

ELECTRODYNAMICS OF MOVING MEDIA

by

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ABSTRACT

This investigation presents an approach that removes the ambiguities usually found in the electrodynamics of moving bodies, as well as the mathematical tool necessary for the development and application of a self-consistent theory.

An analysis of these ambiguities is made, including a brief description of the paths followed in the past for their removal. In addition, the necessity of a relativistic approach even for small velocities is justified.

A brief review of the fundamental concepts of vector and tensor analysis necessary for the development is also made.

After a presentation of the postulates used, a matrix notation which is most convenient for the four-dimensional formulation of the theory is developed. This notation is applied to derive relativistic mechanics and electrodynamics as a consequence of the postulates.

An investigation of the fields in materials and of the influence of motion on their interaction is made, thus clarifying the physical meaning of the different entities in terms of this approach.

The general equations of motion are developed and compared with Minkowski's formulation.

The conclusion is reached that this approach eliminates the ambiguities previously existing and that, by an experimental investigation of the detectable differences in the strains of a body in an electromagnetic field, it will be possible to determine the adequacy of the approach.

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CHAPTER I

INTRODUCTION

1.1 Objective and Introductory Discussion

Electromechanical energy conversion devices rely for their operation on the interaction between fields and moving bodies. Faraday's law is generally sufficient for the analysis of the usual devices, which are based on electromagnetic induction, even though some of the physical details are not considered.^(1,2) This is so because in the conventional electromechanical energy converters only electromotive forces are of interest. However, in trying to shed some light on the general conditions of electromechanical energy conversion, where a more general analysis is required, consideration of the rigorous laws of electrodynamics of moving bodies becomes essential.

In trying to establish a general procedure for analyzing the influence of motion on the electromagnetic fields it was soon realized that the usual classical procedure found in the engineering literature was marred by fundamental ambiguities. As a result, many amusing yet instructive paradoxes may be found, the endless discussion of which still appears in the engineering literature.^(3,4,6)

The situation seems to be better in the physics literature. Due to the fact that the physicist works at a more fundamental level in the quest for the laws of nature, the difficulties above mentioned appeared at an early stage. Thus there appeared the profound possibility of defining an absolute state of motion, which could be detected by electrodynamical means. The negation of this possibility, as given by experiment, initiated the development of the special theory of relativity, the far-reaching conclusions of which have been thoroughly checked. The majority of the paradoxical results of

the electrodynamics of moving bodies then lost their significance and, even though not treated in detail, could be dismissed as meaningless.⁽⁸⁾ The physicist, due to the self-consistency of the theory at a very fundamental level, can confidently expect that a rigorous approach, based on the special theory of relativity, will give no paradoxical results in the solution of these problems.

Unfortunately, this assertion is true only for fields in vacuo. When fields in materials are considered, ambiguities in the definitions of the fields themselves are still admittedly existent, giving uncertain results even for the case of stationary bodies.⁽¹⁰⁾ This does not trouble the physicist of today greatly because of the well-known fact that matter is composed of particles whose interaction ultimately explains its properties. For the regions between particles, electrodynamics in vacuo is necessary and does not involve any ambiguity. For the particles themselves, in whose interior electrodynamics in vacuo would not apply, there is enough evidence for the belief that other important actions have to be considered, the full understanding of which still lies in the future. At the present it seems futile to expect a solution in purely electrodynamical terms, for it is known that very important quantum mechanical aspects play an all-important part in the equilibrium among particles.

Yet the engineer is faced with the problem of studying electromechanical devices in which electrodynamics of moving bodies is a necessary tool, and, in order to conduct a really meaningful quest for the conditions of electromechanical energy conversion, the existing ambiguities have to be eliminated.

With reference to the electromagnetic fields in vacuo this is possible by the use of the special theory of relativity. But for this purpose this theory has to be used as a tool and not as a collection of results. It is

one of the purposes of this work to show that only a few elementary concepts of vector and tensor analysis are necessary for the successful resolution of this situation. In addition, by the use of partitioned matrices, it is possible to keep the usual three-dimensional entities in evidence, avoiding the loss of the physical reasoning due to the mathematical complexities.

It is here attempted to develop the theory of the electromagnetic field in materials from the premise that its only source is electric charge. Thus the interaction between fields and materials is assumed to arise from internal charges.*

A self-consistent development results, giving the majority of accepted conclusions and eliminating the ambiguities mentioned. A measurable difference arises, however, with respect to internal stresses in materials, and this will allow an experimental verification of the adequacy of the above-mentioned hypothesis.

A brief discussion of the classical approach will follow in order to show the necessity of the relativistic approach.

1.2 Principles Used in the Classical Approach

In this section some of the principles and theories used in the past in the analysis of the interaction between fields and moving bodies are stated and discussed. They are not all independent, but they are generally taken as self-consistent. It will be shown that this is not so and that in the resulting contradictions lies the source of most of the usual paradoxes.

The theories and principles to be discussed are the following:

- (a) Classical mechanics. This theory is built on the assumption that the three-dimensional space of the variables x , y and z forms a completely separate entity from the variable t (time).

*This is by no means the unique possibility. Professor L. J. Chu has investigated recently the possibility of considering magnetic charges and currents as additional sources of the electromagnetic field.⁽¹⁶⁾

It is then possible to treat the mathematical entities necessary for the development of the theory as being defined in the three-dimensional space of x , y and z , containing in addition the time t as an independent parameter.

(b) Principle of relativity as applied to systems in uniform motion. This principle is used in the past results from the invariance of the laws of classical mechanics under a Galilean transformation, which is

$$r' = r - vt$$

$$t' = t$$

where r is the position vector of an event happening at time t , and the primes refer to a frame in uniform motion with velocity v with respect to the unprimed frame.

This principle will be briefly called Galilean relativity, meaning invariance of physical (or mechanical) laws with respect to Galilean transformations.

(c) Maxwell's equations. Classically all entities are here considered as three-dimensional, and four equations connecting the field vectors to their sources result. These entities are also functions of the independent parameter t .

(d) Transformations of vector fields. (4,11,15)

$$E' = E + v \times B$$

$$B' = B - \frac{v \times E}{c^2} \quad (\text{sometimes simply } B' = B)$$

where

E = electric field vector

B = magnetic induction vector

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \text{velocity of electromagnetic waves in vacuo}$$

and the primes refer to a system moving with velocity \underline{v} with respect to the unprimed system.

(e) Transformations of the sources of the field vectors. (18)

$$\rho' = \rho$$

$$\underline{j}' = \underline{j} - \rho \underline{v}$$

where

$$\rho = \text{charge density}$$

$$\underline{j} = \text{current density.}$$

The first of these equations results from the conservation of electric charge, which in turn results from Maxwell's equations.

The second one is due to the experimental determination of the equivalence between current and moving charge.

(f) Lorentz's force density.

$$\underline{f} = \rho \underline{E} + \underline{j} \times \underline{B}$$

It is immediately apparent that, due to the transformation law of \underline{j} , this force density, and consequently the force, depends on the state of motion of the reference frame, a fact which is in contradiction with classical mechanics.

Even more fundamental in character is the inconsistency between Gallilean transformations and the transformations of field vectors. In fact, considering a system O' with velocity \underline{v} with respect to a system O , and a system O'' with velocity \underline{u} with respect to O' , according to the Gallilean transformations O'' will have velocity $(\underline{u} + \underline{v})$ with respect to O , but the field vectors will be the following:

$$\underline{E}'' = \underline{E}' + \underline{u} \times \underline{B}' = \underline{E} + \underline{v} \times \underline{B} + \underline{u} \times \left(\underline{B} - \frac{\underline{v} \times \underline{E}}{c^2} \right)$$

$$E'' = E + (u + v) \times B - \frac{1}{c^2} u \times (v \times E)$$

and

$$B'' = B' - \frac{u \times E'}{c^2} = B - \frac{v \times E}{c^2} - \frac{1}{c^2} u \times (E + v \times B)$$

$$B'' = B - \frac{1}{c^2} (u + v) \times E - \frac{1}{c^2} u \times (v \times B)$$

In order to be consistent with Gallilean transformations the last term of each expression should be zero. In particular, for $u = -v$, that is, making the system O'' fixed with respect to O , the following meaningless result is obtained:

$$E = E + \frac{1}{c^2} v \times (v \times E)$$

$$B = B + \frac{1}{c^2} v \times (v \times B)$$

It might be argued that the inconsistencies disappear if terms of the second order in v/c are neglected, and so, no contradictions will arise if only terms up to the first order terms in v/c are considered.⁽²⁾

This is not true, however, of the contradiction existing between the transformation of field vectors and the transformation of sources. In fact, if Maxwell's equations are to satisfy the principle of Gallilean relativity, then they should hold in two different frames O and O' , where O' is moving with constant velocity \underline{v} with respect to O . This will imply an error of the first order in v/c .

Considering for simplicity a case in which the field vectors are time invariant in both frames, then the sources in O' can be obtained by taking $\nabla' \times B'$ and $\nabla' \cdot \epsilon_0 E'$, where the primes refer to quantities and operations in the system O' . But according to the Gallilean transformations as applied

to the time invariant vectors

$$\nabla' \times B' = \nabla \times B'$$

and

$$\nabla' \cdot \epsilon_0 E' = \nabla \cdot \epsilon_0 E$$

This means that the current density should be obtained from:

$$j' = \frac{1}{\mu_0} \nabla \times B' = \frac{1}{\mu_0} \nabla \times \left(B - \frac{1}{c^2} v \times E \right) = j - \frac{1}{\mu_0 c^2} \nabla \times (v \times E) = j - \epsilon_0 \nabla \times (v \times E)$$

The second term can be transformed by the use of the formula:

$$\nabla \times (v \times E) = v(\nabla \cdot E) - E(\nabla \cdot v) + (E \cdot \nabla)v - (v \cdot \nabla)E$$

and since for the case of uniform velocity

$$\nabla \cdot v = 0$$

and

$$(E \cdot \nabla)v = 0$$

with the additional fact that

$$(v \cdot \nabla)E = 0$$

because the field vectors were assumed to be non time varying in O' also.

The current density is then given by

$$j' = j - v(\nabla \cdot \epsilon_0 E) = j - \rho v$$

This shows that the transformation of current density is in agreement with the transformation of field vectors. In addition, it is clear from the derivation that the term ρv is the source corresponding to the term $(v \times E)/c^2$. It is to be concluded that the term $(v \times E)/c^2$ will be negligible if and only if ρv is negligible. That is, the c^2 in the denominator is a consequence of the choice of units, and does not mean that $(v \times E)/c^2$ is always negligible. In other words, this term is not of the second order in v/c .

The charge density is obtained as follows:

$$\rho' = \nabla \cdot (\epsilon_0 E') = \epsilon_0 \nabla \cdot (E + v \times B) = \rho + \epsilon_0 \nabla \cdot (v \times B)$$

The second term can be transformed by the formula

$$\nabla \cdot (v \times B) = B(\nabla \times v) - v \cdot (\nabla \times B)$$

and, since

$$\nabla \times v = 0$$

there follows

$$\epsilon_0 \nabla \cdot (v \times B) = -\epsilon_0 \mu_0 v \cdot j = -\frac{v \cdot j}{c^2}$$

Consequently, the transformation of charge density consistent with the transformation of fields is

$$\rho' = \rho - \frac{v \cdot j}{c^2}$$

and not

$$\rho' = \rho$$

Here again, $(v \cdot j)/c^2$ is a first order term corresponding to the term $v \times B$. Again the c^2 in the denominator of the charge density term is a matter of units and does not imply that this term is, in general, negligible. Only in the cases in which charges of the same sign are present at a given point does this term become a second order correction, because then

$$j = \rho v$$

and

$$\rho' = \rho \left(1 - \frac{v^2}{c^2}\right)$$

However, the existence of charges of different signs may produce situations in which

$$\rho = \sum_i \rho_i$$

is very small, yet

$$j = \sum_i \rho_i v_i$$

is large; consequently, the $(v \cdot j)/c^2$ term is the predominant effect.

This is particularly true in cases involving conduction in metals. At this point it is important to notice that this difficulty can be avoided by accepting the fact that Maxwell's equations are not invariant under Gallilean transformations, that is, that they do not comply with Gallilean relativity.

These equations should then be applicable to a special reference frame, and fields in other frames should be obtained by the transformation equations. This will imply that the $v \times B$ term is directly ascribable to the state of motion of the frame and will be in a sense a measure of its velocity with respect to the preferred frame. In the past these considerations appear to have led to two classical approaches differing widely in philosophy, which, however, can be expected to lead to the same results as far as first approximations are concerned, if applied properly. These approaches are briefly discussed in the next section.

1.3 Brief Historical Background

The two different philosophies referred to stemmed mainly from the differences existing between the approaches of the machinery engineers and the propagation engineers.

In electromechanical engineering the constant necessity of handling the interplay between electric fields and moving circuits tends to make the intuitive picture of fields by lines of forces seem more fundamental than it really is. This is mainly because Faraday's law of induction has to be used generally, and this law is more than a simple consequence of Maxwell's equations for vector fields. As a result, an intuitive feeling about the existence

of an almost material field, pictured by lines of forces, arose. It was just a short step further to attach these field lines to the sources, and from this it seemed quite obvious that they should share, in a rigid way, the motion of their sources. In contraposition, the propagation engineer, having to handle electromagnetic waves (which are strictly a consequence of Maxwell's equations) had to have a more fundamental knowledge about the mathematical aspects of these fields and had to understand clearly that they are defined with respect to an observer. In addition, being familiar with these fields propagating themselves in vacuo, he could not possibly fall into the mistake of attaching them to the sources. On the other hand, this point of view created the idea of a medium (the ether) for the propagation of these fields. There followed the assumption that Maxwell's equations should be true for a frame at rest with respect to this medium, and transformations should be applied in going to any other frame.⁽⁵⁾

The propagation engineer's point of view has been closer to the physicist's, and, since he seldom had to consider materials in motion, he could consider Maxwell's equations to be true in the observer's frame. Thus he did not need to worry about the possibility of different observers and the consequent contradictions.

This allowed him to use the methods of the physicist without regard to the important fact that relativity does not hold for them.

The electromechanical engineer on the contrary, by having to use mechanics, was well aware of this difficulty and was always reluctant in abandoning the principle of relativity. The consideration of the fields as attached to the source led to fewer conceptual difficulties in this respect.⁽⁴⁾

It is fairly easy to make the idea of fields carried by their sources rigorous.

In order to do this it is sufficient to define the field vectors in the rest frame of the source. By rest frame of the source is meant an inertial frame in which the source is at rest at the instant under consideration.

According to this definition, the moving field point of view can be considered as a point of view assuming the validity of Maxwell's equations in the rest frames of the sources, and requiring the use of the transformation laws for the determination of the field vectors in the observer's frame. Of course, superposition has to be used for the case of many sources in different states of motion.

This clearly shows that, in the case of a source whose different points have different velocities (case of rotation, for example), Maxwell's equations have to be used in the rest frame of each point of the source, then transformed to the observer's frame, and finally a summation has to be made in order to obtain the field vectors in the observer's frame. It is clear that the velocity to be used in the transformation formulas has to be the velocity of the source element under consideration. This procedure evidently gives results completely different from those obtained by considering the field lines rigidly carried by the sources. It amounts to taking, for instance,

$$\int_{Vol} v_s \times dB,$$

where v_s is the velocity of the element of source producing dB, instead of $v_f \times B$, where v_f is the velocity of the field vector imagined as rigidly carried by the motion of the source. (5,7)

In this manner, it can be shown that both approaches lead to the same first order results, in the majority of cases, and the predictions of the carried field theory which are wrong (as shown experimentally) can be brought into agreement with experiment.

In physics, both approaches had to be discarded, because of the fact (verified experimentally) that the speed of propagation of electromagnetic waves is independent of both the motion of the observer and of the source. In the ether theory this velocity is independent of the velocity of the source, but depends on the velocity of the observer, and thus can be used to detect his motion. In the carried field theory the velocity of electromagnetic waves in vacuo is dependent on the relative velocity between observer and source. Even though this conclusion does not contradict Gallilean relativity, it is not verified experimentally.

Experiment then pointed to the fact that Maxwell's equations are valid in all inertial frames, independently of their state of motion, and electromagnetic waves in vacuo behave as if they were waves in an ether fixed with respect to any chosen inertial frame. This led to the discovery of the appropriate transformations between inertial reference frames, namely Lorentz transformations, under which Maxwell's equations are invariant. The main contribution of Einstein was to show that this implied a modification of the concepts of space and time.⁽⁸⁾ After that Minkowski indicated the mathematics appropriate for the new concepts of space and time.⁽⁸⁾ Finally, the modifications in the theory of mechanics necessary to satisfy the principle of relativity utilizing Lorentz transformations not only proved to be true experimentally, but induced conceptual changes (like the equivalence of mass and energy, for instance) which opened new paths for the explanation of many obscure and puzzling observations of physics.⁽⁹⁾ Today, it is beyond doubt that Maxwell's equations are a consequence of the behavior of the space-time continuum, the behavior of which cannot be understood in classical theory, and that any law of physics has to comply with the requirements of this kind of relativity, which will herein be briefly called Einstein relativity.

1.4 Preliminary Approach

The realization that the main source of difficulty existing in the carried field theory arose from the misunderstanding of the transformations required for the case of non-uniform motion, opened the possibility of resolving the internal contradictions existing in this viewpoint. Moreover, the rigorous definition of moving fields showed that both points of view, the moving fields and the fixed fields, are actually particular cases of a general approach in which fields may be defined in any convenient state of motion. It was thus expected that both approaches should agree within terms of the first order in v/c . This is in fact true, and both viewpoints predict that the transformations of sources compatible with the field transformations should be

$$j' = j - \rho v$$
$$\rho' = \rho - \frac{v \cdot j}{c^2}$$

In the moving field approach the new term $(v \cdot j)/c^2$ in the transformation of ρ arises from the divergence of the electric field in the primed reference frame. In the fixed field theory, this term can be easily shown to result from the equilibrium of conduction current charges within a conductor in motion through the field. For the case of convection current the situation is not clear at all, unless the charges are considered as carried by particles, and the fields are computed by adding the fields of the individual charges. The fields of the charges have to be calculated by retarded potentials. The final result is equivalent to the use of the term $(v \cdot j)/c^2$.

Incidentally, it is interesting to point out here that a calculation of the field of a particle in motion, by retarded potentials, gives an indication of the space contraction due to the state of motion, which, in the

past, was attributed to an effect of the motion through the ether. This interpretation had to be abandoned because the experimental discovery of the validity of the principle of relativity for electromagnetic phenomena required the same contraction to be attributed to the relative motion, with no reference to an ether.

In order to avoid these difficulties of interpretation and, in addition, to keep the advantages and flexibility afforded by the use of the principle of relativity, a purely phenomenological approach along the lines of the generalized moving field theory was tried. It might be in order to mention, as a clarification, that many problems that require complicated integrations of retarded fields may be easily solved by a transformation, if a convenient frame is chosen.⁽¹¹⁾ It was soon realized that this approach led to the necessity of abandoning the principle of conservation of charge. In addition, ambiguities arose in trying to define the state of motion with respect to a current, unless this definition was restricted to conduction current, an approach which implies the introduction of ideas extraneous to electrodynamics.

This is certainly too much patch work at a fundamental level and requires the abandonment of some very fundamental facts well-established by a rigorous theory. Such a high price is not warranted by the meager advantage of obtaining a self-consistent, approximate theory which does not even remove the contradictions with mechanics. This last statement follows from the observation that force in mechanics does not depend on the state of motion of the inertial frame used, while Lorentz force changes according to this state.

This contradiction was difficult to understand in face of the usual statement that for small velocities Einstein relativity is approximated by Gallilean relativity and that, consequently, a classical approach may be safely used. This difficulty disappeared with the realization that Gallilean

transformations are the limit of Lorentz transformations when c tends to infinity but are not first order approximations in v/c of the Lorentz transformations. It will be shown that in the approximate Lorentz transformations contractions are disregarded, but the interdependence between space and time cannot be disregarded. In order to consider time independent of the state of motion of the reference frame, it is necessary to restrict the transformations of coordinates to a sufficiently small region, while the transformations of the different physical entities may have to satisfy additional restrictive conditions. These additional restrictions are generally not met in electrodynamics.

The consequence of these considerations is that even in an approximate theory time cannot be considered as a parameter but has to be a variable differing between inertial frames. Thus, there will be no gain in intuitive thinking by using an approximate theory. The concepts have to be defined in essentially the same way used in the rigorous theory. In addition, one has to be continually discarding terms of the second order in v/c in order to avoid meaningless results arising from the use of an approximate theory.

The final conclusions reached by this preliminary investigation are the following:

- (a) space and time concepts have to be in agreement with Einstein's relativity even for an approximate theory of electrodynamics of moving media;
- (b) all other physical entities, being defined in this kind of space-time, have to be understood according to this new way of thinking;
- (c) no advantage exists in an approximate approach.

The necessary method of approach is then bound to use Einstein's relativity as a tool and not as a collection of results. This can be

accomplished easily with the help of a few concepts of tensor analysis which lead to the four-dimensional formulation of the laws of physics. Unfortunately, the conciseness of thought and notation afforded by the four-dimensional formulation is gained at the high price of loss of clarity in physical thinking, due to the loss of track of the role of the three-dimensional entities which are ultimately to be used. This disadvantage may be overcome by the use of partitioned matrices which can always keep clear the composition of the four-dimensional entities. The next section will show how these ideas can be used.

1.5 Vectors, Tensors and Matrices^(12,13,14)

As already mentioned, the essential novelty in Einstein's relativity is the fact that time has to be considered as a fourth variable. Thus, associated with every inertial frame of reference there is a particular time which changes when transformations are made to another inertial frame. In addition, the three-dimensional space of the variables x , y and z also changes when changing to an inertial frame in a different state of uniform motion. Consequently, all the physical entities have to be considered as defined in the four-dimensional space of x , y , z and t .

The principle of relativity requires that the laws of physics should be independent of the particular frame of reference in which the physical entities are defined. In other words these laws should be laws among the four-dimensional entities and not simply among particular coordinates of these entities.

The apparent difficulty in establishing these laws and operating with them arises from the lack of geometrical intuition with reference to spaces of dimension larger than three. The situation is analogous to having to operate with the coordinates x , y and z without having any geometrical

intuition about three-dimensional space.

Vector analysis is a construct specially designed to benefit from this geometrical intuition. For instance, once three functions are recognized as coordinates of a vector all of the deductions can be made directly in terms of the vector, making all of the consequences independent of reference frames. This not only affords an advantage in conciseness of notation and thought by uniting three entities into one, but also allows the corresponding three laws in any convenient reference frame to be obtained at once when necessary. From a practical standpoint, this last observation is the most important.

Since all calculations must ultimately be made with the coordinates, a knowledge of their value in every possible frame gives all the necessary information. This knowledge is furnished by giving the coordinates in one particular reference frame and the law of transformation to any other frame. This last procedure is obviously independent of any geometrical intuition and can be used for the complete characterization of the necessary entities. Obviously, this is the procedure to be used in spaces of more than three dimensions.

A brief account of the important considerations and definitions of this procedure as applied to three-dimensional space will now be given in preparation for the development of the method and notation to be used in this investigation.

Scalar - A scalar is an entity characterized by its value at every point of space. Since a point is given by its three coordinates in a given reference frame, a scalar is analytically represented by a function of the three coordinates of the point in the given frame. Its independence of the reference frame chosen implies that a change of reference frame simply induces a substitution of variables in the function characterizing the scalar.

The transformation of reference frame is generally given by the transformation of coordinates of the points of space as follows:

$$x = x(x', y', z')$$

$$y = y(x', y', z')$$

$$z = z(x', y', z')$$

and the substitution gives for the scalar

$$\varphi(x,y,z) = \varphi[x(x',y',z'), y(x',y',z'), z(x',y',z')] = \psi(x',y',z')$$

Vector - A vector is an entity typified by a directed segment. It is given in a reference frame by its three coordinates. In other words, it is given by an ordered set of three functions of x , y and z . When a change in reference frames is made, the three functions have to undergo a transformation equal to the transformation of the coordinates of a directed segment in addition to the substitution of variables.

From now on, for simplicity, the development will be restricted to linear orthogonal transformations, which will be sufficient for this investigation. Linear transformations are conveniently handled by the use of matrices. An orthogonal transformation will be given by

$$r' = r_0 + Rr$$

where

$$r' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \quad r_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and R is a 3×3 matrix of constants such that

$$RR_t = I_3$$

where the index t indicates transposition and I_3 is the 3×3 unit matrix.

From this last relation there follows

$$(\det R)^2 = 1$$

or

$$\det R = \pm 1$$

The positive sign refers to rotations which transform right handed systems into themselves and left handed systems also into themselves. The negative sign indicates transformation from a right handed system to a left handed system or vice versa, as can easily be seen by noticing that such a transformation can be represented by

$$- I_3 P$$

where P is a rotation.

A segment transforms like the differential of the position vector

$$dr' = Rdr$$

There follows that a vector is characterized by the transformation

$$v' = Rv$$

where v' and v are the 3×1 matrices of the coordinates of the vector.

Tensor - In this work only tensors of the second order under orthogonal transformations will be necessary. With this restriction a tensor may be defined as an entity that linearly transforms a vector into another vector. This definition shows that a tensor is independent of the reference frame in which it is represented by a 3×3 matrix T, such that

$$u = Tv$$

where u and v are 3×1 matrices representing vectors. The law of transformation of a tensor is obtained as follows:

Since u and v are vectors

$$u = Tv$$

may be written

$$R_t u' = TR_t v'$$

from which

$$u' = RTR_t v'$$

showing that

$$T' = RTR_t$$

which is the required law.

Important particular tensors are the symmetric tensors, characterized by

$$T = T_t$$

and the antisymmetric tensors, characterized by

$$T = - T_t$$

These properties are independent of the reference frame used, as shown by the fact that from

$$T = \pm T_t$$

there follows

$$T'_t = RT_t R_t = \pm RTR_t = \pm T'$$

Scalars and tensors can be constructed by operations with vectors. In fact the dot product of two vectors is a scalar, as is shown by the following deduction:

$$u' \cdot v' = u'_t v'_t = u_t R_t R_t v = u_t v$$

In this deduction the practice, freely followed from now on, of not making any distinction between entities and their representing matrices was employed.

Conversely, if $u \cdot v$ is a scalar for an arbitrary vector v , then u is a vector. In fact, from

$$u_t v = u'_t v'_t = u'_t R_t v$$

there follows, due to the arbitrariness of v

$$u_t = u'_t R$$

from which

$$u' = Ru$$

The product uv_t is a tensor, as follows from

$$u'v'_t = R(uv_t)R_t$$

Pseudo-scalar - There are important entities called pseudo-scalars which are characterized by their value at a point, and do not change under a rotation transformation. They behave differently though under more general transformations. Even the transformations between right handed and left handed frames can detect this difference. One example of a pseudo-scalar is the volume of the parallelepiped constructed with 3 vectors. It is given by

$$V = \det [abc]$$

where \underline{a} , \underline{b} and \underline{c} are 3×1 matrices representing vectors.

There follows

$$V' = \det \{ R[abc] \} = (\det R)(\det [abc])$$

That is

$$V' = (\det R)V$$

in proof of what was said.

Pseudo-vectors or axial vectors - A pseudo-vector or axial vector is a set of three functions that transforms like a vector under rotation transformations but involves an additional change in sign of all of the components in changing between left handed and right handed systems.

A characteristic example of pseudo-vectors is the cross product of two vectors. In fact, the volume previously considered can be written

$$V = \mathbf{a} \cdot \mathbf{B} = a_t B$$

where

$$B = b \times c$$

There follows

$$a'_t B' = (\det R) a_t B$$

The matrix a_t can be expressed in terms of a'_t . In fact, from

$$a' = Ra$$

there follows

$$a_t = a'_t R$$

and so

$$a'_t B' = a'_t (\det R) RB$$

giving the transformation formula for an axial vector

$$B' = (\det R) RB$$

An antisymmetric tensor has only three independent components, as can be seen by writing its matrix representation explicitly

$$A = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}$$

The result of its application to an arbitrary vector b may be obtained by a cross product of this arbitrary vector with an axial vector having the three independent components of the given tensor.

In fact

$$c = Ab = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_2 a_3 - b_3 a_2 \\ b_3 a_1 - b_1 a_3 \\ b_1 a_2 - b_2 a_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The last 3×1 matrix is the representation of what is called the complement of A , and will be represented by

$${}^c A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Its definition may be written using the components of A and ${}^c A$ as

$${}^c A_i = A_{ijk}$$

where ijk is an even permutation of the numbers 123 .

So with this definition, the relation

$$c = Ab$$

may be written

$$c = b \times A$$

That ${}^c A$ is an axial vector follows from the fact that c and b are vectors.

Pseudo-tensors may be defined in the same way, but they will not be necessary in this work.

All the considerations and definitions made can be extended with no change whatsoever to n dimensional complex spaces, except for those relative to complements. The extension to four dimensions provides the necessary mathematical tool for the development that will follow. It will also be convenient to extend the definition of complement to four-dimensional anti-symmetric tensors in the following way

$${}^c A_{ij} = A_{k\ell}$$

where $ijk\ell$ is an even permutation of the numbers 1234 .

Also, ${}^c A$ will be antisymmetric, as the explicit expressions below indicate.

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}$$

$${}^c A = \begin{bmatrix} 0 & a_{34} & -a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & -a_{13} \\ a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & a_{13} & -a_{12} & 0 \end{bmatrix}$$

The material briefly reviewed in this section is sufficient for the development of the rigorous approach to be discussed in the main part of this investigation.

Some additional comments pertinent to the four-dimensional formulation will be made at the proper places.

CHAPTER II

FUNDAMENTAL POSTULATES AND RELATIVISTIC THEORY

2.1 Fundamental Postulates

The point of view taken in this investigation is that electromagnetic fields describe the interaction of electric charges, and so if materials interact with electromagnetic fields they should be considered as aggregates of charges. It is possible, in this way, to remove all ambiguities in the theory.

It will be seen that a possibility of an experimental verification of the theory exists, since it does not give the usual stresses in magnetized and polarized materials.

The entire mathematical theory rests on five postulates which will be stated and commented on here. The first four postulates refer to the so-called fields in vacuo. According to the fifth postulate adopted in this investigation, materials are considered simply as aggregates of charges and, therefore, fields in materials are simply fields in regions where charge and current densities are different from zero. Consequently, the usual distinction between the case of vacuum and the case of materials loses its significance.

Postulate 1 - Lorentz law of force on a charged particle

$$\vec{F} = q(\vec{E} + v \times \overset{c}{B})$$

where

q = charge upon which force acts

\vec{E} = electric field vector due to all other charges

c

\vec{B} = magnetic field vector due to all other charges

v = velocity of charged particle with respect to the
observer's reference frame

This is a well-established experimental law and can be used for the definition of the electric and magnetic field vectors produced by all other charges when the charge on which the force \mathcal{F} acts does not perturb the distribution of the other charges. The magnetic field axial-vector was indicated as \underline{B} because it is preferable, in this work, to consider the magnetic field as an antisymmetric tensor \underline{B} whose complement is \underline{B} . Then the Lorentz force will be written

$$\mathcal{F} = q(\underline{E} + \underline{B}v)$$

from which the tensor character of \underline{B} immediately follows.

Postulate 2 - A field point of view is allowed. This is the assumption made in a macroscopic theory, where attention is not focused on the large oscillations of the quantities which arise on a microscopic scale. This procedure is equivalent to a smooth extrapolation of the laws verified at a macroscopic scale to zero dimension. The results are only applicable to a macroscopic level. This allows for considerable simplification; however, the discontinuities that arise in the application of this point of view have to be handled by a well-known limiting procedure.

Accordingly, a force per unit volume may be defined and written

$$f = \rho(\underline{E} + \underline{B}v)$$

where ρ is the charge density at the given point. It is now unnecessary to consider \underline{E} and \underline{B} as fields due to the other charges, but as simply the fields of the charge distribution, because the perturbation due to a finite ρ at a given point is an infinitesimal. Surface charges are handled by a well-known limiting procedure. Introducing the definition

$$j = \rho v$$

for current density, the force density may be written

$$f = Bj + \rho E$$

Postulate 3 - Maxwell's equations in vacuo. With the appropriate boundary conditions these equations determine the field vectors by giving their curls and divergences.

$$\nabla \times \frac{c}{\mu_0} B = j + \frac{\partial(\epsilon_0 E)}{\partial t}$$

$$\nabla \times E = - \frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

where

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

They may be rewritten as

$$\frac{1}{\mu_0} (B \nabla - \frac{1}{c^2} \frac{\partial E}{\partial t}) = j$$

$$\frac{c}{E} \nabla + \frac{\partial B}{\partial t} = 0$$

$$\frac{1}{\mu_0 c^2} \nabla_t E = \rho$$

$$\nabla_t B = 0$$

where the product of a function by a differential operator indicates differentiation irrespective of the order in which they are written. This convention is used in order to keep the matrix products convention.

Postulate 4 - Principle of relativity. This principle may be considered as a consequence of the definition of motion. It may be conveniently interpreted as allowing any reference frame to be considered at rest, motion being referred to it. This work will be restricted to considerations of inertial frames.

Postulate 5 - The interaction between materials and electromagnetic fields is due to internal charges. This postulate will be discussed in the chapter dedicated to materials.

The adoption of postulates 1 to 4 implies the use of Lorentz transformations when changing from one inertial frame to another in motion with respect to it. This will be shown in the following section.

2.2 Lorentz Transformations (9,10,11)

From postulate 3 it follows that electromagnetic waves travel with velocity $c = 1/\sqrt{\epsilon_0 \mu_0}$ in vacuo with respect to the observer. From postulate 4 it follows that this should be true in all inertial frames, or else there would be a criterion for distinguishing between them. This can be restated in the following way. Calling r and r' the position vectors in the inertial system O and O' , if propagation of electromagnetic waves is described by $dr^2 - c^2 dt^2 = 0$ in the inertial system O , then it will be described by $dr'^2 - c^2 dt'^2 = 0$ in the inertial system O' . That is, considering the variables ict and ict' instead of t and t' , where $i = \sqrt{-1}$, it can be said that $ds = 0$ in O implies $ds' = 0$ in O' , where

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 + (icdt)^2 = (dr)^2 + (icdt)^2$$

and

$$ds'^2 = (dx')^2 + (dy')^2 + (dz')^2 + (icdt')^2 = (dr')^2 + (icdt')^2$$

On the other hand, if a particle is described as being in rectilinear uniform motion in O , i.e., by a linear relation between x , y , z , and ict , then postulate 4 implies the same type of description in O' , i.e., a linear relation between x' , y' , z' and ict' .

This means that the law of transformation must be a linear one, i.e.,

$$ds' = a + bds$$

where \underline{a} and \underline{b} are constants. But the previous argument gives $\underline{a} = 0$, so

$$ds' = bds$$

Since a space rotation of the reference frames does not change $(dr)^2$ and dt , then it does not change ds . This shows that \underline{b} is independent of the direction of the relative velocity \underline{v} of O' with respect to O . But since O' can describe O as being in motion with velocity $-\underline{v}$ it follows that

$$ds = bds'$$

and so

$$ds = b^2 ds'$$

from which

$$b = \pm 1$$

By continuity arguments it is concluded that \underline{b} has to be either $+1$ or -1 , and this latter possibility has to be discarded because it does not include the identity transformation. Consequently, the transformations between any two inertial frames O and O' have to be such that

$$ds' = ds$$

These are orthogonal transformations in the four-dimensional space of x, y, z and ict . In fact, calling \underline{L} the 4×4 matrix of a linear transformation, one can write

$$\begin{bmatrix} dr' \\ ict' \end{bmatrix} = L \begin{bmatrix} dr \\ ict \end{bmatrix}$$

from which

$$ds'^2 = [dr'_t \quad ict'] \begin{bmatrix} dr' \\ ict' \end{bmatrix} = [dr_t \quad ict] L_t L \begin{bmatrix} dr \\ ict \end{bmatrix}$$

On the other hand

$$ds^2 = [dr_t \quad ict] \begin{bmatrix} dr \\ ict \end{bmatrix}$$

and so

$$[dr_t \quad icdt]L_tL \begin{bmatrix} dr \\ icdt \end{bmatrix} = [dr_t \quad icdt] \begin{bmatrix} dr \\ icdt \end{bmatrix}$$

It is immediately seen that

$$L_tL = I_4$$

(where I_4 is the 4 x 4 unit matrix) is a sufficient condition. That the condition is also necessary follows from the fact that $\begin{bmatrix} dr \\ icdt \end{bmatrix}$ is an arbitrary four-dimensional vector. In fact, making it such that all components are zero except the k^{th} component, which is 1, it follows that

$$(L_tL)_{kk} = 1$$

which means that the diagonal elements of L_tL are equal to 1. Making the $\underline{k}^{\text{th}}$ and the $\underline{\ell}^{\text{th}}$ element of the arbitrary vector equal to 1 and the others equal to zero, there follows

$$(L_tL)_{kk} + (L_tL)_{\ell k} + (L_tL)_{k\ell} + (L_tL)_{\ell\ell} = 2$$

but since

$$(L_tL)_t = L_tL$$

there follows

$$2 + 2(L_tL)_{\ell k} = 2$$

that is

$$(L_tL)_{\ell k} = 0$$

and so

$$L_tL = I_4$$

These transformations are the Lorentz transformations. They include rotations in the three-dimensional space of x, y, z . In fact

$$\begin{bmatrix} \bar{R} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{R}_t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{R}\bar{R}_t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{I}_3 & 0 \\ 0 & 1 \end{bmatrix} = I_4$$

The physical meaning for the case in which time is transformed can be obtained by considering the following simple case in which only one space variable is transformed

$$\begin{bmatrix} dx' \\ dy' \\ dz' \\ icdt' \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & \gamma \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \\ icdt \end{bmatrix} = \begin{bmatrix} dx \\ dy \\ \gamma(dz + icadt) \\ \gamma(-adz + icdt) \end{bmatrix}$$

where

$$\gamma = \frac{1}{\sqrt{1 + a^2}}$$

in order to make the above transformation orthogonal. Considering any point at rest with respect to O' , there follows

$$\begin{bmatrix} dx' \\ dy' \\ dz' \end{bmatrix} = 0 = \begin{bmatrix} dx \\ dy \\ \gamma(dz + icadt) \end{bmatrix}$$

which means that the point is moving along the z direction with velocity $v = -ica$ with respect to O . Or, in other words, the O' system is moving with velocity v with respect to O . Then

$$a = \frac{1}{c} v$$

and

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

It is now easy to relinquish the restriction of motion along the z direction.

Thus

$$\begin{bmatrix} dr'_\perp \\ dr'_\parallel \\ icdt' \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & \gamma \begin{bmatrix} 1 & \frac{i}{c} v \\ -\frac{i}{c} v & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} dr_\perp \\ dr_\parallel \\ icdt \end{bmatrix} = \begin{bmatrix} dr_\perp \\ \gamma(dr_\parallel + \frac{i}{c} v icdt) \\ \gamma(-\frac{i}{c} v dr_\parallel + icdt) \end{bmatrix} = \begin{bmatrix} dr_\perp \\ \gamma(dr_\parallel + \frac{i}{c} v icdt) \\ ic\gamma(dt - \frac{v \cdot dr}{c^2}) \end{bmatrix}$$

where the indexes \perp and \parallel refer to components perpendicular and parallel to \underline{v} . Or, considering the vectors \underline{r} and \underline{r}'

$$\begin{bmatrix} dr' \\ icdt' \end{bmatrix} = \begin{bmatrix} dr_\perp + \gamma dr_\parallel + \frac{i}{c} \gamma v icdt \\ -\frac{i}{c} \gamma v_t dr + \gamma icdt \end{bmatrix} = \begin{bmatrix} dr + (\gamma - 1) \left(\frac{v}{|v|} \cdot dr \right) \frac{v}{|v|} + \frac{i}{c} \gamma v icdt \\ -\frac{i}{c} \gamma v_t dr + \gamma icdt \end{bmatrix}$$

and so

$$\begin{bmatrix} dr' \\ icdt' \end{bmatrix} = \begin{bmatrix} (I_3 + \frac{\gamma - 1}{v^2} v v_t) & \frac{i}{c} \gamma v \\ -\frac{i}{c} \gamma v_t & \gamma \end{bmatrix} \begin{bmatrix} dr \\ icdt \end{bmatrix}$$

This gives the Lorentz transformation matrix

$$L(v) = \begin{bmatrix} (I_3 + \frac{\gamma - 1}{v^2} v v_t) & \frac{i}{c} \gamma v \\ -\frac{i}{c} \gamma v_t & \gamma \end{bmatrix}$$

from which

$$L_t(v) = \begin{bmatrix} (I_3 + \frac{\gamma - 1}{v^2} v v_t) & -\frac{i}{c} \gamma v \\ \frac{i}{c} \gamma v_t & \gamma \end{bmatrix} = L(-v)$$

Since

$$\begin{bmatrix} dr' \\ icdt' \end{bmatrix} = L(v) \begin{bmatrix} dr \\ icdt \end{bmatrix}$$

there follows

$$\begin{bmatrix} dr \\ icdt \end{bmatrix} = L_t(v) \begin{bmatrix} dr' \\ icdt' \end{bmatrix} = L(-v) \begin{bmatrix} dr' \\ icdt' \end{bmatrix}$$

This shows that if 0 sees 0' with velocity \underline{v} , then 0' sees 0 with velocity $-\underline{v}$, as it should. This may be taken as an indication of the fact that if 0' were brought to rest with respect to 0 the corresponding axes would be parallel even though they are not so while in motion (in fact the system 0' is not even orthogonal as seen by 0). This is so because the above transformation was obtained from the case in which the velocity was along one of the axes and the axes were all parallel then.

For the case of the axes of 0' being rotated with respect to 0 by a rotation matrix R, if brought to rest, the pertinent Lorentz transformation is

$$L' = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} vv_t & \frac{i}{c} \gamma v \\ -\frac{i}{c} \gamma v_t & \gamma \end{bmatrix} = \begin{bmatrix} R + \frac{\gamma-1}{v^2} Rvv_t & \frac{i}{c} \gamma Rv \\ -\frac{i}{c} \gamma v_t & \gamma \end{bmatrix}$$

The transformation from 0' to 0 will be

$$L'_t = \begin{bmatrix} R_t + \frac{\gamma-1}{v^2} vv_t R_t & -\frac{i}{c} \gamma v \\ \frac{i}{c} \gamma v_t R_t & \gamma \end{bmatrix}$$

Or, letting

$$u = -Rv$$

from which

$$u_t = -v_t R_t$$

and

$$v = -R_t u, \quad v^2 = v_t v = (-u_t R)(-R_t u) = u^2$$

there follows

$$L'_t = \begin{bmatrix} R_t + \frac{\gamma - 1}{u^2} R_t u u_t & \frac{i}{c} \gamma R_t u \\ -\frac{i}{c} \gamma u_t & \gamma \end{bmatrix}$$

which shows that O' sees O with velocity $u = -Rv$ and rotated by the inverse rotation R_t .

From these considerations the general Lorentz transformation among coordinates results

$$\begin{bmatrix} r' \\ ict' \end{bmatrix} = \begin{bmatrix} r_o \\ ict_o \end{bmatrix} + \begin{bmatrix} R + \frac{\gamma - 1}{v^2} R v v_t & \frac{i}{c} \gamma R v \\ -\frac{i}{c} \gamma v_t & \gamma \end{bmatrix} \begin{bmatrix} r \\ ict \end{bmatrix}$$

2.3 Approximate Lorentz Transformation

The approximate Lorentz transformation, appropriate for small velocities, is obtained by making $\gamma = 1$

$$L_a = \begin{bmatrix} I_3 & \frac{i}{c} v \\ -\frac{i}{c} v_t & 1 \end{bmatrix}$$

which is not the Gallilean transformation. The Gallilean transformation may be obtained by letting c tend to infinity, giving

$$L_\infty = \begin{bmatrix} I_3 & \frac{i}{c} v \\ 0 & 1 \end{bmatrix}$$

It should be mentioned that $\frac{1}{c} v$ has to be kept because it multiplies $icdt$.

The reason why a Gallilean transformation can be used in most cases can be

better understood by considering the transformation in the form

$$dr'_1 = dr_1$$

$$dr'_2 = \gamma(dr_2 - v dt)$$

$$dt' = \gamma\left(dt - \frac{v \cdot dr}{c^2}\right)$$

which, for small v can be written

$$dr' = dr - v dt$$

$$dt' = dt - \frac{v \cdot dr}{c^2}$$

The last term in the last equation can be neglected only for sufficiently small dr , giving the Gallilean transformation. This same approximation is generally valid in mechanics.

In electrodynamics, as will be shown, some of the entities do not allow this additional simplification, and Gallilean transformations cannot be used even for small velocities.

2.4 Four-dimensional Formulation

Postulate 4 implies that all inertial frames are equivalent as far as description of the laws of nature is concerned. So, all physical laws should take the same form in them. This can also be stated as saying that the laws of nature should be form invariant under Lorentz transformations.

Classical mechanics is form invariant under Gallilean transformations and, thus, does not comply with Einstein's relativity which follows from the stated postulates, since they imply Lorentz transformations. It is not difficult to check if a law is invariant under Lorentz transformations, but to modify a law which is not, in such a way as to make it so, is a hopeless task. A more fundamental approach is necessary.

The fundamental difficulty in building a law according to Einstein's relativity arises from the fact that Lorentz transformations create a tie between space and time in such a way that they are no longer independent. The result is that time becomes a variable instead of a parameter, as formerly. The use of the variable ict makes Lorentz transformations formally equal to orthogonal transformations in the space of x , y , z and ict . This allows a generalization of an analogous situation existing in three-dimensional space. In fact, in three-dimensional space it is possible to reason directly with space entities independently of their components in a given reference frame. The laws established among these space entities are obviously true for all orthogonal frames. In other words, the laws among the components in any reference frame are invariant under orthogonal transformations. The natural extension of this procedure is to define all physical entities in the four-dimensional space of x , y , z and ict , and establish all laws directly for these entities.

No direct intuitive way of reasoning is possible in this four-dimensional space, but by use of the general way of definition and transformation already shown it is possible to reason directly with the formal representation of these entities. By the use of partitioned matrices it is possible to keep the three-dimensional entities clearly in evidence and thus follow the physical reasoning through every mathematical derivation.

The entities necessary in this work will be scalars, vectors and tensors in four-dimensional space. The character of the component three-dimensional entities can easily be obtained by performing a three-dimensional space transformation.

Nothing has to be said about scalars, except perhaps that they are not, in general, the same scalars of three-dimensional space.

Consider the four-dimensional vector $V = \begin{bmatrix} \underline{u} \\ \underline{a} \end{bmatrix}$, where \underline{u} is a 3 x 1 matrix and \underline{a} a number.

$$\begin{bmatrix} \underline{u}' \\ \underline{a}' \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{a} \end{bmatrix}$$

from which

$$\underline{u}' = R\underline{u}$$

$$\underline{a}' = \underline{a}$$

This shows that \underline{u} is a three-dimensional vector (which may be called the space component) and \underline{a} is a three-dimensional scalar (which may be called the time component).

Consider now the four-dimensional tensor

$$\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C}_t & \underline{D} \end{bmatrix}$$

where \underline{A} is a 3 x 3 matrix (which may be called the space-space part), \underline{B} a 3 x 1 matrix (which may be called the space-time part), \underline{C} is a 3 x 1 matrix (time-space part) and \underline{D} a 1 x 1 matrix (time-time part). Performing a three-dimensional transformation

$$\begin{bmatrix} \underline{A}' & \underline{B}' \\ \underline{C}'_t & \underline{D}' \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C}_t & \underline{D} \end{bmatrix} \begin{bmatrix} R_t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{R}\underline{A} & \underline{R}\underline{B} \\ \underline{C}_t & \underline{D} \end{bmatrix} \begin{bmatrix} R_t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{R}\underline{A}R_t & \underline{R}\underline{B} \\ \underline{C}_tR_t & \underline{D} \end{bmatrix}$$

The space-space part is transformed according to

$$\underline{A}' = \underline{R}\underline{A}R_t$$

showing that the space-space part is a three-dimensional tensor. Also

$$\underline{B}' = \underline{R}\underline{B} \quad \text{and} \quad \underline{C}' = \underline{R}\underline{C}$$

showing that the space-time and time-space parts are three-dimensional vectors

Finally

$$\underline{D}' = \underline{D}$$

showing that the time-time part is a three-dimensional scalar.

According to this formulation, all entities defined in three-dimensional space are simply components of the four-dimensional entities. They have an invariant meaning as long as transformations between fixed frames are considered; however, they do not have an invariant meaning when transformations between moving frames are considered. So, they have to obey laws which result from the four-dimensional laws in order to comply with the initial postulates. This is easy to accomplish, by simply generalizing the procedure of definition used for three-dimensional entities, as will be shown in the next section.

2.5 Relativistic Mechanics and Maxwell's Equations (9,10,11)

Lorentz transformations, being a consequence of the first four postulates adopted here, imply that relativistic mechanics should be used in this investigation. Since Einstein's relativity arose from the interplay of Maxwell's equations with the principle of relativity, it is to be expected that relativistic mechanics and electrodynamics are closely related. Consequently, it is proper to develop relativistic mechanics and the four-dimensional formulation of electrodynamics simultaneously. This will also be a good opportunity for showing the ease and clarity afforded by the matrix notation adopted in this work.

In order to see what path has to be followed in the development of the four-dimensional formulation of mechanics, it is convenient to consider the way in which the character of the necessary entities was ascertained in classical mechanics.

Starting with the position vector

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and recognizing that time \underline{t} is an invariant in classical mechanics, the vector character of the velocity follows from its definition

$$\frac{d\underline{r}}{d\underline{t}} = \underline{v} = \begin{bmatrix} \frac{dx}{d\underline{t}} \\ \frac{dy}{d\underline{t}} \\ \frac{dz}{d\underline{t}} \end{bmatrix}$$

From the assumption that mass is an invariant, there follows the vector character of momentum

$$\underline{G} = m\underline{v}$$

The fundamental law of mechanics

$$\underline{\mathcal{F}} = \frac{d(m\underline{v})}{d\underline{t}} = \frac{d\underline{G}}{d\underline{t}}$$

is proof of the vector character of force. In the mechanics of deformable bodies, a stress tensor is defined in order to obtain the force \underline{f} per unit area transmitted through an area of normal unit vector \underline{n}

$$\underline{f} = \underline{T}\underline{n}$$

The tensor character of \underline{T} results from this definition and the vector character of \underline{f} and \underline{n} . With these entities and others defined from them, the whole mechanics can be developed.

Following the same path for relativistic mechanics, the starting point is the position vector in four-dimensional space

$$\underline{R} = \begin{bmatrix} \underline{r} \\ i\underline{c}\underline{t} \end{bmatrix}$$

A difficulty immediately arises here. Time \underline{t} , being a variable, destroys the vector character of \underline{R} by differentiation. The best solution is to define a parameter (invariant) closely connected to time. This is done in the following way: The differential of \underline{R} is a four-dimensional vector

$$dR = \begin{bmatrix} dr \\ icdt \end{bmatrix}$$

Its magnitude is a scalar

$$|dR|^2 = |dr|^2 - c^2 dt^2$$

Thus, the following scalar may be defined

$$d\tau^2 = - \frac{|dR|^2}{c^2} = dt^2 - \frac{dr^2}{c^2} = dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

or, finally

$$d\tau = \frac{dt}{\gamma}$$

which shows that τ is a scalar (invariant) closely connected to time. Its physical interpretation can be obtained as follows. A particle in motion is represented in four-dimensional space by the line described by the tip of the position vector \underline{R} . In a reference frame in which the particle is at rest at a given instant (which is called the rest frame of the particle) the previous relation gives

$$d\tau = dt_0$$

where t_0 is the time measured in the rest frame. This is true because the velocity of the particle is $v_0 = 0$, and thus $\gamma = 1$. Therefore, τ is the time measured in the successive rest frames of the particle in motion, or briefly, the time measured in following the particle.

Differentiation with respect to proper time, as τ is called, will not destroy the character of the entities because τ is a scalar by its very definition. On the other hand, since

$$\frac{d}{d\tau} = \frac{d}{dt} \frac{dt}{d\tau} = \gamma \frac{d}{dt}$$

this differentiation will give four-dimensional entities whose components

will be closely related to the three-dimensional entities obtained by differentiation with respect to time. In this way

$$\frac{dR}{d\tau} = \begin{bmatrix} \gamma \frac{dr}{dt} \\ ic\gamma \frac{dt}{dt} \end{bmatrix} = \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix}$$

which is called the four-dimensional velocity. This name is likely to be misleading. It is to be understood that the four-dimensional entities are built with three-dimensional entities, but are not a mere generalization of them. Their usefulness arises from the fact that from the laws established between them three-dimensional laws are deduced. It will be seen later that other entities are built with components of different three-dimensional character, and combine two three-dimensional laws into a single law. In this connection, it is worth mentioning that the use of partitioned matrices is a great help in keeping the role of the three-dimensional entities clearly in evidence.

It is useful to notice that the four-dimensional velocity has constant magnitude.

$$[\gamma v \quad ic\gamma] \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} = \gamma^2 v^2 - c^2 \gamma^2 = -\gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = -c^2$$

A four-dimensional vector having force as one of its components has to be built now, in order to pursue the development. This will be done through the use of the initial postulates, while establishing the four-dimensional form of Maxwell's equations.

One of the consequences of Maxwell's equations is the invariance of charge. In fact

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \frac{\partial(\epsilon_0 E)}{\partial t} = \nabla \cdot \left(\nabla \times \frac{B}{\mu_0} - j \right) = -\nabla \cdot j$$

which may be rewritten

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

This is the so-called equation of continuity, which implies conservation of charge. In fact, for the case of $\mathbf{j} = \rho \mathbf{v}$, there follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v} + \rho \nabla \cdot \mathbf{v} = \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$$

where $d\rho/dt$ is the derivative following a volume element of charge. The term $\nabla \cdot \mathbf{v}$ can be transformed by use of Gauss's theorem. Thus, considering a volume V changing in time, having a bounding surface S

$$dV = \oint_S \mathbf{v} dt \cdot \mathbf{n} dS = dt \oint_S \mathbf{v} \cdot \mathbf{n} dS = dt \int_V (\nabla \cdot \mathbf{v}) dV$$

or

$$\frac{dV}{dt} = \int_V (\nabla \cdot \mathbf{v}) dV$$

For an element of volume δV

$$\frac{d(\delta V)}{dt} = (\nabla \cdot \mathbf{v}) \delta V$$

which substituted in the equation of continuity above gives, after multiplication by δV

$$\frac{d\rho}{dt} \delta V + \rho \frac{d(\delta V)}{dt} = \frac{d}{dt} (\rho \delta V) = 0$$

This shows that an element of charge is conserved in its motion, or in other words, is independent of its velocity. In conclusion

$$\rho \delta V = \text{invariant} = \frac{\rho}{\gamma} (\delta \delta V)$$

On the other hand, the element of four-dimensional volume is also invariant, so

$$icdt\delta V = ic\delta\tau\delta V = \text{invariant}$$

and, since $d\tau$ is invariant,

$$\delta\delta V = \text{invariant}$$

Thus, the conclusion

$$\frac{\rho}{\gamma} = \text{invariant}$$

The result of the multiplication of the four-dimensional velocity, by this invariant is therefore a four-dimensional vector.

$$\frac{\rho}{\gamma} \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} = \begin{bmatrix} \rho v \\ ic\rho \end{bmatrix} = \begin{bmatrix} j \\ ic\rho \end{bmatrix} = \text{four-dimensional vector}$$

This is the so-called four-dimensional current density. The name here may be even more misleading than in the case of the four-dimensional velocity, since this four-dimensional vector is composed of two physically meaningful three-dimensional entities. This is clearly shown in the matrix notation used, which in addition always keeps the positions of the components, and thus shows their role and three-dimensional significance.

It is an easy matter now to establish the four-dimensional electromagnetic field tensor. In order to achieve this it is only necessary to conveniently rewrite the pair of Maxwell's equations containing current density and charge density, using ict instead of t

$$\frac{1}{\mu_0} (B\nabla - \frac{i}{c} \frac{\partial E}{\partial(ict)}) = j$$

$$\frac{1}{\mu_0} \nabla \cdot (\frac{i}{c} E) = ic\rho$$

which may be combined to form four-dimensional entities

$$\frac{1}{\mu_0} \begin{bmatrix} B & -\frac{i}{c} E \\ \frac{i}{c} E_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = \begin{bmatrix} j \\ ic\rho \end{bmatrix}$$

where $\partial_{4t} = \partial/\partial(ict)$, and the index t indicates transposition.

The operator $\square = \begin{bmatrix} \nabla \\ \partial_{4t} \end{bmatrix}$ transforms like a vector, as can be seen by the fact that when applied to a scalar ψ it produces a vector

$$\square\psi = \text{vector}$$

This vector character follows from

$$d\psi = (\square\psi) \cdot dR = \text{scalar}$$

This in turn implies that in the four-dimensional Maxwell's equation written above the field entity when applied to the \square vector produces the four-dimensional current density vector

$$J = \begin{bmatrix} j \\ ic\rho \end{bmatrix}$$

Therefore

$$F = \begin{bmatrix} B & -\frac{i}{c} E \\ \frac{i}{c} E_t & 0 \end{bmatrix}$$

is a four-dimensional tensor, the electromagnetic field tensor, which is seen to be antisymmetric. The previous Maxwell equation may therefore be written as

$$\frac{1}{\mu_0} F \square = J$$

The other pair of equations become

$$\overset{c}{E} \nabla + ic \frac{\partial B}{\partial(ict)} = 0$$

and

$$\nabla_t \overset{c}{B} = 0$$

(where $\overset{c}{E}$ is the antisymmetric pseudo tensor whose complement is \underline{E}) or, in

matrix notation

$$\begin{bmatrix} c & c \\ E & icB \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_{t_4} \end{bmatrix} = 0$$

which on multiplication by $-\frac{i}{c}$ becomes

$$\begin{bmatrix} -\frac{i}{c} c & c \\ E & B \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_{t_4} \end{bmatrix} = 0$$

or

$$F \square = 0$$

where F is the four-dimensional complement of F .

It is now possible to see the four-dimensional character of force density by the use of the formula

$$f = Bj + Ep$$

which in four-dimensional notation is

$$\begin{bmatrix} B & -\frac{i}{c} E \\ \frac{i}{c} E_t & 0 \end{bmatrix} \begin{bmatrix} j \\ icp \end{bmatrix} = \begin{bmatrix} f \\ \frac{i}{c} p \end{bmatrix}$$

where $p = E \cdot j$ is the power density. This is the four-dimensional force density, which is a vector having force density for the space component and power density for the time component.

Integration of the four-dimensional force density vector over a given region of space-time gives a four-dimensional vector which will be shown to be built from the increments of momentum and energy. In fact, this integral when divided by ic can be written

$$\int_{\text{space}} \begin{bmatrix} f \\ \frac{i}{c} p \end{bmatrix} dt dV = \int_{t_1}^{t_2} \begin{bmatrix} \mathcal{F} \\ \frac{i}{c} P \end{bmatrix} dt$$

where \mathcal{F} is the force acting on the three-dimensional volume over which integration was performed and P is the power fed into that same volume. Defining the increment of momentum ΔG as the integral of force over the time $t_2 - t_1$, and calling ΔW the increment of energy of the volume during the same time

$$\int_{t_1}^{t_2} \begin{bmatrix} \mathcal{F} \\ \frac{i}{c} P \end{bmatrix} dt = \Delta \begin{bmatrix} G \\ \frac{i}{c} W \end{bmatrix}$$

This shows that

$$\begin{bmatrix} G \\ \frac{i}{c} W \end{bmatrix} = \text{four-dimensional vector}$$

The classical equation $G = mv$ now has to be generalized in such a way that four-dimensional vectors are equated and the space components go over into the classical equation for small velocities which is known to be true. The most natural generalization is

$$\begin{bmatrix} G \\ \frac{i}{c} W \end{bmatrix} = m_0 \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix}$$

where m_0 is the mass measured in the rest system of the body being considered (and thus is an invariant) giving, therefore, an equation between two four-dimensional vectors. It is possible to retain the classical form by defining mass in general as being

$$m = m_0 \gamma$$

and so

$$\begin{bmatrix} G \\ \frac{1}{c} W \end{bmatrix} = \begin{bmatrix} mv \\ icm \end{bmatrix}$$

The time components give the well-known result

$$W = mc^2$$

meaning that mass is a measure of energy content.

Differentiation of the above equation with respect to proper time gives the following four-dimensional vector equation

$$\begin{bmatrix} \gamma \mathcal{F} \\ \frac{1}{c} \gamma P \end{bmatrix} = \begin{bmatrix} \gamma \frac{d(mv)}{dt} \\ ic \gamma \frac{dm}{dt} \end{bmatrix}$$

which shows that the definition of mass adopted ($m = m_0 \gamma^3$) allows the classical equation

$$\mathcal{F} = \frac{d(mv)}{dt}$$

to be rigorously correct for any inertial reference frame. The above four-dimensional equation can also be written in the following way

$$\begin{bmatrix} \gamma \mathcal{F} \\ \frac{1}{c} \gamma P \end{bmatrix} = \frac{dm_0}{d\tau} \begin{bmatrix} \gamma v \\ ic \gamma \end{bmatrix} + m_0 \frac{d}{d\tau} \begin{bmatrix} \gamma v \\ ic \gamma \end{bmatrix}$$

and, since $\begin{bmatrix} \gamma v \\ ic \gamma \end{bmatrix}$ is a vector of constant magnitude, there follows

$$[\gamma v_t \quad ic \gamma] \frac{d}{d\tau} \begin{bmatrix} \gamma v \\ ic \gamma \end{bmatrix} = 0$$

Consequently,

$$[\gamma v_t \quad ic \gamma] \begin{bmatrix} \gamma \mathcal{F} \\ \frac{1}{c} \gamma P \end{bmatrix} = \frac{dm_0}{d\tau} [\gamma v_t \quad ic \gamma] \begin{bmatrix} \gamma v \\ ic \gamma \end{bmatrix}$$

or, in a different notation,

$$\gamma^2(P - \mathcal{F} \cdot v) = \frac{dm_0}{d\tau} \gamma^2 (c^2 - v^2) = c^2 \frac{dm_0}{d\tau}$$

For purely mechanical systems in which $P = \mathcal{F} \cdot v$ this equation gives

$$\frac{dm_0}{d\tau} = \frac{dm_0}{dt} = 0$$

showing that the rest mass is conserved in the rest system. In this case the four-dimensional force is perpendicular to the four-dimensional velocity.

When the above statement is not true, there is a non-mechanical power

$Q = P - \mathcal{F} \cdot v$, and the rest mass is not constant. In fact

$$\gamma^2 Q = c^2 \frac{dm_0}{d\tau}$$

or

$$\gamma Q = c^2 \frac{dm_0}{dt}$$

which in the rest system gives

$$Q_0 = c^2 \frac{dm_0}{dt_0}$$

showing that the increase in rest mass is due to the non-mechanical power.

The non-mechanical power in any system will be related to the non-mechanical power in the rest system by

$$\gamma^2 Q = Q_0$$

2.6 Transformation Laws

It is useful to see the transformation laws of some of the more important entities developed up to now.

The transformation laws of vectors are typified by the law of transformation of an increment of position vector. The most intuitive formulation

is in terms of components parallel and perpendicular to the relative velocity,
i.e.

$$dr'_\perp = dr_\perp$$

$$dr'_\parallel = \gamma(dr_\parallel - u dt)$$

$$dt' = \gamma\left(dt - \frac{u \cdot dr}{c^2}\right)$$

where \underline{u} is the velocity of O' as measured in O .

For small velocities the following approximations result

$$dr' = dr - u dt$$

$$dt' = dt - \frac{u \cdot dr}{c^2}$$

The usual values of \underline{dr} considered are such that the last equation may again
be approximated by

$$dt' = dt$$

In the case of the velocity, the transformation is obtained in the
following way

$$\gamma_{v', v'_\perp} = \gamma_{v, v_\perp}$$

where

$$\gamma_{v'} = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}}$$

and

$$\gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

thus

$$v'_1 = \frac{\gamma}{\gamma_{v'}} v_1$$

In addition

$$\gamma_{v'} v'_1 = \gamma(\gamma v_{11} - u \gamma_v)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

and, thus

$$v'_{11} = \frac{\gamma \gamma_v}{\gamma_{v'}} (v_{11} - u)$$

The time component gives

$$\gamma_{v'} = \gamma \left(\gamma_v \left(1 - \frac{u \cdot v}{c^2} \right) \right)$$

from which

$$\gamma_{v'} = \gamma \gamma_v \left(1 - \frac{u \cdot v}{c^2} \right)$$

Finally the transformation formulas are obtained

$$v'_1 = \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{u \cdot v}{c^2}} v_1$$

and

$$v'_{11} = \frac{v_{11} - u}{1 - \frac{u \cdot v}{c^2}}$$

For small velocities

$$v' = \frac{v - u}{1 - \frac{u \cdot v}{c^2}}$$

Only for u and v much smaller than c can the following approximation be used.

$$v' = v - u$$

For current and charge densities the following is obtained

$$j'_\perp = j_\perp$$

$$j'_{\parallel} = \gamma(j_{\parallel} - \rho u)$$

$$\rho' = \gamma(\rho - \frac{u \cdot j}{c^2})$$

which is approximated by

$$j' = j - \rho u$$

$$\rho' = \rho - \frac{u \cdot j}{c^2}$$

Further approximations are allowed only in certain kinds of problems. For the case of charges of the same sign, then

$$j = \rho v$$

and

$$\rho' = \rho(1 - \frac{u \cdot v}{c^2})$$

which can be approximated by

$$\rho' = \rho$$

when u and v are much smaller than c.

This approximation can then be used every time a strong predominance of charges of one sign exists at a point. On the contrary, when charges of opposite signs almost balance each other at a given point (as in conduction

problems) the term $(u \cdot j)/c^2$ cannot be neglected. In these cases, the first equation can generally be approximated by

$$j' = j$$

that is, pu may be neglected for small u . As was discussed in the introduction, this corresponds to neglecting $u \times B$ in the first case, and $(u \times E)/c^2$ in the second case, in the field vectors.

In the case of force, the following transformation is obtained

$$\gamma_{v'} \mathcal{F}'_{\perp} = \gamma_v \mathcal{F}_{\perp}$$

giving

$$\mathcal{F}'_{\perp} = \frac{\gamma_v}{\gamma_{v'}} \mathcal{F}_{\perp}$$

or, explicitly,

$$\mathcal{F}'_{\perp} = \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{u \cdot v}{c^2}} \mathcal{F}_{\perp}$$

For the component parallel to u

$$\gamma_{v'} \mathcal{F}'_{\parallel} = \gamma (\gamma_v \mathcal{F}_{\parallel} - \frac{u \gamma_v P}{c^2})$$

from which

$$\mathcal{F}'_{\parallel} = \frac{\gamma \gamma_v}{\gamma_{v'}} (\mathcal{F}_{\parallel} - \frac{Pu}{c^2})$$

or, explicitly,

$$\mathcal{F}'_{\parallel} = \frac{\mathcal{F}_{\parallel} - \frac{Pu}{c^2}}{1 - \frac{u \cdot v}{c^2}}$$

These two transformation formulas for the force have a behavior similar to the transformation formulas of velocity because the force is not a component of four-dimensional force per se but becomes one only after multiplication by a γ factor. In addition the characteristic behavior of time in Einstein's relativity introduces the novel feature of power being a part of the transformed force, and this is still true for the approximate formulas unless power is small enough to make Pu/c^2 negligible. For the power

$$\frac{\gamma_{v'} P'}{c^2} = \gamma \left(\frac{\gamma_v P}{c^2} - \frac{u \cdot \gamma_v \mathcal{F}}{c^2} \right)$$

giving

$$P' = \frac{\gamma \gamma_v}{\gamma_{v'}} (P - u \cdot \mathcal{F})$$

or, explicitly,

$$P' = \frac{P - \mathcal{F} \cdot u}{1 - \frac{u \cdot v}{c^2}}$$

The only novel feature here is the denominator, which is mainly a contraction effect and is negligible for small relative velocities. The following approximations result

$$\mathcal{F}' = \mathcal{F} - \frac{Pu}{c^2}$$

and \underline{P} is usually such that Pu/c^2 can be neglected, yielding

$$\mathcal{F}' = \mathcal{F}$$

and

$$P' = P - \mathcal{F} \cdot u$$

For the force density the transformation is

$$f'_\perp = f_\perp$$

and

$$f'_{||} = \gamma \left(f_{||} - \frac{pu}{c^2} \right)$$

and also

$$\frac{p'}{c^2} = \gamma \left(\frac{p}{c^2} - \frac{f \cdot u}{c^2} \right)$$

or

$$p' = \gamma (p - f \cdot u)$$

Here again the tie between space and time makes force density borrow from power density in its transformation, in addition to the contraction effects which are negligible for small relative velocities. For small velocities

$$f' = f - \frac{pu}{c^2}$$

which can usually be approximated by

$$f' = f$$

and

$$p' = p - f \cdot u$$

The transformations for the field vectors can be obtained from the transformation for the field tensor

$$\begin{bmatrix} 0 & B'_3 & -B'_2 & -\frac{1}{c} E'_1 \\ -B'_3 & 0 & B'_1 & -\frac{1}{c} E'_1 \\ B'_2 & -B'_1 & 0 & -\frac{1}{c} E'_3 \\ \frac{1}{c} E'_1 & \frac{1}{c} E'_2 & \frac{1}{c} E'_3 & 0 \end{bmatrix} =$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \gamma & \frac{1}{c}\gamma u \\ 0 & 0 & -\frac{1}{c}\gamma u & \gamma \end{array} \right] \left[\begin{array}{cc|cc} 0 & B_3 & -B_2 & -\frac{1}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{1}{c}E_2 \\ \hline B_2 & -B_1 & 0 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & \frac{1}{c}E_2 & \frac{1}{c}E_3 & 0 \end{array} \right] L_t$$

which, for brevity, will be written with partitioned matrices composed of 2 x 2 matrices.

$$\begin{bmatrix} A' & -C'_t \\ C' & D' \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} A & -C_t \\ C & D \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & \alpha_t \end{bmatrix} = \begin{bmatrix} A & -C_t \\ \alpha C & \alpha D \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & \alpha_t \end{bmatrix} = \\ = \begin{bmatrix} A & -C_t \alpha_t \\ \alpha C & \alpha D \alpha_t \end{bmatrix}$$

Equating the component matrices, the following transformations result

$$A' = A$$

and so

$$B'_3 = B_3$$

Also

$$\begin{aligned}
 D' &= \frac{1}{c} E'_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \alpha D \alpha_t \\
 &= \gamma \begin{bmatrix} 1 & \frac{1}{c} u \\ -\frac{1}{c} u & 1 \end{bmatrix} \frac{1}{c} E_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \gamma \begin{bmatrix} 1 & -\frac{1}{c} u \\ \frac{1}{c} u & 1 \end{bmatrix} \\
 &= \frac{1}{c} E_3 \gamma^2 \begin{bmatrix} 0 & -(1 - \frac{u^2}{c^2}) \\ 1 - \frac{u^2}{c^2} & 0 \end{bmatrix}
 \end{aligned}$$

which gives the result

$$\frac{i}{c} E'_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{i}{c} E_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

or

$$E'_3 = E_3$$

The perpendicular components appear in the transformation of the 2×2 C matrix.

$$C' = \begin{bmatrix} B'_2 & -B'_1 \\ \frac{i}{c} E'_1 & \frac{i}{c} E'_2 \end{bmatrix} = \alpha C = \gamma \begin{bmatrix} 1 & \frac{i}{c} u \\ -\frac{i}{c} u & 1 \end{bmatrix} \begin{bmatrix} B_2 & -B_1 \\ \frac{i}{c} E_1 & \frac{i}{c} E_2 \end{bmatrix}$$

$$= \gamma \begin{bmatrix} (B_2 - \frac{uE_1}{c^2}) & - (B_1 + \frac{uE_2}{c^2}) \\ \frac{i}{c} (E_1 - uB_2) & \frac{i}{c} (E_2 + uB_1) \end{bmatrix}$$

Putting all components together

$$\begin{bmatrix} E'_1 \\ E'_2 \\ E'_3 \end{bmatrix} = \begin{bmatrix} \gamma E_1 \\ \gamma E_2 \\ E_3 \end{bmatrix} + \gamma \begin{bmatrix} -uB_2 \\ uB_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E_1 \\ \gamma E_2 \\ E_3 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \times \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

or

$$E'_{||} = E_{||}$$

and

$$E'_\perp = \gamma (E_\perp + u \times B)$$

And, for the magnetic field

$$\begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \end{bmatrix} = \begin{bmatrix} \gamma B_1 \\ \gamma B_2 \\ B_3 \end{bmatrix} - \frac{\gamma}{c^2} \begin{bmatrix} -uE_2 \\ uE_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma B_1 \\ \gamma B_2 \\ B_3 \end{bmatrix} - \frac{\gamma}{c^2} \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

or

$$B'_{11} = B_{11}$$

and

$$B'_{\perp} = \gamma \left(B_{\perp} - \frac{u \times E}{c^2} \right)$$

The approximations for small velocities are

$$E' = E + u \times B$$

and

$$B' = B - \frac{u \times E}{c^2}$$

In this way it is easy to see that the transformations of the field vectors are a consequence of space-time behavior and they are in complete harmony with the transformations of the other physical entities. The existence of two types of electric charge, that can neutralize each other as far as production of electric field is concerned, made it possible to detect experimentally terms of the type $\frac{u \times B}{c^2}$ that stood alone clashing with the transformations of other entities at a time in which the wrong transformations were used.

CHAPTER III

FIELDS IN MATERIAL BODIES

3.1 Comments on Postulate 5

Postulates 1 and 2 have assumed that the field vectors are tools for the purpose of studying the interaction between charges. The developments that followed combined these field vectors into a single entity, the electromagnetic field tensor. This is the physically significant entity and its parts, electric and magnetic, are merely aspects seen by an observer, depending on his state of motion. Electric charge is the source of the electromagnetic field tensor, and the four-current density is the appropriate entity to be used in the equations that define the tensor. The necessity of using the mathematically defined four-current density arises from the fact that the relative motion between charge and observer affects the aspects as seen by the observer. It is clear, then, that the three-dimensional current density and magnetic field are due to relativistic effects. In addition, the special structure of space-time unraveled by the principle of relativity makes charge density and electric field depend on relative motion also. It then seems natural to consider electric charge as the only physical cause of the electromagnetic field tensor. Consequently, the interaction between material bodies and electromagnetic fields has to be attributed to the constitutive charges of the material. If any other constituent is assumed to take part in this interaction, then this would require a modification of Maxwell's equations in vacuo in order to account for the influence of this additional constituent as far as fields are concerned.

From the previous development it seems almost unavoidable to assume

that any different constituent would have to be described by a different associated field. This is the point of view followed here. In other words, it is assumed that the only source of electromagnetic field is electric charge. Or, equivalently, the electromagnetic field is the only entity necessary to describe the interaction between charges. These essentially are the considerations that led to Postulate 5.

In considering the interaction between materials and electromagnetic fields, the internal binding between the charges assumes special importance. This binding is closely connected to the structure of matter and cannot be investigated by the macroscopic theory developed here. It depends on the interplay between the fact that charge exists in discrete portions and the principle of exclusion, and thus is strictly under the scope of a microscopic theory, requiring a quantum mechanical approach. The amount of information that can be obtained by the macroscopic approach, however, is sufficient justification for its use.

The influence of the internal binding has to be recognized, however, and assumed to be according to experimental evidence. This does not imply interaction with the electromagnetic field but only a different kind of interaction between the elements of matter superimposed on the electric interaction.

Three different processes are recognized in practice.

(a) Conduction. No binding exists, only a kind of viscous effect associated with the motion of charges. These will be displaced, reaching static equilibrium only when the field is annulled.

(b) Polarization. This process implies internal binding, and static equilibrium is reached when balance exists between the electric forces and the binding forces.

(c) Magnetization. This process is attributed in this approach to internal currents and, again, binding forces have to be assumed because equilibrium is reached for finite values of the internal fields. The existence of constant magnetization implies lossless internal currents in this process.

3.2 Polarization-magnetization Tensor

Conduction, being associated with free charges, is conveniently handled by considering current densities and charge densities because they will be displaced by the electromagnetic fields, no other mechanism being involved.

Polarization and magnetization can best be characterized by different entities, from which the internal current and charge densities can be derived. These entities are polarization and magnetization vectors. It will be shown that they can be combined into a four-dimensional tensor. The influence of motion in these entities results immediately from the transformation laws of current and charge densities. This is one of the advantages of the approach which follows from postulate 5. In addition, there is no possibility of ambiguities about the significance of the field vectors inside materials, since these materials are considered merely as aggregates of charges. Thus, E and B retain their original significance as they did in regions occupied by charges. They will still be the vectors to be used in calculating the force per unit volume on the net charge at a given point.

As will be seen in the development, polarization and magnetization find their ultimate justification in establishing a functional local tie (that characterizes the material) between the entities used in Maxwell's equations as generalized to materials. The functional relation between polarization and magnetization is generally very complex, nonlinear, and even dependent on the past history of the material. However, it may be assumed that a

condition of neutrality can be reached by the application of some convenient field. This means that, in the case of polarized materials, it is possible to write

$$\sum_i \rho_{oi} = 0$$

where the index o indicates this particular neutral state. From

$$\frac{\partial \rho_i}{\partial t} = -\nabla \cdot j_i$$

where

$$j_i = \rho_i v_i$$

there follows

$$\rho_i = \rho_{oi} - \int_{t_0}^{t_1} \nabla \cdot j_i dt$$

where t₁ is the time under consideration and t₀ the time at which the material was neutral. Consequently,

$$\sum_i \rho_i = -\nabla \cdot \int_{t_0}^{t_1} \sum_i j_i dt$$

Polarization may then be defined as

$$P = \int_{t_0}^{t_1} \sum_i j_i dt$$

Thus

$$\sum_i \rho_i = -\nabla \cdot P$$

It may be said that P is a three-dimensional vector whose divergence gives the net charge density at a given instant and at a given point in the material. Its importance arises from the fact that net charge density is the only pertinent quantity in the alteration of the fields.

From the definition it is immediately seen that

$$\frac{\partial P}{\partial t} = \sum_i j_i$$

that is, the partial derivative of P is also important in altering the fields.

The circulating currents that produce magnetization are obviously excluded from these definitions, because for them

$$\nabla \cdot j_k = 0$$

However, this condition allows them to be expressed by

$$j_k = \nabla \times A_k$$

which allows the following to be written.

$$\sum_k j_k = \nabla \times M^c = M \nabla^c$$

where M is the three-dimensional antisymmetric magnetization tensor and M^c is its complementary three-dimensional vector.

These definitions may be put in four-dimensional form, as follows

$$\begin{bmatrix} M & icP \\ -icP_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_t \end{bmatrix} = \begin{bmatrix} j_m \\ ic\rho_m \end{bmatrix}$$

where j_m and ρ_m are the net current and charge densities due to the polarization and magnetization. This shows at once that

$$\begin{bmatrix} M & icP \\ -icP_t & 0 \end{bmatrix}$$

is a four-dimensional tensor, obviously antisymmetric.

3.3 Maxwell's Equations for Materials

As already stated, postulate 5 reduces the study of materials to the case of fields due to charges, as in vacuo. Consequently, the same equations apply, with the only addition that the total current density should be used. In general, it is convenient to keep the bound charges and currents separated from the unbound charges.

The first equation is then written

$$\frac{1}{\mu_0} \begin{bmatrix} B & -\frac{i}{c} E \\ \frac{i}{c} E_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = \begin{bmatrix} j \\ ic\rho \end{bmatrix} + \begin{bmatrix} j_m \\ ic\rho_m \end{bmatrix}$$

where the first four-dimensional current density refers to unbound charges and the second to processes involving binding (polarization and magnetization).

No alteration exists in the second Maxwell equation

$$\begin{bmatrix} c & c \\ -\frac{i}{c} E & B \\ c & 0 \\ -B & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = 0$$

Introducing M and P in place of j_m and ρ_m , there follows

$$\frac{1}{\mu_0} \begin{bmatrix} B & -\frac{i}{c} E \\ \frac{i}{c} E_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = \begin{bmatrix} j \\ ic\rho \end{bmatrix} + \begin{bmatrix} M & icP \\ -icP_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

Maxwell's equations have to be supplemented by the functional relations between fields and magnetization, polarization and conduction in order to determine the field entities. This relation is generally established by measurements in conveniently simplified cases which make Maxwell's equations particularly easy to solve. These are generally stationary body cases.

Considerable simplification can be achieved by a slight modification

of the equations.

$$\begin{bmatrix} \left(\frac{B}{\mu_0} - M\right) & -ic(\mathcal{E}_0 E + P) \\ ic(\mathcal{E}_0 E + P)_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = \begin{bmatrix} j \\ ic\rho \end{bmatrix}$$

which, by the definitions

$$\frac{B}{\mu_0} - M = H$$

$$\mathcal{E}_0 E + P = D$$

may be rewritten as

$$\begin{bmatrix} H & -icD \\ icD_t & 0 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = \begin{bmatrix} j \\ ic\rho \end{bmatrix}$$

According to the point of view followed in this investigation, H and D appear as convenient mathematical entities having the unbound charges as sources.

The relation characterizing the bound processes can be considered to exist directly between the fields and the H and D quantities. In this sense, vacuum can be considered as a particular case of materials characterized by the well-known single linear relation between B and H, and E and D.

The transformations pertinent to P and M result directly from their character as components of a tensor. Rewriting

$$\begin{bmatrix} M & -\frac{1}{c}(-c^2 P) \\ \frac{1}{c}(-c^2 P)_t & 0 \end{bmatrix}$$

then, by analogy with the electromagnetic field tensor

$$-c^2 P'_{||} = -c^2 P_{||} \quad \therefore \quad P'_{||} = P_{||}$$

$$-c^2 P'_{\perp} = \gamma(-c^2 P_{\perp} + v \times M) \quad \therefore \quad P'_{\perp} = \gamma\left(P_{\perp} - \frac{v \times M}{c^2}\right)$$

$$M'_{||} = M_{||} \quad \therefore \quad M'_{||} = M_{||}$$

$$M'_{\perp} = \gamma(M_{\perp} - \frac{v \times (-c^2 P)}{c^2}) \quad \therefore \quad M'_{\perp} = \gamma(M_{\perp} + v \times P)$$

The approximate relations for small velocities are

$$P' = P - \frac{v \times M}{c^2}$$

$$M' = M + v \times P$$

Rewriting the H-D tensor as

$$\begin{bmatrix} H & -\frac{1}{c} (c^2 D) \\ \frac{1}{c} (c^2 D_t) & 0 \end{bmatrix}$$

there follows

$$c^2 D'_{||} = c^2 D_{||} \quad \therefore \quad D'_{||} = D_{||}$$

$$c^2 D'_{\perp} = \gamma(c^2 D_{\perp} + v \times H) \quad \therefore \quad D'_{\perp} = \gamma(D_{\perp} + \frac{v \times H}{c^2})$$

and

$$H'_{||} = H_{||} \quad \therefore \quad H'_{||} = H_{||}$$

$$H'_{\perp} = \gamma(H_{\perp} - \frac{v \times (c^2 D)}{c^2}) \quad \therefore \quad H'_{\perp} = \gamma(H_{\perp} - v \times D)$$

These relations are approximated at small velocities by

$$D' = D + \frac{v \times H}{c^2}$$

$$H' = H - v \times D$$

3.4 Physical Interpretation of the Influence of Motion

The four-dimensional formulation gives equations which comply with relativity. On the other hand, it obscures some of the intuitive meaning usually attached to the different entities. In addition, the relative velocity of the materials is not explicitly indicated. It is thus interesting to look at the intuitive three-dimensional aspect of the field equations showing explicitly the influence of motion. For clarity, the following will be restricted to the approximate equations. In addition, material entities will be introduced in the form usually encountered in experimental measurements. These entities are usually measured with the material at rest, i.e., in the rest frame of the material.

Calling \underline{v} the velocity with respect to the observer of a body at a given point, and indicating the rest quantities by an index \underline{o} , the approximate transformations for the observer's frame are

$$\underline{j} = \underline{j}_o + \rho_o \underline{v}$$

$$\rho = \rho_o + \frac{\underline{j}_o \cdot \underline{v}}{c^2}$$

$$\underline{M} = \underline{M}_o - \underline{v} \times \underline{P}_o$$

$$\underline{P} = \underline{P}_o + \frac{\underline{v} \times \underline{M}_o}{c^2}$$

Consequently,

$$\nabla \times \frac{\underline{B}}{\mu_o} - \frac{\partial(\underline{\epsilon}_o \underline{E})}{\partial t} = \underline{j}_o + \rho_o \underline{v} + \nabla \times \underline{M}_o + \nabla \times (\underline{P}_o \times \underline{v}) + \frac{\partial \underline{P}_o}{\partial t} - \frac{1}{c^2} \frac{\partial(\underline{M}_o \times \underline{v})}{\partial t}$$

and

$$\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_0 + \frac{\mathbf{j}_0 \cdot \mathbf{v}}{c^2} - \nabla \cdot \mathbf{P}_0 + \frac{1}{c^2} \nabla \cdot (\mathbf{M}_0 \times \mathbf{v})$$

In the left side of these equations are the field vectors as seen by the observer and in the right hand side the material entities as measured with the material at rest. This is the way in which the material entities are intuitively pictured and understood. The terms involving the velocity of the material at a given point can then be thought of as apparent material entities resulting from the motion of the material with respect to the observer.

3.5 Material Constants

Even though the above equations clearly show the physical meaning of the material quantities with respect to their effect in producing fields when the material is in motion, their usefulness is severely limited. This follows, because the material entities used are produced by fields in the rest frame of the material. They would be very useful for the case of permanent magnetization and polarization, i.e., in the cases in which these entities can be considered as being independent of the fields, and given as known quantities. In general, this is not the case, and the polarization, magnetization and current and charge densities appear simply as convenient ways of establishing the characteristics of the materials.

It is then usually better to establish the alterations that motion introduces in these characteristic equations. This will be done now for the simple case of homogeneous and isotropic materials in which they can be characterized at a given point at rest by constants.

The characteristic relations will then be written in the rest frame as

$$\mathbf{P}_0 = K \epsilon_0 \mathbf{E}$$

and

$$M_{\parallel} = \frac{B_{\parallel}}{\mu_0 K_m}$$

where K is the usual electric susceptibility, but K_m is used instead of the usual magnetic susceptibility in order to maintain symmetry and to comply with the point of view that B is producing the forces that separate charges and currents. There follows

$$P_{\parallel} = P_{O\parallel} = K \epsilon_0 E_{O\parallel} = K \epsilon_0 E_{\parallel}$$

or

$$P_{\parallel} = \epsilon_0 K E_{\parallel}$$

On the other hand

$$\begin{aligned} P_{\perp} &= \gamma \left(P_{O\perp} + \frac{v \times M_{\parallel}}{c^2} \right) = \gamma \left(\epsilon_0 K E_{O\perp} + \frac{1}{c^2} v \times \frac{B_{O\perp}}{\mu_0 K_m} \right) \\ &= \gamma \left[\epsilon_0 K \gamma (E_{\perp} + v \times B_{\perp}) + \frac{v}{c^2 \mu_0 K_m} \times \gamma (B_{\perp} - \frac{v \times E_{\perp}}{c^2}) \right] \\ &= \gamma^2 \epsilon_0 \left[\left(K + \frac{v^2}{c^2 K_m} \right) E_{\perp} + \left(K + \frac{1}{K_m} \right) v \times B \right] \end{aligned}$$

and thus,

$$P_{\perp} = \epsilon_0 \left[\left(\gamma^2 K + \frac{\gamma^2 - 1}{K_m} \right) E_{\perp} + \gamma^2 \left(K + \frac{1}{K_m} \right) v \times B \right]$$

For the magnetization

$$M_{\parallel} = M_{O\parallel} = \frac{B_{O\parallel}}{\mu_0 K_m} = \frac{B_{\parallel}}{\mu_0 K_m}$$

or

$$M_{\parallel} = \frac{B_{\parallel}}{\mu_0 K_m}$$

In addition

$$\begin{aligned}
 M_{\perp} &= \gamma(M_{O\perp} - v \times P_O) = \gamma\left(\frac{B_{O\perp}}{\mu_o K_m} - v \times \epsilon_o K E_{O\perp}\right) \\
 &= \frac{\gamma}{\mu_o K_m} \gamma\left(B_{\perp} - \frac{v \times E_{\perp}}{c^2}\right) - \gamma \epsilon_o K v \times \gamma(E_{\perp} + v \times B_{\perp}) \\
 &= \frac{\gamma^2}{\mu_o} \left[\frac{1}{K_m} + K\left(1 - \frac{1}{\gamma^2}\right)\right] B_{\perp} - \gamma^2 \epsilon_o \left(K + \frac{1}{K_m}\right) v \times E
 \end{aligned}$$

Consequently

$$M_{\perp} = \frac{1}{\mu_o} \left[\left[\frac{\gamma^2}{K_m} + (\gamma^2 - 1)K \right] B_{\perp} - \gamma^2 \left(K + \frac{1}{K_m}\right) \frac{v \times E}{c^2} \right]$$

The approximate relations for small velocities are

$$P = \epsilon_o [K E + \left(K + \frac{1}{K_m}\right) v \times B]$$

and

$$M = \frac{1}{\mu_o} \left[\frac{B}{K_m} - \left(K + \frac{1}{K_m}\right) \frac{v \times E}{c^2} \right]$$

These equations show that in first approximation the state of motion of a material does not alter its characteristic constants but introduces a new dependence of polarization on B and of magnetization on E in addition to the explicit dependence on v.

If preferred, the relations may be expressed in terms of the relative permittivity ϵ and the relative permeability μ .

The rest frame relations are

$$D_o = \epsilon_o \epsilon E_o \quad \text{and} \quad B_o = \mu_o \mu H_o$$

For an element of material with velocity v

$$D_{\parallel} = D_{o\parallel} = \epsilon_o \epsilon E_{o\parallel} = \epsilon_o \epsilon E_{\parallel}$$

or

$$D_{\parallel} = \epsilon_o \epsilon E_{\parallel}$$

$$\begin{aligned}
 D_{\perp} &= \gamma \left(D_{O\perp} - \frac{v \times H_{O\perp}}{c^2} \right) = \gamma \epsilon_0 \epsilon E_{O\perp} - \frac{\gamma}{c^2} v \times \frac{B_{O\perp}}{\mu_0 \mu} \\
 &= \gamma^2 \epsilon_0 \epsilon (E_{\perp} + v \times B_{\perp}) - \frac{\gamma^2}{c^2 \mu_0 \mu} v \times \left(B_{\perp} - \frac{v \times E_{\perp}}{c^2} \right) \\
 &= \left(\gamma^2 \epsilon_0 \epsilon - \frac{\gamma^2}{c^2 \mu_0 \mu} \frac{v^2}{c^2} \right) E_{\perp} + \left(\gamma^2 \epsilon_0 \epsilon - \frac{\gamma^2}{c^2 \mu_0 \mu} \right) v \times B
 \end{aligned}$$

Consequently

$$D_{\perp} = \epsilon_0 \left[\left(\gamma^2 \epsilon - \frac{\gamma^2}{\mu} - \frac{1}{\mu} \right) E_{\perp} + \gamma^2 \left(\epsilon - \frac{1}{\mu} \right) v \times B \right]$$

For the magnetic fields

$$B_{\parallel} = B_{O\parallel} = \mu_0 \mu H_{O\parallel} = \mu_0 \mu H_{\parallel}$$

or

$$B_{\parallel} = \mu_0 \mu H_{\parallel}$$

In addition

$$\begin{aligned}
 B_{\perp} &= \gamma \left(B_{O\perp} + \frac{v \times E_{\perp}}{c^2} \right) = \gamma \left(\mu_0 \mu H_{O\perp} + \frac{v}{c^2} \times \frac{D_{O\perp}}{\epsilon_0 \epsilon} \right) \\
 &= \gamma^2 \mu_0 \mu (H_{\perp} - v \times D_{\perp}) + \gamma^2 \frac{v}{c^2 \epsilon_0 \epsilon} \times \left(D_{\perp} + \frac{v \times H_{\perp}}{c^2} \right) \\
 &= \mu_0 \left[\gamma^2 \left(\mu - \frac{v^2}{c^2 \epsilon} \right) H_{\perp} - \gamma^2 \left(\mu - \frac{1}{\epsilon} \right) v \times D \right]
 \end{aligned}$$

or

$$B_{\perp} = \mu_0 \left[\left(\gamma^2 \mu - \frac{\gamma^2}{\epsilon} - \frac{1}{\epsilon} \right) H_{\perp} - \gamma^2 \left(\mu - \frac{1}{\epsilon} \right) v \times D \right]$$

The approximate relations are

$$D = \epsilon_0 \left[\epsilon E + \left(\epsilon - \frac{1}{\mu} \right) v \times B \right]$$

and

$$B = \mu_0 \left[\mu H - \left(\mu - \frac{1}{\epsilon} \right) v \times D \right]$$

Here again the effect of motion is to introduce a mixed dependence non-existent for the case of stationary material.

The only case that remains to be examined is that of conductors. The rest frame relation is

$$j_0 = \sigma E_0$$

and from this there follows

$$j_{\perp} = j_{0\perp} = \sigma E_{0\perp} = \sigma \gamma (E_{\perp} + v \times B)$$

and thus

$$j_{\perp} = \sigma \gamma (E_{\perp} + v \times B)$$

In addition

$$\begin{aligned} j_{\parallel} &= \gamma (j_{0\parallel} + \rho_0 v) = \gamma \sigma E_{0\parallel} + \gamma \rho_0 v \\ &= \gamma \sigma E_{\parallel} + \gamma v \gamma \left(\rho - \frac{v j_{\parallel}}{c^2} \right) \end{aligned}$$

Therefore

$$\gamma^2 j_{\parallel} = \gamma \sigma E_{\parallel} + \gamma^2 \rho v$$

or

$$j_{\parallel} = \frac{\sigma}{\gamma} E_{\parallel} + \rho v$$

The following approximate relation results

$$j_{\parallel} = \sigma (E + v \times B) + \rho v$$

No particular new feature appears in this result, because the term $v \times B$ was recognized as necessary to explain experimental results even though it created many difficulties in the classical theory.

CHAPTER IV

FORCES AND ENERGY IN MATERIALS

4.1 Stress-Energy Tensor

The development so far has been restricted to the calculation of the fields and to the investigation of the influence of motion on them. Essentially, fields were considered as tools to be used for the determination of forces on charges. It is well known that the existence of waves and the associated finite velocity of propagation of this action strengthened considerably the belief in a contiguity action. In this sense the fields acquired a physical meaning that transcended their meaning as mathematical tools. If this physical meaning is actually related to some true state of space-time, then forces should be transmitted along the fields through surfaces. Mathematically, this means that instead of calculating the total force on a body (or part of it) by integrating force density over the volume, it should be possible to obtain the same result by integrating a tension over the surface of that volume. On a differential point of view this means that it should be possible to find a tension, the divergence of which is the force density.

This is the aim of the following development, which repeats well-known calculations in order to point out the differences in point of view due to the postulates assumed here. That the field entities used here are the usual ones is shown by the fact that the usual Maxwell's equations were obtained for materials. The differences should then be due to postulate 5.

Considering the force on the total charge and current at a given point, there follows

$$\begin{bmatrix} \mathbf{f} \\ \frac{i}{c} \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & -\frac{1}{c} \mathbf{E} \\ \frac{i}{c} \mathbf{E}_t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{j}_{\text{tot}} \\ i c \rho_{\text{tot}} \end{bmatrix} = \mathbf{F} \mathbf{J}_{\text{tot}}$$

Recalling the first four-dimensional Maxwell's equation

$$\frac{1}{\mu_0} \mathbf{F} \square = \mathbf{J}_{\text{tot}}$$

there follows

$$\begin{bmatrix} \mathbf{f} \\ \frac{i}{c} \mathbf{p} \end{bmatrix} = \frac{1}{\mu_0} \mathbf{F}(\mathbf{F} \square) = [f_i]$$

In index notation this may be written

$$f_i = \frac{1}{\mu_0} \sum_{j,k} F_{ij} \frac{\partial F_{jk}}{\partial x_k} = \frac{1}{\mu_0} \sum_{j,k} \left[\frac{\partial (F_{ij} F_{jk})}{\partial x_k} - \frac{\partial F_{ij}}{\partial x_k} F_{jk} \right]$$

In order to obtain a divergence, the second term has to be modified. Thus

$$\sum_{j,k} \frac{\partial F_{ij}}{\partial x_k} F_{jk} = \sum_{j,k} \frac{\partial F_{ik}}{\partial x_j} F_{kj} = \sum_{j,k} \frac{\partial F_{ki}}{\partial x_j} F_{jk} = \frac{1}{2} \sum_{jk} \left(\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j} \right) F_{jk}$$

The term in parenthesis can be simplified by the use of the second four-dimensional Maxwell's equations. Since this equation involves $\overset{c}{\mathbf{F}}$, the sum referred to has to be written

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j} = \frac{\partial \overset{c}{F}_{kl}}{\partial x_k} + \frac{\partial \overset{c}{F}_{jl}}{\partial x_j}$$

where i, j, k, l is an even permutation of 1, 2, 3, 4. The antisymmetry of $\overset{c}{\mathbf{F}}$ allows the following transformation to be made.

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j} = \frac{\partial \overset{c}{F}_{kl}}{\partial x_k} + \frac{\partial \overset{c}{F}_{jl}}{\partial x_j} + \frac{\partial \overset{c}{F}_{il}}{\partial x_i} - \frac{\partial \overset{c}{F}_{il}}{\partial x_i} + \frac{\partial \overset{c}{F}_{ll}}{\partial x_l} = \sum_m \frac{\partial \overset{c}{F}_{ml}}{\partial x_m} - \frac{\partial \overset{c}{F}_{il}}{\partial x_i}$$

Due to the second four-dimensional Maxwell's equation

$$\sum_m^c \frac{\partial F}{\partial x_m} = 0$$

which, written in index notation, is

$$\sum_m^c \frac{\partial F}{\partial x_m} = 0$$

there follows that

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j} = - \frac{\partial F_{il}}{\partial x_i} = - \frac{\partial F_{jk}}{\partial x_i}$$

which gives

$$\sum_{j,k} \frac{\partial F_{ij}}{\partial x_k} F_{jk} = - \frac{1}{2} \sum_{jk} \frac{\partial F_{jk}}{\partial x_i} F_{jk} = - \frac{1}{4} \sum_{jk} \frac{\partial (F_{jk} F_{jk})}{\partial x_i} = \frac{1}{4} \frac{\partial}{\partial x_i} \sum_{l,m} (F_{lm} F_{ml})$$

This may now be substituted into the force density equation, giving

$$f_i = \frac{1}{\mu_0} \sum_{jk} \left[\frac{\partial (F_{ij} F_{jk})}{\partial x_k} - \frac{1}{4} \frac{\partial}{\partial x_k} \delta_{ik} \sum_{l,m} (F_{lm} F_{ml}) \right]$$

where δ_{ik} is the Kronicker delta. In matrix notation this equation is written

$$\begin{bmatrix} f \\ \frac{1}{c} p \end{bmatrix} = \frac{1}{\mu_0} [FF - \frac{1}{4} \{ \text{Tr}(FF) \} I_4] \square$$

where $\text{Tr}(FF)$ stands for the trace of the matrix (FF) .

In order to obtain a better visualization, the three-dimensional entities will be introduced.

$$FF = \begin{bmatrix} B & -\frac{1}{c} E \\ \frac{1}{c} E_t & 0 \end{bmatrix} \begin{bmatrix} B & -\frac{1}{c} E \\ \frac{1}{c} E_t & 0 \end{bmatrix} = \begin{bmatrix} BB + \frac{1}{c^2} EE_t & -\frac{1}{c} BE \\ \frac{1}{c} E_t B & \frac{1}{c^2} E_t E \end{bmatrix}$$

which can be put in a more common form by the use of \underline{B} in place of \underline{B} .

Recalling that

$$\mathbf{E} \times \mathbf{B} = \mathbf{c} \mathbf{E} \times \mathbf{B}$$

and

$$\mathbf{E}_t \mathbf{B} = -\mathbf{E}_t \mathbf{B}_t = -(\mathbf{E} \times \mathbf{B})_t = -(\mathbf{E} \times \mathbf{B})_t \mathbf{c}$$

and, in addition,

$$\mathbf{B} \mathbf{B} = \mathbf{B} \mathbf{B}_t \mathbf{c} - \mathbf{B}_t \mathbf{B} \mathbf{I}_3 \mathbf{c}$$

there follows

$$\mathbf{F} \mathbf{F} = \begin{bmatrix} \mathbf{c} \mathbf{c} (\mathbf{B} \mathbf{B}_t + \frac{1}{c^2} \mathbf{E} \mathbf{E}_t - \mathbf{B}_t \mathbf{B} \mathbf{I}_3) & -\frac{1}{c} (\mathbf{E} \times \mathbf{B}) \mathbf{c} \\ -\frac{1}{c} (\mathbf{E} \times \mathbf{B})_t \mathbf{c} & \frac{1}{c^2} \mathbf{E}_t \mathbf{E} \end{bmatrix}$$

It is immediately seen that

$$\frac{1}{4} \text{Tr}(\mathbf{F} \mathbf{F}) = -\frac{\mathbf{c} \mathbf{c} \mathbf{B}_t \mathbf{B}}{4} + \frac{2}{4} \frac{1}{c^2} \mathbf{E}_t \mathbf{E} = -\frac{\mathbf{c} \mathbf{c} \mathbf{B}_t \mathbf{B}}{2} + \frac{\mathbf{E}_t \mathbf{E}}{2 c^2}$$

and, thus,

$$\begin{bmatrix} \mathbf{f} \\ \frac{1}{c} \mathbf{p} \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} \left\{ \mathbf{c} \mathbf{c} \mathbf{B} \mathbf{B}_t + \frac{\mathbf{E} \mathbf{E}_t}{c^2} - \left(\frac{\mathbf{c} \mathbf{c} \mathbf{B}_t \mathbf{B}}{2} + \frac{\mathbf{E}_t \mathbf{E}}{2 c^2} \right) \mathbf{I}_3 \right\} & -\frac{1}{c} (\mathbf{E} \times \mathbf{B}) \mathbf{c} \\ -\frac{1}{c} (\mathbf{E} \times \mathbf{B})_t \mathbf{c} & \left(\frac{\mathbf{c} \mathbf{c} \mathbf{B}_t \mathbf{B}}{2} + \frac{\mathbf{E}_t \mathbf{E}}{2 c^2} \right) \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

or, in slightly different notation

$$\begin{bmatrix} f \\ \frac{i}{c} p \end{bmatrix} = \begin{bmatrix} \left\{ \frac{cc}{\mu_0} \frac{B_t B_t}{\mu_0} + \epsilon_0 \frac{E_t E_t}{\mu_0} - W I_3 \right\} \\ - \frac{i}{c} \left(E \times \frac{B}{\mu_0} \right)_t \\ W \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

where

$$W = \frac{cc}{2\mu_0} \frac{B_t B_t}{\mu_0} + \epsilon_0 \frac{E_t E_t}{2} = \text{energy density of the field}$$

The tensor on the right is the four-dimensional stress-energy tensor, which is obviously symmetric. It may be written as

$$\underline{\theta} = \begin{bmatrix} \sigma & -\frac{i}{c} S \\ -\frac{i}{c} S_t & W \end{bmatrix}$$

where

$$\sigma = \frac{cc}{\mu_0} \frac{B_t B_t}{\mu_0} + \epsilon_0 \frac{E_t E_t}{\mu_0} - W I_3$$

is the three-dimensional Maxwell stress-tensor of the field, and

$$S = E \times \frac{B}{\mu_0}$$

is the Poynting vector of the field.

This form of Poynting's vector is usually recognized as Poynting's vector for vacuo. In the approach adopted in this investigation it describes the energy flow to be charged to the field in general. This does not mean that it will give the total energy flow at a given point, since other energy flows may exist inside a material as the next section will show.

4.2 Physical Interpretation of the Stress-Energy Tensor

The four-dimensional equation for the force density can be written as two three-dimensional equations

$$f = \sigma \nabla - \frac{\partial}{\partial t} \left(\frac{S}{c} \right)$$

and

$$p = -\nabla \cdot S - \frac{\partial W}{\partial t}$$

The last equation, when written as

$$-\frac{\partial W}{\partial t} = p + \nabla \cdot S$$

shows that the decrease in energy density of the field at a given point is equal to the power p plus the divergence of S . Consequently, S gives the flow of energy per unit surface in the field, except for a solenoidal vector. Whether p has to be considered as storage of energy in the material or not depends on the particular case at hand, and will be discussed later.

The first equation, when written as

$$\sigma \nabla = f + \frac{\partial}{\partial t} \left(\frac{S}{c} \right)$$

shows that S/c^2 has to be considered as a momentum per unit volume, except for a vector constant in time. This strengthens the interpretation of S as an energy flow (except now for a time invariant solenoidal vector) due to the equivalence of mass and energy.

The force density obtained by this equation

$$f = \sigma \nabla - \frac{\partial}{\partial t} \left(\frac{S}{c} \right)$$

is the force acting on the total net charge and current at the given point, irrespective of whether or not there is material at this point. The different

parts of it can be understood by writing it in terms of the material quantities

$$f = E(\rho + \rho_m) + (j + j_m) \times B = \rho E - (\nabla \cdot P)E + j \times B + \frac{\partial P}{\partial t} \times B + (\nabla \times M) \times B$$

The terms ρE and $j \times B$ give the force on the unbound charge and current, and the others give force on the net bound charge and current. It should be noticed that the forces required to separate charges and currents in the processes of polarization and magnetization are not explicit here, since they are intrinsically included in the establishment of the fields. The only apparent forces are the net resultant forces which cannot be balanced by these internal processes and have to be balanced by inertia and internal stress of the material. These reactions will be discussed later.

The different parts of the power p can be understood by writing its expression in terms of the material quantities

$$p = E \cdot (j + j_m) = E \cdot j + E \cdot \frac{\partial P}{\partial t} + E \cdot (\nabla \times M)$$

The term $E \cdot j$ is the power fed into the free charges. It may be put into kinetic energy in increasing the speed of the charges (convective part) or into joule heat in the conduction currents. Whether this last is stored in the material or flows out as heat depends on the thermodynamical conditions. It will be stored if the process is adiabatic.

The term $E \cdot (\partial P / \partial t)$ is associated with the polarization processes, and again whether it is stored or not depends on the type of material and on the type of thermodynamical conditions. For lossless processes it is stored in the internal mechanism that binds the charges together. In lossy processes, part of it is transformed into heat which may be partly stored with an increase of temperature and partly carried away by thermal conduction.

The same considerations apply to the term $E \cdot (\nabla \times M)^c$, except for the following additional remarks. The existence of time invariant M^c in materials implies that $\nabla \times M^c$ refers to lossless currents. Consequently, if $E \cdot (\nabla \times M)^c \neq 0$, then M^c is time varying.

But a change in M^c implies motion of currents. In fact, the internal currents, being solenoidal, may be pictured as closed tubes, and since

$$\int_S (\nabla \times M)^c dS = \oint M^c \cdot d\ell$$

a changing M^c means a change in the current interlinked with a closed path ℓ , that is, motion into or out of ℓ . On the other hand, for $E \cdot (\nabla \times M)^c \neq 0$ there is a field component along these currents and therefore, these currents have a power content. In other words, there is an energy transport by the motion of these currents. This can be put in evidence by applying the formula

$$\nabla \cdot (E \times M)^c = M^c \cdot (\nabla \times E) - E \cdot (\nabla \times M)^c$$

from which

$$E \cdot (\nabla \times M)^c = M^c \cdot (\nabla \times E) - \nabla \cdot (E \times M)^c$$

The first term may be transformed with the aid of

$$\nabla \times E = - \frac{\partial B^c}{\partial t}$$

and, thus

$$E \cdot (\nabla \times M)^c = - M^c \cdot \frac{\partial B^c}{\partial t} - \nabla \cdot (E \times M)^c$$

showing that $E \times M^c$ is an energy flow per unit area.

This reasoning is not a proof of the validity of this interpretation,

but rather, a plausibility argument stemming from the desire to obtain the usual form of Poynting's theorem. In fact, the power equation may now be written

$$\mathbf{E} \cdot \mathbf{j} + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{M}) = - \nabla \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) - \frac{\partial}{\partial t} \left(\frac{\epsilon_0 \mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2 \mu_0} \right)$$

that is,

$$\mathbf{E} \cdot \mathbf{j} + \mathbf{E} \cdot \frac{\partial}{\partial t} (\mathbf{P} + \epsilon_0 \mathbf{E}) + \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left[\mathbf{E} \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \right] = 0$$

Or, by introducing the vectors $\underline{\mathbf{D}}$ and $\underline{\mathbf{H}}$

$$\mathbf{E} \cdot \mathbf{j} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0$$

It is important to notice here that this result was obtained without any hypothesis about the characteristic relations of the materials. Therefore, this is a perfectly general result. On the other hand, the power $\mathbf{E} \cdot (\partial \mathbf{D} / \partial t)$ is here to be thought of as composed of two parts: the field part $\mathbf{E} \cdot [\partial(\epsilon_0 \mathbf{E}) / \partial t]$ and the material part (storage in the internal binding mechanism) $\mathbf{E} \cdot (\partial \mathbf{P} / \partial t)$. The same applies to $\mathbf{H} \cdot (\partial \mathbf{B} / \partial t)$, which is composed of the field part

$$\frac{\mathbf{B}}{\mu_0} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

and the material part

$$- \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

In the same way, the energy flow per unit area contains two parts:

$\mathbf{E} \times (\mathbf{B} / \mu_0)$ due to the field and $-\mathbf{E} \times \mathbf{M}$ pertaining to the material. It is important to notice that these last interpretations are valid for adiabatic

processes, otherwise part of the energy storages in the materials has to be considered as heat flow.

It is also very important to notice that even for adiabatic processes in moving materials, the fact that the binding mechanism is carried by them makes the energy flow different from $\underline{E} \times \underline{M}^c$. It will be shown later how this case can be handled. These considerations are mentioned here to show that, according to the point of view taken here, the usual form of Poynting's theorem has to be considered as restricted to stationary bodies.

4.3 Comparison with the Usual Stress-Energy Tensors (10)

The stress-energy tensor obtained differs from the tensors formulated in the past for the case of materials. Apart from the differences in the space-time parts, both Minkowski's and Abraham's tensors have the same space-space and time-time part. These parts are Maxwell's three-dimensional stress tensor for materials and energy storage in fields in materials as usually used. It is the purpose of this section to investigate the reason for the differences obtained here. Comparisons will be made mainly with Minkowski's tensor, which is the most accepted.

As shown in Ref. 10, this tensor is obtained by starting with Maxwell's equation in the form

$$G_{\square} = J$$

$$F_{\square} = 0$$

where

$$G = \begin{bmatrix} H & -icD \\ icD_t & 0 \end{bmatrix}$$

and

$$J = \begin{bmatrix} j \\ icp \end{bmatrix}$$

Then the following four-dimensional vector is calculated

$$\begin{bmatrix} f' \\ \frac{1}{c} p' \end{bmatrix} = FJ = F(G \square)$$

According to the point of view adopted in this work, this four-dimensional vector gives the force and power densities acting on the unbound charges and currents, and does not include force and power densities acting on polarized and magnetized materials. The same type of mathematical deductions shown in Sec. 4.1 can then be followed up to the point of obtaining ⁽¹⁰⁾

$$f'_i = \sum_{j,k} \left[\frac{\partial(F_{ij} G_{jk})}{\partial x_k} - \frac{1}{2} \frac{\partial F_{jk}}{\partial x_i} G_{kj} \right]$$

For the case of media with ϵ and μ independent of position (homogeneous media), this can be written

$$f'_i = \sum_{j,k} \left[\frac{\partial(F_{ij} G_{jk})}{\partial x_k} - \frac{1}{4} \frac{\partial(F_{jk} G_{kj})}{\partial x_i} \right]$$

or, in matrix notation

$$\begin{bmatrix} f' \\ \frac{1}{c} p' \end{bmatrix} = [FG - \frac{1}{4} \text{Tr}(FG)] \square$$

In terms of three-dimensional entities this is

$$\begin{bmatrix} f' \\ \frac{1}{c} p' \end{bmatrix} = \begin{bmatrix} \left\{ \begin{matrix} \text{cc} \\ \text{HB}_t + \text{ED}_t - W' I_3 \end{matrix} \right\} & - ic(D \times B) \\ - \frac{1}{c} (E \times H)_t & W' \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

where

$$W' = \frac{D_t E}{2} + \frac{H_t B}{2}$$

The above tensor will only give the forces on the unbound charges and currents

as was already said, and will give zero internal forces in homogeneous polarized and magnetized materials, even for cases in which $\nabla \cdot P \neq 0$ and $\nabla \times M \neq 0$. This obviously follows from the definition of f' . The power density equation will give the usual Poynting's theorem in which field and material's binding mechanism storages and flows are lumped together.

The same tensor is then assumed for the case of non-homogeneous materials, by the following reasoning. (10)

$$f'_i = \sum_{jk} \left[\frac{\partial(F_{ij}G_{jk})}{\partial x_k} - \frac{1}{4} \frac{\partial(F_{jk}G_{kj})}{\partial x_i} \right] + \sum_{jk} \left[\frac{1}{4} F_{jk} \frac{\partial G_{kj}}{\partial x_i} + \frac{1}{4} \frac{\partial F_{jk}}{\partial x_i} G_{kj} - \frac{1}{2} \frac{\partial F_{jk}}{\partial x_i} G_{kj} \right]$$

which may be rewritten as

$$f'_i + \sum_{jk} \left[\frac{1}{4} \frac{\partial F_{jk}}{\partial x_i} G_{kj} - F_{jk} \frac{\partial G_{kj}}{\partial x_i} \right] = \sum_{jk} \frac{\partial}{\partial x_k} \left[F_{ij}G_{jk} - \frac{1}{4} \sum_m (F_{lm}G_{ml}) \right]$$

This then is postulated to be the four-dimensional force density in materials. It is apparent that this is equivalent to postulating

$$f''_i = \frac{1}{4} \sum_{jk} \left(\frac{\partial F_{jk}}{\partial x_i} G_{kj} - F_{jk} \frac{\partial G_{kj}}{\partial x_i} \right)$$

as the four-dimensional force density in the interior of non-homogeneous materials, since f'_i is the four-dimensional force density in the unbound charges. In particular, it will give the tensions at the surfaces of discontinuities. It is then clear that the use of Minkowski's tensor is contradictory to postulate 5 and implies a different interpretation of the interaction between materials and the electromagnetic field.

Even though the difference between the space-time parts is difficult to verify experimentally (as it is for the difference between Abraham's and Minkowski's tensor), the differences in the space-space parts, on the contrary,

do not seem to be difficult to verify experimentally. In fact, this verification can be made by a measurement of deformations in stationary fields. No difference exists between the total forces acting on a body in vacuo, because the result of the integration over the volume of the body has to be equal to the integral of the tensor over a bounding surface in vacuo. Since in vacuo $H = B/\mu_0$ and $D = \epsilon_0 E$, the results coincide for both tensors. On the other hand, the internal stresses are different, thus giving different resultant strains. This measurement will decide whether or not postulate 5 is adequate.

4.4 Inertial Reaction

The previous section developed the force with which the field acts on materials and showed that it can be calculated directly from the fields without any additional entities. These forces will act on the materials producing deformations and motion. Equations are necessary to establish the connection between the field forces and the motion and deformations produced.

These equations will introduce the reaction of the materials as inertial and elastic forces, and the equilibrium between these passive reaction forces and the active forces of the field will establish the state of motion and deformation of the body. In order to understand the inertial reaction in its simplest form, the following development will neglect the internal stresses; that is, this section will be restricted to the so-called incoherent matter. At a later stage, elastic stresses will be introduced and, finally, energy flow in the materials will be considered.

Two kinds of mass densities may be defined, the usual mass density

$$\rho = \frac{\delta m}{\delta V}$$

as the mass per unit volume, and the rest-mass density

$$\lambda_r = \frac{\delta m_o}{\delta V}$$

as the rest-mass per unit volume. In the rest frame these become equal, i.e.,

$$\lambda_o = \lambda_{ro} = \frac{\delta m_o}{\delta V_o}$$

Since

$$\delta m = \gamma \delta m_o$$

and

$$\delta V_o = \gamma \delta V$$

There follows

$$\frac{\delta m_o}{\delta V_o} = \frac{\delta m_o}{\gamma \delta V} = \frac{\delta m}{\gamma^2 \delta V}$$

which shows that

$$\lambda_o = \frac{\lambda_r}{\gamma} = \frac{\lambda}{\gamma^2}$$

The inertial effect is contained in the right hand side of the following equation

$$\begin{bmatrix} \gamma F \\ \frac{1}{c} \gamma P \end{bmatrix} = \frac{d}{d\tau} \left\{ m_o \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} \right\}$$

which for an element of volume δV takes the form

$$\begin{bmatrix} \frac{\gamma F}{\delta V} \\ \frac{1}{c} \gamma \frac{P}{\delta V} \end{bmatrix} \delta V = \frac{d}{d\tau} \left\{ \lambda_o \delta V_o \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} \right\}$$

or

$$\begin{bmatrix} f \\ \frac{i}{c} p \end{bmatrix} \delta v_o = \frac{d}{d\tau} \left\{ \gamma_o \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} \right\} + \gamma_o \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} \frac{d(\delta v_o)}{d\tau}$$

On the other hand, the equation

$$\frac{d(\delta v)}{dt} = (\nabla \cdot v) \delta v$$

becomes in the rest frame of the element

$$\frac{d(\delta v_o)}{dt_o} = \frac{d(\delta v_o)}{d\tau} = (\nabla_o \cdot v_o) \delta v_o$$

but

$$\nabla_o \cdot v_o = [\nabla_{ot} \quad \partial_{o4}] \begin{bmatrix} \gamma_o v_o \\ ic\gamma_o \end{bmatrix} = [\nabla_t \quad \partial_4] \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} = \text{invariant}$$

so that the above may be written

$$\frac{d(\delta v)}{dt} = [\nabla_t \quad \partial_4] \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix}$$

Consequently, the force-density equation may be written

$$f_i = \frac{d}{d\tau} (\gamma_o U_i) + \gamma_o U_i \left(\sum_k \frac{\partial U_k}{\partial x_k} \right)$$

where

$$[f_i] = \begin{bmatrix} f \\ \frac{i}{c} p \end{bmatrix}$$

and

$$[U_i] = \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix}$$

Transforming the first term of the second member

$$f_i = \sum_k \left[\frac{\partial(\gamma_0 U_i)}{\partial x_k} U_k + \gamma_0 U_i \frac{\partial U_k}{\partial x_k} \right]$$

$$= \sum_k \frac{\partial}{\partial x_k} (\gamma_0 U_i U_k)$$

or, in matrix form

$$\begin{bmatrix} f \\ \frac{1}{c} p \end{bmatrix} = \left\{ \gamma_0 \begin{bmatrix} \gamma v \\ ic\gamma \end{bmatrix} \begin{bmatrix} \gamma v_t & ic\gamma \end{bmatrix} \right\} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} =$$

$$= \begin{bmatrix} (\gamma_0 \gamma^2 v v_t) & (ic\gamma_0 \gamma^2 v) \\ ic\gamma_0 \gamma^2 v_t & -\gamma_0 c^2 \gamma^2 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

which gives

$$\begin{bmatrix} f \\ \frac{1}{c} p \end{bmatrix} = \begin{bmatrix} \gamma v v_t & ic\gamma v \\ ic\gamma v_t & -\gamma c^2 \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

Recognizing that γv is the momentum per unit volume

$$\gamma v = g$$

and, from the equivalence between mass and energy, that γc^2 is the energy density

$$\gamma c^2 = h$$

this equation may be written

$$\begin{bmatrix} f \\ \frac{1}{c} p \end{bmatrix} = \begin{bmatrix} g v_t & icg \\ \frac{1}{c} h v_t & -h \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

This four-dimensional equation is equivalent to the two following three-dimensional equations.

$$f = (g v_t) \nabla + \frac{\partial g}{\partial t}$$

which says that the force density is the source of momentum density, and

$$p = \nabla \cdot (hv) + \frac{\partial h}{\partial t}$$

which says that the power density is the source of energy density. For a closed system these equations imply the conservation of momentum and the conservation of energy.

The first equation also implies the conservation of angular momentum in a closed system, due to the symmetry of the space-space part of the four-dimensional inertia tensor. In order to see this, the first equation may be written

$$f_i = \sum_j \frac{\partial (g_i v_j)}{\partial x_j} + \frac{\partial g_i}{\partial t}$$

Consequently

$$\begin{aligned} x_k f_i &= \sum_j x_k \frac{\partial (g_i v_j)}{\partial x_j} + x_k \frac{\partial g_i}{\partial t} \\ &= \sum_j \frac{\partial (x_k g_i v_j)}{\partial x_j} + \frac{\partial (x_k g_i)}{\partial t} - \sum_j g_i v_j \frac{\partial x_k}{\partial x_j} - g_i \frac{\partial x_k}{\partial t} \\ &= \sum_j \frac{\partial (x_k g_i v_j)}{\partial x_j} + \frac{\partial (x_k g_i)}{\partial t} - 2 g_i v_k \end{aligned}$$

Therefore

$$x_k f_i - x_i f_k = \sum_j \frac{\partial}{\partial x_j} [(x_k g_i - x_i g_k) v_j] + \frac{\partial}{\partial t} (x_k g_i - x_i g_k) - 2 (g_i v_k - g_k v_i)$$

Due to the mentioned symmetry, the last term is zero. The quantity

$$x_k f_i - x_i f_k = M_{ki}$$

is recognized as the moment of the force density, and

$$x_k g_i - x_i g_k = m_{ki}$$

as the angular momentum density. The equation may then be written

$$M_{ik} = \sum_j \frac{\partial(m_{ik} v_j)}{\partial x_j} + \frac{\partial m_{ik}}{\partial t}$$

meaning that the moment of force density is the source of angular momentum density. For a closed system this implies the conservation of angular momentum.

For this to hold in any reference frame, it is necessary that $g_{\underline{v}_t}$ be symmetric in any reference frame, and this can be true only if the four-dimensional tensor is symmetric. There follows

$$g = \frac{h}{c^2} v$$

which again shows the equivalence between mass and energy.

It is interesting to see that the four-dimensional inertia tensor could have been obtained directly from the energy density in the rest frame. In fact, the only non-zero component in the rest frame is the time-time component, which is the energy density

$$\begin{bmatrix} 0 & 0 \\ 0 & -h_0 \end{bmatrix}$$

If a reference frame in which the material has a local velocity \underline{v} is now considered, the inertia tensor is obtained by the use of $L(-v)$, because $-\underline{v}$ is the velocity of this frame with respect to the rest frame. Therefore, the inertia tensor will be obtained by the following calculation.

$$\begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} vv_t & -\frac{i}{c}\gamma v \\ \frac{i}{c}\gamma v_t & \gamma \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -h_0 \end{bmatrix} \begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} vv_t & \frac{i}{c}\gamma v \\ -\frac{i}{c}\gamma v_t & \gamma \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\gamma_{h_0}^2}{c^2} v v_t & \frac{i}{c} \gamma_{h_0}^2 v \\ \frac{i}{c} \gamma_{h_0}^2 v_t & -\gamma_{h_0}^2 \end{bmatrix} = \begin{bmatrix} g v_t & i c g \\ \frac{i}{c} h v_t & -h \end{bmatrix}$$

where

$$h = \gamma_{h_0}^2 \quad \text{and} \quad g = \frac{h}{c^2} v .$$

4.5 Internal Stresses

In order to describe the complete behavior of materials, the internal stresses must be considered. These stresses are transmitted through surfaces and are best handled by means of the stress tensor. Since a surface element is completely characterized by the vector $\underline{n} ds$, where \underline{n} is the unit vector normal to the surface and ds its magnitude, the force transmitted can be written

$$f_T ds = T n ds$$

where T is obviously a three-dimensional tensor. The total force over the surface of a given volume is, then

$$\int_S f_T dS = \int_S T n dS = \int_V (T \nabla) dV$$

which shows that the local force per unit volume is given by $T \nabla$. An appropriate choice of the sign of T allows the momentum law to be generalized to

$$f = (g v_t) \nabla + \frac{\partial g}{\partial t} + T \nabla$$

or

$$f = (g v_t + T) \nabla + \frac{\partial g}{\partial t}$$

Since the work per unit time done by these forces is

$$\int_S \mathbf{f}_T \cdot \mathbf{v} dS = \int_S (\mathbf{T}n) \cdot \mathbf{v} dS = \int_S v_t T n dS = \int_S (\mathbf{T}_t \mathbf{v}) \cdot \mathbf{n} dS = \int_V \nabla \cdot (\mathbf{T}_t \mathbf{v}) dV$$

the corresponding power density is $\nabla \cdot (\mathbf{T}_t \mathbf{v})$. The power density equation is then generalized to

$$p = \nabla \cdot (h\mathbf{v}) + \frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{T}_t \mathbf{v})$$

or

$$p = (h v_t + v_t T) \nabla + \frac{\partial h}{\partial t}$$

Combining these two equations, the four-dimensional equation of power and force density results

$$\begin{bmatrix} g v_t + T & i c g \\ \frac{i}{c} (h v_t + v_t T) & -h \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_t \end{bmatrix} = \begin{bmatrix} f \\ \frac{i}{c} p \end{bmatrix}$$

The symmetry condition gives, in this case

$$g = \frac{1}{c^2} (h v + T_t v)$$

which shows that stresses contribute to the momentum, not only due to their contribution to energy density, but also directly through the term $\mathbf{T}_t \mathbf{v}$.

It is instructive to look at the transformation from the rest frame.

In the rest frame the stress-energy tensor has the form

$$\begin{bmatrix} T_o & 0 \\ 0 & -h_o \end{bmatrix}$$

where h_o is the energy density (including elastic energy) and T_o is the stress tensor, which is known to be symmetric. In this frame, energy and stresses can be separated into

$$\begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -h_0 \end{bmatrix}$$

In a frame of reference moving with velocity \underline{v} , in which the material has a velocity \underline{v} , the energy part becomes

$$\begin{bmatrix} g_H v_t & i c g_H \\ \frac{i}{c} h_H v_t & -h_H \end{bmatrix}$$

where

$$h_H = \gamma^2 h_0 \quad \text{and} \quad g_H = \gamma^2 \frac{h_0}{c^2} v .$$

For the stress part, the transformation is more complicated, as is shown below.

$$\begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} v v_t & -\frac{i}{c} \gamma v \\ \frac{i}{c} \gamma v_t & \gamma \end{bmatrix} \begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} v v_t & \frac{i}{c} \gamma v \\ -\frac{i}{c} \gamma v_t & \gamma \end{bmatrix} =$$

$$= \begin{bmatrix} [T_0 + \frac{\gamma-1}{v^2} v v_t T_0 + \frac{\gamma-1}{v^2} T_0 v v_t + \frac{(\gamma-1)^2}{v^4} v v_t T_0 v v_t] & \frac{i}{c} (\gamma T_0 + \gamma \frac{\gamma-1}{v^2} v v_t T_0) v \\ \frac{i}{c} v_t (\gamma T_0 + \gamma \frac{\gamma-1}{v^2} T_0 v v_t) & -\frac{i}{c^2} \gamma^2 v_t T_0 v \end{bmatrix}$$

which can be put into the form

$$\begin{bmatrix} g_T v_t + T & i c g_T \\ \frac{i}{c} (h_T v_t + v_t T) & -h_T \end{bmatrix}$$

by letting

$$h_T = \frac{\gamma^2}{c^2} (v_t T_0 v)$$

$$\varepsilon_T = \frac{\gamma}{c^2} (T_0 + \frac{\gamma-1}{v^2} v_t T_0 v)$$

Therefore

$$\begin{aligned} T &= T_0 + \frac{\gamma-1}{v^2} v v_t T_0 + \frac{\gamma-1}{v^2} T_0 v v_t + \frac{(\gamma-1)^2}{v^4} (v_t T_0 v) v v_t - \varepsilon_T v_t \\ &= T_0 + \frac{\gamma-1}{v^2} v v_t T_0 + [(\frac{\gamma-1}{v^2} - \frac{\gamma}{c^2}) T_0 + (\frac{(\gamma-1)^2}{v^4} - \frac{(\gamma-1)}{c^2 v^2}) v_t T_0 v] v v_t \end{aligned}$$

and, since

$$\frac{\gamma}{c^2} = \frac{\gamma}{v^2} \frac{v^2}{c^2} = \frac{\gamma}{v^2} (1 - \frac{1}{\gamma^2}) = \frac{\gamma - \frac{1}{\gamma}}{v^2}$$

the following law of transformation for the three-dimensional stress tensor results.

$$T = T_0 + \frac{(\gamma-1)}{v^2} v v_t T_0 + \frac{(\frac{1}{\gamma}-1)}{v^2} T_0 v v_t + \frac{(\gamma-1)(\frac{1}{\gamma}-1)}{v^4} (v_t T_0 v) v v_t$$

which shows that \underline{T} is, in general, not symmetric, even though T_0 is.

The following last verification is necessary.

$$\begin{aligned} h_T v_t + v_t T &= \frac{\gamma^2}{c^2} (v_t T_0 v) v_t + v_t T_0 + \frac{\gamma-1}{v^2} v_t v v_t T_0 + \frac{(\frac{1}{\gamma}-1)}{v^2} (v_t T_0 v) v_t \\ &\quad + \frac{(\gamma-1)(\frac{1}{\gamma}-1)}{v^4} (v_t T_0 v) v_t v v_t \\ &= \gamma v_t T_0 + \left[\frac{\gamma^2}{c^2} + \frac{\frac{1}{\gamma}-1}{v^2} + \frac{(\gamma-1)(\frac{1}{\gamma}-1)}{v^2} \right] (v_t T_0 v) v_t \\ &= v_t \left[\gamma T_0 + \frac{\gamma(\gamma-1)}{v^2} T_0 v v_t \right] \end{aligned}$$

which is in accordance with the initial expression.

Incidentally, the case of a non-viscous fluid, in which the stress tensor

in the rest frame may be written as

$$T_0 = pI_3$$

(where p is the internal pressure), is particularly interesting. The transformation formula gives

$$T = pI_3 + \frac{p}{v^2} [(\gamma - 1) + (\frac{1}{\gamma} - 1) + (\gamma - 1)(\frac{1}{\gamma} - 1)]vv_t$$

Therefore

$$T = pI_3$$

which shows that the internal pressure p is a four-dimensional scalar.

The interesting point in the transformation of the three-dimensional stress tensor lies in the realization that, due to the state of motion, the stress contributes not only to the energy and momentum densities, but also directly to the power flow. In fact, $\underline{h_T}$ may be considered as an energy density due to the motion of the surface forces, producing the power flow $\underline{h_T}v$. In addition, there is a power flow $\underline{T_t}v$, and both contribute to momentum (as they should) due to the equivalence of mass and energy.

4.6 Internal Power Flow and Tensor of Material Reaction

Assuming the existence of a non-mechanical power flow characterized in the rest frame by a vector \underline{P}_0^* (which may include heat flow and $E_0 \times M_0^c$) this power flow will also contribute to the momentum. This means that its contribution in the rest frame will be

$$\begin{bmatrix} 0 & \frac{1}{c} P_0^* \\ \frac{1}{c} P_0^* & 0 \end{bmatrix}$$

In motion, the contribution is again obtained by a Lorentz transformation, as shown below.

$$\begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} vv_t & -\frac{1}{c}\gamma v \\ \frac{1}{c}\gamma v_t & \gamma \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{c} P_o^* \\ \frac{1}{c} P_{ot}^* & 0 \end{bmatrix} \begin{bmatrix} I_3 + \frac{\gamma-1}{v^2} vv_t & \frac{1}{c}\gamma v \\ -\frac{1}{c}\gamma v_t & \gamma \end{bmatrix} \\ = \begin{bmatrix} \frac{\gamma}{c^2} v P_{ot}^* + \frac{\gamma}{c^2} P_o^* v_t + \frac{2\gamma(\gamma-1)}{c^2 v^2} (v_t P_o^*) vv_t & \frac{1}{c} \left[\gamma P_o^* + \frac{(2\gamma^2 - \gamma - 1)}{v^2} (v_t P_o^*) v \right] \\ \frac{1}{c} \left[\gamma P_{ot}^* + \frac{(2\gamma^2 - \gamma - 1)}{v^2} (v_t P_o^*) v_t \right] & -2 \frac{\gamma^2}{c^2} v_t P_o^* \end{bmatrix}$$

Introducing the definition

$$h_{P^*} = 2 \frac{\gamma^2}{c^2} v_t P_o^*$$

and

$$g_{P^*} = \frac{1}{c^2} \left[\gamma P_o^* + \frac{(2\gamma^2 - \gamma - 1)}{v^2} (v_t P_o^*) v \right]$$

the space-space term may be put in the form

$$g_{P^*} v_t + \frac{1}{c^2} v P_t^*$$

by letting

$$\frac{1}{c^2} v P_t^* = \frac{1}{c^2} \left[\gamma v P_{ot}^* + \gamma P_o^* v_t + \frac{2\gamma^2 - 2\gamma}{v^2} (v_t P_o^*) vv_t \right] - \frac{1}{c^2} \left[\gamma P_o^* v_t + \frac{2\gamma^2 - \gamma - 1}{v^2} (v_t P_o^*) vv_t \right] \\ = \frac{1}{c^2} \left[\gamma v P_{ot}^* + \frac{1-\gamma}{v^2} (v_t P_o^*) vv_t \right]$$

which implies

$$P^* = \gamma P_o^* + \frac{1-\gamma}{v^2} (v_t P_o^*) v$$

the time-space term then becomes

$$\begin{aligned} \frac{i}{c} [\gamma P_o^* + \frac{2\gamma^2 - \gamma - 1}{v^2} (v_t P_o^*)] v &= \frac{i}{c} [P^* + (\frac{2\gamma^2 - \gamma - 1}{v^2} + \frac{\gamma - 1}{v^2}) (v_t P_o^*)] v \\ &= \frac{i}{c} [P^* + \frac{2(\gamma^2 - 1)}{v^2} \frac{c^2 h_{P^*}}{2\gamma^2} v] = \frac{i}{c} [P^* + h_{P^*} v] \end{aligned}$$

This gives the following form for the non-mechanical power flow.

$$\begin{bmatrix} \mathcal{E}_{P^*} v_t + \frac{1}{c^2} v P_t^* & i c g_{P^*} \\ \frac{i}{c} (h_{P^*} v_t + P_t^*) & - h_{P^*} \end{bmatrix}$$

The final equation including all of the effects may be written in terms of the reaction tensor and the tensor due to the field. Thus

$$\begin{bmatrix} g v_t + T + \frac{1}{c^2} v P_t^* & i c g \\ \frac{i}{c} (h v_t + v_t T + P_t^*) & - h \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix} = \begin{bmatrix} \sigma & - \frac{i}{c} S \\ - \frac{i}{c} S & W \end{bmatrix} \begin{bmatrix} \nabla \\ \partial_4 \end{bmatrix}$$

where the total momentum density is

$$g = g_H + g_T + g_{P^*}$$

and the total energy density is

$$h = h_H + h_T + h_{P^*}$$

This four-dimensional equation embodies the equation of motion of matter in an electromagnetic field and the energy equation.

CHAPTER V

CONCLUDING REMARKS

This investigation has shown that the difficulties in the analysis of electromagnetic fields interacting with moving bodies essentially fall into two categories:

(a) Misunderstanding of the kind of relativity to be used (even in first approximation) in electrodynamics. This was due to the common belief that for small velocities Galilean relativity could be used instead of Einstein's relativity. As a consequence, time was handled as a parameter and the law of transformation of the electric field had to be introduced as an experimental fact clashing with the laws of transformation of the sources of electromagnetic field and making electrodynamics inconsistent with classical mechanics.

This investigation has shown that Galilean relativity is not adequate for the establishment of a set of laws of transformation for the electromagnetic fields and sources self-consistent even in first order terms in v/c . It is now clear that time cannot be considered as invariant, and therefore, care as to be exercised when reasoning intuitively. In order to handle the situation properly, a few concepts of vector and tensor analysis, employing the convenient matrix notation, allowed a simple and clear development of the necessary theory while maintaining the physical meaning in evidence.

(b) Ambiguities in the definition of electromagnetic field vectors in materials. Historically, the influence of materials on electromagnetic fields was taken care of by the introduction of parameters (characteristic of the materials) which accounted for the effects measured at the boundaries.

Even though hypotheses were advanced about the electric origin of these parameters, these hypotheses were not used for the determination of internal force and energy densities. These force and energy densities were obtained from energy conservation and virtual work principles which, strictly speaking, accounted only for the integrated effects. The understanding of the behavior of the field vectors at the surfaces of separation between materials and vacuum allowed a definition of these vectors inside materials by the use of appropriate cavities, which obscured still more their role as force-producing entities. These ambiguities were removed in this investigation by postulate 5, which allowed not only a clear interpretation of the influence of motion, but also gave a very straightforward procedure for handling forces and energies.*

An experimental investigation is now necessary in order to verify the adequacy of this last postulate. Should this postulate not be valid, still the situation is now clear enough for an understanding of where the difficulty lies. There is no reason to believe that essentially the same method of approach cannot be used for handling the development after a convenient modification of postulate 5.

It seems that, in this case, a new source of fields has to be considered in addition to the electric charge. A decision has to be then made about the possibility of the use of electromagnetic fields for the description of the interaction between these new sources, or the possibility of introducing a corresponding new field. This last approach is felt as being the better, even though the more difficult, since this investigation seems to strengthen

*This seems to be the same procedure followed by Professor L. J. Chu in his recent investigation⁽¹⁶⁾ with a different postulate about the sources of the electromagnetic field and a convenient modification of Lorentz force.

the belief that electromagnetic field is purely an aspect of charge interaction.

In any case, the neat and straightforward approach afforded by the notation and method followed here seems to be as important as the results achieved (if not more so), specially considering the amount of additional information put within the reach of the engineer by its use.

BIOGRAPHICAL NOTES

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Father's name: Valente Boffi

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Education

Elementary School, São Paulo, Brazil: March 1928 to November 1929

Gymnasium, São Paulo, Brazil: March 1930 to November 1934

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Degree of Electrical Engineer: December 1940

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January 1954: Voyage to the United States of America in Official Mission of Study.

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REFERENCES

- (1) Electromagnetic Induction, G. I. Cohn, Electrical Engineering, 68, pp. 441.
- (2) Principles of Mathematical Physics (book), W. V. Houston.
- (3) Some Electromagnetic Paradoxes, P. Moon and D. E. Spencer, Jour. Frank. Inst., 260, No. 5, p. 373.
- (4) The Fundamentals of Electromagnetism (book), E. G. Cullwick.
- (5) On Unipolar Induction, E. H. Kennard, Philosophical Magazine, 33, p. 179.
- (6) Slepian, Electrical Engineering, Vol. 67: pp. 58, 187, 337, 530, 653, 771, 904, 978, 1073, 1141. Vol. 68: pp. 667, 145, 877, 613, 784, 862, 308. Vol. 69: pp. 164, 1025, 1086. Vol. 70: pp. 67, 159, 254, 350.
- (7) Unipolar Induction Theory, G. B. Pegram, Physical Review, X, No. 6, p. 591.
- (8) The Principle of Relativity - A collection of papers, A. Einstein, H. A. Lorentz, H. Minkowski, H. Weyl.
- (9) The Meaning of Relativity (book), A. Einstein.
- (10) The Theory of Relativity (book), C. Moller.
- (11) Electrodynamics (book), A. Sommerfeld.
- (12) Vector and Tensor Analysis (class notes), F. D. Murnaghan.
- (13) Tensor Analysis (book), I. S. Sokolnikoff.
- (14) Tensor Analysis of Networks (book), G. Kron.
- (15) Electromagnetic Theory (book), Stratton.
- (16) Electrodynamics (rough draft of unpublished investigation), L. J. Chu.