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EXCITATION OF ION OSCILLATIONS
IN BEAM-PLASMA SYSTEMS

M. T. VLAARDINGERBROEK

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TECHNICAL REPORT 426

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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EXCITATION OF ION OSCILLATIONS IN BEAM-PLASMA SYSTEMS

M. T. Vlaardingerbroek

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Abstract

This report suggests possible ways for further theoretical study of the interaction of an electron beam and the ions in a hot-electron plasma. We obtain the equations for beam-plasma interaction in a filled cylindrical waveguide in which the thermal velocity distribution is longitudinal but otherwise arbitrary. We describe an approximation based on these equations by using only one transverse wave number. A close resemblance is found between the approximate solutions and the theory of plane waves in an infinite beam-plasma system, traveling at an angle with the direction of the beam and the magnetic field and having a certain fixed real transverse wave number. We take advantage of this resemblance and calculate beam- (hot-electron) plasma interaction including a transverse random velocity distribution. We combine the theory of a filamentary beam and a plasma-filled waveguide, using the coupling-of-modes theory, for the description of some of the weak interactions between a thin beam and a finite plasma.

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I. INTRODUCTION

Beam-plasma discharges have been described by Getty and Smullin.^{1, 2} Experimentally, these discharges have been shown to yield a plasma in which the electrons may obtain very high random velocities; 250-keV electrons occur in a beam-plasma discharge excited by a 10 kV-10 amp beam, as is observed from the emitted x-rays.

The ions, however, still seem to be cold (a few electron volts). A logical further development of beam-plasma discharges is to consider the problem of heating the ions. Since impurity radiation is disastrous for the ion temperature, a first experimental attack seems to be to improve the "background vacuum pressure" in the continuously pumped system.³ A second possibility is to investigate the direct interaction of the beam (or possibly a cloud of electrons having a low drift velocity that is somehow generated in the beam-plasma discharge) with the ions in the plasma. The oscillatory ion motion resulting from this interaction may then be randomized through nonlinear effects. (To investigate this mechanism it seems worth while to inquire whether the electron beam carries enough DC energy to drive the ions into nonlinear motion. An ion beam premodulated at ion plasma frequency may also be considered.) A third method is to use ion-cyclotron heating of the beam-plasma discharge. If the external magnetic field in which the discharge is excited is sufficiently high (8000 gauss yields an ion-cyclotron frequency of 560 Mc/sec) the high-frequency field necessary for this heating can be applied through simple condenser plates or coils. A similar experiment is under way with the C-Stellerator at Princeton University.

The subject of this report is mainly aimed at methods for further theoretical investigation of the beam-ion interactions. The existence of these interactions is indicated by the fact that the beam-plasma discharge emits (apart from radiation in the kMc/sec range, very likely caused by electron-plasma oscillations) electromagnetic radiation in the frequency band between 100 Mc/sec and 500 Mc/sec. This must be due to an ion-plasma oscillation, since the frequency is always roughly 50 times as low as the high-frequency electron oscillation

$$\frac{\omega_{pi}}{\omega_{pe}} = \sqrt{\frac{m}{M}} \approx 50,$$

where ω_p is the plasma frequency, m is the electron mass, and M is the ion mass. The excitation mechanism of these oscillations is not clear.

Theoretically, it has been shown by Bers and Briggs that when the electron temperature in a plasma is sufficiently high a beam can interact directly with the ions in a plasma through the same mechanism as that through which it interacts with the electrons in a cold plasma.⁴ The reason is that the ions are unshielded by the electrons at high electron temperatures or, in other words, the random energy of the electrons is so high that they cannot be trapped in the space-charge fields set up by the beam-ion

interaction. Further theoretical study shows that in an infinite beam-plasma system a very strong interaction at the ion frequency occurs when the Debye radius of the electron gas is larger than the wavelength of the ion oscillation. Furthermore, in finite systems a backward-wave interaction occurs.⁴

We shall discuss methods to extend the existing theory, which is based on the quasi-static approximation, and the assumption that the plasma electron temperature (expressed in volts) is much higher than the beam voltage. We shall replace these approximations by less stringent ones and derive dispersion equations for beam-plasma waveguides having arbitrary velocity distribution of the plasma electrons. Furthermore, a model for cyclotron interaction of a thin beam and a plasma-filled waveguide will be discussed. Finally, an experiment showing beam-ion interaction will be described and it will be shown that nonrotational symmetric waves may be responsible for another beam-ion interaction mechanism.

II. EQUATIONS DESCRIBING BEAM-PLASMA INTERACTIONS IN FINITE SYSTEMS

We consider a cylindrical waveguide filled with a number of beams and with an external magnetic field along its axis. For the description of this system we refer to an earlier paper,⁵ or to the work of Serafim.⁶ Here we follow closely the notation and derivations of our earlier paper.⁵

For the AC quantities Maxwell's equations for a space containing n streams of charged particles are

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad (1)$$

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \sum \vec{j}_n \quad (2)$$

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \sum \rho_{1n}, \quad (3)$$

where \vec{E} and \vec{H} are the electric and magnetic AC field strengths, ρ_{1n} and \vec{j}_n denote the contributions of the n^{th} beam to the AC space-charge density and the AC convection-current density.

The relativistic equation of motion for the particles of the n^{th} beam is given by

$$\frac{d}{dt} (m_n K_n \vec{v}_n) = q_n (\vec{E} + \vec{v}_n \times \vec{B}); \quad (4)$$

the beam velocity \vec{v}_n consists of a static velocity \vec{u}_n and an AC velocity \vec{v}_n ; m_n and q_n are the rest mass and the charge of the particles of the n^{th} beam, $K_n = \left(1 - \frac{u_n^2}{c^2}\right)^{-1/2}$, and \vec{B} is the total magnetic field, that is, $\vec{B}_0 + \mu_0 \vec{H}$. Linearizing this equation and taking the drift velocity and the external magnetic field \vec{B}_0 in the z direction, we can write for the separate velocity components:

$$\frac{\partial v_{nr}}{\partial t} + u_n \frac{\partial v_{nr}}{\partial z} = \frac{q_n}{m_n K_n} \{E_r + v_{n\theta} B_0 - \mu_0 u_n H_\theta\} \quad (5)$$

$$\frac{\partial v_{n\theta}}{\partial t} + u_n \frac{\partial v_{n\theta}}{\partial z} = \frac{q_n}{m_n K_n} \{E_\theta - v_{nr} B_0 + \mu_0 u_n H_r\} \quad (6)$$

$$\frac{\partial v_{nz}}{\partial t} + u_n \frac{\partial v_{nz}}{\partial z} = \frac{q_n}{m_n K_n^3} E_z. \quad (7)$$

Equations 5-7 hold for a plasma when taking $u_n = 0$.

The first-order AC convection-current density carried by the n^{th} beam is given by

$$\vec{j}_n = \vec{u}_n \rho_{1n} + \vec{v}_n \rho_{0n}, \quad (8)$$

where ρ_{0n} is its static space-charge density. The continuity equation for the n^{th} beam is

$$\nabla \cdot \vec{j}_n = -\frac{\partial \rho_{1n}}{\partial t}. \quad (9)$$

We now take the z - and t -dependence to be of the form $\exp j(\omega t - \beta z)$. We can then solve for the velocity components from Eqs. 5, 6, and 7:

$$\begin{pmatrix} v_{nr} \\ v_{n\theta} \\ v_{nz} \end{pmatrix} = \frac{j q_n}{m_n K_n (\omega_{\beta n}^2 - \Omega_n^2)} \left[\begin{pmatrix} -\omega_{\beta n} & j\Omega_n & 0 \\ -j\Omega_n & -\omega_{\beta n} & 0 \\ 0 & 0 & -\frac{\omega_{\beta n}^2 - \Omega_n^2}{K_n^2 \omega_{\beta n}} \end{pmatrix} \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix} + u_n \mu_o \begin{pmatrix} j\Omega_n & \omega_{\beta n} & 0 \\ -\omega_{\beta n} & j\Omega_n & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_r \\ H_\theta \\ H_z \end{pmatrix} \right]. \quad (10)$$

In Eq. 10 we have introduced the notation $\omega_{\beta n} = \omega - \beta u_n$ and the cyclotron frequency of the n^{th} beam, $\Omega_n = q_n B_o (m_n K_n)^{-1}$. Applying the conservation-of-charge equation (9), we can determine the AC charge density of the n^{th} beam from (1), (8), and (10) in the form

$$\rho_{1n} = \frac{\omega_{\beta n}^2 \epsilon_o}{(\omega_{\beta n}^2 - \Omega_n^2) \omega_{\beta n}} \left[-\omega_{\beta n} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial}{\partial \theta} E_\theta \right\} + \omega_{\beta n} \mu_o \Omega_n H_z + j\beta \frac{\omega_{\beta n}^2 - \Omega_n^2}{K_n^2 \omega_{\beta n}} E_z + u_n \mu_o \omega_{\beta n} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{r \partial \theta} \right\} \right]. \quad (11)$$

The AC current density of the n^{th} beam, from Eq. 8, is

$$\begin{pmatrix} j_{nr} \\ j_{n\theta} \\ j_{nz} \end{pmatrix} = \frac{j\epsilon_0\omega_{bn}^2}{(\omega_{\beta n}^2 - \Omega_n^2)} \left\{ \begin{pmatrix} -\omega_{\beta n} & j\Omega_n & 0 \\ -j\Omega_n & -\omega_{\beta n} & 0 \\ -j\frac{u_n}{r}\frac{\partial}{\partial r}(r) & -j\frac{u_n}{r}\frac{\partial}{\partial \theta} & -\frac{\omega_{\beta n}^2 - \Omega_n^2}{K_n^2\omega_{\beta n}^2} \end{pmatrix} \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix} \right. \\
\left. + u_n\mu_0 \begin{pmatrix} j\Omega_n & \omega_{\beta n} & 0 \\ -\omega_{\beta n} & j\Omega_n & 0 \\ -j\frac{u_n}{r}\frac{\partial}{\partial \theta} & j\frac{u_n}{r}\frac{\partial}{\partial r}(r) & j\Omega_n \end{pmatrix} \begin{pmatrix} H_r \\ H_\theta \\ H_z \end{pmatrix} \right\}. \quad (12)$$

The matrices now contain differential operators working on the transverse components of the field strength.

We now use Eq. 1 to eliminate \vec{H} from Eq. 12. We therefore write this equation again in tensor form as

$$\vec{H} = \frac{j}{\omega\mu_0} \begin{pmatrix} 0 & j\beta & \frac{1}{r}\frac{\partial}{\partial \theta} \\ -j\beta & 0 & -\frac{\partial}{\partial r} \\ -\frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r}\frac{\partial}{\partial r}(r) & 0 \end{pmatrix} \vec{E} \quad (13)$$

and introduce this form into the last term of Eq. 12:

$$\begin{pmatrix} j_{nr} \\ j_{n\theta} \\ j_{nz} \end{pmatrix} = \frac{j\omega_{bn}^2\epsilon_0}{\omega_{\beta n}^2 - \Omega_n^2} \begin{pmatrix} -\frac{\omega_{\beta n}^2}{\omega^2} & \vdots & j\Omega_n\frac{\omega_{\beta n}}{\omega} & \vdots & -\frac{\Omega_n u_n}{\omega r}\frac{\partial}{\partial \theta} - j\frac{\omega_{\beta n} u_n}{\omega}\frac{\partial}{\partial r} \\ -j\Omega_n\frac{\omega_{\beta n}}{\omega} & \vdots & -\frac{\omega_{\beta n}^2}{\omega} & \vdots & -j\frac{u_n \omega_{\beta n}}{\omega r}\frac{\partial}{\partial \theta} + \frac{\Omega_n u_n}{\omega}\frac{\partial}{\partial r} \\ -j\frac{\omega_{\beta n} u_n}{\omega r}\frac{\partial}{\partial r}(r + \frac{\Omega_n u_n}{\omega r}\frac{\partial}{\partial \theta}) & \vdots & -j\frac{\omega_{\beta n} u_n}{\omega r}\frac{\partial}{\partial \theta} - \frac{\Omega_n u_n}{\omega r}\frac{\partial}{\partial r}(r) & \vdots & P \end{pmatrix} \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix}, \quad (14)$$

where

$$P = -\frac{\omega_{\beta n}^2 - \Omega_n^2}{K_n^2\omega_{\beta n}^2} \omega + \frac{u_n^2}{\omega} \left\{ \frac{1}{r}\frac{\partial}{\partial r} \left(r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \right\}.$$

Having thus expressed the current density in the n^{th} beam in terms of the components of the electric field, we can use Eq. 14 in connection with the well-known wave equation

$$\nabla(\nabla \cdot \vec{E}) - \Delta \vec{E} - \frac{\omega^2}{c^2} \vec{E} = -j\omega\mu_0 \sum_n \vec{j}_n. \quad (15)$$

As will be clear from Eq. 14, addition over the different beams can be done simply by placing a summation sign for each single term. Writing the wave equation (15) in tensor-operator form yields:

$$\begin{pmatrix} -\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \beta^2 - k^2 & \frac{1}{r^2} \frac{\partial^2}{\partial \theta \partial r} (r) & -j\beta \frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial \theta} \right) & -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r + \beta^2 - k^2) \right) & -j\beta \frac{1}{r} \frac{\partial}{\partial \theta} \\ -j\beta \frac{1}{r} \frac{\partial}{\partial r} (r) & -j\beta \frac{1}{r} \frac{\partial}{\partial \theta} & -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - k^2 \right) \end{pmatrix} \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix} = -j\omega\mu_0 \sum_n \vec{j}_n. \quad (16)$$

Substituting Eq. 4 for the right-hand side of Eq. 6, we now find an equation containing only the electric-field components. This equation is of the form

$$\vec{M}_{\text{op}} \vec{E} = 0, \quad (17)$$

where \vec{M}_{op} contains the derivatives in r and θ . We now make an important restriction, namely, we only consider rotational symmetric modes. As we shall show, important instabilities at low frequencies may arise in nonrotational symmetric modes. These modes seem to be entitled to further study, but a detailed study has not yet been made.

We shall restrict ourselves to rotational symmetric modes, $\frac{\partial}{\partial \theta} = 0$. As is known from previous work,^{5,6} solutions of the three partial differential equations as given in Eq. 17 are:

$$E_z = \hat{E}_z J_0(\text{Tr}); \quad E_r = \hat{E}_r J_1(\text{Tr}); \quad E_\theta = \hat{E}_\theta J_1(\text{Tr}). \quad (18)$$

We can introduce this solution into Eq. 7 by writing

$$\vec{E} = \begin{pmatrix} \vec{E}_r \\ E_\theta \\ E_z \end{pmatrix} = \begin{pmatrix} J_1(\text{Tr}) & 0 & 0 \\ 0 & J_1(\text{Tr}) & 0 \\ 0 & 0 & J_0(\text{Tr}) \end{pmatrix} \begin{pmatrix} \hat{E}_r \\ \hat{E}_\theta \\ \hat{E}_z \end{pmatrix}, \quad (19)$$

and a similar equation for \vec{j}_n . Now, expressing Eq. 14 in \vec{j}_n and \vec{E} yields

$$\vec{j}_n = -\frac{i\omega_{bn}^2 \epsilon_0}{\omega_{\beta n}^2 - \Omega_n^2} \begin{pmatrix} -\frac{\omega_{\beta n}^2}{\omega^2} & -j\Omega_n \frac{\omega_{\beta n}}{\omega} & j\frac{\omega_{\beta n} u_n T}{\omega} \\ -j\Omega_n \frac{\omega_{\beta n}}{\omega} & -\frac{\omega_{\beta n}^2}{\omega} & -\frac{\Omega_n u_n T}{\omega} \\ -ju_n T \frac{\omega_{\beta n}}{\omega} & -\frac{\Omega_n u_n T}{\omega} & -\frac{\omega_{\beta n}^2 - \Omega_n^2}{K_n^2 \omega_{\beta n}^2} \frac{u_n^2 T^2}{\omega} \end{pmatrix} \vec{E}. \quad (20)$$

Similarly, Eq. 6 becomes

$$\begin{pmatrix} \beta^2 - k^2 & 0 & j\beta T \\ 0 & T^2 + \beta^2 - k^2 & 0 \\ -j\beta T & 0 & T^2 - k^2 \end{pmatrix} \vec{E} = -j\omega\mu_0 \sum_n \vec{j}_n. \quad (21)$$

Combination of (20) and (21) yields an equation of the form

$$\vec{M}(T) \vec{E} = 0, \quad (22)$$

where \vec{M} is a simple matrix. For the case of a waveguide filled with a plasma of "longitudinally hot" electrons and cold ions, penetrated by an electron beam, we obtain

$$M_{11} = M_{22} - T^2 = \beta^2 - k^2 + \frac{\omega_{pa}^2}{c^2} \int_{-c}^c \frac{(\omega - \beta u)^2 f(u) du}{K_e \{(\omega - \beta u)^2 - \Omega_e^2\}} + \frac{\omega_{pb}^2}{c^2} \frac{\omega_{\beta b}^2}{\omega_{\beta b}^2 - \Omega_b^2} + \frac{\omega_{pi}^2}{c^2} \frac{\omega^2}{\omega^2 - \Omega_i^2} \quad (23)$$

$$M_{33} = T^2 - k^2 + \frac{\omega_{pe}^2}{c^2} \int_{-c}^c \left\{ \frac{\omega^2 f(u)}{K_e^2 (\omega - \beta u)^2} + \frac{u^2 T^2 f(u)}{K_e \{(\omega - \beta u)^2 - \Omega_e^2\}} \right\} du + \frac{\omega_{pi}^2}{c^2} + \frac{\omega_{pb}^2}{c^2} \left(\frac{\omega^2}{K_b^2 \omega_{\beta b}^2} + \frac{u_b^2 T^2}{\omega_{\beta b}^2 - \Omega_b^2} \right). \quad (24)$$

$$M_{12} = M_{21}^* = -j \left\{ \frac{\omega_{pe}^2}{c^2} \int_{-c}^c \frac{\Omega_e (\omega - \beta u) f(u) du}{K_e \{(\omega - \beta u)^2 - \Omega_e^2\}} - \frac{\omega_{pi}^2}{c^2} \frac{\omega \Omega_i}{\omega^2 - \Omega_i^2} - \frac{\omega_{pb}^2}{c^2} \frac{\omega_{\beta b} \Omega_e}{\omega_{\beta b}^2 - \Omega_b^2} \right\} \quad (25)$$

$$M_{31} = M_{13}^* = j \left\{ \beta T - \frac{\omega_{pe}^2}{c^2} \int_{-c}^c \frac{(\omega - \beta u) u T f(u) du}{K_e \{(\omega - \beta u)^2 - \Omega_e^2\}} + \frac{\omega_{pb}^2}{c^2 K_b} \frac{u_o \omega_{\beta b} T}{\omega_{\beta b}^2 - \Omega_b^2} \right\} \quad (26)$$

$$M_{23} = M_{32}^* = \frac{\omega_{pe}^2}{c^2} \int_{-c}^c \frac{\Omega_e u T f(u) du}{\{(\omega - \beta u)^2 - \Omega_e^2\} K_e} + \frac{\omega_{pb}^2}{K_b c^2} \frac{\Omega_b u_o T}{\omega_{\beta b}^2 - \Omega_b^2}, \quad (27)$$

in which the indices i, e, and b denote plasma ions, plasma electrons, and beam electrons. The summation over the different velocities of the plasma electrons is replaced by integrals, where $f(u)$ is the normalized distribution function so that when integrated over all velocities it yields unity. The beam velocity is taken to be u_o . The plasma frequency of the electrons is defined nonrelativistically: $\omega_{pe} = \frac{\eta \rho_{oe}}{\epsilon_o}$, and for completeness relativistic effects are accounted for separately in $K_e = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$. In the beam-plasma frequency, however, relativity is accounted for in the definition given in the earlier paper.⁵

III. APPROXIMATE METHODS FOR THE SOLUTION OF THE EQUATIONS

The requirement for nontrivial solutions of Eq. 12 is

$$\det M = 0, \quad (28)$$

which is the determinantal equation. The dispersion characteristics are found from the determinantal equation, together with the boundary conditions

$$E_r(R) = E_\theta(R) = 0, \quad (29)$$

where R is the radius of the waveguide.

Before treating this boundary condition, we have to realize that Eq. 28 is quadratic in T^2 so that for one set of values for ω and β two values for T arise. Therefore in general the field form of waves in a plasma-filled waveguide is

$$\begin{aligned} E_z &= E_{z1} J_0(T_1 r) + E_{z2} J_0(T_2 r) \\ E_\theta &= E_{\theta 1} J_1(T_1 r) + E_{\theta 2} J_1(T_2 r). \end{aligned} \quad (30)$$

Now we need an expression to relate $E_{\theta n}$ and E_{zn} , which is readily obtained from Eq. 28

$$E_{\theta n} = - \frac{\begin{vmatrix} M_{11} & M_{13} \\ M_{21} & M_{23} \end{vmatrix}_n}{\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix}_n} E_{zn} = C(T_n) E_{zn}, \quad (31)$$

where the index n ($=1, 2$) means that the equation is used after replacing T by T_n . Introducing (31) into (30) and applying (29), we obtain

$$\begin{vmatrix} J_0(T_1 R) & J_0(T_2 R) \\ C(T_1) J_1(T_1 R) & C(T_2) J_1(T_2 R) \end{vmatrix} = 0. \quad (32)$$

The dispersion problem now consists of the simultaneous solution of Eqs. 28 and 32.

The solution of this problem – because of its complexity – is undertaken for the case of a waveguide filled with a cold plasma only.^{7,8} The presence of the beam and the plasma temperature make the problem even more complicated. Therefore we propose an approximation, namely, instead of solving Eqs. 28 and 32 simultaneously, we only solve Eq. 28 after substituting for the value of T that occurs in this equation the value obtained from

$$J_0(T_{01} R) = 0. \quad (33)$$

This approximation is supported by several considerations. In the first place, in the quasi-static approximation only one value of T is used and separation into E- and H-modes is possible. In the exact solution the wave may be predominantly of the H-type or E-type, but it is always a mixture of both. For a predominantly E-wave, one of the T 's, in most practical cases, (phase velocity < light velocity) will be close to T_{01} of Eq. 33. Similarly, for a predominantly H-wave T will be close to T_{11} , given by $J_1(TR) = 0$. If now the dispersion diagrams, as obtained by the suggested approximation, do not depend strongly on T , a number of diagrams with different values of T will give a good impression of the dispersion characteristics of the actual system under study. In the second place, a careful study of the results of the work of Le Mesecq and Camus⁸ shows that for a plasma-filled waveguide, when the phase velocity is smaller than the light velocity, the exact solution follows closely the solution obtained by the suggested approximation.⁹ An advantage of this approximation is that it describes correctly the backward- or forward-wave character of the waves around the highest of the two resonance frequencies ω_{pe} and Ω_{ce} . Quasi statics always gives a backward wave in this band, which may lead to misinterpretation of the nature of beam-plasma interaction in the case of $T_{01}^2 c^2 \approx \omega_{pe}^2$ or ω_{pi}^2 .

Having accepted the approximation suggested above, we obtained the dispersion diagrams of the interaction of an electron beam with a cold plasma (the ions are assumed to be infinitely heavy). Figures 1, 2, and 3 show that the dispersion characteristics are not strongly dependent on T but change in an understandable way with changing T . Figures 4, 5, and 6 show the dispersion characteristics of beam-plasma systems with properties characterizing the start of the beam-plasma discharge (first regime, described by Getty and Smullin¹).

The next step is to solve Eq. 22 for beam-plasma interaction, with the electrons assumed to be hot. The M_{nm} of Eq. 22 are then given by Eqs. 23-27. We now have to specify further the velocity distribution function $f(u)$, and discuss the integrations involved. If we choose for $f(u)$ a square distribution function

$$f(u) = U(u+u_1) - U(u-u_1), \quad (34)$$

U being the unit-step function, effects like Landau damping are excluded. A Maxwellian distribution,

$$f(u) = \frac{1}{\sqrt{2\pi} V_T} e^{-\frac{1}{2}mu^2/kT_e}; \quad V_T = \sqrt{\frac{kT_e}{m}}, \quad (35)$$

includes these effects but the integrals cannot be solved in closed form. A resonance distribution,

$$f(u) = \frac{a/\pi^2}{(u^2+a^2)^2}, \quad (36)$$

may be used for waves having a phase velocity smaller than the mean-square thermal

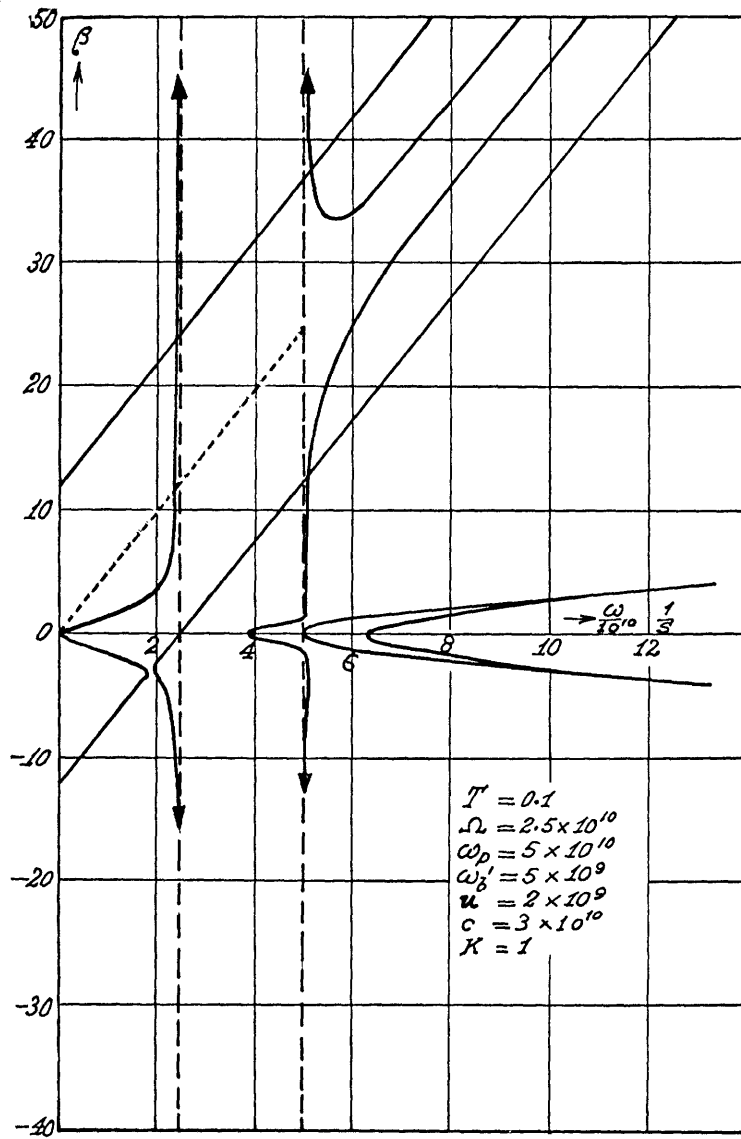


Fig. 1. Beam-plasma dispersion diagram for $T = 0.1$.

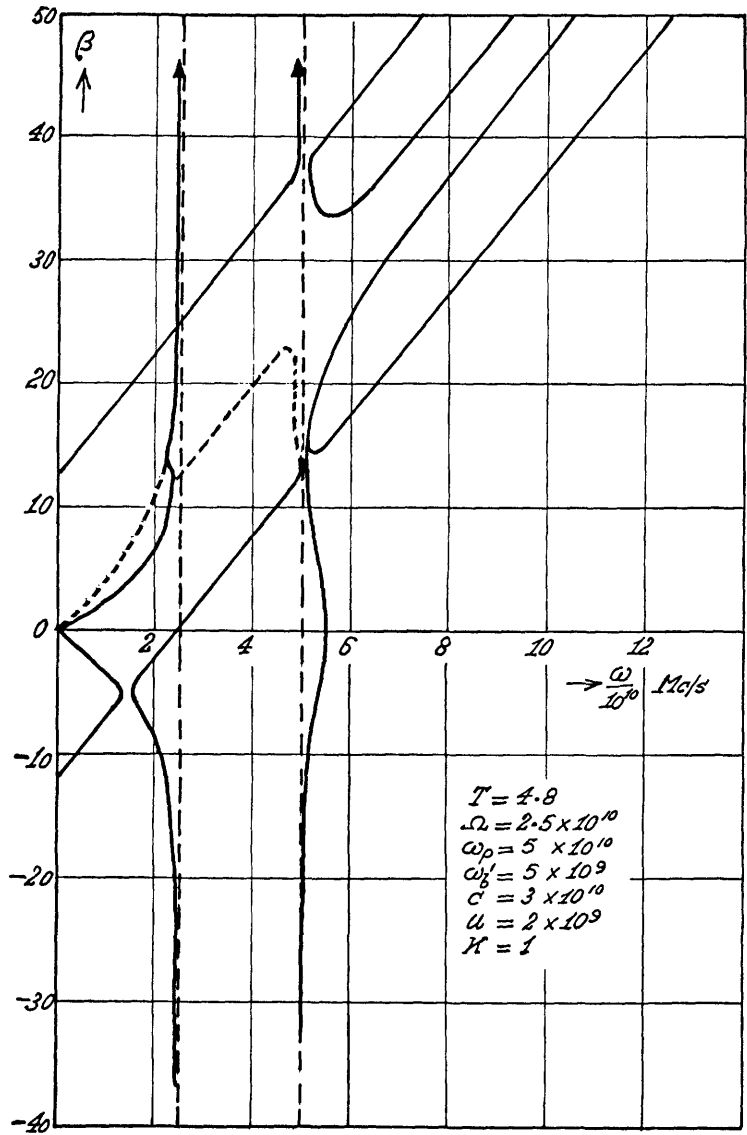


Fig. 2. Beam-plasma dispersion diagram for $T = 4.8$.

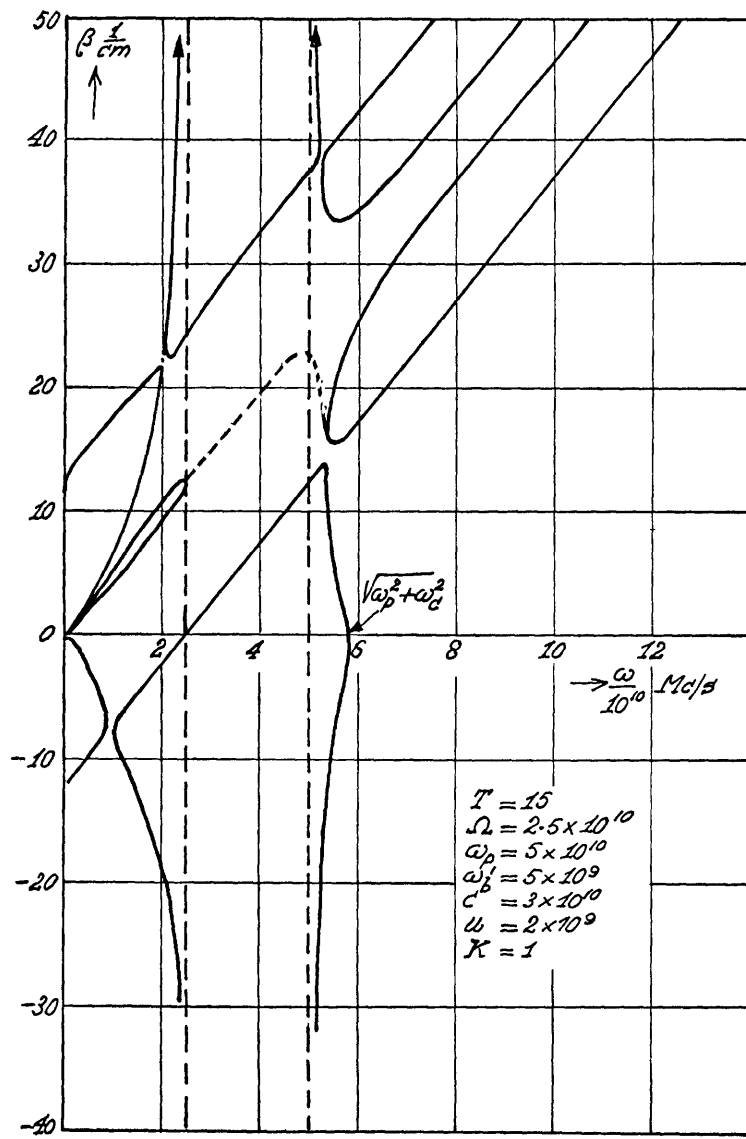


Fig. 3. Beam-plasma dispersion diagram for $T = 15$.

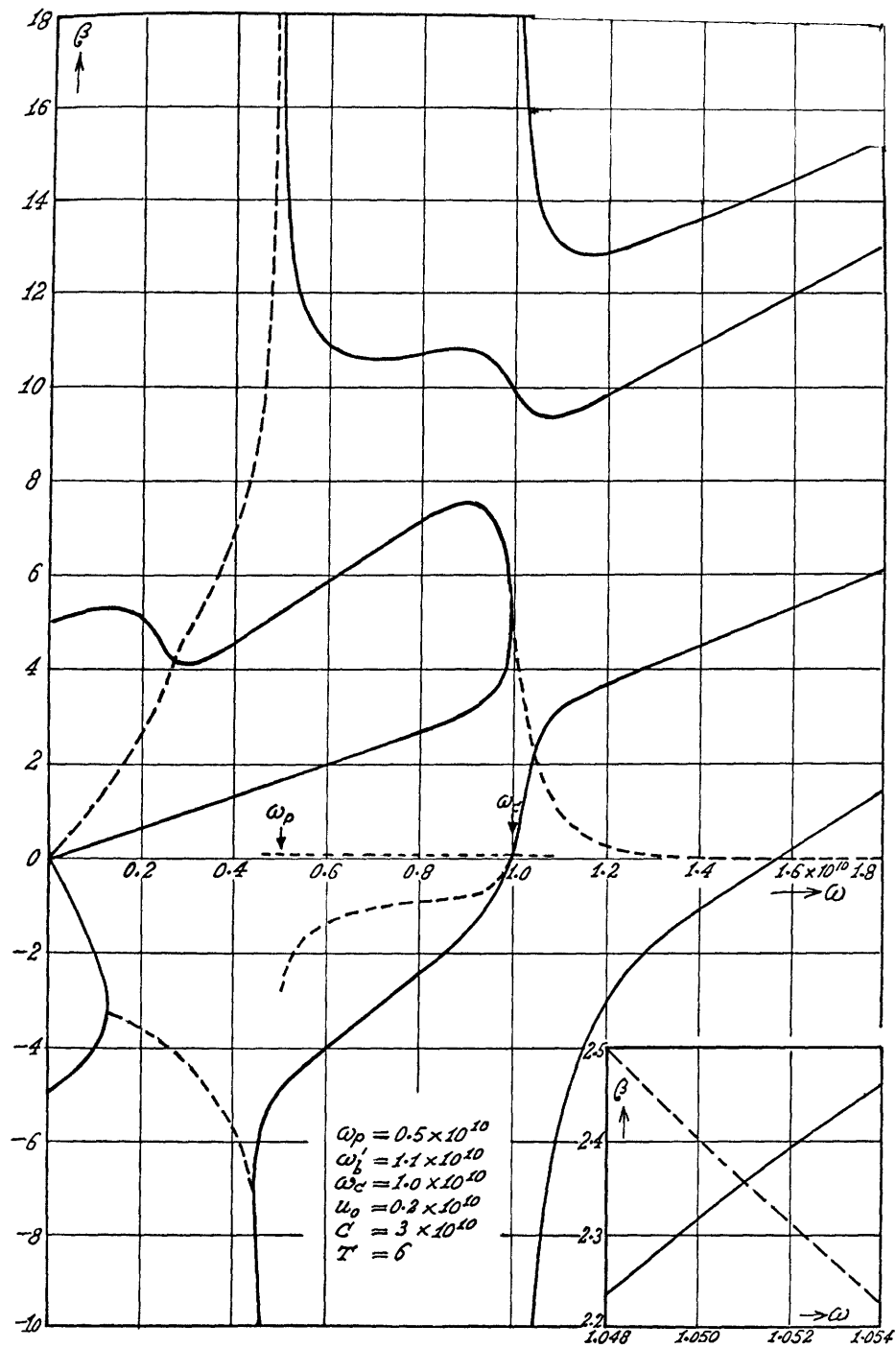


Fig. 4. Beam-plasma dispersion diagram for $u_0 = 0.2 \times 10^{10}$.

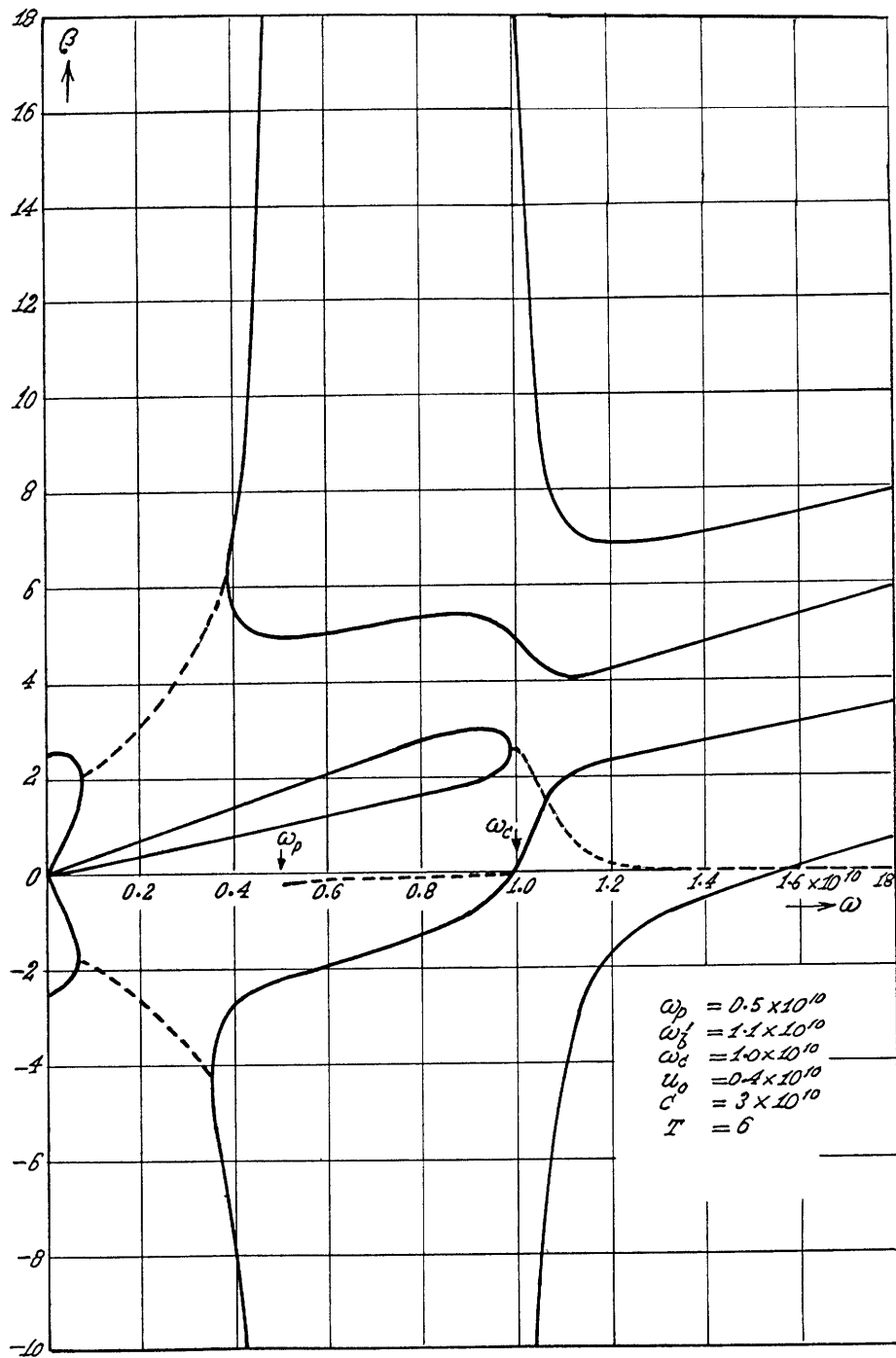


Fig. 5. Beam-plasma dispersion diagram for $u_0 = 0.4 \times 10^{10}$.

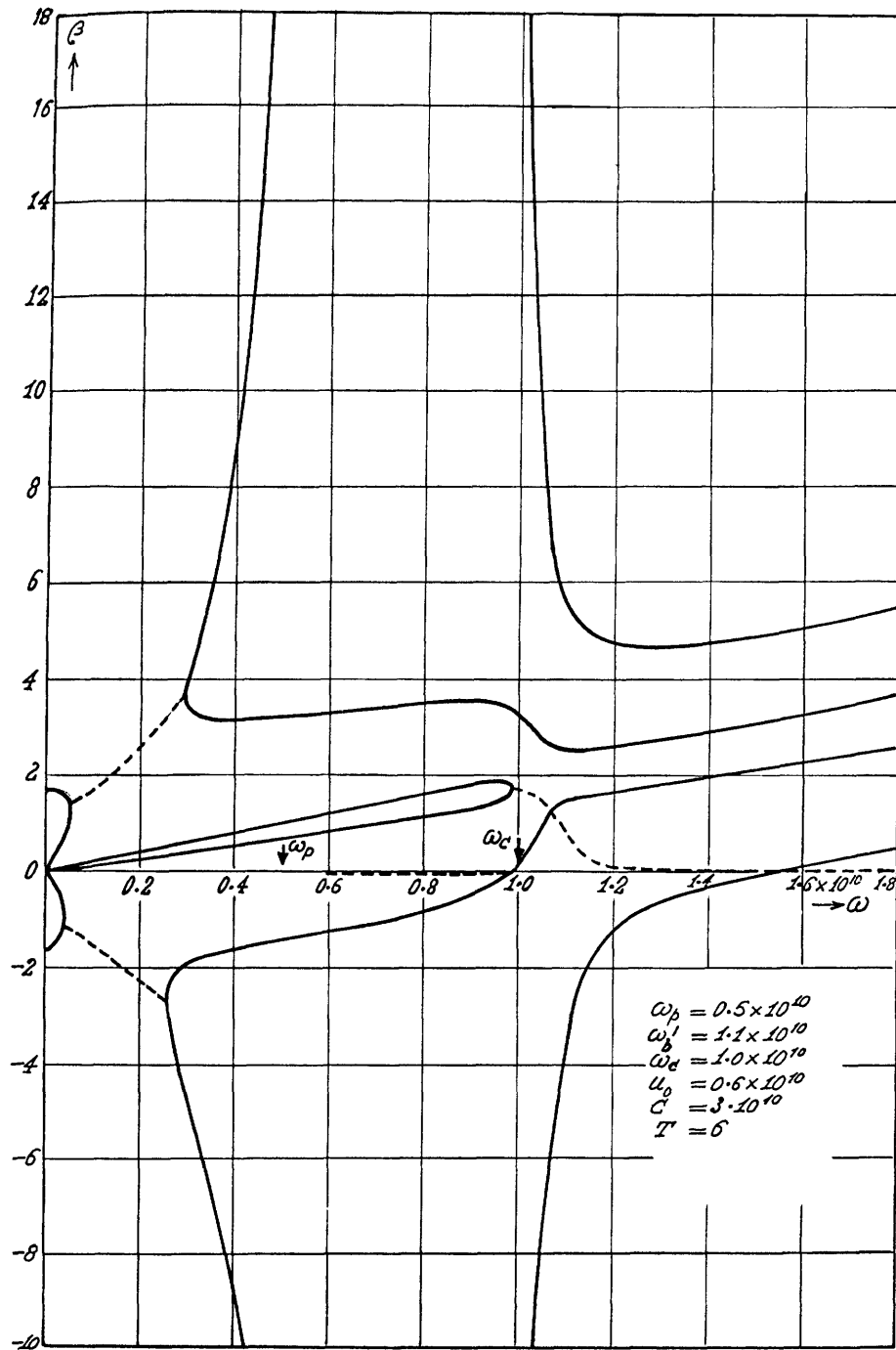


Fig. 6. Beam-plasma dispersion diagram for $u_0 = 0.6 \times 10^{10}$.

velocity. For waves faster than this velocity, the Maxwellian distribution is preferable. Since we did not use quasi statics, it is worth while also to consider relativistic distributions.¹⁰ Furthermore, the theory outlined here can be used in more complicated cases, for example, that of an electron-gas with a double-humped distribution function

$$f(u) = \frac{a}{2\pi^2} \left\{ \frac{1}{\left[(u-u_1)^2 + a^2 \right]^2} + \frac{1}{\left[(u+u_1)^2 + a^2 \right]^2} \right\} \quad (37)$$

interacting with cold ions.¹¹

We now come to the integrations that must be performed along the real axis in the u -plane. We further specify that we look for roots of ω when we assume β to be real. We now encounter a severe difficulty that escaped our attention by assuming the z - and t -dependence to be of the form $\exp\{j(\omega t - \beta z)\}$. We should have taken the Fourier-Laplace transformation

$$\hat{E}_z(\omega, \beta) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dt \hat{E}_z(z, t) e^{-j(\omega t - \beta z)}$$

and similar transformations for the other unknowns with the boundary conditions $E(z, t) = 0$, when $t < 0$ and $E(z, t)$ has a certain prescribed value for $t = 0$. If the transverse dependence of the perturbation "fits" in the mode patterns, we arrive at, instead of Eq. 22,

$$\vec{M}(\omega, \beta) \vec{E}(\omega, \beta) = \vec{B}(0, \beta), \quad (38)$$

where $B(0, \beta)$ is the description of the field at $t = 0$. This can be checked: if we take as the only boundary condition

$$v_{nz}(0, \beta, r, \theta) = \hat{v}_{nz}(0, \beta) J_0(\text{Tr}),$$

$\vec{B}(0, \beta)$ takes the form of a column vector with components $(0, 0, -j\omega \hat{v}_{nz} \omega^{-2})$. Solving (38) for E_z by using Cramer's rule, we obtain an equation of the form

$$\hat{E}_z(\omega, \beta) = \frac{\psi(\omega, \beta)}{|\vec{M}(\omega, \beta)|},$$

where $\psi(\omega, \beta)$ contains also $\hat{v}_{nz}(0, \beta)$. This equation has to be transformed back to the z, t plane by

$$\hat{E}_z(z, t) = \int_{-\infty}^{\infty} d\beta \int_{-\infty - j\gamma}^{\infty - j\gamma} d\omega E_z(\omega, \beta) e^{j(\omega t - \beta z)}.$$

By analogy with the problem of Landau damping, it can be argued that γ is such that the integration goes below all singularities in the ω plane, the singularities being given by $|\vec{M}(\omega, \beta)| = 0$. We now want to move up the integration path to the half-plane given by $\omega_i > 0$. This means that we have to know the poles of the integrals occurring in

Eqs. 23-27 and the analytic continuation of their integrands. This means, by the same arguments used by Landau,¹² that we have to deform the u -integration in such a way that the poles $u = (\omega \pm \Omega) \beta^{-1}$ are always below the integration path, which can be closed through $\omega_1 = \infty$.

The method suggested here has not yet been applied to a special case. Research in this direction is being performed.¹¹

IV. CONNECTION BETWEEN PLANE WAVES AND ROTATIONAL SYMMETRIC WAVES IN WAVEGUIDES AND SOME REMARKS ON HIGHER ORDER MODES

The determinantal equation 28, with M_{mn} given by Eqs. 23-27, appears to have the same form¹³ as the dispersion equation previously given, with T replaced by the transverse wave number β_x . The approximation described in Section III is therefore identical to studying plane waves with a given real transverse wave number through a beam-plasma system. This reminds us that the H_{01} mode in an empty waveguide can also be built up from the interference of a plane wave bouncing back and forth between the low walls of the waveguide. Of course, in our case the effect is much more complicated, but mathematically the correspondence still exists. It is therefore possible to avoid all trouble with cylindrical coordinates and start from the beginning with the consideration of plane waves in an infinite beam-plasma system.

Of course, this correspondence is only proved for rotational symmetric modes. For higher order cylindrical modes with $E_z = A J_n(\text{Tr}) e^{jn\theta}$ further study will be necessary. It seems advisable to perform such a study, since space-charge wave instabilities in transverse direction at the ion-cyclotron frequency may also occur. To understand this¹⁴ one considers the dielectric constant of a cold plasma in transverse direction:

$$\epsilon_{\perp} = -\left(M_{11} - \beta^2\right) \frac{c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}, \quad (39)$$

as can be obtained from Eq. 23. For ϵ_{\perp} to become zero and negative we have to solve

$$\epsilon_{\perp} = 0 \quad \text{or} \quad (\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2) - \omega^2(\omega_{pe}^2 + \omega_{pi}^2) = 0. \quad (40)$$

for $\omega \ll \Omega_e$ and ω_{pe} and $\omega_{pe} \gg \omega_{pi}$ this yields ($\omega \gg \Omega_i$)

$$\omega_p = \frac{\omega_{pi}^2}{\sqrt{1 + \frac{\omega_{pe}^2}{\Omega_e^2}}}. \quad (41)$$

When a beam with a transverse modulation of frequency ω_r or smaller is shot through the plasma (such a beam is in practical cases neutralized by ions) the space-charge variations in the transverse direction will give rise to electric fields. Since $\epsilon < 0$ and $\epsilon_{\perp} E$ is proportional to the space-charge variations involved, the fields will act in such a way as to increase the variation when $\epsilon < 0$. (The normal longitudinal space-charge wave interaction of a beam and a plasma can also be explained in terms of a negative ϵ .)

A final remark: Although the rotational symmetric modes in a waveguide have been shown to be described by the theory of plane waves in an infinite beam-plasma system

with constant real transverse wave number, the study of the dispersion characteristics of plane waves making an angle with the beam constitutes another problem.

V. PLANE WAVES IN INFINITE MEDIA

Having shown the importance of plane waves in the study of finite beam-plasma systems, we shall take advantage of this correspondence by deducing the determinantal equation for plane waves in an infinite beam-plasma system in which the plasma also has a transverse temperature. We shall use the Boltzmann transport equations in our description.

The momentum-conservation equation for the plasma electrons yields

$$j\omega\vec{v}_e + \frac{\vec{v}_T}{\rho_0} \nabla \rho_{1e} + \eta(\vec{v}_e \times \vec{B}_0) = -\eta\vec{E}, \quad (42)$$

where the temperature is described by

$$\vec{v}_T = \begin{pmatrix} v_p^2 & 0 & 0 \\ 0 & v_p^2 & 0 \\ 0 & 0 & v_L^2 \end{pmatrix}.$$

v_p^2 and v_L^2 are the mean-square random motions perpendicular to and along the magnetic field lines, respectively.

Since we consider plane waves of the form

$$\exp j(\omega t - \vec{\beta} \cdot \vec{r}),$$

we may replace ∇ by $-j\vec{\beta}$. Furthermore, without loss of generality, we may assume $\vec{\beta}$ to be located in the z-x plane (\vec{B}_0 and the beam are in the z-direction). Using the equation for mass conservation

$$\nabla \cdot \rho_0 e \vec{v}_e = -j\omega \rho_{1e} \quad (43)$$

to eliminate ρ_{1e} from Eq. 42 and expressing \vec{v}_e in \vec{E} yield

$$\begin{pmatrix} \omega^2 - \beta_x^2 v_p^2 & -j\omega\Omega_e & -v_p^2 \beta_x \beta_z \\ j\omega\Omega_e & \omega^2 & 0 \\ -v_L^2 \beta_x \beta_z & 0 & \omega - \beta_z^2 v_L^2 \end{pmatrix} \vec{v}_e = j\omega\eta\vec{E}. \quad (44)$$

Inversion of this equation, by using

$$\vec{A} \cdot \vec{v}_e = j\omega\eta\vec{E} \rightarrow \vec{v}_e = j\omega\eta \frac{\text{adjoint } \vec{A}}{|\vec{A}|} \vec{E},$$

and calculating the electron current $\vec{j}_e = \rho_{0e} \vec{V}_e$ yield

$$\vec{j}_e = \frac{j\omega \rho_{0e}^2 \epsilon_0}{|A|} \begin{pmatrix} (\omega^2 - \beta_z^2 v_L^2) \omega^2 & \vdots & j\omega \Omega_e (\omega^2 - \beta_z^2 v_L^2) & \vdots & v_L^2 \omega^2 \beta_x \beta_z \\ -j\omega \Omega_e (\omega^2 - \beta_z^2 v_L^2) & \vdots & \omega^2 (\omega^2 - \beta_x^2 v_p^2 - \beta_z^2 v_L^2) & \vdots & -j\omega \Omega_e v_L^2 \beta_x \beta_z \\ \omega^2 v_p^2 \beta_z \beta_x & \vdots & j\omega \Omega_e v_p^2 \beta_x \beta_z & \vdots & \omega^2 (\omega^2 - \beta_x^2 v_p^2 - \Omega_e^2) \end{pmatrix} \vec{E}. \quad (45)$$

Here,

$$|A| = \omega^2 \left\{ (\omega^2 - \beta_z^2 v_L^2) (\omega^2 - \beta_x^2 v_p^2 - \Omega_e^2) - \beta_x^2 \beta_z^2 v_L^2 v_p^2 \right\}.$$

The convection current carried by the plasma ions can be obtained from Eq. 45 by setting the temperature equal to zero and replacing Ω_e by Ω_i and ω_{pe}^2 by ω_{pi}^2 :

$$\vec{j}_i = \frac{j\epsilon_0 \omega_{pi}^2}{\omega^2 - \Omega_i^2} \begin{pmatrix} -\omega & -j\Omega_i & 0 \\ j\Omega_i & -\omega & 0 \\ 0 & 0 & -\frac{\omega^2 - \Omega_i^2}{\omega} \end{pmatrix} \vec{E}. \quad (46)$$

For the beam electrons, for the relation between \vec{j}_b and \vec{E} , we use¹⁵

$$\begin{pmatrix} j_{nx} \\ j_{ny} \\ j_{nz} \end{pmatrix} = \frac{j\epsilon_0 \omega_{bn}^2}{\omega_{\beta n}^2 - \Omega_n^2} \frac{\omega \beta_n}{\omega} \begin{pmatrix} -\omega \beta_n & j\Omega_n & -\beta_T u_n \\ -j\Omega_n & -\omega \beta_n & -j \frac{\Omega_n}{\omega \beta_n} \beta_T u_n \\ -\beta_T u_n & j \frac{\Omega_n}{\omega \beta_n} \beta_T u_n & \frac{\omega^2}{\omega \beta_n} \left(-\frac{\beta_T^2 u_n^2}{\omega^2} - \frac{\omega_{\beta n}^2 - \Omega_n^2}{K_n^2 \omega_{\beta n}^2} \right) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \quad (47)$$

Similarly, we now obtain

$$\begin{pmatrix} \epsilon_{11} - \frac{\beta_z^2}{K^2} & \epsilon_{12} & \epsilon_{13} + \frac{\beta_x \beta_z}{K^2} \\ \epsilon_{21} & \epsilon_{22} - \frac{\beta_x^2 + \beta_z^2}{K^2} & \epsilon_{23} \\ \epsilon_{31} + \frac{\beta_x \beta_z}{K^2} & \epsilon_{32} & \epsilon_{33} - \frac{\beta_x^2}{K^2} \end{pmatrix} \vec{E} = \vec{M} \cdot \vec{E} = 0, \quad (48)$$

where

$$\epsilon_{11} = 1 - \frac{\omega_{\beta b}^2}{\omega^2} \frac{\omega_{pb}^2}{\omega_{\beta b}^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{|A|} \omega^2 (\omega^2 - \beta_z^2 v_L^2) \quad (49)$$

$$\epsilon_{22} = 1 - \frac{\omega_{\beta b}^2}{\omega^2} \frac{\omega_{pb}^2}{\omega_{\beta b}^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{|A|} \omega^2 (\omega^2 - \beta_x^2 v_p^2 - \beta_z^2 v_L^2) \quad (50)$$

$$\epsilon_{33} = 1 - \frac{\omega_{pb}^2}{K^2 \omega_{\beta}^2} - \frac{\omega_{pb}^2}{\omega_{\beta b}^2 - \Omega_e^2} \frac{\beta_x^2 u_o^2}{\omega^2} - \frac{\omega_{pe}^2}{|A|} (\omega^2 - \beta_x^2 v_p^2 - \Omega_e^2) \quad (51)$$

$$\epsilon_{12} = j \frac{\omega_{\beta b} \Omega_e}{\omega^2} \frac{\omega_{pb}^2}{\omega_{\beta b}^2 - \Omega_e^2} - j \frac{\Omega_i}{\omega} \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} + j \frac{\Omega_e \omega}{|A|} (\omega^2 - \beta_z^2 v_L^2) = \epsilon_{21}^* \quad (52)$$

$$\epsilon_{13} = - \frac{\beta_x u_o \omega_{\beta b} \omega_{pb}^2}{(\omega_{\beta b}^2 - \Omega_e^2) \omega^2} - \frac{v_L^2 \omega^2 \beta_x \beta_z \omega_{pe}^2}{|A|} = \epsilon_{31}^* \quad (53)$$

$$\epsilon_{23} = -j \frac{\Omega_e \beta_x u_o}{\omega_{\beta b}^2 - \Omega_e^2} \omega_{pb} - j \frac{\omega_{pe}^2}{|A|^2} \omega \Omega_e v_L \beta_x \beta_z = \epsilon_{32}^* \quad (54)$$

This yields the dispersion equation, since for a nontrivial solution the determinant of the matrix in Eq. 48 must be zero. Although this equation is very lengthy, it can be programmed without any essential trouble by a computer. For simulating a finite beam-plasma system we need only put a fixed value for β_x into the equations. Although this does not give the actual dispersion characteristics, we may expect the same couplings between the different waves to occur so that the same instabilities occur. Equations 48-54 can be easily applied to the case in which a longitudinal temperature only is taken into account by taking V_p to be zero.

We have then described the same case as before, the only difference being that now the dispersion equations are based on the Boltzmann transport equations. Formally, once we have decided to consider plane waves, we could obtain a description based on the Boltzmann equations. The dielectric tensor of a warm plasma, including the finite Larmor radius, as deduced by Stix,¹² can then be used together with the beam dielectric tensor (47). We can combine these equations into the form given in Eq. 48 (which is identical to Eq. 28). A careful study of these equations might give information about the question whether it is possible to couple an electron beam with the harmonics of the cyclotron frequencies. According to Landauer's experiment,¹⁶ there are resonances at these frequencies; this means that coefficients of the dielectric tensor become negative, so that a reactive medium amplification may occur at these frequencies.

A final remark: The determinantal equation (48) appears to be of the fourth degree in T^2 (or β_x^2 as in the case of plane waves) so that more boundary conditions are needed. This problem is not yet solved. When we take the transverse temperature to be zero, of course, we come back to the case of Section III, and we hope that equations 48-54 may serve as a first indication of what happens when transverse temperature becomes important.

VI. INTERACTION BETWEEN A THIN BEAM AND A PLASMA

Besides the models discussed here one might think of another model than that of plane waves in an infinite beam-plasma system, namely the interaction of a thin beam and a plasma. Since we are interested in low-frequency interactions, the wavelength of the waves along the beam is usually much larger than the beam diameter, which makes the thin-beam approximation reasonable. Our model will be a waveguide filled with a plasma in which an electron beam is fired along the axis (z-direction) and may be seen as an extension of an earlier model.¹⁷ The beam is thought of as infinitely thin. The plasma electrons are assumed to have a very high temperature so that they unshield the ions completely and enter into our discussions only as a neutralizing background. With the existing theory of a hot, plane-waveguide we can easily extend the model to finite electron temperatures.

Our method will be to use the coupling-of-modes theory which allows us to solve for the two subsystems separately, both of which are well known.^{18,19} We must be careful, however, because strong interactions are not described by this theory (for example the reactive medium or space-charge wave amplification). Some of these interactions do appear in a model in which an infinite longitudinal magnetic field has been used.²⁰

6.1 WAVES ON THIN BEAMS

A thin beam can carry 6 waves, two longitudinal and four transverse waves. For our purposes, we only need to consider a thin beam streaming along the axis of the waveguide and in the direction of external magnetic field. For the longitudinal waves we know that the dispersion equation is

$$\beta = \frac{\omega \pm F\omega_{pb}}{u_0}. \quad (55)$$

The power flow is given by Haus and Bobroff.²¹ For longitudinal modes this power flow is

$$P_{Lb} = -\frac{1}{2} \operatorname{Re} \frac{u_0 v_b}{\eta} j_1^* = -\frac{1}{2\eta} I_0 \frac{\omega}{|\omega_\beta|^2} v_b v_b^* \operatorname{Re} \omega_\beta, \quad (56)$$

where use has been made of the continuity equation.

For the transverse waves we need only consider the equation of motion because no bunching is involved. If we assume that \vec{r}_T is the displacement of an electron from its unperturbed position, the equation of motion of an electron having a drift velocity u_0 in the direction of the magnetic field is²²

$$\frac{d^2 \vec{r}_T}{dt^2} = \frac{d\vec{r}_T}{dt} \times \vec{\Omega}_e \quad \text{or} \quad (\vec{\Omega}_e = \eta \vec{B}_0)$$

$$\frac{\partial^2 \vec{r}_T}{\partial t^2} + 2u_0 \frac{\partial^2 \vec{r}_T}{\partial t \partial z} + u_0^2 \frac{\partial^2 \vec{r}_T}{\partial z^2} = \left(\frac{\partial \vec{r}_T}{\partial t} + u_0 \frac{\partial \vec{r}_T}{\partial z} \right) \times \vec{\Omega}_e. \quad (57)$$

Assuming \vec{r}_T to be of the form $\vec{r}_T = \vec{r}_T e^{j(\omega t - \beta z)}$, separating Eq. 57 into equations for the x and y components of \vec{r}_T , and defining

$$\begin{aligned} r_+ &= x + jy & \text{or} & & x &= \frac{1}{2}(r_+ + r_-) \\ r_- &= x - jy & & & y &= \frac{1}{2}j(r_+ - r_-) \end{aligned}$$

after adding or subtracting the component equations, we obtain

$$\begin{aligned} (\beta - \beta_e + \beta_c)(\beta - \beta_e) \vec{r}_+ &= 0 \\ (\beta - \beta_e - \beta_c)(\beta - \beta_e) \vec{r}_- &= 0. \end{aligned} \quad (58)$$

The first equation describes the right, and the second the left circularly polarized waves.

Power flow in cyclotron waves ($\beta = \beta_e \pm \beta_c$) and synchronous waves ($\beta = \beta_e$) is given by²¹

$$\begin{aligned} P_{Tb} &= -\frac{1}{2} \text{Re} \left\{ \frac{u_0 v_T}{\eta} - \frac{u_0}{2} (\vec{r}_T \times \eta \vec{B}_0) \right\} (-j\omega \rho_0 \vec{r}_T^*) = \\ &= -\frac{1}{2} \text{Re} \left\{ \frac{u_0 \rho_0}{\eta} \omega \omega_\beta \vec{r}_T \vec{r}_T^* + j \frac{u_0 \rho_0}{2\eta} \Omega_e (\vec{r}_T^* \times \vec{r}_T) \omega \right\}. \end{aligned} \quad (59)$$

For the right circularly polarized waves this power, calculated by using $x = jy$ and $\vec{r}_T = \vec{e}_1 x + \vec{e}_2 y$ (\vec{e}_1 and \vec{e}_2 being unit vectors in the x and y directions) is

$$\begin{aligned} \vec{r}_T \vec{r}_T^* &= 2x^2 \\ \vec{r}_T^* \times \vec{r}_T &= 2j(\text{Re } \vec{r}_T \times \text{Im } \vec{r}_T) = \hat{x}^2 = \frac{1}{2} \vec{r}_T \vec{r}_T^*. \end{aligned}$$

This yields after substitution in Eq. 59

$$P_{Tb} = -\frac{1}{2} i_0 \omega \left(\omega_\beta \mp \frac{\Omega_e}{2} \right) \vec{r}_T \vec{r}_T^* = -\frac{u_0}{2\eta} i_0 \omega \left(\beta - \beta_e \mp \frac{\beta_c}{2} \right) \vec{r}_T \vec{r}_T^*. \quad (60)$$

Now the upper sign holds for the right (+) and the lower sign for the left (-) circularly polarized waves. From this equation we find for the different waves

Wave	(+)	(-)
cyclotron	$-\frac{1}{4} i_0 \omega \Omega_e \vec{r}_T \cdot \vec{r}_T^* = -\frac{i_0}{4\eta} \frac{\omega}{\Omega_e} \vec{v}_T \cdot \vec{v}_T^*$	$+\frac{i_0 \omega \Omega_e}{4} \vec{r}_T \cdot \vec{r}_T^* = \frac{i_0 \omega}{4\eta \Omega_e} \vec{v}_T \cdot \vec{v}_T^*$
synchronous	$\frac{1}{4\eta} i_0 \omega \Omega_e \vec{r}_T \cdot \vec{r}_T^*$	$-\frac{1}{4} i_0 \omega \Omega_e \vec{r}_T \cdot \vec{r}_T^*$

The current associated with the different waves is $\vec{j}_T = \rho_0 \vec{v}_T$ for the cyclotron waves. When the beam is fired along the axis of the waveguide, which coincides with the z-axis, the current will be in the θ -direction and has no r component. The current associated with the synchronous modes is zero, since $|v_T| = j\omega_\beta |r_T|$ and $\omega_\beta = 0$. Therefore these waves do not couple with any waveguide mode. The longitudinal modes have a current $j_z = \rho_0 \frac{\omega}{\omega_\beta} v_z$.

We may now define normalized amplitudes in such a way that the dot product with their complex conjugate gives the power flow. We shall use this result after a short discussion of the quasi-static treatment of waves in an ion-loaded waveguide.

6.2 ION-LOADED WAVEGUIDE

Because of the high electron temperature, we only consider the ions in the waveguide. Using Briggs' techniques,⁴ we can easily extend the theory to finite electron temperatures. The relation between \vec{j} and \vec{E} for ions is given by Eq. 46, where the field \vec{E} is to be derived from a potential, since we assume $\nabla \times \vec{E} = 0$ in the quasi-static approximation

$$\vec{E} = -\nabla \Phi. \quad (61)$$

This potential can be solved from Poisson's equation

$$\nabla^2 \Phi = \frac{\rho_i}{\epsilon_0}, \quad (62)$$

in which ρ_i , the AC ion density, follows from the continuity equation

$$\nabla \cdot \vec{j}_i = -j\omega\rho_i. \quad (63)$$

Combination of Eqs. 46, 62, and 63, by eliminating all variables but Φ , yields

$$\nabla_T^2 \Phi + T^2 \Phi = 0, \quad (64)$$

where

$$T^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} \right) = -\beta^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2} \right). \quad (65)$$

Equation 64 yields for circular cylindrical geometry

$$\Phi_{mn} = A J_n(T_{mn}r) e^{jn\theta} e^{j(\omega t - \beta z)}. \quad (66)$$

The boundary condition is $\Phi = 0$ when $r = R$, so that $T_{nm} = \frac{P_{nm}}{R}$. Here, P_{nm} gives the m^{th} zero of the n^{th} Bessel function. The other components are given by

$$E_r = -\frac{\partial \Phi}{\partial r} = -A T_{mn} J'_n(T_{mn}r) e^{jn\theta}$$

$$\begin{aligned}
E_\theta &= -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = \frac{jn A}{r} J_n(T_{mn} r) e^{jn\theta} \\
E_z &= -\frac{\partial \Phi}{\partial z} = -j\beta A J_n(T_{mn} r) e^{jn\theta}.
\end{aligned} \tag{67}$$

The AC magnetic field is obtained from Maxwell's equations for curl H. Defining the sum of the displacement current and the convection current to be equal to $j\omega\epsilon_0\epsilon_r E$, we have

$$\begin{aligned}
H_{1\theta} &= -\frac{\omega}{\beta} A \left\{ -T_{mn} J_n'(T_{mn} r) \epsilon_{11} - \frac{n}{r} J_n(T_{mn} r) \epsilon_{12} \right\} e^{jn\theta} \\
H_{1r} &= j \frac{\omega}{\beta} A \left\{ T_{mn} J_n'(T_{mn} r) \epsilon_{21} + \frac{n}{r} J_n(T_{mn} r) \epsilon_{22} \right\} e^{jn\theta}.
\end{aligned} \tag{68}$$

The power flow is obtained from

$$P_p = \frac{1}{2} \text{Re} \int_0^{2\pi} \int_0^R (E_r H_\theta^* + E_\theta H_r^*) 2\pi r dr d\theta. \tag{69}$$

6.3 COUPLING-OF-MODES THEORY

We now apply coupling-of-modes theory as described by Haus.¹⁹ Using the same notation, we have for the coupling constant

$$C_{21} = -\frac{1}{4} \frac{\vec{E}_p \cdot \vec{i}_b}{a_p \cdot a_b^*}. \tag{70}$$

Here, C_{21} is the growth rate for the point at which both uncoupled waves have the same phase velocity.²³ Applying this to the cyclotron waves of the beam, we have

$$\begin{aligned}
C_{21} &= \frac{1}{4} \frac{E_\theta j\theta}{a_p \cdot a_b^*} \\
&= \frac{4jn\eta J_n(T_{mn} r) e^{jn\theta} \Omega_e}{u_o \omega r \sqrt{P'_p}},
\end{aligned} \tag{71}$$

where

$$P'_p = -\frac{\omega}{\beta} \pi \text{Re} \int_0^R 2\pi r \left\{ T_{mn} J_n'^2(T_{mn} r) \epsilon_{11} - \frac{n^2}{r^2} J_n^2(T_{mn} r) \epsilon_{22} \right\} dr. \tag{72}$$

From Eq. 71, we see that C_{21} is proportional to $\frac{J_n(T_{mn} r)}{r}$, which quotient is zero for $n > 1$ at $r = 0$. We need the value $r = 0$ because we took an infinitely thin beam. In the next approximation we can let $0 < r \ll R$. For E_θ we then can take $\int_0^{r_b} \frac{J_n(T_{mn} r)}{r} dr$,

in which it is clear that the interaction decreases with increasing order. The frequencies at which the interaction takes place have been given elsewhere for an infinitely wide waveguide.^{17, 24}

VII. CONCLUSIONS AND SUGGESTIONS

Most of the research described here consists of suggestions for further investigation of beam-plasma interactions. For rotational symmetric waves in cylindrical waveguides filled with beam-plasma systems the equations deduced are ready for numerical calculation. We point out, however, that nonrotational symmetric modes may yield interactions between an electron beam and the ions in a plasma so that a study of these interactions is cogent. As a matter of fact, the interaction found by launching an electron beam in a cyclotron discharge is most likely to be of this kind.²⁵

The approximate dispersion equation for rotational symmetric modes in cylindrical waveguides is equal to that of plane waves in an infinite system, provided that the wave vector has a constant real transverse component. When this is recognized, we can include transverse temperature in the description of beam-plasma systems.

Some nonrotational symmetric interactions can be studied by using a thin-beam model. Application of coupling-of-modes theory then yields the increment of the perturbation on the thin beam. The thin-beam approximation makes sense, since at the ion-cyclotron or plasma frequencies the wavelength along the beam is usually much larger than the beam diameter.

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