost complex action on an n step Bott tower* is an action of an n torus, $T^{(n)}$ on each M_i, which is effective on M_n and preserves the maps π_i.

Theorem 1.2 *Let $\{M_i, \pi_i, J^i\}$ be an almost complex Bott tower with a complete torus action of $T^{(n)}$. Suppose there is an equivariant line bundle $\mathbf{L}$, and fix an equivariant connection $\nabla_{\mathbf{L}}$ and hermitian metric h on M_n, then this data determines an equivariant operator*

$$D_{\mathbf{L}} \; : \; \Gamma(\Lambda^{0,even} T^* M \otimes \; \mathbf{L}) \rightarrow \Gamma(\Lambda^{0,odd} T^* M \otimes \; \mathbf{L}) \tag{3.35}$$

and the character $ind_{T^{(n)}}(\; D_{\mathbf{L}} \;)$ has simple multiplicity.

We first need to prove some propositions. Eventually we will establish Theorem 1.2 by induction. Rather than start at $n = 0$, a point, it will be necessary to handle the case $n = 1$ which will be used in the induction step. After proving this elementary step the induction step will be handled as a separate theorem and Theorem 1.2 will follow as a corollary. Let $M_1 = \mathbf{CP}^1$. The group is a circle $T^(1) = S^1$ acting on $\mathbf{CP}^1$ preserving its natural complex structure. Suppose we have an equivariant (smooth) line bundle **L** over $\mathbf{CP}^1$ and an equivariant connection $\nabla_{\mathbf{L}}$. We will need the following:

Proposition 1.1 (Atiyah and Bott) *$\nabla_{\mathbf{L}}$ and J on $\mathbf{CP}^1$ induce a holomorphic structure on* **L**.

Proof. As mentioned in the last section J breaks the complexified cotangent bundle into $T^*(M)^{1,0}$ and $T^*(M)^{0,1}$ which are $+i$ and $-i$ eigenspaces respectively.Suppose we have a *local* non-vanishing section $s \in \Gamma(\mathbf{L})$. Now there is a θ such that

$$\nabla_{\mathbf{L}} s = \theta \otimes s = (\theta^{1,0} + \theta^{0,1}) \otimes s \tag{3.36}$$

using the splitting of the complexified cotangent space. Moreover the connection splits into $\nabla_{\mathbf{L}} = \nabla_{\mathbf{L}}' + \nabla_{\mathbf{L}}''$. What we really need to do is find a locally trivializing section which is "holomorphic." Our notion of a holomorphic section of **L** will mean one that is killed by $\nabla_{\mathbf{L}}''$.

$$\nabla_{\mathbf{L}}'' fs = (\bar{\partial} f + f\theta^{0,1}) \otimes s \tag{3.37}$$

So we should solve the equation,

$$f^{-1}\bar{\partial} f = -\theta^{0,1} \tag{3.38}$$

for f. In the case of a Riemann surface, in particular $\mathbf{CP}^1$, this equation has a solution as explained in [A-B2]. Suppose we had two such solutions s and $\hat{s}$ locally. It would then be the case that there was a smooth function g such that $gs = \hat{s}$. In this case

$$0 = \nabla_{\mathbf{L}}'' \hat{s} = \bar{\partial} g \otimes s + g \nabla_{\mathbf{L}}'' s = \bar{\partial} g \otimes s \tag{3.39}$$

so we conclude that $\bar{\partial} g = 0$, in other words g is holomorphic. Thus we have established a holomorphic structure on **L** . ■

It should be clear that, since S^1 preserves J, it preserves the splitting and since $\nabla_{\mathbf{L}}$ is an equivariant operator, so is $\nabla_{\mathbf{L}}''$. More generally we have an operator on forms

$$(\bar\partial \otimes \nabla_{\mathbf{L}}'')(\varphi \otimes s) = \bar\partial \otimes s + (-1)^{p+q}\varphi \otimes \nabla_{\mathbf{L}}'' s. \tag{3.40}$$

So $\bar\partial \otimes \nabla_{\mathbf{L}}'' = \bar\partial \otimes 1$ for holomorphic frames. Restricting our attention to an almost complex structure on $\mathbf{CP}^1$, $\bar\partial^2 = 0$, since all almost complex structures on $\mathbf{CP}^1$ are integrable. Hence

$$\Gamma(\bigwedge^{0,0} T^* M_1 \otimes \mathbf{L}) \overset{\bar\partial \otimes 1}{\to} \Gamma(\bigwedge^{0,1} T^* M_1 \otimes \mathbf{L}) \overset{\bar\partial \otimes 1}{\to} \Gamma(\bigwedge^{0,2} T^* M_1 \otimes \mathbf{L}) \overset{\bar\partial \otimes 1}{\to} \cdots \tag{3.41}$$

is a complex. In fact this is just the ordinary Dolbeault complex with coefficients in a holomorphic line bundle. Denote the cohomology of this complex by $H^j(M_1, \mathbf{L})$, which agrees with sheaf cohomology with coefficients in holomorphic local sections of $\mathbf{L}$ by the Dolbeault theorem. In the sequence 3.41, $\bigwedge^{0,q} T^* M_1 \otimes \mathbf{L} = 0$, for $q \geq 2$ since dim $\mathbf{CP}^1 = 1$, so $\bar\partial^* = 0$. If the complex is rolled up as explained in Section 1.1.3, $D_{\mathbf{L}} = (\bar\partial + \bar\partial^*) \otimes \nabla_{\mathbf{L}}'' = \bar\partial \otimes 1$. Thus ker $D_{\mathbf{L}} = H^0(M_1, \mathbf{L})$ and coker $D_{\mathbf{L}} = H^1(M_1, \mathbf{L})$ Suppose k is the Chern number of $\mathbf{L}$ on $\mathbf{CP}^1$. In other words k is the number obtained by evaluating the first Chern class of $\mathbf{L}$ on the orientation class of the manifold. The circle acts on $\mathbf{CP}^1$ fixing two points. Pick any other point x and the stabilizer of that point in the circle will be the cyclic group $\mathbf{Z}/l\mathbf{Z}$. Using the orientation we can pick out one of the fixed points as a "north pole," N. We choose N so that at N, S^1 acts on the tangent space as multiplication by $e^{l2\pi i\theta}$, with $l > 0$. The circle also acts on the fiber of $\mathbf{L}$ above N, $\mathbf{L}_N$, by the character $e^{n2\pi i\theta}$. With these numbers fixed the corresponding character at the south pole S, $e^{s2\pi i\theta}$, is determined by the equation $s = n - lk$, (See Figure 1.1)

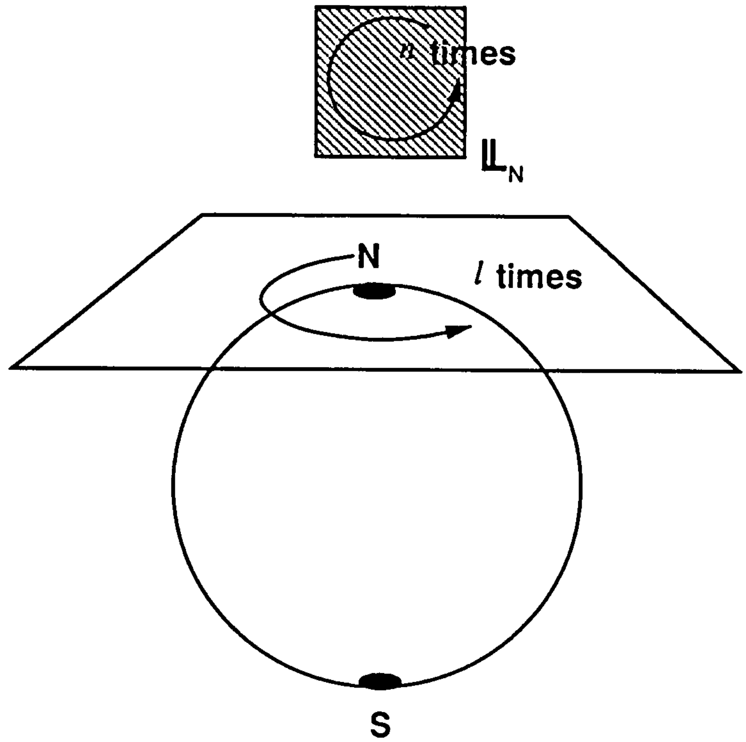


Figure 1.1: The circle action

Proposition 1.2 *If $k \geq -1$ then $H^i(M_1,\ \mathbf{L}) = 0$ for $i > 0$ and*

$$H^0(M_1,\ \mathbf{L}) = \mathbf{C}_{n-lk} \oplus \mathbf{C}_{n-l(k-1)} \oplus \cdots \oplus \mathbf{C}_n. \tag{3.42}$$

while if $k \leq -1$ then $H^i(M_1,\ \mathbf{L}) = 0$ for $i \neq 1$ and

$$H^1(M_1,\ \mathbf{L}) = \mathbf{C}_{n+l} \oplus \mathbf{C}_{n+2l} \oplus \cdots \oplus \mathbf{C}_{n+lk}. \tag{3.43}$$

where $\mathbf{C}_j$ denotes S^1 acting on $\mathbf{C}$ by $e^{2j\pi i\theta}$.

As pointed out above, $H^j(M_1,\ \mathbf{L}) = 0$ for $j \geq 2$ a priori. If $k < 0$ then $\mathbf{L}$ is a negative bundle and $H^0(M_1,\ \mathbf{L}) = 0$ by Kodaira vanishing. Using Serre duality $H^1(M_1,\ \mathbf{L}) = 0$ if $k + 2 > 0$, in other words if $k \geq -1$. Also as noted above

$$\mathrm{ind}_{S^1}(\ D_{\mathbf{L}}\ ) = \mathrm{ch}(\ker\ D_{\mathbf{L}}\ ) - \mathrm{ch}(\mathrm{coker}\ \ D_{\mathbf{L}}\ ) = \mathrm{ch}(H^0(M_1,\ \mathbf{L})) - \mathrm{ch}(H^1(M_1,\ \mathbf{L}). \tag{3.44}$$

So since either $\ker D_{\mathbf{L}} = 0$ or $\operatorname{coker} D_{\mathbf{L}} = 0$ we need only compute the index. The Atiyah-Bott Lefschetz formula (see Section 2.4.2)computes the index from the information at the fixed points.

$$\operatorname{ind}_{S^1}(\ D_{\mathbf{L}}\ ) = \frac{z^n}{1-z^{-l}} + \frac{z^s}{1-z^l} = \frac{z^n}{1-z^{-l}} + \frac{z^{n-lk}}{1-z^l} \tag{3.45}$$

$$= z^{n-lk}\frac{1-z^{l(k+1)}}{1-z^l} = (1+z^l+z^{2l}+\cdots+z^{kl})z^{n-lk} \tag{3.46}$$

The last equality depends on $k \geq 0$. When $k=-1$, $\operatorname{ind}_{S^1}(\ D_{\mathbf{L}}\ ) = 0$. The case where $k<0$ is handled similarly yielding:

$$\operatorname{ind}_{S^1}(\ D_{\mathbf{L}}\ ) = z^{n-lk} + z^{n-l(k-1)} + \cdots + z^n \tag{3.47}$$

if $k \geq 0$ and

$$\operatorname{ind}_{S^1}(\ D_{\mathbf{L}}\ ) = z^{n+l} + z^{n-2l} + \cdots + z^{n+lk} \tag{3.48}$$

if $k \leq -2$. Since the character determines the representation we are done.■

The next step that will be necessary is to understand how to "restrict" the operator we are studying $D_{\mathbf{L}}$ on M_n to the fibers of $\pi_n : M_n \rightarrow M_{n-1}$. More generally suppose we have a rank 2 complex bundle $\mathbf{E}$ over an almost complex manifold $\{M, J_1\}$. Let J_2 be an almost complex structure on $\mathbf{P}(\mathbf{E})$ such that

a) $J_1 T(\pi) = T(\pi) J_2$ where $\pi :\ \mathbf{P}(\mathbf{E}) \rightarrow M$ is the projection.

b) J_2 restricted to a fiber $\mathbf{P}(\mathbf{E}_m) \cong \mathbf{CP}^1$ coincides with the holomorphic structure of $\mathbf{CP}^1$ there.

Assume there is an action of S^1 on $\mathbf{P}(\mathbf{E})$ that is a bundle automorphism that is, the action is a "lift" of the trivial action on M.

c) the action preserves the almost complex structure.

d) the action is a bundle automorphism, that is, the action is a "lift" of the trivial action on M.

Suppose now there is an equivariant line bundle $\mathbf{L}$ over M and an equivariant connection $\nabla_{\mathbf{L}}$. Then we can form the operator $D_{\mathbf{L}} = (\bar{\partial} + \bar{\partial}^*) \otimes \nabla_{\mathbf{L}}''$ as defined in section 1.1.3.

$$D_{\mathbf{L}} : \Gamma(\Lambda^{0,\text{even}} T^* \mathbf{P}(\mathbf{E}) \otimes \mathbf{L}) \to \Gamma(\Lambda^{0,\text{odd}} T^* \mathbf{P}(\mathbf{E}) \otimes \mathbf{L}) \tag{3.49}$$

Let $i : \mathbf{P}(\mathbf{E}_m) \hookrightarrow \mathbf{P}(\mathbf{E})$ be the inclusion of the fiber over $m \in M$. Pulling back the line bundle $i^* \mathbf{L}$, restricting the connection $i^* \nabla_{\mathbf{L}}$, and restricting the almost complex structure J_2 (and the metric,) determines an operator

$$(D_{\mathbf{L}})_m : \Gamma(\Lambda^{0,\text{even}} T^* \mathbf{P}(\mathbf{E}_m) \otimes i^* \mathbf{L}) \to \Gamma(\Lambda^{0,\text{odd}} T^* \mathbf{P}(\mathbf{E}_m) \otimes i^* \mathbf{L}) \tag{3.50}$$

on the fiber over m. As we vary $m \in M$, since restriction and pullbacks are natural, one obtains a family of operators $\mathcal{F}_{\mathbf{L}} \stackrel{\text{def}}{=} \{(D_{\mathbf{L}})_m\}_{m \in M}$. Recall that earlier the index of an operator D was defined as

$$\text{ind}(D) = \dim(\ker D) - \dim(\text{coker} D) \tag{3.51}$$

To generalize this it is more useful to think of this as a virtual difference of vector space rather than a difference of integers, in the language of K-theory we think of $\text{ind}(D) \in K(\text{point})$. Hence if there is a family of operators $\mathcal{F}$ parameterized by a manifold X then for each $x \in X$, the two vector spaces $\ker D_x$ and $\text{coker} D_x$ can be associated with the point. One might hope that each could be used to form a vector bundle. In fact, independently, there is no reason why the dimensions of these vector spaces would be locally constant. The index is locally constant and so one forms the difference "bundle." This is a well defined element of $K(X)$ and will be denoted $\mathcal{I}\text{nd}(\mathcal{F})$. If it happens that either $\ker D_x$ or $\text{coker } D_x$ vanish then the local constancy of the index guarantees that the other forms a well defined vector bundle as x varies over X.

Returning to the problem at hand, if we let $\mathcal{F}_{\mathbf{L}}$ be the family of operators above in 3.50, then

Proposition 1.3 *The index of the family $\mathcal{F}_{\mathbf{L}}$ is a direct sum of line bundles over M (connected), $\sum_{i=1}^{a} \mathbf{L}_i$. If one considers the S^1 index, then $\mathcal{I}nd_{S^1}(\mathcal{F}) = \pm \sum_{i=1}^{a} \lambda_i \mathbf{L}_i$ where λ_i are certain characters of the circle.*

Proof. Restricting to $i^* \mathbf{L} \rightarrow \mathsf{P}(\mathbf{E}_m)$ define k_m, n_m, and l_m as in Proposition 1.2. Note that all of these *integers* are in fact continuous in m. Hence, since M is connected they are globally constant. Suppose $k \geq -1$. Again as in Proposition 1.2, $coker D_m =$ 0 for all $m \in M$. Thus $\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$ is an honest vector bundle.

Two representations of a group can be added via direct sum and multiplied via the tensor product. Denote the Grothendieck group arising from this by $R(G)$, this then is the representation ring of G. This can also be thought of as a subring of the Laurent polynomials in rank(G) variables, generated by characters of the group if G is abelian. For the case $G = S^1$, then $R(S^1) = \mathsf{C}[z, z^{-1}]$. Equivariant K-theory is the Grothendieck group of equivariant virtual bundles. In particular since S^1 acts trivially on M, $K_{S^1}(M) \cong R(S^1) \otimes K(M)$. Thus again using Proposition 1.2 the S^1 action splits the fibers naturally and hence the bundle $\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$ into lines with characters given by the decomposition (3.47). Hence $\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}}) = \sum_{i=1}^{a} \lambda_i \, \mathbf{L}_i$, where the λ_i are characters of the circle determined by 3.47.

To handle the case where $k \leq -1$ one considers $-\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$, which is an honest vector bundle, and uses 3.48 to obtain $\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}}) = -\sum_{i=1}^{a} \lambda_i \, \mathbf{L}_i$.

■

We can now attack the induction step. In the next theorem is an exact analogy of a Leray-Serre spectral sequence argument. It is necessary to use the machinery of the index since cohomology is not well defined for non-integrable almost complex structures. In the theorem, two facts will be needed which will be explained in the next two sections.

Theorem 1.3 *Suppose M, J^M is an almost complex manifold with a rank 2 complex vector bundle* E. *Suppose there is an almost complex structure J^P on $P \equiv \mathsf{P}(\mathbf{E})$ such that if $\pi : \mathsf{P}(\mathbf{E}) \rightarrow M$ is the induced projection, then*

a) $T(\pi)J^P = J^M T(\pi)$

b) The restriction of the almost complex structure to a fiber $J^P \mid_{\mathsf{CP}^1}$ is the ordinary almost complex structure on CP^1 .

Further assume that there is an action of the group $G \times S^1$ on the bundle P and M such that the action *c) preserves π, J^P, and J^M*

d) S^1 fixes M.

Then if we choose an equivariant line bundle $\mathbf{L}$, and equivariant connection on the line bundle and a hermitian structure on P then this determines an operator

$$D_{\mathbf{L}} : \Gamma(\Lambda^{0,even}T^*P \otimes \mathbf{L}) \rightarrow \Gamma(\Lambda^{0,odd}T^*P \otimes \mathbf{L}) \tag{3.52}$$

and there are line bundles $\mathbf{L}_i$, over M, and operators

$$D_{\mathbf{L}_i} : \Gamma(\Lambda^{0,even}T^*M \otimes \mathbf{L}_i) \rightarrow \Gamma(\Lambda^{0,odd}T^*M \otimes \mathbf{L}_i) \tag{3.53}$$

such that

$$ind_{G\times S^1}(D_{\mathbf{L}}) = \pm \sum_{i=1}^{a} \lambda_i ind_G(D_{\mathbf{L}_i}) \tag{3.54}$$

with λ^i weights of the circle S^1.

Proof. The proof depends on essentially two facts. One is Proposition 1.3. The second is the functoriality of the index. This second fact is essentially equivalent to the multiplicitivity axiom. Argument will be given by appealing to the topological index.

Suppose we let $\mathcal{F}_{\mathbf{L}} = \{(D_{\mathbf{L}})_x\}$. Let k be the Chern number when $\mathbf{L}$ is restricted to a single fiber of $P \rightarrow B$. Assume $k \geq -1$. Then $\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$ is an honest vector bundle over M. In fact it is a G-equivariant vector bundle over M, since all of the structures used to define $\mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$ are G-equivariant. Set $\mathbf{V} = \mathcal{I}\mathrm{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$. As a bundle $\mathbf{V}$ splits into $G \times S^1$-equivariant line bundles $\mathbf{V} = \oplus_{i=1}^{a} \mathbf{L}_i$ over M by Proposition 1.3. Here the S^1 acts on $\mathbf{L}_i$ by simple multiplication in the fiber, globally by the character λ_i. Choose G-equivariant connections $\nabla_{\mathbf{L}_i}$ on the line bundles $\mathbf{L}_i$ and form the operators

$$D_{\mathbf{L}_i} = (\bar{\partial} + \bar{\partial}^*)_M \otimes \nabla_{\mathbf{L}_i}, \tag{3.55}$$

where $(\bar{\partial} + \bar{\partial}^*)_M$ is the Dolbeault operator on M. From these connections on $\mathbf{L}_i$ form the direct sum connection on $\mathbf{V}$, $\nabla_{\mathbf{V}}$. Now the functoriality claim which we will need and which we will argue in the next section is that

$$\text{ind}_{G\times S^1}(\, D_{\mathbf{L}}\,) = \text{ind}_{G\times S^1}(\, D_{\mathbf{V}}\,) \tag{3.56}$$

$$= \sum_{i=1}^{a} \lambda_i \,\text{ind}_{G}(\, D_{\mathbf{L}_i}\,) \tag{3.57}$$

The second equality is just an encapsulazation of the simple nature of the S^1 action on $\mathbf{L}$ which was computed in Proposition 1.3. If we consider the case $k < -1$ then $-\,\mathcal{I}\text{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$ is, in this case an honest vector bundle. Letting $\mathbf{V} = -\,\mathcal{I}\text{nd}_{S^1}(\mathcal{F}_{\mathbf{L}})$ our formula becomes

$$\text{ind}_{G\times S^1}(\, D_{\mathbf{L}}\,) = -\sum_{i=1}^{a} \lambda_i \,\text{ind}_{G}(\, D_{\mathbf{L}_i}\,) \tag{3.58}$$

■

Before moving on to the functoriality of the index we can now prove Theorem 1.2.

Proof of Theorem 1.2. The proof is by induction. It is true for $n = 1$ by Proposition 1.2. Suppose it is true for any complete almost complex action of an $n-1$ torus on an $n-1$ step Bott tower. Suppose then we have a complete almost complex action on an n step Bott tower and apply Theorem 1.3. More precisely let M in Theorem 1.3 be M_{n-1} and $P = M_n$. As we will see, Proposition 1.4, in a later section, if we suppose that $T^{(n)}$ acts on M_n with isolated fixed points, then there is an $S^1 \subset T^{(n)}$ that fixes M_{n-1}. Let $G = T^{(n-1)}$. Suppose that the variables $\{z_1, \ldots, z_n\}$ represent the fundamental characters on T^n. In other words if $\{\theta_1, \ldots, \theta_n\}$ are coordinates on T^n, then $z_i = e^{2\pi i\theta_i}$. Applying the theorem we have

$$\text{ind}_{T}(\, D_{\mathbf{L}}\,) = \pm\sum_{i=1}^{a} \lambda_i \,\text{ind}_{T^{(n-1)}}(\, D_{\mathbf{L}_i}\,) \tag{3.59}$$

where $\lambda_i = z_n^q$ for some q. By the inductive assumption, $\text{ind}_{T^{(n-1)}}(\, D_{\mathbf{L}_i}\,)$ is a Laurent polynomial in the variables $\{z_1, \ldots, z_{n-1}\}$ with coefficients +1 or -1. Since the characters λ_i are all independent it follows that

$$\pm\sum_{i=1}^{a} \lambda_i \,\text{ind}_{T^{n-1}}(\, D_{\mathbf{L}_i}\,) \tag{3.60}$$

also has simple multiplicity. ■

As mentioned above this argument is, in some sense, a Leray-Serre spectral sequence argument for indexes of operators. It would be tempting to try to generalize Theorem 1.3, for example, by starting with a rank j complex vector bundle **E** and projectivizing. One problem is that Proposition 1.1 would not apply since the holomorphic structure mentioned here is only guaranteed for complex curves.

1.3.2 The Index and the symbol map

In the next two sections it is argued that the index can be thought of a topological pushforward and is thus functorial. There is a map from elliptic operators (complexes) $\mathcal{E}\ell\ell(M)$ on M to K-theory. Suppose $D : \Gamma(\mathbf{E}) \rightarrow \Gamma(F)$ is elliptic, then its symbol

$$\sigma(D) : \pi^*\mathbf{E} \rightarrow \pi^* F \tag{3.61}$$

defines a class in the K-theory of T^*M. In fact if the operator and the bundles are G equivariant then we have a map $\mathcal{E}\ell\ell_G(M)$ to $K_G(T^*M)$ which we denote as

$$[\sigma(D)] = [\pi^*\mathbf{E}, \pi^* F, \sigma(D)] \in K_G(T^*M). \tag{3.62}$$

One can formulate a notion of a G-index which is a map

$$\text{ind}\ _G^{top} : K_G(T^*M) \rightarrow K_G \cong R(G) \tag{3.63}$$

While there is a purely topological definition of this map the Atiyah-Singer index theorem states that this index coincides with the analytical notion of an index[A-S1]. In other words

$$\text{ind}\ _G^{top}[\sigma(D)] = ind_G^{any} D. \tag{3.64}$$

Moreover suppose $\mathcal{F} = \{D_x\}_{x \in M}$ is a family of operators on a bundle $\psi : P \rightarrow M$, with fiber F. So

$$D_x : \Gamma(\mathbf{E}) \rightarrow \Gamma(F) \tag{3.65}$$

where $\mathbf{E}$ and F are bundles over F_x. If we denote the vertical bundle of $\psi : P \rightarrow M$, by $T_{P/B}$, then

$$[\sigma(\mathcal{F})] \in K_G(T^*_{P/B}) \tag{3.66}$$

and again, in their paper on the index of families of elliptic operators [A-S4], Atiyah and Singer have shown that the topological notion of a family of elliptic operators coincides precisely with the analytical version:

$$\text{Ind}\ _G^{top}[\sigma(\mathcal{F})] = Ind_G^{any}(\mathcal{F}). \tag{3.67}$$

Let M be an almost complex manifold with almost complex structure J_1, and let $\mathbf{E}\rightarrow M$ be a rank two complex vector bundle over M. Suppose that there is an action of G on $\mathbf{P}(\mathbf{E})$ that comes from an action of G on $\mathbf{E}$ and is a lift of an action of G on M. Assume J_2 is an almost complex structure on $\mathbf{P}(\mathbf{E})$ such that if $\mathbf{E}_m$ is the fiber over $m \in M$, then $J_2|_{\mathbf{P}(\mathbf{E}_m)}$ is the standard almost complex structure on $\mathbf{CP}^1$. Now suppose there is an equivariant line bundle $\mathbf{L}\rightarrow\mathbf{P}(\mathbf{E})$ and an equivariant connection on the line bundle $\nabla_{\mathbf{L}}$. Then as in section 1.1 we can form the operator

$$D_{\mathbf{L}} : \Gamma(\Lambda^{0,\text{even}}T^*\mathbf{P}(\mathbf{E})\otimes\mathbf{L}) \rightarrow \Gamma(\Lambda^{0,\text{odd}}T^*\mathbf{P}(\mathbf{E})\otimes\mathbf{L}) \tag{3.68}$$

$$D_{\mathbf{L}} = (\bar{\partial}+\bar{\partial}^*)\otimes\nabla_{\mathbf{L}} \tag{3.69}$$

If we fix a point $m \in M$ and we let $\mathbf{L}|_m$ be the pull back of $\mathbf{L}$ along the inclusion of the fiber. We can then form the operator

$$(D_{\mathbf{L}})_m : \Gamma(\Lambda^{0,\text{even}}T^*\mathbf{P}(\mathbf{E_m})\otimes\mathbf{L}|_m) \rightarrow \Gamma(\Lambda^{0,\text{odd}}T^*\mathbf{P}(\mathbf{E_m})\otimes\mathbf{L}|_m) \tag{3.70}$$

$$(D_{\mathbf{L}})_m = (\bar{\partial}+\bar{\partial}^*)|_m\otimes\nabla_{\mathbf{L}|_m} \tag{3.71}$$

with $(\bar{\partial}+\bar{\partial}^*)|_m$ the Dolbeault operator with respect to the almost complex structure $J_2|_{\mathbf{CP}^1}$. Define the family of operators $\mathcal{F} = \{(D_{\mathbf{L}})_m\}_{m\in M}$.

Let k be the Chern number of $\mathbf{L}$ restricted to a fiber. Recall if $k \geq 0$ then $\mathbf{V} = \mathcal{I}nd_{S^1}\mathcal{F}$ is an honest vector bundle over M. If $k < 0$ then we can set $\mathbf{V} = -\mathcal{I}nd_{S^1}\mathcal{F}$, which is an honest vector bundle, and we can keep track of the sign. Hence we will assume $k \geq 0$. We choose a G-equivariant connection $\nabla_{\mathbf{V}}$ on the G-equivariant vector bundle $\mathbf{V}$. Suppose we consider the almost complex structure J_1 on M. We can then form the operator

$$C_{\mathbf{V}} : \Gamma(\Lambda^{0,\text{even}}T^*M\otimes\mathbf{V})\rightarrow\Gamma(\Lambda^{0,\text{odd}}T^*M\otimes\mathbf{V}) \tag{3.72}$$

$$C_{\mathbf{V}} = (\bar{\partial}+\bar{\partial}^*)\otimes\nabla_{\mathbf{V}} \tag{3.73}$$

where here $\bar{\partial}+\bar{\partial}^*$ is with respect to J_1 on M. The functoriality statement want to justify is that

$$ind_{G\times S^1}(D_{\mathbf{L}}) = ind_{G\times S^1}(C_{\mathbf{V}}) \tag{3.74}$$

Here we mean analytical index but as mentioned in (3.64) the G-index theorem [A-S 1] tells us that we need only prove this for the topological index:

$$ind^{\text{top}}_{G\times S^1}([\sigma(D_{\mathbf{L}})]) = ind^{\text{top}}_{G\times S^1}([\sigma(C_{\mathbf{V}})]). \tag{3.75}$$

Our first step will be to compute the symbols of $D_{\mathbf{L}}$ and $C_{\mathbf{V}}$. If N is some manifold with vector bundles $\mathbf{E}$ and $\mathbf{F}$, then suppose that $A : \Gamma(\mathbf{E}) \rightarrow \Gamma(\mathbf{F})$ is a first order differential operator. If $\pi : T^*N \rightarrow N$ is the projection map then the symbol is a map $\sigma(A) : \pi^*\mathbf{E} \rightarrow \pi^*F$. This can be computed as follows. Let $e \in (\pi^*\mathbf{E})_\eta$ with $\eta \in T^*N$. Take a local section s of $\mathbf{E}$ such that $s(\pi(\eta)) = e$, and take f a function on N such that f vanishes at $\pi(\eta)$ and $df_{\pi(\eta)} = \eta$. Then we have

$$\sigma(A)(\eta, e) = (A(f \cdot s))(\pi(\eta)). \tag{3.76}$$

So the symbol for $A = \bar{\partial} \otimes \nabla_{\mathbf{L}}$ is

$$\sigma(\bar{\partial} \otimes \nabla_{\mathbf{L}}) : \pi^*\Lambda^{0,q}T^*\mathbf{P}(\mathbf{E}) \otimes \mathbf{L} \rightarrow \pi^*\Lambda^{0,q+1}T^*\mathbf{P}(\mathbf{E}) \otimes \mathbf{L} \tag{3.77}$$

with $\sigma(\bar{\partial} \otimes \nabla_{\mathbf{L}})(\eta, \varphi \otimes s) = (\eta^{0,1} \wedge \varphi) \otimes s$, where $\eta = \eta^{1,0} + \eta^{0,1}$, (recall sign convention in 1.16.) The symbol for $D_{\mathbf{L}}$ is

$$\sigma(D_{\mathbf{L}}) : \pi^*\Lambda^{0,\text{even}}T^*\mathbf{P}(\mathbf{E}) \otimes \mathbf{L} \rightarrow \pi^*\Lambda^{0,\text{odd}}T^*\mathbf{P}(\mathbf{E}) \otimes \mathbf{L} \tag{3.78}$$

with $\sigma(D_{\mathbf{L}})(\eta, \varphi \otimes s) = (\eta^{0,1} \wedge \varphi + (\eta^{0,1})^* \neg \varphi) \otimes s$, with $\neg$ being the interior product and the dual, here, is taken with respect to a hermitian metric. Let the operator $D \overset{\text{def}}{=} \bar{\partial} + \bar{\partial}^*$. Thus, when thought of as a classes in K-theory we have

$$\begin{aligned} [\sigma(D_{\mathbf{L}})] &= [\pi^*\Lambda^{0,\text{even}}T^*\mathbf{P}(\mathbf{E}) \otimes \mathbf{L}, \pi^*\Lambda^{0,\text{odd}}T^*\mathbf{P}(\mathbf{E}) \otimes \mathbf{L}, \sigma(D_{\mathbf{L}})] && (3.79) \\ &= [\pi^*\Lambda^{0,\text{even}}T^*\mathbf{P}(\mathbf{E}), \pi^*\Lambda^{0,\text{odd}}T^*\mathbf{P}(\mathbf{E}), \sigma(D)][\pi^*\mathbf{L}] && (3.80) \end{aligned}$$

the multiplication being in K-theory. Note that $[\pi^*\mathbf{L}]$ is not a compactly supported class but when multiplied by $[\sigma(D)]$ the product does have compact support. In a similar manner we can establish that $[\sigma(C_{\mathbf{V}})] = [\sigma(C)][\pi^*\mathbf{V}]$.

1.3.3 Multiplicitivity and Compatibility

Suppose that there is an action of G on $\mathbf{P}(\mathbf{E})$ that comes from an action of G on $\mathbf{E}$ and is a lift of an action of G on M. Choose an equivariant hermitian metric on $\mathbf{E}$. Suppose we denote the bundle of complex orthonormal frames $F(\mathbf{E})$. Thus $F(\mathbf{E})$ is a U(2) principal bundle over M. There is an induced action of G on $F(\mathbf{E})$ as follows: if $f \in F(\mathbf{E})$ is a map $f : \mathbf{C}^2 \rightarrow \mathbf{E}_m$ for $m \in M$, $g \in G$, and $v \in \mathbf{E}_m$, then $(g \cdot f)(v) := g \cdot f(v)$. Essentially by construction

$$F(\mathbf{E}) \times_{U(2)} \mathbf{C}^2 = \mathbf{E}, \tag{3.81}$$

with the identification $(f, v) \mapsto f(v)$. Thus the left action defined by $g \cdot (f, v) = (g \cdot f, v)$ is the same as the left action on $\mathbf{E}$ via the identification. The standard U(2) action on $\mathbf{C}^2$ induces an action on $\mathbf{CP}^1$ and so

$$\mathbf{P}(\mathbf{E}) = \mathbf{P}(F(\mathbf{E}) \times_{U(2)} \mathbf{C}^2) \cong F(\mathbf{E}) \times_{U(2)} \mathbf{CP}^1 \tag{3.82}$$

Again suppose we restrict $\mathbf{L}$ to a fiber $\mathbf{CP}^1$. We can form an operator

$$< D_{\mathbf{L}} >: \Gamma(\Lambda^{0,\mathrm{even}} T^* \mathbf{CP}^1 \otimes \mathbf{L}) \rightarrow \Gamma(\Lambda^{0,\mathrm{odd}} T^* \mathbf{CP}^1 \otimes \mathbf{L}) \tag{3.83}$$

$$< D_{\mathbf{L}} > = (\bar{\partial} + \bar{\partial}^*) \otimes \nabla_{\mathbf{L}} \tag{3.84}$$

This operator is equivariant with respect to the $U(2)$ action so let $V = ind_{U(2)\times S^1}(< D_{\mathbf{L}} >)$. Suppose W is a $G \times H$ module, and suppose P is a right H principal bundle then we can form the vector bundle $P \times_H W$ over M. This extends to a morphism $\mu_P : R(G \times H) \rightarrow K_G(M)$. Thus we can form a vector bundle $F(\mathbf{E}) \times_{U(2)} V$ which is in fact equal to $\mathbf{V}$. Following the notation of [A-S 1] we set $a = [\sigma(C)]$ while we think of $b = [\sigma(\mathcal{F})]$ "=" $[\sigma(< D_{\mathbf{L}} >)]$. The multiplicitivity axiom, (B3) of [A-S 1] says

$$ind_G^{\mathrm{top}}(ab) = ind_G^{\mathrm{top}}(a \cdot \mu_{F(\mathbf{E})}(ind_{G\times H}(b))). \tag{3.85}$$

The product, ab, needs some clarification. If we denote the projection $p : \mathbf{P}(\mathbf{E}) \rightarrow M$, then, using a metric, $T^*\mathbf{P}(\mathbf{E}) = T^*_{\mathbf{P}(\mathbf{E})/M} \oplus p^* T^* M$ where $T^*_{\mathbf{P}(\mathbf{E})/M}$ is the cotangent bundle "along the fibers." If $a \in K_G(T^*M)$ while $b \in K_G(T^*_{\mathbf{P}(\mathbf{E})/M})$ then the product comes from the map

$$K_G(T^*_{\mathbf{P}(\mathbf{E})/M}) \otimes K_G(p^*T^*M) \rightarrow K_G(T^*\mathbf{P}(\mathbf{E})) \tag{3.86}$$

So rewriting (3.85) we obtain

$$ind_G^{\text{top}}([\sigma(C)][\sigma(\mathcal{F})]) = ind_G^{\text{top}}([\sigma(C)][\pi^*V]). \tag{3.87}$$

If we can show that $[\sigma(C)][\sigma(\mathcal{F})] = [\sigma(D_{\mathbf{L}})]$, we will have demonstrated (3.74.) The key condition here is that if $p : \mathbf{P}(\mathbf{E}) \rightarrow M$, then

$$J_1 T(p) = T(p) J_2. \tag{3.88}$$

We assumed that the restriction of J_2 to a fiber defined an almost complex structure on the fiber, although (3.88) insures that. Moreover if we use a hermitian metric on $T^*(\mathbf{P}(\mathbf{E}))$, it can be split in to complex subbundles

$$T^*\mathbf{P}(\mathbf{E}) = T^*_{\mathbf{P}(\mathbf{E})/M} \oplus p^*T^*M. \tag{3.89}$$

We can lift the complex structure J_1 up from the M to p^*T^*M via the map p. Because we assume (3.88), we know that the lifted almost complex structure will agree with the restriction of J_2 to the complex subbundle p^*T^*M. The hermitian metric gives us the following splitting

$$(T^*\mathbf{P}(\mathbf{E}))^{0,1} = \left(T^*_{\mathbf{P}(\mathbf{E})/M}\right)^{0,1} \oplus (p^*T^*M)^{0,1}. \tag{3.90}$$

The compatibility condition further guarantees that $(p^*T^*M)^{0,1} = p^*(T^*M)^{0,1}$.

Suppose W and U are complex vector spaces then taking wedge products over the complex numbers we have that

$$\Lambda(W \oplus U) = \Lambda(W) \otimes \Lambda(U) \tag{3.91}$$

So applying this to the present situation we find that

$$\Lambda^{0,\overset{\text{even}}{\text{odd}}} T^*\mathbf{P}(\mathbf{E}) = \Lambda^{0,\overset{\text{even}}{\text{odd}}} T^*_{\mathbf{P}(\mathbf{E})/M} \otimes \Lambda^{0,\overset{\text{even}}{\text{odd}}} T^*M \tag{3.92}$$

The above identification being compatible with the symbol maps we conclude $[\sigma(C)][\sigma(\mathcal{F})] = [\sigma(D_{\mathbf{L}})]$.

1.4 Towers and actions

Upon studying the definition of the Bott-Tower one may wonder, why we specialize to starting with a complex line bundle to build our complex plane bundle, and under what conditions do Bott-Towers have complete torus actions and equivariant almost complex structures. In this section these questions are addressed. First we define a pre-Bott tower and then we show that a pre-Bott tower supports a complete Torus action if and only if it is a Bott tower. For this section we shall assume that M is simply connected as it will be in pre-Bott and Bott towers.

Definition 1.1 *An n-step* pre-Bott tower *will be a sequence of fiber bundles* $M^n \xrightarrow{\pi_n} M^{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} M^1 \cong \mathbf{CP}^1$ *such that at each stage,* $M^j = \mathsf{P}(\mathbf{E})$*, where* $\mathbf{E}$ *is a complex plane bundle over* M^{j-1}*. A pre-Bott tower is called* sectioned *if each of the bundles* $M^j \xrightarrow{\pi_j} M^{j-1}$ *has a section.*

The main result is the following:

Theorem 1.4 *If $\{M_i, \pi_i\}$ is a pre-Bott tower then the following are equivalent:*

1 $\{M_i, \pi_i\}$ *supports an effective action of an n-torus* $T^{(n)}$

2 $\{M_i, \pi_i\}$ *is sectioned.*

3 $\{M_i, \pi_i\}$ *is a Bott tower*

The proof will be given after a number or propositions are proven.

Proposition 1.4 *Let M be a $2(n-1)-$dimensional compact connected manifold with a rank 2 complex plane bundle over it* $\mathrm{E} \xrightarrow{\pi} M$*. Suppose there is an effective action of an $n-$torus of on the bundle* $\mathsf{P}(\mathrm{E})$ *with isolated fixed points, then there is an* $S^1 \subset T^{(n)}$ *which fixes M.*

Proof. Let $p \in$ be a fixed point. Then $\pi(p) = x \in M$ is a fixed point of M. It follows that this gives us a representation of $T^{(n)}$ on T_xM. Since M is $2(n-1)$ real dimensional $T^{(n)}$ cannot be acting effectively on T_xM. In other words there must be an $S^1 \subset T^{(n)}$ such that S^1 fixes T_xM. We introduce a riemannian metric g on

M. Since $T^{(n)}$ is compact we can average the metric making it $T^{(n)}$ invariant. Let $y \in M$ be any point. We can find a path γ going from x to y which is a geodesic curve. The action of S^1 preserves geodesics since the metric is invariant. Now extend γ maximally (M is compact and hence geodesically complete) to a geodesic $\tilde{\gamma}$ which must either be isometric to $\mathbf{R}$ or S^1. In either case since x and T_xM is fixed by S^1, y is also fixed. This is because there are no orientation preserving isometries of either $\mathbf{R}$ or S^1 that fix a point other than the identity isometry. Hence $S^1 \subset T^{(n)}$ fixes M.■

Next we want to show that the action of $T^{(n)}$ implies the existence of a section:

Proposition 1.5 *Let* $\mathbf{E} \rightarrow M$ *be a rank 2 complex plane bundle over an oriented manifold* M. *Suppose that* $\mathsf{P}(\mathbf{E})$ *supports a non-trivial action of* S^1 *which fixes the base space* M. *Then* $\mathbf{E} \rightarrow M$ *has an equivariant section.*

Proof. Since M is simply connected $\mathsf{P}(\mathbf{E})$ is oriented. Now let $\pi(x)^{-1} = \mathsf{P}(\mathbf{E})_x$ with $x \in M$. Then S^1 acts on $\mathsf{P}(\mathbf{E})_x$ since M is fixed. S^1 cannot fix $\mathsf{P}(\mathbf{E})_x$ since then if $p \in \mathsf{P}(\mathbf{E})_x$ S^1 would fix $T_p\mathsf{P}(\mathbf{E})_x$ and hence by the above argument act trivially. Again we can fix a metric on $\mathsf{P}(\mathbf{E})$ and assume that S^1 acts by isometries. Now we know this action must have a isolated fixed point on $T_p\mathsf{P}(\mathbf{E})_x$. If the fixed point were not isolated S^1 would act trivially. Now we can surround this fixed point with a disk of radius ϵ. The compliment is also a closed equivariant disk. Hence we know that there is exactly one fixed point in this disk. We can compare the actions of S^1 on the tangents spaces to the fixed point. If one rotates positively with respect to the orientation the other must rotate negatively. Call the positive one the north pole at x and denote it n_x. Since $\mathsf{P}(\mathbf{E})$ is an oriented manifold n_x is well defined for all x and continuous in x. The map:

$$\begin{array}{ccc} \mathsf{P}(\mathbf{E}) & \rightarrow & M \\ x & \mapsto n_x & \end{array}$$

is an equivariant section. ■

These two propositions will essentially take care of $1 \Rightarrow 2$ of Theorem 1.4. The next one is addressed at showing $2 \Rightarrow 3$.

Proposition 1.6 *Let* M *be a manifold and* $\pi : \mathbf{E} \rightarrow M$ *be a rank 2 complex plane bundle. If* $\mathsf{P}(\mathbf{E}) \rightarrow M$ *has a section* $s : M \rightarrow \mathsf{P}(\mathbf{E})$, *then there is a natural identification (up to a choice of metric)* $\mathsf{P}(\mathbf{E}) \cong \mathsf{P}(\mathbf{1} \oplus \mathbf{F})$, *where* $\mathbf{1}$ *is the trivial complex line bundle and* $\mathbf{F}$ *is some other complex line bundle.*

Proof. Natural, here, means up to a choice of metric on $\mathbf{E}$. Let $\mathbf{F} \rightarrow \mathsf{P}(\mathbf{E})$ be the line bundle whose fiber at $V \in \mathsf{P}(\mathbf{E})$ is the subspace $V \subset \mathbf{E}_{\pi(V)}$. Suppose we choose a hermitian metric h on $\mathbf{E}$. Then let $\tau_V : \mathbf{E}_V \rightarrow V$ be orthogonal projection onto V. Since we have a section we can pullback $\mathbf{F}$ along s to M,

$$\begin{array}{ccc} s^*\mathbf{F} & \rightarrow & \mathbf{F} \\ \downarrow & & \downarrow \\ M & \xrightarrow{s} & \mathsf{P}(\mathbf{E}). \end{array} \tag{4.93}$$

We have $\psi : \mathbf{E} \rightarrow s^*\mathbf{F} \oplus \mathbf{1}$, defined by $\psi(v) = (\tau(v), \| v - \tau(v) \|)$. This map induces and identification $\mathsf{P}(\mathbf{E}) \cong \mathsf{P}(s^*\mathbf{F} \oplus \mathbf{1})$. ■.

If we use induction we can put these propositions together as follows.

Proof of Theorem 1.4. The statement in trivial for $n = 1$ since all vector bundles over a pont are trivial. Assume $1 \Rightarrow 2$ for all pre-Bott towers of $n-1$ steps with an action of an $n-1$ torus $T^{(n-1)}$. If we have a n step pre-Bott tower $\{M_n, \pi_n\}$ with an action of an n torus $T^{(n)}$. By Proposition 1.4 we can write $T^{(n)} \cong T^{(n-1)} \times S^1$, with $T^{(n)}$ an $n-1$ torus acting on M_{n-1} effectively and S^1 fixing M_{n-1}. Thus we have a section $s : M_{n-1} \rightarrow M_n$ by Proposition 1.5.

To show $2 \Rightarrow 3$ again assume it for all it is true for all pre-Bott towers of $n-1$ steps then the induction step follows from Proposition 1.6. The last step $3 \Rightarrow 1$ is proved once again by induction on n. Supposing that we have the $n-1$ torus action on M_{n-1} then we can lift this action to the line bundle $\mathbf{F} \rightarrow M_{n-1}$, either as in [K] or [H]. Let S^1 act on the trivial bundle $\mathbf{1}$ by multiplication by $e^{2k\pi i\theta}$ where $k \in \mathbf{Z}$ and $k \neq 0$. This induces an effective action of $T^{(n)} = T^{(n-1)} \times S^1$, on $M_n = \mathsf{P}(\mathbf{F} \oplus \mathbf{1})$. ■

We round this section out with two short propositions.

Proposition 1.7 *Let $\{M_n, \pi_n\}$ be a pre-Bott tower, then there is an almost complex structure $\{J_n\}$ such that*

1 *The almost complex structure restricted to a fiber $J_n \mid_{\mathbf{CP}^1}$ is the ordinary complex structure on $\mathbf{CP}^1$.*

2 *The almost complex structure commutes with the projection operators, $J_{n-1}T(\pi_n) = T(\pi_n)J_n$.*

Proof. As is becoming ritual, the proof is by induction. Suppose we have an almost complex structure J_{n-1} on M_{n-1}. Choose a riemannian metric on $M_n = \mathbf{P}(\mathbf{E})$. At a point $V \in \mathbf{P}(\mathbf{E})$, the tangent space splits, $T_V(\mathbf{P}(\mathbf{E})) \cong \mathcal{V} \oplus \mathcal{V}^{\perp}$, into $\mathcal{V}$, the tangents to the fiber and its orthogonal compliment $\mathcal{V}^{\perp}$. If we restrict $T(\pi_n)$ to $\mathcal{V}^{\perp}$ at V then this induces an isomorphism $\mathcal{V}^{\perp} \cong T_{\pi_n(V)}M_{n-1}$. Using this isomorphism we can put an almost complex structure J^a on $\mathcal{V}^{\perp}$ via J_{n-1}. Since the fibers of π_n are naturally isomorphic to $\mathbf{CP}^1$ there is a natural almost complex structure J^b on $\mathcal{V}$. Thus let $J_n = J^a \oplus J^b$. By construction the two properties are satisfied. ∎

Proposition 1.8 *Let $\{M_n, \pi_n\}$ be a Bott tower with a n-torus T, acting, then there is an almost complex structure $\{J_n\}$ such that*

1. *The almost complex structure restricted to a fiber $J_n \mid_{\mathbf{CP}^1}$ is the ordinary complex structure on $\mathbf{CP}^1$.*

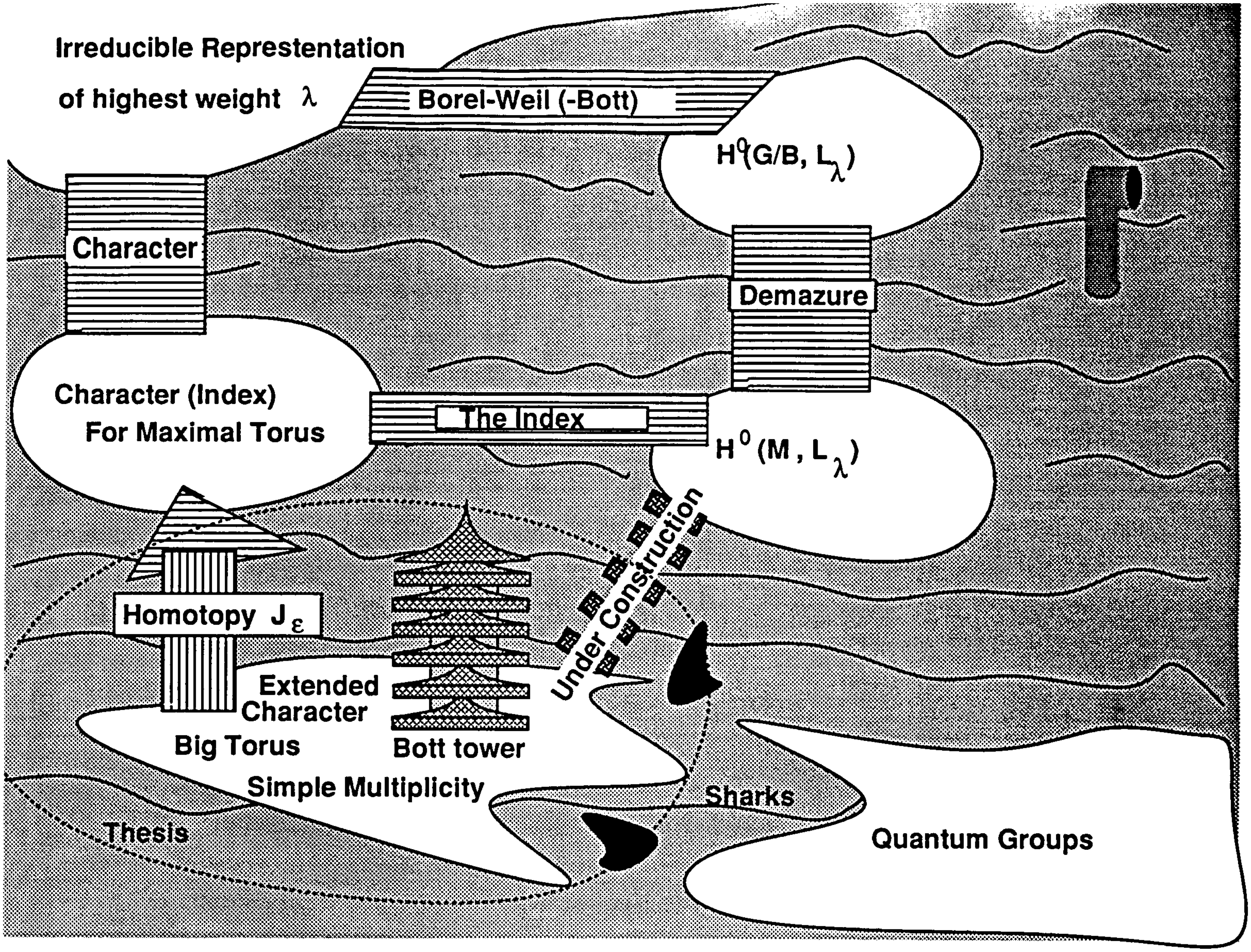
2. *The almost complex structure commutes with the projection operators, $J_{n-1}T(\pi_n) = T(\pi_n)J_n$.*

3. *The almost complex structure is equivariant with respect to T.*

Proof. The proof is exactly as above except that since T is compact we choose an equivariant metric. ∎

Chapter 2

Applications to the Bott-Samelson manifold



Representation Picture

2.1 Bott-Samelson Manifolds

Let K be a compact simple Lie group and fix a maximal torus $T \subset K$. Suppose Δ is the set of roots and Δ_+ is set of positive roots of K. The Bott-Samelson manifolds can be built generally out of an ordered subset of Δ_+. First two different approaches to these orderings are discussed. In one approach a line is chosen from the positive to the negative chamber. In the other approach simple roots are used. This second approach is more convenient when the holomorphic version is examined. The next three subsections review the homogeneous space picture which is used as a model. This is applied to the Bott Samelson manifolds to obtain an almost complex structure preserved by the large torus action, and an equivariant principal connection. This leads to connections on equivariant line bundles. The Bott Samelson manifold is shown to be a Bott tower.

2.1.1 Reduced Expressions and Orderings

In [B-S] Bott-Samelson manifolds are constructed using a line (segment) in the lie algebra of the maximal torus. In other accounts, such as [D] or [J], for example, the spaces are constructed from reduced expressions of the longest element. The function of this section is to give some indication how these two approaches are related. For various reasons it is more useful to use reduced expressions of the longest element of the Weyl group and this approach will be used exclusively. This section can, thus be skipped.

Let $T \subset K$ be a maximal torus, with Lie algebra $\mathfrak{t}$.Let Δ be the set of roots and Δ_+ be the set of positive roots of K. Let $C^+ = \{ \underline{t} \in \mathfrak{t} | \ \alpha(\underline{t}) > 0, \text{for all } \alpha \in \Delta_+\}$ be the positive Weyl chamber. Let $C^- = \{ \underline{t} \in \mathfrak{t} | \ \alpha(\underline{t}) < 0, \text{for all } \alpha \in \Delta_+\}$ be the negative chamber. For example, the root system associated if $K =$ SU(3), is A_2 as follows:

We identify the Lie algebra with its dual via the killing form, so althought the roots live in the dual of the maximal toral subalgebra, $\mathfrak{t}^*$, and the Weyl chambers and walls live in $\mathfrak{t}$, we draw them in the same picture. The positive chamber is shaded. Suppose $\mathbf{L} \subset \mathfrak{t}$ is a line (segment) which begins in the positive chamber and ends in the negative chamber. Suppose $\mathbf{L}$ is in general position. Since $\mathbf{L}$ starts in the positive

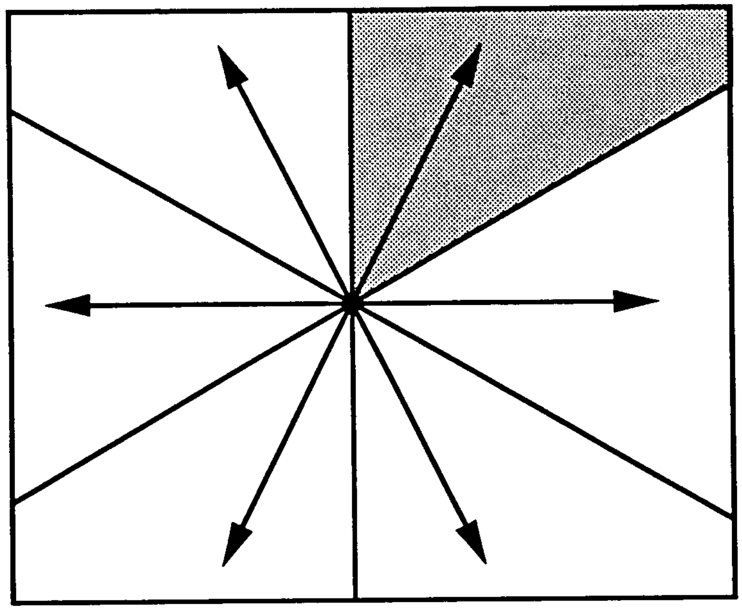


Figure 2.1: The root system of SU(3)

chamber we can choose a point $\underline{p}_0$ on **L**, such that for all $\alpha \in \Delta_+$, $\alpha(\underline{p}_0) > 0$. Suppose n is the number of positive roots. Since **L** ends in the negative chamber there we can find a point $\underline{p}_n$, on **L**, such that for all $\alpha \in \Delta_+$, $\alpha(\underline{p}_n) < 0$. Since **L** is in general position a point $\underline{p}_1$ on **L** may be chosen such that for exactly one root, $\alpha_1 \in \Delta_+$ we have that $\alpha_1(\underline{p}_1) < 0$. Continuing in this manner we find a sequence of points on **L**, $\{\underline{p}_0, \dots \underline{p}_n\}$ an enumeration of the positive roots $\{\alpha_1, \dots, \alpha_n\}$ such that

$$\begin{aligned} \alpha_j(\underline{p}_i) < 0 \text{ if } i \geq j \text{ and} \\ \alpha_j(\underline{p}_i) > 0 \text{ if } i < j. \end{aligned} \tag{1.1}$$

In other words if the hyperplanes $W_\alpha = \{\underline{t} \in \mathfrak{t} | \ \alpha(\underline{t}) = 0\}$ are the walls of the Weyl chambers, then **L** passes each one. As **L** passes though W_α, points on **L** when evaluated on $\alpha \in \mathfrak{t}^*$, go from being positive numbers to being negative numbers. This provides an ordering on the set of positive roots Δ_+, $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ where n is the number of positive roots.

First we should note that α_1 is a simple root. We can see this by noting that $\alpha_j(p_1) > 0$ if $j > 1$ and $\alpha_1(p_1) < 0$ as above. If α_1 is not simple then it must be

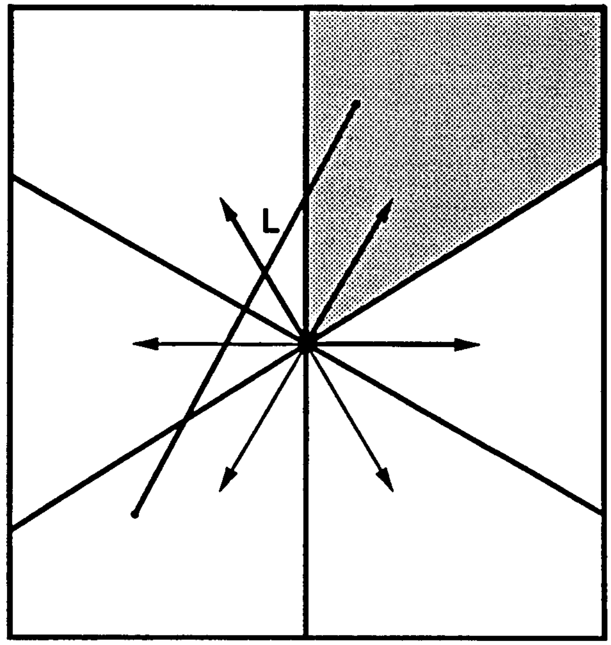


Figure 2.2: The root system of SU(3) with line

the sum of some subset of positive roots in Δ_+, $\alpha_1 = \alpha_{j_1} + \cdots + \alpha_{j_k}$. But then $0 > \alpha_1(p_1) = \alpha_{j_1}(p_1) + \cdots + \alpha_{j_k}(p_1) > 0$ and this contradiction shows α_1 is simple. Now $\{-\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of positive roots associated to the Weyl chamber p_1 is contained in. Continuing in this manner the ordering $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ has the following property:

Property A *For all* j, $\{-\alpha_1, \ldots, -\alpha_{j-1}, \alpha_j, \ldots, \alpha_n\}$ *is a positive root system and that in that system of positive roots* α_j *is a simple root.*

In his letter [B1], Raoul Bott shows that for an ordering of positive roots like the one above we can obtain a reduced expression for the longest element of the Weyl group. Let κ be the killing form on $\mathfrak{k}$, the Lie algebra of K. This is a negative definite form and hence induces a metric $-\kappa$ on $\mathfrak{k}$ and hence on the subspace $\mathfrak{t}$. We have also a metric on $\mathfrak{t}^*$.Denote this metric by $(,)$. For $\alpha \in \Delta$ we define a generator of the Weyl group, $s_\alpha(\gamma) \stackrel{\text{def}}{=} \gamma - \frac{2(\alpha,\gamma)}{(\alpha,\alpha)}$ where γ is in $\mathfrak{t}^*$. If we define the root system $R_j = \{-\alpha_1, \ldots, -\alpha_{j-1}, \alpha_j, \ldots, \alpha_n\}$, then $s_{\alpha_j} R_j = R_{j+1}$. Then $s_{\alpha_n} \ldots s_{\alpha_1} \Delta_+ = \Delta_-$ in other words $s_{\alpha_n} \ldots s_{\alpha_1} = w_0$, where w_0 is the longest element of the Weyl group. We already have that s_{α_1} is a simple reflection (since α is a simple root). Now

$s_{\alpha_1}R_1 = R_2$ and $s_{\alpha_1}^{-1}R_2 = R_1$. Also $s_{\alpha_2}R_2 = R_2 \setminus \{\alpha_2\} \cup \{-\alpha_2\}$. We know that $s_{\alpha_1}^{-1}R_2 \setminus \{\alpha_2\} \subset R_1$ and that $s_{\alpha_1}^{-1}(-\alpha_2) \in -R_1$. Hence we conclude that the Weyl group element $w_2 \stackrel{\text{def}}{=} s_{\alpha_1}^{-1}s_{\alpha_2}s_{\alpha_1}$ operates on R_1 changing a single positive root to a negative root and hence is a simple reflection corresponding to a simple root β_2. In just the same way we define the simple reflections $w_j = (s_{\alpha_{j-1}} \cdots s_{\alpha_1})^{-1}s_{\alpha_j}(s_{\alpha_{j-1}} \cdots s_{\alpha_1})$ This gives us finally that

$$w_1w_2 \cdots w_n = s_{\alpha_n} \cdots s_{\alpha_1} = w_0, \tag{1.2}$$

where w_1 is defined to be s_{α_1}. This takes us from orderings of positive roots with Property A to reduced expressions for the longest element of the Weyl group.

Suppose we have are reduced expression for the longest element of the Weyl group $w_1w_2 \cdots w_n = w_0$. Let α_1 be the simple root associated to w_1 so that $s_{\alpha_1} = w_1$. In general for each $1 \leq j \leq n$, the element

$$(w_1 \cdots w_{j-1})w_j(w_1 \cdots w_{j-1})^{-1} \tag{1.3}$$

is of order two and so corresponds to a reflection of a positive root we call α_j. Let $R_1 = \Delta_+$ and $s_{\alpha_j}R_j = R_{j+1}$. Defined in this way it is clear that R_j is a system of positive roots. We must show that α_j is simple in R_j and that

$$R_j = \{-\alpha_1, \ldots, -\alpha_{j-1}, \alpha_j, \ldots, \alpha_n\}. \tag{1.4}$$

Suppose that this is true for all $j < k$. Assume further that $R_k = \{-\alpha_1, \ldots, -\alpha_{k-1}, \alpha_k, \ldots, \alpha_n\}$. It will be shown that α_k is simple in R_k and that $R_{k+1} = \{-\alpha_1, \ldots -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$. We have that

$$(w_1 \cdots w_{k-1})^{-1}R_k = (w_{\alpha_{k-1}} \cdots w_{\alpha_1})^{-1}R_k = R_1. \tag{1.5}$$

Since w_k corresponds to a simple reflection $w_kR_1 = (R_1 \setminus \{-\gamma\}) \cup \{-\gamma\}$. Thus applying s_{α_k} to R_k, only changes the sign of the root $(w_1 \cdots w_{j-1})\gamma$ and so

$$\alpha_k = (w_1 \cdots w_{j-1})\gamma. \tag{1.6}$$

Hence α_k is simple in R_k and $R_{k+1} = s_{\alpha_k}R_k = \{-\alpha_1, \ldots - \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$.

This demonstrates that there is a bijection between reduced expressions of w_0 and orderings of the positive roots with Property A. One might further wonder whether given an ordering of the positive roots with Property A, one can find a line $\mathbf{L} \subset \mathfrak{t}$ as above, which would realize this ordering. Apparently there are counterexamples for this even for $K = SU(n)$ for sufficiently high n [H.]

2.1.2 Defining the Bott-Samelson Manifolds

Now suppose that $\alpha \in \Delta$ is some root. We can consider the subspace $\mathfrak{t}_\alpha \subset \mathfrak{t}$ with $\mathfrak{t}_\alpha \stackrel{\text{def}}{=} \{ \underline{t} \in \mathfrak{t} | \alpha(\underline{t}) = 0\}$. This is a Lie subalgebra of $\mathfrak{t}$ and hence by the Frobenious integrability theorem we find a group $T_\alpha \subset T$ whose Lie algebra at the identity is $\mathfrak{t}_\alpha$. In other words T_α is the exponentiated version of $\mathfrak{t}_\alpha$. We define $K_\alpha \stackrel{\text{def}}{=} Z(T_\alpha)$ to be the centralizer of T_α in K. Note that since T is a torus containing T_α, $T \subset K_\alpha$.

Proposition 2.1 $K_\alpha/T \cong S^2$

Proof. The killing form gives a natural metric on K. Let $\mathfrak{k}_{(\alpha)}$ be the orthogonal compliment to $\mathfrak{t}_\alpha$ in $\mathfrak{k}_\alpha$. Denote the group associated to the subalgebra $\mathfrak{k}_{(\alpha)}$, $K_{(\alpha)}$. The compliment to $\mathfrak{t}_\alpha$ in $\mathfrak{t}$ will be written as $\mathfrak{t}_{(\alpha)}$ and its associated group $T_{(\alpha)}$. Since $T \cap K_{(\alpha)} = T_{(\alpha)}$ and $K_\alpha = K_{(\alpha)}T$ then the inclusion $K_{(\alpha)} \hookrightarrow K_\alpha$ induces an isomorphism

$$K_{(\alpha)}/T_{(\alpha)} \cong K_\alpha/T \tag{1.7}$$

The rank one semi-simple compact group K_α must be (by the classification) be isomorphic to either $SU(2)$ or $SO(3)$. In either case

$$SU(2)/S^1 \cong SO(3)/S^1 \cong S^2 \tag{1.8}$$

Composing isomorphisms we are done. ■

A very good example to keep in mind throughout this section (and beyond) is $SU(3)$. One might think it would be better to start with $SU(2)$ but for our purposes in this section this is just too simple. We choose a positive set of roots or equivalently a positive Weyl chamber as before. Now we pick a line L from the positive chamber to the negative chamber to give us an ordering on the positive roots. The Lie algebra of $SU(3)$ is the set of traceless skew hermitian 3×3 matrices. The maximal torus of

$SU(3)$ is the set of diagonal 3×3 matrices with complex entries of unit modulus and determinant one. The Lie algebra, then is the set of traceless diagonal 3×3 matrices with arbitrary complex entries:

$$\underline{t} = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \tag{1.9}$$

Now suppose that $\alpha_1(\underline{t}) = t_1 - t_2$, $\alpha_2(\underline{t}) = t_1 - t_3$, and $\alpha_3(\underline{t}) = t_2 - t_3$. Then $\mathfrak{t}_{\alpha_1}$ has the form:

$$\begin{pmatrix} \mathrm{a} & 0 & 0 \\ 0 & \mathrm{a} & 0 \\ 0 & 0 & \mathrm{b} \end{pmatrix} \tag{1.10}$$

with $a, b \in \mathbf{C}$ and $2a+b=0$. T_{α_1} also has this form if we instead assume $a, b \in S^1 \subset \mathbf{C}$ and $a^2 b = 1$. Hence K_{α_1} has the form:

$$\begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} \tag{1.11}$$

with $A \in U(2)$.

Now we return to the general situation. Note that the group $K_{-\alpha_1} = K_{\alpha_1}$. Hence, we need only index by elements of Δ_+. Suppose we have a set $\{\gamma_1, \ldots, \gamma_n\}$ with $\gamma_i \in \Delta_+$. We define

$$\mathcal{K}(\gamma_1, \ldots, \gamma_n) \stackrel{\text{def}}{=} K_{\gamma_1} \times \cdots K_{\gamma_n} \tag{1.12}$$

.

There is a right action of $T \times T \times \cdots \times T = T^n$ on $\mathcal{K}(\gamma_1, \ldots, \gamma_n)$ defined as follows:

$$(k_1, \ldots, k_n) \bullet (t_1, \ldots, t_n) \stackrel{\text{def}}{=} (k_1 t_1, t_1^{-1} k_2 t_2, \ldots, t_{n-1}^{-1} k_n t_n) \tag{1.13}$$

.

Clearly this is a free action and hence the quotient by this action yields a manifold which we will denote $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$. This gives the principal T^n bundle:

$$
\begin{array}{ccc}
T^n & \hookrightarrow & \mathcal{K}(\gamma_1,\ldots,\gamma_n) \\
 & & \downarrow p \\
 & & \mathcal{M}(\gamma_1,\ldots,\gamma_n)
\end{array}
\tag{1.14}
$$

$\mathcal{M}(\gamma_1,\ldots,\gamma_n)$, itself, is naturally an S^2 bundle over $\mathcal{M}(\gamma_1,\ldots,\gamma_{n-1})$. The bundle map comes from considering the natural map

$$
\begin{array}{ccc}
\mathcal{K}(\gamma_1,\ldots,\gamma_n) & \xrightarrow{\iota_n} & \mathcal{K}(\gamma_1,\ldots,\gamma_{n-1}) \\
(k_1,\ldots k_n) & \mapsto & (k_1,\ldots k_{n-1})
\end{array}
\tag{1.15}
$$

This is well defined since $\iota_n((k_1,\ldots k_n)\bullet(t_1,\ldots t_n)) = (k_1,\ldots k_{n-1})\bullet(t_1,\ldots t_{n-1})$. In fact it is useful to represent in the notation of a twisted product . In other words if A is a space with a right action of a group G and B is a space with a left G action then the quotient

$$
A\times_G B \stackrel{\text{def}}{=} \{(a,b)\in A\times B\}/\{(ag,b)\sim(a,gb)\}
\tag{1.16}
$$

.

This notation will be used extensively. Let $A = \mathcal{K}(\gamma_1,\ldots,\gamma_{n-1})$, $B = K_{\gamma_n}/T$ and $G = T^{n-1}$. T^{n-1} acts on $\mathcal{K}(\gamma_1,\ldots,\gamma_{n-1})$ on the right by the standard right action. T^{n-1} acts on K_{γ_n}/T on the left trivially by the first $n-2$ copies of T and by the standard left multiplication induced action with the last T. This allows us to represent $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ clearly as a bundle:

$$
\begin{array}{ccccc}
K_{\gamma_n}/T & \hookrightarrow & \mathcal{K}(\gamma_1,\ldots,\gamma_{n-1})\times_{T^{n-1}} K_{\gamma_n}/T & \cong & \mathcal{M}(\gamma_1,\ldots,\gamma_n) \\
 & & \downarrow \iota_n & & \\
 & & \mathcal{M}(\gamma_1,\ldots,\gamma_{n-1}) & &
\end{array}
\tag{1.17}
$$

It should be noted that this bundle has two "nice" sections. Let $e\in K_{\gamma_n}\subset K$ denote the identity element. Then we can define the map $\sigma_+ : \mathcal{M}(\gamma_1,\ldots,\gamma_{n-1}) \to \mathcal{M}(\gamma_1,\ldots,\gamma_n)$ by $\sigma_+([(k_1,\ldots,k_{n-1})]) \stackrel{\text{def}}{=} [(k_1,\ldots,k_{n-1},e)]$. This is well defined since $\sigma_+([(k_1,\ldots,k_{n-1})\bullet(t_1,\ldots,t_{n-1})]) = [(k_1t_1,\ldots,t_{n-2}^{-1}k_{n-1}t_{n-1},e)] = [(k_1,\ldots,k_{n-1},e)\bullet(t_1,\ldots,t_{n-1},t_{n-1})]$. Similarly suppose s_{γ_n} is the element of the Weyl group associated to the root γ_n. Then we can define another section by the map $\sigma_-([(k_1,\ldots,k_{n-1})]) \stackrel{\text{def}}{=} [(k_1,\ldots,k_{n-1},s_{\gamma_n})]$. Again this is well defined. First note that since s_{γ_n} is in the

Weyl group, $s_{\gamma_n}^{-1} t_n s_{\gamma_n} \in \mathfrak{t}$. Hence $\sigma_-([(k_1,\ldots,k_{n-1}) \bullet (t_1,\ldots,t_{n-1})]) = [(k_1 t_1, \ldots, t_{n-2}^{-1} k_{n-1} t_{n-1}, s_{\gamma_n})]$ $= [(k_1,\ldots,k_{n-1},e) \bullet (t_1,\ldots,t_{n-1}, s_{\gamma_n}^{-1} t_n s_{\gamma_n})]$.

All this should seem reminiscent of the situation in the first section. In fact we will presently show that we are in precisely that situation. Firstly there is a left action on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ by a n dimensional torus produced by Bott, [B1]. Then we can explicitly identify $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ as the projectivization of a complex rank 2 vector bundle over $\mathcal{M}(\gamma_1,\ldots,\gamma_{n-1})$. To define the left action on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ we first define a left action of $\underbrace{T \times \cdots \times T}_{n \text{ times}}$, as follows

$$(s_1,\ldots,s_n) \star (k_1,\ldots,k_n) \stackrel{\text{def}}{=} (s_1 k_1, s_1^{-1} s_2 k_2, \ldots, s_{n-1}^{-1} s_n k_n) \tag{1.18}$$

.

Since T is abelian the left $\star$ action and the right $\bullet$ action commute. Thus the left $\star$ action is well defined on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ though it is, of course, far from being effective. Suppose $(s_1,\ldots,s_n)$ fixes all of $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$. In particular we must have that for all $k_1 \in K_{\gamma_1}$, $k_1^{-1} s_1 k_1 \in T$. Clearly this can only be the case if $s_1 \in T_{\gamma_1}$. Hence we have

$$(s_1 k_1, s_1^{-1} s_2 k_2, \ldots, s_{n-1}^{-1} s_n k_n) \sim (s_1 k_1 s_1^{-1}, s_2 k_2, \ldots, s_{n-1}^{-1} s_n k_n) \sim (k_1, s_2 k_2, \ldots, s_{n-1}^{-1} s_n k_n) \tag{1.19}$$

.

Continuing in this manner we determine that $(s_1,\ldots,s_n) \in T_{\gamma_1} \times \cdots \times T_{\gamma_n}$. Moreover we can see that any thing in this group acts trivially on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$. Hence we define the quotient group $\check{T} \stackrel{\text{def}}{=} T \times \cdots \times T / T_{\gamma_1} \times \cdots \times T_{\gamma_n}$ which acts effectively on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$.

2.1.3 Principal Bundles

First we recall the notation for principal bundles, principal connections and their natural specialization to homogeneous spaces, reductive spaces. It turns out, not surprisingly, that this framework will be very useful for studying the tangent bundle of $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$. It will also allow us to find an almost complex structure which is compatible with the left action defined above.

A principal bundle (P, B, H, π, r) is a fiber bundle $P \xrightarrow{\pi} B$, a right action of a Lie group H, $r : P \times H \to P$ which is free and is a bundle automorphism (in other words it fixes B). As a consequence the fibers of the bundle are isomorphic to H although not naturally. This yields the natural sequence of bundles

$$0 \to T_{P/B} \to TP \to \pi^* TB \to 0 \tag{1.20}$$

.

Here π^*TB is the pullback of the tangent bundle TB along π. $T_{P/B}$ is the called the vertical subbundle of the principal fibration. It is defined, $T_{P/B} \stackrel{\text{def}}{=} \{v \in TP | T(\pi)v = 0\}$. Alternately if we let $o_p : H \to O_p$ be the orbit map which maps H onto its orbit through $p \in P$, then $T_{p\ P/B}$ =image of $o_p = T(o_p)\mathfrak{h}$. Here $\mathfrak{h}$ is the Lie algebra to H. A principal connection is a C^∞ equivariant splitting of the exact bundle sequence 1.20. Equivalently it is a choice of a subbundle $\mathcal{H} \hookrightarrow TP$ such that for $p \in P$

- $\mathcal{H}_p \oplus (T_{P/B})_p = T_pP$
- $T(r_h)\mathcal{H}_p = \mathcal{H}_{p \cdot h}$ where r_h is the right action by $h \in H$ map.
- $\mathcal{H}_p$ varies smoothly with $p \in P$.

Specializing to homogeneous spaces, suppose G is a Lie group and H is a closed subgroup. Then

$$\begin{array}{ccc} H & \hookrightarrow & G \\ & & \downarrow \pi \\ & & G/H \end{array} \tag{1.21}$$

is a principal H bundle over $G/H \stackrel{\text{def}}{=} X$. The right action of H on G is induced by right multiplication in G. TG has a natural trivialization:

$$\begin{array}{ccc} TG & \longrightarrow & G \times \mathfrak{g} \\ v_g & \mapsto & T(l_g)(v_g) \end{array} \tag{1.22}$$

where $v_g \in T_gG$, and l_g is left multiplication by g. Note that since left and right multiplication commute this trivialization is compatible with the right action. In other words the trivializing map in 1.22 is equivariant with respect to the right action. The kernel of the map $T_g\pi : T_gG \longrightarrow T_{gH}X$ is clearly $T(l_g)\mathfrak{h}$ and hence the vertical bundle $T_{G/X} \cong G \times \mathfrak{h}$ again by left translation. TX can be represented as TG modulo an equivalence relation via the canonical surjection $TG \longrightarrow TX$. Suppose $p, q \in TG$ we will say $p \sim q$ iff $T(\pi)p = T(\pi)q$. If $p = (g, \underline{g})$ and $p = (g', \underline{g}')$ with $g, g' \in G$ and $\underline{g}, \underline{g}' \in \mathfrak{g}$, then $p \sim q$ implies that $g' = gh$ for some $h \in H$. Moreeover

$$T(\pi)T(l_g)\ \underline{g} = T(\pi)(g,\ \underline{g}) = T(\pi)T(l_{g'})\ \underline{g}' = T(\pi)T(l_g)T(l_h)\ \underline{g}' \tag{1.23}$$

Now essentially by definition $\pi \circ r_h = \pi$ for all $h \in H$. Hence

$$T(\pi)T(l_g)\ \underline{g} = T(\pi)T(l_g)T(l_h)T(r_{h^{-1}})\ \underline{g}' \tag{1.24}$$

and

$$T(\pi)T(l_g)(\ \underline{g} - \mathrm{Ad}(h)\ \underline{g}') = 0 \tag{1.25}$$

.

Since l_g is a diffeomorphism of G, this means $T(\pi)(\ \underline{g} - \mathrm{Ad}(h)\ \underline{g}') = 0$ which implies that $\underline{g} - \mathrm{Ad}(h)\ \underline{g}' \in \mathfrak{h}$ and at last that $\underline{g}' = \mathrm{Ad}(h^{-1})\ \underline{g} + \underline{h}$ with $\underline{h} \in \mathfrak{h}$. It should also be clear that for any choice of $g \in G$, $h \in H$, $\underline{g} \in \mathfrak{g}$ and $\underline{h} \in \mathfrak{h}$, $(g,\ \underline{g}) \sim (gh, \mathrm{Ad}(h^{-1})\ \underline{g} + \underline{h})$. This shows that in fact

$$TX \cong G \times_H \mathfrak{g}/\mathfrak{h} \tag{1.26}$$

where H acts on G on the right via right multiplication and on $\mathfrak{g}/\mathfrak{h}$ on the left via the adjoint action.

2.1.4 Reductive Spaces

Now suppose we want a principle connection on the principle fiber bundle $G \longrightarrow G/H = X$. Let $g \in G$. Due to the second condition we mentioned in defining a connection (equivariance along the fibers,) once we have chosen a horizontal subspace $\mathcal{H}_g \subset T_gG$, then $\mathcal{H}_{gh}$ is determined for all $h \in H$. In fact since X has a transitive left action we could obtain a connection by choosing a compliment to $\mathfrak{h}$, $\mathcal{H}_e \subset \mathfrak{g}$, provided it satisfied an appropriate compatibility condition with respect to the left and right action. In particular suppose $\mathfrak{m} \subset \mathfrak{g}$ is an invariant subspace (though not necessarily fixed) for the adjoint action of G on $\mathfrak{g}$ restricted to H. Then let $\mathcal{H}_e \stackrel{\text{def}}{=} \mathfrak{m}$ and $\mathcal{H}_g \stackrel{\text{def}}{=} T(l_g)\mathfrak{m}$. First, since $T(l_g)$ is an isomorphism $T_gG \cong T(l_g)\mathfrak{g} \cong T(l_g)(\mathfrak{m} \oplus \mathfrak{h}) \cong \mathcal{H}_g \oplus (T_{G/X})_g$ Secondly

$$\begin{aligned} T(r_h)\mathcal{H}_g &= T(r_h)T(l_g)\mathfrak{m} &&= T(l_g)T(r_h)\mathfrak{m} \\ &= T(l_g)T(l_h)T(l_{h^{-1}})T(r_h)\mathfrak{m} &&= T(l_{gh})Ad(h^{-1})\mathfrak{m} \\ &= T(l_{gh})\mathfrak{m} &&= \mathcal{H}_{gh} \end{aligned} \tag{1.27}$$

since $\mathfrak{m}$ is Ad invariant on H. Lastly since left multiplication varies smoothly with g, this actually does define a connection. We will denote the adjoint action restricted to H by Ad(H). We may rephrase the above as saying that an AdH equivariant splitting of the exact sequence:

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0 \tag{1.28}$$

with $\mathfrak{m}$ the image of $\mathfrak{g}/\mathfrak{h}$, via the splitting, in $\mathfrak{g}$, gives a principal connection on $G \to X$. We can now write

$$TX \cong G \times_H \mathfrak{m} \tag{1.29}$$

.

The triple $(G, H, \mathfrak{m})$ is called a reductive space. Suppose K is a compact Lie group and $T \subset K$ is a maximal torus. Using the compactness of K one can average to get a unique (after normalization) bi-invariant metric. In particular this will be adjoint invariant on $\mathfrak{k}$ so that Ad:$K \to O(\mathfrak{k})$. Now $\mathfrak{t}$ is an invariant subspace of the adjoint action restricted to T. Since T acts on $\mathfrak{k}$ by isometries it must also preserve the orthogonal compliment $\mathfrak{m} \stackrel{\text{def}}{=} \mathfrak{t}^{\perp}$. Hence $(K, T, \mathfrak{m})$ is a reductive space.

2.1.5 The almost complex structure of the flag manifold

In order to understand and compare these two almost complex structures it will be useful to review the homogeneous picture. Recall from the last section that we can decompose $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{m}$. Here $\mathfrak{m}$ is the orthogonal compliment to $\mathfrak{t}$ with respect to the biinvariant metric. Under Ad(T) , $\mathfrak{k}$ decomposes into irreducible pieces, $\mathfrak{k}$

$= \mathfrak{t} \oplus \sum_i \mathfrak{m}_i$, where $\mathrm{Ad}(T)$ acts trivially on $\mathfrak{t}$ (so this is really a sum of one-real-dimensional spaces with $\mathrm{Ad}(T)$ acting trivially.) Each $\mathfrak{m}_i \cong \mathbf{R}^2$ and so there are two natural isomorphisms (isometries) with $\mathbf{C}$. This gives us a total of 2^n equivariant complex structures on $\mathfrak{m}$ and so 2^n homogeneous almost complex structures on

$$T(K/T) \cong K \times_{\mathrm{Ad}(T)} \mathfrak{m} \tag{1.30}$$

Of course only a small number of these are integrable. Suppose J is an equivariant almost complex structure on $\mathfrak{m}$. Then $\mathfrak{m} \otimes \mathbf{C} = \mathfrak{n} \oplus \bar{\mathfrak{n}}$, where here we *define* $\mathfrak{n}$ to be the $-i$ eigenspace and $\bar{\mathfrak{n}}$ to be the $+i$ eigenspace. Since J is $\mathrm{Ad}(T)$ equivariant so are $\bar{\mathfrak{n}}$ and $\mathfrak{n}$. Hence we get a splitting of complex bundles:

$$T(K/T) \otimes \mathbf{C} \cong K \times_{\mathrm{Ad}(T)} \mathfrak{n} \oplus \bar{\mathfrak{n}} \tag{1.31}$$

Define the anti-holomorphic subbundle as $T^{0,1}(K/T) \cong K \times_{\mathrm{Ad}(T)} \mathfrak{n}$. We can now define integrability as follows: suppose for every X and Y, vector fields in the complexification of $T(K/T)$ which lie entirely in the anti-holomorphic subbundle $T^{0,1}(K/T)$, their Lie bracket $[X, Y]$ also lies entirely in $T^{0,1}(K/T)$. The fact that this condition is equivalent to being able to find an honest complex structure with this as its almost complex structure is a hard theorem of Newlander-Nirenberg [N-N], (for a more modern proof [H].) The fortunate thing about the homogeneous space K/T is that all of these computations can be understood in the Lie algebra. This is a situation which will be sorely missed when we return the to Bott-Samelson manifolds.

As before denote $\mathfrak{g} = \mathfrak{k} \otimes \mathbf{C}$. Since $\mathfrak{k} \,/\, \mathfrak{t} \cong \mathfrak{m}$, $\mathfrak{h} \stackrel{\text{def}}{=} \mathfrak{t} \otimes \mathbf{C}$, $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ and since $\bar{\mathfrak{n}}$ is $\mathrm{Ad}(T)$ invariant it follows that $\bar{\mathfrak{n}} \subset \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$. In fact, again by equivariance $\bar{\mathfrak{n}} = \sum_{\alpha \in R} \mathfrak{g}^\alpha$, where $R \subset \Delta$. Moreover since $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$, R should be a closed subset if J has any prayer of being integrable. In other words J integrable implies that R is closed. It will now turn out very useful to define an involution of $\mathfrak{g}$. Suppose $X, Y \in \mathfrak{k}$, then $\sigma(X + iY) \stackrel{\text{def}}{=} X - iY$. σ is therefore just complex conjugation with respect to $\mathfrak{k}$ and is conjugate linear and $\sigma(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$. σ implements an isomorphism $\mathfrak{n} \cong \bar{\mathfrak{n}}$. From this we see that R is unipotent and $\Delta = R \cup -R$. This together with the fact that R is closed tells us that R defines a positive root system $\Delta_+ = R$. This justifies our use of the notation $\mathfrak{n}$ since it is in fact the sum of the negative root spaces with respect to a particular root system. So we have that J integrable implies that R is a positive root system. In fact the reverse is true.

Suppose that we choose a positive root system Δ_+ our problem will be to find a J such that $R = \Delta_+$. Let $\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}^{\alpha}$. By definition σ has $\mathfrak{k}$ as its fixed subspace (real.) Naturally $\mathfrak{m}$ is also fixed under σ while $\sigma(\mathfrak{n})\cap\mathfrak{n} = \bar{\mathfrak{n}}\cap\mathfrak{n} = 0$. Hence $\mathfrak{m}\cap\mathfrak{n} = 0$ and so the inclusion $\mathfrak{m} \hookrightarrow \mathfrak{g}$ implements the real isomorphism $\mathfrak{m} \cong \mathfrak{g} / \mathfrak{b}$. Here we are using the convention that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. The quotient of complex algebras $\mathfrak{g} / \mathfrak{b}$ is a complex vector space. Using this complex structure and the real isomorphism, this induces a complex structure, J on $\mathfrak{m}$. $\mathfrak{g}/\mathfrak{b} \cong \bar{\mathfrak{n}}$ and $\bar{\mathfrak{n}}\otimes\mathbf{C} \cong \bar{\mathfrak{n}}\oplus\mathfrak{n}$ with $\mathfrak{n}$ the anti-holomorphic part of the complexification of $\bar{\mathfrak{n}}$. Therefore with the induced complex structure J on $\mathfrak{m}$ has $\mathfrak{n}$ as its anti-holomorphic structure $R = \Delta_+$.

Suppose we would like to understand this J operator more explicitly. For example, we can write J in terms of a reasonable basis of $\mathfrak{m}$. Let X_α be a nonzero vector in $\mathfrak{g}^{\alpha}$ with $\alpha \in \Delta_+$. Now let $X_{-\alpha} \stackrel{\text{def}}{=} \sigma(X_\alpha) \in \mathfrak{g}^{-\alpha}$. So the set $\{ X_\alpha\}_{\alpha\in\Delta}$ where α runs over all the roots, is a basis for $\mathfrak{n} \oplus \bar{\mathfrak{n}}$. Since σ is an involution this basis is preserved by σ. Now suppose $Y \in \mathfrak{g}$, let $\zeta(Y) = \frac{1}{2}(Y + \sigma(Y))$ then $\sigma(\zeta(Y)) = \zeta(Y)$. The fixed point set of σ is $\mathfrak{k}$ so $\zeta(Y) \in \mathfrak{k}$. In other words if $Y \in i\mathfrak{k}$ then $\zeta(Y)=\frac{1}{2}(Y + \sigma(Y)) = Y$. In other words $\zeta|_{\mathfrak{k}}$ = identity. On the other hand if $Y \in \mathfrak{k}$ then $\zeta(Y)=\frac{1}{2}(Y + \sigma(Y)) = 0$. Now recall that $\mathfrak{m}\otimes \mathbf{C} = \mathfrak{n}\oplus\bar{\mathfrak{n}} = \mathfrak{m}\oplus i\mathfrak{m}$. So it follows that $\zeta|_{\mathfrak{n}\oplus\bar{\mathfrak{n}}}$ is projection onto $\mathfrak{m}$ along $i\mathfrak{m}$ and $\zeta(\mathfrak{n}\oplus\bar{\mathfrak{n}}) = \mathfrak{m}$. We have that $\mathfrak{n}\cap\mathfrak{m} = 0$ or for that matter $\bar{\mathfrak{n}}\cap\mathfrak{m} = 0$. Suppose that $v \in \mathfrak{n}\cap i\mathfrak{m}$. $v \in \mathfrak{n}$ implies that $iv \in \mathfrak{n}$ since $\mathfrak{n}$ is a complex subspace of $\mathfrak{g}$. Thus if $iv \in \mathfrak{m}$ then $iv = 0 = v$. Hence $\mathfrak{n}\cap i\mathfrak{m} = 0$ and $\bar{\mathfrak{n}}\cap i\mathfrak{m} = 0$. From this we conclude that $\zeta|_{\mathfrak{n}} : \mathfrak{n}\to\mathfrak{m}$ is an isomorphism, as is $\zeta|_{\bar{\mathfrak{n}}} : \bar{\mathfrak{n}}\to\mathfrak{m}$. Now $\mathrm{Ad}(T)$ preserves the kernel of ζ, $i\mathfrak{m}$ and restricted to $\mathfrak{m}$, ζ is the identity which is surely equivariant. Moreover $\mathrm{Ad}(p)$ reserves the splitting $\mathfrak{n}\oplus\bar{\mathfrak{n}} = \mathfrak{m}\oplus i\mathfrak{m}$. So ζ is equivariant.

In fact it is precisely the isomorphism $\zeta|_{\bar{\mathfrak{n}}} : \bar{\mathfrak{n}}\to\mathfrak{m}$ that we used, to put an integrable complex structure on . ζ is equivariant so it preserves the splitting $\sum_{\alpha\in\Delta_+} \mathfrak{g}^{-\alpha}$. Since $\zeta|_{\mathfrak{g}^\alpha}$ is an isomorphism, the real basis of $\mathfrak{g}^{\alpha}$, $\{ X_\alpha, i X_\alpha\}$ is mapped to the real basis of $\mathfrak{m}_\alpha$, $\{ X_\alpha + X_{-\alpha},\ iX_\alpha - iX_{-\alpha}\}$. Define

$$e \stackrel{\text{def}}{=} X_\alpha + X_{-\alpha} \text{ and } f \stackrel{\text{def}}{=} iX_\alpha - iX_{-\alpha}. \tag{1.32}$$

Now the complex structure on $\mathfrak{m}$ is defined as the composition of maps $J =$

$\zeta \circ i \circ |_{\mathfrak{g}} \alpha^{-1}$. The map i is complex multiplication by the imaginary number i. It certainly satisfies $J^2 = -Id$. With these definitions $Je = f$ and $Jf = -e$. By extending the above basis to all of $\mathfrak{m}$ we find that

$$J = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \tag{1.33}$$

Actually it would be nice to see this in terms of $\mathfrak{g}/\mathfrak{b}$. First $\mathfrak{m} \hookrightarrow \mathfrak{g}/\mathfrak{b}$, in particular $e = X_\alpha + X_{-\alpha} \mapsto [\ X_\alpha + X_{-\alpha} + \mathfrak{b}\ ]$. Now $i[\ X_\alpha + X_{-\alpha} + \mathfrak{b}] = [i\ X_\alpha + i\ X_{-\alpha} - 2i\ X_{-\alpha} + \mathfrak{b}\ ] = [\ iX_\alpha - iX_{-\alpha} + \mathfrak{b}\ ]$. Since $f = iX_\alpha - iX_{-\alpha} \mapsto [\ iX_\alpha - iX_{-\alpha} + \mathfrak{b}\ ]$, we find again that $Je = f$. The computation that $Jf = -e$ goes in the same manner. This is only a computation of J in $\mathfrak{m} \cong T_{eT}(K/T)$. However since J is $\mathrm{Ad}(T)$ equivariant, the global almost complex structure is just the one at the identity coset eT "pushed around" on K/T by the left translation by the left action of K.

2.1.6 The tangent bundle of Bott-Samelson manifolds

As mentioned earlier $\mathcal{K}(\gamma_1, \ldots, \gamma_n)$ is a T^n principal bundle over $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$. We should keep uppermost in our mind that our right action of T^n is **not** the one induced by the straightforward inclusions $T \hookrightarrow K_\alpha$. Recall that our right action was defined in 1.13 and is somewhat twisted. The effect of this is that unlike in the homogeneous case discussed in the last section, there is no transitive left action on the total space of the principal bundle that commutes with the bundle right action. First we will describe the vertical bundle $\mathcal{V} \stackrel{\text{def}}{=} T(\ \mathcal{K}(\gamma_1, \ldots, \gamma_n)/\ \mathcal{M}(\gamma_1, \ldots, \gamma_n))$ sitting inside $T(\ \mathcal{K}(\gamma_1, \ldots, \gamma_n))$. Then we find a sequence analogous to 1.20. We will describe a principal connection on $\mathcal{K}(\gamma_1, \ldots, \gamma_n)$. This will allow us, just as in the homogeneous case to model the tangent bundle of $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ and give $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ an almost complex structure.

As in the homogeneous case we can trivialize $T(\ \mathcal{K}(\gamma_1, \ldots, \gamma_n))$ via left translation: let $p = (k_1, \ldots, k_n) \in \mathcal{K}(\gamma_1, \ldots, \gamma_n)$ then

$$T(l_{p^{-1}}) \stackrel{\text{def}}{=} T(l_{k_1^{-1}}) \times \cdots \times T(l_{k_1^{-1}}) : T_p(\mathcal{K}(\gamma_1, \ldots, \gamma_n)) \to \mathfrak{k}_{\gamma_1} \oplus \cdots \oplus \mathfrak{k}_{\gamma_n} \tag{1.34}$$

is an isomorphism. So $T(\mathcal{K}(\gamma_1, \ldots, \gamma_n)) \cong \mathcal{K}(\gamma_1, \ldots, \gamma_n) \times \mathfrak{k}_{\gamma_1} \oplus \cdots \oplus \mathfrak{k}_{\gamma_n}$. Now the vertical subspace at p, looks like

$$T(o_p)(\mathfrak{t} \oplus \cdots \oplus \mathfrak{t}) \subset T_p(\mathcal{K}(\gamma_1, \ldots, \gamma_n)) \tag{1.35}$$

where $o_p : T^n \to O_p$ is the orbit map. Suppose $v \in \mathfrak{t} \oplus \cdots \oplus \mathfrak{t}$. Let us take a representative path

$$\begin{aligned} \gamma : (-\epsilon, \epsilon) &\longrightarrow T^n \\ r &\mapsto (t_1(r), \ldots, t_n(r)) \end{aligned} \tag{1.36}$$

such that $t_i(r) \in S^1$ and $\frac{d}{dr}\gamma|_{r=0} = v$. So then $\frac{d}{dr}(p \bullet \gamma)|_{r=0} = T(o_p)v \in T_p(\mathcal{K}(\gamma_1, \ldots, \gamma_n))$. Moreover $\frac{d}{dr} l_{p^{-1}}(p \bullet \gamma)|_{r=0} = T(l_{p^{-1}})T(o_p)v \in \mathfrak{k}_{\gamma_1} \oplus \mathfrak{k}_{\gamma_n}$. Since

$$l_{p^{-1}}(p \bullet \gamma) = l_{p^{-1}}\left((k_1 t_1(r), t_1(r)^{-1} k_2 t_2(r), \ldots, t_{n-1}(r)^{-1} k_n t_n(r))\right) \tag{1.37}$$

$$= (t_1(r), k_2^{-1} t_1(r)^{-1} k_2 t_2(r), \ldots, k_n^{-1} t_{n-1}(r)^{-1} k_n t_n(r)) \tag{1.38}$$

by differentiating we see that if $v = (\underline{\mathtt{t_1}}, \ldots, \underline{\mathtt{t_n}})$ then

$$T(l_{p^{-1}})T(o_p)v = (\underline{\mathtt{t_1}}, \underline{\mathtt{t_2}} - \mathrm{Ad}(k_2^{-1})\, \underline{\mathtt{t_{n-1}}}, \ldots, \underline{\mathtt{t_n}} - \mathrm{Ad}(k_n^{-1})\, \underline{\mathtt{t_{n-1}}}) \tag{1.39}$$

Note that $\underline{\mathtt{t_i}} - \mathrm{Ad}(k_i^{-1})\, \underline{\mathtt{t_{i-1}}} \in \mathfrak{k}_{\gamma_i}$ for all $k_i \in \mathfrak{k}_{\gamma_i}$.

Hence we see that the vertical subbundle is the image of the injective map Υ :

$$\begin{aligned} \Upsilon : \mathcal{K}(\gamma_1, \ldots, \gamma_n) \times \mathfrak{t} \oplus \ldots \oplus \mathfrak{t} &\longrightarrow \mathcal{K}(\gamma_1, \ldots, \gamma_n) \times \mathfrak{k}_{\gamma_1} \oplus \ldots \oplus \mathfrak{k}_{\gamma_n} \\ \left((k_1, \ldots, k_n), (\underline{\mathtt{t_1}}, \ldots, \underline{\mathtt{t_n}})\right) &\mapsto \left((k_1, \ldots, k_n), (\underline{\mathtt{t_1}}, \ldots, \underline{\mathtt{t_n}} - \mathrm{Ad}(k_n^{-1})\, \underline{\mathtt{t_{n-1}}})\right) \end{aligned} \tag{1.40}$$

2.1.7 Choosing a Connection

Now as in the reduced space picture we should choose a connection. That is, before a choice of horizontal subbundle $\mathcal{H} \subset T\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)$ analogous to splitting the sequence 1.20. The relevant sequence in this situation is

$$0 \to T(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)/\,\mathcal{M}(\gamma_1,\ldots,\gamma_n)) \xrightarrow{\Upsilon} T(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)) \to \pi^*T(\,\mathcal{M}(\gamma_1,\ldots,\gamma_n)) \to 0. \tag{1.41}$$

The first term in the sequence is the vertical subundle. Here since Υ is **not** induced by the inclusion $\mathfrak{t} \hookrightarrow \mathfrak{k}_{\gamma_i}$, $\pi^*T(\,\mathcal{M}(\gamma_1,\ldots,\gamma_n))$ is much more complicated than π^*TX in 1.20. Since each of the K_{γ_i} is a compact group, we can find a bi-invariant metric. As before, the orthogonal compliment of $\mathfrak{t}$ in $\mathfrak{k}_{\gamma_i}$, $\mathfrak{m}_{\gamma_i}$ is Ad(T) invariant. At the "identity," $(e,\ldots,e) \in \mathcal{K}(\gamma_1,\ldots,\gamma_n)$ we choose the horizontal space $\mathcal{H}_e \stackrel{\text{def}}{=} \mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n} \subset \mathfrak{k}_{\gamma_1} \oplus \ldots \oplus \mathfrak{k}_{\gamma_n}$. More generally if $p = (k_1 \ldots k_n) \in \mathcal{K}(\gamma_1,\ldots,\gamma_n)$ then we define $\mathcal{H}_p \subset T_p(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n))$, by

$$\mathcal{H}_p \stackrel{\text{def}}{=} (T(l_{k_1}) \times \cdots \times T(l_{k_n}))\, \mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n} \tag{1.42}$$

To show that this is a connection look at $T_p(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)/\,\mathcal{M}(\gamma_1,\ldots,\gamma_n)) \cap \mathcal{H}_p$. Suppose $T_e(o_p)v \in \mathcal{H}_p$ for some $v = (\,\underline{t_1},\ldots,\,\underline{t_n}) \in \mathfrak{t} \oplus \ldots \oplus \mathfrak{t}$. This implies

$$(\,\underline{t_1},\ldots,\,\underline{t_n} - \mathrm{Ad}(k_n^{-1})\,\underline{t_{n-1}}) \in \mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n}. \tag{1.43}$$

In particular $\underline{t_1} \in \mathfrak{m}_{\gamma_1}$, but $\underline{t_1} \in \mathfrak{t}$. So since $\mathfrak{m}_{\gamma_1} \cap \mathfrak{t} = \{0\}$ then $\underline{t_1} = 0$. This in turn implies that $\underline{t_2} \in \mathfrak{m}_{\gamma_2}$ and so again $\underline{t_2} = 0$. Continuing in this manner we find that $T_p(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)/\,\mathcal{M}(\gamma_1,\ldots,\gamma_n)) \cap \mathcal{H}_p = \{0\}$ and hence by a dimension count

$$T_p(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)/\,\mathcal{M}(\gamma_1,\ldots,\gamma_n)) \oplus \mathcal{H}_p = T_p(\,\mathcal{K}(\gamma_1,\ldots,\gamma_n)). \tag{1.44}$$

Lastly we must check that this decomposition is invariant under the right action of T^n and $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$. Let the the map $\hat{r}_t : \mathcal{K}(\gamma_1,\ldots,\gamma_n) \to \mathcal{K}(\gamma_1,\ldots,\gamma_n)$, with $t \in T^n$ be right translation by the twisted right action on $\mathcal{K}(\gamma_1,\ldots,\gamma_n)$. We must check that $T_p(\hat{r}_t)\,\mathcal{H}_p = \mathcal{H}_{p\bullet t}$.

$$(T(r_{t_1}) \times \cdots \times T(l_{t_{n-1}^{-1}}) \circ T(r_{t_n})) \circ (T(l_{k_1}) \times \cdots \times T(l_{k_n}))\, \mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n} \;=$$

$$= \quad (T(l_{k_1 t_1}) \circ T(l_{t_1^{-1}}) \circ T(r_{t_1}) \times \cdots \times T(l_{t_{n-1}^{-1} k_n t_n}) \circ T(l_{t_n^{-1}}) \circ T(r_{t_n}))\ \mathfrak{m}\gamma_1 \oplus \ldots \oplus \mathfrak{m}\gamma_n$$
$$= \quad (T(l_{k_1 t_1}) \times \cdots \times T(l_{t_{n-1}^{-1} k_n t_n})) \circ (\mathrm{Ad}(t_1^{-1}) \times \cdots \times \mathrm{Ad}(t_n^{-1}))\ \mathfrak{m}\gamma_1 \oplus \ldots \oplus \mathfrak{m}\gamma_n$$
$$= \quad \mathcal{H}_{p \bullet t}$$

The last equality follows from the equation $\mathrm{Ad}(t_i^{-1})\mathfrak{m}\gamma_i = \mathfrak{m}\gamma_i$. This shows that, just as in the homogeneous case we can write the tangent bundle as a bundle associated to the priniple bundle $\mathcal{K}(\gamma_1, \ldots, \gamma_n)$:

$$\mathcal{K}(\gamma_1, \ldots, \gamma_n) \times_{\mathrm{Ad}(T) \times \cdots \times \mathrm{Ad}(T)} \mathfrak{m}\gamma_1 \oplus \ldots \oplus \mathfrak{m}\gamma_n \cong T(\mathcal{M}(\gamma_1, \ldots, \gamma_n)) \tag{1.45}$$

In fact the connection $\mathcal{H}$ is compatible with the left action defined above (1.18). Suppose $p \in \mathcal{K}(\gamma_1, \ldots, \gamma_n)$ and $s \in \check{T}$ then define the map $\hat{l}_s : \mathcal{K}(\gamma_1, \ldots, \gamma_n) \rightarrow \mathcal{K}(\gamma_1, \ldots, \gamma_n)$ by $\hat{l}_s \stackrel{\text{def}}{=} (s \star p)$. Then

$$T(\hat{l}_s)\ \mathcal{H}_p$$
$$= \quad \left(T(l_{s_1}) \times \cdots \times T(l_{s_{n-1}^{-1} s_n})\right) \circ ((T(l_{k_1}) \times \cdots \times T(l_{k_n}))\ \mathfrak{m}\gamma_1 \oplus \ldots \oplus \mathfrak{m}\gamma_n$$
$$= \quad (T(l_{s_1 k_1}) \times \cdots \times T(l_{s_{n-1}^{-1} s_n k_n})\ \mathfrak{m}\gamma_1 \oplus \ldots \oplus \mathfrak{m}\gamma_n$$
$$= \quad \mathcal{H}_{s \star p}$$

2.1.8 The Bott Tower Perspective

We would like to show a Bott-Samelson Manifold is a Bott-tower. The induction "liet motif" continues as we begin with the case of a one factor Bott-Samelson manifold.

Proposition 2.2 *There is a natural isomorphism*

$$K\gamma/T \cong \mathsf{P}(\mathfrak{g}^\gamma \oplus \mathsf{C}) \tag{1.46}$$

which is left T equivariant where T acts by left translation on $K\gamma/T$ and by the action induced from the adjoint action of T on $\mathfrak{g}^\gamma$.

Proof. As before in Proposition 2.1, there is a natural isomorphism $K\gamma/T \cong K_{(\gamma)}/T_{(\gamma)}$ which is $T_{(\gamma)}$ equivariant. Once again $T = T\gamma T_{(\gamma)}$. If we extend the $T_{(\gamma)}$

to a T action on $K_{(\gamma)}/T_{(\gamma)}$ with T_γ acting on the left trivially, then the isomorphism is also left T equivariant. Let $\widetilde{K_{(\gamma)}}$ be the universal cover of $K_{(\gamma)}$. If $K_{(\gamma)}$ is simply connected then $\widetilde{K_{(\gamma)}} = K_{(\gamma)}$. In any case there is a natural isomorphism

$$\widetilde{K_{(\gamma)}}/\widetilde{T_{(\gamma)}} \cong K_{(\gamma)}/T_{(\gamma)} \tag{1.47}$$

which again is T equivariant. The simply connected rank one simple compact Lie group $\widetilde{K_{(\gamma)}}$, has a unique complex two dimensional representation V. Restricting to the maximal torus $\widetilde{T_{(\gamma)}}$, V splits in to weight spaces $V \cong \mathfrak{g}^{\frac{1}{2}\gamma} \oplus \mathfrak{g}^{-\frac{1}{2}\gamma}$. Since this is a linear representation there is a naturally induced action of $\widetilde{K_{(\gamma)}}$ on $\mathsf{P}(V)$ Pick a vector in $v \in \mathfrak{g}^{\frac{1}{2}\gamma}$. This choice is unique in $\mathsf{P}(V)$. The action of $\widetilde{K_{(\gamma)}}$ on $\mathsf{P}(V)$ is transitive with stabilizer $\widetilde{T_{(\gamma)}}$. Thus if $O_{[v]}$ is the orbit of $[v]$, then there are natural T equivariant isomorphisms

$$\widetilde{K_{(\gamma)}}/\widetilde{T_{(\gamma)}} \cong O_{[v]} \cong \mathsf{P}(\mathfrak{g}^{\frac{1}{2}\gamma} \oplus \mathfrak{g}^{-\frac{1}{2}\gamma}) \tag{1.48}$$

Now note that $\mathsf{P}(V \otimes \mathfrak{g}^{\frac{1}{2}\gamma}) \cong \mathsf{P}(V)$ so finally $\widetilde{K_{(\gamma)}}/\widetilde{T_{(\gamma)}} \cong \mathsf{P}(\mathfrak{g}^\gamma \oplus \mathbf{C})$. Composing isomorphisms we are done. ■

Theorem 2.1 *$\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ is a Bott tower.*

Proof. As mentioned above the proof is by induction. Assume $\mathcal{M}(\gamma_1, \cdots, \gamma_{n-1})$ is a Bott tower. Let $\mathbf{L} \stackrel{\text{def}}{=} \mathcal{K}(\gamma_1, \ldots, \gamma_{n-1}) \times_{T^n} \mathfrak{g}^\gamma$ be a line bundle over $\mathcal{M}(\gamma_1, \cdots, \gamma_{n-1})$. From the previous proposition

$$\mathsf{P}(\mathbf{L} \oplus \mathbf{C}) \cong \mathcal{K}(\gamma_1, \ldots, \gamma_{n-1}) \times_{T^n} \mathsf{P}(\mathfrak{g}^\gamma \oplus \mathbf{C}) \cong \mathcal{K}(\gamma_1, \ldots, \gamma_{n-1}) \times_{T^n} K_\gamma/T \tag{1.49}$$

From equation 1.17 we know that $\mathcal{K}(\gamma_1, \ldots, \gamma_{n-1}) \times_{T^n} K_\gamma/T \cong \mathcal{M}(\gamma_1, \cdots, \gamma_n)$, so we are done. ■

2.1.9 An Almost Complex Structure

Again, using the homogeneous case as a model, the identification $\mathfrak{m}_\gamma \cong \mathfrak{g}^\gamma \cong \mathbf{C}$ gives us a almost complex structure $(\tilde{J})_{(e,\ldots,e)}$ on the vector space $\mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n}$. Moreover, since these identifications are equivariant under the adjoint action, so is the operator $(\tilde{J})_{(e,\ldots,e)}$. This implies that using the identification (1.45) this induces an almost complex structure on $T(\mathcal{M}(\gamma_1,\ldots,\gamma_n))$ we will call $\tilde{J}$. In the same way an almost complex structure is produced on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$, it can produce one on $\mathcal{M}(\gamma_1,\ldots,\gamma_r)$. Thus we get a family of almost complex structures on the various $\mathcal{M}(\gamma_1,\ldots,\gamma_r)$, as r varies, and these almost complex structures commute with the projections and restricted to the fibers it is the natural complex structure of **CP**. Thus, by a slight abuse of notation $\{\mathcal{M}(\gamma_1,\ldots,\gamma_n), \tilde{J}\}$ is an almost complex structure on an n-step Bott tower. Moreover this almost complex structure is compatible with the left $\star$ action. This follows since the left action on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ lifts to $\mathcal{K}(\gamma_1,\ldots,\gamma_n)$ and $\tilde{J}$ commutes with $\mathrm{Ad}(T)\times\cdots\times\mathrm{Ad}(T)$. Thus we have a complete almost complex action on a Bott tower and we will be able to apply Theorem 1.2 to this case.

One natural question to ask, is whether this almost complex structure is integrable, i.e. does it come from an complex structure. It is not, in general, integrable as is indicated by the appearance +1 and -1 in the extended character formulas.

2.1.10 Coefficient Line bundles

To complete the set up used in section 1. we need to generate "coefficient" line bundles. Define the action of T^n on **C** as follows:

$$t = (t_1,\ldots,t_n) \in T^n, \quad v \in \mathbf{C} \tag{1.50}$$

$$t \cdot v \stackrel{\text{def}}{=} e^{\lambda_1}(t_1)\cdots e^{\lambda_n} v \tag{1.51}$$

where e^{λ_i} are (integral) characters of the torus T. Denote, then, this left T^n module $\mathbf{C}_{\lambda_1}\boxtimes\cdots\boxtimes\mathbf{C}_{\lambda_n}$. Therefore we define the line bundle

$$\mathbf{L}_{\lambda_1,\ldots,\lambda_n} \stackrel{\text{def}}{=} \mathcal{K}(\gamma_1,\ldots,\gamma_n)\times_{T^n}\mathbf{C}_{\lambda_1}\boxtimes\cdots\boxtimes\mathbf{C}_{\lambda_n} \tag{1.52}$$

.

Note that the bundle **L** defined above in Theorem 2.1 $\mathbf{L}\cong\mathbf{L}_{0,\ldots,0,\gamma_n}\oplus\mathbf{L}_{0,\ldots,0}$

2.2 Bott-Samelson varieties and Demazure's Theorem

While the above set up is in the environment of real compact Lie groups, the framework for the Borel-Weil-Bott theorem is that of their associated complex semisimple Lie groups. Our aim here is to relate the index of the operator $D_{\mathbf{L}_\lambda}$ with irreducible highest weight representations.

2.2.1 The complex picture

For simplicity we start again with a simple simply connected compact group K. Let $T \subset K$ be a maximal torus and let $\mathfrak{t}$, and $\mathfrak{k}$ be the Lie algebras of T and K respectively. Complexifying these Lie algebras yield the complex simple Lie algebra $\mathfrak{g} \stackrel{\text{def}}{=} \mathfrak{k} \otimes \mathbf{C}$ and the complex abelian algebra $\mathfrak{h} \stackrel{\text{def}}{=} \mathfrak{t} \otimes \mathbf{C}$. Let G denote the simply connected complex group associated to the Lie algebra $\mathfrak{g}$. Similarly let H denote the complex torus corresponding to the Cartan subalgebra $\mathfrak{h}$. Denote the adjoint action of G on $\mathfrak{g}$ restricted to H, $\mathrm{Ad}(H)$. Under this action there is the root space decomposition $\mathfrak{g} = \sum_{\alpha\in\Delta_+} \mathfrak{g}^{-\alpha} \oplus \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+} \mathfrak{g}^{\alpha}$. Now define the Borel subalgebra $\mathfrak{b} = \sum_{\alpha\in\Delta_+} \mathfrak{g}^{-\alpha} \oplus \mathfrak{h}$. Note that $\mathfrak{b}$ is the direct sum of the Cartan subalgebra and the *negative* root spaces. This sign convention will allow us to work with highest weight representations (alternatively we could define the signs in the other way and work with lowest weight representations.) The corresponding Borel subgroup will be denoted by B.

There is a natural inclusion $K \hookrightarrow G$ and $K \cap B = T$ so this induces a map $K/T \to G/B$ which is in fact a diffeomorphism. Thus, since G/B is a complex manifold, this map gives us a complex structure on K/T.

The root space $\mathfrak{g}^{\alpha}$ for $\alpha \in \Delta$, is trivially a subalgebra. Correspondingly we have the root subgroup, U_α. $R \subset \Delta$ is called a *closed* subset of Δ if $\Delta \cap (\mathbf{N}\alpha + \mathbf{N}\beta) \subset R$ for all $\alpha, \beta \in R$. If $R \subset \Delta$ and $R \cap -R = \emptyset$ then R is called unipotent. Let R be a unipotent set of roots and let U_R denote the subgroup generated by the subgroups $\{U_\alpha\}_{\alpha\in R}$. In particular $B = HU_{-\Delta_+}$

Now suppose $\Sigma \subset \Delta_+$ is the set of simple roots and that $I \subset \Sigma$ is some arbitrary subset. Then if $\mathbf{Z}I$ denotes the lattice defined by I , we define $R_I \stackrel{\text{def}}{=} \Delta \cap \mathbf{Z}I$. Let

$\mathfrak{l}_I \stackrel{\text{def}}{=} \mathfrak{h} \oplus \sum_{\alpha \in R_I} \mathfrak{g}^\alpha$ be the Lie algebra of the Levi factor. The Levi factor L_I as a group, is thus generated by H and the U_α for all $\alpha \in R_I$. The set $R_I^c \stackrel{\text{def}}{=} -\Delta_+ \setminus (-\Delta_+ \cap R_I)$ is unipotent. Finally the parabolic subgroup $P_I \stackrel{\text{def}}{=} L_I U_{R_I^c}$ where $U_{R_I^c}$ is called the unipotent radical of P_I The Borel subgroup is in some sense the minimal parabolic subgroup since $P_\emptyset = B$ and $B \subset P_I$ for all possible I. However since this is perhaps too trivial a case we will call the groups of the form $P_I = P_\alpha$, where $I = \{\alpha\}$, minimal parabolic subgroups.

Note that the inclusion $K_\alpha \hookrightarrow P_\alpha$ induces the isomorphisms $K_\alpha/T \cong P_\alpha/B \cong \mathbf{CP}^1$. We define $\mathcal{P}(\gamma_1, \ldots, \gamma_r) \stackrel{\text{def}}{=} P_{\gamma_1} \times \cdots \times P_{\gamma_r}$. Similar to the right action T^n on $\mathcal{K}(\gamma_1, \ldots, \gamma_n)$ we define a right action of $B^r = B \times \cdots \times B$ on $\mathcal{P}(\gamma_1, \ldots, \gamma_r)$,

$$(p_1, \ldots, p_r) \bullet (b_1, \ldots, b_r) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{r-1}^{-1} p_r b_r) \tag{2.53}$$

where $(p_1, \ldots, p_r) \in \mathcal{P}(\gamma_1, \ldots, \gamma_r)$, and $(b_1, \ldots, b_r) \in B^r$ We have an inclusion map $\mathcal{K}(\gamma_1, \ldots, \gamma_r) \hookrightarrow \mathcal{P}(\gamma_1, \ldots, \gamma_r)$ and the inclusion $T^r \hookrightarrow B^r$ which implement a diffeomorphism on the quotients:

$$\mathcal{M}(\gamma_1, \ldots, \gamma_r) \cong K_{\gamma_1} \times_T \cdots \times_T (K_{\gamma_r}/T) \cong P_{\gamma_1} \times_B \cdots \times_B (P_{\gamma_r}/B) \tag{2.54}$$

This diffeomorphism provides us with the means to put a complex structure on $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$. In fact $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ is a complex projective variety. To see this define a map $\hat{\phi}$

$$\begin{array}{rcl} \hat{\phi} : P_{\gamma_1} \times \cdots \times P_{\gamma_r} & \hookrightarrow & G \times \cdots \times G \cong G^r \\ (p_1, \cdots, p_r) & \mapsto & (p_1, p_1 p_2, \ldots, p_1 \cdots p_r) \end{array} \tag{2.55}$$

If we let B^r act on G^r on the right via componentwise right multiplication and let B^r act on $\mathcal{P}(\gamma_1, \ldots, \gamma_r)$ via 2.53 then this map is equivariant. Hence this yields a map on the quotients, which is an injective map of complex manifolds.

$$\phi : \mathcal{M}(\gamma_1, \ldots, \gamma_r) \hookrightarrow (G/B)^r \tag{2.56}$$

Since G/B is a projective variety this map exhibits $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$ as a projective variety. In fact by identifying the image of ϕ or more particularly of $\hat{\phi}$ one can develop an alternative definition of $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$. Let W be the Weyl group of G, i.e. if

$N(T)$ is the normalizer of the group T, then $W \stackrel{\text{def}}{=} N(T)/T$. If $w \in W$ then $\dot{w}$ will always denote a representative of w in G. Recall the Bruhat decomposition,

$$G = \sum_{w \in W} B\dot{w}B \tag{2.57}$$

with $B\dot{w}B \cap B\dot{w}'B = \emptyset$ when $w \neq w'$ in W. With this notation we identify the image of $\mathcal{P}(\gamma_1, \ldots, \gamma_r)$ as

$$\hat{\phi}(\mathcal{P}(\gamma_1, \ldots, \gamma_r)) = \{(g_1, \ldots, g_r) \in G^r | g_{i-1}^{-1} g_i \in \overline{B\dot{w}_i B}, 1 \leq i \leq r\} \tag{2.58}$$

Here w_i is the reflection with respect to the simple root γ_i and $\overline{B\dot{w}_i B}$ is the closure of $B\dot{w}_i B$ in G. Since $\hat{\phi}$ is 1-1, $\hat{\phi}(\mathcal{P}(\gamma_1, \ldots, \gamma_r))/B^r \cong \mathcal{M}(\gamma_1, \ldots, \gamma_r)$, where the right B^r action is the one inherited from right multiplication in G^r. It should be noted that there is in fact a well defined left B action on $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$. If $b \in B$ and $g_{i-1}^{-1} g_i \in \overline{B\dot{w}_i B}$ then so is $(bg_{i-1})^{-1} bg_i = g_{i-1}^{-1} g_i$. Thus we let B act on the left on G^r by $b \cdot (g_1, \ldots, g_r) = (bg_1, \ldots, bg_r)$, and this preserves $\hat{\phi}(\mathcal{P}(\gamma_1, \ldots, \gamma_r))$. This clearly commute with the left action. Alternately we could act on $\mathcal{P}(\gamma_1, \ldots, \gamma_r)$ by B by multiplying the first component on the left. By the action defined in 2.53 this induces a left action of B on $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$. A very important point is that the left $\star$ action of $\tilde{T}$ does *not* commute with the right action of B^r. This is because B^r is not an abelian group and so unlike in the real compact picture, we so not get a well defined action of an n torus with this complex structure.

2.2.2 Schubert Varieties and Cell Decompositions

Let $\Delta(w) = \{\alpha \in \Delta_+ | -w(\alpha) \in \Delta_+\}$ for $w \in W$. Define $\mathfrak{n}(w) = \sum_{\alpha \in \Delta(w)} \mathfrak{g}^{-\alpha}$ and the corresponding unipotent group $U(w)$. In particular when w_0 is the longest element of W, then $U(w_0) = U$ the unipotent radical of B. If $\gamma \in \Sigma$ is a simple root, and s_γ is the simple reflection in W associated with γ , then $U(s_\gamma) = U_{-\gamma}$. Recall that the Lie algebra of $U_{-\alpha}$ is $\mathfrak{g}^{-\alpha} \cong \mathbf{C}$. We can actually identify $U_{-\alpha}$ and $\mathfrak{g}^{-\alpha}$ via the exponential map. The exponential map sends multiplication in $U_{-\alpha}$ to addition in $\mathbf{C}$. More generally, if $l(w)$ is the length of $w \in W$, the exponential map gives us an identification $U(w) \cong \mathbf{C}^{l(w)}$ but not as groups. Bruhat decomposition induces a cell decomposition of G/B as follows:

$$G/B = \bigcup_{w \in W} B\dot{w}B/B \tag{2.59}$$

Here $U(\dot{w})$ acts freely on the point $\dot{w}B \in G/B$ and $B\dot{w}B/B = U(\dot{w})\dot{w}B/B \cong \mathbf{C}^{l(w)}$. In particular if w_0 is the longest element of the Weyl group then $B\dot{w}_0B/B \cong \mathbf{C}^n$ is an open dense subset of G/B called the "big cell". In a similar manner there is decomposition of $P_\gamma/B \cong \mathbf{CP}^1$ for γ simple. In particular $P_\gamma/B = B/B \cup B\dot{s_\gamma}B/B$. In this case $B\dot{s_\gamma}B/B \cong \mathbf{C}$ is the "big cell." This cell map indicates a method of establishing a cell decomposition of $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$. Suppose that we look at the map

$$\begin{array}{rcl} \hat{f} : \mathbf{C}^r \cong U_{-\gamma_1} \times \cdots \times U_{-\gamma_r} & \hookrightarrow & P_{\gamma_r} \times \cdots \times P_{\gamma_r} \\ (u_1, \ldots, u_r) & \mapsto & (u_1\dot{s}_{\gamma_1}, \ldots, u_r\dot{s}_{\gamma_r}) \end{array} \tag{2.60}$$

Which induces a cell map on the quotient

$$f : U_{-\gamma_1} \times \cdots \times U_{-\gamma_r} \hookrightarrow \mathcal{M}(\gamma_1, \ldots, \gamma_r) \tag{2.61}$$

which is analogous to the big cell. The r, $r-1$ cells are obtained by forming maps such as

$$\begin{array}{rcl} f : U_{-\gamma_1} \times \cdots \times \widehat{U_{-\gamma_i}} \times \cdots \times U_{-\gamma_r} & \hookrightarrow & P_{\gamma_r} \times \cdots \times P_{\gamma_r} \\ (u_1, \ldots, e, \ldots u_r) & \mapsto & (u_1 s_{\gamma_1}, \ldots, e, \ldots, u_r s_{\gamma_r}) \end{array} \tag{2.62}$$

where the hat $\hat{}$ represents omitting the ith factor. To obtain the $r-2$ cells we would omit the ith and jth terms and so forth. In the case of G/B the total number of cells is $|W|$ while $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$ has 2^r cells. Naturally one expects G/B and $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$ to be quite different. We can however establish certain important connections. First suppose we let ϕ_r be the map defined in 2.56 ϕ followed by projection on the last factor.

$$\begin{array}{rcl} \phi_r : \mathcal{M}(\gamma_1, \ldots, \gamma_r) & \rightarrow & G/B \\ (p_1, \ldots, p_r) & \mapsto & p_1 \cdots p_r B \end{array} \tag{2.63}$$

Suppose w_0 is the longest element of the Weyl group. Let $w_0 = w_1 w_2 \ldots w_n$ be a reduced decomposition into simple reflections. In other words $w_1 = s_{\gamma_i}$ where γ_i is some simple root. Then

$$\phi_n(\hat{f}(U_{-\gamma_1} \times \cdots \times U_{-\gamma_r})) = U_{-\gamma_1} w_1 \cdots U_{-\gamma_n} w_n B = Bw_1 \cdots Bw_n B \qquad (2.64)$$

Recall that the Tits properties for reductive (Chevalley) groups tell us that if $s \in \Sigma$, $w \in W$, then $sBw \subset BswB \cup BwB$. Moreover if $l(sw) \geq l(w)$ then $sBw \subset BswB$. Hence 2.64 equals $Bw_1 \cdots w_n B = BwB$, the big cell in G/B. Hence ϕ_n is an isomorphism on a dense open set and hence ϕ_n is a birational map.

2.2.3 A Theorem of Demazure

The fact that there is an isomorphism on a Zariski open set indicates the spaces $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ and G/B are closely related. Let $\mathbf{L} \to G/B$ be a holomorphic line bundle. The result that will be needed later, is that $H^q(G/B, \mathbf{L}) \cong H^q(\mathcal{M}(\gamma_1, \ldots, \gamma_n), \phi_n^* \mathbf{L})$. For positive line bundles this can this can be shown for $q = 0$, with out much difficulty. For $q > 0$ and more general line bundles the proof gets quite involved. The result will be stated with reference but not proven. It should be noted that in this section sheaf cohomology will be used although this will be the same Dolbeault cohomology with coefficients in a holomorphic line bundle by the generalized De Rham theorem.

Let $\mathcal{O}_{\mathcal{M}}$ be the structure sheaf on $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$, in other words the sheaf of holomorphic functions on $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$. Similarly $\mathcal{O}_{G/B}$ be the structure sheaf on G/B. In fact if ϕ_{n*} is interpreted as the pushforward of sheaves, 2.64 is quite nearly equivalent to the statement that $\phi_{n*} \mathcal{O}_{\mathcal{M}} = \mathcal{O}_{G/B}$. In fact it is equivalent since birationality implies that the singular value set is of codimension ≥ 2, which will be reviewed, and the rest will follow by Hartogs's theorem.

Proposition 2.3 *The set of singular values of the map $\phi_n : \mathcal{M}(\gamma_1, \ldots, \gamma_n) \to G/B$ is of codimension ≥ 2.*

Proof. Let $C \subset G/B$ be an irreducible component of the singularity set of the map ϕ_n and suppose C has codimension 1. For $x \in C$ one expects $\dim \phi_n^{-1}(x) \geq 1$. This implies that $\dim \phi_n^{-1}(C) = n$ which would contradict the fact that ϕ_n is an isomorphism on a dense open set. Hence C has codimension 2 or more.■

Suppose $V \subset G/B$ is an open set and suppose $f \in \mathcal{O}_{\mathcal{M}}(V)$ is some holomorphic function defined on V. Let U be the big cell in G/B. Then $\phi_{n*} f$ is well defined on $U \cap V$ since ϕ_n is an isomorphism on $\phi_n^{-1}(U \cap V)$. Since the singular set C has codimension ≥ 2, $\phi_{n*}f$ can be extended uniquely to V, by Hartogs's theorem. This push forward defines a left and a right inverse to the ordinary pullback on the sheaf of holomorphic functions so $\phi_{n*}\ \mathcal{O}_{\mathcal{M}} = \ \mathcal{O}_{G/B}$.

In fact this correspondence is true for sections of line bundles. Suppose $\mathbf{L}$ is a line bundle over G/B. Suppose V is an open set. Since G/B is compact we can find a finite open covering of V $\{U_i\}_{i=1}^k$ such that $\mathbf{L}|_{U_i}$ is trivial. A section $s \in \Gamma(\phi_n^{-1}(V), \phi_n^*\mathbf{L}|_{\phi_n^{-1}(V)})$ hence specifies defined section on $V \setminus (V \cap C)$. This can be extended to $V \cap U_i$ uniquely since $\phi_{n*}\ \mathcal{O}_{\mathcal{M}} = \ \mathcal{O}_{G/B}$. The extension exists and is also unique on overlaps and natural with respect to restriction and thus we obtain a well defined section of $\mathbf{L}|_V$ which we denote $\phi_{n*}s$. This will again be a two sided inverse of the pullback map ϕ_n^*. Hence in particular we have $\Gamma(\ \mathcal{M}(\gamma_1, \ldots, \gamma_n), \phi_n^*\mathbf{L}) \cong \Gamma(G/B, \mathbf{L})$.

In particular suppose $\lambda \in \mathbf{t}^*$ is an integral dominant weight. The integrality guarantees a well defined character $e^\lambda : T \to \mathbf{C}^\times$. This is uniquely extendable to H, the complex torus. $B = UH$ so we can extend e^λ to a character on B by defining $e^\lambda(b) = e^\lambda(h)$, where $b = uh$ with $u \in U$, and $h \in H$. A left action of B on $\mathbf{C}$ can be defined by $e^\lambda(b)a$ for $b \in B$ and $a \in \mathbf{C}$. Denote the resulting module $\mathbf{C}_\lambda$. In this manner define the line bundle $\mathbf{L}_\lambda \stackrel{\text{def}}{=} G \times_B \ \mathbf{C}_\lambda$ and its sheaf of sections $\mathcal{L}(\lambda)$. The above shows that

$$H^0(G/B,\ \mathcal{L}(\lambda)) \cong \Gamma(G/B,\ \mathbf{L}_\lambda) \cong \Gamma(\ \mathcal{M}(\gamma_1, \ldots, \gamma_n),\ \phi_n^*\mathbf{L}_\lambda) \cong\ H^0(G/B,\ \phi_n^*\ \mathcal{L}(\lambda)). \tag{2.65}$$

So far this is just an identification of complex vector spaces. There are, however, left actions of B on G/B and on $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ and the map ϕ_n is equivariant with respect to these actions. Hence 2.65 is an isomorphism of left B modules. Since $H^0(G/B,\ \mathcal{L}(\lambda))$ is also a left G module we can induce a G module structure on $H^0(G/B,\ \phi_n^*\ \mathcal{L}(\lambda))$ via this map. Results along the lines of 2.65 were first announced by Demazure. An error was found in his proof and it was corrected and extended by Anderson, Mehta, Ramanan, Ramanathan and Seshadri. The version quoted here comes from Jantzen's book [J]. The notation has been more or less preserved although Jantzen is working in the context of group schemes. Before the theorem can be stated it is absolutely necessary to introduce more notation which will hardly be used. As mentioned earlier the sets $B\dot{w}B/B$ form a cell decomposition of G/B. While the sets $B\dot{w}B/B$ are affine spaces, their closures in G/B, $X(w) \stackrel{\text{def}}{=} \overline{B\dot{w}B}$ are in general more complicated. In fact they can, in general, be singular. More generally, suppose we have a parabolic subgroup $P = P_I$, as before, with Levi factor L_I. Define the group W_I as the Weyl group of the reductive Lie group L_I. Alternatively we can define it as the subgroup of W generated by the reflections s_α where $\alpha \in R_I$. We now define the variety of partial flags G/P and similar to before, there is a cell decomposition of G/P with cells of the form $B\dot{w}P/P$. Here $w \in W/W_I$. In this case we define the generalized Schubert varieties $X(w)_P$. One may expect that a relation like 2.65 holds for the $X(w)$ and some $\mathcal{M}(\gamma_1, \ldots, \gamma_r)$ and in fact there is the following theorem:

Theorem 2.2 (Demazure et al.)

a) *$X(w)$ is a normal variety*

b) $\phi_{n*}\, \mathcal{O}_{\mathcal{M}_r} = \mathcal{O}_{X(w)_P}$*, and* $R^q \phi_{n*} = 0$

c) *Let $\mathcal{S}$ be locally free sheaf of finite rank on $X(w)$, then there is a natural isomorphism,*

$$H^j(\,X(w)_P,\ \mathcal{S}) \cong H^j(\,\mathcal{M}(\gamma_1, \ldots, \gamma_r), \phi_n^* \,\mathcal{S}) \text{ for all } j \qquad (2.66)$$

d) *Suppose $\lambda \in \mathfrak{t}^*$ is a dominant integral weight, then the restriction map $i :\ H^0(\,\mathcal{M}(\gamma_1, \ldots, \gamma_n),\ \mathcal{L}(\lambda)) \to H^0(\,\mathcal{M}(\gamma_1, \ldots, \gamma_r),\ \mathcal{L}(\lambda))$ is a surjection of B modules and $H^k(\,\mathcal{M}(\gamma_1, \ldots, \gamma_r),\ \mathcal{L}(\lambda)) = 0$ if $j > 0$.*

e) *Suppose $\lambda \in \mathfrak{t}^*$ is a dominant integral weight, and $< \lambda, \alpha^\vee >= 0$ for all $\alpha \in R_I$, then the restriction map $H^0(G/B,\ \mathcal{L}(\lambda)) \to H^0(\,X(w)_P,\ \mathcal{L}(\lambda))$ is a surjection and $H^j(\,X(w)_P,\ \mathcal{L}(\lambda)) = 0$ for $j > 0$.*

Proof. The proof can be found in Jenzen's book [J], it is in Section 14.5 on page 408 although most of the beginning of Chapter 14 develops the necessary Lemmas.

Corollary 2.1 *For λ any weight, $H^j(\,\mathcal{M}(\gamma_1, \ldots, \gamma_n),\ \mathcal{L}(\lambda)) \cong H^j(G/B,\ \mathcal{L}(\lambda))$.*

Proof. This follows directly from part c) of Theorem 2.2 with $P = e$, $w = w_0$ the longest element of the Weyl group, and $\mathcal{S} = \mathcal{L}(\lambda)$.

2.3 Comparing almost complex structures

In Section 2.1 we showed that $\mathcal{M}(\gamma_1,\ldots,\gamma_r)$ has an almost complex structure that is compatible with the action of $\tilde{T}$. The maximal torus $T \subset \tilde{T}$ is compatible with this almost complex structure by restriction. We also have a holomorphic structure on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ defined in Section 2.2 which is compatible with the action of the maximal torus T. What we will show in this section that there is a one parameter family of almost complex structures on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ which are compatible with the action of T. This will be acomplished by explicitly computing the almost complex endomorphism J for both of these structures.

2.3.1 A family of almost complex structures

In the homogeneous case we see two approaches to defining an almost complex structure on K/T which, in the end, yield the same result. One way was to use the fact that $\mathfrak{m} \cong \bar{\mathfrak{n}}$. We moved the complex structure from $\bar{\mathfrak{n}}$ to $\mathfrak{m}$ at [eT] and "pushed it around" on K/T. This might be called the "connection approach." The other way was to use the isomorphism of $K/T \cong G/B$, G/B being a complex manifold. This could be called the homogeneous approach. We saw that because they agree on the tangent space at the identity coset, and because they are invariant under the transitive left action, they agree everywhere. We shall see that these two approaches on $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$, yield different almost complex structures.

Both approaches fit into this principal connection framework. Suppose we have a principal K-bundle $P \rightarrow X$ with base space X. Suppose we have a connection $\mathcal{H}$, ie at a point $p \in P$ we have a splitting

$$T_pP \cong \mathcal{V}_p \oplus \mathcal{H}_p. \tag{3.67}$$

Suppose we can put an almost complex involution J_p on each of the horizontal spaces which commutes with right action. In other words if $T(r_k)$ denotes the differential of right translation by $k \in K$ then we require that $T(r_k)J_p = J_{pk}T(r_k)$. This then gives an almost complex structure on X.

From Section 2.2.1 we realized $\mathcal{M}(\gamma_1,\ldots,\gamma_n)$ as the quotient of a complex group B^n acting freely and holomorphically on the complex manifold $\mathcal{P}(\gamma_1,\ldots,\gamma_n)$. In fact in a complex reinterpretation of Section 2.1.6 $\mathcal{P}(\gamma_1, \ldots, \gamma_n)$ is a principal B^n bundle over $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$. As before we can use left translation to trivialize the tangent space to $\mathcal{P}(\gamma_1, \ldots, \gamma_n)$. Suppose p= $(p_1, \ldots, p_n)$, then

$$T(l_{\mathrm{p}^{-1}}) = T(l_{p_1^{-1}}) \times \cdots \times T(l_{p_n^{-1}}) : T(\mathcal{P}(\gamma_1, \ldots, \gamma_n))_{\mathrm{p}} \rightarrow \mathfrak{p}_{\gamma_1} \oplus \ldots \oplus \mathfrak{p}_{\gamma_n} \tag{3.68}$$

To illustrate the computation of the almost complex structure J coming from the holomorphic structure, we should suppose we only have two factors: $\mathcal{M}_{\gamma_1,\gamma_2}$. Recall that we have the principal bundle $P_{\gamma_1} \times P_{\gamma_2} \rightarrow \mathcal{M}_{\gamma_1,\gamma_2}$. This trivializes using left multiplication, just as in the real case:

$$T(l_{\mathrm{p}^{-1}}) = T(l_{p_1^{-1}}) \times T(l_{p_2^{-1}}) : \mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2} \rightarrow T_{\mathrm{p}}(P_{\gamma_1} \times P_{\gamma_2}) \tag{3.69}$$

with $\mathrm{p} = (p_1, p_2) \in P_{\gamma_1} \times P_{\gamma_2}$. The vertical space of this bundle is also very similar to the real case and the computation is identical. Fix $\mathrm{p} \in P_{\gamma_1} \times P_{\gamma_2}$ and abusing notation let Υ_{p} be defined

$$\begin{array}{rcl} \Upsilon_{\mathrm{p}} : P_{\gamma_1} \times P_{\gamma_2} \times \mathfrak{b} \oplus \mathfrak{b} & \rightarrow & P_{\gamma_1} \times P_{\gamma_2} \times \mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2} \\ ((p_1, p_2), (\underline{b_1}, \underline{b_2})) & \mapsto & ((p_1, p_2), (\underline{b_1}, \underline{b_2} - Ad(p_2^{-1})\, \underline{b_1})) \end{array} \tag{3.70}$$

Then the vertical space $\mathcal{V}_{\mathrm{p}} = \Upsilon_{\mathrm{p}}(\mathfrak{b} \oplus \mathfrak{b})$ and the following is an exact sequence of complex bundles.

$$0 \rightarrow P_{\gamma_1} \times P_{\gamma_2} \times \mathfrak{b} \oplus \mathfrak{b} \xrightarrow{\Upsilon} P_{\gamma_1} \times P_{\gamma_2} \times \mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2} \rightarrow \pi^* T(\mathcal{M}_{\gamma_1,\gamma_2}) \rightarrow 0 \tag{3.71}$$

The inclusion of <u>real</u> vector spaces $\mathfrak{m}_{\gamma_1} \oplus \mathfrak{m}_{\gamma_2} \hookrightarrow \mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2}$, induces an inclusion of bundles $\mathcal{H} \hookrightarrow T(P_{\gamma_1} \times P_{\gamma_2})$. Here as before $\mathcal{H} \stackrel{\mathrm{def}}{=} P_{\gamma_1} \times P_{\gamma_2} \times \mathfrak{m}_{\gamma_1} \oplus \mathfrak{m}_{\gamma_2}$ Now the same argument used to show that $\mathcal{V}_{\mathrm{p}} \cap \mathcal{H}_{\mathrm{p}} = 0$ in the real case works here too. One uses the fact that $\mathfrak{k} \cap \mathfrak{b} = \mathfrak{t}$ and that $\mathfrak{t} \cap \mathfrak{m} = 0$ and proceeds exactly as before. Thus the inclusion $\mathcal{H}_{\mathrm{p}} \hookrightarrow \mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2}$, induces an (real) isomorphism $\mathcal{H}_{\mathrm{p}} \cong (\mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2} / \mathcal{V}_{\mathrm{p}})$. It is though this isomorphism that we give $\mathcal{H}_{\mathrm{p}}$ a complex structure. Suppose that $\underline{\mathrm{p}} \in \mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2}$ To compute J_{p} first multiply $\underline{\mathrm{p}}$ by

i in $\mathfrak{p}_{\gamma_1} \oplus \mathfrak{p}_{\gamma_2}$ then project back on to $\mathcal{H}_{\mathrm{p}}$ along $\mathcal{V}_{\mathrm{p}}$. In particular suppose that $\underline{\mathrm{p}} = (\underline{\mathrm{p_1}}, 0)$. Suppose also that $\mathfrak{p}_{\gamma_i} = \mathfrak{g}^{\gamma_i} \oplus \mathfrak{b}$. We define a projection operator $\eta^+_{\gamma_i} : \mathfrak{p}_i \rightarrow \mathfrak{g}^{\gamma_i}$ with kernel $\mathfrak{b}$. Further we define the projection maps $C_i \stackrel{\text{def}}{=} \frac{1}{2}(\eta^+_{\gamma_i} + \sigma\eta^+_{\gamma_i})$, so $C_i : \mathfrak{p}_i \rightarrow \mathfrak{m}_{\gamma_i}$. Also let $D_i \stackrel{\text{def}}{=} (1 - \frac{1}{2}\eta^+_{\gamma_i}) - \frac{1}{2}\sigma\eta^+_{\gamma_i}$ with $D_i : \mathfrak{p}_i \rightarrow \mathfrak{b}$. With $C_i + D_i = I =$ Identity. With this notation,

$$(i\ \underline{\mathrm{p_1}}, 0) = (C_1 i\ \underline{\mathrm{p_1}}, 0) + (D_1 i\ \underline{\mathrm{p_1}}, 0). \tag{3.72}$$

While $(C_1 i\ \underline{\mathrm{p_1}}, 0) \in \mathfrak{m}_{\gamma_1} \oplus \mathfrak{m}_{\gamma_2}$ we need the second term to be of the form $\Upsilon_{\mathrm{p}}(\underline{\mathrm{b_1}}, \underline{\mathrm{b_2}})$ with $(\underline{\mathrm{b_1}}, \underline{\mathrm{b_2}}) \in \mathfrak{b} \times \mathfrak{b}$. The first component must be $\underline{\mathrm{b_1}} = D_1 i\ \underline{\mathrm{p_1}}$ so we need to find a $\underline{\mathrm{b_2}}$ and a $v \in \mathfrak{m}_{\gamma_2}$ such that

$$\begin{aligned}
(i\ \underline{\mathrm{p_1}}, 0) &= (C_1 i\ \underline{\mathrm{p_1}}, v) + \Upsilon_{\mathrm{p}}(\underline{\mathrm{b_1}}, \underline{\mathrm{b_2}}) \\
&= (C_1 i\ \underline{\mathrm{p_1}} + D_1 i\ \underline{\mathrm{p_1}}, v + \underline{\mathrm{b_2}} - \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}}) \\
&= (i\ \underline{\mathrm{p_1}}, v + \underline{\mathrm{b_2}} - \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}})
\end{aligned} \tag{3.73}$$

from which we glean that $v + \underline{\mathrm{b_2}} = \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}}$. Since v and $\underline{\mathrm{b_2}}$ are in complimentary spaces we conclude that $v = C_2\ \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}}$ and $\underline{\mathrm{b_2}} = D_2\ \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}}$. So

$$(i\ \underline{\mathrm{p_1}}, 0) = (C_1 i\ \underline{\mathrm{p_1}}, C_2\ \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}}) + (D_1 i\ \underline{\mathrm{p_1}}, D_2\ \mathrm{Ad}(p_2^{-1}) D_1 i\ \underline{\mathrm{p_1}}) \tag{3.74}$$

If, on the other hand $\underline{\mathrm{p}} = (0, \underline{\mathrm{p_2}})$ then

$$\begin{aligned}
(0, \underline{\mathrm{p_2}}) &= (0, C_2 i\ \underline{\mathrm{p_2}}) + (0, D_2 i\ \underline{\mathrm{p_2}}) \\
&= (0, C_2 i\ \underline{\mathrm{p_2}}) + \Upsilon_{\mathrm{p}}(0, D_2 i\ \underline{\mathrm{p_2}})
\end{aligned} \tag{3.75}$$

J_p becomes a matrix of operators,

$$J_p = \begin{pmatrix} C_1 i & 0 \\ C_2\ \mathrm{Ad}(p_2^{-1}) D_1 i & C_2 i \end{pmatrix} \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix} \tag{3.76}$$

In the basis defined in equation 1.32, $C_j i e_j = f_j$ and $C_j i f_j = -e_j$ just as in equation 1.33. So

$$J_{jj} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.77}$$

In the general case the form of J with $p \in \mathcal{P}(\gamma_1, \dots, \gamma_n)$ is

$$J = \begin{pmatrix} J_{11} & & 0 \\ \vdots & \ddots & \\ J_{n1} & \cdots & J_{nn} \end{pmatrix} \tag{3.78}$$

with

$$J_{jj} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.79}$$

and more generally, for $i > j$,

$$J_{ij} = C_i \operatorname{Ad}(p_i^{-1}) D_{i-1} \cdots \operatorname{Ad}(p_{j+1}^{-1}) D_j. \tag{3.80}$$

Theorem 2.3 *Let J_ϵ be defined as*

$$J_\epsilon = \begin{pmatrix} J_{11} & & 0 \\ \vdots & \ddots & \\ \epsilon^{n-1} J_{n1} & \cdots & J_{nn} \end{pmatrix} \tag{3.81}$$

with $(J_\epsilon)_{ij} \stackrel{\text{def}}{=} \epsilon^{i-j} J_{ij}$. J_ϵ has four important properties:

1) J_ϵ is right T^n equivariant and hence is a well defined endomorphism on $T(\mathcal{M}(\gamma_1, \dots, \gamma_n))$.

2) J_ϵ is left T equivariant on $T(\mathcal{M}(\gamma_1, \dots, \gamma_n))$.

3) J_ϵ is an almost complex structure for $\epsilon \in \mathbf{R}$.

4) $J_0 = \tilde{J}$ and $J_1 = J$.

Proof. Once we have shown these four properties we will have shown that there is an equivariant homotopy of almost complex structures, through equivariant almost complex structures. Since $T(\mathcal{M}(\gamma_1, \dots, \gamma_n))$ is a direct sum of two-plane bundles, the right T^n invariance and the left T equivariance of J implies the same for J_{ij} and hence for $\epsilon^{i-j} J_{ij}$. This takes care of 1) and 2). For 3) we must show that ${J_\epsilon}^2 = -I$, (with I the identity.) $J^2 = -I$ since it comes from a complex structure. So if $i > j$ then

$$0 = (J^2)_{ij} = \sum_{k=1+j}^{i} J_{ik} J_{kj} \tag{3.82}$$

so it follows that

$$(J\epsilon^2)_{ij} = \sum_{k=1+j}^{i} J_{ik}\epsilon^{i-k} J_{kj}\epsilon^{k-j} = \epsilon^{i-j}(\sum_{k=1+j}^{i} J_{ik} J_{kj}) = 0. \tag{3.83}$$

On the other hand $(J\epsilon^2)_{ii} = (J^2)_{ii} = (J_{ii})^2 = -I\ 4)$ is essentially by definition for J_1 and by inspection for J_0. ■

2.4 Definition and Computation of the Extended Character

In this section the extended character of a finite dimensional representation of a compact Lie group will be defined. Technically this character depends on a reduced expression, however it does not seem to actually depend on the reduced expression and so this will be surpressed. It the extended character can be computed via the Atiyah-Bott fixed point formula and some formulas will be given.

2.4.1 Definition of the Extended Character

Suppose we have a compact Lie group K and an irreducible finite dimensional representation of K with highest weight λ . First suppose w_0 is the longest element of the Weyl group W and that $w_0 = w_1 w_2 \cdots w_n$ is a reduced expression of the longest element w_0, in terms of simple reflections. Then there is a Bott-Samuelson manifold $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ associated to this expression with a line bundle $\mathbf{L}$ associated to the weight λ . This manifold also has an almost complex structure preserved by the left action of an n-dimensional torus $\tilde{T}$ which naturally contains T a maximal torus in K. With a choice of equivariant connection as spelled out in Section 1.1.3 , we construct a twisted Dolbeault operator $\tilde{D}_{\mathbf{L}}$ which is $\tilde{T}$ equivariant. With this notation the extended character $\mathbf{ech}(\lambda)$, is defined as

$$\mathrm{ech}(\lambda) \stackrel{\mathrm{def}}{=} ind_{\tilde{T}}(\tilde{D}_{\mathbf{L}}). \tag{4.84}$$

A priori ech(λ) is dependent on a choice of connection, a reduced expression of the longest element of the Weyl group. While $\tilde{D}_{\mathbf{L}}$ is dependent on a choice of connection, the G index theorem tells us its index is not. While it has not been proved here that $ech(\lambda)$ is independent of the reduced expression, there is certainly some evidence that it is only dependent on a choice of positive Weyl chamber. It shall be written here as if this is the case.The interest in ech(λ) is due to the following theorem.

Theorem 2.4 *Assume λ is a dominant weight. The extended character $ech(\lambda)$ has simple multiplicity and its restriction to the maximal torus T,*

$$ech(\lambda)|_T = ch(\lambda) \tag{4.85}$$

where $ch(\lambda)$ is the ordinary character of the highest weight representation associated to λ , when restricted to the maximal torus T.

Proof. By Theorem 2.1 we know that the $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ is a Bott tower with torus action. Define the line bundle $\mathbf{L}_\lambda$ as

$$\begin{array}{rcl} \mathbf{L}_\lambda & = & \mathcal{K}(\gamma_1, \ldots, \gamma_n) \times_{T^n} \mathbf{C}_0 \boxtimes \cdots \boxtimes \mathbf{C}0 \boxtimes \mathbf{C}_\lambda \\ & & \downarrow \\ & & \mathcal{M}(\gamma_1, \ldots, \gamma_n) \end{array} \tag{4.86}$$

Further we showed in Section 2.1.9 that this Bott tower had an almost complex structure compatible with the complete torus action. Applying Theorem 1.2 we conclude that ech(λ) has simple multiplicity. To prove equation 4.85 we recall that by Theorem 2.3 there is a family of T-equivariant almost complex structures J_ϵ with $J_0 = J$, the almost complex structure associated with the holomorphic structure, and $J_1 = \tilde{J}$ the almost complex structure used to define $\tilde{D}$. We can thus form the operators $(\not{D}_{\mathbf{L}})_\epsilon$, the twisted Dolbeault operators associated to the almost complex structures J_ϵ . These operators will all be T-equivariant and $(D_{\mathbf{L}})_0 = \tilde{D}_{\mathbf{L}}$, which is $\tilde{T}$ equivariant. Hence $\mathrm{ind}_T((D_{\mathbf{L}})_0) = \mathrm{ind}_T((\tilde{D}_{\mathbf{L}}) = ech(\lambda)_T$ by the naturality of the G-index. The G-index of an operator is a homotopy invariant [A-S1] so $ech(\lambda)_T = \mathrm{ind}_T((D_{\mathbf{L}})_0) = \mathrm{ind}_T((D_{\mathbf{L}})_1)$. Since T acts holomorphically on $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$ and as discussed in Section 2.2.3 acts on the cohomology spaces as an endomorphism of vector spaces

$$T_g^j : H^j(\mathcal{M}(\gamma_1, \ldots, \gamma_n), \mathbf{L}_\lambda) \rightarrow H^j(\mathcal{M}(\gamma_1, \ldots, \gamma_n), \mathbf{L}_\lambda) \tag{4.87}$$

with $g \in T$. The Lefschetz number $N(g)$ of this endomorphism T_g is defined

$$N(g) \stackrel{\mathrm{def}}{=} \sum_j traceT_g^j. \tag{4.88}$$

When $g = e$, the identity element then $N(e)$ is just the integer valued index of the elliptic complex. As g varies in T, then $N(g)$ is a character on T. In fact by definition the character $N(g)$ is the T-index of of the Dolbeault complex with coefficients in a holomorphic line bundle. Now the "rolled up" Dolbeault complex two step with an operator which coincides with the Dolbeault twisted by the line bundle $\mathbf{L}_\lambda$ associated with the almost complex structure that comes form the holomorphic structure. Hence we have that $N(g) = \mathrm{ind}_T((\ D_{\mathbf{L}}\ )_1)(g)$. Demazure theorem tells us that in fact $H^j(\ \mathcal{M}(\gamma_1,\ldots,\gamma_n),\ \mathbf{L}_\lambda) \cong H^j(G/B,\ \phi_n^*\ \mathcal{L}(\lambda))$. Now since we choose a dominant weight λ , we know by Borel-Weil that $H^0(G/B,\ \phi_n^*\ \mathcal{L}(\lambda))$ is the irreducible module with highest weight λ and that $H^j(G/B,\ \phi_n^*\ \mathcal{L}(\lambda)) = 0$ for $j > 0$. Thus we find that $ch(\lambda)(g) = N(g) = ech(\lambda)_T(g)$.

2.4.2 The Atiyah-Bott Lefschetz formula

The original proof that the extended character has simple multiplicity was found by studying the form of the character as computed by the Atiyah-Bott fixed point formula. In fact studying the weight in a fiber of the line bundle over a fixed point, as well as the normal exponents, will yield some curious structures. First we recall the theorem of Atiyah and Bott:

Theorem 2.5 (Atiyah and Bott) *Let $\Gamma(E)$ be an elliptic complex over a compact manifold M. Suppose $\rho : M \to M$ is an endomorphism of M whose graph in $M \times M$ is transversal to the diagonal $\Delta : M \to M \times M$. Further let T be an endomorphism of $\Gamma(E)$ induced by a lifting τ of ρ , then if $L(T)$ is the Lefschetz number then*

$$L(T) = \sum_{p \in P} \nu(p) \tag{4.89}$$

Where P is the fixed point set and

$$\nu(p) = \frac{\sum_k (-1)^k trace\ \tau_p^k}{|\det(1 - d\rho_p)|} \tag{4.90}$$

Now more specifically let us assume that M is an almost complex manifold with almost complex structure J. Assume that there is a diffeomorphism $\rho : M \to M$ which preserves J. Suppose we have a line bundle $\mathbf{L} \to M$, and a lifting $\varphi : \rho^*\mathbf{L} \to \mathbf{L}$, of ρ, and a connection $\nabla_{\mathbf{L}}$ which is preserved by the composition $\varphi \circ \rho^*$. As before, then, we can form the operator $D_{\mathbf{L}}$ and there will be induced complex linear automorphisms A^+ on $\ker D_{\mathbf{L}}$ and A^- on $\mathrm{coker}\ D_{\mathbf{L}}$. The Lefschetz number in this case is

$$L(\rho,\varphi) = \text{trace}_{\mathbf{C}}\ A^{+} - \text{trace}_{\mathbf{C}}\ A^{-}. \tag{4.91}$$

Theorem 2.6 *Suppose ρ has isolated fixed points. Denote the Lefschetz number $L(\rho,\varphi)$ given by the action on ker $D_{\mathbf{L}}$ and coker $D_{\mathbf{L}}$. Then*

$$L(\rho,\varphi) = \sum_{p} \frac{\varphi_p}{det_{\mathbf{C}}(1 - T(\rho)')}. \tag{4.92}$$

Proof. As in Section 1.1.2 we can lift ρ to $TM \rightarrow TM$ by $T(\rho)$, and to $T^{*}M\ \rho T^{*}M$ by $T^{*}(\rho)$, the transpose. Further because ρ preserves J, when we extend $T(\rho)$ to $T(\rho)^{*} \otimes 1$ on $T^{*}M \otimes\ \mathbf{C}$, $T(\rho)^{*} \otimes 1 = T(\rho)^{*\prime} \oplus T(\rho)^{*\prime\prime}$ with

$$T(\rho)^{*\prime\prime} : (T^{*}M)^{0,1} \rightarrow (T^{*}M)^{0,1} \tag{4.93}$$

and on the $(0,q)$ forms,

$$\Lambda^{q} T(\rho)^{*\prime\prime} : \Lambda^{0,q} T^{*}M \rightarrow \Lambda^{0,q} T^{*}M. \tag{4.94}$$

Then we have a lifting of ρ to $\Lambda^{0,q}T^{*}M \otimes \mathbf{L}$ via $\Lambda^{q}T(\rho)^{*\prime\prime} \otimes \varphi$. The Atiyah-Bott formula tell us that at a fixed point p

$$\nu(p) = \frac{\sum_k (-1)^k \text{trace}_{\mathbf{C}}\ (\Lambda^k T_p(\rho)^{*\prime\prime} \otimes \varphi)}{|\det_{\mathbf{R}}(1 - T(\rho)_p)|}. \tag{4.95}$$

If A is an endomorphism of a finite dimensional vector space V, $\Lambda^k A$ its kth exterior power and A^{*} its transpose we have

$$\sum (-1)^k \text{trace}(\Lambda^k A) = \det(1 - A) \tag{4.96}$$

and

$$\det(A^{*}) = \det(A). \tag{4.97}$$

From this, and the fact that φ_p is just a complex number,

$$\nu(p) = \varphi_p \frac{\det_{\mathbf{C}}\ (1 - T_p(\rho)^{*\prime\prime})}{|\det_{\mathbf{R}}(1 - T(\rho)_p)|}. \tag{4.98}$$

Let V be a complex vector space thought of as a real vector space with an anti-involution J, $J^2 = -I$. Suppose A is an endomorphism of V which commutes with J. If we complexify $V \otimes \mathbf{C} = V^{1,0} \oplus V^{0,1}$ then $A \otimes 1 = A' \oplus A''$ preserves this splitting. Complex conjugation provides a real isomorphism $V^{1,0} \cong V^{0,1}$ and commutes with $A \otimes 1$ on $V \otimes \mathbf{C}$. It follows that

$$\det_{\mathbf{C}}(1 - A') = \overline{\det_{\mathbf{C}}(1 - A'')}. \tag{4.99}$$

We also have that

$$\begin{aligned} \det_{\mathbf{R}}(1 - A) &= \det_{\mathbf{C}}(1 - 1 \otimes A) & (4.100) \\ &= \det_{\mathbf{C}}(1 - A')\det_{\mathbf{C}}(1 - A') & (4.101) \end{aligned}$$

which is thus positive. Putting these facts together we get finally that

$$\begin{aligned} \nu(p) &= \varphi_p \frac{\det_{\mathbf{C}}\ (1 - T_p(\rho)^{*\prime\prime})}{\det_{\mathbf{C}}(1 - T(\rho)')\det_{\mathbf{C}}(1 - T(\rho)'')} & (4.102) \\ &= \frac{\varphi_p}{\det_{\mathbf{C}}(1 - T(\rho)')} & (4.103) \end{aligned}$$

■

2.4.3 Computation of the Extended Character

It is illustrative to apply the fixed point formula more explicitly. For clarity the computation of the fixed point information on a 2 factor Bott-Samelson variety and the more general formulas given. With the notation used in Section 2.1.9. let $\mathbf{L}_{\lambda_1,\lambda_2}$ be a line bundle over $\mathcal{M}_{\gamma_1,\gamma_2}$. Suppose $[(k_1, k_2)] \in \mathcal{M}_{\gamma_1,\gamma_2}$ is a fixed point. We want to compute how much the fiber over $[(k_1, k_2)]$ rotates as we act on the left. Suppose $(s_1, s_2) \in T \times T$ is an element of the torus that acts on $K_{\gamma_1} \times K_{\gamma_2}$ on the right. Let $(t_1, t_2) \in \tilde{T}$ be an element of the 2-torus which acts on the left. At the fixed point we have

$$\begin{gathered} (t_1 k_1, t_2 t_1^{-1} k_2) = (k_1 s_1, s_1^{-1} k_2 s_2) \\ s_1 = k_1^{-1} t_1 k_1 \\ s_2 = k_2^{-1} k_1^{-1} t_1 k_1 t_1^{-1} t_2 k_2 \end{gathered} \tag{4.104}$$

Recall that $\tilde{T} = T/T_{\gamma_1} \times T/T_{\gamma_2}$. The killing form induces a metric $(,)$ on $\mathfrak{t}^*$. Let $\underline{t_1} \in \mathfrak{t}$ be the vector such that $\beta(\underline{t_1}) = 0$ for all $\beta \in \mathfrak{t}^*$ such that $(\beta, \gamma_1) = 0$ and $\gamma_1(\underline{t_1}) = 2$. Thus $\underline{t_1}$ is an infinitesimal generator of T/T_{γ_1}. Similarly define $\underline{t_2}$ to obtain a basis for $\tilde{\mathfrak{t}}$. Recall also that the fixed points of the left $\tilde{T}$ are $\{(e,e), (\dot{w}_1, e), (e, \dot{w}_2), (\dot{w}_1, \dot{w}_2), \}$ with $\dot{w}_i$ a representative of the non trivial element of the Weyl group of K_{γ_i}. There is a bijection $\kappa : C^2 = \{0,1\} \times \{0,1\} \rightarrow$ *fixed points,*

$$\begin{aligned} \kappa(0,0) &= (e,e) \qquad \kappa(0,1) = (e, \dot{w}_2) \\ \kappa(1,0) &= (\dot{w}_1, e) \quad \kappa(1,1) = (\dot{w}_1, \dot{w}_2). \end{aligned} \tag{4.105}$$

Denote by s_i the reflection in $\mathfrak{t}$ associated to the root γ_i. Let $s_i^{m_i} =$ *identity* if $m_i = 0$ and $s_i^{m_i} = s_i$ if $m_i = 1$. Thus if z_1 and z_2 are coordinates on $\tilde{T}$, the weight in the fiber of $\mathbf{L}_{\lambda_1, \lambda_2}$ at the fixed point $\kappa(m_1, m_2)$ is $z_1^{y_1} z_2^{y_2}$ where

$$\begin{aligned} y_1 &= \lambda_1(s_1^{m_1}\, \underline{t_1}) + \lambda_2(s_2^{m_2}(s_1^{m_1}\, \underline{t_1} - \underline{t_1}) \\ y_2 &= \lambda_2(s_2^{m_2}\, \underline{t_2}) \end{aligned} \tag{4.106}$$

In general the formula for a line bundle $\mathbf{L}_{\lambda_1, \ldots \lambda_2}$ over a Bott-Samelson manifold $\mathcal{M}(\gamma_1, \ldots, \gamma_n)$, is

$$\begin{aligned} y_1 &= \lambda_1(s_1^{m_1}\, \underline{t_1}) + \lambda_2(s_2^{m_2}(s_1^{m^1}\, \underline{t_1} - \underline{t_1}) + \cdots + \lambda_n(s_n^{m_n} \cdots s_2^{m_2}(s_1^{m^1}\, \underline{t_1} - \underline{t_1}) \\ y_2 &= \lambda_2(s_2^{m_2}\, \underline{t_2}) + \lambda_3(s_3^{m_3}(s_2^{m^2}\, \underline{t_2} - \underline{t_2}) + \cdots + \lambda_n(s_n^{m_n} \cdots s_3^{m_3}(s_2^{m^2}\, \underline{t_2} - \underline{t_2}) \\ &\vdots \\ y_n &= \lambda_n(s_n^{m_n}\, \underline{t_n}) \end{aligned} \tag{4.107}$$

We are most interested in those line bundles which are related to representations. Recall that those line bundles which are "pull-backs" of line bundles on flag manifolds (or Schubert varieties) are of the form $\mathbf{L}_{0,\ldots,\lambda}$, so the above formula becomes

$$\begin{aligned} y_1 &= \lambda(s_n^{m_n} \cdots s_2^{m_2}(s_1^{m^1}\, \underline{t_1} - \underline{t_1}) \\ y_2 &= \lambda(s_n^{m_n} \cdots s_3^{m_3}(s_2^{m^2}\, \underline{t_2} - \underline{t_2}) \\ &\vdots \\ y_n &= \lambda(s_n^{m_n}\, \underline{t_n}) \end{aligned} \tag{4.108}$$

To compute the "normal exponents" i.e. the weights of the isotropy representation of the torus at the fixed point, we recall that

$$T(\mathcal{M}(\gamma_1,\ldots,\gamma_n)) \cong \mathcal{K}(\gamma_1,\ldots,\gamma_n) \times_{T^n} \mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n} \tag{4.109}$$

and note that

$$\mathcal{K}(\gamma_1,\ldots,\gamma_n) \times_{T^n} \mathfrak{m}_{\gamma_1} \oplus \ldots \oplus \mathfrak{m}_{\gamma_n} = \mathbf{L}_{\gamma_1,0,\ldots,0} \oplus \mathbf{L}_{0,\gamma_2,\ldots,0} \oplus \cdots \oplus \mathbf{L}_{0,\ldots,0,\gamma_n}. \tag{4.110}$$

The weights, then, at $T_p(\mathcal{M}(\gamma_1,\ldots,\gamma_n))$ with $p = \kappa(m_1,\ldots,m_n)$ are $\{z_1^{x_1^1}\cdots z_n^{x_n^1},\ldots,z_1^{x_1^n}\cdots z_n^{x_n^n}\}$ where

$$\begin{pmatrix} x_1^1 = \gamma_1(s_1^{m_1}\,\underline{t_1}\,) & x_1^2 = \gamma_2(s_2^{m_2}(s_1^{m_1}\,\underline{t_1} - \underline{t_1}\,)) & \cdots & x_1^n = \gamma_n(s_n^{m_n}\cdots s_2^{m_2}(s_1^{m_1}\,\underline{t_1} - \underline{t_1}\,)) \\ x_2^1 = 0 & x_2^2 = \gamma_2(s_2^{m_2}\,\underline{t_2}\,) & \cdots & x_2^n = \gamma_n(s_n^{m_n}\cdots s_2^{m_3}(s_2^{m_2}\,\underline{t_2} - \underline{t_2}\,)) \\ \vdots & \vdots & & \vdots \\ x_n^1 = 0 & x_n^2 = 0 & \cdots & x_n^n = \gamma_n(s_n^{m_n}\,\underline{t_n}\,) \end{pmatrix}$$

2.4.4 Expanding the terms

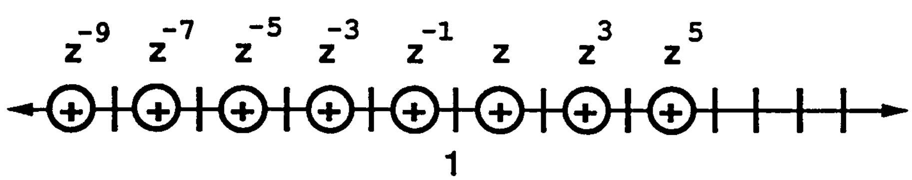
Suppose $n = 1$. In fact suppose $K = SU(2)$, $T = S^1$ and $\lambda(\,\underline{t_1}\; = k =$ Chern number of $\mathbf{L}_\lambda =$ - the character of the fiber at the "north pole," (see page for notation.) Then the fixed point formula reduces to

$$(\mathrm{ch}(\lambda)) = \mathrm{ech}(\lambda) = \frac{z^k}{1+z^{-2}} + \frac{z^{-k}}{1+z^2} \tag{4.111}$$

Expanding the first term about infinity, $(|\, z\, |> 0)$ yields

$$z^k(1 + z^{-2} + z^{-4} + \cdots) = z^k + z^{k-2} + z^{k-4} + \cdots. \tag{4.112}$$

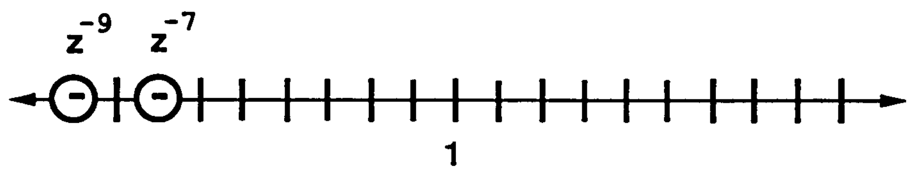
In the notation of the last section $k = y$. Let us think of y as a position vector or origin. In the same spirit $x = -2$ is the generator of a cone in the weight lattice $(2\pi i)\mathbf{Z} \subset i\mathbf{R}$:



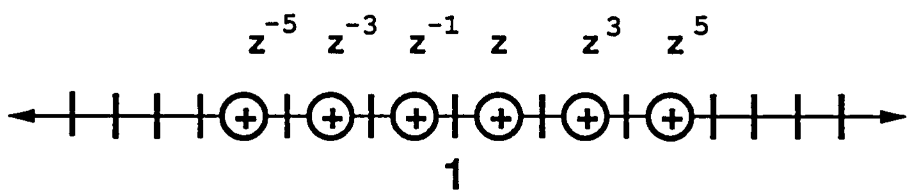
Expanding the second term of 4.111 also about infinity we obtain

$$z^{-k}(-z^{-2} - z^{-4} - \cdots) = -z^{k-2} - z^{k-4} - z^{k-6} - \qquad (4.113)$$

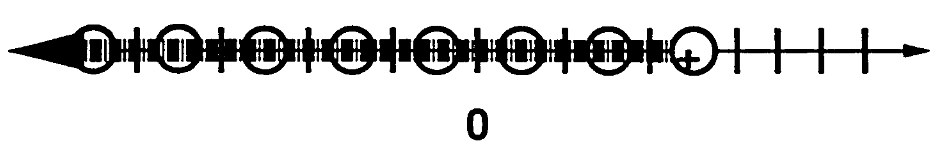
with corresponding picture

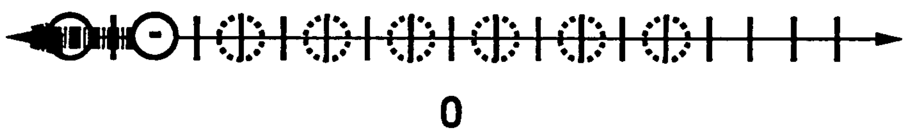


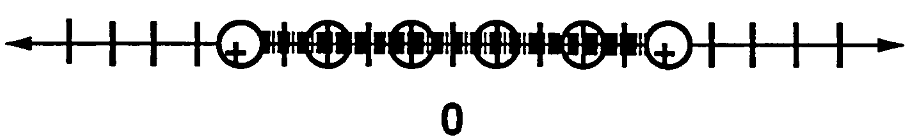
Here we would like to think of the dots as having negative multiplicity. One way to be more rigorous about this is to think of each weight u in the lattice as being a measure $\pm\delta_u$, with support the point u. When we sum the two above terms we get the pictures



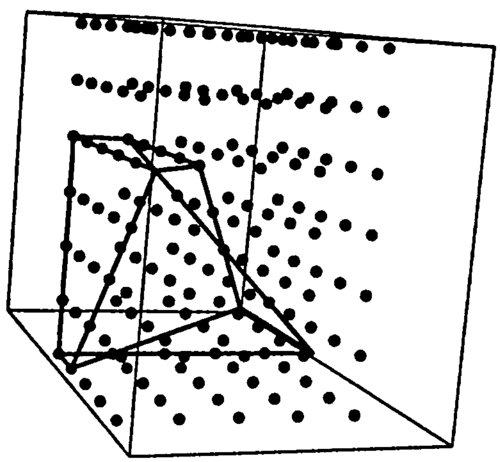
If we shrink the size of our lattice and take larger and larger highest weights we asymptotically approach continuous measure which we can rescale to be Lebesgue measure. The "continuous" version of our three pictures is then

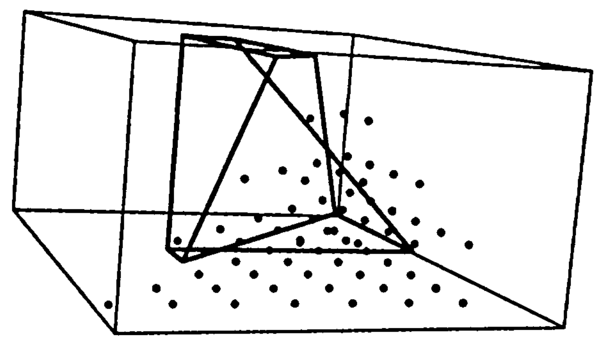


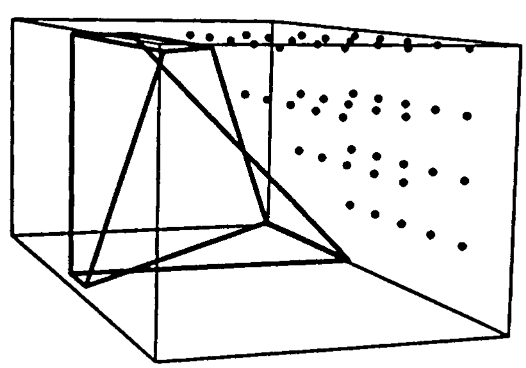

0


0

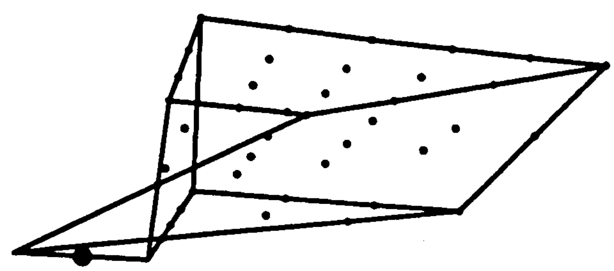
This can, in fact be made more rigorous by introducing some symplectic geometry. The intention here is to be more pictorial than precise. Moving to a much more non-trivial example, assume $K = SU(3)$. In this case there are two simple roots γ_1 and γ_2. A reduced expression for the longest element is $w_1 w_2 w_1$. The Bott-Samelson manifold $\mathcal{M}(\gamma_1, \gamma_2, \gamma_1)$ has 8 fixed points. Here are the weights plotted for the expansion of three of the fixed points:



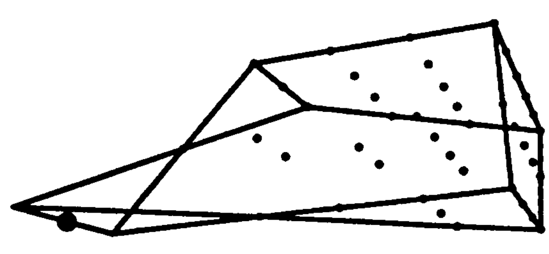




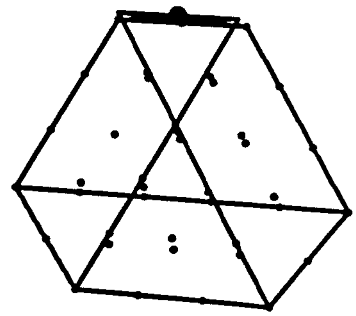
The rectangular box is generated by the drawing program and shows the limits of the plot. The strange shape is the image of a "moment map" associated to the line bundle, connection, manifold and action. The reason for the quotes is that in this context we must work with a degenerate symplectic form. If we sum all the contributions of the fixed points we can again see that there is a strong relation between the weights and the moment map:



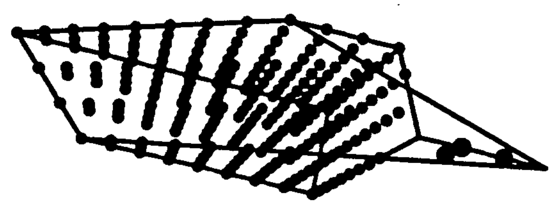
The bold dots above represent -1 multiplicity while the fainter ones represent +1 multiplicity weights. Here is another view from a different angle:

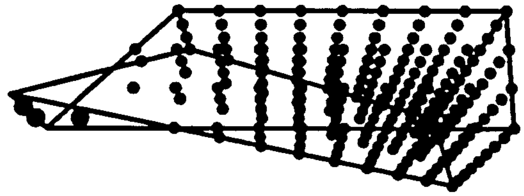


The restriction from the extended torus to the maximal torus yields a projection in the dual Lie algebras. This is the same picture viewed along the plane of projection:



Assymtotically the weight picture approximates this moment map, as we can see in the following two pictures:

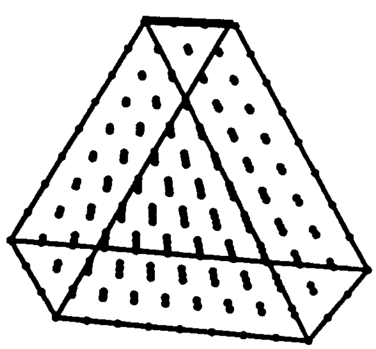


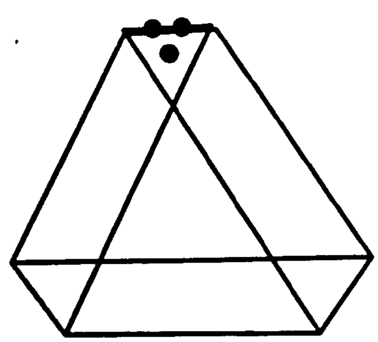


The image of this "moment map" resembles a twisted cube. This is an apt name since we can actually consider it the image of a cube. If we consider the y_i of equations 4.108, as functions on the cube $C^n = \{0,1\} \times \cdots \times \{0,1\}$. We can thus define a map

$$Y : C^n \rightarrow \tilde{\mathfrak{t}} \tag{4.114}$$

This twisted cube has some interesting proporties which there is unfortunately no time expand upon here. It was in fact by studying this shape that an early version of the simple multiplicity theorem was uncovered. Moreover by looking closely one can see a region of positive weights (those with +1 multiplicity) that remain after canceling the negitive weights. These region corresponds, in this example, exactly to a region Lustig produces in his tract on connical basis. It is by better undersanding this region that it is hoped a geometrical "connonical basis" of the representation may be found, and that perhaps Bott towers and quantum groups may be connected (see picture with sharks at the beginning of this chapter.)





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Biographical Note.

Micheal David Grossberg was born on the first of April 1964 in St. Petersburg, Florida. Not liking the nightlife he moved six days later to New York, then Los Angeles, then New York, then Los Angeles, and then New York again with occasional stops in places like Austin, Texas and Enis, Montana. He then went to college in Philadelphia, Pennsylvania, spent a year in Edinburgh, Scotland and then back to Philadelphia. Upon graduating college he entered graduate school at MIT where he learned that beating your head against the wall in exactly the same way for five years is a high price to pay for not having to send out change of address cards. His career plans are to continue a family tradition of going into something his father can't make heads or tails of. He wishes to follow the teachings of his spiritual leader Immam Abdulali and his two-fold "way" of caution and respectibility. He hopes to further his career as a half-baker of rogue ideas with the imminant physo-mathachist Dr. Weinstein (aka Abdul Yaser Sadaam al-Sabaa III) in a picturesque and serene city. Wish me luck.